SFB 823

A portmanteau-type test for detecting serial correlation in locally stationary functional time series

Axel Bücher, Holger Dette,
Florian Heinrichs

Discussion Paper

Nr. 25/2020

SFB 823

A PORTMANTEAU-TYPE TEST FOR DETECTING SERIAL CORRELATION IN LOCALLY STATIONARY FUNCTIONAL TIME SERIES

AXEL BÜCHER, HOLGER DETTE, AND FLORIAN HEINRICHS

Abstract. The Portmanteau test provides the vanilla method for detecting serial correlations in classical univariate time series analysis. The method is extended to the case of observations from a locally stationary functional time series. Asymptotic critical values are obtained by a suitable block multiplier bootstrap procedure. The test is shown to asymptotically hold its level and to be consistent against general alternatives.

Key words: Autocovariance operator, Block multiplier bootstrap, Functional white noise, Time domain test.

1. Introduction

Over the last decades, technological progress allowed to store more and more data. In particular, many time series are recorded with a very high frequency, as for instance intraday prices of stocks or temperature records. In the literature, data of this type is often viewed as functional observations. Due to this development, the field of functional data analysis has been very active recently (see the monographs Bosq, 2000, Ferraty and Vieu, 2006, Horváth and Kokoszka, 2012 and Hsing and Eubank, 2015, among others).

The statistical analysis of functional data simplifies substantially if the observations are serially uncorrelated (or even serially independent). In fact, a huge amount of methodology has been proposed solely for this scenario, whence it is important to validate or reject this assumption in applications. Moreover, in the context of (univariate) financial return data, the absence or insignificance of serial correlation is commonly interpreted as a sign for efficient market prices (Fama, 1970). Likewise, investors may be interested in knowing whether functional counterparts like cumulative intraday returns exhibit significant autocorrelation.

If the observations are not only serially uncorrelated, but also centred and homoscedastic, then the time series is referred to as a functional white noise. Testing for the functional white noise hypothesis has found considerable interest in the recent literature. For instance, inspired by classical portmanteau-type methodology in univariate or multivariate time series analysis (see Box and Pierce, 1970; Hosking, 1980; Hong,

Date: September 15, 2020.

Corresponding author: Axel Bücher.

1996; Peña and Rodríguez, 2002, among others), Gabrys and Kokoszka (2007) propose to apply a multivariate portmanteau test to vectors of a few principal components from the functional time series. Horváth et al. (2013) and Kokoszka et al. (2017) investigate a portmanteau-type test which is based on estimates of the norms of the autocovariance operators. Alternatively, tests in the frequency domain have been proposed as well, which are based on the fact that the spectral density operator of a functional white noise time series is constant. Zhang (2016) proposes a Cramér-von Mises type test based on the functional periodogram and Bagchi et al. (2018) suggest a test based on an estimate of the minimum distance between the spectral density operator and its best approximation by a constant. While this approach estimates the minimal distance directly avoiding estimation of the spectral density operator, Characiejus and Rice (2020) suggest a test which is based on the distance between the estimated spectral density operator and an estimator of the operator calculated under the assumption of an uncorrelated time series.

A common feature of all aforementioned references consists in the fact that the proposed methodology is only applicable under the assumption of a (second order) stationary time series. This paper goes a step further and investigates the problem of testing for uncorrelatedness in possibly non-stationary functional data; in particular, for certain forms of heteroscedasticity. More precisely, we propose a portmanteau type test for locally stationary functional time series, whose critical values may be obtained by a multiplier block bootstrap. As a by-product, if accompanied by a test for constancy of the variance (see, e.g., Bücher et al., 2020), we straightforwardly obtain a test for the null hypothesis of functional white noise as well. Finally, we propose a generalized procedure to test for so-called *relevant* serial correlations, see Section 3.2 for a rigorous definition.

The paper is organized as follows: mathematical preliminaries, including a precise description of the hypotheses, are collected in Section 2. Suitable test statistics are introduced in Section 3, where we also prove weak convergence and validate a bootstrap approximation to obtain suitable critical values. Finite sample results are collected in Section 4, a case study is presented in Section 5 and all proofs are deferred to Section 6.

2. Mathematical Preliminaries

Throughout this document, we deal with objects in $L^p([0,1]^d)$, for different choices of $p \geq 1$ and $d \in \mathbb{N}$. We denote the respective L^p-norms by $\| \cdot \|_{p,d}$, with the special case $\| \cdot \|_{p,1}$ abbreviated by $\| \cdot \|_p$. Further, for functions $f, g \in L^p([0,1])$, we write $(f \otimes g)(x,y) = f(x)g(y)$.

2.1. **Locally stationary time series.** For $t \in \mathbb{Z}$, let $X_t : [0,1] \times \Omega \to \mathbb{R}$ denote a $(\mathcal{B}|_{[0,1]} \otimes \mathcal{A})$-measurable function with $X_t(\cdot, \omega) \in \mathcal{L}^2([0,1])$ for any $\omega \in \Omega$. We can regard $[X_t]$ as a random variable in $L^2([0,1])$ and will denote it by X_t as well. The expected value of $[X_t]$ in $L^2([0,1])$ coincides with the equivalence class of $\tau \mapsto \mu_t(\tau) = \mathbb{E}[X_t(\tau)]$. Similarly, the kernel of the (auto-)covariance operator of $[X_t]$ has a representation in $\mathcal{L}^2([0,1]^2)$ with $c_{X_t}(\tau,\sigma) = \operatorname{Cov}\big(X_t(\tau), X_t(\sigma)\big)$ and $c_{X_t,X_{t+h}}(\tau,\sigma) = \operatorname{Cov}\big(X_t(\tau), X_{t+h}(\sigma)\big)$. We refer to Section 2.1 in Bücher et al. (2020) for technical details.

The sequence $(X_t)_{t\in\mathbb{Z}}$ is called a *functional time series* in $L^2([0,1])$. The sequence is called *stationary* if, for all $q \in \mathbb{N}$ and $h, t_1, \dots, t_q \in \mathbb{Z}$,

$$(X_{t_1+h}, \dots, X_{t_q+h}) \overset{d}{=} (X_{t_1}, \dots, X_{t_q})$$

in $L^2([0,1])^q$. For the definition of a locally stationary functional time series we use a concept introduced by Vogt (2012) and van Delft and Eichler (2018) (see also van Delft et al., 2020; van Delft and Dette, 2020; Bücher et al., 2020). To be precise we call a sequence of functional time series $(X_{t,T})_{t\in\mathbb{Z}}$ indexed by $T \in \mathbb{N}$ a *locally stationary functional time series of order* $\rho > 0$ if there exists, for any $u \in [0,1]$, a stationary functional time series $(X_t^{(u)})_{t\in\mathbb{Z}}$ in $L^2([0,1])$ and an array of real-valued random variables $\{P_{t,T}^{(u)} : t = 1, \dots, T, T \in \mathbb{N}\}$ with $\mathbb{E}|P_{t,T}^{(u)}|^\rho < \infty$ uniformly in $t \in \{1, \dots, T\}, T \in \mathbb{N}$ and $u \in [0,1]$, such that

$$\|X_{t,T} - X_t^{(u)}\|_2 = \left\{ \int_0^1 \{X_{t,T}(\tau) - X_t^{(u)}(\tau)\}^2 \,\mathrm{d}\tau \right\}^{1/2} \le \left(\left|\frac{t}{T} - u\right| + \frac{1}{T} \right) P_{t,T}^{(u)},$$

for any $t \in \{1, \dots, T\}, T \in \mathbb{N}$ and $u \in [0,1]$. Note that in the case $\rho \ge 1$ the approximating family $\{(X_t^{(u)})_{t\in\mathbb{Z}} : u \in [0,1]\}$ is L^2-Lipschitz continuous in the sense that

$$\mathbb{E}\|X_t^{(u)} - X_t^{(v)}\|_2 \le C|u - v|, \tag{2.1}$$

for some constant $C > 0$, by local stationarity of $X_{t,T}$. In the following discussion we assume that $X_{t,T}$ (and hence $X_t^{(u)}$) is centred, i.e. $\mu_{t,T} = \mathbb{E}[X_{t,T}] = 0$ for all $t \in \{1, \dots, T\}$.

2.2. **Serial correlation in locally stationary time series.** In classical (functional) time series analysis, a time series is called uncorrelated if its autocovariances are zero for any lag $h > 0$. In the locally stationary setup, a slightly more subtle version suggests itself: we call a centred locally stationary functional time series of order $\rho \ge 2$ with approximating family of square-integrable stationary time series $\{(X_t^{(u)})_{t\in\mathbb{Z}} : u \in [0,1]\}$ (i.e., $\mathbb{E}[\|X_0^{(u)}\|_2^2] < \infty$ for all u) serially uncorrelated if the hypothesis

$$\bar{H}_0 := H_0^{(1)} \cap H_0^{(2)} \cap \dots \tag{2.2}$$

holds, where the individual hypothesis $H_0^{(h)}$ at lag $h \in \mathbb{N}$ is defined by

$$H_0^{(h)} : \ \|\operatorname{Cov}(X_0^{(u)}, X_h^{(u)})\|_{2,2} = 0 \qquad \text{for all } u \in [0,1]. \tag{2.3}$$

If, additionally, $u \mapsto \operatorname{Var}(X_0^{(u)})$ is constant, then the locally stationary time series will be called functional white noise. As in Remark 1 in Bücher et al. (2020), it may be shown that these definitions are independent of the choice of the approximating family.

