Abraham ARCAVI, Israel

Making Sense of Sensemaking in Mathematics Education

"the ... essentially universal need for meaning, and the need to understand ourselves and the world around us,... [are] the driving force behind all our intellectual activities" (Sfard, 2001, p. 356)

1. Introduction

The key word in the title above is sensemaking. Not many languages have a word that conveys the denotation intended in English. For example, in Spanish (my mother tongue) there is no word for sensemaking, neither there is one in Hebrew. A small survey led me to the following possible translations into German: Sinn machen, Sinn schaffen or preferably (as I was told) Sinnstiftung. In this presentation, I start by illustrating sensemaking through four vignettes, then I present some attempts at definitions and proceed to delineate some instructional principles to promote sensemaking in mathematics classrooms.

2. Four vignettes and a moral

- First vignette: The British mathematician and logician Augustus De Morgan (1806 – 1871) told the following story.

  Diderot paid a visit to Russia at the invitation of Catherine the Second. At that time he was an atheist, or at least talked atheism… His lively sallies on this subject much amused the Empress, and all the younger part of her Court. But some of the older courtiers suggested that it was hardly prudent to allow such unreserved exhibitions. The Empress thought so too, but did not like to muzzle her guest by an express prohibition: so a plot was contrived. The scorner was informed that an eminent mathematician had an algebraic proof of the existence of God, which he would communicate before the whole Court, if agreeable. Diderot gladly consented. The mathematician…was Euler. He came to Diderot, with the gravest air, and in a tone of perfect conviction said, "Monsieur, $\frac{a+b^n}{n}=x$ donc Dieu existe; respondez!" ("Monsieur, (a+bn)/n=x, whence God exists; answer that!"). Diderot, to whom algebra was Hebrew, ... and whom we may suppose to have expected some verbal argument of alleged algebraical closeness, was disconcerted; while peals of laughter sounded on all sides. Next day he asked permission to return to France, which was granted." (De Morgan, 1915, p. 339).

- Second vignette: The following cartoon (distributed anonymously in the web) depicts ways in which students may approach symbol use and manipulation in algebra.

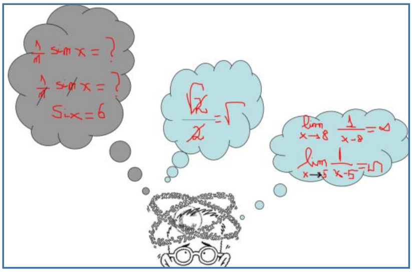


Figure 1: Cartoon depicting symbol manipulation

This is a caricaturesque exaggeration in order to call attention to the potential meaninglessness of algebraic symbols for students. Indeed, there is research evidence that students misapply and/or overgeneralize certain properties, for example: $a + b^2 = a^2 + b^2$ and $2x + 5 \rightarrow 7x$.

- Third vignette:

I thought of a number, I multiplied it by 3, I subtracted 10 and I got 5. What was my number?

This is a rather typical problem given to students in introductory lessons to the concept of equations and how to solve them. Students, even those with difficulties in mathematics, can solve the problem by mentally reversing the operations. Thus, in this case, they first add 10 and then divide by 3 to arrive at the correct solution. Usually teachers (and also researchers), who want to motivate the introduction of algebraic rules for solving linear equations as a general approach to equation solving, decide to challenge students with problems like the following:

I thought of a number, I multiplied it by 3, I subtracted 16 and I got the number I had at the beginning. What was my number?

In this case, the unknown is in both sides of the equation, and mental reversing would not work as easily. The expectation is that the challenge would encourage students' readiness to accept the use symbols as a safe, general and efficient way to solve. In several cases (see Karsenty, Arcavi & Hadas, 2007, pp. 162-3), students may surprise teachers by solving this problem also mentally and rather quickly, for example, by saying "Sixteen is twice eight, out of three times the number sought", and find the solution.

- Fourth vignette: A student facing the equation $\frac{2x+3}{4x+6} = 2$ instead of starting to manipulate the symbols to solve it, he paused and tried to "read through" the symbols. He noticed that the numerator is always one-half of the denominator, therefore this equation cannot have a solution. Then, he tried to 'solve' it anyway, namely to apply the mechanical procedures to see how, in this case, they lead to x=…. He seemed to express the need to feel the way in which the algebra displays the absence of solution. Unfortunately, the algebra is not very forthcoming: technical manipulation yields $x = -1\frac{1}{2}$. Our student was puzzled by the contradiction between his two solutions, and it took him a while to resolve it. What he did was to substitute -1.5 and realized that this is precisely the value which should be excluded from the very beginning (Arcavi, 1995).

