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Rafael Kawka

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# Integrated Modified OLS and Fixed-$b$ Inference for Seasonally Cointegrated Processes

Rafael Kawka

Department of Statistics, TU Dortmund University

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Many economic time series exhibit persistent seasonal patterns. One approach to model this phenomenon is given by models including seasonal unit roots and, if several time series are considered jointly, seasonal cointegration. For quarterly time series, e.g., unit roots may be present at frequencies $\pm\pi/2$ and $\pi$, in addition to the "standard unit root" at frequency zero. Gregoir (2010) has extended the fully modified OLS estimator of Phillips and Hansen (1990) from the cointegrating regression to the seasonally cointegrating regression case. In this paper, we have a similar agenda, in that we undertake the corresponding extension for the IM-OLS estimator of Vogelsang and Wagner (2014). The benefit of the seasonal IM-OLS estimator, or SIM-OLS estimator, is that it forms the basis not only for asymptotic standard inference but also allows for fixed-$b$ inference. The paper furthermore proposes a test for seasonal cointegration at all unit root frequencies. Note here that the cointegrating spaces in general differ across frequencies and have to be estimated separately for each frequency. The theoretical analysis is complemented by a simulation study.

**JEL Classification**: C12, C13, C22, C32

**Keywords**: Seasonal Cointegration, Seasonal Unit Roots, SIM-OLS, Fixed-$b$ Inference, Cointegration Test

## 1. Introduction

Since the seminal paper of Hylleberg et al. (1990) cointegration at frequencies different from zero gained popularity in the econometrics literature. Many theoretical approaches have been developed by extending well known results for the zero frequency case. For instance, Johansen and Schaumburg (1999) developed a cointegration theory in a vector autoregressive framework and Gregoir (1999a,b) introduced a vector error correction model representation with seasonal error correction terms. More recently, Bauer and Wagner (2002, 2005, 2012) developed a state space representation for seasonal unit root processes and showed that the state space framework is suitable for seasonal cointegration analysis. What these approaches all have in common is the fact that they consist of parametric models for which a suitable parameterization has to be found and which are estimated by maximum likelihood techniques. Nonparametric approaches are also known from the literature on cointegration at frequency zero, where only the cointegrating vectors are estimated and all other properties of the underlying processes are regarded as nuisance parameters. Such a method, the so-called fully modified OLS estimator of Phillips and Hansen (1990) was generalized by Gregoir (2010) for seasonally cointegrating regressions. In contrast to the parametric models, this so-called SFM-OLS estimator is based on the nonparametric estimation of the spectral density at the considered frequency. This is usually done using nonparametric kernel density estimators, which in turn require a suitable choice of a kernel function and bandwidth.

Vogelsang and Wagner (2014) presented an estimator, the integrated modified OLS estimator, for the parameters in a cointegrating regression at zero frequency, that does not require the estimation of a spectral density or any other tuning parameters. In this paper we generalize this estimator to cointegrating regressions at arbitrary frequency and label it seasonally integrated modified OLS estimator, SIM-OLS in short.

However, inference based on the SIM-OLS estimator still requires the estimation of the spectral density at the considered frequency. Standard asymptotic theory for kernel estimators of the spectral density requires the bandwidth, $M$ say, to grow with sample size $T$ such that $M/T \to 0$ as $T \to \infty$. Although such assumptions yield consistent spectral density esimators the impact of the bandwidth is not captured in finite sample distributions. Bunzel (2006) and Vogelsang and Wagner (2014) extended asymptotic fixed-$b$ theory from the stationary framework in Kiefer and Vogelsang (2005) to the cointegration framework at frequency zero. Under fixed-$b$ theory the bandwidth $M$ is supposed to be a fixed portion of the sample size, i.e. $M = bT$ for $b \in (0, 1]$ which does not depend on the sample size. We follow this route and develop useful fixed-$b$ results for tests based on SIM-OLS.

In this paper we also present a KPSS-type (cf. Kwiatkowski et al., 1992) test for seasonal cointegration at an arbitrary frequency. This test is similar to the zero frequency cointegration test introduced in Shin (1994) and its extension to seasonal cointegration developed in Gregoir (2010). We show that this test is consistent under standard asymptotic theory but not under fixed-$b$ asymptotics.

The theoretical analysis of the paper is complemented by a simulation study to assess the finite sample performance of the SIM-OLS estimator as well as the tests based upon it. The performance is benchmarked against the results obtained with the SFM-OLS estimator of Gregoir (2010) and the textbook OLS estimator.

The paper is organized as follows: In Section 2 we present the setting, the required assumptions and introduce the SIM-OLS estimator. Section 3 deals with standard inference based on the the new estimator whereas we discuss fixed-$b$ inference in Section 4. In Section 5 we present a residual based test for seasonal cointegration. The finite sample performance of the SIM-OLS estimator and of the resultant test statistics is assessed in Section 6. Section 7 concludes the paper. All proofs are relegated to Appendix A. In Appendix B we recall some basic properties of the complex normal distribution.

Throughout the paper we use the following notation: Weak convergence is denoted by $\Rightarrow$ and convergence to zero in probability is signified by $o_{\mathbb{P}}(1)$, both for $T \to \infty$ if not stated otherwise. Boundedness in probability of a random variable is signified by $\mathcal{O}_{\mathbb{P}}(1)$ and the expected value is denoted by $\mathbb{E}$. The integer part of a real number $x$ is given by $[x]$ and the modulus of a complex number $x = \mathrm{Re}(x) + i\,\mathrm{Im}(x)$ is denoted by $|x|$. We use the notation $\|x\|$ to signify the Frobenius norm. For a (possibly complex valued) matrix $A$ we denote its transpose, complex conjugate and Hermitian transpose by $A'$, $\overline{A}$ and $A^*$, respectively. The Kronecker product is denoted by $\otimes$. For integrals the range of integration is the unit interval if not stated otherwise.

For $x_t$ with $t = 1, \dots, T$ we use $L$ to denote the lag operator, i.e. $Lx_t = x_{t-1}$ and for some frequency $\omega \in (-\pi, \pi]$ we denote the seasonal partial sum by $S^x_{t,\omega} = e^{-i\omega t} \sum_{k=1}^{t} e^{i\omega k} x_k$ and the seasonal first difference by $\nabla_\omega x_t = x_t - e^{-i\omega} x_{t-1}$.

## 2. Cointegrating Regressions and the SIM-OLS Estimator

Consider the scalar time series process $\{y_t\}_{t\in\mathbb{N}}$ generated by

$$y_t = d_t'\delta + x_t'\beta + u_t, \tag{1}$$

where $d_t$ is a $q$-dimensional deterministic component and $\{x_t\}_{t\in\mathbb{N}}$ is a $k$-dimensional vector process that satisfies

$$x_t = e^{-i\omega} x_{t-1} + v_t \tag{2}$$

for some $\omega \in (-\pi, \pi]$, with initial value $x_0$ being $\mathcal{O}_{\mathbb{P}}(1)$. The precise assumptions concerning the deterministic components, the regressors and the regression error processes in the cointegrating regression model (1) that are sufficient to obtain the key theoretical results in this paper are as follows:

**Assumption 1.** The deterministic component can be written as $d_t = f_t e^{-i\omega t}$ for some deterministic sequence $\{f_t\}_{t\in\mathbb{N}}$ and there exists a series of scaling matrices $G_D = G_D(T) \in \mathbb{R}^{q\times q}$ and a $q$-dimensional vector of càdlàg functions $D$ such that

$$\int_0^r D(s)D(s)'\, ds < \infty$$

and

$$\lim_{T\to\infty} T^{-1} G_D^{-1} \sum_{j=1}^{[rT]} f_j = \int_0^r D(s)\, ds$$

for all $r \in (0, 1]$.

**Assumption 2.** The process $\{\eta_t\}_{t\in\mathbb{Z}} = \{[u_t, v_t']'\}_{t\in\mathbb{Z}}$ is a zero mean, real valued stationary process if $\omega \in \{0, \pi\}$ and a zero mean, complex valued stationary process, if $\omega \in (0, \pi)$. Furthermore, $\eta_t$ has an infinite moving average representation of the form

$$\eta_t = \Psi(L)\varepsilon_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},$$

with $\sum_{j=0}^{\infty} j\|\psi_j\| < \infty$ and $\det\{\Psi(e^{i\omega})\} \neq 0$. The sequence $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a martingale difference sequence with respect to its canonical filtration $\mathcal{F}_t = \sigma\{\varepsilon_{t-j}, j \in \mathbb{N}_0\}$ satisfying $\mathbb{E}(\varepsilon_t\varepsilon_t'|\mathcal{F}_{t-1}) = \Sigma_\varepsilon < \infty$ and $\sup_t \mathbb{E}(\|\varepsilon_t\|^{2+\alpha}|\mathcal{F}_{t-1}) < \infty$ with probability one for some $\alpha > 0$.

**Remark 1.** For the construction of the SIM-OLS estimator as well as for standard inference which is discussed in Section 3 we can relax the dimensional restriction and allow $y_t$ to be multivariate. Nevertheless, this restriction is mandatory to develop a fixed-$b$ theory.

The stacked process $\{[y_t, x_t']'\}_{t\in\mathbb{N}}$ usually results from filtering out all unit roots except for $e^{i\omega}$ from a multiple frequency integrated process. An $s$-dimensional process $\{z_t\}_{t\in\mathbb{N}}$ is said to be multiple frequency integrated at frequencies $\omega_1, \ldots, \omega_l$ if

$$\prod_{k=1}^{l} \nabla_{\omega_k} z_t = \mu_t + \xi_t,$$

with initial values being $\mathcal{O}_{\mathbb{P}}(1)$ and where $\{\mu_t\}_{t\in\mathbb{N}}$ is a deterministic sequence and $\{\xi_t\}_{t\in\mathbb{Z}}$ is a stationary process with Wold representation $\xi_t = C(L)\zeta_t$. Thereby, the sequence $\{\zeta_t\}_{t\in\mathbb{Z}}$ is white noise and the moving average polynomial $C(z) = \sum_{j=0}^{\infty} C_j z^j$ satisfies $\sum_{j=0}^{\infty} j\|C_j\| < \infty$ and $C(e^{i\omega_k}) \neq 0$ for all $k = 1, \ldots, l$. The set $\mathcal{U} = \{\omega_1, \ldots, \omega_l\}$ is called *unit root structure.*

**Remark 2.** Every multiple frequency integrated process $\{z_t\}_{t\in\mathbb{N}}$ with unit root structure $\mathcal{U} = \{\omega_1, \ldots, \omega_l\}$ can be decomposed as

$$z_t = \nu_1 z_{t,1} + \ldots + \nu_l z_{t,k}, \tag{3}$$

where $\nu_1, \ldots, \nu_l$ are complex numbers different from zero and where for every $k = 1, \ldots, l$ the process $\{z_{t,k}\}_{t\in\mathbb{N}}$ is defined via

$$z_{t,k} = \prod_{\substack{j=1 \\ j\neq k}}^{l} \nabla_{\omega_j} z_t. \tag{4}$$

Clearly, each $\{z_{t,k}\}_{t\in\mathbb{N}}$ is integrated solely at frequency $\omega_k$. The stacked process $\{[y_t, x_t']'\}_{t\in\mathbb{N}}$ in (1) can be associated with a filtered process of the form (4).