Throughout this paper, we will develop suitable tests for certain hypotheses related to $\bar{H}_0$ and $H_0^{(h)}$ in (2.2) and (2.3), respectively. Following the main principle of classical portmanteau-type tests for detecting serial correlations, we start by fixing a maximum lag $H \in \mathbb{N}$ and to test the hypotheses

$$\bar{H}_0^{(H)} : \|\mathrm{Cov}(X_0^{(u)}, X_h^{(u)})\|_{2,2} = 0 \qquad \text{for all } h \in \{1, \dots, H\} \text{ and } u \in [0,1]. \tag{2.4}$$

Note that $\bar{H}_0 = \bigcap_{H \in \mathbb{N}} \bar{H}_0^{(H)}$.

2.3. **Regularity conditions on the observation scheme.** In order to obtain meaningful asymptotic results, the following regularity conditions will be imposed.

Condition 2.1 (Assumptions on the observations)**.**

(A1) **Local Stationarity.** The observations $X_{1,T}, \dots X_{T,T}$ are an excerpt from a centered locally stationary functional time series $\{(X_{t,T})_{t\in\mathbb{Z}} : T \in \mathbb{N}\}$ of order $\rho = 4$ in $L^2([0,1], \mathbb{R})$, with approximating family of stationary time series $\{(X_t^{(u)})_{t\in\mathbb{Z}} : u \in [0,1]\}$.

(A2) **Moment Condition.** For any $k \in \mathbb{N}$, there exists a constant $C_k < \infty$ such that $\mathbb{E}\|X_{t,T}\|_2^k \le C_k$ and $\mathbb{E}\|X_0^{(u)}\|_2^k \le C_k$ uniformly in $t \in \mathbb{Z}, T \in \mathbb{N}$ and $u \in [0,1]$.

(A3) **Cumulant Condition.** For any $j \in \mathbb{N}$ there is a constant $D_j < \infty$ such that

$$\sum_{t_1,\dots,t_{j-1}=-\infty}^{\infty} \big\| \mathrm{cum}(X_{t_1,T}, \dots, X_{t_j,T}) \big\|_{2,j} \le D_j < \infty,$$

for any $t_j \in \mathbb{Z}$ (for $j = 1$ the condition is to be interpreted as $\|\mathbb{E}X_{t_1,T}\|_2 \le D_1$ for all $t_1 \in \mathbb{Z}$). Further, for $k \in \{2,3,4\}$, there exist functions $\eta_k : \mathbb{Z}^{k-1} \to \mathbb{R}$ satisfying

$$\sum_{t_1,\dots,t_{k-1}=-\infty}^{\infty} (1 + |t_1| + \dots + |t_{k-1}|)\eta_k(t_1, \dots, t_{k-1}) < \infty$$

such that, for any $T \in \mathbb{N}, 1 \le t_1, \dots, t_k \le T, v, u_1, \dots, u_k \in [0,1], h_1, h_2 \in \mathbb{Z}$, $Z_{t,T}^{(u)} \in \{X_{t,T}, X_t^{(u)}\}$, and any $Y_{t,h,T}(\tau_1, \tau_2) \in \{X_{t,T}(\tau_1), X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)\}$, we have

(i) $\| \mathrm{cum}(X_{t_1,T} - X_{t_1}^{(t_1/T)}, Z_{t_2,T}^{(u_2)}, \cdots, Z_{t_k,T}^{(u_k)})\|_{2,k} \le \frac{1}{T}\eta_k(t_2 - t_1, \dots, t_k - t_1)$,

(ii) $\| \mathrm{cum}(X_{t_1}^{(u_1)} - X_{t_1}^{(v)}, Z_{t_2,T}^{(u_2)}, \cdots, Z_{t_k,T}^{(u_k)})\|_{2,k} \le |u_1 - v|\eta_k(t_2 - t_1, \dots, t_k - t_1)$,

(iii) $\| \mathrm{cum}(X_{t_1,T}, \dots, X_{t_k,T})\|_{2,k} \le \eta_k(t_2 - t_1, \cdots, t_k - t_1)$,

(iv) $\int_{[0,1]^2} |\mathrm{cum}\left(Y_{t_1,h_1,T}(\tau), Y_{t_2,h_2,T}(\tau)\right)| \,\mathrm{d}\tau \le \eta_2(t_2 - t_1)$.

Assumption (A1) restricts the non-stationary behaviour of the observations to smooth changes, while the moment condition ensures existence of the cumulants. The cumulant condition originates from classical multivariate time series analysis (see, e. g., Brillinger, 1981). Similar assumptions were made by Lee and Rao (2017) and Aue and van Delft (2020) in the context of non-stationary functional data. Lemma 2 in Bücher et al. (2020) shows that (A3) follows from (A1), (A2) and an additional moment condition, provided that a certain strong mixing condition is met.

3. Testing for serial correlation in locally stationary functional data

3.1. **A test statistic for detecting serial correlation.** In this section, we propose a test statistic for detecting deviations from hypothesis (2.4) and prove a corresponding weak convergence result. A bootstrap device for deriving suitable critical values will be discussed in the subsequent Section 3.3.

The test statistic is based on the following observation: as $X_t^{(u)}$ is centred we may rewrite (observing (2.1)) hypotheses (2.3) and (2.4) as

$$H_0^{(h)} : \|M_h\|_{2,3} = 0 \quad \text{and} \quad \bar{H}_0^{(H)} : \max_{h=1}^{H} \|M_h\|_{2,3} = 0,$$

where

$$M_h(u, \tau_1, \tau_2) = \int_0^u \mathbb{E}[X_0^{(w)}(\tau_1) X_h^{(w)}(\tau_2)] \, \mathrm{d}w.$$

An empirical version of M_h, based on the observations $X_{1,T}, \dots, X_{T,T}$, is provided by the statistic

$$\hat{M}_{h,T}(u, \tau_1, \tau_2) = \frac{1}{T} \sum_{t=1}^{\lfloor uT \rfloor \wedge (T-h)} X_{t,T}(\tau_1) X_{t+h,T}(\tau_2).$$

The next theorem implies consistency of the empirical versions, which suggests to reject the null hypotheses in (2.3) and (2.4) for large values of the statistics

$$\mathcal{S}_{h,T} = \sqrt{T} \|\hat{M}_{h,T}\|_{2,3} \quad \text{and} \quad \bar{\mathcal{S}}_{H,T} = \sqrt{T} \max_{h=1}^{H} \|\hat{M}_{h,T}\|_{2,3},$$

respectively.

Theorem 3.1. *Under Condition 2.1, we have, for any $h \in \mathbb{N}$ as $T \to \infty$*

$$\frac{1}{\sqrt{T}} \mathcal{S}_{h,T} \to \|M_h\|_{2,3}$$

in probability. Moreover, for any $H \in \mathbb{N}$, $h \in \{1, \dots, H\}$ as $T \to \infty$

$$\mathcal{S}_{h,T} \rightsquigarrow \begin{cases} \|\tilde{B}_h\|_{2,3} & \text{under } H_0^{(h)}, \\ +\infty & \text{else,} \end{cases} \quad \text{and} \quad \bar{\mathcal{S}}_{H,T} \rightsquigarrow \begin{cases} \max_{h=1}^{H} \|\tilde{B}_h\|_{2,3} & \text{under } \bar{H}_0^{(H)}, \\ +\infty & \text{else,} \end{cases}$$

where $\tilde{B} = (\tilde{B}_1, \dots, \tilde{B}_H)$ denotes a centred Gaussian variable in $L^2([0,1]^3)^H$, whose covariance operator $C_{\mathbb{B}} : L^2([0,1]^3)^H \to L^2([0,1]^3)^H$ is defined by

$$C_{\mathbb{B}} \begin{pmatrix} f_1 \\ \vdots \\ f_H \end{pmatrix} \begin{pmatrix} (u_1, \tau_{11}, \tau_{12}) \\ \vdots \\ (u_H, \tau_{H1}, \tau_{H2}) \end{pmatrix} = \begin{pmatrix} \sum_{h=1}^H \langle r_{1,h}((u_1, \tau_{11}, \tau_{12}), \cdot), f_h \rangle \\ \vdots \\ \sum_{h=1}^H \langle r_{H,h}((u_H, \tau_{H1}, \tau_{H2}), \cdot), f_h \rangle \end{pmatrix}. \tag{3.1}$$

Here, the kernel function $r_{h,h'}$ is given by

$$\begin{aligned} r_{h,h'}((u, \tau_1, \tau_2), (v, \varphi_1, \varphi_2)) &= \mathrm{Cov}(\tilde{B}_h(u, \tau_1, \tau_2), \tilde{B}_{h'}(v, \varphi_1, \varphi_2)) \\ &= \sum_{k=-\infty}^{\infty} \int_0^{u \wedge v} c_k(w) \, \mathrm{d}w, \end{aligned} \tag{3.2}$$

with

$$c_k(w) = c_k(w,h,h',\tau_1,\tau_2,\varphi_1,\varphi_2) = \mathrm{Cov}\big(X_0^{(w)}(\tau_1)X_h^{(w)}(\tau_2), X_k^{(w)}(\varphi_1)X_{k+h'}^{(w)}(\varphi_2)\big),$$

for any $1 \le h, h' \le H$*. In particular, the infinite sum in* (3.2) *converges.*

It is worthwhile to mention that the distributions of the limiting variables in the previous theorems are not pivotal under the null hypotheses. As a consequence, critical values for respective statistical tests must be estimated, for instance by a plug-in approach or by a suitable bootstrap device. Throughout this paper, we propose a bootstrap approach which will be worked out in Section 3.3 below.