  The four vignettes illustrate instances of sensemaking. The first shows how, sometimes, attempting to react to a simple piece of mathematics which does not make any sense may imply rebelling against a highly respected mathematical authority, and thus running the risk of exposing oneself as an illiterate or unclever person. If we disregard the ludicrous aspect of the second vignette and analyze it deeper, one can discern attempts to make sense, regardless of how clumsy these attempts may seem. These attempts seem to reflect an effort to relate new and unknown material to known rules and to previous knowledge. The third vignette illustrates how strongly some students can cling on to their own idiosyncratic and informal ways at sensemaking and succeed. The fourth vignette shows how making sense implies "reading symbols" first and then finding connection between two very different ways of approaching the same problem in order to search for coherence and settling discordances.

These four vignettes display very different types of reactions to mathematical situations. However, we claim that they share one important common characteristic about human thinking: whether one attempts or shun attempts at sensemaking, and whether these attempts succeed or fail, humans are willing to and do relate reasonably to mathematics (and to themselves with respect to mathematics) and act accordingly.

3. Toward a characterization of sensemaking

The four vignettes above illustrate how sensemaking may be enacted by different people. In this section, I attempt to describe what sensemaking may be. But first, a word of caution: attempts at defining any complex idea relies on other no less complex ideas which require definitions of their own. In our case, such words are 'understanding', 'meaning', 'knowledge', 'connecting',

‘learning’ and more. Due to space limitations, I will not undertake the task of unpacking these words. I will rather assume that our reliance undefined terms (and which have multiple connotations) but still commonly used, may perhaps dispel some of the fog around the elusive idea of sensemaking.
Sensemaking can be seen as the process by which a person assigns meaning to perceived phenomena. It entails connecting new circumstances or new knowledge to existing knowledge, ideas and beliefs. It entails mostly of (some or all) of the following activities (enacted with oneself or with others): explicating intuitions, providing informal explanations, proposing inductive observations and/or formal deductions (see, for example, https://www.nctm.org/store/Products/Focus-in-High-School-Mathematics--Reasoning-and-Sense-Making-in-Algebra/).

Sensemaking requires situational awareness especially when experiencing uncertainty. It includes identification of attributes and qualities and the recognition of patterns. Sensemaking harnesses curiosity, agency, creativity and mental modeling toward the coping sensibly with hazy ideas. It implies acting, thinking or communicating (even when such thinking/acting are not necessarily “correct”). Thus, sensemaking is a loyal companion to learning.

Sensemaking is not necessarily an individual process, it can be eminently social in nature. Sensemaking can be considered as ‘stories’ preserved and shared and thus its audience can be both the speakers themselves and those who listen to it and help shape/reshape such a story, becoming an evolving product of conversations (e.g. Sfard, 2012).

4. Sensemaking and achievements

Sensemaking is a subjective activity and it does not seem to correlate with achievements in mathematics. In this respect diSessa (2000, p.107) claims:

> *“The most disturbing thing I uncovered in a study of bright, motivated, and successful MIT undergraduates years ago was that, although all did well in high school physics and got high marks, almost none felt they really understood the material … teachers just did not work on intuitive judgements and sense making…*”

Conversely, as in the third vignette above, students considered as ‘low achievers’ can informally reason in order to solve a not so simple linear equation in which the unknown is on both sides of the equal sign. From these two testimonies, one can conclude that, at least sometimes, sensemaking does not lead a student to high achievements and conversely, high achieving students may not make sense of the subject in which they “succeed”.

5. A brief interim summary

- All human beings are sensemakers
- Sense making may take many forms
- Sensemaking is a relevant construct for both the cognitive and the socio-cultural perspectives on thinking and learning
- Sensemaking and achievement do not necessarily correlate

The above may suffice to conclude that nurturing sensemaking can and should be an integral part of mathematics education. Thus, how should sensemaking be integrated as a component of mathematical instruction? In the following, we propose some instructional principles on this respect.

6. Some instructional principles

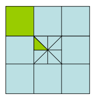
- <u>'Seeing' in different ways</u>

Let us consider the following example from elementary arithmetic (Arcavi, 2015): to calculate $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}$. This exercise which can be solved in several ways, for example:

$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}= \frac{1}{9}+\frac{1}{72}=\frac{8+1}{72}=\frac{1}{8}$ or $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}= \frac{1}{9}\left(1+\frac{1}{8}\right)=\frac{1}{9}\cdot\frac{9}{8}=\frac{1}{8}$.

Also, if one knows the conditions for which the subtraction of two numbers yields the same result as its product, one can calculate as follows: $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{9}+\frac{1}{8}-\frac{1}{9}= \frac{1}{8}$.

In this visual representation, the painted area indicates 1/9 of the whole (the upper painted cell) and 1/8 of 1/9 (the painted triangle in the middle cell).