**Remark 3.** If the stacked process $\{[y_t, x_t']'\}_{t\in\mathbb{N}}$ in (1) is derived by filtering out all unit root frequencies except for $\omega \in (0, \pi)$ from a real valued multiple frequency integrated process $\{z_t\}_{t\in\mathbb{N}}$ then $v_t = \nabla_\omega x_t$ is also real valued. Nevertheless, to avoid notational complexity we assume that all components of $\eta_t$ are complex valued. This simplification has no impact on the results presented in this paper.

An important example for multiple frequency unit root processes are *seasonally integrated* processes which are observed at $S$ equidistant periods per season. For instance, the the quarterly integrated process, $\{z_t\}_{t\in\mathbb{Z}}$ say, has unit root structure $\mathcal{U} = \{0, \pi, \pm\pi/2\}$, i.e. it is defined by

$$(1 - L)(1 + L)(1 + iL)(1 - iL)z_t = \xi_t. \tag{5}$$

The first factor is the usual first difference filter that corresponds to the long-run frequency $\omega_1 = 0$ whereas the remaining three factors correspond to the seasonal frequencies. In particular, the factor $1 + L$ corresponds to the so-called biannual or Nyquist frequency $\omega_2 = \pi$ and the last factors correspond to the complex conjugate pair of annual frequencies $\omega_3 = \pi/2$ and

$\omega_4 = -\pi/2$. By filtering out all unit roots except those at the zero, the biannual and the annual frequencies, respectively, one defines

$$z_{t,1} = (1+L)(1+iL)(1-iL)z_t = z_t + z_{t-1} + z_{t-2} + z_{t-3},$$
$$z_{t,2} = (1-L)(1+iL)(1-iL)z_t = z_t - z_{t-1} + z_{t-2} - z_{t-3},$$
$$z_{t,3} = (1+L)(1-L)(1-iL)z_t = z_t - iz_{t-1} - z_{t-2} + iz_{t-3},$$
$$z_{t,4} = (1+L)(1-L)(1+iL)z_t = z_t + iz_{t-1} - z_{t-2} - iz_{t-3}.$$

Clearly, since

$$(1-L)z_{t,1} = (1+L)z_{t,2} = (1+iL)z_{t,3} = (1-iL)z_{t,4} = \xi_t,$$

it holds that $\{z_{t,k}\}_{t\in\mathbb{N}}$ has unit root structure $\mathcal{U}_k = \{\omega_k\}$ for $k = 1, \ldots, 4$. Furthermore,

$$z_t = \frac{1}{4}\left(z_{t,1} + z_{t,2} + z_{t,3} + z_{t,4}\right),$$

i.e. the coefficients $\nu_k$ in (3) are given by $1/4$ for $k = 1, \ldots, 4$.

A multiple frequency integrated process $\{z_t\}_{t\in\mathbb{N}}$ with unit root structure $\mathcal{U}$ is (statically) cointegrated if there exists $\gamma \in \mathbb{C}^s$, called cointegrating vector, such that the process $\{\gamma' z_t\}_{t\in\mathbb{N}}$ has unit root structure $\mathcal{U}_\gamma \subsetneq \mathcal{U}$. If for some $\omega \in \mathcal{U}$ it holds that $\omega \notin \mathcal{U}_\gamma$ then we say that $\{z_t\}_{t\in\mathbb{N}}$ is *seasonally cointegrated* at frequency $\omega$. The stacked process $\{[y_t, x_t']'\}_{t\in\mathbb{N}}$ in model (1) has unit root structure $\mathcal{U} = \{\omega\}$ and is seasonally cointegrated with cointegrating vector $\gamma = [1, -\beta']'$.

If $\{z_t\}_{t\in\mathbb{N}}$ has a representation of the form (3) then $\gamma \in \mathbb{C}^s$ is a cointegrating vector that annihilates the unit root at frequency $\omega$ if and only if $\gamma' C(e^{i\omega_k}) = 0$. This follows from the so-called Beveridge-Nelson decomposition. In particular, it holds that

$$z_{t,k} = \mu_{t,k} + e^{-i\omega k}(z_{t,0} + \tilde{\xi}_{0,k}) + C(e^{i\omega})S^{\zeta}_{\omega,t} - \tilde{\xi}_{t,k},$$

where $\mu_{t,k}$ is deterministic and results from applying the filter on the right hand side of (4) on the deterministic component $\mu_t$ and where $\{\tilde{\xi}_{t,k}\}_{t\in\mathbb{Z}}$ is a stationary process (see Kawka, 2020, for further details). Hence, it holds that $\{\gamma' z_{t,k}\}_{t\in\mathbb{N}}$ is stationary if and only if $\gamma' C(e^{i\omega}) = 0$ and from (3) we deduce that this is equivalent to $\omega \notin \mathcal{U}_\gamma$, where $\mathcal{U}_\gamma$ denotes the unit root structure of $\{\gamma' z_t\}_{t\in\mathbb{N}}$.

**Remark 4.** The left null space of the matrix $C(e^{i\omega})$ contains all cointegrating vectors at frequency $\omega$ and is therefore called *cointegrating space*. In general, the null spaces of $C(e^{i\omega_1})$ and $C(e^{i\omega_2})$ differ if $\omega_1 \neq \omega_2$. Hence, the cointegrating spaces have to be considered seperately for each frequency.

If a multiple frequency integrated process $\{z_t\}_{t\in\mathbb{N}}$ is real valued then $C(e^{i\omega})$ is complex valued for $\omega \in (0, \pi)$ implying that the corresponding cointegrating vector is usually also complex valued. From $C(e^{-i\omega}) = \overline{C(e^{i\omega})}$ we deduce that if $\{z_t\}_{t\in\mathbb{N}}$ is seasonally cointegrated at frequency $\omega$ with cointegrating vector $\gamma$ then $\{z_t\}_{t\in\mathbb{N}}$ is also seasonally cointegrated at frequency $-\omega$ with cointegrating vector $\overline{\gamma}$. Since complex quantities are hard to interpret in applied research the concept of polynomial cointegration is usually considered for frequencies $\omega \notin \{0, \pi\}$. An $s$-dimensional real valued process $\{z_t\}_{t\in\mathbb{N}}$ with unit root structure $\mathcal{U}$ such that $\omega \in \mathcal{U}$ (and $-\omega \in \mathcal{U}$) is said to be polynomially or dynamically seasonally cointegrated at frequency $\omega$ if there exists a vector polynomial of degree one, $p(L) = \gamma_1 + \gamma_2 L$ with $\gamma_1, \gamma_2 \in \mathbb{R}^s$ say, such that $\{p(L)' z_t\}_{t\in\mathbb{N}}$ has unit root structure $\mathcal{U}_\gamma \subsetneq \mathcal{U}$ with $\omega \notin \mathcal{U}_\gamma$ (and, consequently, $-\omega \notin \mathcal{U}_\gamma$). From Bauer and Wagner (2012, Theorem 4) it follows that a polynomial of degree one is always sufficient to jointly annihilate the unit root pair $\pm\omega$ from $\mathcal{U}$.

**Remark 5.** Let a real valued process $\{z_t\}_{t\in\mathbb{N}}$ be dynamically seasonally cointegrated at frequency $\omega \in (0, \pi)$ with cointegrating polynomial $p(L) = \gamma_1 + \gamma_2 L$. Then, by Gregoir (2010), the process $\{\nabla_{-\omega} z_t\}_{t\in\mathbb{N}}$ is statically cointegrated with cointegrating vector $\gamma = \gamma_1 + e^{i\omega}\gamma_2$. Hence, there is a one-to-one relationship between dynamic cointegration of the real valued process $\{z_t\}_{t\in\mathbb{N}}$ and static cointegration of the complex valued process $\{\nabla_{-\omega} z_t\}_{t\in\mathbb{N}}$.

**Remark 6.** Gregoir (2010) also pointed out that if $\{z_t\}_{t\in\mathbb{N}}$ is dynamically seasonally cointegrated with cointegrating polynomial $p(L) = \gamma_1 + \gamma_2 L$ then $\tilde{p}(L) = -\gamma_2 + (\gamma_1 + 2\cos\omega\gamma_2)L$ is also a cointegrating polynomial. Consequently, every linear combination of $p(L)$ and $\tilde{p}(L)$ is a cointegrating polynomial. To overcome this deficit one can normalize the polynomial by setting the first components of $\gamma_1$ and $\gamma_2$ to unity and zero, respectively. Since polynomial seasonal cointegration of $\{z_t\}_{t\in\mathbb{N}}$ is equivalent to static seasonal cointegration of the filtered process $\{\nabla_{-\omega} z_t\}_{t\in\mathbb{N}}$ with cointegrating vector $\gamma = \gamma_1 + e^{i\omega}\gamma_2$ (cf. Remark 5) such normalization corresponds to normalizing the cointegrating vector $\gamma$ by setting the first component to unity. This normalization is applied in model (1).