3.2. **Detecting relevant serial correlations.** In the previous section, we considered "classical" hypotheses in the sense that we were testing whether the covariance operators up to lag H are exactly equal to zero. However, in concrete applications, hypotheses of this type might rarely be satisfied exactly and it might rather be reasonable to reformulate the null hypothesis in the form that "the norm of the autocovariance operator is small", but not exactly equal to 0. More precisely, given thresholds $\Delta_h > 0$ which may vary with the lag $h \in \{1, \dots, H\}$, we propose to consider the following *relevant hypotheses*

$$\begin{aligned} H_0^{(h,\Delta)} &: \|M_h\|_{2,3} \le \Delta_h, \\ \bar{H}_0^{(H,\Delta)} &: \|M_h\|_{2,3} \le \Delta_h \quad \text{for all } h \in \{1, \dots, H\}, \end{aligned} \tag{3.3}$$

where $H \in \mathbb{N}$ is some fixed constant representing the maximal lag under consideration. The choice of the thresholds Δ_h depends on the specific application and has to be discussed with the practitioner in concrete applications. Although this may be a daunting task, we strongly argue that one should carefully think about it as the classical implicit choice of $\Delta_h = 0$ typically corresponds to an unrealistic null hypothesis in many applications.

Consistency of $\hat{M}_{h,T}$ for M_h suggests to reject the above hypotheses for large values of $\hat{M}_{h,T}$. We propose to consider the "normalized" test statistics

$$\begin{aligned} \mathcal{S}_{h,\Delta_h,T} &= \sqrt{T}(\|\hat{M}_{h,T}\|_{2,3} - \Delta_h)\|\hat{M}_{h,T}\|_{2,3} \ , \\ \bar{\mathcal{S}}_{H,\Delta,T} &= \max_{h=1}^{H} \sqrt{T}(\|\hat{M}_{h,T}\|_{2,3} - \Delta_h)\|\hat{M}_{h,T}\|_{2,3} \ , \end{aligned}$$

whose asymptotic properties are described in the following result. It is worthwhile to mention that related test statistics like $\sqrt{T}(\|\hat{M}_{h,T}\|_{2,3}^2 - \Delta_h^2)$ or $\sqrt{T}(\|\hat{M}_{h,T}\|_{2,3} - \Delta_h)\Delta_h$ may be treated similarly, but that the respective tests exhibited a worse finite-sample performance in an unreported Monte-Carlo simulation study.

Corollary 3.2. *Under Condition 2.1, we have, for any fixed* $H \in \mathbb{N}$ *and for* $T \to \infty$*,*

$$\sqrt{T}\big((\|\hat{M}_{h,T}\|_{2,3} - \|M_h\|_{2,3})\|\hat{M}_{h,T}\|_{2,3}\big)_{h=1,\dots,H} \rightsquigarrow \big(\langle M_h, \tilde{B}_h\rangle\big)_{h=1,\dots,H},$$

where $\tilde{B}_1, \ldots, \tilde{B}_H$ *are defined in Theorem 3.1 and* $\langle f, g \rangle = \int_{[0,1]^3} f(x)g(x)\,\mathrm{d}x$. *As a consequence,*

$$\mathcal{S}_{h,\Delta_h,T} \rightsquigarrow \begin{cases} -\Delta_h \|\tilde{B}_h\|_{2,3} & \text{if } \|M_h\|_{2,3} = 0, \\ -\infty & \text{if } \|M_h\|_{2,3} \in (0, \Delta_h), \\ \langle M_h, \tilde{B}_h \rangle & \text{if } \|M_h\|_{2,3} = \Delta_h, \\ +\infty & \text{if } \|M_h\|_{2,3} > \Delta_h. \end{cases}$$

Moreover,

$$\bar{\mathcal{S}}_{H,\Delta,T} \rightsquigarrow \begin{cases} \max\{\max_{h\in N_H} \langle M_h, \tilde{B}_h \rangle, \max_{h\in O_h} -\Delta_h \|\tilde{B}_h\|_{2,3}\} & \text{if } \bar{H}_0^{(H,\Delta)} \text{ is met}, \\ +\infty & \text{else}, \end{cases}$$

where $N_H = \{h \in \{1, \ldots, H\} : \|M_h\|_{2,3} = \Delta_h\}$, $O_H = \{h \in \{1, \ldots, H\} : \|M_h\|_{2,3} = 0\}$ *and where the maximum over the empty set is interpreted as* $-\infty$.

As in Section 3.1, the limiting distributions under the null hypotheses are not pivotal, whence a bootstrap procedure will be introduced next.

3.3. **Critical values based on bootstrap approximations.** The limiting distributions of the test statistics derived in the previous sections depend in a complicated way on the higher order serial dependence of the underlying approximating family $\{(X_t^{(u)})_{t\in\mathbb{Z}} : u \in [0,1]\}$ and are rather difficult to estimate. To avoid the estimation, we propose a multiplier block bootstrap procedure.

Following Bücher et al. (2020) the bootstrap scheme will be defined in terms of i.i.d. standard normally distributed random variables $\{R_i^{(k)}\}_{i,k\in\mathbb{N}}$ which are independent of $\{(X_{t,T})_{t\in\mathbb{Z}} : T \in \mathbb{N}\}$. Further, let $m = m_T$ and $n = n_T$ denote two block length sequences satisfying one of the following two conditions.

Condition 3.3.

(B1) The block length $m = m_T \in \{1, \ldots, T\}$ tends to infinity and satisfies $m = o(T)$ as $T \to \infty$.

(B2) The block length $n = n(T) \in \{1, \ldots, T\}$ satisfies $m/n = o(1)$ and $mn^2 = o(T^2)$ as $T \to \infty$.

Next, let $K \in \mathbb{N}$ denote the number of bootstrap replications. For $k \in \{1, \ldots, K\}$ and $h \in \{1, \ldots, H\}$, define multiplier bootstrap approximations for

$$B_{h,T}(u, \tau_1, \tau_2) = \sqrt{T}\{\hat{M}_{h,T}(u, \tau_1, \tau_2) - M_h(u, \tau_1, \tau_2)\}$$

as

$$\hat{B}_{h,n,T}^{(k)}(u, \tau_1, \tau_2) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor uT \rfloor \wedge (T-h)} \frac{R_i^{(k)}}{\sqrt{m}} \sum_{t=i}^{(i+m-1)\wedge(T-h)} \{X_{t,T}(\tau_1)X_{t+h,T}(\tau_2) - \hat{\mu}_{t,h,n,T}(\tau_1, \tau_2)\},$$

where

$$\hat{\mu}_{t,h,n,T}(\tau_1, \tau_2) = \frac{1}{\tilde{n}_{t,h}} \sum_{j=\underline{n}_t}^{\bar{n}_{t,h}} X_{t+j,T}(\tau_1) X_{t+j+h,T}(\tau_2)$$

denotes the local empirical product moment of lag h with

$$\bar{n}_{t,h} = n \wedge (T - t - h), \quad \underline{n}_t = -n \vee (1 - t), \quad \tilde{n}_{t,h} = \bar{n}_{t,h} - \underline{n}_t + 1.$$

Note that for $n = T$ we obtain $\hat{\mu}_{t,h,T,T} = \hat{\mu}_{h,T}$ for all $t \in \{1, \dots, T\}$, where

$$\hat{\mu}_{h,T}(\tau_1, \tau_2) = \frac{1}{T - h} \sum_{t=1}^{T-h} X_{t,T}(\tau_1) X_{t+h,T}(\tau_2)$$

denotes the global empirical product moment. Let $\hat{\mathbb{B}}_{n,T}^{(k)} = (\hat{B}_{1,n,T}^{(k)}, \dots, \hat{B}_{H,n,T}^{(k)})$ and $\hat{\mathbb{B}}_T = \sqrt{T}(B_{1,T}, \dots, B_{H,T})$. The following result shows that this multiplier bootstrap is consistent.

