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Re-imagining geometry education in schools

The award of the Fields Medal (the equivalent of the Nobel Prize) to Peter Scholze in 2018 recorded how his work is transforming research in arithmetic geometry (Betti, 2018). Similarly, the award of the 2019 Abel Prize (directly modelled after the Nobel Prize) to Karen Uhlenbeck is for her work on bringing together geometry and analysis. These awards to Scholze and Uhlenbeck underline the importance of what the renowned mathematician Hilbert immortalised as geometry and the imagination (Hilbert & Cohn-Vossen, 1932/1952). Yet in schools, it can be, I would argue, that school geometry is not always experienced by students in ways that could be said to value intuitive imaginings. In this paper, I examine some of the challenges facing the teaching and learning of school geometry from the research literature and from my own research – and the impact of some emerging trends more widely. This leads to the suggestion of some possible routes to re-imagining geometry education in schools.

1. School geometry education

In his 1982 address to the UK Mathematical Association as its President, the prominent mathematician Sir Michael Atiyah argued that “geometry is not so much a branch of mathematics as a way of thinking that permeates all branches” (Atiyah, 1982, p.184). In much the same way, geometry education in school, I would argue, is a central component of the teaching and learning of school mathematics. This is because geometry education supports mathematical thinking in ways that permeates all aspects of school mathematics (whereby learners can encounter, and be involved in, mathematical reasoning, argumentation and proof) as well as other school subjects from art and geography to the sciences.

Despite this key role played by geometry education in school mathematics (and the wider school curriculum), designing geometry education for school students has been, and continues to be, a challenge (Jones, 2000; Jones & Mooney, 2003; Mammana & Villani, 1998; Sinclair et al., 2016). Indeed, the staff of one well-known mathematics curriculum project of the 1960s reflected “Of all the decisions one must make in a curriculum development project with respect to choice of content, usually the most controversial and the least defensible is the decision about geometry (CSMP Project staff, 1971, p.281). Even at that time, there was too much interesting geometry that could be included in school geometry and now there is more. Yet the recognition of the importance of data handling and, more recently, computational

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mathematics, has resulted in a further ‘squeeze’ on what geometry can be included within school mathematics, while, at the same time, increasing the demands on teachers of teaching a wider curriculum.

A central challenge in geometry education is the ‘dual’ nature of geometry in that it is both one of the most practical and reality-related components of mathematics, and, at the same time, an important area of mathematical theory. This means, on the one hand, that geometry can be seen all around – not only in the ways in which it is widely utilised in art, design, architecture, engineering, and so on, but also in nature and the natural world. Simultaneously, on the other hand, geometry is a theoretical field in which geometers and other mathematicians (together with cosmologists and other scientists) work with hypothetical objects in n-dimensional space (using, amongst other things, mathematical visualisation techniques only possible with high-powered computers). Likewise, the work of Scholze illustrates how proposing what he calls ‘perfectoid spaces’ is dramatically expanding the spectrum of methods in mathematics (Betti, 2018) while Uhlenbeck’s founding of modern geometric analysis has produced some of the most dramatic advances in mathematics in recent decades.

While the fount of advances in mathematics, in schools this ‘dual’ nature of geometry presents challenges for teaching and learning geometry. Moreover, there are additional challenges facing school geometry education (and the wider education system). In what follows, I argue that rising to the challenges provides opportunities to re-imagine geometry education in schools.

2. Challenges in school geometry education: research examples

The wealth of research on the range of challenges in geometry education in schools is summarised in recent reviews, including, for example, those by Jones & Tzekaki (2016), Sinclair et al. (2016), and the relevant chapters in Watson et al. (2013).

Such existing research confirms the ways in which visuospatial reasoning in geometry education, and more widely, is a vital component of learners’ successful mathematical thinking and problem solving. The challenge comes in developing classroom experiences that involve dynamic and haptic forms of visuospatial reasoning. This entails moving away from primarily paper-based static representations and making use both of wider (including non-‘Western’) geometric ways of making sense of the world as well as more creative use of digital technologies in providing more opportunities for students to create and reason with dynamic imagery.

From existing research, it is clear that encouraging students to engage in more gesturing and diagramming is vital in school geometry and more widely in school mathematics. The challenges are in identifying the types of gesturing that might be helpful to student learning of geometry and ways of increasing students' experiences with diagramming that both help students develop stronger diagramming practices and provide contexts in which diagramming is valued and productive. Here the increased availability of touchscreen technology, where gesturing and diagramming can reach a new form of interplay, are areas of current research investigation.