It is well known that under Assumption 2 the process $\{\eta_t\}_{t\in\mathbb{Z}}$ possesses a finite spectral density at all frequencies. For $\omega \in (-\pi, \pi]$ we define $\Omega_\omega$ as $2\pi$ times the spectral density at $\omega$, i.e.

$$\Omega_\omega = \begin{bmatrix} \Omega_{\omega,uu} & \Omega_{\omega,uv} \\ \Omega_{\omega,vu} & \Omega_{\omega,vv} \end{bmatrix} = \sum_{j=-\infty}^{\infty} e^{-i\omega j}\mathbb{E}(\eta_t \eta_{t+j}^*), \tag{6}$$

where $\Omega_{\omega,uu}$ is a real number (regardless of $\omega$) and $\Omega_{\omega,uv} = \Omega_{\omega,vu}^*$. Furthermore, it holds that the matrix $\Omega_\omega$ is positive definite, that $\Omega_{\omega,uu}$ is a strictly positive real number and that the block matrix $\Omega_{\omega,vv}$ is positive definite. Hence, the process $\{x_t\}_{t\in\mathbb{N}}$ is integrated in all components but not cointegrated (at frequency $\omega$).

From Chan and Wei (1988, Theorem 2.2) or Kawka (2020, Theorem 1) it is known that under Assumption 2 the following functional central limit theorem holds:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[rT]} e^{i\omega t}\eta_t \Rightarrow \tau_\omega B(r) = \Omega_\omega^{1/2}\tau_\omega W(r), \quad r \in (0, 1]. \tag{7}$$

The definition of $\tau_\omega$ and $W$ depends on the considered frequency $\omega$. If $\omega \in \{0, \pi\}$ it holds that $\tau_\omega = 1$ and $W$ is a real valued $(k+1)$-dimensional standard Wiener process. On the other hand if $\omega \in (0, \pi)$ we have $\tau_\omega = 1/\sqrt{2}$ and $W$ is a complex valued standard Wiener process, i.e. $W(r) = W_{\mathrm{Re}}(r) + iW_{\mathrm{Im}}(r)$ with two independent (real valued) standard vector Wiener processes $W_{\mathrm{Re}}$ and $W_{\mathrm{Im}}$. In any case we label the first component of $W$ by $W_{u\cdot v}$ and the remaining $k$-dimensional vector process by $W_v$.

**Remark 7.** If $\omega \in (0, \pi)$ it holds for every $r \in (0, 1]$ that the random vector $\tau_\omega W(r)$ is circularly symmetrically Gaussian with covariance matrix given by $rI_k$. Consequently, $B(r)$ is also circularly symmetrically Gaussian with covariance matrix given by $r\Omega_\omega$. For further details on complex normal distributions we refer to Appendix B.

We use $\Omega_\omega^{1/2}$ of the Cholesky form

$$\Omega_\omega^{1/2} = \begin{bmatrix} \sigma_{\omega,u\cdot v} & \lambda_{\omega,uv} \\ 0_{k\times 1} & \Omega_{\omega,vv}^{1/2} \end{bmatrix},$$

with $\sigma^2_{\omega,u\cdot v} = \Omega_{\omega,uu} - \Omega_{\omega,uv}\Omega^{-1}_{\omega,vv}\Omega_{\omega,vu}$ and $\lambda_{\omega,uv} = \Omega_{\omega,uv}\Omega^{-1/2^*}_{\omega,vv}$ to rewrite the limiting process in (7) as follows:

$$B(r) = \begin{bmatrix} B_u(r) \\ B_v(r) \end{bmatrix} = \begin{bmatrix} \sigma_{\omega,u\cdot v} W_{u\cdot v}(r) + \lambda_{\omega,uv} W_v(r) \\ \Omega^{1/2}_{\omega,vv} W_v(r) \end{bmatrix}.$$

Defining $B_{u\cdot v}(r) = \sigma_{\omega,u\cdot v} W_{u\cdot v}(r)$ we can further express the process $B_u$ as

$$\begin{aligned} B_u(r) &= B_{u\cdot v}(r) + \Omega_{\omega,uv}\Omega^{-1/2*}_{\omega,vv} W_v(r) \\ &= B_{u\cdot v}(r) + \Omega_{\omega,uv}\Omega^{-1/2*}_{\omega,vv}\Omega^{-1/2}_{\omega,vv}\Omega^{1/2}_{\omega,vv} W_v(r) \\ &= B_{u\cdot v}(r) + \gamma' B_v(r), \end{aligned}$$

with $\gamma = (\Omega_{\omega,uv}\Omega^{-1}_{\omega,vv})' = \overline{\Omega}^{-1}_{\omega,vv}\overline{\Omega}_{\omega,vu}$. Using this decomposition we can state the limiting distribution of the OLS estimator of $\theta = [\delta', \beta']$ in (1). Define $\tilde{x}_t = [d_t', x_t']'$ and let $y = [y_1, \ldots, y_T]'$, $\tilde{X} = [\tilde{x}_1, \ldots, \tilde{x}_T]'$. Then the OLS estimator is given by $\hat{\theta} = (\tilde{X}^* \tilde{X})^{-1} \tilde{X}^* y$. To obtain the limiting distribution of the OLS estimator we define the scaling matrix

$$A_{OLS} = \begin{bmatrix} T^{-1/2} G_D & 0_{q\times k} \\ 0_{k\times q} & T^{-1} I_{k\times k} \end{bmatrix}.$$

Gregoir (2010, Proposition 4.2) and Kawka (2020, Theorem 3 and Corollary 4) generalize the results of Phillips and Durlauf (1986, Theorem 4.1) and Park and Phillips (1988, Theorem 3.1). In particular, under the Assumptions 1 and 2 it holds that

$$\begin{aligned} A^{-1}_{OLS}(\hat{\theta} - \theta) &\Rightarrow \left( \int \overline{J(s)} J(s)' \, ds \right)^{-1} \left( \int \overline{J(s)} \, dB_u(s) + \Delta_{\omega,uv} \right) \\ &= \left( \int \overline{J(s)} J(s)' \, ds \right)^{-1} \left( \int \overline{J(s)} \, dB_{u\cdot v}(s) + \int \overline{J(s)} \, \gamma' dB_v(s) + \Delta_{\omega,uv} \right), \end{aligned} \tag{8}$$

where $J(s) = [D(s)', B_v(s)']'$ and $\Delta_\omega = \sum_{j=0}^{\infty} e^{-i\omega j} \mathbb{E}(\eta_t \eta^*_{t+j})$.

If the process $\{u_t\}_{t\in\mathbb{Z}}$ is intertemporarily uncorrelated with $\{v_t\}_{t\in\mathbb{Z}}$, it holds that $\lambda_{\omega,uv}$ and $\Delta_{\omega,uv}$ are equal to zero. The former implies uncorrelatedness of $B_u$ and $B_v$, which implies, after conditioning on $B_v$, that the limiting distribution is circularly symmetrically normal. In the general case, however, conditioning on $B_v$ does not lead to a nuisance parameter free limiting distribution. This is called *second order bias*, a term proclaimed by Phillips and Hansen (1990) in their seminal paper. To obtain an asymptotically unbiased estimator we generalize the two step procedure of Vogelsang and Wagner (2014). In a first step, computing the seasonal partial sum of both sides of (1) yields

$$S^y_{\omega,t} = S^{d\prime}_{\omega,t}\delta + S^{x\prime}_{\omega,t}\beta + S^u_{\omega,t} = S^{\tilde{x}\prime}_{\omega,t}\theta + S^u_{\omega,t}, \tag{9}$$

where $\tilde{x}_t = [d_t', x_t']'$ and $\theta = [\delta', \beta']'$. By stacking all observations we obtain the matrix representation $S_\omega^y = S_\omega^{\tilde{x}\prime}\theta + S_\omega^u$, where $S_\omega^y = [S_{\omega,1}^y, \ldots, S_{\omega,T}^y]'$ and $S_\omega^{\tilde{x}}$ and $S_\omega^u$ are defined analogously. The OLS estimator of this seasonal partial sum regression is given by

$$\tilde{\theta} = (S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}})^{-1}(S_\omega^{\tilde{x}*} S_\omega^y) = \theta + (S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}})^{-1}(S_\omega^{\tilde{x}*} S_\omega^u).$$

By taking seasonal partial sums the regressors and the error term in equation (1) become $I_\omega(2)$ and $I_\omega(1)$ processes, respectively. The benefit of regression (9) is that the additive bias term $\Delta_{\omega,uv}$, that appears in the limiting distribution of the OLS estimator, vanishes. However, seasonal partial summation does not remove the correlation between the limiting processes. Therefore, we augment regression (9) by adding $x_t$ as additional regressor, i.e.

$$S_{\omega,t}^y = S_{\omega,t}^{d\prime}\delta + S_{\omega,t}^{x\prime}\beta + x_t\gamma + S_{\omega,t}^u = S_{\omega,t}^{\tilde{x}\prime}\theta + S_{\omega,t}^u, \tag{10}$$

with obviously redefined $\theta = [\delta', \beta', \gamma']'$ and $S_{\omega,t}^{\tilde{x}} = [S_{\omega,t}^{d\prime}, S_{\omega,t}^{x\prime}, x_t']'$. This augmentation is independent of the considered frequency $\omega$ and is therefore the same as in the IM-OLS procedure by Vogelsang and Wagner (2014). The OLS estimator of the parameters in regression (10), which we label the *seasonally integrated modified OLS (SIM-OLS)* estimator, is denoted by $\tilde{\theta}$. In order to derive the limiting distribution of the SIM-OLS estimator we define the scaling matrix

$$A_{SIM} = \begin{bmatrix} T^{-1/2}G_D & 0_{q\times k} & 0_{q\times k} \\ 0_{k\times q} & T^{-1}I_{k\times k} & 0_{k\times k} \\ 0_{k\times q} & 0_{k\times k} & I_{k\times k} \end{bmatrix}.$$

The following theorem states the asymptotic distribution of the SIM-OLS estimator $\tilde{\theta}$.