Theorem 3.4. *Suppose that Condition 2.1 is met and let $\tilde{B}^{(1)}, \tilde{B}^{(2)}, \dots$ denote independent copies of $\tilde{B}$. Fix $K, H \in \mathbb{N}$.*

(i) If Condition 3.3 (B1) and (B2) are met, then, as $T \to \infty$,

$$(\hat{\mathbb{B}}_T, \hat{\mathbb{B}}_{n,T}^{(1)}, \dots, \hat{\mathbb{B}}_{n,T}^{(K)}) \rightsquigarrow (\tilde{B}, \tilde{B}^{(1)}, \dots, \tilde{B}^{(K)}).$$

(ii) If Condition 3.3 (B1) is met and if $\operatorname{Cov}(X_0^{(0)}, X_h^{(0)}) = \operatorname{Cov}(X_0^{(w)}, X_h^{(w)})$ for any $w \in [0, 1]$ and $h \in \mathbb{Z}$, then, as $T \to \infty$,

$$(\hat{\mathbb{B}}_T, \hat{\mathbb{B}}_{T,T}^{(1)}, \dots, \hat{\mathbb{B}}_{T,T}^{(K)}) \rightsquigarrow (\tilde{B}, \tilde{B}^{(1)}, \dots, \tilde{B}^{(K)}).$$

It is worthwhile to mention that the assumption on $\operatorname{Cov}(X_0^{(w)}, X_h^{(w)})$ in Theorem 3.4(ii) is met provided that $X_{t,T} = X_t$ for some stationary time series $(X_t)_{t \in \mathbb{Z}}$. In such a situation (for instance to be validated by a stationarity test in practice), using the bootstrap scheme with $n = T$ over the one with n satisfying Condition 3.3 (B2) typically results in better finite sample results, see Section 4 for more details.

Subsequently, we reconsider the problem of testing for serial uncorrelation of a locally stationary time series using classical and relevant hypotheses. For the sake of brevity, we only treat the hypotheses $\bar{H}_0^{(H)}$ and $\bar{H}_0^{(H,\Delta)}$, which are defined in (2.4) and (3.3), respectively and involve multiple lags. For this purpose we consider the following bootstrap approximations of the respective test statistics

$$\bar{\mathcal{S}}_{H,n,T}^{(k)} = \max_{h=1}^{H} \|\hat{\mathbb{B}}_{h,n,T}^{(k)}\|_{2,3}$$

for the classical hypotheses and

$$\bar{\mathcal{S}}_{H,n,T,\mathrm{rel}}^{(k)} = \max_{h=1}^{H} \langle \hat{M}_{h,T}, \hat{\mathbb{B}}_{h,n,T}^{(k)} \rangle$$

for the relevant hypotheses. Finally, we propose to reject the classical hypothesis (2.4) whenever

$$\bar{p}_{H,n,K,T} = \frac{1}{K} \sum_{k=1}^{K} \mathbb{1}\left(\bar{\mathcal{S}}_{H,n,T}^{(k)} \geq \bar{\mathcal{S}}_{H,T}\right) < \alpha \ . \tag{3.4}$$

Similarly, the relevant hypothesis (3.3) is rejected whenever

$$\bar{p}_{H,n,K,T,\text{rel}} = \frac{1}{K} \sum_{k=1}^{K} \mathbb{1}\left(\bar{\mathcal{S}}_{H,n,T,\text{rel}}^{(k)} \geq \bar{\mathcal{S}}_{H,\Delta,T}\right) < \alpha. \tag{3.5}$$

Corollary 3.5. *Fix $\alpha \in (0,1)$, suppose that Condition 2.1 is met and let $K = K_T \to \infty$.*

(i) If Condition 3.3 (B1) and (B2) hold, then the decision rule (3.4) defines a consistent asymptotic level α test for the classical hypotheses (2.4), that is

$$\lim_{T\to\infty} \mathbb{P}(\bar{p}_{H,n,K,T} < \alpha) = \begin{cases} \alpha & \text{under } \bar{H}_0^{(H)}, \\ 1 & \text{else.} \end{cases}$$

Similarly, for $\alpha < 1/2$, the decision rule (3.5) for the relevant hypotheses (3.3) satisfies

$$\begin{aligned} \lim_{T\to\infty} \mathbb{P}(\bar{p}_{H,n,K,T,\text{rel}} < \alpha) = 0 \qquad & \text{if } \|M_h\|_{2,3} < \Delta_h \text{ for all } h \in \{1,\dots,H\}, \\ \limsup_{T\to\infty} \mathbb{P}(\bar{p}_{H,n,K,T,\text{rel}} < \alpha) \leq \alpha \qquad & \text{if } \bar{H}_0^{(H,\Delta)} \cap R \text{ is met,} \\ \lim_{T\to\infty} \mathbb{P}(\bar{p}_{H,n,K,T,\text{rel}} < \alpha) = 1 \qquad & \text{else,} \end{aligned} \tag{3.6}$$

where R denotes the set of all models from the null hypothesis $\bar{H}_0^{(H,\Delta)}$ for which $\|M_h\|_{2,3} = \Delta_h$ for some $h \in \{1,\dots,H\}$ and for which $\operatorname{Var}(\langle M_h, \tilde{B}_h\rangle) > 0$ for each such h. In (3.6), the value α is attained if $\|M_h\|_{2,3} = \Delta_h$ for all $h \in \{1,\dots,H\}$.

(ii) If Condition 3.3 (B1) is met and if $\operatorname{Cov}(X_0^{(0)}, X_h^{(0)}) = \operatorname{Cov}(X_0^{(w)}, X_h^{(w)})$ for any $w \in [0,1]$ and $h \in \mathbb{N}_0$, then the same assertions as in (i) are met for $n = T$.

The restriction to $\alpha < 1/2$ for the test defined by (3.5) is needed to make sure that the contribution from $\max_{h\in O_H} -\Delta_h \|\tilde{B}_h\|_{2,3}$ in Corollary 3.2 is negligible (see Section 6 for details).

4. Monte Carlo Simulations

A large scale Monte Carlo simulation study was performed to analyse the finite-sample properties of the proposed tests. The major goal of the study was to analyse the level approximation and the power of the tests for hypotheses of the form $\bar{H}_0^{(H)}$ and $\bar{H}_0^{(H,\Delta)}$, with $H \in \{1,\dots,4\}$. Moreover, we also provide a comparison with existing tests for white noise / no serial correlation in the stationary setup, both for tests in the time domain (Kokoszka et al., 2017) and in the frequency domain (Zhang, 2016; Bagchi et al., 2018; Characiejus and Rice, 2020).

4.1. **Models.** We start by employing the same (stationary) models as in Zhang (2016) and Bagchi et al. (2018). In particular, for the null hypothesis of serial uncorrelation for any lag h, we consider: Model (N_1), an i.i.d. sequence of Brownian motions; Model (N_2), an i.i.d. sequence of Brownian bridges; and Model (N_3), data from a FARCH(1) process defined by

$$X_t(\tau) = B_t(\tau)\sqrt{\tau + \int_0^1 c_\psi \exp\Big(\frac{\tau^2+\sigma^2}{2}\Big) X_{t-1}^2(\sigma)\,\mathrm{d}\sigma},$$

where $(B_t)_{t\in\mathbb{Z}}$ denotes an i.i.d. sequence of Brownian motions and $c_\psi = 0.3418$. Under the alternative, we consider the FAR(1) model given by

$$X_t = \rho(X_{t-1} - \mu) + \varepsilon_t,$$

where ρ denotes an integral operator $\rho(f) = \int_0^1 K(\cdot,\sigma) f(\sigma)\,\mathrm{d}\sigma,\ f \in L^2([0,1])$, for a given kernel $K \in L^2([0,1]^2)$ and a sequence of centred, i.i.d. innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$ in $L^2([0,1])$. We consider the following choices for K and ε_t:

$$\begin{aligned}
&(A_1) \quad K(\tau,\sigma) = c_g \exp\big((\tau^2+\sigma^2)/2\big),\ \varepsilon_t \text{ i.i.d. Brownian motions,}\\
&(A_2) \quad K(\tau,\sigma) = c_g \exp\big((\tau^2+\sigma^2)/2\big),\ \varepsilon_t \text{ i.i.d. Brownian bridges,}\\
&(A_3) \quad K(\tau,\sigma) = c_w \min(\tau,\sigma),\ \varepsilon_t \text{ i.i.d. Brownian motions,}\\
&(A_4) \quad K(\tau,\sigma) = c_w \min(\tau,\sigma),\ \varepsilon_t \text{ i.i.d. Brownian bridges,}
\end{aligned}$$

where c_g and c_w are chosen such that the Hilbert-Schmidt norm of the ρ is 0.3.

Note that the above models are stationary. Since our proposed methodology allows for smooth changes in the distribution of the underlying stochastic processes as well, we additionally consider the following heteroscedastic locally stationary models:

$$\begin{aligned}
&(N_4) \quad X_{t,T} = \sigma(t/T)B_t,\\
&(A_5) \quad X_{t,T} = \rho(X_{t-1,T}) + \sigma(t/T)B_t,\\
&(A_6) \quad X_{t,T} = \sigma(t/T)\rho(X_{t-1,T}) + B_t,
\end{aligned}$$

where $(B_t)_{t\in\mathbb{Z}}$ denotes an i.i.d. sequence of Brownian motions, $\sigma(x) = x + 1/2$ and ρ is defined as in model (A_1). For model (N_4), the null hypothesis holds true, whereas the alternative is true for models (A_5) and (A_6).

