Pfister Numbers over Rigid Fields, under Field Extensions and Related Concepts over Formally Real Fields

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Dissertation

Pfister Numbers over Rigid Fields, under Field Extensions and Related Concepts over Formally Real Fields

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1. Preface

It seems that any deeper investigation of the algebraic theory of quadratic forms requires a thorough knowledge of the theory of Pfister forms.
W. Scharlau, Quadratic and Hermitian Forms, page 142.

The above quotation pretty much sums up the main motivation for the upcoming thesis in just one sentence. The main goal for an algebraist working in the theory of quadratic forms is to understand the Witt ring of a field. This seems to be a really tough task, since even nowadays, approximately 90 years after the invention of the Witt ring, there are a lot of people researching in the beautiful area of quadratic forms to solve many different interesting problems concerning the Witt ring.
As it seems to be pretty hard to study that Witt ring of a field in all its completeness, many researchers started to investigate the filtration of the Witt ring given by the ideals I^nF that are generated by the so-called Pfister forms. To understand the complexity of these ideals, a common way is to take a form φ lying in this ideal and to look for the minimal number of Pfister forms that are actually needed to represent the Witt class of φ. This minimal number is called the *Pfister number* of φ and the main object in this thesis.
For $n \in \{1, 2\}$, the problem to determine Pfister numbers is completely solved. For $n = 3$, we only have results for forms of small dimension. For forms of dimension 16 in I^3F and higher, only some few partial results are known and even these require some heavy tools from related areas such as algebraic geometry, algebraic groups or even category theory. For $n \geq 4$, only a few very special cases can be handled so far. It is thus convenient to first consider some easy cases rather than trying to solve the problem in all its generality. This is where this thesis starts. We will consider fields that are easy enough to develop special calculation techniques so that we can determine Pfister numbers of forms over such fields.
Based on this research, some related problems occurred that are also approached.

We now would like to give a quick overview of the thesis.
The upcoming chapter is used to fix notation and recapturing standard results from the algebraic theory of quadratic forms that will be essential in the sequel.

In the third chapter, we give a short introduction to the current state of research concerning the main problem and give some first further results that are obtained by adapting known techniques.

Chapter 4 is completely devoted to so-called rigid fields. When restricting to this interesting class of fields, we are able to exactly determine the Pfister numbers for low-dimensional forms lying in an arbitrary high power of the fundamental ideal. We further provide upper bounds for the Pfister number for any form of higher dimension.

We change our point of view to ask a modified question in the next chapter. Given a field extension E/F and some form in I^nE, what can we say about its Pfister number if we presume knowing the behaviour of Pfister numbers over F? In fact, for several important types of field extensions, including quadratic extensions, we can express the Pfister numbers over the bigger field in terms of Pfister numbers over the base field.

The starting point for Chapter 6 is Karim Becher's article [Bec04]. Here, he used so called supreme Pfister forms to bound invariants that are related to Pfister numbers. Also with the upcoming chapter in mind, we adapted this concept to formally real fields. Even though we cannot deduce some further facts on the Pfister numbers itself, the developed theory leads to some examples of fields with interesting properties that reoccur in the last chapter.

In Chapter 7, we extend the usual question to real fields while considering signature ideals and further generalizations of the fundamental ideal. To do so, we restrict ourselves to Pfister forms that are compatible with a given set of orderings. We try to generalize known results and provide examples in those cases where the results cannot be transferred to our new setting.

Finally, in the appendix, we provide some important results from related theories that we use as tools in the main text. Primarily, these are results that are easy but technical consequences of well-known theorems that are rather non-standard.

I would like to express my gratitude to my advisor Prof. Dr. Detlev Hoffmann for creating a pleasant working atmosphere and giving helpful advice whenever needed. Further I would like to thank my colleagues, especially Marco Sobiech, for helpful dialogues, both mathematically and non-mathematically. In addition I would like to thank Prof. Dr. Karim Becher for his interest in my work and inspiring discussions. Finally, I would like to thank my friends and my family and particularly my wife for supporting me on my way.

2. Important Results in the Theory of Quadratic Forms

2.1. Basic Notation and General Facts

In this chapter, we will introduce some basic notation and record some fundamental results that will be used later. The notation is rather standard and compatible with at least one of the main sources [Lam05], [EKM08] or [Sch85]. Further, we will give some side facts to provide a wider framework for the reader's convenience.
We assume all fields to be of characteristic not 2. By a *quadratic form* over F or just *form* for short, we always mean a finite dimensional non-degenerate quadratic form. Such forms can be diagonalized. As usual, we write $\langle a_1, \dots, a_n \rangle$ with suitable $a_1, \dots, a_n \in F^*$ for such a diagonalization. Orthogonal sum and tensor product of quadratic forms are as usually denoted by $\perp$ respective $\otimes$. The n-fold orthogonal sum of a quadratic form φ for $n \in \mathbb{N}$ is denoted by $n \times \varphi := \varphi \perp \dots \perp \varphi$.
The *Witt ring* of F, whose elements are in 1-1 correspondence to the isometry classes of anisotropic forms over F, is denoted by WF. By abuse of notation, we will often use the same symbol for a quadratic form, its anisotropic part and its Witt class in situations where confusion is unlikely. Similarly, $\dim(\varphi)$ will either mean the *dimension* of the underlying vector space or the value $\dim(\varphi) \mod 2$, depending on what we are working with.
We will use the symbol $\cong$ to denote *isometry* of quadratic forms, while the equality sign $=$ is used to indicate equality in the Witt ring. When we would like to emphasize that we are working in the Witt ring, we also use the usual symbols $+, -$ and $\cdot$ for addition, subtraction and multiplication respectively rather than the symbols for the appropriate operation for the forms itself.
As usual, if ψ is a *subform* of φ, we will write $\psi \subseteq \varphi$. Since the determinant of a quadratic form is not an invariant of its Witt class, we will sometimes need its signed version denoted by $d_\pm : WF \to F^*/F^{*2}$.
We write $\mathbb{H}$ for the *hyperbolic plane*. The *Witt index* of φ, i.e. the number of hyperbolic planes occurring in a Witt decomposition, will be denoted as $i_W(\varphi)$. For the Witt index of an orthogonal sum of two forms, we have the following estimate.

Proposition 2.1.1 ([Lam05, Chapter I. Exercise 16 (2)]):
Let φ_1, φ_2 be (regular) quadratic forms over some field F. Then, for the orthogonal sum, we have

$$i_W(\varphi_1 \perp \varphi_2) \leq i_W(\varphi_1) + \dim(\varphi_2).$$

By applying the above, we can easily deduce a criterion for a form to be a subform of a given quadratic form.

Corollary 2.1.2 ([Lam05, Chapter I. Exercise 16 (3)]):
Let ψ, φ be quadratic forms over F. Then ψ is a subform of φ, i.e. $\psi \subseteq \varphi$ if and only if we have $i_W(\varphi \perp -\psi) \geq \dim \psi$.

We denote the set of all nonzero elements *represented* by φ by $D_F(\varphi)$. As the set $D_F(\varphi)$ is obviously closed under multiplication with squares, we will often identify $D_F(\varphi)$ with its canonical image in F^*/F^{*2}.
Using a basic dimension argument involving orthogonal complements, we can get a criterion for a non-zero field element to be represented by a quadratic form. In fact, the set of represented elements of a quadratic form is precisely the set of elements that can be chosen as an entry for some diagonalization.

Proposition 2.1.3 (Representation Criterion, [Lam05, Chapter I. 2.3]):
Let φ be a quadratic form of dimension $n \in \mathbb{N}$ over F and $a \in F^*$. We then have $a \in D_F(\varphi)$ if and only if there are $b_2, \ldots, b_n \in F^*$ with $\varphi \cong \langle a, b_2, \ldots, b_n \rangle$.

An important result for classifying Witt rings is the following one. It will be used frequently in a later chapter. It is due to C.M. Cordes and D.K. Harrison.

Theorem 2.1.4 ([Cor73, Proposition 2.2, Theorem 2.3]):
Let F, E be fields. We then have $WF \cong WE$ if and only if there is a group isomorphism $\varphi : F^*/F^{*2} \to E^*/E^{*2}$ with

(a) $\varphi(-1_F) = -1_E$, and

(b) for any $a, b \in F^*$, we have $b \in D_F(\langle 1, a \rangle)$ if and only if $\varphi(b) \in D_E(\langle 1, \varphi(a) \rangle)$.

Further, if such a group isomorphism exists, it induces an isomorphism $WF \to WE$ by

$$\langle x_1, \ldots, x_n \rangle \mapsto \langle \varphi(x_1), \ldots, \varphi(x_n) \rangle.$$

In particular, if the Witt rings of F and E are isomorphic, there is an isomorphism that sends unary forms to unary forms.

2.2. Pfister Forms and Function Fields

An important class of quadratic forms is given by the *Pfister forms.* For any $a \in F^*$, we define $\langle\langle a\rangle\rangle := \langle 1, -a\rangle$ and call this a *1-fold Pfister form.* For $a_1, \dots, a_n \in F^*$, we define $\langle\langle a_1, \dots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$, a so called *n-fold Pfister form.* Additionally we define $\langle 1\rangle$ to be the unique 0-fold Pfister form.. For a field F and $n \in \mathbb{N}_0$, we define

$$P_nF := \{\varphi \mid \varphi \text{ is isometric to an } n-\text{fold Pfister form}\}$$

respectively

$$GP_nF := \{\varphi \mid \varphi \text{ is similar to an } n-\text{fold Pfister form}\},$$

It is easy to see that P_1F generates the so-called *fundamental ideal*

$$IF := \{\varphi \in WF \mid \dim \varphi \equiv 0 \mod 2\}$$

both as an additive group and as an ideal. Therefore the ideal $I^nF := (IF)^n$ is additively generated by GP_nF.

Let now $\pi = \langle\langle a_1, \dots, a_n\rangle\rangle$ be an n-fold Pfister form. The so-called *slots* $a_1, \dots, a_n$ are not unique. It is often helpful to know how to manipulate the slots or to know which elements can be chosen as a slot. We will now collect some results concerning this problem. Obviously, π represents 1 so we can write $\langle 1\rangle \perp \pi'$ for some form π' over F according to the Representation Criterion. By Witt's Cancellation Law, π' is unique up to isometry. It is thus convenient to call π' the *pure part* of π.

Theorem 2.2.1 (Pure Subform Theorem, [Lam05, Chapter X. 1.5]):
Let π an n-fold Pfister form and $a \in F^*$. Then, a can be chosen as a slot for π (i.e. there are $b_2, \dots, b_n \in F^*$ with $\pi \cong \langle\langle a, b_2, \dots, b_n\rangle\rangle$) if and only if we have $-a \in D_F(\pi')$.

As a main step in the proof of the Pure Subform Theorem, we have the following calculation rule.

Proposition 2.2.2 ([Lam05, Chapter X. Proposition 1.6]):
Let $n \in \mathbb{N}$ be an integer and $a_1, \dots, a_n \in F^*$. We further consider the quadratic form $\tau = \langle\langle a_1, \dots, a_{n-1}\rangle\rangle$ and an element $y \in D_F(\tau)$. We then have

$$\langle\langle a_1, \dots, a_{n-1}, a_n\rangle\rangle \cong \langle\langle a_1, \dots, a_{n-1}, a_n y\rangle\rangle.$$

Another important property of Pfister forms is the roundness, i.e. the set of represented values and the set of similarity factors coincide:

Theorem 2.2.3 ([Lam05, Chapter X. Theorem 1.8]):
Let $\pi \in P_nF$ be a Pfister form for some $n \in \mathbb{N}_0$. Then π is round, i.e. we have $D_F(\pi) = G_F(\pi)$.

If a form is similar to some n-fold Pfister form, we can decide whether it is even isometric to a Pfister form by its set of represented values:

Corollary 2.2.4:
Let φ with $1 \in D_F(\varphi)$ be similar to an n-fold Pfister form π. We then have an isometry $\varphi \cong \pi$, i.e. φ is even isometric to an n-fold Pfister form.

Proof:
If we have $a\varphi \cong \pi$ for some $a \in F^*$, we have $a \in D_F(a\varphi) = D_F(\pi) = G_F(\pi)$, which then implies

$$\pi \cong a\pi \cong a^2\varphi \cong \varphi.$$

□

The roundness of Pfister forms readily implies the following well-known assertion.

Corollary 2.2.5:
Let $n \in \mathbb{N}_0$ be an integer and $\pi_1, \pi_2 \in P_nF$ be n-fold Pfister forms. Then, π_1, π_2 are similar if and only if π_1, π_2 are isometric.

As another consequence, the isotropy behaviour of Pfister forms is as extreme as possible in the following sense.

Corollary 2.2.6 ([EKM08, Corollary 9.10]):
Any Pfister form is either anisotropic or hyperbolic.

In the literature, forms that are similar to some n-fold Pfister forms are called *general* n-fold Pfister forms, but we will call such forms again n-fold Pfister forms for short as the scaling is not relevant for us most of the time. Further, if the foldness does not matter, we will just say *Pfister form* without mentioning the foldness to keep things short.

An important tool in the algebraic theory of quadratic forms is the use of the function field of a quadratic form φ. To have the function field well defined, we need $n := \dim\varphi \geq 2$ and $\varphi \ncong \mathbb{H}$ which we will assume in the whole thesis whenever we talk about function field extensions.
The *function field extension* $F(\varphi)$ we are working with is defined to be

$$F(\varphi) := \operatorname{Quot}\left(F[X_1, \ldots, X_{n-1}]/(\varphi(X_1, \ldots, X_{n-1}, 1))\right).$$

If we have $\varphi \cong \langle a_1, \ldots, a_n\rangle$, we have

$$F(\varphi) \cong F(X_2, \ldots, X_{n-1})\left(\sqrt{-\frac{1}{a_1}(a_2X_2^2 + \ldots + a_{n-1}X_{n-1}^2 + a_n)}\right).$$

As an important special case, for $\varphi = \langle\langle d\rangle\rangle$ for some $d \in F^* \setminus F^{*2}$, we have $F(\varphi) = F(\sqrt{d})$.
The function field does not depend on scaling, i.e. for any $a \in F^*$, we have $F(\varphi) = F(a\varphi)$.
Further a quadratic form φ with $\dim\varphi \geq 3$ is isotropic if and only if $F(\varphi)/F$ is purely transcendental.
As an important application of the function field machinery, we get the following fundamental theorem. It is so important that the German name *Hauptsatz* is used even in the international literature.

Theorem 2.2.7 (Arason-Pfister Hauptsatz, [AP71, Hauptsatz, Korollar 3]):
Let $\varphi \in I^nF$ be an anisotropic form of positive dimension. We then have $\dim\varphi \geq 2^n$. We further have $\dim\varphi = 2^n$ if and only if φ is similar to an n-fold Pfister form.

An extension of the Arason-Pfister Hauptsatz is the following result. Its known proofs all use advanced algebro-geometric tools that we do not introduce in this thesis.

Theorem 2.2.8 (Holes Theorem, [EKM08, Corollary 82.2]):
Let $\varphi \in I^n F$ be an anisotropic quadratic form with $2^n < \dim \varphi < 2^{n+1}$. Then there is some $k \in \{1, \dots, n\}$ such that $\dim \varphi = 2^{n+1} - 2^k$.

Definition 2.2.9:
A quadratic form φ is called a *Pfister neighbor*, if there is a Pfister form π such that φ is similar to a subform of π and if $2 \dim \varphi > \dim \pi$.
If φ is further of the form $a\sigma \perp b\tau$ for some $a, b \in F^*$, a Pfister form σ and some $\tau \subseteq \sigma$ with $\dim \tau \geq 1$, then φ is called a *special Pfister neighbor*.

If φ is a Pfister neighbor, it can be easily shown that the corresponding Pfister form as in the definition is uniquely determined, see [Lam05, Chapter X. Proposition 4.17]. Therefore, we will refer to it as the *associated Pfister form*. Further, there is a strong connection concerning the isotropy behaviour of φ and π. The upcoming well known result is a straightforward consequence of Theorem 2.2.3, Corollary 2.2.6 and Proposition 2.1.1.

Proposition 2.2.10:
Let φ be a Pfister neighbor with associated Pfister form π. Then the following are equivalent:

(i) φ is isotropic;

(ii) π is isotropic;

(iii) π is hyperbolic.

As a first classification result for Pfister neighbors, we have the following.

Proposition 2.2.11 ([Lam05, Chapter X. Example 4.18 (3)]):
Let $\pi \in P_n F$ be a Pfister form for some $n \in \mathbb{N}$ and φ be a Pfister neighbor of π of dimension $2^n - 1$. Then φ is similar to the pure part π', i.e. all Pfister neighbors of codimension 1 of π are similar.

Furthermore the Pfister neighbors of dimension 5 can be classified. It turns out that they are all special Pfister neighbors.

Proposition 2.2.12 ([Lam05, Chapter X. Propositon 4.19]):
Let φ be a quadratic form of dimension 5. Then the following are equivalent:

(i) φ is a Pfister neighbor;

(ii) there is some $\sigma \in GP_2 F$ with $\sigma \subseteq \varphi$.

A central concept that comes along whith studying Pfister forms is that of linkage that we will now introduce.

Theorem and Definition 2.2.13 ([Hof96, Lemma 3.2]):
Let $\sigma \in P_n F$ and $\pi \in P_m F$ be anisotropic Pfister forms for some $m, n \in \mathbb{N}$ with $1 \leq m \leq n$ and $a, b \in F^*$. We then have

$$i := i_W(a\sigma \perp b\pi) \in \{0\} \cup \{2^r \mid 0 \leq r \leq m\}.$$

Further, we have $i \geq 1$ if and only if there is some $x \in F^*$ with

$$(a\sigma \perp b\pi)_{\mathrm{an}} \cong x(\sigma \perp -\pi)_{\mathrm{an}}.$$

If $i = 2^r \geq 1$ then there exist $\alpha \in P_r F, \sigma_1 \in P_{n-r} F$ and $\pi_1 \in P_{m-r} F$ such that we have

$$\sigma \cong \alpha \otimes \sigma_1 \text{ and } \pi \cong \alpha \otimes \pi_1.$$

In this case, r is called the *linkage number* of σ and π and α is called a *link* of σ and π. If we have $n = m$ and $r \geq n - 1$, we say σ and π are *linked*.

Definition 2.2.14:
Let $n \in \mathbb{N}$ be an integer with $n \geq 2$. A field F is called *n-linked*, if any two n-fold Pfister forms are linked. A field F is called *linked*, if F is 2-linked.

Example 2.2.15:
Local and global fields (defined as usual in algebraic number theory) are linked by [Lam05, Chapter VI. Corollary 3.6]. In particular, $\mathbb{Q}$ and its finite extensions, the p-adic numbers $\mathbb{Q}_p$ for any prime p and finite fields and its extensions of transcendence degree at most 1 are linked.

A more general concept is that of so-called *twisted Pfister forms* that were firstly introduced in [Hof96]. As these forms will appear several times in the sequel, we would like to give the formal definition here.

Definition 2.2.16:
An anisotropic quadratic form φ over some field F is called a *twisted Pfister form*, if there are some $n, m \in \mathbb{N}$ with $n > m$ and anisotropic Pfister forms $\pi \in GP_n F, \sigma \in GP_m F$ with $\dim \varphi = 2^n$ and $\varphi = \pi + \sigma \in WF$.

2.3. The Fundamental Ideal and Related Objects

In this section, we would like to collect some further facts about the important ideals I^nF that are generated by the set GP_nF of n-fold Pfister forms. As already seen, IF is the kernel of the map $\dim : WF \to \mathbb{Z}/2\mathbb{Z}$. In particular, we have

$$WF/IF \cong \mathbb{Z}/2\mathbb{Z}.$$

We can extend this further. A straightforward check yields

$$I^2F = \{\varphi \in IF \mid d_\pm(\varphi) = 1\}$$

which then implies

$$IF/I^2F \cong F^*/F^{*2}.$$

To have a further insight into higher powers of the fundamental ideal and its quotients, a lot of further techniques are necessary. To give some details for the description of I^3F, we have to introduce the *Clifford invariant* of a quadratic form φ defined on a vector space V that we will now define.
We first consider the tensor algebra

$$T(V) := \bigoplus_{n\in\mathbb{N}_0} V^{\otimes n},$$

where $V^{\otimes n}$ denotes the n-fold tensor product $V \otimes_F \ldots \otimes_F V$ with itself with the convention $V^{\otimes 0} := F$, and in $T(V)$ the two sided ideal $I(V,\varphi)$ generated by elements of the form $x \otimes x - \varphi(x)$. The algebra

$$C(V,\varphi) := T(V)/I(V,\varphi)$$

is called the *Clifford algebra of* φ. The Clifford algebra of φ is an invariant of the isometry class of φ, has dimension $2^{\dim V}$ and has a canonical $\mathbb{Z}/2\mathbb{Z}$-graduation as $I(V,\varphi)$ is a graded ideal. It can be shown that

$$c(\varphi) := \begin{cases} [C(\varphi)], & \text{if } \dim\varphi \equiv 0 \mod 2 \\ [C_0(\varphi)], & \text{if } \dim\varphi \equiv 1 \mod 2 \end{cases}$$

lie in the Brauer group Br(F) of F, and are again an invariant of the isometry class of φ, called the *Clifford invariant of* φ. Some basic calculations show that

$$c : I^2F \to \mathrm{Br}_2(F) = \{[A] \in \mathrm{Br}(F) \mid\mid [A]^2 = 1\}$$

is a well-defined group homomorphism with $I^3F \subseteq \ker(c)$.
In fact, using some more advanced techniques, equality can be shown.

Theorem 2.3.1 (Merkurjev's Theorem, [EKM08, Theorem 44.1]):
We have

$$I^3F := \{\varphi \in I^2F \mid c(\varphi) = 1\}.$$

In particular, we have

$$I^2F/I^3F \cong \mathrm{Br}_2(F).$$

An important calculation rule that will be frequently used is the identity

$$c\,(x\langle\langle a,b\rangle\rangle) = (a,b)_F$$

for any $a,b,x \in F^*$, where $(a,b)_F$ denotes the quaternion algebra with slots a,b, i.e. the 4-dimensional F-algebra with basis $1,i,j,k$ fulfilling the relations

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

As I^2F is generated by 2-fold Pfister forms and using the above identity, we have the following important consequence of Merkurjev's Theorem:

Corollary 2.3.2:
The 2-torsion part $\mathrm{Br}_2(F)$ of the Brauer group is generated by the classes of quaternion algebras.

Without giving further details, we would like to say that the above can even generalized to all higher powers. All the quotients of $I^nF/I^{n+1}F$ we presented so far can be interpreted as certain cohomology groups, commonly denoted by $H^n(F,\mathbb{Z}/2\mathbb{Z})$ that have natural generalizations to higher n. These higher cohomology groups are in fact isomorphic to the respective $I^nF/I^{n+1}F$. This is part of the famous *Milnor Conjecture*, first stated in [Mil70] and finally proved in [Voe03, Theorem 7.5].

2.4. Quadratic Forms over Complete Discrete Valuation Fields

For the whole chapter, let F be a field and $v : F \to \mathbb{Z}$ a normalized discrete valuation with uniformizer π. As in the whole thesis, we assume the characteristic of F to be not equal to two even though it is not necessary for all statements that do not involve quadratic forms. We further fix the notation for the valuation ring $\mathcal{O}$, its unique maximal ideal $\mathfrak{m}$ and its units $\mathcal{U}$. Finally, we have the residue field $K = \mathcal{O}/\mathfrak{m}$ which we assume to be of characteristic not 2.

From now on, we will assume the valuation to be complete. The next results will show how strong the connection of quadratic form theory over a complete discrete valuation field and over its residue class field are. To start we will show how to compute the square class group.

Proposition 2.4.1 ([Lam05, Chapter VI. Proposition 1.3]):
We have an isomorphism $F^*/F^{*2} \cong K^*/K^{*2} \times \{1, \pi\}$. We further have $\mathcal{U}/\mathcal{U}^2 \cong K^*/K^{*2}$ given by $uU^2 \mapsto (u + \mathfrak{m})K^{*2}$.

A quadratic form φ over F is called *unimodular*, if it has a diagonalization $\varphi \cong \langle u_1, \dots, u_n \rangle$ with $u_1, \dots, u_n \in \mathcal{U}$. We can then consider its residue class form $\langle \overline{u_1}, \dots, \overline{u_n} \rangle$ where $\overline{u_i}$ stands for the nonzero coset of u_i in the residue field. We denote the residue class form with a bar, i.e. we write $\overline{\varphi} = \langle \overline{u_1}, \dots, \overline{u_n} \rangle$.
Proposition 2.4.1 readily implies that for each quadratic form φ over F, there are unimodular forms φ_1, φ_2 with $\varphi \cong \varphi_1 \perp \pi\varphi_2$. We define $\partial_1(\varphi) := \overline{\varphi_1}$ and $\partial_2(\varphi) := \overline{\varphi_2}$ to be the first respectively second residue class forms of F. It is well known that these are well-defined in WK.
Since we assume the valuation to be complete, taking residue class forms gives rise to a group homomorphism $WF \cong WK \oplus WK$ and even a ring isomorphism $WF \cong WK[C]$ with a cyclic group C of order 2 due to a theorem of Springer, see [Lam05, Chapter VI. Corollary 1.7].
A crucial step in the proof of these results is to construct the inverse maps. This is done by *lifting* forms over K to forms over F as follows. If $\langle x_1, \dots, x_n \rangle$ is a form over K, we can find $u_1, \dots, u_n \in \mathcal{U} \subseteq F^{*2}$ such that u_iU^2 is a preimage of x_iK^{*2} under the isomorphism $\mathcal{U}/\mathcal{U}^2 \cong K^*/K^{*2}$ introduced in Proposition 2.4.1. The square classes in F^*/F^{*2} of these u_i are uniquely determined as can be seen by using the other isomorphism in Proposition 2.4.1. Using this among others, it can be shown that the isometry class of the form $\langle u_1, \dots, u_n \rangle$ is uniquely determined by the isometry class of $\langle x_1, \dots, x_n \rangle$ and is thus called the *lift* of $\langle x_1, \dots, x_n \rangle$.

We will now give a criterion to check whether a given form is anisotropic. It is a well known fact that this can be checked by just checking if both residue class forms are anisotropic.

Proposition 2.4.2 ([Lam05, Chapter VI. Proposition 1.9 (2)]):
A quadratic form φ decomposed in its residue class forms $\varphi = \varphi_1 \perp \pi\varphi_2$ is anisotropic over F if and only if both $\partial_1(\varphi)$ and $\partial_2(\varphi)$ are anisotropic over K.

Corollary 2.4.3:
For the Witt index, we have $i_W(\varphi_1 \perp \pi\varphi_2) = i_W(\overline{\varphi_1}) + i_W(\overline{\varphi_2})$.

As two forms are isometric if and only if their difference is hyperbolic, we can directly deduce the following:

Corollary 2.4.4:
Two quadratic forms over F are isometric if and only if they have the same dimension and their first and second residue class forms respectively are both Witt-equivalent over K. In particular, a form over F is hyperbolic if both residue class forms are hyperbolic.

In some cases, we can determine the dimensions of the residue class forms of forms that are divisible by a one-fold Pfister form.

Lemma 2.4.5:
Let $\varphi = \langle\langle a\rangle\rangle \otimes \psi$ be an anisotropic form disivible by the Pfister form $\langle\langle a\rangle\rangle$, where we have $v(a) \equiv 1 \mod 2$. We then have $\dim \partial_1(\varphi) = \dim \partial_2(\varphi) = \frac{\dim \varphi}{2} = \dim \psi$, where the $\partial_i(\varphi)$ are chosen to be anisotropic representatives of the residue class forms for $i \in \{1, 2\}$.

Proof:
It obviously suffices to show $\dim \partial_1(\varphi) = \dim \psi$. Thus the claim follows by considering the following equalities:

$$\begin{aligned}
\dim \partial_1(\varphi) &= \dim \partial_1(\psi \perp -a\psi) \\
&= \dim(\partial_1(\psi) \perp \partial_1(-a\psi)) && (\partial_1 \text{ is a homomorphism}) \\
&= \dim(\partial_1(\psi)) + \dim(\partial_1(-a\psi)) \\
&= \dim(\partial_1(\psi)) + (\dim \psi - \dim(\partial_1(\psi)) && (v(a) \text{ odd}) \\
&= \dim \psi.
\end{aligned}$$

□

As a special case of a field with a complete discrete valuation, we will consider the field $F((t))$ of formal Laurent series over the field F. Its valuation is given by

$$v\left(\sum_{k=n}^{\infty} a_k \pi^k\right) := n,$$

where $n \in \mathbb{Z}$ is chosen as the least index with $a_n \neq 0$. The residue class field is isomorphic to F and t is a uniformizing element. We will consider the residue class field as F, it is then in particular a subfield of the discretely valued field. Huge parts of this thesis will deal with (iterated) Laurent series extensions.

Proposition 2.4.6:
Let $a \in F^* \setminus F^{*2}$ be an element in F that is not a square. We then have a field isomorphism $F((t))(\sqrt{a}) \cong F(\sqrt{a})((t))$ given by

$$\sum_{n\in\mathbb{Z}} a_n t^n + \left(\sum_{n\in\mathbb{Z}} b_n t^n\right)\sqrt{a} \mapsto \sum_{n\in\mathbb{Z}} \left(a_n + b_n\sqrt{a}\right) t^n$$

Proof:
This is just a standard computation whose details are left to the reader. $\square$

Proposition 2.4.7 ([Lam05, Chapter VI. Exercise 3]):
Let $\varphi \cong \varphi_1 \perp t\varphi_2$ be a quadratic form defined over the Laurent series field $F((t))$ with residue class forms φ_1 and φ_2. For the set of represented elements, we have the equality

$$D_{F((t))}(\varphi_1 \perp t\varphi_2) = D_F(\varphi_1) \cup tD_F(\varphi_2),$$

where we identify $D_F(\varphi_1)$ respective $D_F(\varphi_2)$ with its canonical images in $F((t))$.

A somewhat more involved construction is the construction of an iterated Laurent series extension with an arbitrary set of Laurent variables:

Definition 2.4.8:
Let F be a field and I an arbitrary index set. We define $F((t_i))_{i\in I}$ to be the direct limit of all fields $F((t_{i_1}))\cdots((t_{i_r}))$ for $r \in \mathbb{N}_0$ and $i_1, \ldots, i_r \in I$ with $i_1 < i_2 < \cdots < i_r$ for an arbitrary fixed well ordering $<$ on I.

The observations that are stated in the following remark are essential for the sequel. These are frequently used reduction strategies to handle quadratic forms.

Remark 2.4.9:
Let K be a field of characteristic not 2, $F = K((t_1))((t_2))$ and φ a quadratic form over F. We then have the choice if we want to consider φ as a form over F or as a form over $E := K((t_2))((t_1))$ as the $\mathbb{F}_2$-linear map $\Phi : F^*/F^{*2} \to E^*/E^{*2}$ defined by

$$\begin{aligned} aF^{*2} &\mapsto aE^{*2} \quad \text{for all } a \in K^*;\\ t_1F^{*2} &\mapsto t_1E^{*2};\\ t_2F^{*2} &\mapsto t_2E^{*2} \end{aligned}$$

is a group isomorphism with

$$\Phi(-1) = -1 \text{ and } \Phi(D_F(\langle x_1, \ldots, x_n\rangle)) = D_E\big(\langle \Phi(x_1), \ldots, \Phi(x_n)\rangle\big) \tag{2.1}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in F^*$, see Theorem 2.1.4. It is an isomorphism as if $\{a_i \mid i \in I\}$ is a system of representatives of K^*/K^{*2}, then $\{a_i, a_it_1, a_it_2, a_it_1t_2 \mid i \in I\}$ is a system of representatives of both F^*/F^{*2} and E^*/E^{*2}. Further the first identity in (2.1) is clear and the second one follows readily by Proposition 2.4.7. We would like to emphasize the fact that we have $F \neq E$ and that Φ is not an identity map even though it looks like one, especially when abusing the notation and identifying a non zero element of the field with its square class.
We now further assume that we have $\varphi \in I^nF$ for some $n \in \mathbb{N}$. As Φ induces a ring isomorphism that obviously takes 1-fold Pfister forms over F to 1-fold Pfister forms over E, the Pfister number of φ over E is lower than or equal to the Pfister number of φ over F. As we can argue the other way round with Φ^{-1} we can even say that both Pfister numbers coincide.
As the symmetric group S_n for $n \geq 2$ is generated by transpositions of the form $(k\ \ k+1)$ for all $k \in \{1, \ldots, n-1\}$, we can extend the above to handle the field $K((t_1))\cdots((t_n))$ by reordering the Laurent variables in an appropriate way.
As a next step, we will show how we can reduce to the case of finitely many Laurent

variables. To do so, we now consider the field $F' := K((t_i))_{i\in I}$ for an arbitrary index set I. If we have a quadratic form φ over F', we can choose a diagonalisation of φ. In this diagonalisation, only finitely many Laurent variables can occur. We denote these by $s_1, \dots, s_n$ and assume that we have $s_i < s_j$ if and only if $i < j$. We can then consider φ as a form over the field $K((s_1))\cdots((s_n))$ which can be embedded in the direct limit F'.

As a last trick, we would like to mention that we can always change the uniformizing element in some ways. For example, we have $K((t)) = K((at))$ for all $a \in K^*$, so for example $K((t_1))((t_2)) = K((t_1))((t_1t_2))$.

We will use these facts without mentioning them explicitly several times in the sequel. The main idea while using this is that quadratic forms are good to manage if they have a well understood subform. It is thus convenient to reorder the Laurent variables such that one gets a residue form of small dimension.

2.5. Field Extensions

Let E/F be a field extension and b be a symmetric bilinear form defined over some F-vector space V. Then $E \otimes_F V =: V_E$ is an E-vector space and we have a bilinear map b_E on V_E defined by

$$b_E\left(\sum_{i=1}^{n} x_i \otimes v_i, \sum_{j=1}^{n} y_j \otimes w_j\right) := \sum_{i,j} x_i y_j b(v_i, w_j)$$

for all $n, m \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_m \in E$ and $v_1, \ldots, v_n, w_1, \ldots, w_m \in V$. If q is the quadratic form defined by b, the quadratic form q_E defined by b_E is given by

$$q_E(z) = b_E(z, z)$$

for all $z \in V_E$. One can readily check that we have a ring homomorphism $WF \to WE$ defined by $q \mapsto q_E$. We will denote this homomorphism by $r_{E/F}$. Further, if $v_1, \ldots, v_n$ is an F-basis of the vector space V, then $1 \otimes v_1, \ldots, 1 \otimes v_n$ is an E-basis of V_E and the Gram matrix of b is the same as the Gram matrix of b_E with respect to the particular basis.

Theorem 2.5.1 ([Lam05, Chapter VII. Theorem 3.1]):
Let φ be an anisotropic form over F and $E = F(\sqrt{a})$ a quadratic field extension. Then φ_E is isotropic if and only if φ contains a subform similar to $\langle\langle a \rangle\rangle$.

Using induction and the above result we can even determine the *Witt kernel* of a quadratic field extension. In general, the Witt kernel of a field extension E/F is defined as

$$W(E/F) := \ker(WF \to WE) = \{\varphi \in WF \mid \varphi_E = 0 \in WE\},$$

i.e. the ideal of those forms in WF that become hyperbolic after scalar extension to E.

Theorem 2.5.2 ([Lam05, Chapter VII. Theorem 3.2]):
Let φ be an anisotropic form over F and $a \in F^* \setminus F^{*2}$. For any $m \in \mathbb{N}$ we have $i_W(\varphi_{F(\sqrt{a})}) \geq m$ if and only if there is some form ψ over F of dimension $\dim \psi = m$ such that $\langle\langle a \rangle\rangle \otimes \psi \subseteq \varphi$ is a subform of φ. In particular $\varphi_{F(\sqrt{a})}$ is hyberbolic if and only if φ is divisible by $\langle\langle a \rangle\rangle$ and we have

$$W\left(F\left(\sqrt{a}\right)/F\right) = \langle\langle a \rangle\rangle WF.$$

A similar assertion is the following one. We would like to lay emphasise on the important difference that the upcoming statement is just about Witt classes and not about the forms itself.

Corollary 2.5.3 ([Lam05, Chapter XII. Theorem 4.3]):
For $a, b \in F^* \setminus F^{*2}$, we have $W\Big(F\left(\sqrt{a}, \sqrt{b}\right)/F\Big) = \langle\langle a \rangle\rangle WF + \langle\langle b \rangle\rangle WF$.

Another important class of field extensions is the case of finite field extensions of odd degree.

Theorem 2.5.4 (Springer's Theorem, [Lam05, Chapter VII. 2.6, 2.7]):
Let φ be an anisotropic quadratic form and E/F be a field extension of odd degree. Then, φ_K is anisotropic. In particular we have $W(E/F) = \{0\}$ and the map $WF \to WK$ induced by scalar extension is injective.

For a field extension E/F, we considered only the canonical map $WF \to WE$ yet. One natural question to ask is whether there is some useful map $WE \to WF$. In fact, for finite field extensions, there is a huge class of such maps that can be all constructed in a similar manner, using the *Scharlau transfer*. To introduce the construction, we would like to recall that we can consider the bigger field E as a vector space over the ground field F. As also F is an F-vector space, we can thus talk about F-linear maps $E \to F$. Thus, if q is an E-quadratic form on an E-vector space V and s a non-zero F-linear map $E \to F$, we can consider the F-quadratic form $s \circ q : V \to F$ that will be denoted by $s_*(q)$ and will be called the *transfer* of q via s. We will collect some well known results on the transfer in the upcoming theorem.

Theorem 2.5.5 ([Lam05, Chapter VII. 1.1, 1.2, 1,4, 1.5]):
Let F, E, s, q be as above. We then have:

(a) if q is regular, so is $s_*(q)$.

(b) $\dim s_*(q) = [E : F] \cdot \dim q$.

(c) if q is hyperbolic, so is $s_*(q)$.

(d) $s_* : WE \to WF$ defines a group homomorphism.

One map that yields remarkable results in this context is the field trace. We will come back to the meaning of the transfer induced by the field trace in a later chapter. In the sequel we will instead deal with the case of a quadratic extension $E = F(\sqrt{a})$ and some nontrivial linear map $s : E \to F$ fulfilling $s(1) = 0$. This notation will be fixed for the rest of the section.
Combining [Lam05, Chapter VII. Theorem 3.4] with [EKM08, Lemma 34.18] and considering the dimension, we readily get the following computation rule for the transfer of one-dimensional forms.

Proposition 2.5.6:
For any $x \in E^*$ and s as above, there is some $d \in F^*$ with $s_*(\langle x\rangle) = d\langle 1, -N_{E/F}(x)\rangle$, where $N_{E/F} : E \to F$ denotes the field norm.

Theorem 2.5.7 (Exact Triangle, [EKM08, Corollary 34.12]):
For the quadratic extension $E = F(\sqrt{a})$, let $t : WF \to WF$ denote the map given by multiplication with $\langle\langle a\rangle\rangle$. We have an exact triangle

$$\begin{tikzcd} & WE \arrow[dr, "s_*"] & \\ WF \arrow[ur, "r_{E/F}"] & & WF \arrow[ll, "t"'] \end{tikzcd}$$

By considering the proof for the exact triangle theorem for unary forms in detail, one can get additional information on the square class groups.

Theorem 2.5.8 ([Lam05, Chapter VII. Theorem 3.8]):
By abuse of notation, let $r_{E/F}$ and $N_{E/F}$ denote the induced homomorphisms by scalar extension respective field norm for the appropriate square class groups. Further, let $\varepsilon : F^*/F^{*2} \to \mathrm{Br}(F)$ be the homomorphism defined by $\varepsilon(b) := (a,b)_F$. We then have an exact sequence

$$1 \longrightarrow \{F^{*2}, aF^{*2}\} \xrightarrow{\iota} F^*/F^{*2} \xrightarrow{r_{E/F}} E^*/E^{*2} \xrightarrow{N_{E/F}} F^*/F^{*2} \xrightarrow{\varepsilon} \mathrm{Br}(F)$$

where the first nontrival map is given by inclusion.

An easy computation using the well known Frobenius reciprocity formula for the Scharlau transfer (that will not be needed itself and will thus not be formulated explicitely) yields:

Corollary 2.5.9 ([EKM08, Corollary 34.17]):
We have $s_*(I^nE) \subseteq I^nF$.

By the above result, we thus can consider the transfer map as a map $I^nE \to I^nF$ for any $n \in \mathbb{N}_0$. It would be nice to have a similar exactness result as in Theorem 2.5.7. In fact, as a consequence of the Milnor Conjecture, we have the following.

Theorem 2.5.10 ([EKM08, Theorem 40.3]):
The following sequences are exact for any $n \in \mathbb{N}_0$:

$$\cdots \xrightarrow{s_*} I^{n-1}F \xrightarrow{\cdot\langle\langle a\rangle\rangle} I^nF \xrightarrow{r_{E/F}} I^nE \xrightarrow{s_*} I^nF \xrightarrow{\cdot\langle\langle a\rangle\rangle} I^{n+1}F \longrightarrow \cdots$$

and

$$\cdots \xrightarrow{s_*} \overline{I}^{n-1}F \xrightarrow{\cdot\langle\langle a\rangle\rangle} \overline{I}^nF \xrightarrow{r_{E/F}} \overline{I}^nE \xrightarrow{s_*} \overline{I}^nF \xrightarrow{\cdot\langle\langle a\rangle\rangle} \overline{I}^{n+1}F \longrightarrow \cdots$$

where we use the bar to denote the factor group $\overline{I}^nF := I^nF/I^{n+1}F$.

We now turn to transcendental field extensions. The following result is an easy consequence of [Lam05, Chapter IX. Lemma 1.1].

Proposition 2.5.11:
Let φ be a quadratic form over F and K/F a purely transcendental field extension. Then φ is isotropic if and only if φ_K is isotropic.

Proposition 2.5.12:
Let φ be a quadratic form that lies in $I^nF(X)$ or $I^nF((t))$ defined over F. Then, there is a unique preimage $\psi \in WF$ under the canonical map $r_{F(X)/F}$ respectively $r_{F((t))/F}$ and it fulfills $\psi \in I^nF$.

Proof:
We will denote the map induced by scalar extension in both cases by r. The existence and uniqueness of some $\psi \in WF$ with $r(\psi) = \varphi$ is clear as φ is defined over F and r is known to be injective, see e.g. [Lam05, Chapter IX. Lemma 1.1] respectively Section 2.4
As φ has a preimage in I^nF because of [EKM08, Theorem 21.1] and Theorem 2.5.13 respectively [EKM08, Exercise 19.15], the claim follows. □

The valuations on $F(X)$ that are trivial on F are well studied, see [Sti09, Section 1.2]. For each monic irreducible polynomial $\pi \in F[X]$, we have a π-adic valuation v_π that is defined via $v_\pi(f) = n$ if $f \in F(X)^*$ has a representation $f(X) = \pi(X)^n \cdot \frac{p(X)}{q(X)}$ with $n \in \mathbb{Z}$ and $\pi(X) \nmid p(X), q(X) \in F[X] \setminus \{0\}$. Further, we have the valuation v_∞ induced by the degree: for $f(X) = \frac{p(X)}{q(X)} \in F(X)^*$, we have $v_\infty(f) = \deg(q) - \deg(p)$. It is known that these are all valuations on $F(X)$ that are trivial on F. We will set $P = \{\pi \in F[X] \mid \pi \text{ is monic and irreducible}\} \cup \{\infty\}$

We can now consider the completion of these valuations and get residue class maps as in Section 2.4. We denote the second residue class homomorphism for the valuation induced by π respectively the degree by ∂_π respectively ∂_∞. For some monic irreducible polynomial $\pi \in F[X]$ as above and $f \in F[X]$ prime to π, we have

$$\partial_\pi(\langle f \rangle) = 0 \quad \text{and} \quad \partial_\pi(\langle \pi f \rangle) = \langle \overline{f} \rangle,$$

where $\overline{f}$ denotes the residue class of f in $F[X]/(\pi(X))$. If f has leading coefficient a and degree d, we have

$$v_\infty(\langle f \rangle) = \begin{cases} a, & \text{if } d \text{ is odd} \\ 0, & \text{otherwise} \end{cases}.$$

Both formulae follow directly from the definition.

For all $v \in P$, the residue class field F_v is a field extension of F. If $v = \pi$ is a polynomial of degree n, an F-basis of $F_v = F[X]/(\pi(X))$ is given by $1, \overline{X}, \ldots, \overline{X^{n-1}}$. We thus have a linear map $s^{(v)} : F_v \to F$ defined on the basis via

$$\overline{X^k} \mapsto \begin{cases} 1, & \text{if } k = n-1 \\ 0, & \text{otherwise} \end{cases},$$

whose transfer map will be denoted by $s_*^{(v)} : WF_v \to WF$.

For $v = \infty$, we have $F_v = F$ and we define $s_*^{(v)} : WF \to WF$ to be $-\mathrm{id}_{WF}$. With all this notation in mind. we can formulate the next important result.