**Theorem 1.** *Let $\{y_t\}_{t\in\mathbb{N}}$ and $\{x_t\}_{t\in\mathbb{N}}$ be generated according to* (1) *and* (2) *with Assumptions 1 and 2 in place. Then, with $\overline{\gamma} = \Omega_{\omega,vv}^{-1}\Omega_{\omega,vu}$, it holds that*

$$\begin{aligned} A_{SIM}^{-1}(\tilde{\theta} - \theta) &\Rightarrow \sigma_{\omega,u\cdot v}\left(\overline{\Pi}_\omega \int \overline{g(s)} g(s)'\, ds \Pi_\omega'\right)^{-1} \overline{\Pi}_\omega \int \overline{g(s)} W_{u\cdot v}(s)\, ds \\ &= \sigma_{\omega,u\cdot v}(\Pi_\omega')^{-1}\left(\int \overline{g(s)} g(s)'\, ds\right)^{-1} \int \left[\overline{G(1)} - \overline{G(s)}\right] dW_{u\cdot v}(s) \\ &= \Psi_\omega, \end{aligned}$$

*where*

$$\Pi_\omega = \begin{bmatrix} \tau_\omega^{-1} I_{q\times q} & 0_{q\times k} & 0_{q\times k} \\ 0_{k\times q} & \Omega_{\omega,vv}^{1/2} & 0_{k\times k} \\ 0_{k\times q} & 0_{k\times k} & \Omega_{\omega,vv}^{1/2} \end{bmatrix}, \quad g(r) = \begin{bmatrix} \int_0^r f(s)\, ds \\ \int_0^r W_v(s)\, ds \\ W_v(r) \end{bmatrix}, \quad G(r) = \int_0^r g(s)\, ds.$$

Note that if $\omega = 0$ the result obviously coincides with the limiting distribution of the IM-OLS estimator of Vogelsang and Wagner (2014). Conditional on $B_v$, it holds that $\Psi_\omega$ is Gaussian (if

$\omega \in \{0, \pi\}$) or circularly symmetrically Gaussian (if $\omega \in (0, \pi)$). In both cases the conditional covariance matrix is given by

$$
\begin{aligned}
V_{SIM} = \sigma^2_{\omega,u\cdot v}(\Pi'_\omega)^{-1} &\left(\int \overline{g(s)} g(s)'\, ds\right)^{-1} \\
\times &\left(\int \left[\overline{G(1)} - \overline{G(s)}\right] [G(1) - G(s)]'\, ds\right) \\
\times &\left(\int \overline{g(s)} g(s)'\, ds\right)^{-1} (\overline{\Pi}_\omega)^{-1},
\end{aligned}
\tag{11}
$$

where in the former case all quantities are real valued and complex conjugation is redundant. From the properties of the complex normal distribution (cf. Appendix B) we immediately deduce the following result.

**Corollary 1.** *If $\omega \in (0, \pi)$ it holds, conditional on $B_v$, that $\mathrm{Re}(\Psi_\omega)$ and $\mathrm{Im}(\Psi_\omega)$ are jointly zero mean Gaussian with conditional covariance matrix given by*

$$
V_{SIM,\mathbb{R}} = \frac{1}{2} \begin{bmatrix} \mathrm{Re}(V_{SIM}) & -\mathrm{Im}(V_{SIM}) \\ \mathrm{Im}(V_{SIM}) & \mathrm{Re}(V_{SIM}) \end{bmatrix}.
$$

## 3. Standard Inference using SIM-OLS

In this section we discuss hypothesis testing based on the SIM-OLS estimator. If the data generating process is real valued, i.e. if $\omega \in \{0, \pi\}$, we consider Wald-type tests for multiple linear hypotheses of the form

$$
H_0 : R\theta = r, \tag{12}
$$

where $R \in \mathbb{R}^{l \times (q+2k)}$ with full rank $l$ and some vector $r \in \mathbb{R}^l$. If the data generating process is complex valued, i.e. if $\omega \in (0, \pi)$, we consider hypotheses of the form (12) with $R \in \mathbb{C}^{l \times (q+2k)}$ with full rank $l$ and $r \in \mathbb{C}^l$.

Since the components of $\tilde{\theta}$ converge at different rates we require an additional assumption on $R$ for the Wald-type statistics to be asymptotically chi-squared distributed. We adopt the following sufficient condition from Vogelsang and Wagner (2014). There exists a nonsingular scaling matrix $A_R \in \mathbb{R}^{l \times l}$ such that

$$
\lim_{T \to \infty} A_R^{-1} R A_{SIM} = R_A, \tag{13}
$$

where $R_A$ has full rank $l$. Note that the matrix $A_R$ typically contains positive powers of the sample size $T$ but it is not necessarily diagonal.

Let $\check{\sigma}^2_{\omega,u\cdot v}$ denote a consistent estimator of $\sigma^2_{\omega,u\cdot v}$. Then the asymptotic conditional covariance matrix of the SIM-OLS estimator, $V_{SIM}$, given in (11), entails estimators of the form

$$
\begin{aligned}
\check{V}_{SIM} &= \check{\sigma}^2_{\omega,u\cdot v} A_{SIM}^{-1} (S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}})^{-1} (C^* C) (S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}})^{-1} A_{SIM}^{-1} \\
&= \check{\sigma}^2_{\omega,u\cdot v} (T^{-2} A_{SIM} S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}} A_{SIM})^{-1} (T^{-4} A_{SIM} C^* C A_{SIM}) (T^{-2} A_{SIM} S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}} A_{SIM})^{-1},
\end{aligned}
\tag{14}
$$

where $C$ is defined by stacking

$$c_t = e^{i\omega T} S_{\omega,T}^{S_\omega^{\tilde{x}}} - e^{i\omega(t-1)} S_{\omega,t-1}^{S_\omega^{\tilde{x}}}.$$

To calculate $\check{V}_{SIM}$ we need a suitable estimator for $\sigma^2_{\omega,u\cdot v}$. Therefore, we first estimate $\Omega_\omega$ using a traditional kernel estimator of the form

$$\hat{\Omega}_\omega = \frac{1}{T}\sum_{t=2}^{T}\sum_{s=2}^{T} e^{-i\omega(t-s)} k\left(\frac{t-s}{M_T}\right)\hat{\eta}_t\hat{\eta}_s^*, \tag{15}$$

where $\hat{\eta}_t = [\hat{u}_t, \nabla_\omega x_t']'$ with $\hat{u}_t$ denoting the OLS residuals of (1). After partitioning $\hat{\Omega}_\omega$ in the same way as in (6), an estimator for $\sigma^2_{\omega,u\cdot v}$ is given by

$$\hat{\sigma}^2_{\omega,u\cdot v} = \hat{\Omega}_{\omega,uu} - \hat{\Omega}_{\omega,uv}\hat{\Omega}^{-1}_{\omega,vv}\hat{\Omega}_{\omega,vu}. \tag{16}$$

The function $k : \mathbb{R} \to \mathbb{R}$ in (15) is the kernel weighting function and $M_T$ is the bandwidth parameter. We impose the following assumptions on the kernel and the bandwidth.

**Assumption 3.** The kernel function $k : \mathbb{R} \to \mathbb{R}$ is even, bounded and continuous. Furthermore, the function $\tilde{k} : \mathbb{R} \to \mathbb{R}$, defined by $\tilde{k}(x) = \sup_{y \geq |x|} |k(y)|$ is integrable on $\mathbb{R}$.

**Assumption 4.** The bandwidth $M_T$ satisfies $M_T^{-1} \to 0$ and $M_T/\sqrt{T} \to 0$ as $T \to \infty$.

Under Assumptions 3 and 4 the population quantities $\Omega_\omega$ and $\sigma^2_{\omega,u\cdot v}$ are consistently estimated by $\hat{\Omega}_\omega$ and $\hat{\sigma}^2_{\omega,u\cdot v}$, respectively. This is shown in Jansson (2002) for the zero frequency case and can be generalized for arbitrary frequencies.

The Wald-type test statistic is defined as

$$\hat{W} = \tau_\omega^{-2}(R\tilde{\theta} - r)^* \left[RA_{SIM}\hat{V}_{SIM}A_{SIM}R^*\right]^{-1}(R\tilde{\theta} - r). \tag{17}$$

where we use the $\hat{V}_{SIM}$ to indicate that $\hat{\sigma}^2_{\omega,u\cdot v}$ is plugged into $\check{\sigma}^2_{\omega,u\cdot v}$ in the definition of the general estimator (14). Motivated by Corollary 1, we can use the continuous mapping theorem as well as standard arguments to deduce the limiting distribution of the test statistic $\hat{W}$.

**Theorem 2.** *Let $\{y_t\}_{t\in\mathbb{N}}$ and $\{x_t\}_{t\in\mathbb{N}}$ be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and let the estimation of $\Omega_\omega$ be performed under Assumptions 3 and 4. If the matrix $R$ fulfills* (13) *then*

$$\hat{W} \Rightarrow \chi^2_m,$$

*where $\chi^2_m$ is a chi-squared distributed random variable with $m$ degrees of freedom. It holds that $m = l$ if $\omega \in \{0, \pi\}$ and $m = 2l$ otherwise.*

If $\omega \in (0, \pi)$ Corollary 1 allows us to generalize this result to test multiple linear hypothesis of the form

$$H_0 : R_\mathbb{R}\theta_\mathbb{R} = r_\mathbb{R},$$

with $\theta_{\mathbb{R}} = [\mathrm{Re}(\theta)', \mathrm{Im}(\theta)']'$, $r_{\mathbb{R}} \in \mathbb{R}^h$ and $R_{\mathbb{R}} \in \mathbb{R}^{h \times 2(q+2k)}$ with full rank $h$. Assumption (13) is generalized as follows. There exists a regular scaling matrix $A_{R,\mathbb{R}} \in \mathbb{R}^{h \times h}$ such that

$$\lim_{T \to \infty} A_{R,\mathbb{R}} R_{\mathbb{R}} A_{SIM,\mathbb{R}} = R_{A,\mathbb{R}}, \tag{18}$$

with $R_{A,\mathbb{R}}$ being nonsingular and $A_{SIM,\mathbb{R}} = I_2 \otimes A_{SIM}$.