4.2. **Details on the implementation.** For the comparison with the tests by Zhang (2016) and Bagchi et al. (2018) (results in Table 1) and the evaluation of the finite-sample properties under non-stationarity (results in Tables 3 and 5), the data was simulated on an equidistant grid of size 1000 on the interval $[0,1]$. For the comparison with the tests by Kokoszka et al. (2017) and Characiejus and Rice (2020) (results in Table 2), the size of the grid was chosen as 100 to accommodate the computational complexity of the tests. For the latter two tests, we relied on their implementation in the R-package `wwntests` by Petoukhov (2020).

Model	$\bar{H}_0^{(1)}$	$\bar{H}_0^{(2)}$	$\bar{H}_0^{(3)}$	$\bar{H}_0^{(4)}$	(B)	(Z)	$\bar{H}_0^{(1)}$	$\bar{H}_0^{(2)}$	$\bar{H}_0^{(3)}$	$\bar{H}_0^{(4)}$	(B)	(Z)
Panel A: $T = 128$							*Panel C:* $T = 512$					
(N_1)	7.3	6.3	5.9	5.6	1.8	4.2	6.1	5.4	6.0	5.1	2.8	4.7
(N_2)	4.9	4.4	4.2	4.2	1.1	5.4	5.9	4.8	4.6	4.8	1.9	5.9
(N_3)	5.4	4.5	4.2	3.8	4.7	5.9	4.3	5.0	4.7	4.0	6.3	4.9
(A_1)	99.8	99.5	99.3	99.1	66.5	83.7	100.0	100.0	100.0	100.0	99.3	99.5
(A_2)	98.4	97.9	97.1	96.5	51.7	83.1	100.0	100.0	100.0	100.0	98.3	99.8
(A_3)	99.8	99.7	99.7	99.7	84.9	68.3	100.0	100.0	100.0	100.0	100.0	98.7
(A_4)	91.8	88.5	85.7	82.7	37.0	65.8	100.0	100.0	100.0	100.0	90.4	100.0
Panel B: $T = 256$							*Panel D:* $T = 1024$					
(N_1)	5.3	6.1	5.9	5.1	1.9	4.2	5.5	6.3	5.7	5.4	3.5	4.9
(N_2)	4.4	5.3	4.8	4.2	1.4	4.8	5.5	5.7	5.3	5.4	3.5	5.1
(N_3)	5.0	4.4	4.0	4.0	6.0	5.2	4.3	3.9	3.7	3.6	7.6	4.8
(A_1)	100.0	100.0	100.0	100.0	91.5	99.2	100.0	100.0	100.0	100.0	100.0	100.0
(A_2)	99.9	99.9	99.9	99.9	84.4	99.5	100.0	100.0	100.0	100.0	99.9	99.8
(A_3)	100.0	100.0	100.0	100.0	99.1	98.2	100.0	100.0	100.0	100.0	100.0	100.0
(A_4)	99.6	99.4	99.3	99.2	65.9	99.1	100.0	100.0	100.0	100.0	99.6	100.0

TABLE 1. *Empirical rejection rates of test* (3.4) *for the classical hypotheses* (2.4) *in the case of stationary models, for various values of the maximal lag H in $\bar{H}_0^{(H)}$. The columns denoted by (B) and (Z) correspond to the tests of Bagchi et al. (2018) and Zhang (2016), respectively.*

For computational reasons, we reduced the dimension by projecting the generated data onto the subspace of $L^2([0,1])$ spanned by the first $D = 17$ functions of the Fourier basis $\{\psi_n\}_{n\in\mathbb{N}_0}$, where, for $n \in \mathbb{N}$,

$$\psi_0 \equiv 1, \quad \psi_{2n-1}(\tau) = \sqrt{2}\sin(2\pi n\tau), \quad \psi_{2n}(\tau) = \sqrt{2}\cos(2\pi n\tau)$$

to calculate the proposed test statistic.

For the calculation of the bootstrap quantiles, we employed the data driven choice of the block length m explained in Bücher et al. (2020). In the context of stationary processes (models (N_1)–(N_3) and (A_1)–(A_4)), it is natural to consider global estimators in the bootstrap procedure and we chose the bandwidth $n = T$. In fact, preliminary simulations suggested that this choice of n leads to better finite sample behavior. For the non-stationary models however, this choice is not reasonable and we used local estimators in order to avoid a possible bias. In this setting, we chose the bandwidth $n = \lfloor T^{2/3} \rfloor$, satisfying Condition 3.3 (B2). The number of bootstrap replicates was chosen as 200 and each model was simulated 1000 times.

4.3. **Results for the classical hypotheses.** In the following, we denote by (B) and (Z) the tests proposed by Bagchi et al. (2018) and Zhang (2016), respectively. (M_H), $H \in \{1, 2, 3\}$, denotes the multiple-lag test at lag H proposed by Kokoszka et al. (2017). Finally, (Spec_s) and (Spec_a) denote the spectral test as proposed by Characiejus and Rice (2020), with static and adaptive bandwidth, respectively. The empirical rejection rates of test (3.4) for the stationary models (N_1)–(N_3) and (A_1)–(A_4) are shown in

Model	$\bar{H}_0^{(1)}$	$\bar{H}_0^{(2)}$	$\bar{H}_0^{(3)}$	$\bar{H}_0^{(4)}$	(M_1)	(M_2)	(M_3)	(Spec_s)	(Spec_a)
Panel A: $T = 100$									
(N_1)	6.7	6.0	6.2	5.7	5.4	4.8	6.7	5.1	5.7
(N_2)	5.1	4.5	4.8	4.4	3.0	3.5	3.7	4.3	5.0
(N_3)	5.3	5.8	4.9	5.2	4.3	4.4	5.4	18.8	21.8
(A_1)	97.5	96.3	95.4	94.6	96.6	92.3	88.0	100.0	99.7
(A_2)	95.3	92.7	91.2	89.3	89.9	78.8	69.1	99.8	98.9
Panel B: $T = 200$									
(N_1)	4.5	4.3	4.6	4.3	5.4	5.4	5.4	4.6	5.5
(N_2)	4.6	5.1	4.3	4.0	3.1	3.4	3.8	4.7	4.6
(N_3)	3.2	3.2	3.2	3.2	5.1	5.3	6.3	26.2	28.8
(A_1)	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
(A_2)	100.0	99.9	99.9	99.8	100.0	100.0	98.7	100.0	100.0
Panel C: $T = 300$									
(N_1)	5.7	4.9	4.5	4.8	6.0	4.4	6.9	4.7	5.8
(N_2)	5.3	4.5	4.1	4.4	5.3	5.2	4.6	5.2	7.3
(N_3)	4.6	4.0	3.7	3.6	5.4	5.3	6.1	25.3	28.8
(A_1)	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
(A_2)	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

TABLE 2. *Empirical rejection rates of test* (3.4) *for the classical hypotheses* (2.4) *in the case of stationary models, for various values of the maximal lag H in $\bar{H}_0^{(H)}$. The columns denoted by* $(\mathrm{M}_i), i \in \{1, 2, 3\}$, *and* $(\mathrm{Spec}_i), i \in \{a, s\}$, *correspond to the tests of Kokoszka et al. (2017) and Characiejus and Rice (2020), respectively.*

Tables 1 and 2. We observe that the level approximation of the new test (3.4) is very accurate for all scenarios under consideration, and that the power is larger than for the competitors from the literature, in particular for small samples. A partial explanation for this observation consists in the fact that tests based in frequency domain formulate the white noise hypothesis in terms of the spectral density operator and therefore implicitly consider the auto-covariance operators at any lag h. Although the power of test (3.4) slightly decreases with increasing H, it decreases slower than the power of the multiple-lag time domain test by Kokoszka et al. (2017). The type I errors of the tests (Spec_s) and (Spec_a) seem to exceed the level of 5% for model (N_3). This difficulty might arise from the fact that the data is uncorrelated but dependent. In contrast, the level approximation of the proposed tests seems to be more accurate.

The empirical rejection rates of test (3.4) for the locally stationary models (N_4), (A_5) and (A_6) are shown in Table 3, for different sample sizes. We observe a reasonable approximation of the nominal level and high power under the non-stationary alternatives.