Theorem 2.5.13 ([EKM08, Corollary 21.7]):
We have an exact sequence

$$0 \longrightarrow I^n F \xrightarrow{r_{F(X)/F}} I^n F(X) \xrightarrow{\partial} \bigoplus_{v \in P} I^{n-1} F_v \xrightarrow{s} I^{n-1} F \longrightarrow 0,$$

where the sum ranges over all valuations described above and the maps are defined via $\partial = (\partial_v)_{v \in P}$ and $s = \sum\limits_{v \in P} s_*^{(v)}$.

To close this section, we will present a result concerning the function field extension. As $F(\sqrt{a})$ is nothing but the function field $F(\langle\langle a \rangle\rangle)$, Theorem 2.5.2 is just a special case of the following result. However, we decided to present the statements about quadratic field extensions separately as they can be proved ad hoc and because Theorem 2.5.1 does not generalise to higher Pfister forms.

Theorem 2.5.14 ([Lam05, Chapter X. Theorem 4.11, Corollary 4.13]):
Let π be an anisotropic n-fold Pfister form for some $n \in \mathbb{N}$ and φ an anisotropic form. The following statements are equivalent:

(i) $\varphi_{F(\pi)}$ is hyperbolic;

(ii) there is some form ψ with $\varphi \cong \pi \otimes \psi$;

(iii) we have $\varphi = \pi \cdot \psi \in WF$ for some quadratic form ψ over F.

In particular, for the Witt kernel, we have $W(F(\pi)/F) = \pi WF$ (which also holds if we allow π to be isotropic).
If φ is in addition also a Pfister form, the above are further equivalent to

(iv) $\pi \subseteq \varphi$;

(v) $\varphi \cong \pi \otimes \tau$ for some Pfister form τ.

3. Structural Results for the Powers of IF

3.1. Known Pfister Numbers

It is well known that the fundamental ideal is generated by the one-fold Pfister forms both as an additive group and as an ideal. Thus the n-th power $I^nF := (IF)^n$ is generated by the set of n-fold Pfister forms over F both as an additive group and as an ideal. In order to study the Witt ring, it is reasonable to study the fundamental ideal and its powers. We will study the complexity of a given form in I^nF by finding a small set of Pfister forms that can be used to represent the Witt class of the given form. This will be made more precise in the following definition.

Definition 3.1.1:
We define the *n-Pfister number* of a quadratic form $\varphi \in I^nF$ to be

$$GP_n(\varphi) := \min\{k \in \mathbb{N} \mid \text{there are } \pi_1, \dots, \pi_k \in GP_nF \text{ with } \varphi = \pi_1 + \dots + \pi_k \in WF\}.$$

For a subset $S \subseteq WF$ and an integer $d \in \mathbb{N}$, we define

$$GP_n(S, d) := \sup\{GP_n(\varphi) \mid \varphi \in S \cap I^nF, \dim\varphi \leq d\}.$$

Additionally, we define the shortcuts

$$GP_n(F, d) := GP_n(WF, d) \quad \text{and} \quad GP_n(S) := \bigcup_{d \in \mathbb{N}} GP_n(S, d).$$

We further define the *unscaled n-Pfister number* of φ to be

$$P_n(\varphi) := \min\{k \in \mathbb{N} \mid \text{there are } \varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\} \text{ and } \pi_1, \dots, \pi_k \in P_nF \text{ with } \varphi = \varepsilon_1\pi_1 + \dots + \varepsilon_k\pi_k \in WF\}.$$

If the integer n is clear from the context, we will often just say *(unscaled) Pfister number*.

The main task in this thesis is now to calculate Pfister numbers in terms of invariants of a given form. As this seems to be a quite tough task, we will often be satisfied with upper or lower bounds. We will concentrate on the scaled version, as we have the following correspondence between both versions.

Proposition 3.1.2:
For any quadratic form φ over F and any $n \in \mathbb{N}$, we have $P_n(\varphi) \leq 2 \cdot GP_n(\varphi)$.

Proof:
For any $a, x_1, \dots, x_n \in F^*$, we have

$$\begin{aligned} a\langle\langle x_1, \dots, x_n\rangle\rangle &= \langle\langle x_1, \dots, x_{n-1}\rangle\rangle \otimes (a\langle\langle x_n\rangle\rangle) \\ &= \langle\langle x_1, \dots, x_{n-1}\rangle\rangle \otimes (\langle 1, a\rangle \perp -\langle 1, ax_n\rangle) \\ &= \langle\langle x_1, \dots, x_{n-1}, -a\rangle\rangle \perp -\langle\langle x_1, \dots, x_{n-1}, -ax_n\rangle\rangle \end{aligned}$$

which then readily implies the assertion. □

It is clear that any form in IF of dimension $2d$ is isometric to a sum of d forms in GP_1F. Further an easy induction yields the following result. The induction base is given by the Arason-Pfister Hauptsatz.

Proposition 3.1.3 ([Lam05, Chapter X. Exercise 4]):
Let $\varphi \in I^2F$ be a form of dimension $\dim\varphi \in \mathbb{N}$. Then φ is Witt equivalent to a sum of at most $\frac{\dim\varphi}{2} - 1$ forms in GP_2F.

The bound is sharp, as the following example shows.

Example 3.1.4:
Let K be a field and $F := K((X_1))\dots((X_n))$ for some $n \in \mathbb{N}$ with $n \geq 2$. According to (the proof of) [PST09, Theorem 2.2] (in which the assumption that -1 is a square is only needed to assure that the upcoming forms lie in I^2F and can be omitted by adding a sign as below), we see that we have

$$P_2F(\langle 1, X_1, \dots, X_n, (-1)^{\frac{n+2}{2}} X_1 \cdot \ldots \cdot X_n\rangle) = n - 1 \text{ if } n \text{ is even}$$

and

$$P_2F(\langle X_1, \dots, X_n, (-1)^{\frac{n+1}{2}} X_1 \cdot \ldots \cdot X_n\rangle) = n - 1 \text{ if } n \text{ is odd.}$$

Thus, Proposition 3.1.2 implies

$$GP_2F(\langle 1, X_1, \dots, X_n, (-1)^{\frac{n+2}{2}} X_1 \cdot \ldots \cdot X_n\rangle) \geq \frac{n-1}{2}$$

and

$$GP_2F(\langle X_1, \dots, X_n, (-1)^{\frac{n+1}{2}} X_1 \cdot \ldots \cdot X_n\rangle) \geq \frac{n-1}{2}. \tag{3.1}$$

In the case where n is even, using that the Pfister number is always an integer, we even get

$$GP_2F(\langle 1, X_1, \dots, X_n, (-1)^{\frac{n+2}{2}} X_1 \cdot \ldots \cdot X_n\rangle) \geq \frac{n}{2}. \tag{3.2}$$

As the reverse inequalities hold by Proposition 3.1.3, we have equalities both in (3.1) and (3.2). Of course, since the values of GP_2F are invariant under scaling and since we can redefine the indeterminates, we can restrict ourselves to the case where n is even and just consider the form

$$\varphi := \langle 1, X_1, \dots, X_n, (-1)^{\frac{n+2}{2}} X_1 \cdot \ldots \cdot X_n\rangle \in I^2F$$

with $\dim\varphi = n + 2$ and

$$GP_2(\varphi) = \frac{n}{2},$$

which is the biggest possible value. This form will also be referred to as the *generic (rigid)* I^2-form of dimension $n + 2$.

We will now record the known Pfister numbers for forms in I^3F of low dimension. Note that the dimensions up to 10 are already covered by the Arason-Pfister Hauptsatz and the Holes Theorem. As a further side fact, we would like to recall that the 10-dimensional case was proved by Pfister before the Holes Theorem was proved using just techniques from the algebraic theory of quadratic forms and not any algebro-geometric tools. We further would like to introduce the notion of an *Albert form* over F, which are defined to be 6-dimensional forms in I^2F.

Theorem 3.1.5 ([Pfi66, Satz 14, Zusatz]):
Let $\varphi \in I^3F$ be a quadratic form over F with $\dim\varphi = 12$. Then there are $x \in F^*$ and an Albert form $\alpha \in WF$ with $\varphi \cong \langle\langle x\rangle\rangle \otimes \alpha$.

As any Albert form can be written as the sum of two GP_2-forms by Proposition 3.1.3, we get the following:

Corollary 3.1.6:
For any $\varphi \in I^3F$ with $\dim\varphi = 12$, there are $\pi_1, \pi_2 \in GP_3F$ with $\varphi = \pi_1 + \pi_2 \in WF$.

Before stating the next result, we would like to recall the definition of the field trace. For a finite field extension E/F, the field trace $\mathrm{tr} : E \to F$ maps an element $x \in E$ to the trace (in the sense of linear algebra) of the F-linear map $a \mapsto xa$. The following result is due to Rost and describes 14-dimensional forms in I^3 using the Scharlau transfer of the field trace. Its proof uses a Galois cohomological representation of I^3F using spinor groups.

Theorem 3.1.7 ([Ros99b, Seite 4]):
Let $\varphi \in I^3F$ be a quadratic form with $\dim\varphi = 14$. Then there is some $a \in F^*$ and some $\pi \in P_3F(\sqrt{a})$ such that we have $\varphi = \mathrm{tr}_*(\sqrt{a}\pi')$.

An easy consequence of the representation as a transfer is the following.

Corollary 3.1.8 ([Kar95, Corollary 1.3]):
For any φ as above, there is some Albert form α over F with $\alpha \subseteq \varphi$.

The existence of Albert forms is a main ingredient in the proof of the upper bound for the Pfister number of 14-dimensional forms in I^3F.

Proposition 3.1.9 ([HT98, Proposition 2.3] or [IK00, Proposition 17.2]):
Let $\varphi \in I^3F$ be a quadratic form over F with $\dim\varphi = 14$. Then φ is Witt equivalent to a sum of 3 GP_3-forms. Further the following are equivalent:

(i) there are $\tau_1, \tau_2 \in P_3F$ and $s_1, s_2 \in F^*$ such that φ is Witt equivalent to $s_1\tau_1 \perp s_2\tau_2$;

(ii) there are $\tau_1, \tau_2 \in P_3F$ and $s \in F^*$ such that φ is isometric to $s(\tau_1' \perp -\tau_2')$;

(iii) there is some $\sigma \in GP_2F$ with $\sigma \subseteq \varphi$.

From both [HT98] and [IK00], it is known that for any field F there is a connection between forms of dimension 8 in I^2F and forms of dimension 14 in I^3F.
Further it is known that there are examples of such forms that have 3-Pfister number exactly 3, i.e. the bound in Proposition 3.1.9 is sharp.

Similarly 8-dimensional forms in I^2F whose Clifford invariant has index 4 can have 2-Pfister number 2 or 3 and both values occur , see [HT98, Example 6.3] or [IK00, Theorem 16.7, 17.3]. In [HT98], the *easy* fields were given a name:

Definition 3.1.10:
A field F is called a $D(8)$*-field*, if any 8-dimensional form in I^2F whose Clifford invariant has index 4 is Witt equivalent to a sum of 2 forms in GP_2F.
The field F is called a $D(14)$*-field* if any 14-dimensional form in I^3F is Witt equivalent to a sum of two forms in GP_3F.

As the complement of α in Corollary 3.1.8 is an 8-dimensional I^2F-form, it seems likely that there is some connection involving the properties $D(8)$ and $D(14)$. As an example, we have the following when considering Laurent series extensions.

Proposition 3.1.11 ([HT98, Theorem 3.4, 4.1, 4.4]):
Let F be a field and $K := F((t))$ be the field of formal Laurent series over F.

(a) If F is a $D(8)$-field, it is a $D(14)$-field.

(b) If K has property $D(8)$, so does F.

(c) If K is a $D(14)$-field, F has property $D(8)$ and $D(14)$

(d) if F is a $D(8)$-field, K is a $D(14)$-field.

For I^3, there are no complete results for forms of dimension at least 16. Up to now, there are only lower bounds that we will now repeat here.

Theorem 3.1.12 ([Kar17, Theorem 0.1]):
For any field F there is a field extension E/F such that there is a quadratic form $\varphi \in I^3E$ of dimension $\dim\varphi = 16$ such that $GP_3(\varphi) \geq 4$.

Theorem 3.1.13 ([BRV18, Theorem 1.1]):
Let F be a field and $n \geq 2$ be an even integer. Then there is a field extension E/F and an n-dimensional quadratic form $\varphi \in I^3E$ such that $P_3(\varphi) \geq \frac{2^{(n+4)/4} - n - 2}{7}$.

As an example for easy fields in which we can determine the Pfister numbers precisely, we have n-linked fields, which are a straight forward generalization of the usual linked fields.

Theorem 3.1.14 ([Hof95]):
For any field F and any $n \geq 2$, the following are equivalent:

(i) for any anisotropic form $\varphi \in I^nF$ there are $\pi \in P_{n-1}F$ and $\tau \in IF$ with $\varphi \cong \pi \otimes \tau$;

(ii) any anisotropic form $\varphi \in I^nF$ is isometric to a sum of forms in GP_nF;

(iii) F is n-linked, i.e. for any $\pi_1, \pi_2 \in P_nF$, there is some $\pi \in P_{n-1}F$ and $a, b \in F^*$ with $\varphi \cong \langle\langle a\rangle\rangle \otimes \sigma$ and $\psi \cong \langle\langle b\rangle\rangle \otimes \sigma$.

The second statement in the above result immediately implies the following:

Corollary 3.1.15:
Let F be an n-linked field and $\varphi \in I^n F$ an anisotropic quadratic form with $\dim \varphi = d$. Then, d is divisible by 2^n and φ is isometric to a sum of $\frac{d}{2^n}$ elements in $GP_n F$.

Example 3.1.16:
As an easy consequence of the Holes Theorem, any field F with Hasse number $\tilde{u}(F) < 2^n + 2^{n-1}$ is n-linked. Further $\mathbb{Q}$ and $\mathbb{R}((t))$ are 2-linked (we could even replace $\mathbb{R}$ by any real closed field).

3.2. Auxiliary Results for Forms in I^nF

In this section, we will record some technical results that will be used here and then in the sequel. At first, we record a technique to change the field we are working with that will be essential in the sequel.

Remark 3.2.1:
Let F, E be fields with isomorphic Witt rings $WF \cong WE$. Then there is some group homomorphism $\sigma : F^*/F^{*2} \to E^*/E^{*2}$ as in Theorem 2.1.4 that canonically induces an isomorphism of the respective Witt rings. It is clear that this isomorphism sends n-fold Pfister forms to n-fold Pfister forms and vice versa. Thus, if we have a representation $\varphi = \pi_1 + \ldots + \pi_k$ for some $\pi_1, \ldots, \pi_k \in GP_nF$, we also have $\sigma(\varphi) = \sigma(\pi_1) + \ldots + \sigma(\pi_k)$ with n-fold Pfister forms $\sigma(\pi_1), \ldots, \sigma(\pi_k) \in GP_nE$. As we also can argue the other way round, we have $GP_n(\varphi) = GP_n(\sigma(\varphi))$.

Lemma 3.2.2:
Let $n, m \in \mathbb{N}$ be integers with $n \geq 2$ and m odd, and $\varphi \in I^nF$ be a quadratic form of dimension $2^{n-1}m$ that is divisible by a form $\pi \in GP_{n-1}F$. Then π and thus also φ are hyperbolic.

Proof:
We write $\varphi \cong \pi \otimes \langle a_1, \ldots, a_{2k+1}\rangle$ for suitable $a_1, \ldots, a_{2k+1} \in F^*$, where $m = 2k+1$. We then have an isometry

$$\varphi \cong \perp_{\ell=0}^{k-1} \pi \otimes \langle a_{2\ell+1}, a_{2\ell+2}\rangle \perp a_{2k+1}\pi$$

As both φ and the big sum on the right side of the equation lie in I^nF, we also have $a_{2k+1}\pi \in I^nF$. By the Arason-Pfister Hauptsatz, the latter form has to be hyperbolic. But this can only be the case if π is hyperbolic. This readily implies φ to be hyperbolic. $\square$

As another fact we can classify Pfister neighbors of dimension 6. These are exactly those 6-dimensional forms that are divisible by some 1-fold Pfister form. This result is known, see [Kne77, p. 10] or [Hof98c, Theorem 4.1] for a formulation that is more consistent with our proposition. We include a proof for the reader's convenience.

Proposition 3.2.3:
Let φ be a quadratic form of dimension 6. Then φ is a Pfister neighbor if and only if φ is divisible by a binary form.

Proof:
If we have $\varphi \cong \beta \otimes \langle x, y, z\rangle$, it is a subform of $\beta \otimes \langle x, y, z, xyz\rangle \in GP_3F$ and thus a Pfister neighbor.
Let now φ be a Pfister neighbor. After scaling, we may assume the existence of some $a, x \in F^*$ such that $\pi := x\langle\langle a\rangle\rangle \perp -\varphi \in P_3F$. If π is isotropic and hence hyperbolic, φ is Witt equivalent to $x\langle\langle a\rangle\rangle$, i.e. we have $\varphi \cong x\langle\langle a\rangle\rangle \perp \mathbb{H} \perp \mathbb{H} \cong x\langle\langle a\rangle\rangle \otimes \langle 1, 1, -1\rangle$.
If π is anisotropic, we have $a \notin F^{*2}$. Since we have $\pi, x\langle\langle a\rangle\rangle \in W(F(\sqrt{a})/F)$, we also have $\varphi \in W(F(\sqrt{a})/F)$. Thus, the assertions follows from Theorem 2.5.2. $\square$

The next lemma describes some weak kind of normal form for Pfister forms.

Lemma 3.2.4:
Let π be an anisotropic n-fold Pfister form. Then, there is some $k \leq n$ and $x_1, \ldots, x_k \in F^*$ linearly independent in the $\mathbb{F}_2$ vector space F^*/F^{*2} such that we have

$$\pi \cong \langle\langle x_1, \ldots, x_k, -1, -1, \ldots, -1 \rangle\rangle.$$

Proof:
We write $\pi = \langle\langle a_1, \ldots, a_\ell, \underbrace{-1, \ldots, -1}_{n-\ell \text{ times}} \rangle\rangle$ for some $\ell \leq n$ and $a_1, \ldots, a_\ell \in F^*$. As we assume π to be anisotropic, none of the a_i is a square. If $a_1, \ldots, a_\ell$ are linearly independent, we are done. Otherwise, after renumbering, we have

$$a_\ell \in \operatorname{span}\{a_1, \ldots, a_{\ell-1}\} \setminus F^{*2}.$$

As we have

$$\langle\langle a_1, \ldots, a_{\ell-1} \rangle\rangle = \perp_{\lambda_1, \ldots, \lambda_\ell \in \{0,1\}} \langle (-1)^{\sum_{i=1}^{\ell-1} \lambda_i} \prod_{i=1}^{\ell-1} a_i^{\lambda_i} \rangle$$

by definition of Pfister forms, this implies

$$a_\ell \in D_F(\langle\langle a_1, \ldots, a_{\ell-1} \rangle\rangle') \text{ or } -a_\ell \in D_F(\langle\langle a_1, \ldots, a_{\ell-1} \rangle\rangle'),$$

where the apostrophe denotes the pure part of the Pfister form. In the first case, using Theorem 2.2.1, we have

$$\pi = \langle\langle a_1, \ldots, a_\ell, -1, \ldots, -1 \rangle\rangle = \langle\langle -a_\ell, b_1, \ldots, b_{\ell-2}, a_\ell, -1, \ldots, -1 \rangle\rangle$$

for some $b_1, \ldots, b_{\ell-2} \in F^*$. As both a_ℓ and $-a_\ell$ occur as slots of π in this representation simultaneously, this implies π to be isotropic, a contradiction.
We thus have $-a_\ell \in D_F(\langle\langle a_1, \ldots, a_{\ell-1} \rangle\rangle')$. Using the isometry

$$\langle\langle a_\ell, a_\ell \rangle\rangle = \langle 1, -a_\ell, -a_\ell, 1 \rangle \cong \langle 1, 1, -a_\ell, -a_\ell \rangle = \langle\langle -1, a_\ell \rangle\rangle$$

and again Theorem 2.2.1, we get the existence of $c_1, \ldots, c_{\ell-2} \in F^*$ with

$$\pi = \langle\langle a_\ell, c_1, \ldots, c_{\ell-2}, \underbrace{-1, \ldots, -1}_{n-\ell+1 \text{ times}} \rangle\rangle.$$

Iterating the above procedure will now yield the result. $\square$

As a side fact, we want to remark that the integer k in the above lemma is in general not unique. For example, over the field of formal Laurent series over the rational numbers $\mathbb{Q}((t))$, we have $\langle\langle t, -2, -1 \rangle\rangle = \langle\langle t, -1, -1 \rangle\rangle$ with linear independent sets $\{t, -2\}$ respective $\{t\}$.

We will close this section with two results for forms that are divisible by some I^n-form, but lie in a higher power of the fundamental ideal.

Lemma 3.2.5:
Let $m, n \in \mathbb{N}$ be integers and φ, σ, τ be quadratic forms with $\varphi \in I^m F$ and $\varphi \otimes (\sigma \perp \tau) \in I^{m+n}$. Further let there be some quadratic form τ' with $\sigma \perp \tau' \in I^n F$ and $\dim(\tau \perp -\tau')_{\text{an}} < \frac{2^{m+n}}{\dim \varphi}$. In WF we then have

$$\varphi \otimes (\sigma \perp \tau) = \varphi \otimes (\sigma \perp \tau').$$

In particular, we have

$$GP_{n+m}(\varphi \otimes (\sigma \perp \tau)) \leq GP_m(\varphi) \cdot GP_n(\sigma \perp \tau').$$

Proof:
We will show that the difference of these two forms is hyperbolic which certainly implies the first assertion. In the Witt ring it is given by

$$\varphi \otimes (\tau \perp -\tau')$$

and lies in $I^{m+n}F$ as it is the difference of two forms in $I^{m+n}F$. By assumption, its anisotropic dimension is strictly smaller than 2^{m+n}. Now the Arason-Pfister Hauptsatz implies $\varphi \otimes (\tau \perp -\tau')$ to be hyperbolic and the first claim is established. The second claim then follows readily from the first one by considering minimal representations of φ respectively $\sigma \perp \tau'$ as sums of m-fold respectively n-fold Pfister forms. □

If we strenthen the assumptions, we can say a little bit more.

Lemma 3.2.6:
Let $n \in \mathbb{N}$ be an integer with $n \geq 3$ and $\varphi \in I^nF$ be an anisotropic quadratic form of positive dimension that is divisible by some $\pi \in GP_{n-2}F$. Then, we have $2 \mid \frac{\dim \varphi}{\dim \pi}$ and there is some $\sigma \in I^2F$ with $\varphi \cong \pi \otimes \sigma$ in WF.
In particular, we have

$$GP_n(\varphi) \leq GP_2(\sigma) \leq \frac{\dim \sigma - 2}{2}.$$

Proof:
By assumption, we can find a quadratic form τ over F with $\varphi \cong \pi \otimes \tau$. We first assume further that $2 \mid \dim \tau = \frac{\dim \varphi}{\dim \pi}$ and will show at the and of the proof that this is necessary. If we have $\tau \in I^2F$, we are done. Otherwise we find a suitable $x \in F^*$ and a quadratic form τ' over F with $\tau \cong \tau' \perp \langle x \rangle$. For $d := \pm \det \tau'$, we put $\sigma := \tau' \perp \langle d \rangle$. For a suitable choice of the sign depending on $\dim \tau'$, we have $\sigma \in I^2F$. In WF we then have

$$\begin{aligned}\varphi &= \psi \otimes \tau = \pi \otimes (\tau' \perp \langle x \rangle) \\ &= \pi \otimes (\tau' \perp \langle d \rangle \perp \langle x, -d \rangle) = \pi \otimes \sigma \perp \pi \otimes \langle x, -d \rangle.\end{aligned}$$

As we have $\varphi, \pi \otimes \sigma \in I^nF$, we also have $\pi \otimes \langle x, -d \rangle \in I^nF$. But the Arason-Pfister Hauptsatz now implies the latter form to be hyperbolic. We thus have $\varphi = \pi \otimes \sigma$ in WF and as the dimensions of both forms coincide, we even have the isometry $\varphi \cong \pi \otimes \sigma$.
If $\dim \tau$ were odd with determinant $d := \det \tau$, we would have

$$\varphi = \pi \otimes \tau = \pi \otimes (\tau \perp \langle \pm d \rangle \perp \langle \mp d \rangle) = \pi \otimes (\tau \perp \langle \pm d \rangle) \perp \mp d\pi.$$

As above, for a suitable choice of the sign, we have $\sigma := \tau \perp \langle \pm d \rangle \in I^2F$ and thus $\pi \otimes \sigma \in I^nF$. As above, this implies $\mp d\pi$ to be hyperbolic. Hence π would be hyperbolic, contradiction the hypothesis.
As now the first claim is established, the second one is clear. □

3.3. Upper Bounds for Forms with Good Subforms

A common strategy to find upper bounds for the Pfister number of some given form is to use well understood subforms to find a representation as a sum of Pfister forms. For forms in I^nF, subforms in $I^{n-1}F$ often are a good starting point, see e.g. the proof of [HT98, Proposition 2.3]. Here we collect some upper bounds for the Pfister number of forms with a subform of good shape. Some of these results will be used in a later chapter for forms over some fields that are to be specified.
It should be said that in general, it is known that I^3-forms of dimension at most 14 always contain some proper subform in I^2F as can be readily verified with the results in Section 3.1. It is further known that for any even integer $d \geq 18$, there is a field F and an I^3-form of dimension d that does not contain any subform in I^2F. For dimension 16, this is still an open problem, see [CM14, Theorem 4.2].

Proposition 3.3.1:
Let $\varphi \in I^3F$ be an anisotropic quadratic form with $\dim \varphi = 16$. We further presume the existence of some $\sigma, \tau \in GP_2F$ with $\sigma \perp \tau \subseteq \varphi$. Then φ is Witt equivalent to a sum of at most three elements in GP_3F.

Proof:
By our assumption, we have $\varphi \cong \sigma \perp \tau \perp \langle w \rangle \perp \psi$ for some $w \in F^*$ and a 7-dimensional quadratic form ψ over F. We choose $x, y, z \in F^*$ such that $\langle w, x, y, z \rangle$ is similar to σ. This implies in particular $\sigma \perp \langle w, x, y, z \rangle \in GP_3F$. In WF we thus have

$$\varphi = \sigma + \tau + \langle w \rangle + \psi = (\sigma \perp \langle w, x, y, z \rangle) + (\tau \perp \langle -x, -y, -z \rangle \perp \psi).$$

Since we have $\varphi, \sigma \perp \langle w, x, y, z \rangle \in I^3F$, we also have $\tau \perp \langle -x, -y, -z \rangle \perp \psi \in I^3F$. Further we have $\dim(\tau \perp \langle -x, -y, -z \rangle \perp \psi) = 14$ and this form contains $\tau \in GP_2F$ as a subform. Thus $\tau \perp \langle -x, -y, -z \rangle \perp \psi$ is Witt equivalent to a sum of at most two GP_3F-forms by Proposition 3.1.9 and the conclusion follows. □

Having developed further theory we can even show that the above upper bound cannot be improved, see Example 4.3.7.

Corollary 3.3.2:
Let $\varphi \in I^3F$ be an anisotropic form with $\dim \varphi = 16$, such that there is a quadratic field extension E/F with $i_W(\varphi_E) \geq 3$. Then φ is Witt equivalent to a sum of at most 3 forms in GP_3F.

Proof:
Let E/F be a quadratic extension as stated above. We then have $\varphi_E \in I^3E$ with $\dim(\varphi_E)_{\mathrm{an}} \leq 10$. Because of the Holes Theorem we even have $\dim(\varphi_E)_{\mathrm{an}} \leq 8$. By Theorem 2.5.2, φ thus has a subform of the shape

$$\langle\langle a \rangle\rangle \otimes \langle x, y, z, w \rangle = \underbrace{\langle\langle a \rangle\rangle \otimes \langle x, y \rangle}_{=:\sigma} \perp \underbrace{\langle\langle a \rangle\rangle \otimes \langle z, w \rangle}_{=:\tau}.$$

As we obviously have $\sigma, \tau \in GP_2F$, the conclusion follows by Proposition 3.3.1. □

For forms as above, we can be even more precise and get the following.

Proposition 3.3.3:
Let $\varphi \in I^3F$ be a quadratic form of dimension $\dim \varphi = 16$. We further assume there are $a \in F^*$ and some forms σ, ψ over F with $\dim \sigma = 4, \dim \psi = 8$ and $\varphi \cong \langle\langle a \rangle\rangle \otimes \sigma \perp \psi$.

Then ψ is divisible by a binary form. In particular, we have $\varphi_K = 0$ for some multiquadratic extension K/F of degree ≤ 4.

Proof:
Since we have $\langle\langle a\rangle\rangle \otimes \sigma \in I^2F$ and $\varphi \in I^3F \subseteq I^2F$, we have $\psi \in I^2F$. Using the same argument for the third power of the fundamental ideal, we further know $\langle\langle a\rangle\rangle \otimes \sigma \in I^3F$ if and only if $\psi \in I^3F$. In this case, both forms would be 3-fold Pfister forms by the Arason-Pfister Hauptsatz, hence divisible by a binary form.
Otherwise we have $\operatorname{ind}\Big(c\big(\langle\langle a\rangle\rangle \otimes \sigma\big)\Big) = 2$ by [HT98, Proposition 2.5]. As we have $\langle\langle a\rangle\rangle \otimes \sigma \perp \psi \in I^3F$, i.e. $c\big(\langle\langle a\rangle\rangle \otimes \sigma \perp \psi\big) = 1$, it follows $c\big(\langle\langle a\rangle\rangle \otimes \sigma\big) = c(\psi)$ which then implies $\operatorname{ind}\big(c(\psi)\big) = 2$. Now [HT98, Proposition 2.5] and Theorem 2.5.2 imply ψ to be divisible by a binary form. □

Proposition 3.3.4:
Let $\varphi \in I^mF$ be similar to a twisted Pfister form of dimension 2^n for some $m, n \in \mathbb{N}$ with $1 \leq m < n$. Then φ is isometric to a sum of 2^{n-m} forms in GP_mF.

Proof:
As φ is a twisted Pfister form, there are $s \in F^*, \sigma \in P_nF$ and $\pi \in P_mF$ with $\varphi = s(\sigma - \pi) \in WF$. After scaling, we may assume $s = 1$. By Theorem 2.2.13, there exists an $(m-1)$-fold Pfister form $\alpha \in GP_{m-1}F$ and $x_1, \ldots, x_{2^{n-m+1}}$ with

$$\varphi \cong \alpha \otimes \langle x_1, \ldots, x_{2^{n-m+1}}\rangle.$$

Putting $\pi_k := \alpha \otimes \langle x_{2k-1}, x_{2k}\rangle \in GP_mF$ for $k \in \{1, \ldots, 2^{n-m}\}$, we obtain

$$\varphi = \pi_1 + \ldots + \pi_{2^{n-m}} \in WF$$

and the conclusion follows. □

Proposition 3.3.5:
Let $\psi \in I^nF$ and $a \in F^*$. For the form $\varphi := \langle\langle a\rangle\rangle \otimes \psi \in I^{n+1}F$, we have

$$GP_{n+1}(\varphi) \leq GP_n(\psi).$$

Proof:
It is clear that we have $\varphi \in I^{n+1}F$ as we have $\psi \in I^nF$ and $\langle\langle a\rangle\rangle \in IF$. Let now $\pi_1, \ldots, \pi_k \in GP_nF$ with $\psi = \pi_1 + \ldots + \pi_k \in WF$. We then have

$$\varphi = \langle\langle a\rangle\rangle \otimes \pi_1 + \ldots + \langle\langle a\rangle\rangle \otimes \pi_k \in WF$$

with $\langle\langle a\rangle\rangle \otimes \pi_\ell \in GP_{n+1}F$ for all $\ell \in \{1, \ldots, k\}$ as desired. □

Many of the results here heavily depend on the field we are working over. If the ground field is easier to handle, we can weaken the assumptions or get stronger conclusions. As an example, for a $D(14)$-field, we just need a single 2-fold Pfister form as a subform to get the same upper bound as in Proposition 3.3.1.

Lemma 3.3.6:
Let F be a $D(14)$-field and $\varphi \in I^3F$ be an anisotropic quadratic form of dimension 16. We further assume there is some $\sigma \in GP_2F$ with $\sigma \subseteq \varphi$. Then φ is Witt equivalent to a sum of at most three elements in GP_3F

Proof:
We write $\varphi = \sigma \perp \langle w \rangle \perp \psi$ for some $w \in F^*$ and a suitable quadratic form ψ of dimension 11 over F. As in the proof of Proposition 3.3.1, we can find $x, y, z \in F^*$ such that we have $\sigma \perp \langle w, x, y, z \rangle \in GP_3F$. In WF we thus have

$$\varphi = \sigma \perp \langle w \rangle \perp \psi = \sigma \perp \langle w, x, y, z \rangle \perp \psi \perp \langle -x, -y, -z \rangle.$$

We further have $\dim(\psi \perp \langle -x, -y, -z \rangle) = 14$ so that this form is Witt equivalent to a sum of at most two GP_3F-forms by assumption. This obviously implies the assertion. □

3.4. Pfister Numbers of Complete Discrete Valuation Fields

In this section, let F be a field equipped with a complete discrete valuation with residue field K. In this case, all forms over K can be lifted to forms over F in a canonical way as explained in Section 2.4. In fact, they even can be lifted in a way that respects the powers of the fundamental ideal in the sense of the following result.

Lemma 3.4.1 ([EKM08, Exercise 19.15]):
For all $n \in \mathbb{N}$ we have a split exact sequence

$$0 \longrightarrow I^n K \longrightarrow I^n F \longrightarrow I^{n-1} K \longrightarrow 0$$

where the maps are given by lifting and taking the second residue class form.

Remark 3.4.2:
One step in the proof of the fact that taking residue class forms in the above result is a well defined map $I^n F \to I^{n-1} K$ is to see that any n-fold Pfister form can be written such that at most one slot is a uniformizing element and all the other slots are units in the valuation ring. This is a direct consequence of the isometry

$$\langle\langle a, b \rangle\rangle \cong \langle\langle a, -ab \rangle\rangle,$$

that can readily be verified.

Corollary 3.4.3:
Let $\varphi \in I^n F$ be a unimodular form. Then, the n-Pfister number of φ over F and of $\overline{\varphi}$ over K coincide.

Proof:
If we have $\overline{\varphi} = \overline{\pi}_1 + \ldots + \overline{\pi}_k$ for some Pfister forms $\overline{\pi}_1, \ldots, \overline{\pi}_k \in GP_n K$, we can lift them to get a representation $\varphi = \pi_1 + \ldots + \pi_k$ by Lemma 3.4.1.
For the converse, we fix a uniformizing element t and consider a representation

$$\varphi = \pi_1 + \ldots + \pi_k + \tilde{\pi}_1 \otimes \langle\langle c_1 t \rangle\rangle + \ldots + \tilde{\pi}_\ell \otimes \langle\langle c_\ell t \rangle\rangle + t\hat{\pi}_1 + \ldots + t\hat{\pi}_m$$

with unimodular forms $\pi_1, \ldots, \pi_k, \hat{\pi}_1, \ldots, \hat{\pi}_m \in GP_n F$ and $\pi_1, \ldots, \pi_\ell \in GP_{n-1} F$ and $c_1, \ldots, c_\ell \in F^*$, see Remark 3.4.2. By comparing both residue class forms, we see that in WF, we have equalities

$$\varphi = \pi_1 + \ldots + \pi_k + c_1\tilde{\pi}_1 + \ldots + c_\ell\tilde{\pi}_\ell \text{ and } c_1\tilde{\pi}_1 + \ldots + c_\ell\tilde{\pi}_\ell = \hat{\pi}_1 + \ldots + \pi_m.$$

This implies

$$\varphi = \pi_1 + \ldots + \pi_k + \hat{\pi}_1 + \ldots + \hat{\pi}_m,$$

where all forms are unimodular. Thus, the claim follows. □

With a similar idea, we can prove the following equality of Pfister forms.

Proposition 3.4.4:
Let $\psi \in I^{n-1} K$ be a unimodular form and $\varphi := \langle\langle t \rangle\rangle \otimes \psi$ for some uniformizer t. We then have $GP_{n-1}(\psi) = GP_n(\varphi)$.

Proof:
The inequality $GP_{n-1}(\psi) \geq GP_n(\varphi)$ follows directly from Proposition 3.3.5. For the converse, we consider a representation

$$\varphi = \pi_1 + \ldots + \pi_k + \tilde{\pi}_1 \otimes \langle\langle c_1 t\rangle\rangle + \ldots + \tilde{\pi}_\ell \otimes \langle\langle c_\ell t\rangle\rangle + t\hat{\pi}_1 + \ldots + t\hat{\pi}_m$$

as in the proof of Corollary 3.4.3. After comparing residue class forms, we see that we have

$$\pi_1 + \ldots + \pi_k + \tilde{\pi}_1 + \ldots + \tilde{\pi}_\ell = \psi = -\hat{\pi}_1 - \ldots - \hat{\pi}_m + c_1\tilde{\pi}_1 + \ldots + c_\ell\tilde{\pi}_\ell.$$

These are representations of ψ as a sum of $2k + \ell$ respectively $2m + \ell$ forms in $GP_{n-1}F$. If we had $k + \ell + m < GP_{n-1}F(\psi)$ one of the terms $2k + \ell$ and $2m + \ell$ would also be strictly smaller than $GP_{n-1}F(\psi)$, a contradiction. Thus, we have $GP_{n-1}(\psi) \leq GP_n(\varphi)$ and the proof is complete. $\square$

Lemma 3.4.1 further leads us to the following result:

Proposition 3.4.5:
Let $\varphi \in I^nF$ be a quadratic form such that both residue class forms are not hyperbolic. Then there is uniformizer π, unimodular forms $\sigma \in I^nF$ and $\tau \in I^{n-1}F$ with $\varphi = \sigma \perp \langle\langle -\pi\rangle\rangle \otimes \tau \in WF$ and $\dim\sigma < \dim\varphi$.

Proof:
We denote the first respectively second residue class forms of φ with respect to some uniformizing element π with φ_1 respectively φ_2. We then have

$$\varphi = \varphi_1 \perp \pi\varphi_2 = \varphi_1 \perp -\varphi_2 \perp \varphi_2 \perp \pi\varphi_2 = \varphi_1 \perp -\varphi_2 \perp \langle\langle -\pi\rangle\rangle \otimes \varphi_2. \tag{3.3}$$

After multiplying π with some unit of the valuation ring, i.e. changing the uniformizer, we can assume $D_F(\varphi_1) \cap D_F(\varphi_2) \neq \varnothing$. Then the form $\varphi_1 \perp -\varphi_2$ is isotropic. If we choose

$$\sigma := (\varphi_1 \perp -\varphi_2)_{\text{an}} \text{ and } \tau := \varphi_2,$$

we have $\dim\sigma < \dim\varphi$ and $\tau \in I^{n-1}F$ by Lemma 3.4.1. Finally (3.3) implies $\varphi \equiv \varphi_1 \perp -\varphi_2 \mod I^nF$, which then leads to $\sigma := (\varphi_1 \perp -\varphi_2)_{\text{an}} \in I^nF$. $\square$

With the above result, we are now in the position to bound the Pfister numbers of forms over a complete discrete valuation field in terms of Pfister numbers over the associated residue class field. As a first step, we record the following special case which follows directly by Proposition 3.4.5.

Corollary 3.4.6:
Let φ be as in Proposition 3.4.5. Then its n-Pfister number is bounded by

$$GP_n(K, \dim(\varphi) - 2) + GP_{n-1}\left(K, \frac{1}{2}\dim(\varphi)\right).$$

Proof:
We use the notation as in the proof of Proposition 3.4.5. Since the Pfister number of any form is invariant under scaling, we can assume $\dim\varphi_2 \leq \frac{1}{2}\dim\varphi$. We thus get

$\sigma \in I^n F$ and $\tau \in I^{n-1} F$ such that we have a representation $\varphi = \sigma + \langle\langle -\pi \rangle\rangle \otimes \tau$ in the Witt ring WF with some suitable uniformizer π and

$$\dim \sigma \leq \dim \varphi - 2 \text{ and } \dim \tau \leq \frac{1}{2} \dim \varphi,$$

where the first inequality can be assumed by Proposition 3.4.5 since both residue forms are not hyperbolic. By Corollary 3.4.3 and Proposition 3.3.5, we have

$$GP_n\left(\langle\langle -\pi \rangle\rangle \otimes \tau\right) \leq GP_{n-1}\left(K, \frac{1}{2}\dim(\varphi)\right)$$

and the result now follows. □

As the main result of this section, we have the following:

Theorem 3.4.7:
Let F be complete discrete valuation field such that the characteristic of the residue class field K is not equal to 2. Then for all $n \in \mathbb{N}$ and all $d \in 2\mathbb{N}$, we have

$$GP_n(F, d) \leq \max\left\{GP_n(K, d-2) + GP_{n-1}\left(K, \frac{d}{2}\right), GP_n(K, d)\right\}.$$

Proof:
For any d-dimensional quadratic form $\varphi \in I^n F$, either one of its residue class forms is not hyperbolic or φ is similar to an unimodular form. The claim now follows by Corollary 3.4.6 and Corollary 3.4.3. □

3.5. Forms in I^nF of Dimension $2^n + 2^{n-1}$

According to the Arason-Pfister Hauptsatz and the Holes Theorem, the first dimension for forms in I^nF that has to be studied is $2^n + 2^{n-1}$: lower dimensional forms in I^nF are either hyberbolic or Witt equivalent to an anisotropic form in GP_nF. It is thus convenient to have a closer look at those forms.
We start this section by proving a generalization of [Hof98a, Proposition 4.1] and parts of the just mentioned result for arbitrary powers of the fundamental ideal. We will mainly use the same techniques as in the original article, but we will further use the Holes Theorem that was not known when [Hof98a] was published.
Before stating and proving the result, we would like to note that the case $n = 2$ is trivial and that the case $n = 3$ can essentially be found in [Pfi66].

Proposition 3.5.1:
Let $\varphi \in I^nF$ be an anisotropic form of dimension $\dim\varphi = 2^n + 2^{n-1}$ for some $n \in \mathbb{N}$ with $n \geq 2$. Then the following are equivalent:

(i) there are $\pi_1, \pi_2 \in GP_nF$ with $\varphi = \pi_1 + \pi_2 \in WF$, i.e. $GP_n(\varphi) = 2$;

(ii) there is some $\pi \in GP_{n-2}F$ and an Albert form α with $\varphi \cong \pi \otimes \alpha$;

(iii) there are $\sigma_1, \sigma_2, \sigma_3 \in GP_{n-1}F$ with $\varphi \cong \sigma_1 \perp \sigma_2 \perp \sigma_3$, i.e. $GP_{n-1}(\varphi) = 3$;

(iv) there is some $\sigma \in GP_{n-1}F$ with $\sigma \subseteq \varphi$;

(v) there is some Pfister neighbor $\psi \subseteq \varphi$ of dimension $2^{n-1} + 1$.

Proof:
By the remark above, we only need to consider the case $n \geq 4$.

(i)⇒(ii): According to Theorem 2.2.13 the n-fold Pfister forms that π_1 respectively π_2 are similar to have linkage number $n-2$, i.e. there is some $\pi \in GP_{n-2}F$ that divides both π_1 and π_2 and thus φ. By Theorem 2.5.14 we have $\varphi \cong \pi \otimes \sigma$ for some quadratic form α of dimension 6. As we have $n \geq 4$, we can assume $\alpha \in I^2F$ by Lemma 3.2.6.

(ii)⇒(iii): We decompose $\alpha = \alpha_1 \perp \alpha_2 \perp \alpha_3$ with binary forms $\alpha_1, \alpha_2, \alpha_3$. We now just have to put $\sigma_i := \pi \otimes \alpha_i$ for $i \in \{1, 2, 3\}$.

(iii)⇒(iv): This is trivial.

(iv)⇒(v): We write $\varphi = \sigma \perp \langle x, \ldots\rangle$ for some suitable $x \in F^*$. Then $\psi := \sigma \perp \langle x\rangle$ is a Pfister neighbor of dimension $2^{n-1} + 1$ of the Pfister form $\sigma \perp x\sigma$.

(v)⇒(i): As ψ is a Pfister neighbor of dimension $2^{n-1} + 1$, there is some $\pi_1 \in GP_nF$ with $\psi \subseteq \pi_1$. We then have $\pi_2 := (\varphi \perp -\pi)_{\mathrm{an}} \in I^nF$ and this form is of dimension at most $2^n + 2^{n-1} + 2^n - 2\cdot(2^{n-1} + 1) = 2^n + 2^{n-1} - 2$. By the Holes Theorem and the Arason-Pfister Hauptsatz we have $\pi_2 \in GP_nF$ and the conclusion follows.

□

Remark 3.5.2:
For another proof of the equivalence of (ii) and (iii), please further note [Lam05, Chapter X, Linkage Theorem 6.22].