**Corollary 2.** *Let for $\omega \in (0, \pi)$ the processes $\{y_t\}_{t \in \mathbb{N}}$ and $\{x_t\}_{t \in \mathbb{N}}$ be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and let the estimation of $\Omega_\omega$ be performed under Assumptions 3 and 4. Define $\tilde{\theta}_{\mathbb{R}} = [\mathrm{Re}(\tilde{\theta})', \mathrm{Im}(\tilde{\theta})']'$ and*

$$\hat{V}_{SIM,\mathbb{R}} = \frac{1}{2} \begin{bmatrix} \mathrm{Re}(\hat{V}_{SIM}) & -\mathrm{Im}(\hat{V}_{SIM}) \\ \mathrm{Im}(\hat{V}_{SIM}) & \mathrm{Re}(\hat{V}_{SIM}) \end{bmatrix}.$$

*If the matrix $R_{\mathbb{R}}$ fulfills* (18) *then*

$$\hat{W}_{\mathbb{R}} = (R_{\mathbb{R}} \tilde{\theta}_{\mathbb{R}} - r_{\mathbb{R}})' \left[ R_{\mathbb{R}} A_{SIM,\mathbb{R}} \hat{V}_{SIM,\mathbb{R}} A_{SIM,\mathbb{R}} R'_{RR} \right]^{-1} (R_{\mathbb{R}} \tilde{\theta}_{\mathbb{R}} - r_{\mathbb{R}}) \Rightarrow \chi^2_h, \tag{19}$$

*where $\chi^2_h$ is a chi-squared distributed random variable with $h$ degrees of freedom. If $h = 1$, it holds that*

$$\hat{t}_{\mathbb{R}} = \frac{R_{\mathbb{R}} \tilde{\theta}_{\mathbb{R}} - r_{\mathbb{R}}}{\sqrt{R_{\mathbb{R}} A_{SIM,\mathbb{R}} \hat{V}_{SIM,\mathbb{R}} A_{SIM,\mathbb{R}} R'_{\mathbb{R}}}} \Rightarrow Z, \tag{20}$$

*where $Z$ has a standard normal distribution.*

**Remark 8.** If $\{[\tilde{y}_t, \tilde{x}'_t]'\}_{t \in \mathbb{N}}$ is a real valued process such that the process $\{[y_t, x'_t]'\}_{t \in \mathbb{N}}$ considered in Corollary 2 fulfills $[y_t, x'_t]' = \nabla_{-\omega}[\tilde{y}_t, \tilde{x}'_t]'$ then, by Remarks 5 and 6 $\{[\tilde{y}_t, \tilde{x}'_t]'\}_{t \in \mathbb{N}}$ is dynamically seasonally cointegrated and it holds that

$$\tilde{y}_t = \tilde{d}'_t \tilde{\delta} + \tilde{x}'_t \tilde{\alpha}_1 + \tilde{x}'_{t-1} \tilde{\alpha}_2 + \vartheta_t,$$

where $\tilde{d}_t$ is a real valued deterministic component, $\{\vartheta_t\}_{t \in \mathbb{Z}}$ is a real valued stationary process, $\tilde{\delta} \in \mathbb{R}^q$ and $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{R}^k$. From the one-to-one relationship $\tilde{\alpha}_1 + e^{i\omega} \tilde{\alpha}_2 = \beta$ we deduce that $\tilde{\alpha}_2 = 0$ if and only if $\mathrm{Im}(\beta) = 0$. Hence, we can test the null hypothesis of static seasonal cointegration of $\{[\tilde{y}_t, \tilde{x}'_t]'\}_{t \in \mathbb{Z}}$ against the alternative hypothesis of dynamic seasonal cointegration using the Wald-type test in Corollary 2 with $A_{R,\mathbb{R}} = [0_{k \times 2(k+q)}, I_{k \times k}, 0_{k \times k}]$ and $r_{\mathbb{R}} = 0_{k \times 1}$.

## 4. Fixed-$b$ Inference Using SIM-OLS

In the previous section we constructed test statistics that make use of a consistent estimator for $\Omega_\omega$ and, hence, a consistent estimator for $\sigma^2_{\omega,u \cdot v}$. Assumptions 3 and 4 guarantee that $\hat{\Omega}_\omega$, given in (15), is consistent. However, as already mentioned in the introduction, relying solely on the consistency of $\hat{\Omega}_\omega$ does not reflect the impact of the choice of the kernel function and the bandwidth parameter on the finite sample distribution of $\hat{\Omega}_\omega$ and, consequently, on the limiting distribution $\Psi_\omega$. Therefore, in this section we present a fixed-$b$ estimator for $\sigma^2_{\omega,u \cdot v}$ which has limiting distribution that is proportional to $\sigma^2_{\omega,u \cdot v}$ and which is independent of $\Psi_\omega$. We use this estimator to construct $t$- and Wald-type test statistics that have asymptotic nuisance parameter free limiting distributions and for which critical values can be tabulated for hypothesis testing.

A natural candidate for a fixed-$b$ estimator for $\sigma^2_{\omega,u\cdot v}$ is given by

$$\tilde{\sigma}^2_{\omega,u\cdot v} = \frac{1}{T}\sum_{t=2}^{T}\sum_{s=2}^{T} e^{-i\omega(t-s)} k\left(\frac{t-s}{M}\right)\nabla_\omega \tilde{S}^u_{\omega,t}\nabla_\omega \tilde{S}^{u*}_{\omega,s}, \tag{21}$$

where the bandwidth $M$ is given by $M = bT$ for some fixed (independent of $T$) $b \in (0,1]$ and where $\tilde{S}^u_{\omega,t} = S^y_{\omega,t} - S^{\tilde{x}\prime}_{\omega,t}\tilde{\theta}$ denote the residuals from the SIM-OLS regression (10). We first present an invariance principle for these residuals and show that the resulting limiting process is correlated with the limiting distribution of the SIM-OLS estimator $\tilde{\theta}$.

**Lemma 1.** *Let $\{y_t\}_{t\in\mathbb{N}}$ and $\{x_t\}_{t\in\mathbb{N}}$ be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and let $\tilde{S}^u_{\omega,t}$ denote the OLS residuals from regression* (10)*. Then, the following invariance principle holds true:*

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{[rT]} e^{i\omega t}\nabla_\omega \tilde{S}^u_{\omega,t} \Rightarrow \tau_\omega \sigma_{\omega,u\cdot v}\tilde{P}(r), \quad r\in(0,1],$$

*with*

$$\tilde{P}(r) = W_{u\cdot v}(r) - g(r)'\left(\int \overline{g(s)}g(s)'\,ds\right)^{-1}\int\left[\overline{G(1)} - \overline{G(s)}\right] dW_{u\cdot v}(s),$$

*where $g(r)$ and $G(r)$ are defined in Theorem 1. Furthermore, conditional upon $W_v$, it holds that $\tilde{P}(r)$ is correlated with $\Psi_\omega$, the limiting distribution of the SIM-OLS estimator.*

The correlation between $\tilde{P}(r)$ and $\Psi_\omega$ causes $\tilde{\sigma}^2_{\omega,u\cdot v}$ to be correlated with $\Psi_\omega$. This makes the fixed-$b$ estimator infeasible. To overcome this issue we generalize the approach of Vogelsang and Wagner (2014) by adjusting regression (10) in order to get proper modified residuals. Consider

$$S^y_{\omega,t} = S^{d\prime}_{\omega,t}\delta + S^{x\prime}_{\omega,t}\beta + x_t'\gamma + z_t'\kappa + S^u_{\omega,t}, \tag{22}$$

where

$$z_t = te^{-i\omega(t-1)}e^{i\omega T}S^\xi_{\omega,T} - S^{S^\xi_\omega}_{\omega,t-1}, \quad \xi_t = [S^{d\prime}_{\omega,t}, S^{x\prime}_{\omega,t}, x_t']'. \tag{23}$$

The OLS residuals from this regression are denoted by $\tilde{\tilde{S}}^u_{\omega,t}$. It will turn out that these residuals fulfill an invariance principle such that the limiting process is independent of $\Psi_\omega$. To prove this statement, we first derive the limiting distribution of the OLS estimator of (22).

**Proposition 1.** *Let $\{y_t\}_{t\in\mathbb{N}}$ and $\{x_t\}_{t\in\mathbb{N}}$ be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and consider the OLS estimator for $\theta = [\delta', \beta', \Omega_{uv}\Omega_{vv}^{-1}, 0]'$ in regression* (22)*, denoted by $\tilde{\tilde{\theta}}$. Then, it holds that*

$$A_M^{-1}(\tilde{\tilde{\theta}} - \theta) \Rightarrow \sigma_{\omega,u\cdot v}(\Pi'_{\omega,M})^{-1}\left(\int\overline{h(s)}h(s)'\,ds\right)^{-1}\int\left[\overline{H(1)} - \overline{H(s)}\right] dW_{u\cdot v}(s),$$

*where* $\Pi_{\omega,M} = I_2 \otimes \Pi_\omega$,

$$A_M = \begin{bmatrix} 1 & 0 \\ 0 & T^{-2} \end{bmatrix} \otimes A_{SIM}, \quad h(r) = \begin{bmatrix} g(r) \\ \int_0^r [G(1) - G(s)]\, ds \end{bmatrix}, \quad H(r) = \int_0^r h(s)\, ds.$$

With this proposition in place we are now able to extend Lemma 1 and derive an invariance principle for the residuals $\tilde{\tilde{S}}^u_{\omega,t}$ with limiting process being uncorrelated with the limiting distribution of the SIM-OLS estimator.