4.4. **Results for relevant hypotheses.** We conclude this section with a brief discussion of the performance of the proposed test (3.5) for the relevant hypotheses (3.3). For

Model	$\bar{H}_0^{(1)}$	$\bar{H}_0^{(2)}$	$\bar{H}_0^{(3)}$	$\bar{H}_0^{(4)}$	Model	$\bar{H}_0^{(1)}$	$\bar{H}_0^{(2)}$	$\bar{H}_0^{(3)}$	$\bar{H}_0^{(4)}$
Panel A: $T = 128$					*Panel C: $T = 512$*				
(N_4)	6.5	6.2	5.5	5.5	(N_4)	6.5	6.8	5.6	4.9
(A_5)	99.0	98.6	98.6	98.6	(A_5)	100.0	100.0	100.0	100.0
(A_6)	99.4	98.6	98.4	98.1	(A_6)	100.0	100.0	100.0	100.0
Panel B: $T = 256$					*Panel D: $T = 1024$*				
(N_4)	7.3	5.7	5.5	4.9	(N_4)	6.9	6.5	6.5	5.0
(A_5)	100.0	100.0	100.0	100.0	(A_5)	100.0	100.0	100.0	100.0
(A_6)	100.0	100.0	100.0	100.0	(A_6)	100.0	100.0	100.0	100.0

TABLE 3. *Empirical rejection rates of test (3.4) for the classical hypotheses (2.4) in the case of locally stationary models, for various values for the maximal lag H in $\bar{H}_0^{(H)}$.*

Model	$h = 1$	$h = 2$	$h = 3$	$h = 4$
(A_1)	0.1419 (*$5.00 \cdot 10^{-5}$*)	0.0689 (*$3.93 \cdot 10^{-5}$*)	0.0336 (*$3.41 \cdot 10^{-5}$*)	0.0169 (*$2.94 \cdot 10^{-5}$*)
(A_2)	0.0283 (*$1.94 \cdot 10^{-6}$*)	0.0138 (*$1.46 \cdot 10^{-6}$*)	0.0069 (*$1.20 \cdot 10^{-6}$*)	0.0037 (*$0.92 \cdot 10^{-6}$*)
(A_3)	0.1996 (*$2.51 \cdot 10^{-4}$*)	0.1220 (*$2.02 \cdot 10^{-4}$*)	0.0755 (*$1.71 \cdot 10^{-4}$*)	0.0468 (*$1.50 \cdot 10^{-4}$*)
(A_4)	0.0235 (*$4.40 \cdot 10^{-6}$*)	0.0117 (*$3.03 \cdot 10^{-6}$*)	0.0070 (*$2.15 \cdot 10^{-6}$*)	0.0048 (*$1.51 \cdot 10^{-6}$*)

TABLE 4. *Theoretical values of $\|M_h\|_{2,3}$, obtained by simulation. The numbers in brackets correspond to the empirical variance of the simulation.*

this purpose we have calculated the quantities $\|M_h\|_{2,3}$ for the models (A_1)–(A_4) by a numerical simulation (specifically, we simulated 10,000 time series of length $T = 2,000$, projected them on a Fourier basis of dimension $D = 101$, calculated for each time series the quantity $\|\hat{M}_{h,T}\|$, for $h \in \{1, \dots, 4\}$, and used the respective means as an approximation for $\|M_h\|$). The results can be found in Table 4. For the simulation experiment, we chose hypotheses corresponding to $\Delta = \Delta_{h,w} = w\|M_h\|_{2,3}$ with $w \in \{0.4+i/10 : i = 1, \dots, 11\}$ and $h = 1, \dots, 4$, such that the null hypotheses are met for $w \geq 1$ and the alternative hypotheses are met for $w < 1$. The results can be found in Table 5, where we omit the results for $H \in \{2, 3\}$ since they are qualitatively similar to the cases $H \in \{1, 4\}$. Again, we observe convincing level approximations and good power properties.

5. Case Study

Functional data arises naturally when time series are recorded with a very high frequency. To illustrate the proposed methodology, we consider intraday prices of various stocks. More specifically, we consider prices over the time span from February 2016 to January 2020, where each observation corresponds to the intraday price at a given day. In particular, let $P_t(x_j)$, $t \in \{1, \dots, T\}, j \in \{1, \dots, m\}$ denote the price of a share,

Model	$H \setminus w$	0.5	0.6	0.7	0.8	0.9	**1.0**	1.1	1.2	1.3	1.4	1.5
Panel A: $T = 128$												
(A_1)	1	58.1	39.4	25.2	13.2	8.1	**4.4**	1.8	0.9	0.2	0.0	0.0
	4	58.6	40.3	26.1	14.3	9.3	**5.3**	2.6	1.3	0.6	0.3	0.2
(A_2)	1	72.7	50.3	31.0	16.7	8.8	**3.9**	2.2	1.3	0.3	0.2	0.2
	4	75.0	53.7	35.3	20.2	11.7	**6.2**	4.2	3.0	1.9	1.8	1.5
(A_3)	1	57.9	40.1	22.6	13.2	6.9	**3.6**	1.8	0.6	0.0	0.0	0.0
	4	57.5	40.4	22.6	13.6	7.1	**3.9**	2.0	0.7	0.0	0.0	0.0
(A_4)	1	61.0	42.3	27.2	15.7	7.7	**4.7**	3.0	1.4	0.7	0.2	0.2
	4	67.7	49.2	34.2	21.6	12.4	**8.1**	5.4	3.3	2.4	2.0	1.4
Panel B: $T = 256$												
(A_1)	1	87.4	66.4	42.6	21.8	10.7	**4.4**	1.4	0.2	0.0	0.0	0.0
	4	87.3	66.3	42.8	22.0	10.9	**4.5**	1.4	0.2	0.0	0.0	0.0
(A_2)	1	91.7	75.7	51.9	28.4	12.5	**5.4**	1.8	0.4	0.0	0.0	0.0
	4	92.0	76.0	52.7	29.1	12.9	**5.8**	2.2	0.5	0.1	0.1	0.1
(A_3)	1	87.1	65.5	41.1	20.8	8.9	**3.6**	0.9	0.2	0.0	0.0	0.0
	4	87.1	65.6	41.2	20.8	8.9	**3.6**	0.9	0.2	0.0	0.0	0.0
(A_4)	1	82.8	64.0	41.9	22.8	12.6	**5.9**	1.8	0.5	0.0	0.0	0.0
	4	83.5	66.1	43.6	23.8	13.5	**6.7**	2.4	0.8	0.1	0.1	0.1
Panel C: $T = 512$												
(A_1)	1	97.4	88.6	68.2	39.8	16.6	**5.3**	1.0	0.3	0.0	0.0	0.0
	4	97.4	88.6	68.3	39.8	16.8	**5.5**	1.2	0.5	0.0	0.0	0.0
(A_2)	1	99.7	95.2	73.5	40.2	16.3	**4.4**	1.0	0.2	0.0	0.0	0.0
	4	99.7	95.2	73.3	40.4	16.5	**4.5**	0.9	0.2	0.0	0.0	0.0
(A_3)	1	98.1	88.2	64.1	37.0	13.4	**4.2**	0.6	0.2	0.0	0.0	0.0
	4	98.1	88.2	64.0	37.1	13.5	**4.2**	0.7	0.3	0.0	0.0	0.0
(A_4)	1	97.3	87.1	60.1	32.9	14.2	**4.1**	1.0	0.2	0.0	0.0	0.0
	4	97.3	87.3	60.0	33.1	14.6	**4.0**	0.9	0.2	0.0	0.0	0.0
Panel D: $T = 1024$												
(A_1)	1	100.0	99.7	89.6	60.3	23.8	**5.1**	0.7	0.2	0.0	0.0	0.0
	4	100.0	99.7	89.6	60.3	23.8	**5.1**	0.8	0.2	0.0	0.0	0.0
(A_2)	1	100.0	99.8	93.9	64.8	25.9	**5.0**	0.5	0.0	0.0	0.0	0.0
	4	100.0	99.8	93.9	64.7	26.0	**5.1**	0.6	0.0	0.0	0.0	0.0
(A_3)	1	100.0	99.6	87.0	53.8	18.6	**3.6**	0.5	0.1	0.0	0.0	0.0
	4	100.0	99.6	87.0	53.8	18.6	**3.6**	0.6	0.1	0.0	0.0	0.0
(A_4)	1	100.0	98.2	85.5	52.5	21.2	**5.0**	0.6	0.0	0.0	0.0	0.0
	4	100.0	98.2	85.5	52.6	21.2	**5.2**	0.7	0.0	0.0	0.0	0.0

TABLE 5. *Empirical rejection rates of the test* (3.5) *for the relevant hypotheses* (3.3) *in the case of stationary models.*

observed at time points x_j at day t. The lengths T of the considered time series depend on the different stocks as for some days observations are missing.