To conclude this section, we would like to study the forms of dimension $2^n + 2^{n-1}$ in I^nF under field extensions. To do so, we first introduce the following term.

Definition 3.5.3:
We say that a field F has property $P(n)$ if every anisotropic quadratic form $\varphi \in I^nF$ of dimension $2^n + 2^{n-1}$ satisfies the equivalent conditions in Proposition 3.5.1.

The notation $P(n)$ alludes to the fact that the property describes the n-Pfister number of certain form. As we know that any field has property $P(n)$ for $n \in \{2, 3\}$, we will exclude these cases in the following (even though the proof works for $n = 3$).

Theorem 3.5.4:
Let F be a field and $n \geq 4$ be an integer. Then $F((t))$ satisfies $P(n)$ if and only if F satisfies $P(n)$ and $P(n-1)$.

Proof:
First let $F((t))$ satisfy $P(n)$. Let $\varphi \in I^nF$ be a form of dimension $2^n + 2^{n-1}$. We then have $\varphi \in I^nF((t))$. Thus there are Pfister forms $\pi_1, \pi_2 \in GP_nF((t))$ such that we have $\varphi = \pi_1 + \pi_2 \in WF((t))$. By Proposition 3.4.3, φ satisfies Proposition 3.5.1 (i).
To show $P(n-1)$, let now $\psi \in I^{n-1}F$ be an anisotropic form of dimension $2^{n-1} + 2^{n-2}$. Then $\psi \otimes \langle\langle t \rangle\rangle$ is an anisotropic form of dimension $2^n + 2^{n-1}$, see Proposition 2.4.2, and obviously lies in $I^nF((t))$. It is thus Witt equivalent to the sum of two n-fold Pfister forms over $F((t))$, and therefore satisfies Proposition 3.5.1 (i). By Proposition 3.4.4, we know that already ψ satisfies Proposition 3.5.1 (i) which concludes the first part of the proof.
Let now F satisfy $P(n)$ and $P(n-1)$. We choose an anisotropic quadratic form $\varphi \in I^nF((t))$ of dimension $2^n + 2^{n-1}$. If φ is similar to a form defined over F, it is similar to a form in I^nF of the same dimension according to Proposition 2.5.12 and this case is done.
Otherwise we consider the residue forms $\varphi = \varphi_1 \perp t\varphi_2$, where we can assume after scaling the inequalities $0 < \dim \varphi_2 \leq \dim \varphi_1$. As we have $\varphi_1, \varphi_2 \in I^{n-1}F$ by Lemma 3.4.1, we have $\dim \varphi_2 \in \{2^{n-1}, 2^{n-1} + 2^{n-2}\}$ according to the Holes Theorem. If $\dim \varphi_2 = 2^{n-1}$, Proposition 3.5.1 (iv) is fulfilled.
Otherwise we have $\dim \varphi_1 = \dim \varphi_2 = 2^{n-1} + 2^{n-2}$ and $\varphi_1, \varphi_2 \in I^{n-1}F$. As we assume F to satisfy $P(n-1)$, there is some $\pi \in GP_{n-3}F$ and an Albert form α over F with $\varphi_1 \cong \pi \otimes \alpha$. As Pfister forms become hyperbolic over their function field, the form $(\varphi_1)_{F(\pi)}$ is also hyperbolic. We thus have

$$\dim(\varphi_{F(\pi)})_{\text{an}} \leq \dim\big((\varphi_2)_{F(\pi)}\big)_{\text{an}} = 2^{n-1} + 2^{n-2} < 2^n.$$

By the Arason-Pfister Hauptsatz, $\varphi_{F(\pi)} \in I^nF(\pi)$ is hyperbolic. Thus φ_2 becomes hyperbolic over $F(\pi)$ as well and is thus divisible by π, see Theorem 2.5.14. Using Lemma 3.2.6, we thus have a representation $\varphi_2 \cong \pi \otimes \beta$ with an Albert form β. As we clearly have

$$\varphi_1 = \pi \otimes \alpha \equiv \pi \otimes \beta = \varphi_2 \mod I^nF,$$

the Holes Theorem and [Hof99, Proposition 1] imply that $\pi \otimes \alpha$ is similar to $\pi \otimes \beta$.

Thus, there is some $x \in F^*$ with

$$\varphi \cong \varphi_1 \perp t\varphi_2 \cong \pi \otimes \alpha \perp t\pi \otimes \beta \cong \pi \otimes \alpha \perp -xt\pi \otimes \alpha = \langle\langle xt \rangle\rangle \otimes \pi \otimes \alpha$$

i.e. Proposition 3.5.1 (ii) is fulfilled with $\pi \otimes \langle\langle xt \rangle\rangle \in GP_{n-2}F((t))$ and Albert form α. □

Corollary 3.5.5:
Let E be a linked field, I be an arbitrary index set and $F := E((t_i))_{i \in I}$. Then F has property $P(n)$ for all $n \geq 2$.

Proof:
For $n \in \{2, 3\}$, there is nothing more to show. With Proposition 3.A.2 and Remark 2.4.9 in mind, it is enough to verify the property $P(n)$ for all $n \in \mathbb{N}$ with $n \geq 3$ for linked fields. But this property is trivially fulfilled as over these fields, there are no anisotropic forms of dimension $2^n + 2^{n-1}$ in $I^n F$, see Corollary 3.1.15 □

We will now introduce another property concerning quadratic forms in $I^n F$ of dimension at most $2^n + 2^{n-1}$ that correlates with the property $P(n)$.

Definition 3.5.6:
Let $n \in \mathbb{N}$ be an integer with $n \geq 2$. We say a field F has property $\mathrm{Sim}(n)$ if any two non-zero anisotropic forms in $I^n F$ of dimension at most $2^n + 2^{n-1}$ that are congruent modulo $I^{n+1}F$ are similar.

It is well known that any field satisfies property $\mathrm{Sim}(2)$, see [Lam05, Chapter XII. Theorem 2.9].

Corollary 3.5.7:
Let n be an integer with $n \geq 2$ and let F be a field that satisfies $P(n)$ and $P(n+1)$. Then F satisfies $\mathrm{Sim}(n)$.

Proof:
Let $\varphi_1, \varphi_2 \in I^n F$ be anisotropic forms with $\dim \varphi_1 = 2^n + 2^{n-1} = \dim \varphi_2$ and $\varphi_1 \equiv \varphi_2 \mod I^{n+1}F$. If we have $\dim \varphi_1 = \dim \varphi_2 = 2^n$, we have $\varphi_1, \varphi_2 \in GP_n F$ and the result follows from [Lam05, Chapter X. Corollary 5.4]. For the remaining cases, we consider the form $\varphi := \varphi_1 \perp t\varphi_2$ over the field $K := F((t))$. It is an anisotropic form of dimension $2^{n+1} + 2^n$ and lies in $I^{n+1}K$ as we have

$$\varphi = \varphi_1 + t\varphi_2 \equiv \varphi_1 - \varphi_2 + \langle\langle -t \rangle\rangle \otimes \varphi_2 \equiv 0 \mod I^n K$$

by hypothesis. The Holes Theorem now implies that we necessarily have $\dim \varphi_1 = 2^n + 2^{n-1} = \dim \varphi_2$. By Theorem 3.5.4, we have $\varphi \cong \pi \otimes \alpha$ for some $\pi \in P_{n-1}K$ and some Albert form α over K. As φ_1 and φ_2 are exactly the residue class forms of φ concerning the t-adic valuation and as they are both of the same dimension, one of the residue class forms of α has to be zero: otherwise, one of its residue class forms would have dimension 4 and the other one dimension 2 which would then readily lead to a contradiction. After eventually multiplying some slot of π with t, we may assume α to be an Albert form over F and π to be of the form $\pi \cong \langle\langle xt \rangle\rangle \otimes \pi_0$ for some $x \in F^*$ and some Pfister form $\pi_0 \in P_{n-2}F$. By comparing residue class forms, we thus obtain

$$\varphi_1 = \pi_0 \otimes \alpha \quad \text{ and } \quad \varphi_2 = -x\pi_0 \otimes \alpha.$$

□

3.A. Appendix: Another approach to forms of dimension $2^n + 2^{n-1}$ in I^n

In collaboration with K. Becher, an alternative version of Theorem 3.5.4 could be proved. We will provide a proof in the appendix to this chapter. The crux is to study the behaviour of the property $\mathrm{Sim}(n)$ when going up from a field F to its Laurent series extension $F((t))$, i.e. the proof here works the other way round than the one presented in the previous section. This approach can even be generalized to the theory of abstract Witt rings, but as this is not important for the rest of the thesis, we will not present the most general proof we found here.

Proposition 3.A.1:
Let $n \geq 3$ be an integer. If F fulfils $\mathrm{Sim}(n-1)$ and $\mathrm{Sim}(n)$ then $F((t))$ fulfils $\mathrm{Sim}(n)$.

Proof:
Let $\varphi_i = \alpha_i \perp t\beta_i \in I^nF((t))$ be anisotropic forms with $2 \cdot \dim(\beta_i) \leq \dim(\varphi) \leq 2^n + 2^{n-1}$ (possibly after scaling with t) decomposed into residue class forms. We then have

$$(\alpha_1 \perp -\alpha_2) \perp t(\beta_1 \perp -\beta_2) = \varphi_1 \perp -\varphi_2 \in I^{n+1}F((t)),$$

which leads to

$$\alpha_1 \equiv \alpha_2 \equiv \beta_1 \equiv \beta_2 \mod I^nF.$$

As we assume F to fulfil $\mathrm{Sim}(n-1)$, there is some $y \in F^*$ with $\beta_2 \cong y\beta_1$. In particular, this implies $\beta_1 = 0$ if and only if $\beta_2 = 0$. Thus the case $\beta_1 = 0$ is clear as we assume F to fulfil $\mathrm{Sim}(n)$, so that we can exclude this case in the sequel.
For $\dim(\varphi_1) = \dim(\varphi_2) = 2^n$, i.e. $\varphi_1, \varphi_2 \in GP_n(F((t)))$, the result is well known, see [Lam05, Chapter X. Corollary 5.4].
In the remaining cases, in $WF((t))$ we have

$$\alpha_1 - y\alpha_2 = \varphi_1 - y\varphi_2 = \varphi_1 - \varphi_2 + \langle\langle y\rangle\rangle \cdot \varphi_2 \equiv \varphi_1 - \varphi_2 \equiv 0 \mod I^{n+1}F((t)). \tag{3.4}$$

If we have $\dim(\varphi_1) \neq \dim(\varphi_2)$, the form $\alpha_1 \perp -y\alpha_2$ has dimension at most $2^n + 2^{n-1} < 2^{n+1}$ and is thus hyperbolic by the Arason-Pfister Hauptsatz.
So let finally be $\dim(\varphi_1) = \dim(\varphi_2) = 2^n + 2^{n+1}$ and let $\beta_1 \neq 0 \neq \beta_2$. We have $\dim(\beta_1) = \dim(\beta_2) \in \{2^{n-1}, \frac{3}{2} \cdot 2^{n-1}\}$. We will now show the existence of some $x \in F^*$ with $\alpha_2 \cong x\alpha_1$ for both cases separately.
In the latter case, we have $\dim(\alpha_1) = \dim(\alpha_2) = \frac{3}{2} \cdot 2^{n-1}$ and $\alpha_1 \equiv \alpha_2 \mod I^nF$, so the existence of x follows from $\mathrm{Sim}(n-1)$ for F. In the former case, we have $\dim(\alpha_1) = \dim(\alpha_2) = 2^n$; i.e these forms are twisted Pfister forms by [Hof96, Proposition 3.11 (i)]. As (3.4) implies $\alpha_1 \equiv y\alpha_2 \mod I^{n+1}F$, the existence of x follows from [Hof98b, Proposition 2.8]. In $WF((t))$ we further have

$$\begin{aligned} t(\beta_1 - x\beta_2) &= \varphi_1 - x\varphi_2 \\ &= \varphi_1 - \varphi_2 + \langle\langle x\rangle\rangle \cdot \varphi_2. \end{aligned}$$

We thus have $\beta_1 - x\beta_2 \in I^{n+1}F((t))$. But as this form has dimension at most $3 \cdot 2^{n-1} < 2^{n+1}$, it is hyperbolic by the Arason-Pfister Hauptsatz. We thus have $\beta_1 \cong x\beta_2$, which is equivalent to $\beta_2 \cong x\beta_1$. Summarising, we have

$$x\varphi_1 \cong x\alpha_1 \perp tx\beta_1 \cong \alpha_2 \perp t\beta_2 \cong \varphi_2,$$

as claimed. $\square$

Proposition 3.A.2:
If F has $P(n-1), P(n)$ and $\mathrm{Sim}(n-1)$ for some $n \geq 3$ then $F((t))$ has property $P(n)$.

Proof:
Let $\varphi = \varphi_1 \perp t\varphi_2 \in I^n F((t))$ be a form of dimension $\frac{3}{2} \cdot 2^n$ with $\dim(\varphi_2) \leq \frac{1}{2} \cdot \dim(\varphi) = \frac{3}{2} \cdot 2^{k-1}$ decomposed into residue class forms. If we have $\varphi_2 = 0$, we even have $\varphi \in I^n F$ and the result is clear. If we have (possibly after scaling) $\varphi_2 \in P_{n-1}F$, we choose some $x \in D_F(\varphi_1)$. We then have

$$\varphi = \varphi_1 + t\varphi_2 = \varphi_1 - x\varphi_2 + x\varphi_2 + t\varphi_2 = (\varphi_1 - x\varphi_2) + \langle x, t\rangle \otimes \varphi_2.$$

As the anisotropic part of the first summand has dimension smaller than $\dim(\varphi)$ by choice of x, it is in $GP_n(F)$ by the Holes Theorem. It is further clear that the second summand lies in $GP_n(F((t)))$ so this case is done.
In the last case, we have $\varphi_2 \in I^{n-1}F, \dim(\varphi_2) = \frac{3}{2} \cdot 2^{n-1}$. As we have

$$\varphi_1 \equiv \varphi_2 \mod I^n F$$

with $\varphi_1 \in I^{n-1}F$ and $\dim(\varphi_1) = \frac{3}{2} \cdot 2^{n-1}$, we have $\varphi_1 \cong a\varphi_2$ for some $a \in F^*$ as F fulfills $\mathrm{Sim}(n-1)$. This now implies

$$\varphi = \varphi_1 \perp t\varphi_2 = a\varphi_2 \perp t\varphi_2 = \langle a, t\rangle \otimes \varphi_2,$$

which implies the assertion. □

Of course we can deduce Corollary 3.5.5 from Proposition 3.A.2 as before.

4. Rigid Fields

4.1. Introduction to the Theory of Rigid Fields

Inspired by the work of M. Raczek [Rac13], we will prove upper bounds for the Pfister number of so called rigid fields. Using similar arguments, we generalize a lot of the arguments used in the just cited article. In the theory of quadratic forms, rigid fields are of interest because of several reasons. Firstly, they are simple enough to handle to build up a theory that already started in the late 1970s, see [War78]. As an example, there are a lot of interesting Galois-theoretic results available for rigid fields. Furthermore, nonreal rigid fields with a finite number of square classes are examples of the so called $\overline{C}$-fields. These are extreme examples as these are those fields that have the maximal number of anisotropic quadratic forms that can occur, when considering nonreal fields with finitely many square classes, see [Lam05, Chapter XI., Theorem 7.10, 7.14, Definition 7.16].

Definition 4.1.1:
A field F is called *rigid*, if, for any binary anisotropic quadratic form β over F, we have $|D_F(\beta)| \leq 2$.

Example 4.1.2:
As the square class groups of finite fields or euclidean fields consist of only two elements, these fields are rigid. Over a quadratically closed field there are no binary anisotropic forms. Thus quadratically closed fields are rigid as well.

We will now give a characterization of rigid fields that will be useful in the sequel.

Theorem 4.1.3 ([War78, Theorems 1.5, 1.8, 1.9]):
For a field F the following are equivalent:

(i) F is rigid;

(ii) we have an isomorphism $WF \cong (\mathbb{Z}/n\mathbb{Z})[G]$ with $n \in \{0, 2, 4\}$ and G a group of exponent 2;

(iii) we have an isomorphism $WF \cong (\mathbb{Z}/n\mathbb{Z})[H]$ with either $n = 2$ and $H = F^*/F^{*2}$ or $n \in \{0, 4\}$ and $H \subseteq F^*/F^{*2}$ a subgroup with $-1 \notin H$ and $[F^*/F^{*2} : H] = 2$;

(iv) for any anisotropic form φ, we have $|D_F(\varphi)| \leq \dim \varphi$;

(v) for any quadratic field extension K/F, the image of the inclusion map $\iota : F^*/F^{*2} \to K^*/K^{*2}$ has index ≤ 2.

An important field invariant when studying quadratic forms is the so called *level* of a field, in symbols $s(F)$. It is defined as the least number n of squares such that -1 is a sum of n squares or ∞ if no such integer exists or equivalently the least integer n such that $(n+1) \times \langle 1 \rangle$ is isotropic. It is well known that the level is either ∞ or a power of 2, see [Lam05, Chapter XI. Pfister's Level Theorem]. We thus see that rigid fields always have level 1,2, or ∞.

Recall that a field is called pythagorean if any sum of squares is square. Following [EL72], we introduce the following name for formally real rigid fields.

Corollary and Definition 4.1.4:
If F is a formally real rigid field, it is pythagorean. A formally real rigid field F is also called *superpythagorean*.

Proof:
If F is formally real and rigid, its Witt ring is isomorphic to $\mathbb{Z}[G]$ for some group G of exponent 2. We thus have $W_tF = \{0\}$ which is equivalent to F being pythagorean by [Lam05, Chapter VIII., Theorem 4.1 (1)]. □

The above characterization together with Springer's theorem for complete discrete valuation fields motivate us to build the following prototypes of rigid fields in which we can calculate reasonably well and such that these fields realize any possible Witt ring of rigid fields.

Corollary 4.1.5:
Let F be a rigid field. Then there is a field $K \in \{\mathbb{F}_3, \mathbb{R}, \mathbb{C}\}$ and an index set I with

$$WF \cong WK((t_i))_{i \in I}.$$

Proof:
According to Theorem 4.1.3 (ii), we have $WF \cong \mathbb{Z}/n\mathbb{Z}[G]$ for some $n \in \{0,2,4\}$ and some group G of exponent 2.

We choose the field K as shown in the adjacent table:

n	0	2	4
K	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{F}_3$

It is well known that we then have $WK \cong \mathbb{Z}/n\mathbb{Z}$. As G is of exponent 2, it can be seen as a vector space over the fields with two elements $\mathbb{F}_2$ and thus has an $\mathbb{F}_2$-basis $(g_i)_{i \in I}$ for some index set I. We now consider the field $E := K((t_i))_{i \in I}$. We then have

$$WE \cong \mathbb{Z}/n\mathbb{Z}[G]$$

as in the proof of [War78, Lemma 1.6] (this is essentially a direct limit argument using Springer's Theorem on complete discrete valuation fields mentioned at the beginning of Section 2.4). □

The above result further allows us to always work in explicitly given fields if we want to study rigid fields in general. We will fill in the details in the next remark for future reference.

Remark 4.1.6:
As Witt rings are isomorphic if and only if there is an isomorphism of the respective square class groups as described in Theorem 2.1.4, the study of quadratic forms over rigid fields can thus be restricted to study quadratic forms over fields of the form $K((t_i))_{i \in I}$ for a field $K \in \{\mathbb{F}_3, \mathbb{R}, \mathbb{C}\}$ and some index set I, which can be assumed to

be well-ordered due to the well-ordering theorem.
If we want to study a concrete form, it is often even possible to only consider the case that I is finite as the direct limit $K((t_i))_{i\in I}$ can be regarded as the union of the fields $K((t_{i_1}))\cdots((t_{i_r}))$ for some $r \in \mathbb{N}_0$ and $i_1, \dots, i_r \in I$ with $i_1 < \dots < i_r$, see again the proof of [War78, Lemma 1.6]. Thus, if a quadratic form φ over E is given, we can take any diagonalization of φ. In this diagonalization, only finitely many Laurent-variables can occur, say these are $t_{j_1}, \dots t_{j_m}$ with $j_1 < \dots < j_m$. Then, φ is already defined over $E' := K((t_{j_1}))\cdots((t_{j_m}))$ and we can work over this field. For example, the Pfister number of φ over E' is bigger than or equal to the Pfister number of φ over E as we have $E' \subseteq E$. Thus the task of finding upper bounds for the Pfister numbers over arbitrary rigid fields is reduced to the task of finding upper bounds for the Pfister numbers over fields of the form $K((t_1))\cdots((t_n))$ for some $n \in \mathbb{N}$ and $K \in \{\mathbb{F}_3, \mathbb{R}, \mathbb{C}\}$.

The following corollary will be the key idea to determine asymptotic upper bounds for the Pfister numbers. Its proof combines the above theory with the tools that were developed before over fields equipped with a discrete valuation.

Corollary 4.1.7:
Let $\varphi \in I^n F$ be a quadratic form over some rigid field F that represents 1 and an element $a \notin \pm D_F(s(F) \times \langle 1\rangle)$, where we interpret $D_F(\infty \times \langle 1\rangle)$ as $\bigcup_{n\in\mathbb{N}} D_F(n \times \langle 1\rangle)$. Then there are quadratic forms $\sigma \in I^n F, \tau \in I^{n-1}F$ with $\dim \sigma < \dim \varphi$ and some $t \in F^*$ with $\varphi = \sigma \perp \langle\langle t\rangle\rangle \otimes \tau$.

Proof:
Using Remark 4.1.6 and Remark 2.4.9, we are reduced to the case where we have $F = K((t_1))\cdots((t_n))$ for some $n \in \mathbb{N}$ with $a = t_n$. But then, the assertion readily follows from Proposition 3.4.5 and Lemma 3.4.1 as both residue class forms for $a = t_n$ are non-hyperbolic by assumption. □

We would like to remark that our above result can be applied in particular to rigid fields F with $s(F) = 1$. When specialising to the case $n = 3$, we get the main results from [Rac13, Lemma 1.5], the starting point for the calculation of Pfister numbers in the just cited article.

Remark 4.1.8:
If we have an isomorphism as in Theorem 4.1.3 (iii), it can be realized as follows. For $a \in H$ and $[k] \in \mathbb{Z}/n\mathbb{Z}$ the element $k \cdot a$ is mapped to $k \times \langle a\rangle$ if we have $k \in \mathbb{N}_0$ and to $-k\langle -a\rangle$ otherwise. This additive extension of this rule gives rise to a ring isomorphism. For $s(F) = 2$, notice that we have

$$3 \times \langle a\rangle = \langle a, a, a\rangle \cong \langle -a, -a, a\rangle = \langle -a\rangle$$

in the Witt ring.

As usual it may be helpful to study the behaviour of a given quadratic form under field extensions. Thus the following result is essential for us.

Theorem 4.1.9 ([War78, Corollary 2.8]):
Let F be a rigid field and K/F a quadratic field extension. Then K is also a rigid field.

For later reference, we will now discuss the possible diagonalizations of anisotropic binary forms over rigid fields in detail.

Proposition 4.1.10:
Let F be a rigid field and $\beta = \langle x, y\rangle$ be an anisotropic binary form over F. By abuse of terminology, we say that two diagonalizations of a quadratic form are the same if they only differ by multiplying some entries by a square. We then have one of the following cases:

- $s(F) = 1$, x, y represent different square classes and $\langle x, y\rangle$ and $\langle y, x\rangle$ are the only diagonalizations of β;
- $s(F) = 2$, x, y represent different square classes and $\langle x, y\rangle$ and $\langle y, x\rangle$ are the only diagonalizations of β;
- $s(F) = 2$, x, y represent the same square classes and $\langle x, x\rangle$ and $\langle -x, -x\rangle$ are the only diagonalizations of β
- $s(F) = \infty$, x, y represent different square classes and $\langle x, y\rangle$ and $\langle y, x\rangle$ are the only diagonalizations of β;
- $s(F) = \infty$, x, y represent the same square classes and $\langle x, x\rangle$ is the only diagonalization of β.

Proof:
We first note that in general, for any $a \in F^*$, we cannot have a and $-a$ in the same diagonalization of an anisotropic quadratic form. In the sequel, we use several times the fact that any entry of a diagonalization is represented by the form. Finally, if x, y represent different square classes, we clearly have $D_F(\beta) = \{x, y\}$ because F is rigid.
If we have $s(F) = 1$ we have $x = -x$ in F^*/F^{*2}. It is thus clear that x, y have to represent different square classes. As F is rigid we have $D_F(\beta) = \{x, y\}$ and by the above remarks, this case follows.
For $a \in F^*$, we have $D_F(\langle a, a\rangle) = \{a, -a\}$ if $s(F) = 2$ and $D_F(\langle a, a\rangle) = \{a\}$ if $s(F) = \infty$ by Corollary 4.1.4. Thus, if x, y represent different square classes, they both have to occur in any diagonalization of β. This readily implies that $\langle x, y\rangle$ and $\langle y, x\rangle$ are the only diagonalizations of β in the respective cases.
So let now x, y represent the same square class. If we have $s(F) = 2$, it follows by the remarks at the beginning of the proof that $\langle x, x\rangle$ and $\langle -x, -x\rangle$ are the only diagonalizations of β.
Finally, if we have $s(F) = \infty$, Corollary 4.1.4 implies that $\langle x, x\rangle$ is the only diagonalization of β. □

As a corollary, we will now see what makes the theory of quadratic forms over rigid fields much easier than the general case: if one diagonalization of a given form is known, it is easy to determine all the others.

Corollary 4.1.11:
Let φ be an anisotropic form over a rigid field F. If we have $s(F) \in \{1, \infty\}$ the diagonalization of φ is unique up to permuting the entries and multiplying them with squares. If we have $s(F) = 2$, the diagonalization of φ is unique up to permuting

the entries, multiplying them with squares and replacing subforms of the form $\langle x, x\rangle$ for some $x \in F^*$ by $\langle -x, -x\rangle$.

Proof:
It is clear that any of the operations in the statement of the proposition describes isometries of quadratic forms. Further it is well known that two quadratic forms are isometric if and only if they are chain equivalent, see [Lam05, Chapter I. Chain Equivalence Theorem 5.2]. The conclusion thus readily follows from Proposition 4.1.10. □

Corollary 4.1.12:
Let φ, ψ be quadratic forms over a rigid field F such that $\varphi \perp \psi$ is anisotropic. We then have

$$D_F(\varphi \perp \psi) = \begin{cases} D_F(\varphi) \cup D_F(\psi), & \text{if } s(F) \in \{1, \infty\} \\ D_F(\varphi) \cup D_F(\psi) \cup \{x \in F^* \mid -x \in D_F(\varphi) \cap D_F(\psi)\}, & \text{if } s(F) = 2. \end{cases}$$

Proof:
It is well known that we have

$$D_F(\varphi \perp \psi) = \bigcup_{x \in D_F(\varphi), y \in D_F(\psi)} D_F(\langle x, y\rangle),$$

see for example [Lam05, Chapter I. exercise 20]. As the elements that are represented by a quadratic form are exactly those that can occur in a diagonalization, the claim now readily follows from Proposition 4.1.10. □

In the following, we will record some technical results in order to study how hyperbolic planes can occur in the sum of three quadratic forms over rigid fields.

Lemma 4.1.13:
Let F be a rigid field and $\varphi_1, \varphi_2, \varphi_3$ be anisotropic quadratic forms over F such that $\varphi_1 \perp \varphi_2$ is anisotropic as well. Then $\varphi_1 \perp \varphi_2 \perp \varphi_3$ is isotropic if and only if one of the following cases occurs:

(1) at least one of the forms $\varphi_1 \perp \varphi_3$ and $\varphi_2 \perp \varphi_3$ is isotropic.

(2) we have $s(F) = 2$ and $D_F(\varphi_1) \cap D_F(\varphi_2) \cap D_F(\varphi_3) \neq \emptyset$.

Proof:
The form $(\varphi_1 \perp \varphi_2) \perp \varphi_3$ is isotropic if and only if there is some $x \in D_F(\varphi_1 \perp \varphi_2) \cap -D(\varphi_3)$. As we have determined the value set $D_F(\varphi_1 \perp \varphi_2)$ in Corollary 4.1.12, the claim readily follows by the validity of the following three easy equivalences for some x as above:

$$\begin{aligned} x \in D_F(\varphi_1) &\iff \varphi_1 \perp \varphi_3 \text{ is isotropic} \\ x \in D_F(\varphi_2) &\iff \varphi_2 \perp \varphi_3 \text{ is isotropic} \\ -x \in D_F(\varphi_1) \cap D_F(\varphi_2) &\iff -x \in D_F(\varphi_1) \cap D_F(\varphi_2) \cap D_F(\varphi_3). \end{aligned}$$

□

Lemma 4.1.14:
Let F be a rigid field and φ_1, φ_2 be quadratic forms over F such that the orthogonal sum $\varphi_1 \perp \varphi_2$ is anisotropic. Further let $\psi \subseteq \varphi_1 \perp \varphi_2$ be a subform of $\varphi_1 \perp \varphi_2$. Then there are quadratic forms ψ_1, ψ_2, ψ_3 over F such that we have $\psi \cong \psi_1 \perp \psi_2 \perp \psi_3$ and the forms ψ_1, ψ_2, ψ_3 fulfil the following:

(a) $\psi_1 \subseteq \varphi_1, \psi_2 \subseteq \varphi_2$;

(b) $\big(D_F(\varphi_1) \cup D_F(\varphi_2)\big) \cap D_F(\psi_3) = \varnothing$;

(c) if we have $s(F) \neq 2$, we further have $\psi_3 = 0$;

(d) for any $x \in F^*$, the form $\langle x, x\rangle$ is not a subform of ψ_3.

Proof:
We prove the assertion by induction on $\dim \psi$, the initial step $\dim \psi = 0$ being trivial. We thus assume $\dim \psi > 0$ in the following. We will first show that we can decompose $\psi \cong \psi_1 \perp \psi_2 \perp \psi_3$ such that (a), (b) and (c) are fulfilled and finally that any such decomposition fulfils (d) as well.
If we have

$$D_F(\psi) \cap \big(D_F(\varphi_1) \cup D_F(\varphi_2)\big) = \varnothing,$$

we must have $s(F) = 2$ by Corollary 4.1.12 and we can put $\psi_3 = \psi$ and $\psi_1 = 0 = \psi_2$. Otherwise we choose an arbitrary $x \in D_F(\psi) \cap \big(D_F(\varphi_1) \cup D_F(\varphi_2)\big)$ and write $\psi \cong \langle x\rangle \perp \psi'$ for some suitable form ψ' over F. After renumbering we can assume without loss of generality that we have $x \in D_F(\varphi_1)$. In particular there is a form φ_1' such that we have $\varphi_1 \cong \langle x\rangle \perp \varphi_1'$. Using Witt's Cancellation Theorem, we see that ψ' is a subform of $\varphi_1' \perp \varphi_2$.
By induction hypothesis there are quadratic forms $\psi_1' \subseteq \varphi_1', \psi_2' \subseteq \varphi_2$ and ψ_3' with $\big(D_F(\varphi_1') \cup D_F(\varphi_2)\big) \cap D_F(\psi_3') = \varnothing$, such that we have $\psi' \cong \psi_1' \perp \psi_2' \perp \psi_3'$.
We now put

$$\psi_1 := \psi_1' \perp \langle x\rangle, \quad \psi_2 := \psi_2', \quad \psi_3 := \psi_3'.$$

Obviously, we have $\psi \cong \psi_1 \perp \psi_2 \perp \psi_3$ and $\psi_1 \subseteq \varphi_1$ und $\psi_2 \subseteq \varphi_2$. We will now prove $\big(D_F(\varphi_1) \cup D_F(\varphi_2)\big) \cap D_F(\psi_3) = \varnothing$.
At first, we note that we have

$$D_F(\varphi_1) = \begin{cases} D_F(\varphi_1') \cup \{x\}, & \text{if } s(F) = 1 \\ D_F(\varphi_1') \cup \{x\}, & \text{if } s(F) = 2 \text{ and } x \notin D_F(\varphi_1') \\ D_F(\varphi_1') \cup \{-x\}, & \text{if } s(F) = 2 \text{ and } x \in D_F(\varphi_1') \\ D_F(\varphi_1') \cup \{x\}, & \text{if } s(F) = \infty \text{ and } x \notin D_F(\varphi_1') \\ D_F(\varphi_1'), & \text{if } s(F) = \infty \text{ and } x \in D_F(\varphi_1'). \end{cases}$$

As we have $\big(D_F(\varphi_1') \cup D_F(\varphi_2)\big) \cap D_F(\psi_3') = \varnothing$ by induction hypothesis, the last case is clear. Since $\psi \cong \psi_1 \perp \psi_2 \perp \psi_3$ with $x \in D_F(\psi_1)$ is anisotropic, we further cannot have $-x \in D_F(\psi_3)$. Thus, the first and the third case are done.
For the remaining two cases, we have to exclude $x \in D_F(\psi_3)$. Assume the contrary. Since we have $\psi_3 = \psi_3'$, the induction hypothesis yields $x \notin D_F(\varphi_1') \cup D_F(\varphi_2)$. But

$\psi_3 = \psi_3'$ is a subform of $\varphi_1' \perp \varphi_2$ so we have $x \in D_F(\varphi_1' \perp \varphi_2)$. As F is rigid, this is only possible if we have $s(F) = 2$ and additionally $-x \in D_F(\varphi_1') \cap D_F(\varphi_2)$, see Corollary 4.1.12. But this is impossible since then, $\varphi_1 = \langle x \rangle \perp \varphi_1'$ would be isotropic. Thus (b) holds.
To prove (c), we now assume $s(F) \neq 2$. It is then enough to remark that we have $D_F(\varphi_1 \perp \varphi_2) = D_F(\varphi_1) \cup D_F(\varphi_2)$ by Corollary 4.1.12. Thus the first case in the induction step never occurs and we get $\psi_3 = 0$ automatically by proceeding as described above.
Finally, for (d), we can assume that we have $s(F) = 2$ according to (c). If we had $\langle z, z \rangle \subseteq \psi_3$ for some $z \in F^*$, we would have $z, -z \in D_F(\psi_3) \subseteq D_F(\psi) \subseteq D_F(\varphi_1 \perp \varphi_2)$. As we have

$$D_F(\varphi_1 \perp \varphi_2) = D_F(\varphi_1) \cup D_F(\varphi_2) \cup \{-x \mid x \in D_F(\varphi_1) \cap D_F(\varphi_2)\}$$

by Corollary 4.1.12 this would contradict the fact that we have

$$\big(D_F(\varphi_1) \cup D_F(\varphi_2)\big) \cap D_F(\psi_3) = \varnothing$$

and the conclusion follows. □

As a strengthening of the above results, we get the following consequence which gives us a precise description of how three quadratic forms over a rigid field have to be related such that their sum has a prescribed Witt index.

Corollary 4.1.15:
Let F be a rigid field and $\varphi_1, \varphi_2, \varphi_3$ be anisotropic forms over F such that $\varphi_1 \perp \varphi_2$ is anisotropic as well. Further let $m \in \mathbb{N}$ be an integer. We then have $i_W(\varphi_1 \perp \varphi_2 \perp \varphi_3) \geq m$ if and only if one of the following cases holds:

- we have $s(F) \neq 2$ and there are quadratic forms $\psi_1 \subseteq \varphi_1, \psi_2 \subseteq \varphi_2$ over F such that

$$\dim(\psi_1 \perp \psi_2) \geq m \quad \text{and} \quad -\psi_1 \perp -\psi_2 \subseteq \varphi_3;$$

  or

- we have $s(F) = 2$ and there are quadratic forms $\psi_1 \subseteq \varphi_1, \psi_2 \subseteq \varphi_2$ over F and $x_1, \dots, x_r \in F^* \setminus (D_F(\varphi_1) \cup D_F(\varphi_2))$ representing pairwise different square classes such that

$$\begin{aligned} -\psi_1 \perp -\psi_2 \perp -\langle x_1, \dots, x_r \rangle &\subseteq \varphi_3 \\ \langle x_1, \dots, x_r \rangle &\subseteq (\varphi_1 \perp -\psi_1)_{\mathrm{an}} \perp (\varphi_2 \perp -\psi_2)_{\mathrm{an}}, \\ \dim \psi_1 + \dim \psi_2 + r &\geq m. \end{aligned}$$

Proof:
By an easy induction on the integer m using the uniqueness of the Witt decomposition and the anisotropy of $\varphi_1 \perp \varphi_2$, we have $i_W(\varphi_1 \perp \varphi_2 \perp \varphi_3) \geq m$ if and only if there is some quadratic form ψ over F of dimension at least m such that

we have $-\psi \subseteq \varphi_3$ and $\psi \subseteq \varphi_1 \perp \varphi_2$.
Thus, to show the if part, it is enough to remark that we can choose

$$\psi := \begin{cases} \psi_1 \perp \psi_2, & \text{if } s(F) \neq 2 \\ \psi_1 \perp \psi_2 \perp \langle x_1, \dots, x_r \rangle, & \text{if } s(F) = 2 \end{cases}$$

as such a form. To show the only if part, let ψ be given as above. We separate the cases $s(F) \neq 2$ and $s(F) = 2$. If we have $s(F) \neq 2$, Lemma 4.1.14 yields that we have a decomposition $\psi = \psi_1 \perp \psi_2$ and for these ψ_1, ψ_2, the requirements are obviously fulfilled.
So let now $s(F) = 2$. We apply Lemma 4.1.14 again and get a decomposition $\psi = \psi_1 \perp \psi_2 \perp \psi_3$, where we can write $\psi_3 = \langle x_1, \dots, x_r \rangle$ for some $r \in \mathbb{N}$ and $x_1, \dots, x_r \in F^*$ representing different square classes. As the other properties are readily seen to be satisfied, it remains to show that we have $\langle x_1, \dots, x_r \rangle \subseteq (\varphi_1 \perp -\psi_1)_{\text{an}} \perp (\varphi_2 \perp -\psi_2)_{\text{an}}$ As ψ_i is a subform of φ_i for $i \in \{1, 2\}$ and $\varphi_1 \perp \varphi_2$ is anisotropic, the latter form is isometric to $(\varphi_1 \perp \varphi_2 \perp -\psi_1 \perp -\psi_2)_{\text{an}}$. Since we have

$$\psi = \psi_1 \perp \psi_2 \perp \langle x_1, \dots, x_r \rangle \subseteq \varphi_1 \perp \varphi_2$$

we get the desired subform relation as an easy consequence of Witt's Cancellation Theorem. □

Lemma 4.1.16:
Let F be a rigid field of level $s(F) = 2$ and let $x_1, \dots, x_r \in F^*$ represent pairwise different square classes such that the quadratic form $\langle x_1, \dots, x_r \rangle$ is anisotropic. Further, let φ, ψ be quadratic forms over F such that $\varphi \perp \psi$ is anisotropic and such that we have $x_i \notin D_F(\varphi) \cup D_F(\psi)$, but $x_i \in D_F(\varphi \perp \psi)$ for all $i \in \{1, \dots, r\}$. We then have both

$$-\langle x_1, \dots, x_r \rangle \subseteq \varphi \quad \text{and} \quad -\langle x_1, \dots, x_r \rangle \subseteq \psi.$$

Proof:
As we have $\langle x_1, \dots, x_r \rangle \subseteq \varphi \perp \psi$ but $x_i \notin D_F(\varphi) \cup D_F(\psi)$ for all $i \in \{1, \dots, r\}$, Lemma 4.1.12 implies $-x_i \in D_F(\varphi) \cap D_F(\psi)$. Thus, the induction base is clear by the Representation Criterion. So let now $r \geq 2$. By the above, we further have representations

$$\varphi = \langle -x_1 \rangle \perp \varphi' \quad \text{and} \quad \psi = \langle -x_1 \rangle \perp \psi'.$$

We thus have

$$\varphi \perp \psi \cong \langle x_1, x_1 \rangle \perp \varphi' \perp \psi'$$

and Lemma 4.1.12 then implies that we have a disjoint union

$$D_F(\varphi \perp \psi) = D_F(\langle x_1, x_1 \rangle \perp \varphi' \perp \psi') = \{\pm x_1\} \cup D_F(\varphi' \perp \psi').$$

Since the form $\langle x_1, \dots, x_r \rangle$ is anisotropic and the x_i represent different square classes, we have $x_2, \dots, x_r \notin \{\pm x_1\}$. We thus have $x_2, \dots, x_r \in D_F(\varphi' \perp \psi')$.
It is clear that we still have $x_i \notin D_F(\varphi') \cup D_F(\psi')$ for all $i \in \{2, \dots, r\}$ as these are subforms of φ respective ψ. By induction hypothesis, we have

$$-\langle x_2, \dots, x_r \rangle \subseteq \varphi' \quad \text{and} \quad -\langle x_2, \dots, x_r \rangle \subseteq \psi'$$

which then implies the assertion. □

4.2. 14-dimensional I^3-forms and 8-dimensional I^2-forms

We already discussed the connection between 14-dimensional I^3-forms and 8-dimensional I^2-forms in Section 3.1. In this section, we will study both types over rigid fields since the results obtained here will help us to classify 16-dimensional forms in the third power of the fundamental ideal I^3F. We will see that rigid fields fulfil both $D(8)$ and $D(14)$. We will now prove the validity of the latter to get the former property as an easy consequence.

Proposition 4.2.1:
Let F be a rigid field and $\varphi \in I^3F$ be an anisotropic 14-dimensional quadratic form. Then we have $\varphi = \pi_1 + \pi_2 \in WF$ for some $\pi_1, \pi_2 \in GP_3F$, i.e. F is a $D(14)$-field.

Proof:
By Proposition 3.1.9, we know that φ is Witt equivalent to a sum of three forms in GP_3F. We choose such a representation $\varphi = \pi_1 + \pi_2 + \pi_3$ such that we have

$$\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} \leq \dim(\pi_i \perp \pi_j)_{\mathrm{an}}$$

for any $i, j \in \{1, 2, 3\}$ with $i \neq j$. We will distinguish between the possible values that can occur. As we have $(\pi_1 \perp \pi_2)_{\mathrm{an}} \in I^3F$, the Holes Theorem implies $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} \in \{0, 8, 12, 14, 16\}$.

dim = 0: This contradicts the fact that we have $\dim \varphi = 14$.

dim = 8: In this case, we have $(\pi_1 \perp \pi_2)_{\mathrm{an}} \in GP_3F$ according to the Arason-Pfister Hauptsatz and the claim follows.

dim = 12: Here Theorem 2.2.13 implies that π_1 and π_2 are both divisible by the same binary Pfister form $\langle\langle a \rangle\rangle$ for some $a \in F^*$. In particular $(\pi_1 \perp \pi_2)_{F(\sqrt{a})}$ is hyperbolic, which then implies

$$\dim(\varphi_{F(\sqrt{a})})_{\mathrm{an}} \leq \dim\left((\pi_3)_{F(\sqrt{a})}\right)_{\mathrm{an}} \leq 8.$$

Thus φ has a form in GP_2F as a subform. Finally Proposition 3.1.9 then implies φ to be Witt equivalent to a sum of two forms in GP_3F.

dim = 14: According to Proposition 3.1.9 we can assume, possibly after a scaling, that we have $\pi_1, \pi_2 \in P_3F$ and $(\pi_1 \perp \pi_2)_{\mathrm{an}} = \pi_1' \perp -\pi_2'$, where the prime symbol denotes the pure part of the respective Pfister form as usual.
Further we have $i_W((\pi_1 \perp \pi_2)_{\mathrm{an}} \perp \pi_3) = 4$. This implies the existence of a quadratic form ψ over F with $\dim \psi = 4$, $-\psi \subseteq \pi_3$ and $\psi \subseteq \pi_1' \perp -\pi_2'$. We now decompose $\psi = \psi_1 \perp \psi_2 \perp \psi_3$ as in Lemma 4.1.14. We then have $\dim \psi_1, \dim \psi_2 \leq 1$ since if we had say $\dim \psi_1 \geq 2$, we would have

$$\dim(\pi_1 \perp \pi_3)_{\mathrm{an}} \leq \dim \pi_1 + \dim \pi_3 - 2 \dim \psi_1 \leq 8 + 8 - 2 \cdot 2 = 12,$$

contradicting the minimality of $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}}$. In particular, we must have $s(F) = 2$.
Thus we have $\dim \psi_3 \geq 2$. According to Lemma 4.1.14 (b) and (d) there

are $x, y \in D_F(\pi_1' \perp -\pi_2') \setminus (D_F(\pi_1') \cup D_F(-\pi_2'))$ that represent different square classes and are represented by ψ_3. Now Lemma 4.1.16 implies

$$-\langle x, y\rangle \subseteq \pi_1' \quad \text{and} \quad \langle x, y\rangle \subseteq \pi_2'.$$

This implies that both π_1 and π_2 become isotropic (hence hyperbolic) over $F(\sqrt{-xy})$. Since this is equivalent to π_1, π_2 having a common slot, this contradicts $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} = 14$ because of Theorem 2.2.13.

dim = 16: Just as above in the case $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} = 14$, we can deduce that the Pfister forms that π_1 respectively π_2 are similar to have a common slot. Thus, as in the case $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} = 12$, we see that φ contains a subform in GP_2F and is thus Witt equivalent to a sum of two forms in GP_3F according to Proposition 3.1.9 again.