**Lemma 2.** *Let* $\{y_t\}_{t\in\mathbb{N}}$ *and* $\{x_t\}_{t\in\mathbb{N}}$ *be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and let* $\tilde{\tilde{S}}^u_{\omega,t}$ *denote the OLS residuals from regression* (22)*. Then, the following invariance principle holds true:*

$$\frac{1}{\sqrt{T}} \sum_{t=2}^{[rT]} e^{i\omega t} \nabla_\omega \tilde{\tilde{S}}^u_{\omega,t} \Rightarrow \tau_\omega \sigma_{\omega,u\cdot v} \tilde{\tilde{P}}(r), \quad r \in (0,1],$$

*with*

$$\tilde{\tilde{P}}(r) = W_{u\cdot v}(r) - h(r)' \left( \int \overline{h(s)} h(s)'\, ds \right)^{-1} \int \left[ \overline{H(1)} - \overline{H(s)} \right] dW_{u\cdot v}(s),$$

*where*

$$h(r) = \begin{bmatrix} g(r) \\ \int_0^r [G(1) - G(s)]\, ds \end{bmatrix}, \quad H(r) = \int_0^r h(s)\, ds.$$

*Furthermore, conditional upon* $W_v$*, it holds that the limiting distribution of the SIM-OLS estimator* $\tilde{\theta}$*, i.e.* $\Psi_\omega$*, is uncorrelated with* $\tilde{\tilde{P}}(r)$*.*

Consider the the fixed-$b$ estimator for $\sigma^2_{\omega,u\cdot v}$ based on the OLS residuals from the augmented regression (22), $\tilde{\tilde{S}}^u_{\omega,t}$, which is given by

$$\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v} = \frac{1}{T} \sum_{t=2}^{T} \sum_{s=2}^{T} e^{-i\omega(t-s)} k\left( \frac{|t-s|}{M} \right) \nabla_\omega \tilde{\tilde{S}}^u_{\omega,t} \nabla_\omega \tilde{\tilde{S}}^{u*}_{\omega,s}. \tag{24}$$

In order to describe the fixed-$b$ limiting distribution of $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$ we define the functional $Q(P)$ for some stochastic process $P$ as follows. If the kernel function $k$ is twice continuously differentiable and satisfies $k(0) = 1$ we define

$$\begin{aligned} Q(P) = &-\frac{1}{b^2} \int_0^1 \int_0^1 \ddot{k}\left( \frac{r-s}{b} \right) P(r) P(s)^*\, dr\, ds \\ &+ \frac{1}{b} \int_0^1 \dot{k}\left( \frac{1-r}{b} \right) \left( P(1)P(r)^* + P(r)P(1)^* \right) dr + P(1)P(1)^*, \end{aligned}$$

with $\dot{k}$ and $\ddot{k}$ denoting the first and second derivatives of the kernel function $k$. This case covers for instance the Parzen, Daniell, Bohman or Quadratic Spectral kernels. If $k$ is the Bartlett kernel, i.e., $k(x) = 1 - |x|$ for $|x| \leq 1$ and zero otherwise, then we define

$$\begin{aligned} Q(P) = \frac{2}{b}\int_0^1 P(r)P(r)^* \, dr - \frac{1}{b}\int_0^{1-b} (P(r)P(r+b)^* + P(r+b)P(r)^*) \, dr \\ - \frac{1}{b}\int_{1-b}^1 (P(1)P(r)^* + P(r)P(1)^*) \, ds + P(1)P(1)^*. \end{aligned}$$

With the definition of $Q(P)$ in place we can generalize the main result of Hashimzade and Vogelsang (2007).

**Proposition 2.** *Let $\{\zeta_t\}_{t\in\mathbb{Z}}$ be a stochastic process that fulfills a functional central limit theorem of the form*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\zeta_t \Rightarrow P(r), \quad r \in (0,1]$$

*and let $\check{\sigma}^2(\zeta)$ be defined as*

$$\check{\sigma}^2(\zeta) = \frac{1}{T}\sum_{t=2}^{T}\sum_{s=2}^{T} e^{-i\omega(t-s)} k\left(\frac{|t-s|}{M}\right)\zeta_t\zeta_s^*,$$

*with $k$ being either the Bartlett kernel or a twice continuously differentiable function that satisfies $k(0) = 1$. The bandwidth $M$ is defined by $M = bT$ with $b \in (0,1]$ (independent of $T$). Then, as $T \to \infty$, it holds that*

$$\check{\sigma}^2(\zeta) \Rightarrow Q(P).$$

The random variable $Q(P)$ is real valued regardless of $P$. To verify this, we only have to show that $Q(P)^* = Q(P)$, which is trivial in the second (Bartlett) case and follows by a simple application of Fubini's Theorem in the first (continuously differentiable kernel) case. If $P$ is complex valued, we can easily derive the explicit form of $Q(P)$ in terms of the real and imaginary parts of $P$.

**Corollary 3.** *Assume that the limiting process $P$ in Proposition 2 is complex valued, i.e. $P(r) = P_1(r) + iP_2(r)$. If $k$ is twice continuously differentiable with $k(0) = 1$ then it holds that*

$$\begin{aligned} Q(P) = -\frac{1}{b^2}\sum_{i=1}^{2}\int_0^1\int_0^1 \ddot{k}\left(\frac{r-s}{b}\right) P_i(r)P_i(s) \, dr \, ds \\ + \frac{2}{b}\sum_{i=1}^{2} P_i(1)\int_0^1 \dot{k}\left(\frac{1-r}{b}\right) P_i(r) \, dr + \sum_{i=1}^{2} P_i(1)^2. \end{aligned}$$

**Table 1:** Fixed-$b$ asymptotic critical values (Bartlett kernel) for Wald-type test in a cointegrating regression with $\omega \in (0, \pi)$, two stochastic regressors and a deterministic component $d_t = e^{-i\omega t}$.

| $b$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.10 | 0.12 | 0.14 | 0.16 | 0.18 | 0.20 |
|---|---|---|---|---|---|---|---|---|---|---|
| 90% | 3.0634 | 3.8762 | 4.9396 | 6.3171 | 7.9979 | 9.9553 | 12.1136 | 14.3180 | 16.4318 | 18.4679 |
| 95% | 4.3852 | 5.5978 | 7.1601 | 9.1890 | 11.6590 | 14.4925 | 17.5678 | 20.8195 | 23.9641 | 26.8486 |
| 99% | 7.7324 | 10.0245 | 12.9840 | 16.9634 | 21.6155 | 27.0050 | 32.6865 | 38.7724 | 44.6733 | 49.6200 |
| $b$ | 0.22 | 0.24 | 0.26 | 0.28 | 0.30 | 0.32 | 0.34 | 0.36 | 0.38 | 0.40 |
| 90% | 20.1669 | 21.6858 | 23.0169 | 24.2313 | 25.4073 | 26.5820 | 27.8476 | 28.9854 | 30.1214 | 31.3966 |
| 95% | 29.3439 | 31.5896 | 33.4533 | 35.2848 | 37.1084 | 38.8372 | 40.4885 | 42.2277 | 44.0080 | 45.8518 |
| 99% | 54.5059 | 58.2774 | 62.1846 | 65.8995 | 69.0014 | 72.6234 | 76.0563 | 78.9366 | 82.2768 | 86.2823 |
| $b$ | 0.42 | 0.44 | 0.46 | 0.48 | 0.50 | 0.52 | 0.54 | 0.56 | 0.58 | 0.60 |
| 90% | 32.6977 | 33.9286 | 35.1146 | 36.4037 | 37.6047 | 38.8282 | 39.9611 | 41.0836 | 42.1169 | 43.2176 |
| 95% | 47.7578 | 49.6898 | 51.4977 | 53.3345 | 55.2138 | 56.8521 | 58.4270 | 60.1834 | 61.9210 | 63.7502 |
| 99% | 89.9133 | 93.3340 | 97.4772 | 101.4402 | 104.9144 | 108.6211 | 111.1449 | 114.4210 | 117.2866 | 120.7045 |
| $b$ | 0.62 | 0.64 | 0.66 | 0.68 | 0.70 | 0.72 | 0.74 | 0.76 | 0.78 | 0.80 |
| 90% | 44.2168 | 45.2873 | 46.3237 | 47.2857 | 48.2643 | 49.3693 | 50.4373 | 51.2936 | 52.2023 | 53.1395 |
| 95% | 65.2041 | 66.7245 | 68.3009 | 69.8652 | 71.1557 | 72.8208 | 74.3809 | 75.7272 | 77.1777 | 78.5499 |
| 99% | 123.9202 | 127.4599 | 130.6415 | 133.7872 | 136.9260 | 139.6952 | 142.6479 | 146.1120 | 149.5329 | 152.1242 |
| $b$ | 0.82 | 0.84 | 0.86 | 0.88 | 0.90 | 0.92 | 0.94 | 0.96 | 0.98 | 1.00 |
| 90% | 54.0310 | 54.9834 | 55.8812 | 56.8973 | 57.7842 | 58.7284 | 59.5111 | 60.4007 | 61.1553 | 62.0011 |
| 95% | 79.9602 | 81.4841 | 82.9412 | 84.2464 | 85.5845 | 86.9878 | 88.3384 | 89.7000 | 90.9166 | 92.2464 |
| 99% | 154.2709 | 156.9221 | 159.9765 | 162.9743 | 166.3192 | 169.4175 | 172.3358 | 175.1685 | 178.0476 | 180.9433 |

Note: Critical values for the $t$-tests are obtained by taking the positive square root.

*If $k$ is the Bartlett kernel it holds that*

$$Q(P) = \frac{2}{b}\sum_{i=1}^{2}\int_0^1 P_i^2(r)\,dr - \frac{2}{b}\sum_{i=1}^{2}\int_0^{1-b} P_i(r)P_i(r+b)\,dr - \frac{2}{b}\sum_{i=1}^{2} P_i(1)\int_{1-b}^{1} P_i(r)\,dr + \sum_{i=1}^{2} P_i^2(1).$$

Combining Lemma 1, Lemma 2 and Proposition 2 we can now deduce the limiting distribution of the fixed-$b$ estimators of $\sigma^2_{\omega,u\cdot v}$.

**Proposition 3.** *Let $\{y_t\}_{t\in\mathbb{N}}$ and $\{x_t\}_{t\in\mathbb{N}}$ be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and let $\tilde{\sigma}^2_{\omega,u\cdot v}$ and $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$ be the fixed-b estimators for $\sigma^2_{\omega,u\cdot v}$, defined in* (21) *and* (24)*, with $M = bT$ where $b \in (0,1]$ being independent of the sample size. Then it holds that $\tilde{\sigma}^2_{\omega,u\cdot v} \Rightarrow \sigma^2_{\omega,u\cdot v}\tau^2_\omega Q(\tilde{P})$ and $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v} \Rightarrow \sigma^2_{\omega,u\cdot v}\tau^2_\omega Q(\tilde{\tilde{P}})$, where the limiting processes $\tilde{P}$ and $\tilde{\tilde{P}}$ are defined in Lemma 1 and 2, respectively.*

We deduce that $Q(\tilde{\tilde{P}})$ and $\Psi_\omega$ are independent whereas $Q(\tilde{P})$ and $\Psi_\omega$ are not. Hence, replacing $\check{\sigma}^2_{\omega,u\cdot v}$ in the definition of $\check{V}_{SIM}$ in (17) by $\tilde{\tilde{\sigma}}_{\omega,u\cdot v}$ leads to a fixed-$b$ Wald-type statistic which is asymptotically pivotal.