Gabrys et al. (2010) define intradaily cumulative returns as

$$R_t(x_j) = 100\{\log P_t(x_j) - \log P_t(x_1)\}, \quad j \in \{1, \dots, m\},\ t \in \{1, \dots, T\}.$$

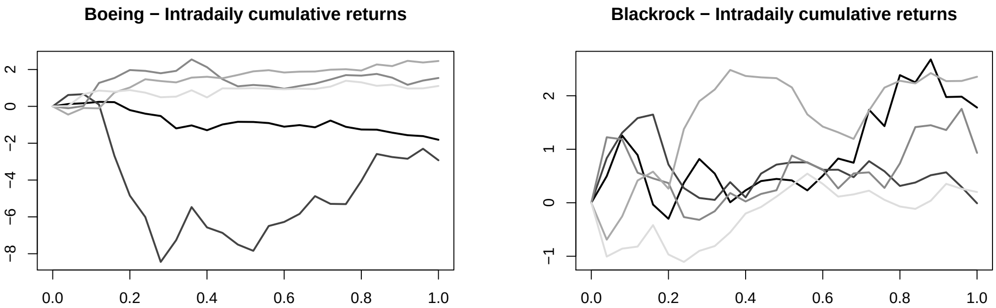

FIGURE 1. *Intradaily cumulative returns of Boeing and Blackrock from 8th to 12th of February 2016, where the x-axis corresponds to rescaled time and the y-axis denotes returns.*

Stock	$\bar{H}_0^{(1)}$	$\bar{H}_0^{(2)}$	$\bar{H}_0^{(3)}$	$\bar{H}_0^{(4)}$	T
Bank of America	80.6	80.1	79.7	79.4	982
Blackrock	98.4	98.3	98.3	98.3	822
Boeing	82.3	81.9	81.4	81.1	984
Goldman Sachs	74.5	73.9	73.6	73.3	990
JP Morgan	93.3	93.2	93.0	92.9	982

TABLE 6. *p-values of the (combined) tests for the respective null hypotheses in percent.*

Throughout, we consider $R_t(\cdot)$ as an L^2-function. Some exemplary intradaily cumulative return curves are displayed in Figure 1. The results of our testing procedure for detecting possible serial correlations can be found in Table 6, where we employed $K = 1000$ bootstrap replicates and considered up to $H = 4$ lags. The null hypotheses of serial correlation cannot be rejected at level $\alpha = 0.05$, as the p-values clearly exceed α. Thus, our results match the common assumption of uncorrelatedness in the literature.

6. PROOFS

Proof of Theorem 3.1. We prove that for any $H \in \mathbb{N}$ and as $T \to \infty$,

$$\sqrt{T}(\hat{M}_{1,T} - M_1, \dots, \hat{M}_{H,T} - M_H) \rightsquigarrow \tilde{B} := (\tilde{B}_1, \dots, \tilde{B}_H), \tag{6.1}$$

where $\tilde{B}$ denotes a centred Gaussian variable in $L^2([0,1]^3)^H$, with covariance operator given by (3.1). The statement is then a consequence of the continuous mapping theorem.

By Theorem 1 of Bücher et al. (2020), the vector $\sqrt{T}(\hat{M}_{1,T} - \mathbb{E}\hat{M}_{1,T}, \dots, \hat{M}_{H,T} - \mathbb{E}\hat{M}_{H,T})$ converges weakly to a vector of centred Gaussian variables $\tilde{B} = (\tilde{B}_1, \dots, \tilde{B}_H)$ in $L^2([0,1]^3)^H$. Thus, (6.1) follows from Slutsky's lemma, once we have shown that $\lim_{T\to\infty} \sqrt{T}\|\mathbb{E}\hat{M}_{h,T} - M_h\|_{2,3} = 0$ for any $h \in \mathbb{N}$. For the latter purpose, invoke the triangle inequality to obtain

$$\begin{aligned}
&\sqrt{T}\|\mathbb{E}\hat{M}_{h,T} - M_h\|_{2,3} \\
&= \sqrt{T}\Big(\int_0^1 \Big\| \frac{1}{T} \sum_{t=1}^{\lfloor uT \rfloor \wedge (T-h)} \mathbb{E}[X_{t,T} \otimes X_{t+h,T}] - \int_0^u \mathbb{E}[X_0^{(w)} \otimes X_h^{(w)}] \, \mathrm{d}w \Big\|_{2,2}^2 \mathrm{d}u \Big)^{1/2} \\
&= \sqrt{T}\Big(\int_0^1 \Big\| \sum_{t=1}^{\lfloor uT \rfloor \wedge (T-h)} \int_{\frac{t-1}{T}}^{\frac{t}{T}} \mathbb{E}[X_{t,T} \otimes X_{t+h,T}] - \mathbb{E}[X_t^{(w)} \otimes X_{t+h}^{(w)}] \, \mathrm{d}w \\
&\qquad\qquad - \int_{T^{-1}\{\lfloor uT \rfloor \wedge (T-h)\}}^u \mathbb{E}[X_0^{(w)} \otimes X_h^{(w)}] \, \mathrm{d}w \Big\|_{2,2}^2 \mathrm{d}u \Big)^{1/2} \\
&\le \sqrt{T}\Big(\int_0^1 \Big\{ \sum_{t=1}^{\lfloor uT \rfloor \wedge (T-h)} \Big\| \int_{\frac{t-1}{T}}^{\frac{t}{T}} \mathbb{E}[X_{t,T} \otimes X_{t+h,T} - X_t^{(w)} \otimes X_{t+h}^{(w)}] \, \mathrm{d}w \Big\|_{2,2} \\
&\qquad\qquad + \Big\| \int_{T^{-1}\{\lfloor uT \rfloor \wedge (T-h)\}}^u \mathbb{E}[X_0^{(w)} \otimes X_h^{(w)}] \, \mathrm{d}w \Big\|_{2,2} \Big\}^2 \mathrm{d}u \Big)^{1/2}.
\end{aligned}$$

The integral from $T^{-1}\{\lfloor uT \rfloor \wedge (T-h)\}$ to u at the right-hand side is of order $1/T$. Further, by Jensen's inequality and local stationarity,

$$\begin{aligned}
&\Big\| \int_{\frac{t-1}{T}}^{\frac{t}{T}} \mathbb{E}[X_{t,T} \otimes X_{t+h,T} - X_t^{(w)} \otimes X_{t+h}^{(w)}] \, \mathrm{d}w \Big\|_{2,2} \\
&\qquad\qquad \le \int_{\frac{t-1}{T}}^{\frac{t}{T}} \|\mathbb{E}[X_{t,T} \otimes X_{t+h,T} - X_t^{(w)} \otimes X_{t+h}^{(w)}]\|_{2,2} \, \mathrm{d}w \le \frac{C}{T^2}
\end{aligned}$$

for some constant $C > 0$. Thus, it follows

$$\sqrt{T}\|\mathbb{E}\hat{M}_{h,T} - M_h\|_{2,3} = O(T^{-1/2}),$$

which completes the proof of the theorem. □

Proof of Corollary 3.2. If $\|M_h\|_{2,3} = 0$ for some $h \in \{1, \dots H\}$, then $\sqrt{T}(\|\hat{M}_{h,T}\|_{2,3} - \|M_h\|_{2,3})\|\hat{M}_{h,T}\|_{2,3}$ converges to zero in probability by Theorem 3.1 and Slutsky's lemma. Hence, it is sufficient to assume that $\|M_h\|_{2,3} \neq 0$ for all $h \in \{1, \dots H\}$. We then obtain

$$\sqrt{T}(\|\hat{M}_{h,T}\|_{2,3} - \|M_h\|_{2,3})_{h=1,\dots,H} \rightsquigarrow \Big(\frac{\langle M_h, \tilde{B}_h \rangle}{\|M_h\|_{2,3}} \Big)_{h=1,\dots,H},$$

from the functional delta method (Theorem 3.9.4 in van der Vaart and Wellner, 1996), applied to the functional in Proposition 6.1 below. Apply Slutsky's lemma to conclude. □

Proposition 6.1. *The function $\Phi := \|\cdot\|_{2,3}$ from $L^2([0,1]^3)$ to $\mathbb{R}$ is Hadamard-differentiable in any M with $\|M\|_{2,3} > 0$, with derivative $\Phi'_M(h) = \frac{\langle M,h \rangle}{\|M\|_{2,3}}$ in direction $h \in L^2([0,1]^3)$.*

Proof. For any sequences $h_n \to h$ with $h_n \in L^2([0,1]^3)$ and $t_n \to 0$ with $t_n \in \mathbb{R} \setminus \{0\}$, it holds

$$\begin{aligned}\frac{\|M + t_n h_n\|_{2,3}^2 - \|M\|_{2,3}^2}{t_n} &= \frac{1}{t_n} \int_{[0,1]^3} 2M(x) t_n h_n(x) + t_n^2 h_n^2(x)\,\mathrm{d}x \\ &= \int_{[0,1]^3} 2M(x) h_n(x)\,\mathrm{d}x + t_n \int_{[0,1]^3} h_n^2(x)\,\mathrm{d}x,\end{aligned}$$

which converges to $2\int_{[0,1]^3} M(x)h(x)\,\mathrm{d}x = 2\langle M, h\rangle$. The square root function in $\mathbb{R}$ is Hadamard-differentiable at $x > 0$ with derivative $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$. By the chain rule for Hadamard-differentiable functions (Lemma 3.9.3 in van der Vaart and Wellner, 1996), the Hadamard-derivative of Φ is given by $\Phi'_M(h) = \frac{\langle M,h\rangle}{\|M\|_{2,3}}$. □