□

Because of the strong connection of the two types of forms studied here, we can easily deduce the following as announced before:

Corollary 4.2.2:
Rigid fields are $D(8)$-fields.

Proof:
Since F is rigid, so is $F((t))$ according to [War78, Examples 1.11 (iv)]. As we have shown in Proposition 4.2.1, $F((t))$ is a $D(14)$-field. By [HT98, Theorem 4.1], this implies F to be a $D(8)$-field. □

It would be interesting to prove $D(8)$ directly, such that we can get $D(14)$ by [HT98, Theorem 4.4].

4.3. 16-dimensional I^3-forms

We are able to classify those 16-dimensional forms in I^3F for rigid fields that are Witt equivalent to a sum of at most three forms in GP_3F. The result is a strengthening of Lemma 3.3.6 in the special case of rigid fields. Its proof uses the same techniques as the proof of Proposition 4.2.1. At the end of the section, we will see that any 16-dimensional form in I^3F satisfies the following equivalent conditions.

Proposition 4.3.1:
Let F be a rigid field and $\varphi \in I^3F$ be an anisotropic quadratic form with $\dim\varphi = 16$. Then the following are equivalent:

(i) φ is isometric to a sum of 4 forms in GP_2F;

(ii) φ contains a subform in GP_2F;

(iii) φ is Witt equivalent to a sum of at most at most 3 forms in GP_3F.

Proof:
The implication (i) $\Rightarrow$ (ii) is trivial and the implication (ii) $\Rightarrow$ (iii) follows directly by Lemma 3.3.6 and Proposition 4.2.1.
For the implication (iii) $\Rightarrow$ (i), let now $\varphi = \pi_1 + \pi_2 + \pi_3 \in WF$ with $\pi_1, \pi_2, \pi_3 \in GP_3F$. We further assume that $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}}$ is minimal under all such representations.
Similar to the proof of Proposition 4.2.1, we are readily reduced to the cases in which we have $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} \in \{8, 12, 14, 16\}$.
dim = 8: If we have $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} = 8$ then $(\pi_1 + \pi_2)_{\mathrm{an}}$ is isometric to some $\pi \in GP_3F$ according to the Arason-Pfister Hauptsatz. Thus, we have $\varphi \cong \pi \perp \pi_3$ and the conclusion follows.
dim = 12: If we have $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} = 12$, then $(\pi_1 + \pi_2)_{\mathrm{an}}$ is divisible by a binary form $\langle\langle a\rangle\rangle$ due to Theorem 3.1.5. Thus, we have $i_W(\varphi_{F(\sqrt{a})}) \geq 4$ and we can write $\varphi \cong \langle\langle a\rangle\rangle \otimes \sigma \perp \psi$ with some 4-dimensional form σ and some 8-dimensional form ψ because of Theorem 2.5.2. According to [Kne77, Example 9.12] $\langle\langle a\rangle\rangle \otimes \sigma$ is an 8-dimensional form in I^2F, whose Clifford invariant has index at most 2. In WF we therefore have

$$\psi = \varphi - \langle\langle a\rangle\rangle \otimes \sigma \in I^2F$$

which then implies

$$c(\psi) = c(\varphi)c(\langle\langle a\rangle\rangle \otimes \sigma) = c(\langle\langle a\rangle\rangle \otimes \sigma).$$

Using [Kne77, Example 9.12] again, we see that ψ is divisible by a binary form as well. As 8-dimensional forms that are divisible by a binary form are isometric to a sum of two forms in GP_2F, we are done in this case.
dim = 14: So let now $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} = 14$. According to Proposition 3.1.9 we can assume we have $(\pi_1 \perp \pi_2)_{\mathrm{an}} \cong \pi_1' \perp -\pi_2'$, possibly after a scaling. We further have $i_W(\pi_1' \perp -\pi_2' \perp \pi_3) = 3$, such that there is some 3-dimensional form $\psi \subseteq \pi_1' \perp -\pi_2'$ with $-\psi \subseteq \pi_3$.
We decompose $\psi = \psi_1 \perp \psi_2 \perp \psi_3$ as in Lemma 4.1.14. Because of the minimality of $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}}$, we have $\dim\psi_1 \leq 1$ and $\dim\psi_2 \leq 1$.

As in the case of dimension 14 in the proof of Proposition 4.2.1, we can see that $\dim \psi_3 \geq 2$ would contradict Linkage theory, see 2.2.13. As the dimensions of ψ_1, ψ_2 and ψ_3 have to sum up to 3, we get

$$\dim \psi_1 = \dim \psi_2 = \dim \psi_3 = 1.$$

Thus φ contains a 10-dimensional subform that is the orthogonal sum of a 5-dimensional subform of π_1 and a 5-dimensional subform of π_2. Both of these forms are Pfister neighbors that contain a subform in GP_2F according to Proposition 2.2.12. Thus φ has a decomposition $\varphi = \sigma \perp \tau$, where σ is isometric to a sum of 2 forms in GP_2F. We thus have $\sigma \in I^2F$ and the Clifford invariant of σ has index at most 4. As in the case dim = 12, these properties also hold for τ. Applying Corollary 4.2.2 now gives us that τ is also isometric to a sum of two forms in GP_2F which finishes this case.
dim = 16: Here, we are reasoning just as in the latter case and use the same terminology for all upcoming forms etc. We have $\dim \psi = 4$. Because of the minimality of $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}}$, we even have $\psi_1 = 0 = \psi_2$. As in the case dim = 14 above (i.e. as in the proof of Proposition 4.2.1), we see that the Pfister forms that are similar to π_1 respectively π_2 have a common slot, so that $\pi_1 \perp \pi_2$ is divisible by a binary form $\langle\langle a \rangle\rangle$. Now the conclusion follows as in the case dim = 12. $\square$

Our next goal is to study 16-dimensional form in I^3F in more detail in order to prove that each such form satisfies the equivalent conditions of Proposition 4.3.1. To do so, we need the next technical lemma.

Lemma 4.3.2:
Let F be a rigid field and φ_1, φ_2 be two anisotropic quadratic forms over F, such that $\varphi_1 \perp \varphi_2$ is an anisotropic form in I^3F of dimension 14. Then, for any $t \in F^*$, the form $\varphi_1 \perp t\varphi_2$ contains a subform in GP_2F.

Proof:
We show that one of the forms φ_1 and φ_2 already contains a subform in GP_2F or that there is some binary form that is similar to both a subform of φ_1 and a subform of φ_2. This obviously implies the assertion.
Since F is a rigid field, F is a $D(14)$-field by Proposition 4.2.1. Therefore, after a possible scaling, we may assume that we have $\pi_1, \pi_2 \in P_3F$ with

$$\varphi_1 \perp \varphi_2 \cong \pi_1' \perp -\pi_2'.$$

We remark that π_1, π_2 cannot have a common slot due to Theorem 2.2.13.
As π_1, π_2 are 3-fold Pfister forms, we can choose $a, a' \in F^*$ and 3-dimensional forms σ, σ' over F such that we have

$$\psi := \langle\langle a \rangle\rangle \otimes \sigma \subseteq \pi_1', \quad \psi' := \langle\langle a' \rangle\rangle \otimes \sigma' \subseteq -\pi_2',$$

see Proposition 3.2.3. In particular $\psi \perp \psi'$ is also a subform of $\varphi_1 \perp \varphi_2$. We now decompose $\psi \cong \psi_1 \perp \psi_2 \perp \psi_3$ and $\psi' \cong \psi_1' \perp \psi_2' \perp \psi_3'$ according to Lemma 4.1.14. We will now proof the assertion while distinguishing the possible dimensions of these subforms:

Case 1: $\dim \psi_3 = 0$ or $\dim \psi_3' = 0$:
According to the symmetry of the statement, it is enough to consider the case

$\dim \psi_3 = 0$. Further we can assume $\dim \psi_1 \geq \dim \psi_2$, possibly after renumbering the φ_i. As we clearly have $\dim \psi_1 + \dim \psi_2 = 6$ the latter implies $\dim \psi_1 \geq 3$.

If we have $\dim \psi_1 \geq 5$, it follows, using Proposition 2.1.1 and the fact that $\psi_{F(\sqrt{a})}$ is hyperbolic, that ψ_1 already contains a four dimensional subform that is divisible by $\langle\langle a\rangle\rangle$, i.e. a form in GP_2F.

If we have $\dim \psi_1 = 4$ we can use the same arguments as above to get that ψ_1 becomes isotropic over $F(\sqrt{a})$ which then implies that ψ_1 is similar to $\langle\langle a\rangle\rangle \perp \sigma$ with some quadratic form σ of dimension 2. We then have that $\sigma \perp \psi_2$ is divisible by $\langle\langle a\rangle\rangle$, i.e. a form in GP_2F. Using [Lam05, Chapter X. Corollary 5.4] one readily sees that this is only possible if σ and ψ_2 are similar which concludes this case.

If $\dim \psi_1 = 3$ and ψ_1 becomes isotropic over $F(\sqrt{a})$, then so does ψ_2 according to Proposition 2.1.1 as ψ becomes hyperbolic over $F(\sqrt{a})$. Thus, both ψ_1 and ψ_2 contain a subform similar to $\langle\langle a\rangle\rangle$ according to Theorem 2.5.2 and this case is done.
Otherwise ψ_1 and ψ_2 are quadratic forms of dimension 3 that stay anisotropic over $K := F(\sqrt{a})$ but fulfil $(\psi_1)_K \cong -(\psi_2)_K$. By Theorem 4.1.9 K is a rigid field, too. Using Proposition 4.1.11 we see that the diagonalization of $(\psi_1)_K$ is either unique up to multiplying its entries with squares and permuting the entries or we have $s(K) = 2$ (and thus also $s(F) = 2$ as can readily seen using [War78, Theorem 2.7]) and $(\psi_1)_K = \langle x, x, y\rangle$ for some $x, y \in F^*$.
In the first case, we write $(\psi_1)_K = \langle x, y, z\rangle$ for suitable $x, y, z \in F^*$ representing pairwise different square classes in K. Using Theorem 2.5.8, we see that we have

$$\psi_1 = \langle a^{i_1}x, a^{j_1}y, a^{k_1}z\rangle \text{ and } \psi_2 = -\langle a^{i_2}x, a^{j_2}y, a^{k_2}z\rangle$$

for some $i_1, i_2, j_1, j_2, k_1, k_2 \in \{0, 1\}$. After renaming x, y, z, the pigeon hole principle implies that we have either $i_1 = i_2$ and $j_1 = j_2$ or $i_1 \neq i_2$ and $j_1 \neq j_2$. In both cases $\langle a^{i_1}x, a^{j_1}y\rangle$ and $-\langle a^{i_2}x, a^{j_2}y\rangle$ are similar so that this case is done.
In the second case we argue the same way. We get that ψ_1 is isometric to one of the following forms on the left for some $i \in \{0, 1\}$ and ψ_2 is isometric to one of the forms on the right for some $j \in \{0, 1\}$:

$$\begin{array}{rr}
\langle x, x, a^i y\rangle \cong \langle -x, -x, a^i y\rangle & \langle x, x, -a^j y\rangle \cong \langle -x, -x, -a^j y\rangle \\
\langle ax, ax, a^i y\rangle \cong \langle -ax, -ax, a^i y\rangle & \langle ax, ax, -a^j y\rangle \cong \langle -ax, -ax, -a^j y\rangle \\
\langle -x, -ax, a^i y\rangle & \langle -x, -ax, -a^j y\rangle \\
\langle x, ax, a^i y\rangle & \langle x, ax, -a^j y\rangle
\end{array}$$

Thus a binary form that is similar to both a subform of ψ_1 and a subform of ψ_2 can be found in the upcoming table in which all cases with $\psi_1 \ncong -\psi_2$ (that case being clear) are considered.

	$\langle x, x, -ay\rangle$	$\langle ax, ax, -ay\rangle$	$\langle -x, -ax, -ay\rangle$	$\langle x, ax, -ay\rangle$
$\langle x, x, y\rangle$	$\langle x, x\rangle$	$\langle x, x\rangle$	$\langle x, y\rangle$	$\langle -x, y\rangle$
$\langle ax, ax, y\rangle$	$\langle x, x\rangle$	$\langle x, x\rangle$	$\langle ax, y\rangle$	$\langle -ax, y\rangle$
$\langle -x, -ax, y\rangle$	$\langle -ax, y\rangle$	$\langle -x, y\rangle$	$\langle -x, -ax\rangle$	$\langle -x, -ax\rangle$
$\langle x, ax, y\rangle$	$\langle ax, y\rangle$	$\langle x, y\rangle$	$\langle x, ax\rangle$	$\langle x, ax\rangle$

Case 2: $\dim \psi_3 \geq 2$ or $\dim \psi'_3 \geq 2$:

It is again enough to consider the case $\dim \psi_3 \geq 2$. Because of Lemma 4.1.14 (d) there are $x, y \in F^*$ representing different square classes with $\psi_3 = \langle x, y, \ldots \rangle$. Because of Lemma 4.1.14 (b) we have $x, y \in D_F(\psi_3) \subseteq D_F(\varphi_1 \perp \varphi_2)$ but $x, y \notin D_F(\varphi_1) \cup D_F(\varphi_2)$. Now, Lemma 4.1.16 implies both $\varphi_1 = \langle -x, -y, \ldots \rangle$ and $\varphi_2 = \langle -x, -y, \ldots \rangle$. According to the statement at the beginning of the proof, this case is done.

Case 3: $\dim \psi_3 = 1 = \dim \psi'_3$:

If we have $\psi_3 = \langle x \rangle \ncong \langle y \rangle = \psi'_3$ for some $x, y \in F^*$, we can argue as in the last case using Lemma 4.1.16 to get $\varphi_1 = \langle -x, -y, \ldots \rangle, \varphi_2 = \langle -x, -y, \ldots \rangle$ and we are done.

Otherwise we have $\psi_3 = \langle x \rangle = \psi'_3$, so we can write $\varphi_1 = \nu_1 \perp \langle -x \rangle$ and $\varphi_2 = \nu_2 \perp \langle -x \rangle$. We further choose orthogonal complements of $\langle x \rangle$ in π'_1 respectively $-\pi'_2$. As in the beginning of the proof, we can write them as a product of a Pfister form and a ternary form, i.e. we have

$$\pi'_1 = \langle\langle b \rangle\rangle \otimes \tau \perp \langle x \rangle \text{ and } -\pi'_2 = \langle\langle b' \rangle\rangle \otimes \tau' \perp \langle x \rangle$$

for some ternary forms τ, τ' and $b, b' \in F^*$. We have a chain of isometries

$$\begin{aligned} \nu_1 \perp \langle x \rangle \perp \nu_2 \perp \langle x \rangle &\cong \nu_1 \perp \langle -x \rangle \perp \nu_2 \perp \langle -x \rangle \\ &\cong \varphi_1 \perp \varphi_2 \\ &\cong \pi'_1 \perp -\pi'_2 \\ &\cong \langle\langle b \rangle\rangle \otimes \tau \perp \langle x \rangle \perp \langle\langle b' \rangle\rangle \otimes \tau' \perp \langle x \rangle. \end{aligned}$$

Witt's cancellation law now implies $\langle\langle b \rangle\rangle \otimes \tau \perp \langle\langle b' \rangle\rangle \otimes \tau' \cong \nu_1 \perp \nu_2$.

We now apply the above argument for $\langle\langle b \rangle\rangle \otimes \tau$ and $\langle\langle b' \rangle\rangle \otimes \tau'$ as subforms of $\nu_1 \perp \nu_2$. Note that all arguments used above stay valid as we did not use any specific information on φ_1, φ_2 but only of the chosen subforms ψ, ψ'.

If we are in case 1 or 2 for $b, b', \tau, \tau', \nu_1, \nu_2$ we are done as we have already seen. If we are again in case 3 for $b, b', \tau, \tau', \nu_1, \nu_2$, we get the existence of some $y \in F^*$ represented by both π'_1 and $-\pi'_2$. This would imply π_1 and π_2 to have $-xy$ as a common slot similar as in the case dim = 14 in Proposition 4.2.1, which we excluded at the beginning of the proof. Thus we are done.

□

Theorem 4.3.3:

Let F be a rigid field and $\varphi \in I^3F$ be an anisotropic quadratic form over F of dimension 16. Then φ is Witt equivalent to a sum of at most three forms in GP_3F.

Proof:

We will show that φ contains a subform in GP_2F so that the conclusion then follows by Proposition 4.3.1. After scaling, we can assume $1 \in D_F(\varphi)$. If φ is isometric to $16 \times \langle 1 \rangle$ (which is only possible if F is superpythagorean), the assertion is clear. Otherwise there is some $n \in \mathbb{N}$ such that we can assume φ to be defined over the field $K((t_1))\cdots((t_n))$ and that φ has a decomposition into residue class forms $\varphi \cong \varphi_1 \perp t_n\varphi_2$

such that both residue class forms have positive dimension. As mentioned in Remark 2.4.9 we can replace the uniformizer t_n with at_n for any $a \in K((t_1))\cdots((t_{n-1}))^*$. By doing so, we also get $a\varphi_2$ as the second residue class form instead of φ_2. We may thus assume $D_F(\varphi_1) \cap D_F(\varphi_2) \neq \varnothing$, i.e. $\sigma := (\varphi_1 \perp -\varphi_2)_{\mathrm{an}}$ has dimension at most 14. If we have $\dim \sigma \leq 12$, there is some binary form β that is a subform of both φ_1 and φ_2, so that $\beta \otimes \langle 1, t_n \rangle \in GP_2F$ is a subform of φ.
If we have $\dim \sigma = 14$, there is some $x \in F^*$ and quadratic forms ψ_1, ψ_2 such that we have

$$\varphi_1 \cong \langle x \rangle \perp \psi_1 \text{ and } \varphi_2 \cong \langle x \rangle \perp \psi_2.$$

As in the proof of Proposition 3.4.5, we have $\sigma \cong \psi_1 \perp -\psi_2 \in I^3F$ (in fact, our σ here has exactly the same role as the σ in the above mentioned result). As we have $\dim \sigma = 14$, it contains a subform lying in GP_2F according to Proposition 4.2.1. By Lemma 4.3.2 the form $\psi_1 \perp t_n\psi_2$ also contains a GP_2-subform, which then trivially implies

$$\varphi \cong \psi_1 \perp \langle x \rangle \perp t_n(\psi_2 \perp \langle x \rangle)$$

to have a subform in GP_2F, which concludes the proof. □

Example 4.3.4:
The bound in Theorem 4.3.3 is sharp as the following example shows. Let $K \in \{\mathbb{F}_3, \mathbb{R}, \mathbb{C}\}$ and $F = K((a))((b))((c))((d))((e))((f))$. We first construct an 8-dimensional form in I^2F that is not Witt equivalent to a sum of 2 forms in GP_2F. To do so, we can consider

$$\psi := \langle 1, a, b, c, d, e, f, abcdef \rangle \in I^2F,$$

which is the generic 8-dimensional form in I^2F and fulfills $GP_2(\psi) = 3$ by Example 3.1.4. Then, $\varphi := \psi \otimes \langle\langle t \rangle\rangle \in I^3F((t))$ fulfils $GP_3(\varphi) = 3$ by Proposition 3.4.4.

Another common way to measure the complexity of a quadratic form is to study its splitting behaviour over multiquadratic field extensions. There are 16-dimensional I^3-forms over non-rigid fields that do not split over multiquadratic extensions of degree ≤ 8, see [Kar17, Theorem 2.1]. For rigid fields, the situation is much less involved.

Proposition 4.3.5:
Let φ be a 16-dimensional form in I^3F. Then φ splits over some biquadratic extension of F, i.e. there are $a, b \in F^*$ such that $\varphi_{F(\sqrt{a},\sqrt{b})}$ is hyperbolic.

Proof:
According to Theorem 4.3.3 and Proposition 4.3.1 we can write $\varphi = \psi \perp \sigma$ where we have $\sigma \in GP_2F$. We choose $a \in F$ such that $\sigma_{F(\sqrt{a})}$ is isotropic hence hyperbolic. If $\psi_{F(\sqrt{a})}$ is isotropic then it is hyperbolic or Witt equivalent to a form in $GP_3F(\sqrt{a})$ that is defined over F as quadratic extensions are excellent, see [Lam05, Chapter XII. Proposition 4.4]. In both cases the assertion is clear.
Otherwise $\psi_{F(\sqrt{a})}$ is an anisotropic, 12-dimensional form in $I^3F(\sqrt{a})$ and hence divisible by a binary Pfisterform $\langle\langle b \rangle\rangle$ for some $b \in K^*$. By [War78, Theorem 1.9], the square class of b in $F(\sqrt{a})$ has a representative of the form z or $z\sqrt{a}$ for some $z \in F^*$. We are done if we can exclude the latter case. As $F(\sqrt{a})$ is also a rigid field by Theorem 4.1.9, we know how two diagonalizations of the same form can differ by

Proposition 4.1.11. As ψ is defined over F, we can thus deduce that we must have $b \in F^*$. □

Example 4.3.6:
Proposition 4.3.5 is sharp in the sense that in general, forms over dimension 16 in I^3F over a rigid field F will not split over a quadratic extension. As an example, we can consider the 16-dimensional form $\langle\langle a,b,c\rangle\rangle \perp \langle\langle d,e,f\rangle\rangle$ over the field $F := K((a))((b))((c))((d))((e))((f))$ where we can choose $K \in \{\mathbb{R},\mathbb{C},\mathbb{F}_3\}$.

Example 4.3.7:
To conclude this section, we will show that the characterisation in Proposition 4.3.1 does not generalize to arbitrary fields. To be precise, we will construct a 16-dimensional form in I^3F for a suitable field F that has Pfister number 3 but is not isometric to a sum of four forms in GP_2F. Over the field $F := \mathbb{Q}(x)((t_1))\cdots((t_4))$ we consider the forms

$$\begin{aligned}
\psi_1 &:= \langle x, -(x+4)\rangle \perp -t_1\langle 1, -(x+4)\rangle,\\
\psi_2 &:= \langle x, -(x+1)\rangle \perp -2t_1\langle 1, -(x+1)\rangle\\
\rho_1 &:= \langle 1, -x, -t_1t_2(x+4), t_1t_2x(x+4)\rangle = \langle\langle x, t_1t_2(x+4)\rangle\rangle,\\
\rho_2 &:= \langle 1, -x(x+1)(x+4), 2t_1x(x+2), -2t_1(x+1)(x+2)(x+4)\rangle\\
&= \langle\langle x(x+1)(x+4), -2t_1x(x+2)\rangle\rangle.
\end{aligned}$$

and finally build the form $\varphi := \psi_1 \perp -t_2\psi_2 \perp t_4(\rho_1 \perp t_3\rho_2)$. In the sequel we will use a lot of facts shown in [HT98, Example 6.3]. At first we know that $\varphi_1 := \psi_1 \perp -t_2\psi_2$ is anisotropic, lies in I^2F and does not contain a subform in GP_2F. Further, $\varphi_2 := \rho_1 \perp t_3\rho_2$ is also an anisotropic form in I^2F that has the same Clifford invariant. We thus have $\varphi \in I^3F$ with $\dim\varphi = 16$ and φ is anisotropic. By Proposition 3.3.1 we further know that φ has 3-Pfister number at most 3. By showing that φ is not isometric to a sum of four forms in GP_2F, it will further be clear that we even have an equality $GP_3(\varphi) = 3$.
Similarly as in Lemma 4.1.14, we can show by an induction argument that any form $\psi \subseteq \varphi$ has a decomposition $\psi \cong \psi_1 \perp t_4\psi_2$ with $\psi_1 \subseteq \varphi_1, \psi_2 \subseteq \varphi_2$. Thus, if φ would be isometric to an orthogonal sum of four GP_2-forms, there has to be a $\sigma \in GP_2F$ that can be decomposed into $\sigma_1 \perp t_4\sigma_2$ with $\sigma_1 \subseteq \varphi_1, \sigma_2 \subseteq \varphi_2$ and $\sigma_1 \neq 0 \neq \sigma_2$ (as φ_1 does not contain any subform in GP_2F itself). By Lemma 3.4.1 we have $\sigma_1, \sigma_2 \in IF$ which then implies $\dim\sigma_1 = \dim\sigma_2 = 2$. As we have $\det\sigma = 1$, we have $\det\sigma_1 = \det\sigma_2$. Analysing the decomposition of φ, we see this can only happen if we have $\sigma_2 \subseteq \rho_1$ or $\sigma_2 \subseteq t_3\rho_2$. We will assume $\sigma_2 \subseteq \rho_1$, the other case is similar.
We now choose an $a \in F^*$ such that σ_2 becomes isotropic (hence hyperbolic) over $F(\sqrt{a})$. In fact, by the choice of σ_2, we can even choose $a \in \mathbb{Q}(x)((t_1))((t_2))^*$. Then, as Pfister forms are either anisotropic or hyperbolic and σ_1 is similar to σ_2, both σ_1 and ρ_1 become hyperbolic over $F(\sqrt{a})$. This implies $i_W(\varphi_{F(\sqrt{a})}) \geq 3$ and thus, by the Holes Theorem, even $i_W(\varphi_{F(\sqrt{a})}) \geq 4$. By the choice of a, the t_4-adic valuation has an extension to $F(\sqrt{a})$ and t_4 still is a uniformizer, see Corollary A.1.3. Thus by Corollary 2.4.3 the inequality $i_W(\varphi_{F(\sqrt{a})}) \geq 4$ can only be fulfilled if

(a) $(\rho_2)_{F(\sqrt{a})}$ is isotropic or

(b) we have $i_W((\varphi_1)_{F(\sqrt{a})}) \geq 2$.

In case (a), the Pfister forms that ρ_1 respectively ρ_2 are similar to would have a common slot, but this was excluded in [HT98, Example 6.3].
But case (b) would imply the existence of a subform of φ_1 lying in GP_2F, a contradiction. Thus the proof is complete.

4.4. Forms in I^nF of Dimension $2^n + 2^{n-1}$

If not stated otherwise, let $n \geq 4$ in this section. As already discussed at the beginning of Section 3.5, forms of dimension $2^n + 2^{n-1}$ in I^nF are forms of the smallest dimension that we are interested in and whose structure is not known yet. The validity of [Hof98a, Conjecture 2] for rigid fields is now an easy consequence.

Corollary 4.4.1:
Let F be a rigid field. Then F has property $P(n)$ for all $n \geq 2$.

Proof:
For $n \in \{2, 3\}$ there is nothing more to show. So let now $n \geq 4$. With Theorem 3.5.4, Remark 2.4.9 and Corollary 4.1.5 in mind, it is enough to verify the property $P(n)$ for all $n \geq 3$ for the fields $\mathbb{F}_3, \mathbb{R}, \mathbb{C}$, but this is clear. $\square$

Corollary 4.4.2:
Let F be a rigid field. Then F has property $\mathrm{Sim}(n)$ for all $n \geq 2$.

Proof:
This follows directly from Corollary 4.4.1 and Corollary 3.5.7. $\square$

We will further give another proof for Corollary 4.4.1. This proof is way more technical, but uses the more typical arguments for rigid fields. Here, as usual, the case that the considered field has level $s(F) = 2$ is the most technical one and requires some preliminary results we will start with.

Lemma 4.4.3:
Let F be a rigid field with $s(F) = 2$. Further let $\sigma \in I^3F$ be an anisotropic form of dimension $\dim \sigma = 14$, such that we have $\dim(\langle\langle -1\rangle\rangle \otimes \sigma)_{\mathrm{an}} = 24$. Then there is some $\tau \in I^3F$ with $\dim \tau = 12$ such that in WF, we have

$$\langle\langle -1\rangle\rangle \otimes \sigma = \langle\langle -1\rangle\rangle \otimes \tau.$$

Proof:
After scaling, using Proposition 4.2.1 and Proposition 3.1.9 we can assume that we have an isometry

$$\sigma \cong \pi_1' \perp -\pi_2'$$

for some $\pi_1, \pi_2 \in P_3F$.
As $s(F) = 2$ implies $\langle\langle -1, -1\rangle\rangle$ to be hyperbolic and since we assume $\dim(\langle\langle -1\rangle\rangle \otimes \sigma)_{\mathrm{an}} = 24$ we know that -1 cannot be chosen as a slot for one of the π_i. Since we further know that $\langle\langle -1\rangle\rangle \otimes \sigma = 2 \times \sigma$ is isotropic, there is some $x \in D_F(\sigma) \cap -D_F(\sigma)$.
As we have $x \in D_F(\sigma)$, Corollary 4.1.12 implies that we have one of the following cases on the left side and analogously, $x \in D_F(-\sigma)$, i.e. $-x \in D_F(\sigma)$, implies that we further have one the following cases on the right side:

(1) $x \in D_F(\pi_1')$	(I) $-x \in D_F(\pi_1')$
(2) $x \in D_F(-\pi_2')$	(II) $-x \in D_F(-\pi_2')$
(3) $-x \in D_F(\pi_1') \cap D_F(-\pi_2')$	(III) $x \in D_F(\pi_1') \cap D_F(-\pi_2')$.

Here, we cannot have (1) and (II) respectively (2) and (I) at the same time since then, π_1, π_2 would have a common slot due to Theorem 2.2.1, but as we have $\dim \sigma = 14$ and σ anisotropic, this would contradict the Linkage theory, see Theorem 2.2.13.
Further, we cannot have (1) and (I) respectively (2) and (II) simultaneously since otherwise, we would have (say) $x, -x \in D_F(\pi_1')$. Using Corollary 4.1.12, it is easy to see that we would then have

$$\pi_1 \cong \langle 1, -x, -x, , \ldots \rangle \cong \langle\langle x, x, \ldots \rangle\rangle \cong \langle\langle -1, x, \ldots \rangle\rangle$$

by [Lam05, Chapter X. Corollary 1.11], which would contradict the above mentioned fact that -1 cannot be chosen as a slot for π_1.
We thus know that at least one the cases (3) or (III) occurs. Using the symmetry, we can assume without loss of generality that (III) occurs. Therefore, there are some $a, b, \alpha, \beta \in F^*$ with

$$\pi_1 = \langle\langle -x, a, b \rangle\rangle, \qquad \pi_2 = \langle\langle x, \alpha, \beta \rangle\rangle.$$

Using the isometry $\langle\langle -1, x \rangle\rangle \cong \langle\langle -1, -x \rangle\rangle$ (recall that we have $s(F) = 2$), we have

$$\langle\langle -1 \rangle\rangle \otimes \sigma = \langle\langle -1, -x \rangle\rangle \otimes (\langle\langle a, b \rangle\rangle \perp -\langle\langle \alpha, \beta \rangle\rangle) =\in WF.$$

We can thus put

$$\tau := \langle\langle -x \rangle\rangle \otimes \langle -a, -b, ab, \alpha, \beta, -\alpha\beta \rangle \in I^3F$$

and the claim follows. □

Lemma 4.4.4:
Let F be a rigid field with $s(F) = 2$ and $\psi \in I^2F$ with $\dim \psi = 12$ such that we have $\langle\langle -1 \rangle\rangle \otimes \psi \in I^4F$ and $\dim(\langle\langle -1 \rangle\rangle \otimes \psi)_{\text{an}} = 24$. We then have $\operatorname{ind}(c(\psi)) \in \{1, 2\}$.

Proof:
First recall the isometry in Theorem 2.3.1, which identifies $c(\psi) \in \operatorname{Br}_2(F)$ with $\overline{\psi} \in I^2F/I^3F$, where the bar denotes the corresponding class in the factor group. By abuse of notation, we will use the bar also for classes in other quotients of this form if appropriate.
We now consider the field extension $E = F(\sqrt{-1})$ of F. As we have

$$\langle\langle -1 \rangle\rangle \otimes \overline{\psi} = \overline{0} \in I^3F/I^4F,$$

Theorem 2.5.10 implies the existence of some $\overline{\rho} \in I^2E/I^3E$ with $s_*(\overline{\rho}) = \overline{\psi} \in I^2F/I^3F$. As F is a rigid field with $s(F) = 2$, for the field norm $N_{E/F} : E \to F$, we have

$$N_{E/F}(x + y\sqrt{-1}) = x^2 + y^2 \in \pm F^{*2}$$

for all $x, y \in F$ not both equal to 0. Thus Proposition 2.5.6 implies that $\overline{\psi}$ is divisible by the class of the Pfister form $\langle\langle -1 \rangle\rangle$.
In particular we have $\overline{\psi_E} = \overline{0}$ and therefore $c(\psi_E) = 1 \in \operatorname{Br}(E)$. Now [GS06, Corollary 4.5.11] implies

$$\operatorname{ind}(c(\psi)) \mid [E : F] \cdot \operatorname{ind}(c(\psi_E)) = 2 \cdot 1 = 2,$$

as desired. □

Theorem 4.4.5:
Let F be a rigid field and $\varphi \in I^nF$ be an anisotropic form with $d := \dim\varphi = 2^n + 2^{n-1}$. Then φ satisfies the equivalent conditions in Proposition 3.5.1.

Proof:
We will prove the statement by induction on n. For $n = 2$ there is nothing to show and the case $n = 3$ is covered by Theorem 3.1.5 independently of the field (which has characteristic not 2). We thus assume $n \geq 4$.
After scaling, we can further assume $1 \in D_F(\varphi)$. As we can have $d \times \langle 1\rangle \in I^nF$ only if $d \times \langle 1\rangle$ is isotropic which we excluded and using the usual reduction techniques, we can assume φ be defined over $F = K((t))$ for some rigid field K with $\varphi = \varphi_1 \perp t\varphi_2$ and $\varphi_1 \neq 0 \neq \varphi_2$, where φ_1, φ_2 are the residue forms. Since we have $\varphi_1, \varphi_2 \in I^{n-1}K$ by Lemma 3.4.1, the Holes Theorem implies

$$\dim\varphi_1, \dim\varphi_2 \in \{2^{n-1}, 2^{n-1} + 2^{n-2}, 2^n\}.$$

If one of the dimensions is 2^{n-1}, Proposition 3.5.1 (iv) is fulfilled and we are done. Otherwise we have

$$\dim\varphi_1 = \dim\varphi_2 = 2^{n-1} + 2^{n-2}$$

and $\varphi_1, \varphi_2 \in I^{n-1}F$. Using the induction hypothesis, we can argue as in the second part of the proof of Proposition 3.5.4 to get a representation

$$\varphi \cong \rho \otimes (\alpha \perp \beta)$$

with $\rho \in P_{n-3}F$ and two Albert forms α, β over F. According to Lemma 3.2.4 we write

$$\rho = \langle\langle x_1, \ldots, x_m, -1, \ldots, -1\rangle\rangle \tag{4.1}$$

with $x_1, \ldots, x_m$ linearly independent in F^*/F^{*2}.
We now distinguish between the cases $s(F) = 1, 2$ or ∞.

$s(F) = 1$: As -1 is a square, we cannot have any -1 as a slot in the representation (4.1) as otherwise ρ would be hyperbolic. We thus have $m = n-3 \geq 1$ i.e. ρ has a slot $x_1 \neq -1$. According to Remark 2.4.9 and Remark 3.2.1 we can assume K to be $K'((x_1))$ for some field K' and thus consider the residue class forms concerning x_1, which lie in $I^{n-1}K'$ due to Lemma 3.4.1. We therefore have

$$\langle\langle x_2, \ldots, x_m\rangle\rangle \otimes (\alpha \perp \beta) \in I^{n-1}F$$

and the claim follows now by induction.

$s(F) = 2$: Here we have $m \in \{n-4, n-3\}$. If we have $m \geq 1$, we can argue as in the case $s(F) = 1$. So we just have to deal with the case $m = n-4 = 0$, i.e. $n = 4$. We then have

$$\langle\langle -1\rangle\rangle \otimes (\alpha \perp \beta) \in I^4F.$$

Now, Lemma 4.4.4 implies $\operatorname{ind}(c(\alpha \perp \beta)) \in \{1, 2\}$. In the case $\operatorname{ind}(c(\alpha \perp \beta)) = 1$, i.e. $\alpha \perp \beta \in I^3F$, we are already done in view of Theorem 3.1.5. Otherwise, we choose some $x \in D_F(\alpha \perp \beta)$ and some $\sigma \in P_2F$ with

$c(\sigma) = c(\alpha \perp \beta)$. We then have $\psi := (\alpha \perp \beta \perp -x\sigma)_{\text{an}} \in I^3F$ with $\dim \psi \le 14$. In WF we further have

$$\langle\langle -1\rangle\rangle \otimes x\sigma = \langle\langle -1\rangle\rangle \otimes (\alpha \perp \beta) - \langle\langle -1\rangle\rangle \otimes \psi \in I^4F,$$

the Arason-Pfister Hauptsatz implies $\langle\langle -1\rangle\rangle \otimes x\sigma$ to be hyperbolic which then leads to

$$\langle\langle -1\rangle\rangle \otimes (\alpha \perp \beta) = \langle\langle -1\rangle\rangle \otimes \psi.$$

Finally, there is some $\tau \in I^3F$ with $\dim \tau = 12$ and

$$\langle\langle -1\rangle\rangle \otimes \psi = \langle\langle -1\rangle\rangle \otimes \tau \in WF$$

due to Lemma 4.4.3. Now Theorem 3.1.5 yields the result.

$s(F) = \infty$: If we have $m \ge 1$, we can argue as above. Otherwise, we get $\alpha \perp \beta \in I^3F$ by [EKM08, Corollary 41.10].

□

Remark 4.4.6:
There is another proof for the above in the case $s(F) = 2$. Above, we reduced the situation to the case where have $n = 4$. Thus we could have used the other equivalent statements of [Hof98a, Proposition 4.1] that are not covered in Proposition 3.5.1.
A faster way to prove Theorem 4.4.5 would then to remark that our form becomes hyperbolic over a quadratic extension, which then is equivalent to what we wanted to show.
Then, we would not have needed to use Lemma 4.4.3 und Lemma 4.4.4, but in the proof of [Hof98a, Proposition 4.1], some other nontrivial facts were used. Since our two auxiliary results just use the usual techniques to calculate in Witt rings over rigid fields and Merkurjev's Theorem, we decided to include this proof even if it is a bit longer.

4.5. Asymptotic Pfister Numbers

In this section, we will study the growth of Pfister numbers for forms of increasing dimension. As a fixed field can be too small to have anisotropic forms of all dimensions, which is a necessary assumption to talk about meaningful lower bounds, we will allow rigid field extensions while finding lower bounds as can be seen in the upcoming Proposition.

Proposition 4.5.1:
Let F be a rigid field. Then, there is some field extension E/F such that E is a rigid field and for any integer $d \geq 8$, we have

$$GP_3(E,d) \geq \left\lfloor \frac{d}{4} \right\rfloor - 1. \tag{4.2}$$

Proof:
As the term on the right sight of (4.2) increases monotonously when d grows, we may assume that d is even. According to Corollary 4.1.5 and passing to a field extension, we may further assume $F = K((t_i))_{i \in I}$ for some algebraically closed field K and some infinite index set I. To simplify notation, we assume $\mathbb{N} \subseteq I$. We define the integer n to be

$$n := 2 \cdot \left\lfloor \frac{d}{4} \right\rfloor - 2 = \begin{cases} \frac{d}{2} - 2, & \text{if } d \equiv 0 \mod 4 \\ \frac{d}{2} - 3, & \text{if } d \equiv 2 \mod 4 \end{cases}.$$

Note that n is even in both cases. By Example 3.1.4, using Corollary 5.4.1 and induction (recall the definition of $K((t_i))_{i \in I}$ as a direct limit, see Corollary 4.1.5 again), for $\psi := \langle 1, t_1, \ldots, t_n, (-1)^{\frac{n+2}{2}} t_1 \cdot \ldots \cdot t_n \rangle \in I^2 F$, we have

$$GP_2(\psi) = \frac{n}{2}.$$

Now, for the form $\varphi := \langle\langle t_{n+1} \rangle\rangle \otimes \psi \in I^3 F$, which is of dimension

$$2(n+2) \leq 2\left(\frac{d}{2} - 2 + 2\right) = d,$$

we have

$$GP_3(\varphi) = \frac{n}{2} = \left\lfloor \frac{d}{4} \right\rfloor - 1.$$

by Proposition 3.4.4 and the conclusion follows. □

Furthermore we are already in a good position to determine an upper bound for the 3-Pfister number. Our main ingredient is Corollary 4.1.7, which was proved with valuation theory.

Theorem 4.5.2:
Let F be a rigid field. For all even $d \in \mathbb{N}_0$, we have

$$GP_3(F,d) \leq \frac{d^2}{16}.$$

If we further have $d \geq 16$, we even have

$$GP_3(F,d) \leq \frac{d^2}{16} - \frac{d}{2} - \frac{82 - 2 \cdot (-1)^{\frac{d}{2}}}{16}.$$

Proof:
We will implicitly use that the functions $d \mapsto GP_3(d)$ and $d \mapsto \frac{d^2}{16} - \frac{d}{2} - \frac{82-2\cdot(-1)^{\frac{d}{2}}}{16}$ are monotonically increasing on the set of even integers ≥ 16 without referring to this fact explicitly. We use induction on d. We already know the following inequalities

$$GP_3(F,d) = 0 \text{ for all even } d < 8, \quad GP_3(F,8) = GP_3(F,10) = 1,$$
$$GP_3(F,12) = 2, \quad GP_3(F,14) = 2, \quad GP_3(F,16) = 3,$$

that are all compatible with the assertion. As we obviously have the inequality

$$\frac{d^2}{16} - \frac{d}{2} - \frac{82 - 2\cdot(-1)^{\frac{d}{2}}}{16} \leq \frac{d^2}{16},$$

we only have to show the second bound.
If a form $\varphi \in I^3F$ of dimension $d \geq 16$ is similar to $d \times \langle 1 \rangle$ it is Witt equivalent (in fact even isometric) to a sum of $\frac{d}{8}$ elements in GP_3F and we are done. Otherwise we can bound $GP_3(\varphi)$ according to Corollary 4.1.7 by

$$GP_3(F, d-2) + GP_2(F,k),$$

where k is the biggest integer $\leq \frac{d}{2}$ that is divisible by two, i.e. we have

$$k = \frac{d}{2} - \frac{1}{2} + (-1)^{\frac{d}{2}} \cdot \frac{1}{2} = 2 \cdot \left\lfloor \frac{d}{4} \right\rfloor,$$

as we can assume the form τ in Corollary 4.1.7 to be of dimension at most $\leq \frac{d}{2}$ after a possible scaling with a uniformizer (note that τ is the second residue class form). By Proposition 3.1.3 we thus know

$$\begin{aligned} GP_2(F,k) &\leq GP_2\left(F, \frac{d}{2} - \frac{1}{2} + (-1)^{\frac{d}{2}} \cdot \frac{1}{2}\right) \\ &= \frac{\frac{d}{2} - \frac{1}{2} + (-1)^{\frac{d}{2}} \cdot \frac{1}{2}}{2} - 1 = \frac{d}{4} - \frac{5}{4} + (-1)^{\frac{d}{2}} \cdot \frac{1}{4}, \end{aligned}$$

which leads to

$$GP_3(F,d) \leq GP_3(F,d-2) + \frac{d}{4} - \frac{5}{4} + (-1)^{\frac{d}{2}} \cdot \frac{1}{4}. \tag{4.3}$$

We now put $n := \frac{d}{2} - 8$, which is equivalent to $d = 2n + 16$, and consider for $n \in \mathbb{N}$ the recurrence relation

$$a_n = a_{n-1} + \frac{n}{2} + \frac{11}{4} + (-1)^n \cdot \frac{1}{4},$$

which was build by replacing the inequality with an equality in (4.3). For $a_0 = 3$ (corresponding to $GP_3(F,16) = 3$) this relation has the unique solution

$$a_n = \frac{1}{8}\big(2n(n+12) + (-1)^n + 23\big) = \frac{d^2}{16} - \frac{d}{2} - \frac{82 - 2\cdot(-1)^{\frac{d}{2}}}{16}.$$

By construction this is an upper bound for $GP_3(\varphi)$ and the proof is complete. $\square$

Remark 4.5.3:
For non-rigid fields, the 3-Pfister number of quadratic forms may grow exponentially in terms of the dimension, see [BRV18, Theorem 1.1] (with Proposition 3.1.2 in mind).

We can use the above result with an induction to also get upper bounds for the n-Pfister numbers of forms in I^nF for any $n \geq 4$. We will estimate a little bit coarser to get more succinct bounds. We will further use the following number theoretic result due to Jacob I. Bernoulli [Ber13].

Theorem 4.5.4 ([IR90, Chapter 15, Theorem 1]):
Let $m \in \mathbb{N}$ be an integer. Then there is some polynomial $p \in \mathbb{Q}[X]$ of degree $\deg(p) = m+1$ such that

$$1^m + 2^m + \ldots + n^m = p(n)$$

for all $n \in \mathbb{N}$.