**Theorem 3.** *Let* $\{y_t\}_{t\in\mathbb{N}}$ *and* $\{x_t\}_{t\in\mathbb{N}}$ *be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and let* $\tilde{\tilde{W}}$ *denote the Wald-type test statistic* (17) *with* $\check{\sigma}^2_{\omega,u\cdot v}$ *being replaced by the fixed-b estimator* $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$*, defined in* (24)*. If the hypothesis matrix* $R$ *satisfies* (13) *and if the bandwidth* $M$ *is given by* $M = bT$ *with fixed* $b \in (0, 1]$ *then*

$$\tilde{\tilde{W}} \Rightarrow \frac{\chi^2_m}{Q(\tilde{\tilde{P}})},$$

*where* $\chi^2_m$ *is a chi-squared distribution with* $m$ *degrees of freedom that is independent of the denominator* $Q(\tilde{\tilde{P}})$*. Thereby,* $m$ *is equal to* $l$ *if* $\omega \in \{0, \pi\}$ *and to* $2l$ *otherwise.*

The limiting distribution of $\tilde{\tilde{W}}$ obviously depends on the explicit form of $\tilde{\tilde{P}}$. Hence, critical values have to be simulated taking into account the number of regressors that contain a unit root at frequency $\omega$, the explicit form of the deterministic terms, the chosen kernel function and the value of $b$. For $\omega \in (0, \pi)$, the kernel function being the Bartlett kernel and $d_t = e^{-i\omega t}$ the critical values of the test statistic are displayed in Table 1.

We define the test statistics $\tilde{\tilde{W}}_{\mathbb{R}}$ and $\tilde{\tilde{t}}_{\mathbb{R}}$ in the same way as in the previous section with $\check{\sigma}^2_{\omega,u\cdot v}$ being replaced by $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$. The following corollary characterizes the asymptotic distributions of the test statistics when considering hypothesis tests on the real and imaginary parts of $\theta$.

**Corollary 4.** *Let for* $\omega \in (0, \pi)$ *the processes* $\{y_t\}_{t\in\mathbb{N}}$ *and* $\{x_t\}_{t\in\mathbb{N}}$ *be generated according to* (1) *and* (2) *under Assumptions 1 and 2 and let* $\tilde{\tilde{W}}_{\mathbb{R}}$ *and* $\tilde{\tilde{t}}_{\mathbb{R}}$ *denote the Wald- and t-type test statistics from* (19) *and* (20)*, respectively, with* $\hat{\sigma}^2_{\omega,u\cdot v}$ *being replaced by the fixed-b estimator* $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$*, defined in* (24)*. If the matrix* $R_{\mathbb{R}}$ *fulfills* (18) *and if the bandwidth* $M$ *is given by* $M = bT$ *with fixed* $b \in (0, 1]$ *then*

$$\tilde{\tilde{W}}_{\mathbb{R}} \Rightarrow \frac{\chi^2_h}{Q(\tilde{\tilde{P}})},$$

*where* $\chi^2_m$ *is a chi-squared distributed random variable with* $h$ *degrees of freedom that is independent of the denominator* $Q(\tilde{\tilde{P}})$*. If* $h = 1$*, it holds that*

$$\tilde{\tilde{t}}_{\mathbb{R}} \Rightarrow \frac{Z}{\sqrt{Q(\tilde{\tilde{P}})}}$$

*where* $Z$ *has a standard normal distribution and is independent of* $Q(\tilde{\tilde{P}})$*.*

## 5. Testing for Cointegration using SIM-OLS

In this section we introduce a residual based tests for cointegration at frequency $\omega$ in the spirit of Kwiatkowski et al. (1992) and Shin (1994). Therefore, we consider $\{y_t\}_{t\in\mathbb{N}}$ generated according to

$$\begin{aligned} y_t &= d_t'\delta + x_t'\beta + m_t, \\ m_t &= s_t + u_t, \end{aligned} \tag{25}$$

where $d_t$ is a $q$-dimensional deterministic component that satisfies Assumption 1. The processes $\{x_t\}_{t\in\mathbb{N}}$ is a $k$-dimensional vector process and $\{s_t\}_{t\in\mathbb{N}}$ is a scalar process, both satisfying

$$\begin{aligned} x_t &= e^{-i\omega} x_{t-1} + v_t, \\ s_t &= e^{-i\omega} s_{t-1} + w_t, \end{aligned} \tag{26}$$

for some frequency $\omega \in (-\pi, \pi]$ with initial values $x_0$ and $s_0$ being $\mathcal{O}_{\mathbb{P}}(1)$. The following assumption on the stacked error process $\{[u_t, v_t', w_t]'\}_{t\in\mathbb{Z}}$ is sufficient for the derivation of the subsequent results.

**Assumption 5.** The process $\{\eta_t^a\}_{t\in\mathbb{Z}}$ with $\eta_t^a = [u_t, v_t', w_t]'$ is zero mean stationary with $2\pi$ times the spectral density matrix at frequency $\omega$ given by

$$\Omega_\omega^a = \sum_{j=-\infty}^{\infty} e^{-i\omega j} \mathbb{E}(\eta_t^a \eta_{t+j}^{a*}) = \begin{bmatrix} \Omega_\omega & 0 \\ 0 & \sigma^2_{\omega,w} \end{bmatrix}, \tag{27}$$

where $\Omega_\omega$ is defined in (6). Furthermore, $\{\eta_t^a\}_{t\in\mathbb{Z}}$ fulfills the following functional central limit theorem:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[rT]} e^{i\omega t} \eta_t^a \Rightarrow B^a(r) = \begin{bmatrix} B(r) \\ \sigma_{\omega,w} \tau_\omega W_w(r) \end{bmatrix}, \quad r \in (0,1],$$

where $B$ is a vector Brownian motion, defined in (7), and $W_w$ a standard (complex) Wiener process which is independent of $B$.

The test procedures are based on the SIM-OLS regression

$$S^y_{\omega,t} = S^{d\prime}_{\omega,t}\delta + S^{x\prime}_{\omega,t}\beta + x_t\gamma + S^m_{\omega,t}, \tag{28}$$

and on the augmented SIM-OLS regression

$$S^y_{\omega,t} = S^{d\prime}_{\omega,t}\delta + S^{x\prime}_{\omega,t}\beta + x_t'\gamma + z_t'\kappa + S^m_{\omega,t}, \tag{29}$$

where $z_t$ is defined in (23). We denote the OLS residuals of (28) and (29) by $\tilde{S}^m_{\omega,t}$ and $\tilde{\tilde{S}}^m_{\omega,t}$, respectively. The test statistics are given by

$$\tilde{K} = \frac{1}{T^2 \hat{\sigma}^2_{\omega,u\cdot v}} \sum_{t=1}^{T} |\tilde{S}^m_{\omega,t}|^2 \tag{30}$$

and

$$\tilde{\tilde{K}} = \frac{1}{T^2 \hat{\sigma}^2_{\omega,u\cdot v}} \sum_{t=1}^{T} |\tilde{\tilde{S}}^m_{\omega,t}|^2. \tag{31}$$

Thereby, $\hat{\sigma}^2_{\omega,u\cdot v}$ is defined as in (16) with $\hat{\eta}_t = [\hat{m}_t, \nabla_\omega x_t']'$, where $\hat{m}_t$ denote OLS residuals from (25). We can investigate whether $\{[y_t, x_t']'\}_{t\in\mathbb{N}}$ is seasonally cointegrated by testing the null hypothesis $H_0 : \sigma^2_{\omega,w} = 0$.

Under $H_0$ we can easily derive the limiting distributions of the test statistics. In fact, these follow immediately from Lemma 1 and Lemma 2 in conjunction with the continuous mapping theorem.

**Table 2:** Critical values for the cointegration tests with $\omega \in (0, \pi)$. The three block-columns correspond to the cases without deterministic component, with constant oscillation and with constant and linearly increasing oscillation.

| | $d_t = \emptyset$ | | | $d_t = e^{-i\omega t}$ | | | $d_t = [-e^{-i\omega t}, te^{-i\omega t}]'$ | | |
|---|---|---|---|---|---|---|---|---|---|
| Level: | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| Panel A: Critical values of $\tilde{K}$ | | | | | | | | | |
| $k = 1$ | 0.1950 | 0.2612 | 0.4745 | 0.0742 | 0.0873 | 0.1204 | 0.0495 | 0.0565 | 0.0755 |
| $k = 2$ | 0.0724 | 0.0887 | 0.1341 | 0.0448 | 0.0508 | 0.0659 | 0.0344 | 0.0388 | 0.0487 |
| $k = 3$ | 0.0437 | 0.0508 | 0.0682 | 0.0314 | 0.0356 | 0.0448 | 0.0258 | 0.0287 | 0.0350 |
| $k = 4$ | 0.0307 | 0.0349 | 0.0455 | 0.0239 | 0.0268 | 0.0327 | 0.0206 | 0.0228 | 0.0274 |
| $k = 5$ | 0.0233 | 0.0262 | 0.0324 | 0.0195 | 0.0213 | 0.0254 | 0.0171 | 0.0187 | 0.0221 |
| Panel B: Critical values of $\tilde{\tilde{K}}$ | | | | | | | | | |
| $k = 1$ | 0.0517 | 0.0595 | 0.0783 | 0.0307 | 0.0345 | 0.0441 | 0.0209 | 0.0231 | 0.0273 |
| $k = 2$ | 0.0239 | 0.0264 | 0.0322 | 0.0180 | 0.0198 | 0.0233 | 0.0141 | 0.0152 | 0.0179 |
| $k = 3$ | 0.0153 | 0.0165 | 0.0195 | 0.0126 | 0.0137 | 0.0159 | 0.0105 | 0.0112 | 0.0129 |
| $k = 4$ | 0.0111 | 0.0120 | 0.0139 | 0.0096 | 0.0103 | 0.0117 | 0.0083 | 0.0088 | 0.0100 |
| $k = 5$ | 0.0087 | 0.0093 | 0.0105 | 0.0078 | 0.0083 | 0.0093 | 0.0069 | 0.0073 | 0.0081 |

Note: Critical values for the 50%, 25%, 20%, 2.5% and 0.5% levels are available upon request.