Within these wider challenges there continue to be the challenges of teaching geometrical ideas of symmetry, invariance, transformation, similarity, and congruence, and the extent to which the teaching of such ideas is across both 2D (plane) and 3D (solid) geometry (and perhaps other geometries such as spherical or projective geometry). Research continues to focus on the challenge of identifying ways of developing the capabilities of students (at different ages) with geometric defining, reasoning, and proving – including, for example, ways of ensuring the interrelationships are more explicit between geometric measurement formulae and related geometrical theorems.

Three recent projects with which I have been involved illustrate the range of the challenges and the way in which research is responding.

In the first project, the focus was on students' visuospatial reasoning across primary and lower secondary school ages with students tackling problems involving 2D representation of 3D shapes (see, for example, Fujita, et al., online first). In one component, and with data from a total of 1357 students from grade 4 (aged 10) to grade 9 (age 15), the project examined how the students visualised shapes in given geometric diagrams and made use of properties of shapes to reason to solve each problem. The findings were that using either spatial visualisation or property-based spatial analytic reasoning was not enough for the problems that required more than one step of reasoning, but also that these two skills needed to be harmonised by domain-specific knowledge in order for students to overcome the perceptual appearance (or 'look') of the given diagram. This suggests both primary and secondary school students would benefit from more opportunities to exercise not only their spatial reasoning skills but also consolidate and use their existing domain-specific knowledge of geometry for productive reasoning in geometry.

Given the well-established difficulties that students at the secondary school level (and beyond) have in learning about proof and proving, the second project involved designing, and using, a web-based learning support system (available in Japanese, English and Chinese) intended for students who are just starting to tackle deductive proving in geometry using congruency (see, for example, Miyazaki, et al., 2017). The system was designed to enable students to access the study of proofs in geometry by tackling proof problems where they can ‘drag’ sides, angles and triangles from the figural diagram of the problem to on-screen cells within a ‘flow-chart’ proof format. When doing so, the system automatically converts the figural elements to their symbolic form and identifies any of four kinds of errors in the learners’ proof attempts, providing relevant feedback on-screen. To date, the web-based learning support system has been used to help students develop the strategic knowledge of how to construct alternative proofs to the same geometric problem and how to avoid circular arguments in geometric proofs.

The third project focused on formulating, and testing, a set of task design principles for supporting students’ heuristic refutation (i.e. the re-vising of conjectures/proofs through addressing counterexamples) when tackling geometry tasks designed to be tackled, in part, by using dynamic geometry software (see, for example, Komatsu & Jones, 2019). Three design principles were established: using tasks whose conditions are purposefully implicit; providing tools that enhance the production of counterexamples; and increasing students’ recognition of contradictions. To date, empirical studies have showed how using tasks designed using these principles have enabled students to engage successfully in heuristic refutation.

3. Challenges to school geometry, and to education more widely

The challenges noted above are, to a certain extent, primarily ‘internal’ to geometry education in that the challenges derive, for the most part, from current classroom practices and current curriculum specifications. Although no doubt the challenges of, for example, developing students’ visuospatial reasoning, their gesturing and diagramming, and their deductive proving and refuting, relate to concerns in mathematics education that are wider than solely geometry education, there are emerging wider challenges that demand new attention and new approaches.

One challenge is the balance of attention to various components of mathematics in students’ curriculum experience. For example, in the UK there have been education policy developments over recent years that are, or are threatening to, narrow students’ experience of geometry education. In 2013, the UK Government developed guidelines on what are called ‘core mathematics’ qualifications (these are post-16 mathematics qualifications that provide preparation for the quantitative skills that students need for many university courses that do not require full preparation in mathematics). The ‘core mathematics’ qualifications are expected to draw on mathematical content that encompasses number, algebra, probability and statistics; geometry (and measures) is notable by its absence. More recently, in 2018, the UK Government published draft new Early Learning Goals (ELGs) as part of a review of the UK education Early Years Foundation Stage (EYFS) for students up to five years of age. The new draft ELGs omitted the ‘shape, space and measures’ (i.e. geometry) target (as well as omitting any mention of problem-solving and reasoning). These decisions by the UK Government to exclude geometry (and measures) from the ‘core mathematics’ qualifications and from the draft new Early Learning Goals demon-strates a lack of appreciation of the vital importance of geometry in a balanced education in mathematics and is a significant challenge to the continuing importance of geometry (and measures) education. This issue of the balance of attention to various components of mathematics in students’ curriculum experience has implications for geometry education.

While the balance of attention to various components of mathematics in students’ curriculum experience is a somewhat ‘internal’ challenge for geometry education, there are three wider challenges for geometry education (and for education more widely) to be considered.