Using the distributive rule and the above result several times, we immediately get the following consequence:

Corollary 4.5.5:
Let $q \in \mathbb{Q}[X]$ be a polynomial of degree $\deg(q) = m$. Then there is some polynomial $p \in \mathbb{Q}[X]$ of degree $m+1$ such that we have

$$q(1) + q(2) + \ldots + q(n) = p(n)$$

for all $n \in \mathbb{N}$.

The main result of this chapter is the following which states that Pfister numbers over all rigid fields can only increase polynomially. For non-rigid fields, it is not even known if the Pfister numbers are finite, see [BRV18, Remark 4.3].

Theorem 4.5.6:
Let $n \geq 3$ be an integer. Then there is some polynomial $p \in \mathbb{Q}[X]$ of degree $n-1$ whose associated function $\mathbb{R}_{\geq 0} \to \mathbb{R}$ is increasing, nonnegative and fulfils

$$GP_n(d, F) \leq p(d)$$

for all rigid fields F and all even integers $d \geq 2^n$.

Proof:
We prove this by induction on n, where the induction base $n = 3$ is covered by Theorem 4.5.2. So let now $n \geq 4$ and let $q_{n-1} \in \mathbb{Q}[X]$ be the polynomial as described in the statement for $n-1$ that exists due to the induction hypothesis and let $p_{n-1} \in \mathbb{Q}[X]$ be the polynomial of degree $n-1$ with

$$q_{n-1}(1) + \ldots q_{n-1}(k) = p_{n-1}(k) \tag{4.4}$$

for all $k \in \mathbb{N}$ that exists by Corollary 4.5.5. Obviously, the function

$$\mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto p_{n-1}(x)$$

is increasing and nonnegative as the function defined by q_{n-1} is so.

Just as in the proof of Theorem 4.5.2 we have

$$GP_n(d,F) \leq GP_n(d-2,F) + GP_{n-1}\left(\frac{d}{2} - \frac{1}{2} + \frac{1}{2} \cdot (-1)^{\frac{d}{2}}, F\right)$$

which is - using the same argument again - lower than or equal to

$$GP_n(d-4,F) + GP_{n-1}\left(\frac{d-2}{2} - \frac{1}{2} + \frac{1}{2} \cdot (-1)^{\frac{d-2}{2}}, F\right) + GP_{n-1}\left(\frac{d}{2} - \frac{1}{2} + \frac{1}{2} \cdot (-1)^{\frac{d}{2}}, F\right).$$

Iterating this process, we get a sum of expressions of the form $GP_{n-1}(k,F)$ with $2^{n-1} \leq k \leq \frac{d}{2}$ - each of these summands occuring at most 2 times - and one summand of the form $GP_n(2^n, F)$.
As we have $GP_n(2^n, F) = 1$ according to the Arason-Pfister Hauptsatz, we thus get the upper bound

$$\begin{aligned} 1 + 2 \sum_{k=2^{n-1}, 2|k}^{\lfloor \frac{d}{2} \rfloor} GP_{n-1}(k,F) &\leq 1 + 2 \sum_{k=2^{n-1}, 2|k}^{\lfloor \frac{d}{2} \rfloor} q_{n-1}(k) \\ &\leq 1 + 2 \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} q_{n-1}(k) \\ &= 1 + 2p_{n-1}\left(\lfloor \frac{d}{2} \rfloor\right) \leq 1 + 2p_{n-1}\left(\frac{d}{2}\right). \end{aligned}$$

It is thus easy to see that the polynomial $p_n(X) := 1 + 2p_{n-1}\left(\frac{X}{2}\right)$ does the job. $\square$

4.6. Symbol Length in $I^nF/I^{n+1}F$

A problem that is similar to that of calculating Pfister numbers is to determine the so called *symbol length* in the quotients $I^nF/I^{n+1}F$, i.e. the maximal number $m \in \mathbb{N}$ such that each element in $I^nF/I^{n+1}F$ can be written as the sum of at most m cosets that have an n-fold Pfister form as a representative. Here the case $n = 2$ is of particular interest as the results can be interpreted as results for the 2-torsion part of the Brauer group due to Merkurjev's Theorem. If we restrict ourselves to the case of rigid fields with a finite number of square classes, we can deduce upper bounds for the symbol length in the quotients from our earlier results.
Here, the case of superpythagorean fields is more involved as we can have forms of arbitrarily large dimension. The main idea of the proofs in this section is inspired by [Bec04, 7.2 Theorem].

Corollary 4.6.1:
Let F be a rigid field with $s(F) \in \{1,2\}$ and $2^m = q := |F^*/F^{*2}| < \infty$ for some $m \geq 4$. Then the symbol length in $I^nF/I^{n+1}F$ for $n \geq 3$ is bounded by $\mathcal{O}(2^{(n-1)(m-1)})$

Proof:
We first note that, according to [Lam05, Chapter XI, Theorem 7.19 (1) und (2)], there is some unique anisotropic universal π over F. By [Bec04, 3.2 Proposition, 3.3 Corollary] we know that each anisotropic form over F is a subform of π and π is an m-fold Pfister form. Thus, we can assume $n \leq m$ since otherwise, we have $I^nF = 0$ and the result is clear.
So let $0 \neq \overline{\varphi} \in I^nF/I^{n+1}F$ be an element of the factor group and $\varphi \in I^nF$ an anisotropic representative of this coset. By the choice of π, there is some quadratic form ψ over F with $\pi \cong \varphi \perp \psi$. Now $\pi \in I^mF \subseteq I^nF$ and $\varphi \in I^nF$ imply $\psi \in I^nF$.
We further have

$$\varphi \equiv \varphi \perp \pi \equiv \varphi \perp \varphi \perp \psi \equiv \psi \mod I^{n+1}F,$$

i.e. $\overline{\varphi} = \overline{\psi}$. As we have $\pi = \varphi \perp \psi$ we can deduce that we have $\dim\varphi \leq 2^{m-1}$ or $\dim\psi \leq 2^{m-1}$.
By Theorem 4.5.6 we know there is some polynomial $p \in \mathbb{Q}[X]$ of degree $n-1$ such that one of the representatives φ, ψ can be written as a sum of at most $p(2^{m-1})$ n-fold Pfister forms and the claim follows. □

Corollary 4.6.2:
Let F be a rigid field with $s(F) = \infty$, i.e. a superpythagorean field, and $|F^*/F^{*2}| = 2^m$ for some $m \in \mathbb{N}$. Then the symbol length in $I^nF/I^{n+1}F$ for $n \geq 3$ is bounded by $\mathcal{O}(2^{(n-1)(m+n-1)})$.

Proof:
As in the proof of Corollary 4.6.1 we consider an anisotropic representative φ of some coset $\overline{\varphi} \in I^nF/I^{n+1}F$ and want to construct another representative of small dimension. To do so, we take a diagonalization of φ and remark at first that there are at most 2^{m-1} different square classes in this diagonalization, since otherwise φ would be isotropic.
We can further assume that no square class occurs more than $2^{n+1}-1$ times because otherwise φ would have a subform that is similar to the $(n-1)$-fold Pfisterform $\langle\langle -1,\ldots,-1\rangle\rangle$. Finally, we can assume that there is at most one square class that

occurs at least 2^n times, since otherwise, φ would have a subform of the form $a\langle\langle x,-1,\dots,-1\rangle\rangle \in GP_{n+1}F$ for some $a,x \in F^*$. We can thus find a representative of $\overline{\varphi}$ of dimensional at most $2^{n+1}-1+(2^{m-1}-1)\cdot(2^n-1) = 2^{m+n-1}+2^n-2^{m-1}$ and the claim now follows as in the last step of Corollary 4.6.1 using Theorem 4.5.6. □

With similar arguments as above, but using the explicit bounds in Theorem 4.5.2 for $n=3$, we even get the following:

Corollary 4.6.3:
Let F be a rigid field with finite square class group with cardinality $|F^*/F^{*2}| = 2^m$ for some $m \geq 3$. Then the symbol length in I^3F/I^4F is bounded by

$$\begin{gathered} 2^{(m-1)^2-4}, \text{ if } s(F) \in \{1,2\} \\ (2^m-2^{m-3}+2)^2, \text{ if } s(F) = \infty. \end{gathered}$$

For $m<3$ the symbol lengths are bounded by

$$\begin{gathered} 0, \text{ if } s(F) \in \{1,2\} \\ 1, \text{ if } s(F) = \infty. \end{gathered}$$

Proof:
For the case $m=3$, there is nothing left to show after the above remarks and the proofs of Corollary 4.6.1 and Corollary 4.6.2.
If we have $s(F) \in \{1,2\}$ and $m<3$ there are no anisotropic forms in I^3F so that this case is clear.
For $m=1$ and $s(F)=\infty$ we obviously have $I^3F/I^4F = \{\overline{0}, \overline{\langle\langle -1,-1,-1\rangle\rangle}\}$.
If we have $m=2$ and $s(F)=\infty$, then F is equivalent with respect to quadratic forms to $\mathbb{R}((t))$. Thus, the claim in this case follows by Example 3.1.16, as we only have to consider forms of dimension at most 22 according to the proof of Corollary 4.6.2, using that we have among others

$$\langle\langle -1,-1,-1\rangle\rangle \perp \langle\langle -t,-1,-1\rangle\rangle \equiv \langle\langle t,-1,-1\rangle\rangle \mod I^4F.$$

□

Without going into technical details and introducing all the objects, we would like to state the analogous results for related objects such as Milnor-K-groups and Galois cohomology for those readers who are familiar with these structures. For the proof, it is then enough to recall the validity of the Milnor conjecture, see Section 2.3.

Corollary 4.6.4:
Under the same assumptions as in Corollary 4.6.1, Corollary 4.6.2 respectively Corollary 4.6.3, we get the same upper bounds for the symbol lengths in the Milnor-K-groups modulo 2, usually denoted by k_nF, and in the Galois cohomology groups $H^3(F,\mathbb{Z}/2\mathbb{Z})$.

We will now sketch another approach to obtain upper bounds for the symbol length and compare this one to known results.
Let F be a field of level $s(F) = 1$ with $V := F^*/F^{*2}$ and $|V| = 2^m$ for some $m \in \mathbb{N}$. Then V is an $\mathbb{F}_2$ vector space of dimension m and thus has a basis $B = \{b_1, \dots, b_m\}$. Let $n \in \mathbb{N}$ be an integer with $1 \le n \le m$. For a subset $U = \{u_1, \dots, u_n\} \in \mathcal{P}_n(B)$ of B consisting of exactly n elements, we define $\pi_U := \langle\langle u_1, \dots, u_n\rangle\rangle \in P_nF$.
As we have

$$\langle\langle xy, x_2, \dots, x_n\rangle\rangle = \langle\langle x, x_2, \dots, x_n\rangle\rangle + \langle\langle y, x_2, \dots, x_n\rangle\rangle \in WF \tag{4.5}$$

and $2 = 0 \in WF$ for all $\pi \in P_nF$, any anisotropic Pfisterform $\pi \in P_nF$ has a representation

$$\pi = \sum_{U \in C} \pi_u$$

for some $C \subseteq \mathcal{P}_n(B)$. As I^nF is generated additively by n-fold Pfister forms, for any $\varphi \in I^nF$, we have

$$\varphi = \sum_{U \in C} \pi_u$$

for some suitably chosen $C \subseteq \mathcal{P}_n(B)$.
We can thus bound the unscaled n-Pfister number for any form over F and thus the n-symbol length by $|\mathcal{P}_n(B)| = \binom{m}{n}$.

We can refine these techniques a little bit more, but we restrict ourselves to the easy case $n = 3$.
For all $x, y \in B$ with $x \neq y$, we only need a unique $z \in F^*$ such that $\pi_{\{x,y,z\}}$ occurs in a representation $\varphi = \sum \pi$ of minimal length due to the calculation in (4.5). Note that we cannot guarantee $z \in B$ anymore.
We write $m = 2k$ or $m = 2k+1$ for some suitable $k \in \mathbb{N}$, according to whether m is odd or even, and consider the sets $B_1 := \{b_1, \dots, b_k\}, B_2 := \{b_{k+1}, \dots, b_m\}$. By the pigeon hole principle, for any $U = \{u_1, u_2, u_3\} \in \mathcal{P}_3(B)$, we have, possibly after renumbering the u_i, either $u_1, u_2 \in B_1$ or $u_1, u_2 \in B_2$. There are $\binom{k}{2}$ possibilities for the first case and $\binom{m-k}{2}$ for the second case. Combining with the above, we can therefore bound the 3-Pfister number and the symbol length by $\binom{k}{2} + \binom{m-k}{2}$. For example, for $m = 5$, i.e. $k = 2$, we get $\binom{2}{2} + \binom{3}{2} = 1 + 3 = 4$.
It is even known that the symbol length in this case is given by 2, see [Kah05, Proposition 2.3 (a), (c)] and [BH04, Theorem 1.1, Lemma 2.1].
We now drop the extra assumption $n = 3$ again. When combining [Kah05] and [BH04], we obtain $\frac{m^{n-1}}{2(n-1)!} + f_n(m)$ as an upper bound for the n-symbol length with a polynomial f_n of degree at most $n - 2$. The above approach can be generalized to obtain the same upper bound and even gives us an explicit formula for the polynomial f_n. The details will be given in a future paper.

5. Pfister Numbers under Field Extensions

5.1. General Facts and Reduction Techniques

For this chapter, we fix an integer $n \in \mathbb{N}$. If have a field extension K/F and a quadratic form $\varphi \in I^n K$ that is defined over F, we ask whether there are connections between the Pfister number of φ and the Pfister numbers of a suitably chosen preimage under the canonical map $r_{K/F} : WF \to WK$ for various kinds of field extensions.

Lemma 5.1.1:
Let $\varphi \in I^n F$ and K/F be any field extension. We have $GP_n(\varphi_K) \leq GP_n(\varphi)$.

Proof:
Write $\varphi = \pi_1 + \ldots + \pi_\ell$ in WF for some $\pi_1, \ldots, \pi_\ell \in GP_n F$ and $\ell \in \mathbb{N}_0$ as small as possible. As we have

$$\varphi_K = (\pi_1)_K + \ldots + (\pi_\ell)_K$$

in WK the assumption follows. □

By the next result, we can restrict ourselves always to finitely generated field extensions.

Lemma 5.1.2:
Let K/F be an arbitrary field extension and $\varphi \in I^n K$ anisotropic. There is a finitely generated field extension E/F with $E \subseteq K$ and a $\psi \in I^n E$ with $\psi_K \cong \varphi$.

Proof:
There exist $k \in \mathbb{N}, a_1, \ldots, a_k \in K, \pi_1, \ldots, \pi_k \in P_n K$ with $\varphi = a_1\pi_1 + \ldots + a_k\pi_k$ in WK. For $\ell \in \{1, \ldots, k\}$ write $\pi_\ell = \langle\langle a_{\ell 1}, \ldots, a_{\ell n}\rangle\rangle$ for some $a_{\ell 1}, \ldots, a_{\ell n}$. Consider the field

$$E_0 := F(\{a_i, a_{\ell,j} \mid i, j \in \{1, \ldots, n\}, \ell \in \{1, \ldots, k\}\}),$$

which is finitely generated over F, and the form

$$\psi_0 := (a_1\pi_1 \perp \ldots \perp a_k\pi_k)_{\mathrm{an}} \in I^n E_0.$$

We definitely have $(\psi_0)_K = \varphi$ in WK, so we are done if we further have $\dim\psi_0 = \dim\varphi$. Otherwise $(\psi_0)_K$ is isotropic. Then there are $x_1, \ldots, x_{\dim\psi_0} \in K$ not all equal to 0 with

$$\psi_0(x_1, \ldots, x_{\dim\psi_0}) = 0.$$

We can replace E_0 and ψ_0 by

$$E_1 := E_0(x_1, \ldots, x_{\dim\psi_0}), \qquad \psi_1 := ((\psi_0)_{E_1})_{\mathrm{an}}$$

respectively. Repeating this last step several times, E_k and ψ_k for some $k \leq \frac{\dim \psi_0 - \dim \varphi}{2}$ will have the desired properties. $\square$

Corollary 5.1.3:
Let K/F be an arbitrary field extension. Let $d \in \mathbb{N}$ be an integer. Suppose there is an $m \in \mathbb{N}$ such that for every finitely generated field extension E/F with $E \subseteq K$, we have $GP_n(E, d) \leq m$. We then also have $GP_n(K, d) \leq m$.

Proof:
For $\varphi \in I^n K$ choose an intermediate field $F \subseteq E \subseteq K$ that is finitely generated over F and $\psi \in I^n E$ with $\psi_K \cong \varphi$ according to Lemma 5.1.2. As ψ can be written as the sum of no more than m elements in $GP_n E$, the conclusion follows by Lemma 5.1.1. $\square$

Corollary 5.1.4:
Let K be an algebraic extension of $\mathbb{Q}$. Then K is linked.

Proof:
Let $\pi_1, \pi_2 \in P_2 K$ and consider the form $\varphi := (\pi_1 \perp -\pi_2)_{\mathrm{an}}$. By Lemma 5.1.2 we can choose a finitely generated extension E of $\mathbb{Q}$ in K and a form $\psi \in I^2 E$ such that $\psi_K \cong \varphi$. We have $\dim \psi = \dim \varphi \in \{0, 4, 6\}$ by the Holes Theorem and by choice of φ. According to Example 2.2.15, E is a linked field so there are no anisotropic forms of dimension 6 in $I^2 E$. We thus have $\dim \varphi = \dim \psi \leq 4$ which means that π_1 and π_2 have a common slot by linkage theory, see Theorem 2.2.13. $\square$

Remark 5.1.5:
A crucial ingredient in the above proof was the fact that any finite extension of the rational numbers is linked as well. As the same also holds for the rational function fields over finite fields $\mathbb{F}_q$ with $q = p^n$ for some odd prime p and an integer $n \in \mathbb{N}$ or the field $\mathbb{R}(X)$ (see [Gen89, 5.1 iii) and v)]), we can deduce the same for arbitrary algebraic extensions of these fields.

Example 5.1.6:
As the pythagorean closure of a field is an algebraic extension, Corollary 5.1.4 yields that $\mathbb{Q}_{\mathrm{py}}$ is a linked field as well. In particular we can apply Corollary 3.1.15 to determine the Pfister numbers.

5.2. Quadratic Extensions

We fix a field F. Let E/F be a quadratic field extension. As we always assume $\mathrm{char}F \neq 2$ we can write $E = F(\sqrt{a})$ for some $a \in F^* \setminus F^{*2}$. We would like to study forms $\varphi \in I^3E$ that are defined over the ground field F. Quadratic forms over E that are defined over F are precisely those forms that lie in the kernel of the Scharlau transfer induced by a non trivial linear map $s : E \to F$ with $s(1) = 0$ by Theorem 2.5.7. Using Theorem 2.5.10 we can even say that we find $\psi \in I^3F$ such that we have $\psi_E = \varphi$ in WE. A natural question to ask is weather we can even have $\psi_E \cong \varphi$. We will show that this cannot be fulfilled in general, which is equivalent to the existence of some φ as above such that any preimage in I^3F has larger dimension than φ itself. Regarding this characterization it is reasonable to find upper bounds for the dimension of ψ in dependance of $\dim\varphi$. This question will be answered completely in this section.

Proposition 5.2.1:
Let $\varphi \in I^3E$ be an anisotropic form that is defined over F. Then there is some $\psi \in I^3F$ with $\dim\psi \leq \dim\varphi + 2$ and $\psi_E = \varphi$ in WE, i.e. for any $d \in \mathbb{N}$, we have $GP_3(\mathrm{im}(r_{E/F}), d) \leq GP_3(F, d+2)$.

Proof:
We take an arbitrary preimage of φ in WF which can be written in the form $\sigma \perp \langle\langle a\rangle\rangle \otimes \tau$ for some forms σ, τ over F such that σ_E is anisotropic according to Theorem 2.5.1. We clearly have $\sigma_E \cong \varphi$ which particularly implies $\dim\sigma = \dim\varphi$ and thus $\sigma \in IF$.
If $\sigma \notin I^2F$ Theorem 2.5.8 yields $d_\pm(\sigma) = a$. In this case we choose $x \in D_F(\sigma)$ and consider the form $\tau := (\sigma \perp -x\langle\langle a\rangle\rangle)_{\mathrm{an}}$ which lies in I^2F by construction, satisfies $\dim\tau = \dim\sigma$ and we have $\tau_E = \varphi$ in WE. In particular $\tau_E \in I^3E$ so that $c(\tau_E)$ is trivial.
If we already have $\tau \in I^3F$ we are done. Otherwise [GS06, Corollary 4.5.11] yields

$$\mathrm{ind}_F(c(\tau)) \mid [E : F] \cdot \mathrm{ind}_E(c(\tau_E)) = 2 \cdot 1 = 2.$$

As we excluded the case $\tau \in I^3F$ we actually have an equality $\mathrm{ind}_F(c(\tau)) = 2$. By [GS06, Proposition 1.2.3] we can find an (anisotropic) $\rho \in P_2F$ with $c(\tau) = c(\rho)$. For any $y \in D_F(\tau)$ the form $\psi := (\tau \perp -y\rho)_{\mathrm{an}}$ has all the desired properties and the conclusion follows. □

As a direct consequence we get the following:

Corollary 5.2.2:
Let $\varphi \in I^3E$ be defined over F with $\dim\varphi = d$. Then we have $GP_3(\varphi) \leq GP_3(F, d+2)$.

Proof:
If we choose ψ as in Proposition 5.2.1 the assertion follows by Lemma 5.1.1. □

A well studied case of quadratic extensions is the case of $F(\sqrt{-1})$ where F is pythagorean. Here we get a more precise statement.

Corollary 5.2.3:
Let F be a pythagorean field and $E = F(\sqrt{-1})$. For all $d \in \mathbb{N}_0$ we then have

$$GP_3(E, d) \leq GP_3(F, d+2).$$

Proof:
For all $x, y \in F$ not both equal to 0, we have $N_{E/F}(x + y\sqrt{-1}) = x^2 + y^2 = z^2 \in F^{*2}$ for some $z \in F^*$ as F is pythagorean. According to Proposition 2.5.6 there is some $d \in F^*$ for which we have

$$s_*(\langle x + y\sqrt{-1}\rangle) = d\langle 1, -N_{E/F}(x + iy)\rangle \cong d\langle 1, -z^2\rangle \cong \mathbb{H}.$$

We thus have $s_* = 0$. Theorem 2.5.10 yields that any quadratic form over E is already defined over F and has a preimage in the same power of the fundamental ideal. The assertion then follows by Corollary 5.2.2. □

In the following example we will show that the estimate in Proposition 5.2.1 is best possible.

Example 5.2.4:
We consider the field $F := \mathbb{Q}(x)((t_1))((t_2))((t_3))$. According to [HT98, Example 6.3, Corollary 6.2]

$$\begin{aligned}\varphi := \langle -x, -t_1t_2(x+4), t_1t_2x(x+4), x(x+1)(x+4), 2t_1(x+1)(x+2)(x+4), -2t_1x(x+2)\rangle \\ \perp t_3\Big(\langle x, -(x+4)\rangle \perp -t_1\langle 1, -(x+4)\rangle \perp -t_2\big(\langle x, -(x+1)\rangle \perp -2t_1\langle 1, -(x+1)\rangle\big)\Big)\end{aligned}$$

is a 14-dimensional form in I^3F that is not Witt equivalent to the sum of two forms in GP_3F. As the second and third entry just differ by $-x$ multiplicatively, φ is Witt equivalent to

$$\begin{aligned}&\langle -1, (x+1)(x+4), 2t_1(x+1)(x+2)(x+4), -2t_1(x+2)\rangle \\ &\perp t_3\Big(\langle 1, -(x+4)\rangle \perp -t_1\langle 1, -(x+4)\rangle \perp -t_2\big(\langle 1, -(x+1)\rangle \perp -2t_1\langle 1, -(x+1)\rangle\big)\Big) \\ &= -\underbrace{\langle\langle (x+1)(x+4), -2t_1(x+2)\rangle\rangle}_{=:\psi_1} \perp t_3\underbrace{\langle\langle x+4, t_1\rangle\rangle}_{=:\psi_2} \perp -t_2t_3\underbrace{\langle\langle x+1, 2t_1\rangle\rangle}_{=:\psi_3}. \end{aligned} \tag{5.1}$$

over $E := F(\sqrt{x}) \cong \mathbb{Q}(\sqrt{x})((t_1))((t_2))((t_3))$.
The anisotropic part of this form has dimension 12 as can be readily checked by an iterated application of Proposition 2.4.2 for the natural valuations with uniformizers t_3, t_2 and t_1 respectively (in this order) that can be extended to E and still have uniformizers t_3, t_2, t_1 respectively, see Corollary A.1.3: we finally have to check the anisotropy of binary forms over $\mathbb{Q}(\sqrt{x})$. This can be done by checking that the determinant is not equal to -1 which is clear in the upcoming cases.
Since we have

$$\sqrt{x} = 1 \cdot \left(\frac{1}{2}(\sqrt{x} + 2)\right)^2 - (x+4) \cdot \left(\frac{1}{2}\right)^2 \in D_F(\langle\langle x+4\rangle\rangle),$$

Proposition 2.2.2 now yields

$$\psi_2 \cong \langle\langle x+4, t_1\sqrt{x}\rangle\rangle. \tag{5.2}$$

A similar calculation shows $\psi_3 \cong \langle\langle x+1, t_1\sqrt{x}\rangle\rangle$. In $WE(\sqrt{t_1\sqrt{x}}) = WF(\sqrt{t_1\sqrt{x}})$ we therefore have

$$-\psi_1 = \varphi \in I^3E(\sqrt{t_1\sqrt{x}}).$$

By the Arason-Pfister Hauptsatz ψ_1 becomes hyperbolic over $E(\sqrt{t_1\sqrt{x}})$, but we have already seen that ψ_1 is anisotropic over E. Now Theorem 2.5.14 implies $\psi_1 \cong \langle\langle t_1\sqrt{x}_1, z\rangle\rangle$ for some $z \in E^*$.
Putting all this together in (5.1) we get

$$\varphi_E = \langle\langle t_1\sqrt{x}\rangle\rangle \otimes \big(t_3\langle\langle x+4\rangle\rangle \perp -t_2t_3\langle\langle x+1\rangle\rangle \perp -\langle\langle z\rangle\rangle\big).$$

Imitating the proof of Lemma 3.2.6 we see that we can choose $z = (x+1)(x+4)$. In WE we thus have the representations

$$\begin{aligned}\varphi_E &= \langle\langle t_1\sqrt{x}\rangle\rangle \otimes \big(t_3\langle\langle x+4\rangle\rangle \perp -t_2t_3\langle\langle x+1\rangle\rangle \perp -\langle\langle (x+1)(x+4)\rangle\rangle\big) && (5.3)\\ &= -\langle\langle t_1\sqrt{x}, x+1, -t_2t_3\rangle\rangle + t_3\langle\langle t_1\sqrt{x}, x+4, t_3(x+1)\rangle\rangle && (5.4)\\ &= -\langle\langle 2t_1, x+1, -t_2t_3\rangle\rangle + t_3\langle\langle t_1, x+4, t_3(x+1)\rangle\rangle. && (5.5)\end{aligned}$$

We shall now show that $(\varphi_E)_{\mathrm{an}}$ is not divisible by a Pfister form of the shape $\langle\langle a\rangle\rangle$ with $a \in F^*$.
So we take any $a \in E^*$ such that $(\varphi_E)_{\mathrm{an}}$ is divisible by $\langle\langle a\rangle\rangle$. Lemma 2.4.5 yields that the valuation of a with respect to t_3 has to be even. In particular, we can assume a to lie in $\mathbb{Q}(x)((t_1))((t_2))$. By Corollary A.1.3, t_3 is still a uniformizer of the valuation of E. As $\varphi_{E(\sqrt{a})}$ is hyperbolic the above implies that both residue class forms have to be hyperbolic over $E(\sqrt{a})$, see Corollary 2.4.4.
As the first residue class form given by $-\langle\langle (x+1)(x+4), t_1\sqrt{x}\rangle\rangle$, which is equal to $-(\psi_2 + (x+4)\psi_3)$ in WE as can be readily seen using (5.2) and the following representation of ψ_3, is defined over $\mathbb{Q}(x)((t_1))$, we can further assume a to lie in $\mathbb{Q}(x)((t_1))$.
We now consider the second residue class form. By the above and arguing as before, we see that both of its residue class forms concerning the t_2-valuation, i.e. ψ_2 and ψ_3, become hyperbolic over E, i.e. a is a common slot of ψ_2 and ψ_3. Such an element has to be an element of $E^* \smallsetminus F^*$ according to [HT98, Example 6.3] as we wanted to show.
As 12-dimensional forms in I^3F are divisible by a binary form according to Theorem 3.1.5 we can deduce there is no form $\psi \in I^3F$ with $\dim\psi = 12$ and $\psi_E = \varphi_E$. In particular the bound in Proposition 5.2.1 is sharp.

Regarding (5.5) we see that φ_E has at least a preimage in I^3F, which has the same 3-Pfister number as φ_E itself. This leads to the following question:

Question 5.2.5:
Let $n \in \mathbb{N}$ be an integer and $\varphi \in I^nE$ be defined over F. Does there always exist a $\psi \in I^nF$ with $\psi_E = \varphi$ in WE and $GP_n(\varphi) = GP_n(\psi)$?

5.3. Odd Degree Extensions

Let K/F be a field extension of odd degree. By Theorem 2.5.4 the map $r_{K/F} : WF \to WK$ induced by scalar extension is injective, thus an embedding. As before, we would like to study forms in some power of the fundamental ideal in the bigger field WK that are defined over F. In contrast to the case of quadratic field extensions, here, we have unique preimages because of the injectivity of $r_{K/F}$. We thus would like to connect I^nK with I^nF for any $n \in \mathbb{N}$ via $r_{K/F}$.

Lemma 5.3.1:
Let K/F be an odd degree extension and $n \in \mathbb{N}$. We then have $I^nF = r_{K/F}^{-1}(I^nK)$.

Proof:
It is clear that we have $r_{K/F}(I^nF) \subseteq I^nK$. For the converse, let now $\varphi \in WF$ with $\varphi_K \in I^nK$. By [Sch85, Chapter 2, 5.6 Theorem, 5.8 Theorem] there is some F-linear function $s : K \to F$ with $s_* \circ r_{K/F} = \mathrm{id}_{WF}$. We then have

$$\varphi = s_*(\varphi_K) \in s_*(I^nK) \subseteq I^nF$$

by [Ara75, Satz 3.3]. □

The following result is essentially due to Rost, but to our knowledge, it has never been stated that explicitly for the calculation of Pfister numbers before.

Corollary 5.3.2:
Let K/F be an odd degree extension and $\varphi \in I^nK$ a quadratic form that is defined over F. Then its unique preimage φ_F is in I^nF and we have $GP_n(\varphi) = GP_n(\varphi_F)$.

Proof:
By Lemma 5.3.1, we know that we have $\varphi_F \in I^nF$. The conclusion thus follows immediately by Lemma 5.1.1 and [Ros99a, Corollary 1]. □

If we have a purely inseparable extension, we can say a little bit more. For the reader's convenience, we include a proof of the following well known result.

Proposition 5.3.3 ([EKM08, Exercise 18.8]):
Let K/F be a finite purely inseparable field extension. We then have a canonical isomorphism $WF \cong WK$ given by $r_{K/F}$.

Proof:
As $r_{K/F}$ is injective as remarked in the beginning of the section, it is enough to show that it is also surjective. To do so, we only have to show that each unary quadratic form is already defined over F, i.e. any element in K^*/K^{*2} has a representative in F. So let now $x \in K^*$. If we have $x \in F$, the assertion is clear. Otherwise, we have a nontrivial field extension. As fields of characteristic 0 are perfect, this can only be the case if we have $\operatorname{char} F = p > 0$ and, as we always exclude the case of characteristic two, we even have $\operatorname{char} F > 2$. Since K/F is purely inseparable, there is some $n \in \mathbb{N}$ with $x^{p^n} \in F$. As p is odd, so is p^n. Since the square class group is always a group of exponent two, we have $xK^{*2} = x^{p^n}K^{*2}$ and the claim follows. □

Corollary 5.3.4:
Let K/F be a purely inseparable field extension. For any $d, n \in \mathbb{N}$ we have

$$GP_n(F, d) = GP_n(K, d).$$

5.4. Transcendental Extensions

In this section we will study transcendental field extensions. As the rational function field can be embedded in the field of formal Laurent series, which is a complete discrete valuation field, we can use many techniques that we have introduced before.

Corollary 5.4.1:
Let $\varphi \in I^n F$ and E be a field with $F(t) \subseteq E \subseteq F((t))$. We then have $GP_n(\varphi) = GP_n(\varphi_E)$.

Proof:
As the Pfister number can only decrease when going up to a field extension according to Lemma 5.1.1, it is enough to show the inequality

$$GP_n(\varphi) \leq GP_n(\varphi_E)$$

for $E = F((t))$. But this follows directly from Proposition 3.4.3. □

As a kind of self-strengthening of Proposition 3.4.4, we can enlarge the domain of possible field extensions for which the conclusion of the just mentioned proposition holds in the case of transcendental field extensions.

Corollary 5.4.2:
Let F be an intermediate field $K(t) \subseteq F \subseteq K((t))$ for some field K and let $\psi \in I^{n-1}K$ be a quadratic form. For the form $\varphi := \langle\langle t\rangle\rangle \otimes \psi \in I^n F$ we have $GP_{n-1}(\psi) = GP_n(\varphi)$.

Proof:
As in the proof of Proposition 3.4.4 we certainly have the inequality

$$GP_n(\varphi) \leq GP_{n-1}(\psi).$$

For the opposite inequality, we consider the field extension $E := K((t))$ of F. Using Lemma 5.1.1, Proposition 3.4.4 and Corollary 5.4.1 we have

$$GP_n(\varphi) \geq GP_n(\varphi_E) = GP_{n-1}(\psi_E) = GP_{n-1}(\psi),$$

which concludes the proof. □

By an obvious induction using the above proposition, we get the following result:

Corollary 5.4.3:
Let $n \in \mathbb{N}$ be an integer, K a field and F a field extension of K with $K(t_1, \ldots, t_n) \subseteq F \subseteq K((t_1))\cdots((t_n))$. For a form $\psi \in I^k F$, let $\varphi := \langle\langle t_1, \ldots, t_n\rangle\rangle \otimes \psi$. We then have

$$GP_k(\psi) = GP_{n+k}(\varphi).$$

Instead of checking that the form $\langle\langle t_1, \ldots, t_n\rangle\rangle$ is a divisor of the form φ in Corollary 5.4.3, we can apply the following criterion.

Proposition 5.4.4:
Let K be a field and F be the field of iterated laurent series $F = K((t_1))\ldots((t_n))$. Further let $\varphi \in WF$ be divisible by the Pfister forms $\langle\langle t_1\rangle\rangle, \ldots, \langle\langle t_n\rangle\rangle$. Then φ is divisible by the n-fold Pfister form $\langle\langle t_1, \ldots, t_n\rangle\rangle$.
Further, if φ is in addition divisible by $\langle\langle x\rangle\rangle$ for some $x \in K$, then φ is divisible by $\langle\langle x, t_1 \ldots, t_n\rangle\rangle$.

Proof:
We prove both assertions simultaneously by induction on n. The induction base, i.e. the cases $n = 1$ in the first case respectively $n = 0$ are trivial. For the induction step, we write $\varphi = \langle\langle t_n \rangle\rangle \otimes \psi = \psi \perp -t_n\psi$ for some quadratic form $\psi \in WK((t_1))\dots((t_{n-1}))$. For any $a \in \{\sqrt{t_1}, \dots, \sqrt{t_{n-1}}\}$ respectively $a \in \{\sqrt{x}, \sqrt{t_1}, \dots, \sqrt{t_{n-1}}\}$, the t_n-valuation has an extension to $F(a)$ such that t_n is still a uniformizing element due to Corollary A.1.3. As $\varphi_{F(a)}$ is hyperbolic by assumption, we see that $\psi_{F(a)}$ is also hyperbolic by comparing residue class forms. Thus, ψ is divisible by the forms $\langle\langle t_1 \rangle\rangle, \dots, \langle\langle t_{n-1} \rangle\rangle$ (and where appropriate in addition by $\langle\langle x \rangle\rangle$) by Theorem 2.5.2. By induction hypothesis, ψ is divisible by $\langle\langle t_1, \dots, t_{n-1} \rangle\rangle$ respective $\langle\langle x, t_1, \dots, t_{n-1} \rangle\rangle$ and the claim follows. □

Remark 5.4.5:
If we had introduced the notation of the basic part, we would have been able to give a slightly more general formulation of the above result. For an arbitrary field F, we have the same conclusion if we just consider elements $t_1, \dots, t_n$ that represent independent classes in the $\mathbb{F}_2$ vector space $F^*/A(F)$, where $A(F)$ denotes the basic part. For further information, see [BCW80], especially the paragraphs after Theorem 5.

In Proposition 5.4.4 it is important that all the elements $t_1, \dots, t_n$ are assumed to be rigid as the following example shows.

Example 5.4.6:
We consider the form

$$\varphi := \langle\langle -1, -1 \rangle\rangle = \langle 1, 1, 1, 1 \rangle \cong \langle\langle -1, -2 \rangle\rangle \cong \langle\langle -1, -5 \rangle\rangle,$$

over the field of rational numbers $\mathbb{Q}$. The above mentioned isometries are readily verified using the fact that both 2 and 5 are sums of 2 squares in $\mathbb{Q}$.
Thus φ is divisible by both $\langle\langle -2 \rangle\rangle$ and $\langle\langle -5 \rangle\rangle$, but it is not divisible by $\langle\langle -2, -5 \rangle\rangle \cong \langle 1, 2 \rangle \perp 5\langle 1, 2 \rangle$ as can easily be verified using the Hasse-Minkowski Theorem: if φ was divisible by $\langle\langle -2, -5 \rangle\rangle$, these two forms would be isometric as their dimensions coincide. But this is not the case as $\langle 1, 2 \rangle$ is anisotropic over $\mathbb{F}_5$, and thus $\langle 1, 2 \rangle \perp 5\langle 1, 2 \rangle$ is anisotropic over the 5-adic numbers $\mathbb{Q}_5$, but $\langle 1, 1, 1, 1 \rangle$ is an unimodular form of dimension 4, hence isotropic over $\mathbb{Q}_5$ (in fact, it is even hyperbolic over $\mathbb{Q}_5$ as an isotropic Pfister form).

Corollary 5.4.7:
Let F be a field and $\varphi := \langle\langle a \rangle\rangle \otimes \sigma \in I^n F$ be an anisotropic form for some $n \in \mathbb{N}$ and some $a \in F^* \smallsetminus F^{*2}$ with $|D_F(\langle\langle a \rangle\rangle)| = 2 = |D_F(\langle\langle -a \rangle\rangle)|$. Then there is some $\tau \in I^{n-1}F$ with $\dim \tau = \dim \sigma$ and $GP_n(\varphi) = GP_{n-1}(\tau)$.

Proof:
By [BCW80, (the proofs of) Theorem 2 and Theorem 4] with Remark 3.2.1 in mind, we are reduced to the case where we have $F = K((t))$ for some field K and $a = t$. By multiplying the entries of σ in an arbitrary diagonalization with suitable powers of a, we can get a form τ that is defined over K. We then clearly have $\langle\langle a \rangle\rangle \otimes \sigma \cong \langle\langle a \rangle\rangle \otimes \tau$ and τ is the residue class form with respect to the a-adic valuation on F.
The conclusion thus follows from Lemma 3.4.1 and Corollary 5.4.2. □

Reducing to the Ground Field

For the rest of the section, we consider a field F and its rational function field $E = F(X)$. We further fix $n \in \mathbb{N}$. We want to recall the exact sequence from Theorem 2.5.13 and will use the same terminology. Our aim now is to establish techniques to put the Pfister numbers over E down to those over F using the above sequence. The main ideas follow the pattern of [BR13]. We first introduce some further notation.

Definition and Remark 5.4.8:
Let $\varphi \in I^n E$. We define the *support* of φ to be

$$\operatorname{supp}(\varphi) := \left\{ v \mid \left(\partial(\varphi)\right)_v \neq 0 \right\}.$$

Note that $\operatorname{supp}(\varphi)$ is a finite set by Theorem 2.5.13. Further, we define the *degree* of φ to be

$$\deg(\varphi) := \sum_{v \in \operatorname{supp}(\varphi)} [F_v : F].$$

Valuations v with $[F_v : F] = 1$ are also called *rational points*.

We would like to give information on forms in $I^n E$ of small degree or with managable support. At first, it is clear that the degree of a given form $\varphi \in I^n E$ is equal to zero if and only if we even have $\varphi \in I^n F$ by the exactness of the sequence in Theorem 2.5.13. If we have $\deg(\varphi) \neq 0$, we even have $\deg(\varphi) \geq 2$ due to the following proposition.

Proposition 5.4.9:
Let $\varphi \in I^n E$ be a quadratic form. We then have $\deg(\varphi) \neq 1$.

Proof:
If we had $\deg(\varphi) = 1$ there would be a unique valuation v on E with $\left(\partial(\varphi)\right)_v \neq 0$ and this one would further fulfil $[F_v : F] = 1$. The latter would imply the linear functional $s^{(v)} : F_v \to F$ to be the identity. Thus, the corresponding transfer map $s_*^{(v)} : WF_v \to WF$ is a ring isomorphism. We would thus get

$$s \circ \partial(\varphi) = s_*^{(v)}\big(\underbrace{\partial_v(\varphi)}_{\neq 0} \big) \neq 0,$$

contradicting Theorem 2.5.13. □

We will now discuss the case that we have degree 2 in detail. There are two possibilities: the support can consist either of two rational points or of one valuation whose corresponding residue class is an extension of degree 2 of our base field F. We will start with the latter case.

Proposition 5.4.10:
Let $\varphi \in I^n E$ be a quadratic form of degree $\deg(\varphi) = 2$ with $\operatorname{supp}(\varphi) = \{v\}$, where v is induced by an irreducible polynomial of degree 2. Then, there is some $\psi \in I^{n-1} F$ with $\partial(\langle\langle p \rangle\rangle \otimes \psi) = \partial(\varphi)$.
For $n \in \{1, 2, 3\}$, we can further assume $\dim \psi = \dim \partial_v(\varphi)$, for $n = 4$, we can assume $\dim \psi \leq \dim \partial_v(\varphi) + 2$.

Proof:
As only one place is relevant by hypothesis, we may write by abuse of notation $\partial(\varphi) = \varphi_v \in I^{n-1}F_v$. As F_v/F is a quadratic field extension and we have

$$0 = s \circ \partial(\varphi) = s_*^{(v)} \circ \partial_v(\varphi) = s_*^{(v)}(\varphi_v),$$

Theorem 2.5.10 implies the existence of some $\sigma \in I^{n-1}F$ with $\sigma_{F_v} = \varphi_v$. According to (the proof of) Proposition 5.2.1, we can assume σ to satisfy the assertion about the dimension. We now define ψ to be $\psi := -\sigma \in I^{n-1}F$. Be the definition of the components of ∂, it is clear that we have $\partial(\langle\langle p\rangle\rangle \otimes \psi) = \partial(\varphi)$ (just consider the valuations induced by p, by ∞ and simultaneously those induced by other polynomials $q \neq p$ separately). □

Corollary 5.4.11:
With the notations as in Proposition 5.4.10, we have

$$GP_n(\varphi) \le GP_{n-1}(\psi) + GP_n(F, \dim(\varphi \perp -\langle\langle p\rangle\rangle \otimes \psi)).$$

Proof:
As we have $\partial(\varphi) = \partial(\langle\langle p\rangle\rangle \otimes \psi)$, the form $\varphi \perp -\langle\langle p\rangle\rangle \otimes \psi$ is defined over F due to Theorem 2.5.13 and thus comes from a form lying in I^nF due to Proposition 2.5.12. As we clearly have $GP_n(\langle\langle p\rangle\rangle \otimes \psi) \le GP_{n-1}(\psi)$, the claim follows. □

We now consider the case of a support consisting of two rational points.

Proposition 5.4.12:
Let $\varphi \in I^nE$ be a quadratic form of degree $\deg(\varphi) = 2$ with $\mathrm{supp}(\varphi) = \{v_1, v_2\}$, where v_1, v_2 are rational points and v_1 is induced by a monic linear polynomial p (which we may assume without loss of generality after renumbering the v_i). Then, there is some form $\psi \in I^{n-1}F$ of dimension $\dim\psi = \dim\partial_{v_1}(\varphi)$ such that there is a binary form β with $\partial(\beta \otimes \psi) = \partial(\varphi)$.

Proof:
Let $\psi := \partial_p(\varphi)$ denote the residue class form for $v_1 = v_p$. We have $\psi \in I^{n-1}F$ by Proposition 2.5.12.
We now have either $v_2 = v_\infty$ or $v_2 = v_q$ for a monic linear polynomial $p \neq q \in F[X]$.