Proof of Theorem 3.4. (i) can be deduced directly from Theorem 2 of Bücher et al. (2020). For (ii) note that by Theorem C.3 of the supplementary material of the latter article, it holds $(\hat{\mathbb{B}}_T, \mathbb{B}_T^{(1)}, \dots, \mathbb{B}_T^{(K)}) \rightsquigarrow (\tilde{B}, \tilde{B}^{(1)}, \dots, \tilde{B}^{(K)})$, where $\mathbb{B}_T^{(k)} = (\tilde{B}_{T,1}^{(k)}, \dots, \tilde{B}_{T,H}^{(k)})$ and

$$\begin{aligned}&\tilde{B}_{T,h}^{(k)}(u,\tau_1,\tau_2) \\ &\quad = \frac{1}{\sqrt{T}} \sum_{i=1}^{\lfloor uT\rfloor \wedge (T-h)} \frac{R_i^{(k)}}{\sqrt{m}} \sum_{t=i}^{(i+m-1)\wedge(T-h)} X_{t,T}(\tau_1)X_{t+h,T}(\tau_2) - \mathbb{E}[X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)].\end{aligned}$$

Note that for $u < 1$ it holds $\lfloor uT\rfloor + m - 1 \le T - h$, for any sufficiently large $T \in \mathbb{N}$. Thus, rewrite

$$\begin{aligned}&\hat{B}_{h,T,T}^{(k)}(u,\tau_1,\tau_2) = \tilde{B}_{T,h}^{(k)}(u,\tau_1,\tau_2) \\ &+ \sqrt{\frac{m}{T}} \sum_{i=1}^{\lfloor uT\rfloor} R_i^{(k)} \Big(\frac{1}{T-h} \sum_{t=1}^{T-h} \mathbb{E}[X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)] - X_{t,T}(\tau_1)X_{t+h,T}(\tau_2) \Big) + \mathcal{O}_{\mathbb{P}}\Big(\sqrt{\tfrac{m}{T}}\Big).\end{aligned}$$

For the second term on the right-hand side of the latter display, it holds by independence of the random variables $R_i^{(k)}$,

$$\begin{aligned}&\mathbb{E}\Big\| \sqrt{\frac{m}{T}} \sum_{i=1}^{\lfloor \cdot T\rfloor \wedge (T-h)} R_i^{(k)} \Big(\frac{1}{T-h} \sum_{t=1}^{T-h} \mathbb{E}[X_{t,T} \otimes X_{t+h,T}] - X_{t,T} \otimes X_{t+h,T} \Big) \Big\|_{2,3}^2 \\ &\le \int_{[0,1]^2} m\mathbb{E}\Big[\Big(\frac{1}{T-h} \sum_{t=1}^{T-h} \mathbb{E}[X_{t,T}(\tau_1)X_{t+h,T}(\tau_2)] - X_{t,T}(\tau_1)X_{t+h,T}(\tau_2) \Big)^2 \Big] \,\mathrm{d}(\tau_1,\tau_2) \\ &= \frac{m}{(T-h)^2} \sum_{t_1,t_2=1}^{T-h} \int_{[0,1]^2} \mathrm{Cov}(X_{t_1,T}(\tau_1)X_{t_1+h,T}(\tau_2), X_{t_2,T}(\tau_1)X_{t_2+h,T}(\tau_2))\,\mathrm{d}(\tau_1,\tau_2),\end{aligned}$$

which is of order $O(m/T)$ by the same arguments as in the proof of Theorem 2 of Bücher et al. (2020). Thus, $\hat{\mathbb{B}}_{T,T}^{(k)} = \mathbb{B}_T^{(k)} + O_{\mathbb{P}}(\sqrt{m/T})$ and (ii) follows. □

Proof of Corollary 3.5. The assertions for the null hypothesis $H_0^{(H)}$ follow from Theorem 3.4 and Corollary 4.3 in Bücher and Kojadinovic (2019). The null hypothesis $H_0^{(H,\Delta)}$

may be treated by similar arguments as in the last-named corollary, observing that the weak limit of $\bar{\mathcal{S}}_{H,\Delta,T}$ is stochastically bounded by $\max_{h=1}^{H}\langle M_h, \tilde{B}_h\rangle$ on the positive real line. The assertions regarding the alternative hypotheses follow from divergence to infinity of the test statistics and stochastic boundedness of the bootstrap statistics. □

Acknowledgements

Financial support by the German Research Foundation is gratefully acknowledged. Collaborative Research Center "Statistical modelling of nonlinear dynamic processes" (SFB 823, Teilprojekt A1, A7 and C1) of the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) is gratefully acknowledged. Further, the research of H. Dette was partially supported by the German Research Foundation under Germany's Excellence Strategy - EXC 2092 CASA - 390781972.

The authors are grateful to V. Characiejus and G. Rice for providing their software and to N. Jumpertz for help with the data set.

References

Aue, A. and A. van Delft (2020). Testing for stationarity of functional time series in the frequency domain. *To appear in: Annals of Statistics*.

Bagchi, P., V. Characiejus, and H. Dette (2018). A simple test for white noise in functional time series. *Journal of Time Series Analysis 39*(1), 54–74.

Bosq, D. (2000). *Linear processes in function spaces*, Volume 149 of *Lecture Notes in Statistics*. Springer-Verlag, New York. Theory and applications.

Box, G. E. P. and D. A. Pierce (1970). Distribution of residual autocorrelations in autoregressive-integrated moving average time series models. *Journal of the American Statistical Association 65*(332), 1509–1526.

Brillinger, D. (1981). *Time Series: Data Analysis and Theory*. Holden Day, Inc., San Francisco.

Bücher, A., H. Dette, and F. Heinrichs (2020). Detecting deviations from second-order stationarity in locally stationary functional time series. *Annals of the Institute of Statistical Mathematics 72*(4), 1055–1094.

Bücher, A. and I. Kojadinovic (2019). A note on conditional versus joint unconditional weak convergence in bootstrap consistency results. *J. Theoret. Probab. 32*(3), 1145–1165.

Characiejus, V. and G. Rice (2020). A general white noise test based on kernel lag-window estimates of the spectral density operator. *Econometrics and Statistics 13*, 175–196.

Fama, E. F. (1970). Efficient capital markets: A review of theory and empirical work. *The Journal of Finance 25*(2), 383–417.

Ferraty, F. and P. Vieu (2006). *Nonparametric Functional Data Analysis: Theory and Practice*. New York: Springer.

Gabrys, R., L. Horváth, and P. Kokoszka (2010). Tests for error correlation in the functional linear model. *Journal of the American Statistical Association 105*(491), 1113–1125.

Gabrys, R. and P. Kokoszka (2007). Portmanteau test of independence for functional observations. *J. Amer. Statist. Assoc. 102*(480), 1338–1348.

Hong, Y. (1996). Consistent testing for serial correlation of unknown form. *Econometrica 64*(4), 837–864.

Horváth, L. and P. Kokoszka (2012). *Inference for functional data with applications.* Springer Series in Statistics. Springer, New York.

Horváth, L., M. Hušková, and G. Rice (2013). Test of independence for functional data. *Journal of Multivariate Analysis 117*, 100 – 119.

Hosking, J. R. M. (1980). The multivariate portmanteau statistic. *Journal of the American Statistical Association 75*(371), 602–608.

Hsing, T. and R. Eubank (2015). *Theoretical Foundations of Functional Data Analysis, with an Introduction to linear Operators.* New York: Wiley.

Kokoszka, P., G. Rice, and H. L. Shang (2017). Inference for the autocovariance of a functional time series under conditional heteroscedasticity. *Journal of Multivariate Analysis 162*, 32–50.

Lee, J. and S. S. Rao (2017). A note on general quadratic forms of nonstationary stochastic processes. *Statistics 51*(5), 949–968.

Peña, D. and J. Rodríguez (2002). A powerful portmanteau test of lack of fit for time series. *Journal of the American Statistical Association 97*(458), 601–610.

Petoukhov, D. (2020). *wwntests: Hypothesis Tests for Functional Time Series.* R package version 1.0.1.

van Delft, A., V. Characiejus, and H. Dette (2020). A nonparametric test for stationarity in functional time series. *Statistica Sinica, to appear, DOI: 10.5705/ss.202018.0320*.

van Delft, A. and H. Dette (2020). A similarity measure for second order properties of non-stationary functional time series with applications to clustering and testing. *Bernoulli, to appear; arXiv:1810.08292*.

van Delft, A. and M. Eichler (2018). Locally stationary functional time series. *Electron. J. Statist. 12*(1), 107–170.

van der Vaart, A. and J. Wellner (1996). *Weak Convergence and Empirical Processes*, Volume 1 of *Springer series in statistics.* Springer Science+Business Media New York.

Vogt, M. (2012). Nonparametric regression for locally stationary time series. *The Annals of Statistics 40*, 2601–2633.

Zhang, X. (2016). White noise testing and model diagnostic checking for functional time series. *Journal of Econometrics 194*(1), 76–95.

(Axel Bücher) Heinrich-Heine-Universität Düsseldorf, Mathematisches Institut, Universitätsstr. 1, 40225 Düsseldorf, Germany.

Email address: `axel.buecher@hhu.de`

(Holger Dette, Florian Heinrichs) Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstr. 150, 44780 Bochum, Germany.

Email address: holger.dette@rub.de, florian.heinrichs@rub.de