Depending on how we look at this diagram, it offers a representation of either the left hand side of the equality (the conjunction of two pieces, the upper left colored cell and the small triangle within the central cell, namely the addition $\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}$) or its right hand side (the eighth part of a whole, namely the result of the calculation, since the whole can be regarded as assembled out of eight combinations of one square cell and a small triangle attached to it). This geometrical representation may 'make more sense' to many students than a concatenation of formal operations, since it visually connects the operation to the meaning of a fraction as part of a whole and provides a global insight of what is being calculated and how.). In other words, the figure may resonate with our inner feeling of understanding because of its visual nature, it represents both the calculation (a process), its result (a product) and its explanation.

In a similar spirit, Papert (1980, p. 144) poses the string around the earth problem which elicits a compelling intuitive sense running counter to the simple algebraic solution, and potentially hindering sensemaking. An eye-opening geometrical representation offered by Papert can be very helpful in conciliating the intuition with the surprising algebraic result.

There are many more examples from school mathematics (e.g. Arcavi, 2003) in which displaying a visual representation of a formal symbolic procedure/property may nurture the habit, the joy of and our trust in sensemaking. Thus a general instructional principle is to help students to see and to experience the same mathematical phenomenon in different ways in order to facilitate sensemaking by connecting ideas/perspectives.

- <u>Establishing social norms</u>

  Given the function f(x)=1/x. P is a point on its graph (in the first quadrant). A tangent line through P creates (with the axes) a right-angled triangle. What should the coordinates of P be, in order for the hypotenuse of that triangle to be maximum/minimum?

A student started to solve this problem by sketching a graph, and guided by it, he found the derivative of the function in order to create the equation of the tangent line. He then looked for the coordinates of the intersection points with the axes and wrote the expression for the segment (the hypotenuse of the triangle). This is a rather laborious procedure which he performed correctly. At this point in the solution, he stopped to decide whether he was looking for maximum or a minimum of that expression. He looked at the graph, this time not as an orientation for what to do but as a tool for reasoning. He noticed (showing it with his hands) that when P slides (towards either very small or very large values of x), the hypotenuse grows indefinitely (thus no maximum), and since the situation is symmetrical, there should be a minimum at (1,1). Problem solved! When asked why he did not start with such a reasoning instead of his laborious alternative, he replied: "*I have a friend who always does that [plays with the problem and makes sense of it], after such an effort, he usually has neither time nor energy to do the symbols, he does not get credit for what he may have done and fails the exams. If I don't have to, I do only the symbols, which is what the teacher and the exam want*."

This reply may illustrate student perceptions of classroom norms which implicitly or explicitly suppress attempts at sensemaking. Classroom norms should openly nurture and reward sensemaking of the kinds this student was capable of enacting but consciously discarded as not appropriate to what he perceived was expected of him.

- Promoting agency and autonomy

26 sheep and 10 goats are on a ship. How old is the ship's captain?

The first vignette above refers to the extent of which mathematics can be intimidating even to scholarly people. Similarly, this problem (quoted by Verschaffel et al., 2000) illustrates a breach of the "didactical contract" (e.g. Brousseau et al., 2014). Students are used to the habit that any problem posed in a mathematics class can and must be solved by applying a known operation or an equation. Indeed, many students did so with this problem. I suggest that what is at stake here is that students did not harness their sensemaking to assert their autonomy and reject the legitimacy of the problem, since the "contract" established that problems in mathematics have ways to be solved by procedures learned in class. Classroom should legitimize students questioning, the expression of their misunderstandings and puzzlements in order to elucidate what is not clear and thus to support the development of autonomous sensemakers of mathematics.

- Nurturing intellectual patience

Tobias (1990) investigated why successful scholars in the humanities would not become science scholars. Among other things, she found that part of the answer has to do with the capability (or cognitive style) to live with partial understandings for long periods of time, until meanings are connected and a large picture emerges, a style more typical of science students than of those who excel in the humanities. Recognizing this difference in cognitive styles and approaches to learning may nurture sensemaking by supporting intellectual patience towards partial understandings and confidence that further actions (not totally clear at the beginning) will have its future rewards. The expectation of instant understanding by both students and teachers, and the non-recognition of different cognitive styles may be an enemy of sensemaking.

- Providing a sense of purpose

Students should be presented with purposeful tasks and engaging problems with meaningful outcome for them. Advancing purposefulness has the potential to support student empowerment (e.g. Arcavi, 2008). Empowerment and purposefulness promote ownership of one's activities, which are not merely dictated by external sources of authority (teachers, textbooks) but rather driven by one's decision when and how it makes sense to use certain knowledge.

Coda

We need additional research that will characterize, analyze both sensemaking and the 'tools' to nurture it, for example: perception, intuition, visualization, language, colloquial language, analogies, daily experience, sensible thinking, the use of representations and idiosyncratic strategies, and communication and discussion among students. In my view, this has a strong potential to advance mathematics teaching and learning.

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