$v = v_\infty$: Consider the form $-\langle\langle p\rangle\rangle \otimes \psi$. We obviously have

$$\partial_p(-\langle\langle p\rangle\rangle \otimes \psi) = \psi = \partial_\infty(-\langle\langle p\rangle\rangle \otimes \psi).$$

Now it is clear that we have $\mathrm{supp}(-\langle\langle p\rangle\rangle \otimes \psi) = \{v_p, v_\infty\} = \mathrm{supp}(\varphi)$.
As we have $s \circ \partial(\varphi) = 0$ due to Theorem 2.5.13 and $\mathrm{supp}(\varphi) = \{v_p, v_\infty\}$ with $s_*^{(\infty)} = -\mathrm{id}_{WF} = -s_*^{(p)}$, we can calculate

$$\partial_\infty(\varphi) = -s_*^{(\infty)}(\partial_\infty(\varphi)) = s_*^{(p)}(\partial_\infty(\varphi)) = \partial_\infty(\varphi) = \psi,$$

which finishes the proof in this case with $\beta = -\langle\langle p\rangle\rangle$.

$v = v_q$: Similarly as above, we get

$$\partial_q(\varphi) = -\partial_p(\varphi) = -\psi.$$

We now set $\beta := \langle p, -q\rangle$. We then have

$$\partial_p(\beta \otimes \psi) = \psi = -\partial_q(\beta \otimes \psi).$$

As we further have $\partial_\infty(\beta \otimes \psi) = 0$ and $\operatorname{supp}(\beta \otimes \psi) \subseteq \{v_p, v_q, v_\infty\}$, the claim follows.

□

As several times before in this thesis, it is convenient to have as sharp bounds for all upcoming dimensions as possible. Further, we already know that scaling a form does not change its Pfister number, but the dimension of the second residue class form may change after scaling so that scaling a form suitably may improve our situation. We thus study how the support of a quadratic form can change under scaling. As every square class in $F(X)^*/F(X)^{*2}$ has a representative that is the product of an element in F^* and some pairwise different irreducible monic polynomials, it is sufficient to study the scaling with irreducible monic polynomials. So let now $p \in F[X]$ be such a polynomial.
For $q \in F[X]$ another irreducible polynomial, we have $q \in \operatorname{supp}(\varphi)$ if and only if $q \in \operatorname{supp}(p\varphi)$: in fact, studying the residue class map shows that $\partial_q(\varphi)$ and $\partial_q(p\varphi)$ are similar.
If we have $\dim \partial_p(\varphi) = \dim \varphi$, we can work instead with $p\varphi$ as we have

$$\operatorname{supp}(p\varphi) \subseteq (\operatorname{supp}(\varphi) \smallsetminus \{p\}) \cup \{\infty\}$$

so that this form seems to be easier to handle. We will thus assume $\dim \partial_p(\varphi) < \dim \varphi$ for all $p \in \operatorname{supp}(\varphi)$ from now on.
Lastly, we have to consider v_∞. If the degree of the polynomial p is even, the definition of ∂_∞ directly shows $\partial_\infty(\varphi) = \partial_\infty(p\varphi)$. But if $\deg p$ is odd, we may have $\partial_\infty(\varphi) = 0$ but $\partial_\infty(p\varphi) \neq 0$ as the example $\varphi = \langle 1\rangle$ with $p(X) = X$ shows.
For a form φ as in Proposition 5.4.12 that fulfils the extra assumption $\dim \partial_p(\varphi) < \dim \varphi$ for all $p \in \operatorname{supp}(\varphi)$ introduced above, this does not matter:
If we have $\operatorname{supp}(\varphi) = \{p, \infty\}$, we have $p \in \operatorname{supp}(p\varphi) \subseteq \{p, \infty\}$ and as $\deg(p\varphi)$ cannot be equal to 1 because of Proposition 5.4.9, we even have equality.
If we have $\operatorname{supp}(\varphi) = \{p, q\}$ for two different linear polynomials $p, q \in F[X]$ we either have that one of the residue class forms has a dimension lower than or equal to than $\frac{\dim \varphi}{2}$ or we can work instead with $pq\varphi$ and determine its Pfister number.
We thus get the following result.

Corollary 5.4.13:
With the notations as in Proposition 5.4.12 we have

$$\begin{aligned} GP_n(\varphi) &\leq GP_{n-1}(\psi) + GP_n(F, \dim(\varphi \perp -\beta \otimes \psi)) \\ &\leq GP_{n-1}\left(F, \frac{\dim \varphi}{2}\right) + GP_n(F, 2\dim(\varphi)). \end{aligned}$$

Proof:
The first bound can be shown exactly as in Corollary 5.4.11. The second bound is clear by the paragraph above. □

We can now extend this result to the case of forms of arbitrary degree with support consisting only of rational points.

Corollary 5.4.14:
Let $\varphi \in I^n E$ be a quadratic form whose support consists only of rational points. We then have

$$GP_n(\varphi) \leq m \cdot GP_{n-1}\left(F, \frac{\dim \varphi}{2}\right) + GP_n(F, (m+1)\dim \varphi).$$

where m denotes the number of rational points in $\operatorname{supp}(\varphi)$ unequal to v_∞.

Proof:
Let $p_1, \ldots, p_m$ be those polynomials such that $v_{p_1}, \ldots, v_{p_m} \in \operatorname{supp}(\varphi)$ and denote by φ_k the respective residue class form. After a scaling, we may assume $\dim \varphi_k \leq \frac{\dim \varphi}{2}$ for all $k \in \{1, \ldots, m\}$. As the calculation in Proposition 5.4.12 shows, the support of the form

$$\varphi \perp \mathop{\perp}_{k=1}^{m} \langle\langle p_k \rangle\rangle \otimes \varphi_k$$

is contained in $\{\infty\}$. By Proposition 5.4.9, the form thus has empty support, and is therefore defined over F. Considering the dimensions, we then see that its Pfister number is bounded by

$$m \cdot GP_{n-1}\left(F, \frac{\dim \varphi}{2}\right) + GP_n(F, (m+1)\dim \varphi)$$

as claimed. □

Remark 5.4.15:
The above strategy cannot be extended to forms with support that does not only contain rational points. The problem is that the residue class forms do not have to be defined over F and thus may contain other polynomials so that we cannot guarantee the degree to decrease.
We want to note further that the above bounds cannot be expected to be sharp, but it is an interesting fact that the Pfister number can be bounded in some cases by the Pfister number of related forms over the base field.

6. Supreme Torsion Forms

6.1. Introduction

Our aim is to develop a theory analogous to the one in [Bec04] in the case of a formally real field. For the whole chapter let F be a formally real field that is not pythagorean. By abuse of notation we call a quadratic form over F torsion if its Witt class is torsion in WF.

Definition 6.1.1:
Let φ be a quadratic form over F. We call φ a *supreme torsion form* if φ is an anisotropic torsion form such that every anisotropic torsion form over F is similiar to a subform of φ.

Example 6.1.2:

(a) We consider a formally real field F with square class group $\{1, -1, 2, -2\}$. Such a field exists by [Lam05, II. Remark 5.3] and can be constructed by a modification of the Gross-Fischer construction. For the Witt group we have $WF \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the only nontrivial torsion form is given by $\langle\langle 2 \rangle\rangle = \langle 1, -2 \rangle$ which then trivially is a supreme torsion form.

(b) See [Lam05, Chapter II. Example 5.4 and page 45]: Let $F = \mathbb{Q}_3 \cap R$, where R is a real closed subfield of the algebraic closure $\overline{\mathbb{Q}_3}$ of the 3-adic numbers. Then F has square class basis $\{-1, 2, 3\}$ and 3 is not a sum of two squares. The torsion subgroup W_tF is given by

$$\{\mathbb{H}, \langle 1, -2 \rangle, \langle 1, -3 \rangle, \langle 2, -6 \rangle, \langle 1, -6 \rangle, \langle -3, 6 \rangle, \langle 2, -3 \rangle, \langle 1, 1, -3, -3 \rangle\}.$$

One readily checks that $\langle 1, 1, -3, -3 \rangle$ is a supreme torsion form.

(c) See [Lam05, Chapter II. Example 5.7, and page 46]: Analogously to the above example we can take $F = \mathbb{Q}_5 \cap R$ where R is a real closed subfield of the algebraic closure $\overline{\mathbb{Q}_5}$ of the 5-adic numbers. Then F has square class basis $\{-1, 2, 5\}$. The torsion subgroup W_tF is given by

$$\{\mathbb{H}, \langle 1, -2 \rangle, \langle 1, -5 \rangle, \langle 1, -10 \rangle, \langle 2, -5 \rangle, \langle 2, -10 \rangle, \langle 5, -10 \rangle, \langle 1, -2, 5, -10 \rangle\}$$

and $\langle 1, -2, 5, -10 \rangle$ is a supreme torsion form as can be readily verified.

Before stating the next result, we would like to recall the definition of the u-invariant for formally real fields F. By definition, we have

$$u(F) := \sup\{\dim \varphi \mid \varphi \text{ is an anisotropic torsion form}\} \in \mathbb{N} \cup \{\infty\}.$$

Lemma 6.1.3:
If F admits a supreme torsion form φ, then F has finite u-invariant $u(F) = \dim\varphi$.

Proof:
As every anisotropic torsion form over F is similar to a subform of φ, we have $u(F) \leq \dim\varphi$. But φ is anisotropic by definition of a supreme torsion form, so we have $u(F) = \dim\varphi$. □

Proposition 6.1.4:
Let φ be a supreme torsion form. Then φ is a Pfister form and every anisotropic torsion form is a subform of φ. In particular φ is unique up to isometry.

Proof:
We choose an anisotropic torsion Pfister form π of maximal dimension. Such a form exists as F is not pythagorean so there are two dimensional torsion forms, therefore an 1-fold torsion Pfister form, and every torsion form of dimension greater than $\dim\varphi$ has to be isotropic according to Lemma 6.1.3. Then for every $a \in F^*$, $\pi \otimes \langle\langle a \rangle\rangle$ is a torsion form. Because of the maximality of π, for any $a \in F^*$, the form $\pi \otimes \langle\langle a \rangle\rangle$ is isotropic and therefore as a Pfister form hyperbolic. That means we have $a\pi \cong \pi$, i.e. $a \in G_F(\pi) = D_F(\pi)$. Thus π is universal so that π cannot be similar to a proper subform of any anisotropic form. Thus π is similar to φ. Because π is universal and round as a Pfister form, it is therefore isometric to φ. So if ψ is an anisotropic torsion form and $a \in F^*$ such that $a\psi \subseteq \varphi$, we have

$$\psi \cong a^2\psi \subseteq a\varphi \cong \varphi$$

as $\varphi \cong \pi$ is round and universal as shown above. □

In view of the above result, we will often call a supreme torsion form *the* supreme torsion form in the sequel.

Proposition 6.1.5:
Let φ be the supreme torsion form over F. Then φ is the only anisotropic universal torsion form.

Proof:
Let ψ be an anisotropic universal torsion form. Then ψ is isometric to a subform of φ by Proposition 6.1.4. But as a universal form cannot be a proper subform of any anisotropic form we get $\psi \cong \varphi$. □

Considering the above proposition and [Bec04, 3.3 Corollary], it is natural to ask the following:

Question 6.1.6:
Let $u(F) < \infty$ and φ be the only anisotropic universal torsion form. Is φ the supreme torsion form?

To construct examples of every possible size we study the behaviour of supreme forms under Laurent series extensions.

Proposition 6.1.7:
Let $\varphi \in P_nF$ be the supreme torsion form over F. Then $\varphi \otimes \langle\langle t \rangle\rangle$ is the supreme torsion form over $F((t))$.

If conversely $\psi = \psi_1 \perp -t\psi_2$ with residue class forms $\psi_1, -\psi_2$ is the supreme torsion form over $F((t))$ we have $\psi_1 \cong \psi_2$ and this form is the supreme torsion form over F.

Proof:
It follows directly from Proposition A.2.9 that a form over $F((t))$ is torsion if and only if both residue class forms of the given form are torsion over F. In particular $\varphi \otimes \langle\langle t\rangle\rangle$ is torsion. It is further anisotropic by Proposition 2.4.2. If $\psi = \psi_1 \perp -t\psi_2$ is a torsion form over $F((t))$ we have that both ψ_1 and ψ_2 are subforms of φ as φ is the supreme torsion form over F. It is then clear that ψ is a subform of $\varphi \otimes \langle\langle t\rangle\rangle = \varphi \perp -t\varphi$. For the converse let τ be an anisotropic torsion form over F. Then $\tau \otimes \langle\langle t\rangle\rangle$ is an anisotropic torsion form over $F((t))$ according to Proposition 2.4.2 which is therefore a subform of the supreme torsion form ψ. By Corollary 2.1.2 this is equivalent to

$$i_W(\psi \perp -\tau \otimes \langle\langle t\rangle\rangle) \geq \dim(\tau \otimes \langle\langle t\rangle\rangle) = 2\dim\tau.$$

Using Proposition 2.4.3 we have

$$\begin{aligned} i_W(\psi \perp -\tau \otimes \langle\langle t\rangle\rangle) &= i_W(\psi_1 \perp -\tau \perp -t(\psi_2 \perp \tau)) \\ &= \underbrace{i_W(\psi_1 \perp -\tau)}_{\leq \dim\tau} + \underbrace{i_W(\psi_2 \perp \tau)}_{\leq \dim\tau} \leq 2\dim\tau. \end{aligned}$$

We thus have

$$i_W(\psi_1 \perp -\tau) = \dim\tau = i_W(\psi_2 \perp \tau)$$

which means $\tau \subseteq \psi_1$ and $-\tau \subseteq \psi_2$ respectively again by Corollary 2.1.2. As τ is an arbitrary anisotropic torsion form we see that both ψ_1 and ψ_2 are supreme torsion forms over F. As supreme torsion forms are unique up to isometry by Proposition 6.1.4, we finally see $\psi_1 \cong \psi_2$. □

Example 6.1.8:
Combining Example 6.1.2 with Proposition 6.1.7 we can construct a field F_n with an n-fold Pfister form as a supreme torsion form for any $n \in \mathbb{N}$. To do so let F_1 be the field as in Example 6.1.2 (a) and define F_n to be $F_n := F_1((t_1))\cdots((t_{n-1}))$ for $n \geq 2$.

Proposition 6.1.9:
Let $n \in \mathbb{N}$ be an integer and π be the supreme torsion n-fold Pfister form over F. We then have:

(a) I_t^{n+1} is trivial,

(b) π is the unique anisotropic torsion n-fold Pfister form over F.

(c) Every anisotropic form in $I_t^{n-1}F$ is either isometric to π or similar to an $(n-1)$-fold Pfister form

(d) If φ is a nonhyperbolic torsion form, there is a Pfister form ψ over F such that $\varphi \otimes \psi$ is Witt equivalent to π. If we have moreover $\varphi \in I^kF$ and $\dim\varphi < 2^{k+1}$ for some $k \in \{1, \ldots, n\}$, then ψ is an $(n-k)$-fold Pfister form.

Proof:

(a) Is clear with Lemma 6.1.3 and the Arason-Pfister Hauptsatz.

(b) Every anisotropic torsion n-fold Pfister form over F is isometric to a subform of π because of Proposition 6.1.4. Considering the dimensions we get that such a form is isometric to π.

(c) Let $\varphi \in I_t^{n-1}F \setminus \{0\}$ be an anisotropic torsion form with $\varphi \neq \pi$. According to Proposition 6.1.4, φ is thus a subform of π. We can therefore write $\pi \cong \varphi \perp \psi$. As $\pi, \varphi \in I_t^{n-1}F$ we further have $\psi \in I_t^{n-1}F \setminus \{0\}$. The Arason-Pfister Hauptsatz yields $\dim \varphi = \dim \psi = 2^{n-1}$ and therefore φ and ψ have to be similar to $(n-1)$-fold Pfister forms respectively. More precisely [Lam05, Chapter X. Corollary 5.4] readily implies that φ is even similar to ψ such that we have $\pi \cong x\varphi \otimes a$ for suitable $a, x \in F^*$.

(d) As φ is not hyperbolic, every multiple of φ is torsion and $u(F)$ is finite, we can choose an m-fold Pfister form ψ such that $\varphi \otimes \psi$ is not hyperbolic with $m \in \mathbb{N}_0$ maximal. Let $\rho := (\varphi \otimes \psi)_{\text{an}}$ denote the anisotropic part of $\varphi \otimes \psi$. Because of the maximality of m, for every $a \in F^*$, the form $\varphi \otimes \psi \otimes \langle\langle a \rangle\rangle$ is hyperbolic, which is equivalent to $\rho \cong a\rho$. This implies ρ to be universal. Proposition 6.1.5 implies $\rho \cong \pi$ which implies $\varphi \otimes \psi = \pi \in WF$.
If we have $\varphi \in I_t^k F$ with $\dim \varphi < 2^{k+1}$, we get
$$2^n = \dim(\pi) \leq \dim(\varphi \otimes \psi) < 2^{m+k+1}$$
which implies $m \geq n-k$. We also have $m \leq n-k$ as otherwise we would get $\varphi \otimes \psi \in I_t^{n+1}$ contradicting (a).

□

The next result is an analogue to Kneser's Lemma, see [Lam05, Chapter XI. 6.5].

Proposition 6.1.10:
Let φ be a non universal torsion form and β binary torsion form over F. We then have $D_F(\varphi) \subsetneq D_F(\varphi \perp \beta)$.

Proof:
We have $\beta \cong \langle a, -xa \rangle$ for some $a \in F^*$ and $x \in \sum F^{*2}$. Thus $x = e_1^2 + \ldots + e_n^2$ for some $e_1, \ldots, e_n \in F^*$, where we assume n to be minimal so that any sum $e_1^2 + \ldots + e_k^2$ with $k < n$ is not 0. Now assume $D_F(\varphi) = D_F(\varphi \perp \beta)$. We first show per induction that $a(e_1^2 + \ldots + e_k^2) \in D_F(\varphi)$ for all $k \in \{1, \ldots, n\}$. For $k = 1$ this follows from the equality $D_F(\varphi) = D_F(\varphi \perp \beta)$ as clearly $a \in D_F(\beta) \subseteq D_F(\varphi \perp \beta)$. If $a(e_1^2 + \ldots + e_{k-1}^2) \in D_F(\varphi)$ we have
$$a(e_1^2 + \ldots + e_k^2) = a(e_1^2 + \ldots + e_{k-1}^2) + ae_k^2 \in (D_F(\varphi) + D_F(\beta)) \setminus \{0\} \subseteq D_F(\varphi \perp \beta) = D_F(\varphi)$$
as claimed.
In particular, it follows that $xa = (e_1^2 + \ldots + e_n^2)a \in D_F(\varphi)$ so that $D_F(\varphi \perp \beta)$ is isotropic, hence universal. As we have $D_F(\varphi) = D_F(\varphi \perp \beta) = F^*$ it follows that φ is universal, contradicting the hypothesis. □

Question 6.1.11:
Does the conclusion of Proposition 6.1.10 also hold if we do not assume β to be binary?

6.2. 2-real-maximality

We develop a theory similar to that in [Bec06b, Section 3]. We would like to draw attention to the fact that the basic definitions differ, but the results will sometimes read the same.

Definition 6.2.1:
A field extension K/F is called *totally positive* if every ordering of F extends to an ordering of K.

In contrast to our definition, K. Becher defines a field extension K/F as totally positive if every semi-ordering of F extends to a semi-ordering of K. As any ordering is a semi-ordering, we deal with a weaker property than K. Becher in his article.

Example 6.2.2:

(a) Finite extensions of odd degree are totally positive due to Corollary A.2.7.

(b) Purely transcendental extensions are totally positive by Example A.2.8

(c) In Proposition A.2.9, we have seen that $F((t))/F$ is totally positive for a real field F.

(d) The function field exension $F(\varphi)/F$ is totally positive if and only if φ is totally indefinite, see Theorem A.2.10.

As another easy example we remark the following for later reference. This result also justifies the used terminology.

Lemma 6.2.3:
Let $a \in F^* \setminus F^{*2}$. The quadratic extension $F(\sqrt{a})/F$ is totally positive if and only a is totally positive, i.e. we have $a \in \sum F^{*2}$.

Proof:
By Proposition A.2.6 we see that $F(\sqrt{a})/F$ is totally real if and only if $a \in P$ for all $P \in X_F$. By Artins theorem this is equivalent to $a \in \sum F^{*2}$. □

As a first step we will study how the term totally positive behaves in the case of towers of field extensions.

Lemma 6.2.4:
Let $K/E/F$ be a tower of field extensions. We then have:

(a) If K/E and E/F both are totally positive then so is K/F.

(b) If K/F is totally positive, so is E/F.

Proof:

(a) This is obviously true.

(b) Every ordering of F can be extended to an ordering of $K \supseteq E$ by hypothesis which then can be restricted to an ordering on E by Proposition A.2.5, which clearly extends the given ordering.

□

As the most important examples of totally positive field extensions, we have the following generalisation of Lemma 6.2.3.

Proposition 6.2.5:
If $K \subseteq F_{\text{py}}$ is a subfield of the pythagorean closure of F then K/F is totally positive.

Proof:
By Lemma 6.2.4 (b) it is enough to show that F_{py}/F is totally positive. This is shown for example in [Lam05, Chapter VIII. Corollary 4.6]. We further give an alternative proof using techniques that will reoccur in the next section.
By [EKM08, Lemma 31.16] F_{py} is the direct limit of all field extensions E/F such that there is a tower of field extensions $F = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = E$ with $F_k = F_{k-1}(\sqrt{z_{k-1}})$, where $z_{k-1} = 1 + x_{k-1}^2$ for some $x_{k-1} \in F_{k-1}$. Since each F_k/F_{k-1} is totally positive by Lemma 6.2.3, so is E/F according to Lemma 6.2.4 (a). Since the property of being totally positive is preserved under direct limits by Proposition A.2.13, the conclusion follows. $\square$

Remark 6.2.6:
We will now compare our version of totally positive field extensions with the strong version of totally positive field extensions introduced by K. Becher, to which we will refer as *strongly totally positive*. As mentioned above, since any ordering is an semi-ordering, strongly totally positive field extensions are always totally positive. A quadratic extension is strongly totally positive if and only if it is totally positive due to Lemma 6.2.3 and [Bec06b, Proposition 3.2].
Further, for any subfield E of the pythagorean closure of F, the extension E/F is strongly totally positive due to Proposition 6.2.5 and [Bec06b, Corollary 3.3]. We finally consider the case of the function field extension of an anisotropic form φ of dimension at least 3. The extension $F(\varphi)/F$ is strongly totally positive if and only if $n \times \varphi$ is isotropic for some $n \in \mathbb{N}$ by [Bec06b, Theorem 3.4]. Over so-called *SAP fields* any form is totally indefinite if and only if some suitable multiple is isotropic by [Pre84, (9.1) Theorem], where the if-part obviously holds over any field. We therefore obtain that both concepts coincide over SAP fields. But by [Pre84, (2.12) Theorem], for any SAP field F, there is a form of the shape $\langle 1, a, b, -ab\rangle$ for suitable $a, b \in F^*$ that is totally indefinite but $n \times \langle 1, a, b, -ab\rangle$ is anisotropic for all $n \in \mathbb{N}$. As a concrete example, we can consider the field $F = \mathbb{R}((s))((t))$ and $\varphi = \langle 1, s, t, -st\rangle$. Then φ is of the desired shape but clearly, for any $n \in \mathbb{N}$, $n \times \varphi$ is anisotropic. Thus $F(\varphi)/F$ is a totally positive field extension that is not strongly totally positive.

With the following example we will disprove the reverse implications in the above results.

Example 6.2.7:
Let $F = \mathbb{Q}, E = \mathbb{Q}(\sqrt{2}), K = E(\sqrt{\sqrt{2}}) = \mathbb{Q}(\sqrt[4]{2})$. As K is real and $F = \mathbb{Q}$ has a unique ordering K/F is totally positive. Now Lemma 6.2.4 (b) implies E/F to be totally positive. But K/E is not totally positive as the ordering of E in which $\sqrt{2} < 0$ (see Proposition A.2.6) cannot be extended to K. So K/F (and E/F) totally positive does not in general imply K/E to be totally positive for a tower of field extensions $K/E/F$, the other implication in Lemma 6.2.4 (a) is therefore false.
We further have $F_{\text{py}} = E_{\text{py}}$ and $\sqrt{2}$ is not totally positive in E. We thus see

$K \notin E_{\text{py}} = F_{\text{py}}$ so that the other implication in Proposition 6.2.5 is false as well.

As we have just seen that totally positive field extensions do not behave well in general when enlarging the base field, it seems natural to refine the term. We will see that this refinement is exactly the modification we need.

Definition 6.2.8:
A field extension K/F is called *hereditarily totally positive* if K/E is totally positive for all intermediate fields $F \subseteq E \subseteq K$.

Example 6.2.9:
Let E/F be a field extension of odd degree and K an intermediate field. As we have $[E : F] = [E : K] \cdot [K : F]$, the extension E/K is also of odd degree and therefore totally real, see [Sch85, Chapter 3, 1.10 Theorem (ii)]. Therefore odd degree extensions are hereditarily totally real.

Proposition 6.2.10:
Let $K \subseteq F_2$ be a subfield of the quadratic closure of F. Then K/F is hereditarily totally positive if and only if we have $K \subseteq F_{\text{py}}$.

Proof:
We assume first $K \subseteq F_{\text{py}}$ and let E be an intermediate field $K/E/F$. We then have $E \subseteq K \subseteq F_{\text{py}} = K_{\text{py}}$ and the implication then follows by Proposition 6.2.5.
For the converse we consider the field $E := K \cap F_{\text{py}}$. Since we are done if we have $E = K$ we now assume $E \neq K$. As K/E is an algebraic extension within the quadratic closure of F, there is some quadratic extension of E within K by [Mor96, Section 18, Problem 1]. As we have $\operatorname{char} E \neq 2$, using a well known fact in elementary field theory yields the existence of some $a \in E$ such that we have $E \subsetneq E(\sqrt{a}) \subseteq K$. By the choice of a we have $\sqrt{a} \notin F_{\text{py}} = E_{\text{py}}$. Thus the binary form $\langle 1, -a \rangle$ is not isometric to the hyperbolic plane over E_{py}, which means that $\langle 1, -a \rangle$ is not a torsion form in WE by Theorem A.2.4. By Pfister's local global principle this means that a is not totally positive in E. So there is an ordering P on E in which a is negative. By Proposition A.2.6 P does not have any extension to $E(\sqrt{a})$ and therefore cannot have any extension to $K \supseteq E(\sqrt{a})$. Thus K/E is not totally positive, a contradiction. □

Lemma 6.2.11:
If E/F is hereditarily totally positive field extension then E/F is algebraic.

Proof:
Let E/F be a field extension that is not algebraic. Then there is some $x \in E$ that is transcendental over F. Consider the purely transcendental field extension $K := F(x^2)$ of F which is a proper subfield of E. By [Lam05, Chapter VIII. Examples 1.13 (C)] there is some ordering on K in which x^2 is negative. Such an ordering cannot be extended to an ordering on E since x^2 is a square in E and therefore totally positive in E. □

Corollary 6.2.12:
Let φ be a quadratic form over F such that the function field $F(\varphi)$ is defined, i.e. $\dim \varphi \geq 2$ and if $\dim \varphi = 2$ we assume φ to be anisotropic. The function field extension $F(\varphi)/F$ is hereditarily totally positive if and only of φ is a non hyperbolic

binary torsion form.

Proof:
We must have $\dim\varphi = 2$, φ anisotropic since otherwise, $F(\varphi)/F$ is not an algebraic extension contradicting Lemma 6.2.11. We therefore have $\varphi \cong b\langle 1, -a\rangle$ for some $a, b \in F^*$ and $F(\varphi) = F(\sqrt{a})$. In this case, the only proper subfield of $F(\varphi)$ containing F is F itself. Thus $F(\varphi)/F$ is hereditarily totally positive if and only if $F(\varphi)/F$ is totally positive. By Lemma 6.2.3 this is the case if and only if a is totally positive, which is equivalent to φ being torsion. $\square$

As a last step in this section, we would like to deduce a necessary condition for a quadratic form to be a supreme torsion form using the above theory of hereditarily totally positive field extensions.

Definition 6.2.13:
Let φ be a form over F and E/F a formally real non-pythagorean field extension. Then φ is called *2-real-maximal* over E if φ_E is anisotropic but φ_K is isotropic for every nontrivial hereditarily totally positive 2-extension K/E.

Proposition 6.2.14:
Let φ be an anisotropic torsion form over F. Then there exists a hereditarily totally positive 2-extension K/F such that φ is 2-real-maximal over K.

Proof:
We define $\mathcal{C}$ to be the class

$$\mathcal{C} := \{K \subseteq F_{\text{py}} \mid K/F \text{ is a field extension, } \varphi_K \text{ is anisotropic}\}.$$

We clearly have $F \in \mathcal{C}$. Further every chain in $\mathcal{C}$ has its union as an upper bound. Thus Zorn's lemma implies the existence of a maximal element $K \in \mathcal{C}$. As φ is torsion and we require φ_K to be anisotropic, we have $K \subsetneq F_{\text{py}}$. If now E/K is a nontrivial hereditarily totally positive 2-extension we have $E \subseteq K_{\text{py}} = F_{\text{py}}$ by Proposition 6.2.10. Since we have chosen $K \in \mathcal{C}$ to be maximal, φ_E has to be isotropic, i.e. φ is 2-real maximal over K. $\square$

Remark 6.2.15:
In Proposition 6.2.14 we required φ to be a torsion form to guarantee that the constructed field K is not pythagorean, in line with our general assumption that we want to work only over nonpythagorean fields. Over a pythagorean field the theory of 2-real-maximality would not be fruitful as any form would trivially be 2-real-maximal. As an aside we further remark that [Bec06b, Example 3.6 and Lemma 3.1] implies the existence of a totally indefinite form over a field F that is still anisotropic over F_{py} (and hence no torsion form over F), such that this form would lead to $K = F_{\text{py}}$ in Proposition 6.2.14 when imitating the proof.

Proposition 6.2.16:
Let φ be an anisotropic quadratic form over F. Then the following are equivalent:

(i) The form φ is 2-real maximal over F;

(ii) Every anisotropic binary torsion form is similar to a subform of φ.

Proof:

(i) $\Rightarrow$ (ii): Let β be an anisotropic binary torsion form. Then β is similar to $\langle 1, -a\rangle$ for some $a \in \sum F^{*2} \setminus F^{*2}$. We therefore have $F(\sqrt{a}) \subseteq F_{\text{py}}$. As φ is 2-real maximal over F, $\varphi_{F(\sqrt{a})}$ has to be isotropic. The assertion now follows from Theorem 2.5.1.

(ii) $\Rightarrow$ (i): Let K/F be a nontrivial hereditarily totally positive 2-extension. According to Proposition 6.2.10 we have $K \subseteq F_{\text{py}}$. As in the proof of this proposition K contains a quadratic extension of F. That means there is an $a \in F^*$ such that $\langle 1, -a\rangle$ is not isotropic over F but over K and therefore also over $F_{\text{py}} \supseteq K$. This means that $\langle 1, -a\rangle \in W(F_{\text{py}}/F) = W_tF$ is a binary torsion form over F and thus a subform of φ. So φ_K is isotropic.

□

Corollary 6.2.17:
If φ is the supreme torsion form over F then φ is 2-real-maximal over F.

Proof:
This is clear since φ obviously satisfies Proposition 6.2.16 (ii). □

The other implication in Corollary 6.2.17 is not true. We will give two different reasons to disprove this:
Firstly, we will show that we can construct a field such that there is more than one Pfister form that is 2-real maximal over this field simultaneously, see Proposition 6.3.10. Since supreme torsion forms are unique up to isometry by Proposition 6.1.4, this will finish the first argument.
Secondly, we can give an example of a field with a 2-real maximal form that is not even a torsion form. This will be done in Example 6.3.14.

As supreme torsion forms are always Pfister forms and 2-real maximal, it is convenient to have a criterion for 2-real-maximality specifically for Pfister forms. This is done in the following using the set of represented elements of the pure part of the given Pfister form.

Proposition 6.2.18:
Let π be an anisotropic Pfister form. Then φ is 2-real-maximal over F if and only if we have $-\sum F^{*2} \setminus -F^{*2} \subseteq D_F(\pi')$.

Proof:
For the if-part, let $\langle a, b\rangle$ be a binary anisotropic torsion form. We then have

$$ab \in -\sum F^{*2} \setminus -F^{*2} \subseteq D_F(\pi').$$

We thus have $\langle 1, ab\rangle \subseteq \pi$. As $\langle 1, ab\rangle$ is similar to $\langle a, b\rangle$, this implication follows in view of Proposition 6.2.16.
For the opposite direction, let $a \in -\sum F^{*2} \setminus -F^{*2}$. As the form $\langle 1, a\rangle$ is torsion and anisotropic by the choice of a, it is similar to a subform of π by Proposition 6.2.16, i.e. there is some $x \in F^*$ with $x\langle 1, a\rangle \subseteq \pi$. This in particular implies $x \in D_F(\pi) = G_F(\pi)$, so that we even get $\langle 1, a\rangle \subseteq \pi$. Witt Cancellation now yields $\langle a\rangle \subseteq \pi'$, i.e. $a \in D_F(\pi')$.

□

6.3. Construction of Examples

The aim of this section is to introduce methods to construct fields with supreme torsions forms or fields that have a given set of forms that are 2-real maximal and which fulfil several additional properties in order to fill the gaps of the former section.

Definition 6.3.1:
Let $\mathcal{C}$ be a class of field extensions of a field F. The class $\mathcal{C}$ is called *admissible* if the following holds:

(AD1) The class $\mathcal{C}$ is not empty.

(AD2) The class $\mathcal{C}$ is closed under direct limits (in the category of field extensions of F).

(AD3) If $K \in \mathcal{C}$ and $E \subseteq K$ is a subfield of K with $F \subseteq E$ then we have $E \in \mathcal{C}$.

Remark 6.3.2:
Recall that, for a direct system $(E_i)_{i\in I}$ of field extensions of F with direct limit $E = \varinjlim_{i\in I} E_i$, we have embeddings $\varphi_i : E_i \to E$. Replacing E_i with $\varphi_i(E_i)$ for all $i \in I$ and F with $\varphi_i(F)$ for some $i \in I$ (in fact, we have $\varphi_i(F) = \varphi_j(F)$ for all $i, j \in I$ as can easily be shown), we can thus assume without loss of generality $F \subseteq E$ and even $E_i \subseteq E$ for all $i \in I$ as all studied objects can also be studied over isomorphic fields. Thus, it is justified to speak about field extensions when it comes to direct limits. We will often use the above implicitly to simplify the notation.

As a first step we will provide several examples of admissible field extensions that will be used later.

Lemma 6.3.3:
Let X be a set of anisotropic quadratic forms over F and $\mathcal{C}$ be the class of field extensions of F such that every form $\varphi \in X$ stays anisotropic. Then $\mathcal{C}$ is admissible.

Proof:
As X consists of anisotropic quadratic forms we have $F \in \mathcal{C}$, thus (AD1). To verify (AD2) let $E = \varinjlim E_i$ with $E_i \in \mathcal{C}$ and $\varphi \in X$. If φ_E was isotropic with isotropic vector $x \in E^{\dim\varphi}$, this vector x would already be defined over some E_j with $j \in I$. Thus φ would be isotropic over E_j, a contradiction to $E_j \in \mathcal{C}$.
Lastly if E is a field extension of F such that φ_E is isotropic for some $\varphi \in X$, then φ_K is clearly isotropic for every field extension K/E. Since this is equivalent to (AD3), the conclusion follows. □

Corollary 6.3.4:
The class of all formally real field extensions of F is admissible.

Proof:
This follows by applying Lemma 6.3.3 to the set $X = \{n \times \langle 1\rangle \mid n \in \mathbb{N}\}$. □

Lemma 6.3.5:
Let F be a real field and $X \subseteq X_F$ be a subset of orderings of F. Then the class $\mathcal{C}$ of field extensions extending the orderings in X is admissible. In particular the class

of totally positive field extensions of F is admissible.

Proof:
We clearly have $F \in \mathcal{C}$ so we have (AD1). Further Proposition A.2.13 implies the validity of (AD2). Lastly Proposition A.2.5 directly yields (AD3).
To show that the class of totally positive field extensions of F is admissible, we just have to apply the above to the case $X = X_F$. □

Remark 6.3.6:
Of course Lemma 6.3.5 can also be used to imply Corollary 6.3.4.

Lemma 6.3.7:
Let F be a real field and $\mathcal{C}$ be the class of hereditarily totally positive field extensions of F. Then $\mathcal{C}$ is admissible.

Proof:
We clearly have $F \in \mathcal{C}$, thus (AD1). Further (AD3) holds in the light of Lemma 6.2.4 (b).
For (AD2) let $(E_i)_{i\in I}$ be a directed system of fields in $\mathcal{C}$ and E its direct limit in the category of field extensions of F. Let K be an intermediate field, i.e. we have a tower of field extensions $E/K/F$. Let $P \in X_K$ be an ordering of K and $a_1, \dots, a_n \in P$ for some $n \in \mathbb{N}$. Let $x_1, \dots, x_n \in E$ be such that $a_1x_1^2 + \dots + a_nx_n^2 = 0$. Then, there is some $i \in I$ such that we have $a_k, x_k \in E_i$ for all $k \in \{1, \dots, n\}$.
We now consider the field $K_i := E_i \cap K$. This field is ordered by $K_i \cap P$. As E_i/F is an hereditarily totally positive extension by assumption, we can extend this ordering of K_i to an ordering P_i of E_i. Thus, the form $\langle a_1, \dots, a_n\rangle$ is anisotropic over E_i by Proposition A.2.11, i.e. we have $x_1 = \dots = x_n = 0$. Thus, the form is even anisotropic over E. Using Proposition A.2.11 again, we see that P has an extension to E and hence E/K is totally positive. As K was an arbitrary subfield of E, the extension E/F is hereditarily totally positive and the proof is complete. □

The next result shows that we can combine the requirements of different admissible classes of field extensions to get a new admissible class.

Lemma 6.3.8:
Let I be a nonempty index set and $\mathcal{C}_i$ be an admissible class of field extensions of the field F for any $i \in I$. Then $\bigcap_{i\in I}\mathcal{C}_i$ is admissible.

Proof:
As (AD1) and (AD3) together imply $F \in \mathcal{C}_i$ for all $i \in I$ the class $\bigcap_{i\in I}\mathcal{C}_i$ is not empty. The validity of (AD2) and (AD3) is clear. □

The crucial step to construct fields with supreme torsion forms is the following:

Theorem 6.3.9 ([Bec04, 6.1 Theorem]):
Let $\mathcal{C}$ be an admissible class of field extensions of F. There exists a field $K \in \mathcal{C}$ such that $K(\varphi) \notin \mathcal{C}$ for any anisotropic quadratic form φ over K of dimension at least 2.

The proof of the above result uses a modification of Merkurjev's function field techniques. To be more precise, the field K in Theorem 6.3.9 is constructed as a direct limit of iterated function field extensions. This fact will be used in an upcoming incidental remark.

A well known variation of the u-invariant of a field F that we need to recall before stating the next result is the *Hasse number* defined as

$$\tilde{u}(F) := \max\{\dim\varphi \mid \varphi \text{ is totally indefinite}\}$$

or ∞ if no such maximum exists.

Proposition 6.3.10:
Let $n \in \mathbb{N}$ be an integer with $n \geq 2$ and P be a non-empty set of anisotropic torsion n-fold Pfister forms over F. There exists a field extension K/F such that

(a) the field K is formally real.

(b) for every $\pi \in P$ the form π_K is anisotropic .

(c) we have $\tilde{u}(K) = u(K) = 2^n$. In particular $I_t^{n+1}K$ is trivial.

(d) every anisotropic form over K that is indefinite at at least one ordering of K is a subform of π_K for some $\pi \in P$.

(e) any anisotropic n-fold Pfister form over K is 2-real maximal over K.

Proof:
Let $\mathcal{C}$ be the class of field extensions K/F such that K is real and π_K is anisotropic for every $\pi \in P$. This class is admissable due to Lemma 6.3.3, Corollary 6.3.4 and Lemma 6.3.8. By Theorem 6.3.9 we get the existence of a field $K \in \mathcal{C}$ such that $K(\varphi) \notin \mathcal{C}$ for every anisotropic quadratic form φ over K of dimension at least 2.
As $K \in \mathcal{C}$, it satisfies (a) and (b).
Now let φ be a quadratic form that is indefinite at at least one ordering of K. Then $K(\varphi)$ is a real field by Theorem A.2.10. Thus $\pi_{K(\varphi)}$ has to be isotropic and as a Pfister form therefore hyperbolic for some $\pi \in P$. By [Lam05, Chapter X. Corollary 4.9] there exists some $a \in F^*$ with $a\varphi \subseteq \pi$. In particular $\dim\varphi \leq \dim\pi$, so we get (c). This further implies that π_K is universal for every $\pi \in K$. Since a form is similar to a subform of a universal round form iff it is a subform of this form itself, we get (d).
Now let π be any anisotropic torsion n-fold Pfister form over K and $a \in \sum F^{*2} \setminus F^{*2}$. As $n \geq 2$, the torsion form $\psi := \langle a \rangle \perp \pi'$ is not similar to any Pfisterform and therefore has to be isotropic. That means $-a \in D_K(\pi')$, so we have $\langle 1, -a \rangle \subseteq \pi$. The assertion follows by Proposition 6.2.16. □

The next result can be proved in a similar way as Proposition 6.3.10 using a slightly modified admissible class of field extensions. We therefore will not give a complete proof but only the class of field extensions that can be used. We leave the details to the reader.

Proposition 6.3.11:
Let $n \in \mathbb{N}$ be an integer with $n \geq 2$ and P be a set of anisotropic torsion n-fold Pfister forms over F. There exists a field extension K/F such that

(a) The field extension K/F is totally positive.

(b) For every $\pi \in P$ the form π_K is anisotropic .

(c) We have $\tilde{u}(K) = u(K) = 2^n$. In particular $I_t^{n+1}K$ is trivial.

(d) Every anisotropic totally indefinite form over K is a subform of π_K for some $\pi \in P$.

(e) Any anisotropic n-fold Pfister form over K is 2-real maximal over K.

Proof:
This follows by considering the class

$$\mathcal{C} := \{K/F \mid K/F \text{ is totally positive}, \pi_k \text{ is anisotropic for every } \pi \in P\},$$

which is admissible by Lemma 6.3.3, Corollary 6.3.5 and Lemma 6.3.8. □

Remark 6.3.12:
We can give another proof for Proposition 6.2.14 using the just developed techniques. To do so, given an anisotropic torsion Pfister form φ over the field F, consider the class of field extensions

$$\mathcal{C} := \{E/F \mid E/F \text{ is a hereditarily totally positive extension, } \varphi \text{ anisotropic}\}.$$

As the proof of Theorem 6.3.9 just uses the direct limit of several function field extensions, with Corollary 6.2.12 in mind, one readily sees that the resulting field will be a 2-extension.

Corollary 6.3.13:
Let $\pi \in I_t^n F$ for some $n \in \mathbb{N}$ with $n \geq 2$. Then there is a formally real field extension K/F such that π_K is the supreme torsion form over K.

Proof:
This is a direct consequence of Proposition 6.3.10 or Proposition 6.3.11 with $P = \{\pi\}$. □

With the next example we will finish the discussion at the end of Section 6.2.

Example 6.3.14:
We consider the form $\varphi = \langle 1, 1, -3\rangle$ over the field $\mathbb{Q}$ of rational numbers. It is clear that φ is anisotropic and totally indefinite, but not a torsion form. We further consider the class of field extensions

$$\mathcal{C} := \{K/\mathbb{Q} \mid K/\mathbb{Q} \text{ is totally positive, } \varphi_K \text{ is anisotropic}\}.$$

This class is admissible by Lemma 6.3.3, Lemma 6.3.5 and Lemma 6.3.8. By Theorem 6.3.9 there is some field $F \in \mathcal{C}$ such that $F(\psi) \notin \mathcal{C}$ for every anisotropic quadratic form ψ over F of dimension at least 2. Note that F cannot be pythagorean as otherwise φ_F would be isotropic. Thus there is some $a \in \sum F^{*2} \setminus F^{*2}$. We then have $F(\pi) \notin \mathcal{C}$, where π is the binary torsion form $\pi = \langle 1, -a\rangle$. Using Lemma 6.2.4 (a), Lemma 6.2.3 and the isomorphism $F(\pi) \cong F(\sqrt{a})$, we see that $F(\pi)/\mathbb{Q}$ is totally positive. But as we have $F(\pi) \notin \mathcal{C}$, the form $\varphi_{F(\pi)}$ has to be isotropic which means that π is similar to a subform of φ_F, see Theorem 2.5.1. In view of Proposition 6.2.16, we see that we have constructed a field F such that φ is 2-real maximal over F, but φ_F is not a torsion form.