One challenge is the dramatic increase in the worldwide prevalence of myopia (nearsightedness in which nearby objects look in focus but objects farther away are blurry). On current rates of growth, more than half of the population in over half of all countries across the world are predicted to be myopic by 2050. Research shows a strong link between myopia and education, including that longer duration of education is a causal risk factor for myopia (Wong, et al., 2020). On the more positive side, recent studies are reporting that the time children spend engaged in outdoor activities is negatively associated with their risk of myopia and that greater time spent outdoors is associated with a significantly lower myopia prevalence and reduced risk of myopia onset in childhood (op cit.). This issue of greater time spent outdoors being associated with a significantly lower myopia prevalence and reduced risk has implications for geometry education.

A challenge that is almost existential is the incidence of the worldwide Coronavirus pandemic that has led to schools being closed and school education being curtailed. Across the world, parents have been asked to home-school their children and teachers have been asked to provide online learning. Whether the pandemic (if, or when, it recedes) will have changed everything, or changed nothing, remains to be seen. Perhaps there will be moves to change the way that education is structured or promote the idea that education should be completely online. Such moves to restructure education, or move it entirely online, have implications for geometry education.

The challenge that is more certainly existential is climate change, the long-term alteration of temperature and typical weather patterns that is leading to weather extremes, wildfires, the expansion of deserts, sea-level rise, and more – all of which is threatening the continuing existence of various ecosystems around the world. Responding to this challenge is likely to entail raising awareness of the mathematics behind climate science. This issue raising awareness of the mathematics behind climate science of has implications for geometry education.

4. Possible ways of re-imagining school geometry education

Much existing research in geometry education has, as noted above, focused to a certain extent, on issues that are 'internal' to geometry education and, as such, reflect, for the most part, current classroom practices and current curriculum specifications. Re-imagining school geometry education is not something that is likely to be easy or straightforward, and the nature of the challenges is likely to influence the possible ways of re-imagining school geometry education.

One route to re-imagining school geometry education is through more inventive and creative use of digital technology. Use of digital technology has the potential to enable mathematics teachers to use or create more interactive, engaging and flexible learning materials for geometry education. This might necessitate changes being made to the school mathematics curriculum so that it might be more possible to utilise more fully the potential of digital technologies for geometry education. Issues remain, however, about the extent to which the digital 'revolution' has been overhyped as something that can cure all teaching ills, and equity issues remain of key importance such as the extent to which all teachers and learners of mathematics have access to the latest digital technologies (Jones, 2020).

Given the increase in the worldwide prevalence of myopia, and the evidence that greater time spent outdoors is associated with a significantly lower prevalence and reduced risk, then greater use of outside space for geometry education is another route to re-imagining school geometry education. Greater time spent outdoors is related to ways of reducing the threat of the Coronavirus pandemic in that the virus contagion is reduced outdoors.

The existential challenge of climate change, and the likely response across education of devising way of raising awareness of the mathematics behind climate science suggests an increased attention to geometrical modelling within geometry education. In this, diagrams, and diagramming, are likely to play an important role as geometric figures are seen as geometric models of situations (both real and imagined). The modelling of problems geometrically may motivate students and provide them with resources to develop solutions. This could be a way of re-imagining geometry education in schools that values students' intuitive imaginings.

5. Summary and conclusion

Geometry is full of intriguing, and frequently extendable, possibilities. As the mathematician and educator Sawyer (1977, p.12) put it in his elegant essay on geometry education "In the subject matter of geometry we suffer from an embarrassment of riches. We have so many tools for the discussion of geometric problems – Euclid, transformations, coordinates, matrices, calculus…". In some ways, this 'embarrassment of riches' is itself a challenge for geometry education because within a finite time for geometry in the school schedule, not all these riches can be included.

Notwithstanding the issue of this 'embarrassment of riches', there are, as set out above, other challenges for research in geometry education. Of the ones that are, to a certain extent, primarily 'internal' to geometry education and derive, for the most part, from current classroom practices and current curriculum specifications, these are being addressed by existing research. It is where much of my own research is directed.

Beyond such 'internal' challenges, there are greater challenges. One, the issue of the balance of attention to various components of mathematics in students' curriculum experience, is also somewhat 'internal' but it is one what provides opportunities for innovative research. The other, rather substantial challenges of dramatic worldwide increases in rates of myopia, of the incidence of the worldwide Coronavirus pandemic, and the existential threat posed by climate change demand new thinking.

This new thinking on ways of re-imagining geometry education are, I would say, likely to entail more inventive and creative use of digital technology, greater use of outside space for geometry education, and increased attention to geometrical modelling within geometry education. Re-imagining geometry education in schools is going to take collective efforts. This paper is an invitation to join this collective effort to re-imagine geometry education in schools in ways that value intuitive imaginings.

Dedication: In remembrance of the sad passing in recent times of geometers Michael Atiyah (1929 -2019), John Conway (1937- 2020), David Henderson (1939-2018).

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