6.4. Correlations with Invariants

While studying fields it is a natural task to find correlations between invariants of the given field. Several results in [Pfi66] deal with this question. One main problem is to find lower bounds for the number of square classes $|F^*/F^{*2}|$ in dependence of other invariants. In [Pfi66] this is done using the level $s(F)$ for nonreal fields and, for formally real fields, using the Pythagoras number $p(F)$ defined as the smallest positive integer n such that any sum of squares is a sum of n squares or ∞ if no such integer exists. For a given formally real field, in order to show the estimate

$$|F^*/F^{*2}| \geq 2 \cdot 2^{\frac{t(t+1)}{2}}$$

for $t := \lfloor \log_2 p(F) \rfloor$, the idea in [Pfi66, Satz 25] was to find lower bounds for the quotients $[D_F(2^n) : D_F(2^{n-1})]$ for $n \in \{1, \ldots, t+1\}$. Pfister showed that we have $[D_F(2^{t+1}) : D_F(2^t)] \geq 2$. The main purpose in this section is to give an example to show that this bound is sharp.

Proposition 6.4.1:
Let $n \in \mathbb{N}$ be an integer $\varphi \in P_n F$ the supreme torsion form over F and $p(F) > 2^{n-1}$. Then we have $[D_F(\infty) : D_F(2^{n-1})] = 2$.

Proof:
As we have $p(F) > 2^{n-1}$, we clearly have $[D_F(\infty) : D_F(2^{n-1})] \geq 2$, see also [Pfi66, Satz 25, proof of Satz 18 d)]. So let now x, y be representives of nontrivial classes of $D_F(\infty)/D_F(2^{n-1})$. Then $2^{n-1} \times \langle 1 \rangle \perp \langle -x \rangle$ and $2^{n-1} \times \langle 1 \rangle \perp \langle -y \rangle$ are anisotropic Pfister neighbors of the n-fold Pfister forms $2^{n-1} \times \langle\langle x \rangle\rangle, 2^{n-1} \times \langle\langle y \rangle\rangle$ respectively which therefore both have to be isometric to φ. Witt cancellation yields $2^{n-1} \times \langle -x \rangle \cong 2^{n-1} \times \langle -y \rangle$ which is equivalent to

$$2^{n-1} \times \langle 1 \rangle \cong 2^{n-1} \times \langle xy \rangle.$$

Thus we obtain $xy \in D_F(2^{n-1})$ which means that x and y represent the same class in $D_F(\infty)/D_F(2^{n-1})$. $\square$

Remark 6.4.2:
The above proof shows in particular that in the situation of Proposition 6.4.1 the supreme torsion form is given by

$$\langle\langle x, -1, \ldots, -1 \rangle\rangle$$

for an arbitrary element $x \in D_F(\infty) \setminus D_F(2^{n-1})$, i.e. an element of length greater than 2^{n-1}.

Example 6.4.3:
To construct a field K fulfilling the assumptions of Proposition 6.4.1 for a given $n \in \mathbb{N}$ with $n \geq 2$ we can start with any field with pythagoras number greater than 2^{n-1}. Specifically, we can choose $\mathbb{R}(X_1, \ldots, X_{2^{n-1}})$ by [Lam05, Chapter IX. Corollary 2.4]. We then take an element x of finite length greater than 2^{n-1}. In the concrete example we can take $x = 1 + X_1^2 + \ldots + X_{2^{n-1}}^2$. Finally we get the desired field K by applying Corollary 6.3.13 to the given field and the n-fold Pfister form $\pi := \langle\langle x, -1, \ldots, -1 \rangle\rangle$

which by construction is obviously an anisotropic torsion form.
As π_K is anisotropic, the element x has to be of length greater than 2^{n-1} which implies $p(K) > 2^{n-1}$, as desired.

Corollary 6.4.4:
Let F be a real field with finite Pythagoras number $p(F)$ fulfilling $2^{n-1} < p(F) \leq 2^n$ for some $n \in \mathbb{N}$. We then have $[D_F(\infty) : D_F(2^{n-1})] \geq 2$ and there are fields for which equality holds.

Proof:
The estimate is clear and due to Pfister as mentioned above and fields for which the estimate is an equality are constructed in Example 6.4.3. $\square$

As another invariant, we consider the *height* of F, in symbols $h(F)$, i.e. the exponent of W_tF. As we only deal with formally real fields, this is the smallest 2-power 2^k with $2^k \geq p(F)$ if $p(F) \in \mathbb{N}$ or infinity otherwise, see [Lam05, Chapter XI. Theorem 5.6 (1)]. The existence of supreme torsion forms has influence on the height as we see in the final result of this chapter.

Theorem 6.4.5:
Let F be a field with a supreme torsion form $\pi \in P_n(F)$ for some $n \in \mathbb{N}$. We then have $h(F) \leq 2^n$. We have equality if and only if $2^{n-1} \times \langle 1 \rangle \subseteq \pi$.

Proof:
For the upper bound it is enough to show $p(F) \leq 2^n$. So let now x be a non-zero sum of squares. The torsion Pfister form $2^n \times \langle 1, -x \rangle$ cannot be a subform of the supreme torsion form due to the dimensions of the respective forms and thus has to be isotropic, hence even hyperbolic. Thus its Pfister neighbor $2^n \times \langle 1 \rangle \perp \langle -x \rangle$ is isotropic, which means that x is a sum of at most 2^n squares as desired.
For the equivalence, we start with the case $h(F) = 2^n$. We then have $p(F) > 2^{n-1}$, i.e. there is some $x \in \sum F^{*2}$ that is not a sum of 2^{n-1} squares. Thus the Pfister neighbor $2^{n-1} \times \langle 1 \rangle \perp \langle -x \rangle$ is anisotropic and therefore so is its associated Pfister form $2^{n-1} \times \langle 1, -x \rangle$. The latter form therefore is an anisotropic n-fold torsion Pfister form and has to be the supreme torsion form π. As this forms has $2^{n-1} \times \langle 1 \rangle$ as a subform, this implication is done.
For the other implication, we now assume $2^{n-1} \times \langle 1 \rangle \subseteq \pi$. Let $x \in F^*$ such that $2^{n-1} \times \langle 1 \rangle \perp \langle -x \rangle \subseteq \pi$. By considering the signature of this form and using Artins Theorem, we obtain $x \in \sum F^{*2}$. But x cannot be a sum of at most 2^{n-1} squares because otherwise, the above form and thus also π would be isotropic. This implies $2^{n-1} < p(F)$. With $h(F) \leq 2^n$ and the description of the height in terms of the Pythagoras number given above, we are done. $\square$

7. Signature Ideals

7.1. Compatibility with I^n and X-decomposition

We would like to develop a theory for Pfister numbers that are compatible with a given set of orderings $X \subseteq X_F$ in a sense that will be made concrete at the beginning of the upcoming section. Here our aim is to find requirements for X such that we get statements analogous to the known results concerning the usual Pfister numbers and which can be specialized to these results in as many cases as possible. As a main motivation, the reader may think of the signature ideals $I^n_P F$ for some ordering $P \in X_F$ or the torsion part $I^n_t F$ in the n-th power of the fundamental ideal.

Definition 7.1.1:
For a subset $X \subseteq X_F$ of orderings of F, we define $I_X F$ to be the ideal in WF generated by the binary forms of the form $\langle\langle a \rangle\rangle$ with $a \in \bigcap_{P \in X} P$. If we have $X = X_T$ for some preordering T, we also write $I_T F$ for short; for $X = \{P\}$ for some $P \in X_F$, we also write $I_P F$.
We further set $I^n_X F := I^n F \cap I_X F$ and use analogous shortcuts as above.

For the rest of this chapter, our main interest will lie on the study of $I^n_X F$.
As a convention, we define the empty intersection in the above definition to be F^* so that we get $I_\varnothing F = IF$, the fundamental ideal. If we have $X = X_T$ for some preordering T, we have $I_T F = \ker(\mathrm{sgn}|_{X_T})$ due to Pfisters Local-Global principle. These are the only examples that can occur. Before we make that precise, recall Proposition A.2.1 which tells us that for any $X \subseteq X_F$, the intersection $\bigcap_{P \in X} P$ is a preordering.

Proposition 7.1.2:
Let $\varphi \in I_X F$ be a form for some subset $X \subseteq X_F$ and $T := \bigcap_{P \in X} P$. We then even have $\varphi \in I_T F$. In particular, we have $I_X F = I_T F$.

Proof:
By definition of $I_X F$ we have a Witt equivalence

$$\varphi = \beta_1 \perp \ldots \perp \beta_r \tag{7.1}$$

for some binary forms $\beta_k \in I_X F$. Every β_k has the form

$$\beta_k \cong a_k \langle 1, -t_k \rangle$$

with some t_k that is positive with respect to every ordering $P \in X$ and thus has to lie in the intersection $\bigcap_{P \in X} P = T$. Therefore we even have $\beta_k \in I_T F$ for all $k \in \{1, \ldots, r\}$, which also implies $\varphi \in I_T F$. As the inclusion $I_T F \subseteq I_X F$ is trivial, this shows that we have

$$I_X F = I_T F.$$

□

If $X \subseteq X_F$ is not given by X_T for some preordering T, the equality

$$I_X F = \ker(\operatorname{sgn}|_X)$$

fails in general as the following example shows. Note that we obviously always have $I_X F \subseteq \ker(\operatorname{sgn}|_X)$.

Example 7.1.3:
Let $F = \mathbb{R}((s_1))((s_2))((t_1))((t_2))$ be the field of iterated Laurent series over the field $\mathbb{R}$ of real numbers.

Further let $X \subseteq X_F$ be the set of orderings defined by the following table, where + means that the respective element is contained in the ordering and − means that it is not contained. Note that by Proposition A.2.9, the information given in the table is enough to describe the orderings and that each described ordering does exist.

	s_1	s_2	t_1	t_2
P_1	+	+	+	+
P_2	+	+	+	−
P_3	+	+	−	+
P_4	+	−	+	+
P_5	+	−	+	−
P_6	+	−	−	+

It is then clear that we have

$$\bigcap_{P \in X} P = F^{*2} \cup s_1 F^{*2}. \tag{7.2}$$

We consider the anisotropic Albert form

$$\alpha := (\langle\langle s_1, s_2\rangle\rangle - \langle\langle t_1, t_2\rangle\rangle)_{\mathrm{an}} = \langle -s_1, -s_2, s_1 s_2, t_1, t_2, -t_1 t_2\rangle.$$

An iterated use of Proposition 2.4.2 yields that α is indeed anisotropic. As we have $s_1 \in \bigcap\limits_{P \in X} P$, we obviously have $\operatorname{sgn}_P(\langle -s_1, -s_2, s_1 s_2\rangle) = -1$ for all $P \in X$. As there is no ordering in X containing both $-t_1$ and $-t_2$, we further have $\operatorname{sgn}_P(\langle t_1, t_2, -t_1 t_2\rangle) = 1$ for all $P \in X$. We thus have $\alpha \in I_X F = \ker(\operatorname{sgn}|_X)$.
Denote by $T := \bigcap\limits_{P \in X} P$ the intersection of the orderings we consider here. We further consider the ordering P with $s_1, s_2, -t_1, -t_2 \in P$. We have $P \in X_T$ as can readily be seen using the fact that we have $s_1 \in P$. We then have $\operatorname{sgn}_P \alpha = -4$, which implies

$$\alpha \in \ker(\operatorname{sgn}|_X) \subsetneq \ker(\operatorname{sgn}|_T) = I_T F = I_X F,$$

where the equalities follow by Pfister's Local-Global Principle and Proposition 7.1.2.

For any $X \subseteq X_F$ and $n \in \mathbb{N}$ we have the obvious relations

$$(I_X F)^n \subseteq I^{n-1} F \cdot I_X F \subseteq I_X^n F. \tag{7.3}$$

It is clear that $(I_X F)^n$ is generated as an ideal by Pfister forms $\langle\langle a_1, \ldots, a_n\rangle\rangle$ with $a_1, \ldots, a_n \in \bigcap\limits_{P \in X} P$. We further have:

Proposition 7.1.4:
Let $X \subseteq X_F$ be a a set of orderings of F and $n \in \mathbb{N}$. Then $I^{n-1}F \cdot I_X F$ is generated as an ideal by Pfister forms of the shape $\langle\langle a, b_1, \dots, b_{n-1}\rangle\rangle$ with $a \in \bigcap_{P \in X} P$ and $b_1, \dots, b_{n-1} \in F^*$.

Proof:
This is clear since $I^{n-1}F$ is generated by $(n-1)$-fold Pfister forms and $I_X F$ is generated by Pfister forms $\langle\langle a\rangle\rangle$ with $a \in \bigcap_{P \in X} P$. □

Notation: For $X \subseteq F$ we define $GP_n F_X$ to be the set of all quadratic forms similar to forms of the shape $\langle\langle a, b_1, \dots, b_{n-1}\rangle\rangle$ with $a \in \bigcap_{P \in X} P$ and $b_1, \dots, b_{n-1} \in F^*$.
We note that there is no canonical set of generators for $I^n_X F$ in general, but in the sequel we will only consider the case in which we have $I^{n-1}F \cdot I_X F = I^n_X F$.
Thus we will give a name for the case in which we even have equality in some of the inclusions in (7.3).

Definition 7.1.5:
Let $X \subseteq X_F$ be a set of orderings of F and $n \in \mathbb{N}$. We say that X is *compatible with* $I^n F$, if the equality $I^{n-1}F \cdot I_X F = I^n_X F$ holds. We call X *strongly compatible with* $I^n F$, if the equality $(I_X F)^n = I^n_X F$ holds.

Of course any $X \subseteq X_F$ is trivially compatible with IF and if X is strongly compatible with $I^n F$ for some $n \in \mathbb{N}$, it is also compatible with $I^n F$ in view of (7.3).

As a fundamental example for compatibility we have the following result, which is a consequence of the validity of the Milnor conjecture:

Theorem 7.1.6 ([AE01, Corollary 2.7]):
Let T be a preordering of the field F. Then X_T is compatible with $I^n F$ for every $n \in \mathbb{N}$.

As an easy consequence we have the following cases which were the main motivation to study this problem:

Corollary 7.1.7:
For every $n \in \mathbb{N}$, we have

(a) $I^n_t F = I^{n-1}F \cdot I_t F$;

(b) $I^n_P F = I^{n-1}F \cdot I_P F$ for every ordering $P \in X_F$.

Proof:
This follows using Theorem 7.1.6 with $T = \sum F^*$ for (a) respectively $T = P$ for (b). □

In the case of (b), we can say even more.

Lemma 7.1.8:
Let $P \in X_F$ be an ordering. We then have equalities

$$(I_P F)^n = I^{n-1}F \cdot I_P F = I^n_P F,$$

i.e. $\{P\}$ is strongly compatible with $I^n F$ for all $n \in \mathbb{N}$.

Proof:
In view of (7.3), we just have to show $(I_P F)^n = I_P^n F$. As we have already shown in Corollary 7.1.7 (b) that P is compatible with $I^n F$, it is enough to show that every generator as in Proposition 7.1.4 is in $(I_P F)^n$. For $n = 1$ there is nothing to show. So first assume $n = 2$ and let $a \in P$ and $b \in F^*$. As exactly one of the elements $b, -ab$ is in P, this case follows by considering the isometry

$$\langle\langle a, -ab\rangle\rangle = \langle 1, -a, ab, -a^2 b\rangle \cong \langle 1, -a, -b, ab\rangle = \langle\langle a, b\rangle\rangle.$$

For $n \geq 3$ an obvious induction yields the conclusion. $\square$

Example 7.1.9:
Let F be the field in Example 6.1.2 (a) and $K = F((t))$. Then $\langle\langle 2\rangle\rangle$ is the only anisotropic torsion 1-fold Pfister form over K, but $2 \times \langle\langle 2\rangle\rangle = \langle\langle 2, 2\rangle\rangle$ is hyperbolic. In particular $(I_t K)^2$ is trivial. But we clearly have $\langle\langle 2, t\rangle\rangle \in I_t^2 K$. Since this form is anisotropic by Proposition 2.4.2, we therefore have an example of a field K for which the strict inclusion $(I_t K)^2 \subsetneq IF \cdot I_t F = I_t^2 K$ holds.

As we have already seen, only the compatible case is interesting for us as otherwise, no good results can be expected as we do not even have a canonical sets of generators for $I_X^n F$. There is another property that we need and that we will now define. Its importance will be discussed in detail in Example 7.1.13.

Definition 7.1.10:
Let $X \subseteq X_F$ be a subset of orderings on F and $\varphi \in I_X F$ be a quadratic form over F. We say that φ has *X-decomposition* if we have an isometry

$$\varphi \cong \beta_1 \perp \ldots \perp \beta_r$$

for some $r \in \mathbb{N}$ and binary quadratic forms β_k with $\operatorname{sgn}_P \beta_k = 0$ for all $P \in X$ and all $r \in \{1, \ldots, r\}$. We say that F has *X-decomposition* if every quadratic form $\varphi \in I_X F$ over F has X-decomposition.

Remark 7.1.11:
(a) In [AP77], Arason and Pfister introduced the property "strongly balanced" ("stark ausgeglichen") for a quadratic form φ over a formally real field F. This is exactly the property of having X_F-decomposition. Further F has X-decomposition if and only if F has property $A(F)$ in the language of [AP77].

(b) One readily sees that for any $n \in \mathbb{N}$ and $X \subseteq X_F$, a form $\pi \in P_n F \cap I_X F$ has X-decomposition if and only if we have $\pi \in GP_n F_X$.

Proposition 7.1.12:
Let $X = \{P\} \subseteq X_F$ be a singleton set consisting of one ordering of F. Then F has X-decomposition.

Proof:
Let $\varphi \in I_P F$ be a quadratic form with $\operatorname{sgn}_P \varphi = 0$. By definition of sgn_P there is an $m \in \mathbb{N}$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in P$ such that we have an isometry

$$\varphi \cong \langle a_1, -b_1, \ldots, a_m, -b_m\rangle$$

Putting $\beta_k = \langle a_k, -b_k\rangle$ for $k \in \{1, \ldots, m\}$, the assumption follows. $\square$

Example 7.1.13:
By [AP77, Beispiel 1], the quadratic form $\langle\langle -X, 1+Y^2+3X\rangle\rangle$ over the rational function field $F = \mathbb{Q}(X,Y)$ is a torsion form that does not contain any binary torsion form as a subform. In particular it cannot have any totally positive element as a slot and is therefore not isometric to a form in $GP_nF_{X_F}$. We can further deduce that F does not have X_F-decomposition.

This example gives insight into the complexity of the problem of finding the least integer k such that a given torsion form can be written as a sum of k binary torsion forms. Further, in this generality, it seems to be pretty hard to give answers to the extended question for the torsion part of the higher powers of the fundamental ideal. But we will see in the next section that the combination of compatibility with I^nF and X-decomposition will yield a foundation to develop a lot of theory.

Proposition 7.1.14:
Let $X \subseteq X_F$ be a subset of orderings on F and φ, ψ two quadratic forms over F having X-decomposition and let σ be any form over F. Then $\varphi \perp \psi$ and $\varphi \otimes \sigma$ have X-decomposition.

Proof:
Let the respective X-decomposition of φ, ψ be given by

$$\varphi \cong \beta_1 \perp \ldots \perp \beta_r, \quad \psi \cong \gamma_1 \perp \ldots \perp \gamma_s$$

for some $r, s \in \mathbb{N}_0$ and binary forms $\beta_1, \ldots, \beta_r, \gamma_1, \ldots, \gamma_s \in I_XF$. Then the orthogonal sum $\varphi \perp \psi$ has the X-decomposition

$$\varphi \perp \psi \cong \beta_1 \perp \ldots \perp \beta_r \perp \gamma_1 \perp \ldots \perp \gamma_s.$$

If φ has an X-decomposition as above and $\sigma \cong \langle a_1, \ldots, a_t\rangle$ is any form over F, the tensor product has the X-decomposition

$$\varphi \otimes \sigma \cong \mathop{\perp}_{k=1}^{t} \big(a_k\beta_1 \perp \ldots \perp a_k\beta_r\big),$$

which concludes the proof. □

We will now proof a reduction criterion analogous to that one in [AP77, Satz 4]. The proof is the same except for the fact that we are working in a more general setting and thus have to be a bit more careful and use other formulations every now and then.

Lemma 7.1.15:
Let $X \subseteq X_F$ a subset of the space of orderings, $\varphi \in I_XF$ be a quadratic form having X-decomposition and $a \in D_F(\varphi)$ be an element represented by φ. We further assume that every form in $P_2F \cap I_XF$ has X-decomposition, i.e. we have $P_2F \cap I_XF = P_2F_X$. Then φ has an X-decomposition

$$\varphi \cong \beta_1 \perp \ldots \perp \beta_r$$

such that $a \in D_F(\beta_1)$.

Proof:
Since φ has X-decomposition we can write $\varphi = b_1\langle 1, -u_1\rangle \perp \ldots \perp b_r\langle 1, -u_r\rangle$ for some $b_1, \ldots, b_r \in F^*$ and $u_1, \ldots, u_r \in \bigcap_{P\in X} P$. We use induction on r, the case $r = 1$ being trivial.
For $r = 2$ we write $\varphi \cong \langle a\rangle \perp \psi$ for some form ψ of dimension 3. We now consider the form

$$\sigma := \langle au_1u_2\rangle \perp \psi. \tag{7.4}$$

As we have $u_1, u_2 \in \bigcap_{P\in X} P$, we have $\operatorname{sgn}_P(a) = \operatorname{sgn}_P(au_1u_2)$ for all $P \in X$, which then implies

$$\operatorname{sgn}_P(\sigma) = \operatorname{sgn}_P(\varphi) = 0 \text{ for all } P \in X.$$

Modulo squares, we further have

$$\det \sigma = au_1u_2 \cdot \det \psi = au_1u_2 a \det \varphi = (au_1u_2)^2$$

so that we get $\sigma \in GP_2F \cap I_XF$. Thus we can write $\sigma = au_1u_2\pi$ for some $\pi \in P_2F \cap I_XF = P_2F_X$ with X-decomposition

$$\pi = \langle\langle x, y\rangle\rangle \text{ with } x \in \bigcap_{P\in X} P.$$

Plugging in that representation, comparing with (7.4) and using Witt cancellation yields

$$\psi \cong \langle -au_1u_2x, -au_1u_2y, au_1u_2xy\rangle.$$

We thus have

$$\varphi \cong a\langle 1, -u_1u_2x\rangle \perp -ayu_1u_2\langle 1, -x\rangle,$$

proving the assertion.
Let now $r \geq 3$. If a is already represented by $\psi := b_1\langle 1, -u_1\rangle \perp \ldots \perp b_{r-1}\langle 1, -u_{r-1}\rangle$ or $b_r\langle 1, -u_r\rangle$ we are done by the case $r = 1$ or induction hypothesis. Otherwise we have $a = x + y$ for $x \in D_F(\psi), y \in D_F(b_r\langle 1, -u_r\rangle)$. Applying the induction hypothesis to ψ yields an isometry

$$\psi \cong x\langle 1, -u_1'\rangle \perp b_2'\langle 1, -u_2'\rangle \perp \ldots b_{r-1}'\langle 1, -u_{r-1}'\rangle$$

with some $u_1', \ldots, u_{r-1}' \in \bigcap_{P\in X} P$ and $b_2', \ldots, b_{r-1}' \in F^*$. Applying the case $r = 2$ to the subform $x\langle 1, -u_1'\rangle \perp b_r\langle 1, -u_r\rangle$ will now yield the conclusion. $\square$

Theorem 7.1.16:
Let $X \subseteq X_F$ be a subset of orderings. The field F has X-decomposition if and only if every form in $P_2F \cap I_XF$ has X-decomposition.

Proof:
The only if part is trivial. For the converse let now $\varphi \in I_XF$. As any hyperbolic form obviously has X-decomposition for any $X \subseteq X_F$, we may assume that φ is anisotropic by using Witt decomposition. By definition of I_XF, we have

$$\varphi = \psi := b_1\langle 1, -u_1\rangle \perp \ldots \perp b_r\langle 1, -u_r\rangle$$

in WF for some $b_1, \ldots, b_r \in F^*$ and $u_1, \ldots, u_r \in \bigcap_{P \in X} P$. If we even have an isometry $\varphi \cong \psi$, we are done. Otherwise, as φ is anisotropic, the form ψ has to be isotropic. Thus, there is some $c \in F^*$ such that

$$c \in D_F(b_1\langle 1, -u_1\rangle \perp \ldots \perp b_{r-1}\langle 1, -u_{r-1}\rangle) \cap -D_F(b_r\langle 1, -u_r\rangle).$$

By Lemma 7.1.15 we can thus assume $b_{r-1} = c$ and $b_r = -c$. This implies

$$\psi \cong b_1\langle 1, -u_1\rangle \perp \ldots \perp b_{r-2}\langle 1, -u_{r-2}\rangle \perp c\langle 1, -u_{r-1}, -1, u_r\rangle,$$

the last form being Witt equivalent to

$$b_1\langle 1, -u_1\rangle \perp \ldots \perp b_{r-2}\langle 1, -u_{r-2}\rangle \perp -u_{r-1}c\langle 1, -u_{r-1}u_r\rangle.$$

As this form has X-decomposition, iterating this procedure will yield the result. □

If we consider an SAP preordering T and the set X_T, we can strengthen the above result, see the upcoming Theorem 7.1.19. It is not clear if SAP is really necessary. We first need some preliminaries. But before stating it, we would like to recall that we can regard any quadratic form as a T-form in the sense of the reduced theory of forms concerning a preordering T, see [Lam83, Chapter 1], especially [Lam83, Theorem 1.26]. Thus we can use the machinery of T-forms. In particular, we can say that a quadratic form φ is *T-isotropic*, i.e. there is some diagonalization $\varphi \cong \langle a_1, \ldots, a_n\rangle$ and some $t_1, \ldots, t_n \in T$ not all equal to zero with $a_1t_1 + \ldots + a_nt_n = 0$.

Corollary 7.1.17:
Let T be a preordering of F which is SAP and φ a T-indefinite quadratic form over F, i.e. φ is indefinite with respect to every ordering $P \in X_T$. Then φ is T-isotropic.

Proof:
This follows directly from [Lam83, Theorems 16.2 and 17.12] □

The next auxiliary result is a generalization of the needed facts from [Bec06a, 4.1, 4.2] to the current situation.

Lemma 7.1.18:
Let T be a preordering of F and φ and ψ be quadratic forms over F with $\dim\varphi = \dim\psi$ and $\mathrm{sgn}_P(\varphi) = \mathrm{sgn}_P(\psi)$ for all $P \in X_T$. We further assume that any 3-dimensional T-indefinite quadratic form contains a binary T-indefinite form as a quadratic subform. Then φ contains a binary T-indefinite form as a quadratic subform if and only if ψ does.

Proof:
It is obviously sufficient to show one implication. So let φ contain a binary T-indefinite form as a subform. The assumption implies φ and ψ to be chain-T-equivalent by [Lam83, Theorem 1.28]. We thus have to show that the property of having a binary T-indefinite subform is preserved under the basic transformations. As two of them even describe isometries of quadratic forms, we are reduced to the case that we have $\varphi = \varphi' \perp \langle a\rangle$ and $\psi = \varphi' \perp \langle ta\rangle$ for some $t \in T$. Let $\alpha \subseteq \varphi$ be a binary T-indefinite quadratic subform. We will now show that there is some binary quadratic subform $\beta \subseteq \varphi'$ such that $\beta \perp \langle a\rangle$ contains this T-indefinite form α. To do so, let V be the vector space φ is defined on, $V = V_{\varphi'} \oplus V_a$ a

decomposition for the orthogonal sum $\varphi = \varphi' \perp \langle a \rangle$ and $V_\alpha \subseteq V$ the subspace over which α is defined. Let further v be a basis of V_a, i.e. an arbitrary vector in $V_a \smallsetminus \{0\}$.
If we have $V_\alpha \subseteq V_{\varphi'}$, i.e. $\alpha \subseteq \varphi'$, we can take $\beta := \alpha$.
If we have $v \in V_\alpha$, we can extend v with some vector $w \in V_a^\perp = V_{\varphi'}$ to get an orthogonal basis for V_α. As φ' is non-degenerate, we further find $u \in V_{\varphi'} \smallsetminus \{0\}$ such that for $W := \operatorname{span}(u, w)$, the quadratic form $\varphi'|_W$ is a non-degenerate form of dimension 2 that fulfils the desired properties.
In the remaining case, i.e. if $\alpha \nsubseteq \varphi'$ and $v \notin V_\alpha$, we find $u, w \in V_{\varphi'} \smallsetminus \{0\}$ such that $(u+v, w+v)$ is an orthogonal basis for V_α. As we have $v \notin V_\alpha$, the system (u, w) is linearly independent. An easy check shows that $\varphi|_{\operatorname{span}(u,w)}$ is a non-degenerate binary subform of φ' and can be chosen as β in this remaining case.
As $\beta \perp \langle a \rangle$ contains the T-indefinite form α and is therefore T-indefinite itself, so is $\beta \perp \langle ta \rangle$. Thus, this form also contains a binary T-indefinite subform by hypothesis. As $\beta \perp \langle ta \rangle$ is a subform of ψ, the claim follows. □

The next result is the announced generalization of Theorem 7.1.16 in the SAP case.

Theorem 7.1.19:
Let F be a field and T be a preordering of F. We consider the following statements:

(i) every form φ over F can be decomposed into $\varphi \cong \gamma \perp \beta_1 \perp \ldots \perp \beta_r$ with $r \in \mathbb{N}_0$, γ is T-anisotropic and binary forms $\beta_1, \ldots, \beta_r \in I_T F$;

(ii) every form in $I_T F$ has X_T-decomposition;

(iii) every form in $P_2 F \cap I_T F$ has X_T-decomposition;

(iv) every 3-dimensional T-indefinite form over F contains a binary T-indefinite subform.

The statements (ii), (iii) and (iv) are equivalent and we have (iv) $\Rightarrow$ (i).
If further T is SAP, then all the above statements are equivalent.

Proof:
We have (ii) $\iff$ (iii) by Theorem 7.1.16. For the implication (iii) $\Rightarrow$ (iv), consider a 3-dimensional T-indefinite form φ. After scaling we may assume that we have $\varphi \cong \langle 1, -a, -b \rangle$. We then have $\langle 1, -a, -b, ab \rangle \in P_2 F \cap I_T F$ which then contains a binary T-indefinite form $d\langle 1, -t \rangle$. We thus have $t \in \bigcap_{P \in X_T} P$ and, as Pfister forms are round, we may assume $d = 1$. Then φ has a subform of the shape $\langle -t, -s, st \rangle$ for some $s \in F^*$, which contains the binary T-indefinite form $\langle -s, st \rangle$. As Pfister neighbors of codimension 1 of the same Pfister form are similar, see Proposition 2.2.11, φ contains a binary T-indefinite form as well.
For (iv) $\Rightarrow$ (iii), let $\varphi \in P_2 F \cap I_T F$ and φ' be its pure part. Clearly, φ' is T-indefinite and thus contains a binary T-indefinite subform. Therefore, φ also contains this binary T-indefinite subform and thus has X_T-decomposition.
We now have the equivalence of the statements (ii), (iii) and (iv). For the implication (iv) $\Rightarrow$ (i), let φ be any form over F. If φ is T-anisotropic, we are done by putting $\gamma := \varphi$ and $r := 0$.
So let now φ be T-isotropic. Using [Lam83, Corollary 1.20] and Lemma 7.1.18 we see that φ contains a binary T-indefinite form $\beta \in I_T F$, i.e. we have $\varphi \cong \varphi' \perp \beta$ for

some quadratic form φ'. Iterating the above procedure with φ' will yield the claim. Lastly, if T is SAP, the implication (i) $\Rightarrow$ (ii) is clear using Corollary 7.1.17 as forms in $I_T F$ are exactly the T-hyperbolic forms and we have a Witt decomposition for T-forms, see [Lam83, Corollary 1.21]. □

In view of the above results, some questions occur naturally.

Question 7.1.20:

(a) Does the implication (i) $\Rightarrow$ (ii) in Theorem 7.1.19 also hold for fields that are not SAP?

(b) Does X_T-decomposition imply T to be SAP?

In Proposition 7.1.2 we have already seen that for compatibility, it is enough to consider subsets of the shape X_T. For the decomposition property, the same applies.

Proposition 7.1.21:
Let $X \subseteq X_F$ be a subset of the space of orderings of F such that F has X-decomposition. Let $T := \bigcap_{P \in X}$ be the intersection of the orderings in P. If F has X-decomposition, then F has X_T-decomposition.

Proof:
This follows as in the proof of Proposition 7.1.2, replacing *Witt equivalence* with *isometry.* □

Remark 7.1.22:
We would like to include a topological argument to see that a form φ with X-decomposition also has $\overline{X}$-decomposition, where $\overline{X}$ denotes the topological closure of X with respect to the Harrison topology.
To see this, let $(P_n)_{n \in \mathbb{N}}$ be a sequence in X that converges to some $P \in X_F$. It is enough to show that for every binary form β over F with $\operatorname{sgn}_{P_n} \beta = 0$ for all $n \in \mathbb{N}$, we also have $\operatorname{sgn}_P \beta = 0$. But this follows directly from the fact that the map

$$X_F \to \mathbb{Z}, \quad Q \mapsto \operatorname{sgn}_Q \beta$$

is continuous when X_F is equipped with the Harrison topology and $\mathbb{Z}$ is equipped with the discrete topology, cf. [Lam05, Chapter VIII. Proposition 6.6].

Of course, as we have

$$\overline{X} \subseteq \bigcap_{a \in T} \{P \in X \mid a \in P\} = X_T,$$

where $T = \bigcap_{P \in X} P$ denotes the intersection of all $P \in X$, this is just a weaker version of Proposition 7.1.21.
As another side fact, we would like to give an example for which we have a proper inclusion $\overline{X} \subsetneq X_T$, where T is defined again as $T = \bigcap_{P \in X} P$. Consider the field $F = \mathbb{R}((s))((t))$. The orderings of F are given in the following table:

	s	t
P_1	+	+
P_2	+	−
P_3	−	−
P_4	−	+

One readily sees that we have $T := P_1 \cap P_2 \cap P_3 = F^{*2}$, which then implies $X_T = X_F$. As X_F is a boolean space, every singleton set is closed and thus, so is $X = \{P_1, P_2, P_3\}$, but we have

$$\overline{X} = X \subsetneq X_F = X_T.$$

7.2. Transfer of Known Results

For the whole section, let $X \subseteq X_F$ be such that F has X-decomposition. Recall that this implies that we do not have to differ between $P_n F_X$ and $P_n F \cap I_X$ for all $n \in \mathbb{N}$. Whenever we talk about $I_X^n F$ for some $n \in \mathbb{N}$ we further assume that X is compatible with $I^n F$.
In view of Proposition 7.1.21 we may assume without loss of generality that we have $X = X_T$ for some preordering T of F and the assumption of compatibility with $I^n F$ is vacuous by Theorem 7.1.6.
In this section we study the following problem: Given $\varphi \in I_X^n F$, what is the least integer $k \in \mathbb{N}$ such that there are $\pi_1, \ldots, \pi_k \in GP_n F_X$ so that we have

$$\varphi = \pi_1 + \ldots + \pi_k \in WF?$$

This minimal k will be called the *Pfister number of φ with respect to X.*
We will mostly transfer the known results for Pfister numbers to our setting. Most of the proofs use the same or at least similar arguments as the original ones and just make use of more subtle calculations.

Proposition 7.2.1:
Let $\varphi \in I_X^n F$ for some $n \in \mathbb{N}$ be a form of dimension 2^n. We then have $\varphi \in GP_n F_X$.

Proof:
We write $\varphi = \beta_1 \perp \ldots \perp \beta_{2^{n-1}}$ for some binary forms in $I_X F$ compatible with the definition of X-decomposibility. Let $a, b \in F^*$ be such that $\beta_1 \cong \langle a, b\rangle$. We thus have

$$\varphi \sim a\varphi \cong \langle 1, ab, \ldots\rangle.$$

As we have

$$0 = \operatorname{sgn}_P \langle a, b\rangle = \operatorname{sgn}_P(a) + \operatorname{sgn}_P(b)$$

for all $P \in X$, we have $-ab \in \bigcap_{P \in X} P$. Now according to the Arason-Pfister Hauptsatz $a\varphi$ is a general n-fold Pfister form representing 1 and therefore isometric to a Pfister form by Corollary 2.2.4. Using Theorem 2.2.1, we see $a\varphi \cong \langle\langle -ab, \ldots\rangle\rangle$ and the claim follows. □

The above result and Theorem 7.1.16 together give reason to believe that the concept of X-compatibility and X-decomposition are the right concepts to generalize the results on Pfister numbers to our setting.
As further important examples of quadratic forms of dimension 2^n, twisted Pfister forms will be the next forms to study.

Proposition 7.2.2:
Let $\varphi \in I^m F_X$ be a twisted Pfister form of dimension 2^n for some $1 \le m < n$, i.e. there are $\pi \in GP_n F$ an $\sigma \in GP_m F$ such that $\varphi \cong (\pi \perp \sigma)_{\mathrm{an}}$. We then have $\pi \in GP_n F_X$ and $\sigma \in GP_m F_X$.

Proof:
Let $P \in X$. As π is an n-fold Pfister form, we have $\operatorname{sgn}_P \pi \in \{0, \pm 2^n\}$ and similarly $\operatorname{sgn}_P \sigma \in \{0, \pm 2^m\}$. Since we have $m \neq n$ and

$$0 = \operatorname{sgn}_P \varphi = \operatorname{sgn}_P \pi + \operatorname{sgn}_P \sigma,$$

we get $\operatorname{sgn}_P \pi = 0 = \operatorname{sgn}_P \sigma$. Now, the conclusion follows in by Proposition 7.2.1. □

Proposition 7.2.3:
Let $\varphi \in I_X F$ be a quadratic form of dimension $\dim \varphi = 2m$. Then φ is isometric to a sum of m forms in $GP_1 F_X$.

Proof:
This follows directly from F having X-decomposition. □

Proposition 7.2.4:
Let $\varphi \in I_X^2 F$ be an anisotropic form of dimension $\dim \varphi = 2m$ for some $m \geq 2$. Then φ is Witt equivalent to a sum of $m-1$ elements in $GP_2 F_X$.

Proof:
We use induction on m. The case $m = 2$ is already covered by Proposition 7.2.1. So assume now $m > 2$, i.e. $\dim \varphi \geq 6$. We write $\varphi \cong \beta_1 \perp \ldots \perp \beta_m$ with binary forms β_k such that $\operatorname{sgn}_P \beta_k = 0$ for all $P \in X$. Choose $x \in F^*$ such that the form $\psi := \beta_1 \perp \ldots \perp \beta_{m-1} \perp x\beta_m$ is isotropic. In WF the equality

$$\varphi = \psi + \underbrace{\langle\langle x \rangle\rangle \otimes \beta_m}_{\in I_X^2 F} \tag{7.5}$$

holds. We thus have $\psi_{\text{an}} \in I_X^2 F$ with $\dim \psi_{\text{an}} \leq 2m-2$. By induction hypothesis ψ_{an} and therefore also ψ is Witt equivalent to a sum of $m-2$ elements in $GP_2 F_X$. The claim now follows by (7.5) because of $\langle\langle x \rangle\rangle \otimes \beta_m \in GP_2 F_X$. □

The characterization for 8-dimensional forms in $I^2 F$ whose Clifford Invariant has index 4 concerning whether they can be written as a sum of two $GP_2 F$-forms or not as in [IK00, Proposition 16.4] can be transferred.

Proposition 7.2.5:
Let $\varphi \in I_X^2 F$ with $\dim \varphi = 8$. Then the following are equivalent:

(i) There is some $\sigma \in GP_2 F_X$ such that $\sigma \subseteq \varphi$;

(ii) There are $\sigma, \tau \in GP_2 F_X$ such that $\varphi \cong \sigma \perp \tau$.

If we have further $\operatorname{ind} c(\varphi) = 4$ and α is an Albert form with $c(\alpha) = c(\varphi)$, the above statements are also equivalent to

(iii) There is some $a \in \bigcap_{P \in X} P$ such that both $\varphi_{F(\sqrt{a})}$ and $\alpha_{F(\sqrt{a})}$ are isotropic.

Proof:
(i) $\Rightarrow$ (ii): This is clear by putting $\tau := (\varphi \perp -\sigma)_{\text{an}}$ and the reverse implication is obvious.
So let now $\operatorname{ind} c(\varphi) = 4$ and α be an Albert form with $c(\alpha) = c(\varphi)$.

(ii) $\Rightarrow$ (iii): Let $\sigma_1, \tau_1 \in P_2 F_X$ be similar to σ respectively τ. Then α is similar to the form $\sigma_1' \perp -\tau_1'$. Further it is clear that σ_1' contains a binary form in $I_X F$, which is therefore similar to $\langle 1, -a \rangle$ for some $a \in \bigcap_{P \in X} P$. This a already does the job.

(iii) $\Rightarrow$ (i): We consider the form $\psi := (\varphi_{F(\sqrt{a})})_{\mathrm{an}} \in I^2F(\sqrt{a})$. If we have $\dim\psi \leq 4$, we are obviously done in view of Theorem 2.5.2. We will show that this is always the case.
To do so, we now assume $\dim\psi = 6$, so ψ is an anisotropic Albert form. Using [Lam05, Chapter XII. Proposition 2.5] we have

$$4 = \operatorname{ind} c(\psi) = \operatorname{ind} c(\varphi_{F(\sqrt{a})}) = \operatorname{ind} c(\alpha_{F(\sqrt{a})}).$$

But then, $\alpha_{F(\sqrt{a})}$ has to be anisotropic again by [Lam05, Chapter XII. Proposition 2.5], a contradiction.

□

Even though we will see in Example 7.3.5 that the classification for 8-dimensional forms in I^2F whose Clifford invariant has index 2 as those 8-dimensional forms that are divisible by a binary form, cannot be transferred to our context, we can show that 8-dimensional forms in I^2_XF whose Clifford invariant has index 2 are at least isometric to a sum of two forms in GP_2F_X.

Proposition 7.2.6:
Let $\varphi \in I^2_XF$ with $\dim\varphi = 8$ and $\operatorname{ind} c(\varphi) = 2$. Then φ is isometric to a sum of two forms in GP_2F_X.

Proof:
It is commonly known that such a φ is a twisted Pfister form, see [Hof98c, Theorem 4.1]. By Proposition 7.2.2 we have $\varphi = \pi + \sigma$ for some $\pi \in GP_3F_X$ and $\sigma \in GP_2F_X$ in the Witt ring WF. Thus there is some $a \in \bigcap_{P\in X} P$ such that $\pi_{F(\sqrt{a})}$ is isotropic, hence hyperbolic. In particular we have

$$i_W(\varphi_{F(\sqrt{a})}) \geq 2.$$

This implies the existence of 4-dimensional subform of φ that is divisible by $\langle\langle a\rangle\rangle$ by Theorem 2.5.2. The last statement is implies that φ has a subform in GP_2F_X, which concludes the proof by Proposition 7.2.5. □

Now we turn to I^3_XF. We first note that we already covered 8-dimensional forms by Proposition 7.2.1. As we have $I^3_XF \subseteq I^3F$, we do not have to consider forms of dimensions < 8 or 10 by the Arason-Pfister Hauptsatz and Theorem 3.1.5 respectively. We thus turn to 12-dimensional forms in I^3_XF.

Proposition 7.2.7:
Let $\varphi \in I^3_XF$ be a quadratic form over F with $\dim\varphi = 12$. Then there is some $\sigma \in GP_2F_X$ such that $\sigma \subseteq \varphi$ and φ is Witt equivalent to a sum of two elements in GP_3F_X.

Proof:
As F has X-decomposition we can assume after a scaling that there is some $a \in \bigcap_{P\in X} P$ such that $\langle 1, -a\rangle \subseteq \varphi$. We thus have

$$\dim(\varphi_{F(\sqrt{a})})_{\mathrm{an}} \leq 10$$

and as there are no anisotropic quadratic forms in $I^3F(\sqrt{a})$ according to Theorem 3.1.5, we even have

$$\dim(\varphi_{F(\sqrt{a})})_{\mathrm{an}} \leq 8.$$

By Theorem 2.5.2, this means that there is some $b \in F^*$ such that $\langle\langle a, b\rangle\rangle$ is similar to a subform of φ, proving the first assertion.
Possibly after another scaling, we can assume $\langle\langle a, b\rangle\rangle \perp \langle -c\rangle \subseteq \varphi$ for some $c \in F^*$. As we have

$$\dim(\varphi \perp -\langle\langle a, b, c\rangle\rangle)_{\mathrm{an}} \leq 12 + 8 - 2 \cdot 5 = 10,$$

we can argue as above with Theorem 3.1.5 and get that $\varphi \perp -\langle\langle a, b, c\rangle\rangle$ is Witt equivalent to a form $\pi \in GP_3F$. As we have $\operatorname{sgn}_P \varphi = 0 = \operatorname{sgn}_P\langle\langle a, b, c\rangle\rangle$ for all $P \in X$, we even have $\pi \in GP_3F_X$ by Proposition 7.2.1, which finishes the proof. □

If we further assume the field to be SAP, we can deduce the following. By now, it is not clear if we can drop this additional assumption.

Theorem 7.2.8:
Let T be a preordering of F that is SAP and $X = X_T$. Further let $\varphi \in I^3_X F$ be a quadratic form over F of dimension $\dim \varphi = 14$. Then φ is Witt equivalent to a sum of 3 forms in GP_3F_X.

Proof:
At first, we know by Corollary 3.1.8 that there is some Albert form α over F and an 8-dimension form ψ over F such that we have a decomposition $\varphi \cong \psi \perp \alpha$. As we have $\alpha \in I^2F, \varphi \in I^3F \subseteq I^2F$, we clearly also have $\psi \in I^2F$.
Let $P \in X$ be an ordering. Since $\alpha \in I^2F$ we have $4 \mid \operatorname{sgn}_P \alpha$. With $|\operatorname{sgn}_P \alpha| \leq 6 = \dim \alpha$ this implies

$$\operatorname{sgn}_P \alpha \in \{-4, 0, 4\}. \tag{7.6}$$

In particular α is indefinite with respect to any ordering $P \in X$ and is therefore T-isotropic by Corollary 7.1.17. By Theorem 7.1.19 we know that α contains a binary form in I_TF. Therefore, imitating the proof of Proposition 7.2.4 and possibly after a scaling, we can write $\alpha \cong \sigma_1' \perp -\sigma_2'$ with $\sigma_1 \in P_2F$ and $\sigma_2 \in P_2F_X$. Note that this implies

$$\operatorname{sgn}_P \sigma_1 = \operatorname{sgn}_P \alpha \in \{0, 4\} \tag{7.7}$$

for all $P \in X$.
For any $P \in X$, we thus get

$$\operatorname{sgn}_P \psi = \operatorname{sgn}_P \varphi - \operatorname{sgn}_P \alpha = -\operatorname{sgn}_P \alpha \in \{-4, 0\}. \tag{7.8}$$

Reasoning as above gives a decomposition

$$\psi \cong \psi_1 \perp \beta_1 \perp \beta_2$$

with some binary forms $\beta_1, \beta_2 \in I_XF$. Then (7.8) implies $\operatorname{sgn}_P \psi_1 = -\operatorname{sgn}_P \sigma_1$ for all $P \in X$.
We now choose $x \in F^*$ such that $-x \in D_F(\psi_1)$, write $\psi = \langle -x\rangle \perp \psi'$ and consider the form $\psi' \perp x\sigma_1'$. We have

$$\psi' \perp x\sigma_1' \equiv \psi \perp x\sigma_1 \equiv \varphi \perp -\alpha \perp x\sigma_1 \equiv \sigma_2 \perp -\sigma_1 \perp x\sigma_1 \equiv \sigma_2 \mod I^3F.$$

According to [HT98, Corollary 2.2 (i)] there are some $\pi_3 \in GP_3F$ and $y \in F^*$ such that we have $\psi' \perp x\sigma_1' = \pi_3 + y\sigma_2$ in WF. We further consider the forms

$$\pi_1 := \langle\langle x\rangle\rangle \otimes \sigma_1, \quad \pi_2 := \langle\langle y\rangle\rangle \otimes \sigma_2.$$

Obviously, these are contained in GP_3F. As we have $\operatorname{sgn}_P \sigma_2 = 0$ for all $P \in X$, we even have $\pi_2 \in GP_3F_X$. For those $X \in P$ with $\operatorname{sgn}_P \sigma_1 = 0$, we obviously have $\operatorname{sgn}_P \pi_1 = 0$. For those $P \in X$ with $\operatorname{sgn}_P \sigma_1 = 4$, we have $x \in P$ because of the choice of x which implies $\operatorname{sgn}_P \pi_1 = 0$. Thus we also have $\pi_1 \in GP_3F_X$.
In WF we have

$$\pi_1 - \pi_2 + \pi_3 = \langle\langle x\rangle\rangle \otimes \sigma_1 - \langle\langle y\rangle\rangle \otimes \sigma_2 + \pi_3 = \sigma_1 - x\sigma_1 - \sigma_2 + y\sigma_2 + \pi_3 = \sigma_1 - x\sigma_1 - \sigma_2 + \psi + x\sigma_1 = \varphi.$$

We thus have $0 = \operatorname{sgn}\varphi = \operatorname{sgn}\pi_1 - \operatorname{sgn}\pi_2 + \operatorname{sgn}\pi_3 = \operatorname{sgn}\pi_3$, which means $\pi_3 \in GP_3F_X$ and the claim follows. $\square$

Proposition 7.2.9:
Let $\varphi \in I^3_X F$ be a 14-dimensional quadratic form. Then the following are equivalent:

(i) there is a $\sigma \in GP_2F_X$ that is a subform of φ;

(ii) we have $\varphi = s_1\tau_1 + s_2\tau_2 \in WF$ for some $s_1, s_2 \in F^*$ and $\tau_1, \tau_2 \in GP_3F_X$;

(iii) we have $\varphi \cong s(\tau_1' \perp -\tau_2')$ for some $s \in F^*$ and $\tau_1, \tau_2 \in GP_3F_X$.

Proof:
(i) $\Rightarrow$ (ii): As we have $\sigma \in GP_2F_X$ there is some $a \in \bigcap_{P\in X} P$, such that $\sigma_{F(\sqrt{a})}$ is isotropic and therefore hyperbolic. As there are no anisotropic forms of dimension 10 in $I^3F \supseteq I^3_P F$ due to Theorem 3.1.5, we have $i_W(\varphi_{F(\sqrt{a})}) \geq 3$. Because of Theorem 2.5.2 there are $x, y, z \in F^*$ such that $\langle\langle a\rangle\rangle \otimes \langle x, y, z\rangle$ is a subform of φ.

We denote $\pi_1 := \langle\langle a\rangle\rangle \otimes \langle x, y, z, xyz\rangle$. As we have chosen $a \in \bigcap_{P\in X} P$ we have $\pi_1 \in GP_3F_X$.

Further let $\pi_2 := (\varphi \perp -\pi_1)_{\mathrm{an}}$. We will show that we have $\pi_2 \in GP_3F_X$. As we have

$$0 = \operatorname{sgn}_P \varphi = \operatorname{sgn}_P(\pi_1 \perp -\pi_1 \perp \varphi) = \operatorname{sgn}_P(\pi_1 \perp \pi_2) = \operatorname{sgn}_P \pi_1 + \operatorname{sgn}_P \pi_2 = \operatorname{sgn}_P \pi_2$$

for all $P \in X$ and with Proposition 7.2.1 in mind, it just remains to show $\pi_2 \in GP_3F$.

Considering the Dimensions of φ and π_1 we know that π_2 is not hyperbolic and since we have $\varphi, \pi_1 \in I^3_X F$ we have $\pi_2 \in I^3_X F$. By construction φ and π_1 have a 6-dimensional common subform. We therefore have

$$\dim \pi_2 = \dim(\varphi \perp -\pi_1)_{\mathrm{an}} \leq 14 + 8 - 2 \cdot 6 = 10$$

and again by Theorem 3.1.5, even $\dim \pi_2 \leq 8$. As π_2 is a nonhyerbolic form in I^3F, the Arason Pfister Hauptsatz implies at first $\dim \pi_2 = 8$ and then $\pi_2 \in GP_3F$, finishing the proof as explained above.

(ii) $\Rightarrow$ (iii): As we have $\dim \varphi = 14 < 16 = \dim(s_1\tau_1 \perp s_2\tau_2)$, the latter form has to be isotropic, i.e. there are $a \in D_F(\tau_1), b \in D_F(\tau_2)$ with $s_1a + s_2b = 0$. As Pfister forms are round, we have

$$\varphi = s_1\tau_1 \perp s_2\tau_2 = s_1a\tau_1 \perp s_2b\tau_2 = s(\tau_1 \perp -\tau_2) = s(\tau_1' \perp -\tau_2') \in WF$$

with $s := s_1 a = -s_2 b$. As the latter form has dimension 14, we even have the isometry $\varphi \cong s(\tau_1' \perp -\tau_2')$.

(iii) $\Rightarrow$ (i): This implication is clear since τ_1' (and also τ_2') contains a subform in $GP_2 F_X$.

$\square$

7.3. Counterexamples

In this section we will collect several examples to illustrate the results that cannot be transferred to the more general setting that we are dealing with here.

Before stating the next lemma, we would like to recall that by Proposition A.2.9, any ordering of a field F has exactly two extensions to the field of formal Laurent series $F((t))$, one in which t is positive and one in which t is negative.

Lemma 7.3.1:
Let $X \subseteq X_F$ be a subset of orderings and $K := F((t))$ be the field extension of formal Laurent series. Further let $Y \subseteq X_K$ be the set of all extensions of all the orderings in X to orderings on K.
Then F has X-decomposition if and only if K has Y-decomposition.

Proof:
We first assume that K has Y-decomposition. As any $\varphi \in I_X F$ can be viewed as a form in $I_Y K$ due to the choice of Y, we get an X-decomposition of φ by considering a Y-decomposition of φ.
Next assume that F has X-decomposition and consider a form $\psi \cong \psi_1 \perp t\psi_2 \in I_X F$ with residue class forms ψ_1, ψ_2 defined over F.
Now let $P \in X$ be an ordering with extensions $P_1, P_2 \in Y$. We then have

$$0 = \operatorname{sgn}_{P_1}(\psi) = \operatorname{sgn}_{P_1}(\psi_1) + \operatorname{sgn}_{P_1}(t)\operatorname{sgn}_{P_1}(\psi_2)$$

and

$$0 = \operatorname{sgn}_{P_2}(\psi) = \operatorname{sgn}_{P_2}(\psi_1) + \operatorname{sgn}_{P_2}(t)\operatorname{sgn}_{P_2}(\psi_2).$$

As P_1 and P_2 coincide on F and we have $\operatorname{sgn}_{P_1}(t) = -\operatorname{sgn}_{P_2}(t)$, combining both equations yields

$$0 = 2\operatorname{sgn}_{P_1}(t)\operatorname{sgn}_{P_1}(\psi_2) = 2\operatorname{sgn}_{P_1}(t)\operatorname{sgn}_{P}(\psi_2),$$

which then implies $\operatorname{sgn}_P(\psi_2) = 0$. As we have chosen an arbitrary $P \in X$, this readily implies $\psi_2 \in I_X F$ which then implies $\psi_1 \in I_X F$. Putting together X-decompositions for ψ_1, ψ_2 respectively, we get a Y-decomposition for ψ as desired. □

As a direct consequence of Lemma 7.3.1 and Proposition 7.1.12, we have the following result that will be used several times in the sequel.

Corollary 7.3.2:
Let K be a field with a unique ordering, $n \in \mathbb{N}$ and $F = K((t_1))\cdots((t_n))$. Then F has X_F-decomposition.

In general, for a form $\varphi \in I_X^n F$, the minimal number of forms in $GP_n F_X$ can be bigger than the minimal number of forms in $GP_n F$ to write φ as a sum of those.

Example 7.3.3:
We consider the field of iterated formal Laurent series

$$F = \mathbb{R}((a_1))((a_2))((b_1))((b_2))((t))$$

with the ordering $P \in X_F$ such that we have $a_1, a_2, b_1, b_2, -t > 0$. Then the quadratic form

$$\begin{aligned}\varphi &:= \langle\langle -a_1, -a_2\rangle\rangle \perp t\langle\langle -b_1, -b_2\rangle\rangle \\ &= \langle\langle -t, -a_1\rangle\rangle + a_2\langle\langle a_1a_2t, a_1b_1\rangle\rangle + a_1a_2\langle\langle -ta_1a_2b_2, -b_1\rangle\rangle \in I^2_X F\end{aligned}$$

will give us an example of a form such that the usual Pfister number and the Pfister number with respect to $X = \{P\}$ will differ.

By the above φ is a sum of two elements in GP_2F. Because of the uniqueness of the diagonalization over rigid fields (up to permutation of the entries and multiplication with squares, see Proposition 4.1.11), an easy check shows that φ has no subform in GP_2F_X and cannot be Witt equivalent to a sum of two forms in GP_2F_X as it then would even be isometric to a sum of two forms in GP_2F_X.

We will briefly describe how to run the check effectively: in the given diagonalization, choose 2 positive entries and one negative entry. The product of these three entries will not occur in the complement of the so constructed three dimensional subform of φ. By doing so there are $\binom{4}{2} \cdot 4 = 24$ forms that have to been checked.

Remark 7.3.4:
The above example can easily be extended to higher powers of the fundamental ideal. To do so, let $n \geq 3$ and consider the field $F_n := F((t_1))\cdots((t_{n-2}))$ and the form

$$\varphi_n := \varphi \otimes \langle\langle t_1, \ldots, t_{n-2}\rangle\rangle \in I^n F$$

where F and φ are defined as above. As φ is isometric to a sum of two forms in GP_2F, the form φ_n is easily seen to be isometric to a sum of two forms in GP_nF. Let now P_n be the extension of the ordering P in Example 7.3.3 such that every upcoming t_i is negative. Then a repeated use of a slight modification of the argument in Proposition 3.4.4 will show that φ_n is not Witt equivalent to a sum of two forms in GP_nF_X where $X = \{P\}$.

It is commonly known that 8-dimensional forms in I^2F whose Clifford invariant has index ≤ 2 are exactly those forms of dimension 8 that are divisible by a form in GPF, see [Hof98c, Theorem 4.1]. This cannot be transferred to our setting, even if we assume that the Clifford invariant has one slot in $\bigcap_{P \in X} P$, as the following example shows.

Example 7.3.5:
Let $F = \mathbb{Q}((s))((t))$, which has X_F-decomposition by Lemma 7.3.2, and

$$\varphi = 2\langle\langle 6, t\rangle\rangle \perp -s\langle\langle 3, t\rangle\rangle = \langle 2, -3, -s, 3s, -2t, 3t, st, -3st\rangle = (\langle\langle 3, s, t\rangle\rangle - \langle\langle 2, t\rangle\rangle)_{\mathrm{an}} \in I^2_t F.$$

We see this last equality by a standard computation using Proposition 2.4.2. As we have

$$\varphi \equiv \langle\langle 2, t\rangle\rangle \mod I^3F,$$

the Clifford invariant $c(\varphi) = (2, t)_F$, which is of index 2, can be written with one slot in $\sum F^{*2}$, but we will now show that φ is not divisible by a form in GPF_{X_F}. To do so, first not that we have

$$\bigcap_{P \in X_F} P = \bigcup_{q \in \sum \mathbb{Q}^{*2}} qF^{*2}$$

by Proposition A.2.9. Thus it is enough to show that $\varphi_{F(\sqrt{d})}$ is not hyperbolic for all $d \in \sum \mathbb{Q}^{*2} = \mathbb{Q}_{>0}$. With the isomorphism $F(\sqrt{d}) \cong \mathbb{Q}(\sqrt{d})((s))((t))$ in mind (see Proposition 2.4.6), this can be checked by considering the residue class forms. One then readily sees that if $\varphi_{F(\sqrt{d})}$ were hyperbolic, we then would have

$$\langle\langle 3\rangle\rangle, \langle\langle 6\rangle\rangle \in W(F(\sqrt{d})/F),$$

a contradiction.

If on the other hand for an arbitrary field K and a subset $X \subseteq X_K$, we have $\psi \in I^2K$ with $\psi \cong \langle\langle a\rangle\rangle \otimes \sigma$ for some $a \in \bigcap\limits_{P\in X} P$ and a form σ over K with $\dim\sigma = 4$, analysing the proofs of [Kne77, Corollary 9.8, Example 9.12], one readily sees $\operatorname{ind} c(\varphi) \leq 2$ and a can be chosen as a slot of the (possibly split) quaternion algebra $c(\varphi)$.

Example 7.3.6:
Let $F = \mathbb{Q}((t_1))((t_2))((s))$ be the iterated Laurent series extension of $\mathbb{Q}$ in three variables, which has X_F-decomposition by Lemma 7.3.2. Then

$$\varphi = \langle\langle s\rangle\rangle \otimes \langle -2, -t_1, 2t_1, 3, t_2, -3t_2\rangle = \langle\langle s, 2, t_1\rangle\rangle - \langle\langle s, 3, t_2\rangle\rangle \in WF$$

is a 12-dimensional Form in I_t^3F that is not divisible by a binary torsion form, which can be seen similarly as in Example 7.3.5: if φ were divisible by a binary torsion form, this would imply the existence of some $d \in \mathbb{Q}^*$ such that $\langle\langle 2\rangle\rangle, \langle\langle 3\rangle\rangle, \langle\langle 6\rangle\rangle \in W(F\sqrt{d}/F)$, a contradiction.

A. Foundations of Related Areas

A.1. Extensions of Valuations

In this appendix, we will collect some basic facts about valuation theory that can be found in any book about the topic, for example [Neu92]. After introducing some standard notation, we will focus on the behaviour of valuations under field extensions to state a technical fact that will be used frequently in some sections of the main part of this thesis.
Let F be a field. A *(discrete) valuation* on F is a map $v : F \to \mathbb{R} \cup \{\infty\}$ with

(V1) $v(x) = \infty \iff x = 0$

(V2) for all $x, y \in F$, we have $v(xy) = v(x) + v(y)$

(V3) for all $x, y \in F$, we have $v(x + y) \geq \min\{v(x), v(y)\}$,

with the convention $a + \infty = \infty + a = \infty + \infty = \infty$ for all $a \in \mathbb{R}$ such that there is some $s \in \mathbb{R}_{>0}$ with $s = \min\{v(x) \mid x \in F, v(x) > 0\}$. If we have $s = 1$, the valuation v is said to be *normalized*. We then have $v(F^*) = s\mathbb{Z}$ and an element $\pi \in F^*$ with $v(\pi) = s$ is called a *uniformizing element* or *uniformizer* for short. We will often assume the equality $s = 1$, which can be assumed without loss of generality in most cases. We recall the well-known fact that a discrete valuation induces a metric d on F via

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \left(\frac{1}{2}\right)^{v(x-y)}, & \text{if } x \neq y \end{cases}$$

and thus a topology. We can therefore use topological language to describe the valuation. In particular, we can talk about *complete* valuations, i.e. valuations, whose induced topology is complete.
Further we have some important objects related to the valuation: the *valuation ring* $\mathcal{O} = \{x \in F \mid v(x) \geq 0\}$, its unique maximal ideal $\mathfrak{m} = \{x \in F \mid v(x) > 0\}$ and its units $\mathcal{U} = \{v(x) = 0\}$. As just mentioned, $\mathfrak{m}$ is the only maximal ideal in $\mathcal{O}$, i.e. $\mathcal{O}$ is a local ring, so that we have $\mathcal{U} = \mathcal{O} \setminus \mathfrak{m}$. Finally, we have the residue field $F_v = \mathcal{O}/\mathfrak{m}$.
We would like to recall that any element $a \in F^*$ with $v(a) = n \in \mathbb{Z}$ has a unique representation

$$a = \sum_{k=n}^{\infty} r_k \pi^k \tag{A.1}$$

with $r_k \in R, r_n \neq 0$, where R is a system of representants of the cosets of $\mathfrak{m}$ in $\mathcal{O}$ with $0 \in R$ and π is a fixed uniformizer, see [Neu92, Kapitel II. (4.4) Satz].

Let now F be a field equipped with such a valuation v and E/F be a finite field extension. We will now study valuations on E that extend the given valuation v, i.e. valuations $w : E \to \mathbb{R} \cup \{\infty\}$ with $w|_F = v$. Our main interest lies in the case that the valuation v is complete. At first, we will state a theorem for extensions of complete discrete valuation fields, the only case needed in this thesis.

Theorem A.1.1 ([Neu92, Kapitel II. (4.8) Theorem]):
Let F be a complete discrete valuation field with normalized valuation v and E/F a finite field extension of degree n. Then there is a unique extension w of v on E. In this case, E is also complete and the extension is given by

$$w : E \to \frac{1}{n}\mathbb{Z}, \quad x \mapsto \frac{1}{n} v(N_{E/F}(x)),$$

where $N_{E/F}$ denotes the field norm $E \to F, x \mapsto \det(\alpha \mapsto \alpha x)$.

If v is a valuation on F that has an extension w to E, the residue class field E_w is a field extension of F_v. It is often useful to know if a uniformizer for v is still a uniformizer for w. This can in some cases be shown using the following important theorem.

Theorem A.1.2 (Fundamental Equation of Valuation Theory, [Neu92, (8.5) Satz]):
Let E/F be a separable field extension and v a discrete valuation of F. We then have

$$\sum_{\substack{w \text{ extends } v \\ \text{to } E}} e_w f_w = [E : F],$$

where $e_w := [w(E^*) : v(F^*)]$ is the *ramification index* of the extension w of v and $f_w := [E_w : F_v]$ is the *inertia degree.*

If (F, v) is a complete discrete valuation field and E/F is a separable quadratic field extension, there exists a unique extension w of v on E. A uniformizing element for v is a uniformizer for w if and only if $e_w := [w(E^*) : v(F^*)] = 1$. According to Theorem A.1.2 this is the case if and only if we have $f_w := [\overline{E} : \overline{F}] = 2$.
If $\mathrm{char}(F) \neq 2$ we can write $E = F(\sqrt{a})$ for some $a \in F^* \setminus F^{*2}$. We will see that we can describe the situation explicitly and that it just depends on $v(a)$. Before stating the result we would like to recall that we have $N_{E/F}(x + y\sqrt{a}) = x^2 - ay^2$ in this case.

Corollary A.1.3:
Let F be a field equipped with a complete discrete valuation with uniformizer π and $E = F(\sqrt{a})$ for some $a \in F^* \setminus F^{*2}$. Then E is a complete discrete valuation field with a valuation w that extends the given valuation v of F. If $v(a)$ is even then π is a uniformizer for w and we have $E_w = F_v(\sqrt{\overline{a}})$ with $\overline{a} = a + \mathfrak{m} \in F_v = \mathcal{O} + \mathfrak{m}$.
If $v(a)$ is odd then π is not a uniformizer for w but $\sqrt{a}$ is a uniformizer for w and we have $E_w = F_v$.

Proof:
Because of Theorem A.1.1 it is clear that E is a complete discrete valuation field as well. After normalizing the valuation and modifying a by a square we may assume $v(a) \in \{0, 1\}$. If $v(a) = 1$ then

$$w(\sqrt{a}) = \frac{1}{2} v(N_{E/F}(\sqrt{a})) = \frac{1}{2} v(-a) = \frac{1}{2}$$

and

$$w(\pi) = \frac{1}{2}v(N_{E/F}(\pi)) = \frac{1}{2}v(\pi^2) = 1,$$

which means that $\sqrt{a}$ is uniformizer for w but π is not. In particular we have $e_w = 2$. Now, as seen above, Theorem A.1.2 yields $[E_w : F_v] = 1$ which is equivalent to $E_w = F_v$.
To show that π is still a uniformizer in the case $v(a) = 0$, it is enough to show that $v(N_{E/F}(x+y\sqrt{a}))$ is even for all $x, y \in F$ not both equal to 0. If $v(x) \neq v(y)$ we have

$$v(x^2) = 2v(x) \neq 2v(y) = v(y^2) = v(a) + v(y^2) = v(ay^2)$$

and therefore $v(N_{E/F}(x + y\sqrt{a})) = v(x^2 - ay^2) = \min\{2v(x), 2v(y)\}$ which is even (see [Neu92, page 124, Bemerkung]). Otherwise we have

$$x = \sum_{k=n}^{\infty} x_k \pi^k, \qquad y = \sum_{k=n}^{\infty} y_k \pi^k$$

with n, x_k, y_k as in (A.1). We then have

$$x^2 - ay^2 = (x_n^2 - ay_n^2)\pi^{2n} + \sum_{k=2n+1}^{\infty} z_k \pi^k$$

for some $z_k \in F$ as in (A.1). As $a \notin F^{*2}$ we have $x_n^2 - ay_n^2 \neq 0$ and therefore $v(N_{E/F}(x + y\sqrt{a})) = 2n$ is even. Here, according to Theorem A.1.2, E_w is a quadratic field extension of F_v. In E_w we have $\overline{\sqrt{a}} \neq 0$ because we have $w(\sqrt{a}) = \frac{1}{2}v(a^2) = v(a) = 0$. As $\overline{\sqrt{a}}$ clearly is a square root of $\overline{a}$, the result follows. □

A.2. Orderings of Fields

In this paragraph, we will fix some notation on formally real fields and recall some basic results. We will focus on results that deal with orderings under field extensions as these are frequently used in Chapter 6. In this exposition, we mainly follow [Lam83, Chapter 1] and [Lam05, Chapter VIII].
A field F is called *(formally) real* if -1 cannot be written as a sum of squares in F. Otherwise, F is called *non real*. A field F is formally real if and only if it admits an *ordering*, i.e. a subset $P \subsetneq F^*$ that is closed under addition and multiplication and fulfils $P \cup -P = F^*$ due to the famous Artin-Schreier Theorem. Those elements $x \in P$ are then called *positive* and elements in $-P$ are called *negative*. It is easy to see that P contains all non-zero sums of squares and we have $P \cap -P = \varnothing$. As usual, we will denote the set of all orderings on F as X_F.
Any ordering always comes with its related signature

$$\mathrm{sgn}_P : F^* \to \{\pm 1\}, a \mapsto \begin{cases} 1, & \text{if } a \in P \\ -1, & \text{if } a \notin P \end{cases},$$

which is a surjective group homomorphism. As orderings can be described equally well by their signature, we often interchange both terms depending on the situation. To indicate which kind we are using, we use big Latin letters such as P for the set of positive elements in F^* and small Greek letters like α for the signature map. For example, it is convenient to consider X_F as a subset of all maps $F^* \to \{\pm 1\}$. As the latter can be identified with $\{\pm 1\}^{F^*}$ which can be equipped with the product topology, where every factor $\{\pm 1\}$ is equipped with the discrete topology, we can consider X_F with the induced subspace topology, called the *Harrison topology*. A subbasis of clopen sets is given by the sets $H(a) := \{P \in X_F \mid a \in P\}$ for all $a \in F^*$.
A more general concept is that of a *preordering*, that is a subset $T \subsetneq F^*$ that is closed under addition and multiplication with $F^{*2} \subseteq T$. Of course, any ordering is a preordering. We further have the following:

Proposition A.2.1:
Let $(P_i)_{i \in I}$ a non-empty system of preorderings of F. Then, $\bigcap_{i \in I} P_i$ is a preordering of F.

In particular, the set of elements that are positive with respect to all orderings is a preordering. This set is computed in the following important Theorem.

Theorem A.2.2 ([Lam05, VIII. 1.12 Artins Theorem]):
In a formally real field F, an element $a \in F^*$ is totally positive, i.e. positive with respect to every $\alpha \in X_F$, if and only if a is sum of squares. In particular, we have

$$\bigcap_{P \in X_F} P = \left\{ \sum_{k=1}^{n} x_k \mid n \in \mathbb{N}, x_k \in F^* \right\} =: \sum F^{*2}.$$

As another standard notation, we put the set of all orderings that contain a fixed preordering T as

$$X_T := \{P \in X_F \mid T \subseteq P\}.$$

For an ordering P, the above defined signature map can be extended to a surjective homomorphism $WF \to \mathbb{Z}$, where the Witt class of a diagonalized form $\langle a_1, \dots, a_n \rangle$ is mapped to $\mathrm{sgn}_P(a_1) + \ldots + \mathrm{sgn}_P(a_n)$. This is well-defined by an application of Sylvester's law of inertia. By yet another generalization, we even get a homomorphism

$$\begin{aligned} \mathrm{sgn} : WF &\to \prod_{P \in X_F} \mathbb{Z} \\ [\varphi] &\mapsto (\mathrm{sgn}_P(\varphi))_{P \in X_F} . \end{aligned}$$

The following famous result computes the kernel of the signature map and several of its restrictions.

Theorem A.2.3 (Pfister's Local-Global Principle, [Lam83, Theorem 1.26]):
For every formally real field F and preordering T, we have

$$\mathrm{kern}(\mathrm{sgn}|_{X_T}) = \left\{ \sum_{k=1}^{n} \langle\langle t_k \rangle\rangle \otimes \varphi_k \mid n \in \mathbb{N}, t_1, \dots, t_n \in T, \varphi_1, \dots, \varphi_n \in WF \right\}.$$

In particular, we have $\mathrm{kern}(\mathrm{sgn}) = W_t F$, the torsion part of the Witt ring.

Using the above, we can readily deduce the following description of the torsion part of the Witt ring as the Witt kernel of the pythagorean closure. By definition, the *pythagorean closure* is the intersection of all *pythagorean fields* inside of a fixed algebraic closure, i.e. of all those fields in which any sum of squares is a square itself.

Theorem A.2.4 ([Lam05, Chapter VIII. Theorem 4.10]):
Let F be a formally real field. The Witt kernel $W(F_{\mathrm{py}}/F)$ is given by the torsion subgroup of WF, i.e. we have $W(F_{\mathrm{py}}/F) = W_t F$.

The rest of this section will now deal with orderings under field extensions. We first answer the question what we can say about subfields of an ordered field.

Proposition A.2.5 ([Lam05, page 272]):
Let K/F be a field extension with K formally real and $P \in X_K$. We then have $P \cap F^* \in X_F$. Expressed for a signature map $\alpha \in X_K$, we have $\alpha|_{F^*} \in X_F$.

All the remaining results will now deal with the opposite question: given a field extension K/F, under what circumstances can an ordering $P \in X_F$ be extended to K, i.e. to an ordering P_K of K with $P_K \cap F^* = P$? We will collect criteria for several important types of field extensions, starting with quadratic extensions.

Proposition A.2.6 ([Sch85, Chapter 3. 1.10 Theorem (i), 1.11 Remark]):
Let $P \in X_F$ be an ordering on the formally real field F and $a \in F^* \setminus F^{*2}$. The ordering P has an extension to $F(\sqrt{a})$ if and only if $a \in P$. In this case α has exactly two extensions on $F(\sqrt{a})$, one with $\sqrt{a}$ positive and one with $\sqrt{a}$ negative.

As another case of finite extensions, we will now deal with odd degree extensions.

Corollary A.2.7 ([Lam05, Chapter VIII. 7.10 (1)]):
Let F be a formally real field $P \in X_F$ an ordering on F and E/F be an odd degree extension. Then P can be extended to an ordering on E.

We will now turn to transcendental field extensions. The first case is the one of a rational function field, followed by the Laurent series extension.

A.2.8 Example ([Lam05, Chapter VIII. Example 1.13 (C)]):
Let P be an ordering on F and $E = F(X)$ the rational function field in one variable over F. Then E has infinitely many extensions of P. If we consider an element $f(X) = \frac{g(X)}{h(X)} \in E \setminus \{0\}$, we can for example define f to be positive if the product of the leading coefficients of g and h is positive. If we call that ordering P_1, we can get another ordering by $P_2 = \sigma(P_1)$, where $\sigma : E \to E$ is the F-linear automorphism defined by $x \mapsto -x$. In particular, we always have an ordering in which X is positive and one in which X is negative. Of course, the way we constructed P_2 can be generalized for any F-automorphism of E.

Proposition A.2.9 ([Lam05, Chapter VIII. Proposition 4.11 (1)]):
Let $P \in X_F$ be an ordering on a formally real field F. Then P can be extended in precisely two ways to an ordering on $F((t))$, one with t positive and one with t negative.

The last standard extension occurring in quadratic form theory that we study here is the one of a function field. The extendibility can be checked directly via the signature of the form we are looking at.

Theorem A.2.10 ([Lam05, Chapter XIII. Theorem 3.1]):
An ordering P on a field F can be extended to $F(\varphi)$ if and only if φ is indefinite at P. In particular, if $\varphi \in W_tF$, then $F(\varphi)$ is a real field in which every ordering of F has an extension.

We have another general criterion for the extendability of an ordering that just uses quadratic form theory.

Proposition A.2.11 ([Lam05, Chapter VIII. Corollary 9.8]):
Let F be a formally real field with ordering P and K/F be a field extension. Then P can be extended to K if and only if for any $n \in \mathbb{N}$ and any $a_1, \dots, a_n \in P$, the quadratic form $\langle a_1, \dots, a_n \rangle$ is anisotropic over K.

The following result uses some general facts about direct and inverse limits as can be found in [Bra16, Chapter 6, especially Sections 6.2, 6.4, 6.5]. The result itself is a slight generalization of [Cra75, Lemma 6]. We could prove it by reducing to the case of the just cited lemma (see Remark 6.3.2), but as we need to refer to some arguments in the proof, we will sketch another proof and fill in a few important details.

Proposition A.2.12 ([Cla, Exercise 15.35]):
Let (F_i, f_{ij}) be a directed system indexed by an index set I of formally real fields F_i with respective space of orderings $X_i \neq \varnothing$. Then the field $F := \varinjlim F_i$ is formally real with space of ordering isomorphic to $\varprojlim X_i$ as a topological space.

Proof:
We denote the canonical injections $F_i \to F$ by f_i. By a slight abuse of notation we will further denote the upcoming homomorphisms and its respective restrictions by the same symbol.

As the f_{ij} are injective ring homomorphisms they induce a direct system of groups

$$(F_i^*, f_{ij})$$

indexed by I. Further the f_i induce group homomorphisms

$$f_i : F_i^* \to F^*.$$

By a routine check of the universal property of the direct limit, we see that (F^*, f_i) is the direct limit of the system (F_i^*, f_{ij}). We can therefore identify $\mathrm{Hom}(F^*, \{\pm 1\})$ using the f_i with

$$\mathrm{Hom}(\varinjlim F_i^*, \{\pm 1\}).$$

The latter can be identified with

$$\varprojlim \mathrm{Hom}(F_i^*, \{\pm 1\}),$$

the maps of the inverse system given by

$$g_{ij} : \mathrm{Hom}(F_j^*, \{\pm 1\}) \to \mathrm{Hom}(F_i^*, \{\pm 1\})$$
$$\alpha \mapsto \alpha \circ f_{ij}$$

for $i, j \in I$ with $i \leq j$, where $\alpha \in \mathrm{Hom}(\varinjlim F_i^*, \{\pm 1\})$ is identified with $(\alpha_i)_{i \in I}$ with $\alpha_i(x_i) := \alpha(f_i(x))$, see [Bra16, Satz 6.5.5].

Let now $\alpha \in X_F \subseteq \mathrm{Hom}(F^*, \{\pm 1\})$ be an ordering of F. With the above identifications α corresponds to $(\alpha_i)_{i \in I} \in \varprojlim \mathrm{Hom}(F_i^*, \{\pm 1\})$, where $\alpha_i : F_i^* \to \{\pm 1\}$ is defined by $\alpha_i(x) := \alpha(f_i(x))$ for $x \in F_i^*$. We fix $i, j \in I$ with $i \leq j$. As $\alpha \in X_F$ and f_i is induced by a ring homomorphism it is now clear that we have $\alpha_i \in X_{F_i}$ for all $i \in I$.

Conversely an element $(\beta_i)_{i \in I} \in \varprojlim X_{F_i} \subseteq \varprojlim \mathrm{Hom}(F_i^*, \{\pm 1\})$ corresponds to $\beta \in \mathrm{Hom}(F^*, \{\pm 1\})$ defined by $\beta(f_i(x)) := \beta_i(f_i(x))$ for $x \in F_i$ (recall that every element in F has a preimage in some F_i). A standard calculation using the same arguments as above now shows $\beta \in X_F$.

Now it can be shown as in the proof of [Cra75, Lemma 6] that the map $X_F \to \varprojlim X_i, \alpha \mapsto (\alpha)_{i \in I}$ as defined above is bijective. □

The proof of the following result is an adaption and refinement of the proof of [RZ00, Proposition 1.1.10].

Proposition A.2.13:
Let (F_i, f_{ij}) be a directed system indexed by an index set I of formally real fields F_i with respective space of orderings $X_i \neq \varnothing$. Let $j \in I$ and $\varnothing \neq X \subseteq X_j$ such that for all $k \in I$ with $j \leq k$, every ordering in X has an extension to F_k. Then every ordering in X has an extension to $F := \varinjlim F_i$.

Proof:
By Proposition A.2.12 we know that X_F is isomorphic to the inverse limit $\varprojlim X_i$. As the homomorphisms in our direct system (F_i) are given by inclusion, analysing the proof of the above result yields that the homomorphisms in this inverse system are given by the restrictions of the orderings, denoted by $\varphi_{ij} : X_j \to X_i$ for $i, j \in I$ with $i \leq j$.

Let now $I' := \{k \in I \mid k \geq j\}$ be the cofinal subset of I of indices greater than or equal to j (in the given ordering of I).
For every $\alpha \in X$ and every $k \in I'$, the set $Y_k = \varphi_{jk}^{-1}(\alpha)$ is not empty by hypothesis. As the φ_{jk} are continuous by [Lam05, Corollary, page 272] and $\{\alpha\} \subseteq X_j$ is compact as a singleton set in a boolean space, the Y_k are all nonempty and compact. As subspaces of boolean spaces, the Y_k are further totally disconnected and Hausdorff. Since the φ_{ik} are just given by restriction we clearly have $\varphi_{ik}(Y_k) \subseteq Y_i$ for all $i, k \in I'$ with $i \leq k$. We thus have an inverse subsystem (Y_i, φ_{ik}, I') of (X_i, φ_{ik}, I') consisting of compact Hausdorff totally disconnected topological spaces, whose inverse limit $\varinjlim_{I'} Y_{i'} \subseteq \varinjlim_{I'} X_{i'}$ is not empty according to [RZ00, Proposition 1.1.3].
As I' is cofinal in I, we have a canonical isomorphism

$$\varinjlim_{I} X_i \to \varinjlim_{I'} X_{i'}$$

by [RZ00, Lemma 1.1.9]. By the choice of the Y_k every preimage of any element of $\varinjlim_{I'} Y_{i'}$ corresponds to an ordering of F extending α. $\square$

Bibliography

[AE01] Jón Kr. Arason and Richard Elman. Powers of the fundamental ideal in the Witt ring. J. Algebra, 239(1):150–160, 2001.

[AP71] Jón Kristinn Arason and Albrecht Pfister. Beweis des Krullschen Durchschnittsatzes für den Wittring. Invent. Math., 12:173–176, 1971.

[AP77] Jón K. Arason and Albrecht Pfister. Zur Theorie der quadratischen Formen über formalreellen Körpern. Math. Z., 153(3):289–296, 1977.

[Ara75] Jón Kr. Arason. Cohomologische invarianten quadratischer Formen. J. Algebra, 36(3):448–491, 1975.

[BCW80] Lawrence Berman, Craig Cordes, and Roger Ware. Quadratic forms, rigid elements, and formal power series fields. J. Algebra, 66(1):123–133, 1980.

[Bec04] Karim Johannes Becher. Supreme Pfister forms. Comm. Algebra, 32(1):217–241, 2004.

[Bec06a] Karim Johannes Becher. Minimal weakly isotropic forms. Math. Z., 252(1):91–102, 2006.

[Bec06b] Karim Johannes Becher. Totally positive extensions and weakly isotropic forms. Manuscripta Math., 120(1):83–90, 2006.

[Ber13] Jacques Bernoulli. Ars conjectandi, opus posthumum. Accedit Tractatus de seriebus infinitis, et epistola Gallicè scripta de ludo pilæ reticularis -. Freuburg i.B., 1713.

[BH04] Karim Johannes Becher and Detlev W. Hoffmann. Symbol lengths in Milnor K-theory. Homology Homotopy Appl., 6(1):17–31, 2004.

[BR13] Karim Johannes Becher and Mélanie Raczek. On the second K-group of a rational function field. Pacific J. Math., 262(1):1–9, 2013.

[Bra16] Martin Brandenburg. Einführung in die Kategorientheorie - Mit ausführlichen Erklärungen und zahlreichen Beispielen. Springer-Verlag, Berlin Heidelberg New York, 2. aufl. edition, 2016.

[BRV18] Patrick Brosnan, Zinovy Reichstein, and Angelo Vistoli. Essential dimension in mixed characteristic. Doc. Math., 23:1587–1600, 2018.

[Cla] Pete L. Clark. Field theory. Website. Online available at http://math.uga.edu/~pete/FieldTheory.pdf; retrieved at 21.10.2019.

[CM14] Vladimir Chernousov and Alexander Merkurjev. Essential dimension of spinor and Clifford groups. Algebra Number Theory, 8(2):457–472, 2014.

[Cor73] Craig M. Cordes. The Witt group and the equivalence of fields with respect to quadratic forms. J. Algebra, 26:400–421, 1973.

[Cra75] Thomas C. Craven. The Boolean space of orderings of a field. Trans. Amer. Math. Soc., 209:225–235, 1975.

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. The algebraic and geometric theory of quadratic forms, volume 56 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2008.

[EL72] Richard Elman and T. Y. Lam. Quadratic forms over formally real fields and pythagorean fields. Amer. J. Math., 94:1155–1194, 1972.

[Gen89] Enzo R. Gentile. On linked fields. Rev. Un. Mat. Argentina, 35:67–81 (1991), 1989.

[GS06] Philippe Gille and Tamás Szamuely. Central simple algebras and Galois cohomology, volume 101 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2006.

[Hof95] Detlev W. Hoffmann. A note on simple decomposition of quadratic forms over linked fields. J. Algebra, 175(2):728–731, 1995.

[Hof96] Detlev W. Hoffmann. Twisted Pfister forms. Doc. Math., 1:No. 03, 67–102, 1996.

[Hof98a] Detlev W. Hoffmann. On the dimensions of anisotropic quadratic forms in I^4. Invent. Math., 131(1):185–198, 1998.

[Hof98b] Detlev W. Hoffmann. Similarity of quadratic forms and half-neighbors. J. Algebra, 204(1):255–280, 1998.

[Hof98c] Detlev W. Hoffmann. Splitting patterns and invariants of quadratic forms. Math. Nachr., 190:149–168, 1998.

[Hof99] Detlev W. Hoffmann. On a conjecture of Izhboldin on similarity of quadratic forms. Doc. Math., 4:61–64, 1999.

[HT98] Detlev W. Hoffmann and Jean-Pierre Tignol. On 14-dimensional quadratic forms in I^3, 8-dimensional forms in I^2, and the common value property. Doc. Math., 3:189–214, 1998.

[IK00] Oleg T. Izhboldin and Nikita A. Karpenko. Some new examples in the theory of quadratic forms. Math. Z., 234(4):647–695, 2000.

[IR90] Kenneth Ireland and Michael Rosen. A classical introduction to modern number theory, volume 84 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1990.

[Kah05] Bruno Kahn. On “horizontal” invariants attached to quadratic forms. In Algebra and number theory, pages 21–33. Hindustan Book Agency, Delhi, 2005.

[Kar95] N. Karpenko. Chow groups of quadrics and index reduction formula. Nova J. Algebra Geom., 3(4):357–379, 1995.

[Kar17] Nikita A. Karpenko. Around 16-dimensional quadratic forms in I_q^3. Math. Z., 285(1-2):433–444, 2017.

[Kne77] Manfred Knebusch. Generic splitting of quadratic forms. II. Proc. London Math. Soc. (3), 34(1):1–31, 1977.

[Lam83] Tsit Yuen Lam. Orderings, valuations and quadratic forms, volume 52 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1983.

[Lam05] T. Y. Lam. Introduction to quadratic forms over fields, volume 67 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2005.

[Mil70] John Milnor. Algebraic K-theory and quadratic forms. Invent. Math., 9:318–344, 1969/70.

[Mor96] Patrick Morandi. Field and Galois theory, volume 167 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1996.

[Neu92] Jürgen Neukirch. Algebraische Zahlentheorie. Springer-Verlag, Berlin, 1992.

[Pfi66] Albrecht Pfister. Quadratische Formen in beliebigen Körpern. Invent. Math., 1:116–132, 1966.

[Pre84] Alexander Prestel. Lectures on formally real fields, volume 1093 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1984.

[PST09] R. Parimala, V. Suresh, and J.-P. Tignol. On the Pfister number of quadratic forms. In Quadratic forms—algebra, arithmetic, and geometry, volume 493 of Contemp. Math., pages 327–338. Amer. Math. Soc., Providence, RI, 2009.

[Rac13] Mélanie Raczek. On the 3-Pfister number of quadratic forms. Comm. Algebra, 41(1):342–360, 2013.

[Ros99a] Markus Rost. A descent property for Pfister forms. J. Ramanujan Math. Soc., 14(1):55–63, 1999.

[Ros99b] Markus Rost. On 14-dimensional quadratic forms, their spinors, and the difference of two octonion algebras. Website, 1999. Online available at https://www.math.uni-bielefeld.de/~rost/data/14-dim.pdf; retrieved at 21.10.2019.

[RZ00] Luis Ribes and Pavel Zalesskii. Profinite groups, volume 40 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000.

[Sch85] Winfried Scharlau. Quadratic and Hermitian forms, volume 270 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1985.

[Sti09] Henning Stichtenoth. Algebraic function fields and codes, volume 254 of Graduate Texts in Mathematics. Springer-Verlag, Berlin, second edition, 2009.

[Voe03] Vladimir Voevodsky. Motivic cohomology with $\mathbf{Z}/2$-coefficients. Publ. Math. Inst. Hautes Études Sci., (98):59–104, 2003.

[War78] Roger Ware. When are Witt rings group rings? II. Pacific J. Math., 76(2):541–564, 1978.