**Theorem 4.** *Let $\{y_t\}_{t\in\mathbb{N}}$ be generated according to* (25) *with $\{x_t\}_{t\in\mathbb{N}}$ and $\{s_t\}_{t\in\mathbb{N}}$ being generated according to* (2) *and* (26) *under Assumption 5 with $\sigma^2_{\omega,w} = 0$. If the kernel function and the bandwidth parameter in the definition of $\hat{\sigma}^2_{\omega,u\cdot v}$ satisfy Assumptions 3 and 4, then*

$$\tilde{K} \Rightarrow \tau^2_\omega \int \tilde{P}(s)\tilde{P}(s)^*\, ds,$$
$$\tilde{\tilde{K}} \Rightarrow \tau^2_\omega \int \tilde{\tilde{P}}(s)\tilde{\tilde{P}}(s)^*\, ds,$$

*where the limiting processes $\tilde{P}$ and $\tilde{\tilde{P}}$ are defined in Lemma 1 and Lemma 2, respectively.*

**Remark 9.** The limiting distributions depend on the frequency, on the number of regressors in (25) and on the specific form of the deterministic components. Asymptotic critical values for both tests are depicted in Table 2 for a typical set of regressors integrated at some frequency $\omega \in (0, \pi)$.

Next, we investigate the consistency of the tests. Therefore, we derive the limiting distributions of the test statistics $\tilde{K}$ and $\tilde{\tilde{K}}$ under the alternative hypothesis $H_1 : \sigma^2_{\omega,w} > 0$. As an preliminary result, which may be of interest on its own, we first present the asymptotic distribution of $\hat{\sigma}^2_{\omega,u\cdot v}$ under the alternative hypothesis.

**Lemma 3.** *Let $\{y_t\}_{t\in\mathbb{N}}$ be generated according to* (25) *with $\{x_t\}_{t\in\mathbb{N}}$ and $\{s_t\}_{t\in\mathbb{N}}$ being generated according to* (2) *and* (26) *under Assumption 5 with $\sigma^2_{\omega,w} > 0$. Then, with Assumptions 3 and 4 in place, it holds that*

$$\frac{1}{TM_T}\hat{\sigma}^2_{\omega,u\cdot v} \Rightarrow \int_{-\infty}^{\infty} k(s)\, ds \int U(s)U(s)^*\, ds,$$

*with*

$$U(r) = B_w(r) - J(r)' \left( \int \overline{J(s)} J(s)'\, ds \right)^{-1} \int \overline{J(s)} B_w(s)\, ds,$$

*where $J$ is defined in* (8)*.*

After establishing the limiting distribution of $\hat{\sigma}^2_{\omega,u\cdot v}$ for strictly positive $\sigma^2_{\omega,w}$ we are now able to derive the limiting distributions of the test statistics $\tilde{K}$ and $\tilde{\tilde{K}}$ under the alternative hypothesis.

**Proposition 4.** *Let $\{y_t\}_{t\in\mathbb{N}}$ be generated according to* (25) *with $\{x_t\}_{t\in\mathbb{N}}$ and $\{s_t\}_{t\in\mathbb{N}}$ being generated according to* (2) *and* (26) *under Assumption 5 with $\sigma^2_{\omega,w} > 0$. Then, with Assumptions 3 and 4 in place, it holds that*

$$\frac{M_T}{T}\tilde{K} \Rightarrow \frac{\int \tilde{R}(s)\tilde{R}(s)^*\, ds}{\int_{-\infty}^{\infty} k(s)\, ds\, \int U(s)U(s)^*\, ds},$$
$$\frac{M_T}{T}\tilde{\tilde{K}} \Rightarrow \frac{\int \tilde{\tilde{R}}(s)\tilde{\tilde{R}}(s)^*\, ds}{\int_{-\infty}^{\infty} k(s)\, ds\, \int U(s)U(s)^*\, ds},$$

*where*

$$\tilde{R}(r) = \int_0^r B_w(s)\, ds - g(r)' \left( \int \overline{g(s)} g(s)'\, ds \right)^{-1} \int \overline{g(s)} \int_0^s B_w(u)\, du\, dr,$$
$$\tilde{\tilde{R}}(r) = \int_0^r B_w(s)\, ds - h(r)' \left( \int \overline{h(s)} h(s)'\, ds \right)^{-1} \int \overline{h(s)} \int_0^s B_w(u)\, du\, dr.$$

*Thereby, $g$ and $h$ are defined in Theorem 1 and Proposition 1, respectively.*

It follows that the test statistics $\tilde{K}$ and $\tilde{\tilde{K}}$ are consistent since $T/M_T$ diverges as $T \to \infty$. Note that this particular rate of divergence is the same as in the tests of Kwiatkowski et al. (1992) and Shin (1994).

Motivated by the results in the previous sections we also introduce fixed-$b$ versions of the KPSS statistics, labeled by $\tilde{K}^b$ and $\tilde{\tilde{K}}^b$, where $\hat{\sigma}^2_{\omega,u\cdot v}$ is replaced by

$$\tilde{\sigma}^2_{\omega,u\cdot v} = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} e^{-i\omega(t-s)} k\left(\frac{t-s}{bT}\right) \nabla_\omega \tilde{S}^m_{\omega,t} \nabla_\omega \tilde{S}^{m*}_{\omega,s},$$

and

$$\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v} = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} e^{-i\omega(t-s)} k\left(\frac{t-s}{bT}\right) \nabla_\omega \tilde{\tilde{S}}^m_{\omega,t} \nabla_\omega \tilde{\tilde{S}}^{m*}_{\omega,s},$$

respectively. The limiting distributions of the test statistics under the null hypothesis are given in the following theorem.

**Theorem 5.** *Let $\{y_t\}_{t\in\mathbb{N}}$ be generated according to* (25) *with $\{x_t\}_{t\in\mathbb{N}}$ and $\{s_t\}_{t\in\mathbb{N}}$ being generated according to* (2) *and* (26) *under Assumption 5 with $\sigma^2_{\omega,w} = 0$. Furthermore, let $\tilde{K}^b$ and*

$\tilde{\tilde{K}}^b$ *denote the fixed-b versions of the test statistics* (30) *and* (31) *with fixed* $b \in (0,1]$. *Then, it holds that*

$$\tilde{K}^b \Rightarrow \frac{\tau_\omega^2}{Q(\tilde{P})} \int \tilde{P}(s)\tilde{P}(s)^* \, ds,$$
$$\tilde{\tilde{K}}^b \Rightarrow \frac{\tau_\omega^2}{Q(\tilde{\tilde{P}})} \int \tilde{\tilde{P}}(s)\tilde{\tilde{P}}(s)^* \, ds,$$

*where* $Q$ *is defined in Section 4 and* $\tilde{P}$ *and* $\tilde{\tilde{P}}$ *are defined in Lemma 1 and 2, respectively.*

Although the limiting distributions of the fixed-$b$ test statistics are free of nuisance parameters the tests are not consistent. In particular, the following theorem states that $\tilde{K}^b$ and $\tilde{\tilde{K}}^b$ converge to a non-degenerative distribution under the alternative hypothesis with the same rate as under the null hypothesis.

**Proposition 5.** *Let* $\{y_t\}_{t\in\mathbb{N}}$ *be generated according to* (25) *with* $\{x_t\}_{t\in\mathbb{N}}$ *and* $\{s_t\}_{t\in\mathbb{N}}$ *being generated according to* (2) *and* (26) *under Assumption 5 with* $\sigma^2_{\omega,w} > 0$. *Furthermore, let* $\tilde{K}^b$ *and* $\tilde{\tilde{K}}^b$ *denote the fixed-b versions of the test statistics* (30) *and* (31) *with fixed* $b \in (0,1]$. *Then, it holds that*

$$\tilde{K}^b \Rightarrow \frac{1}{Q(\tilde{R})} \int \tilde{R}(s)\tilde{R}(s)^* \, ds,$$
$$\tilde{\tilde{K}}^b \Rightarrow \frac{1}{Q(\tilde{\tilde{R}})} \int \tilde{\tilde{R}}(s)\tilde{\tilde{R}}(s)^* \, ds,$$

*where* $Q$ *is defined in Section 4 and where* $\tilde{R}$ *and* $\tilde{\tilde{R}}$ *are defined in Proposition 4.*

Proposition 5 implies that the power of fixed-b tests does not converge to unity with increasing sample size. Hence, we do not recommend to use this test in applied work since the test decision is based solely on asymptotic critical values.

## 6. Simulation Study

We compare, in this section, the finite sample performance of the SIM-OLS estimator with the SFM-OLS estimator of Gregoir (2010) by means of a simulation study. The results are benchmarked against the (complex) OLS estimator. We use the following data generating process:

$$y_t = e^{-i\omega t}\delta + x_{1,t}\beta_1 + x_{2,t}\beta_2 + u_t, \tag{32}$$
$$x_{k,t} = e^{-i\omega} x_{k,t-1} + \varepsilon_{k,t}, \quad x_{k,0} = 0, \quad k = 1, 2, \tag{33}$$

with

$$u_t = \rho_1 e^{-i\omega} u_{t-1} + \rho_2(\varepsilon_{1,t-1} + \varepsilon_{2,t-1}) + \varepsilon_{3,t}, \quad u_0 = 0,$$

**Table 3:** Finite sample bias and RMSE for the real part of $\beta_1$, $\rho_2 = 0$, $T = 100$.

| $\rho_3$ | $\rho_1$ | OLS | SIM-OLS | SFM-OLS, Bartlett kernel | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $M = 6$ | 10 | 30 | 50 | 70 | 90 | 100 | NW |
| Panel A: Bias | | | | | | | | | | | |
| 0.0 | 0.0 | 0.0004 | -0.0005 | 0.0002 | 0.0001 | 0.0004 | 0.0004 | 0.0004 | 0.0005 | 0.0005 | 0.0003 |
| | 0.3 | 0.0006 | -0.0007 | 0.0002 | 0.0001 | 0.0005 | 0.0005 | 0.0006 | 0.0006 | 0.0006 | 0.0004 |
| | 0.6 | 0.0007 | -0.0012 | 0.0002 | 0.0000 | 0.0006 | 0.0006 | 0.0007 | 0.0007 | 0.0007 | 0.0006 |
| | 0.9 | 0.0015 | -0.0006 | 0.0010 | 0.0007 | 0.0013 | 0.0011 | 0.0013 | 0.0014 | 0.0014 | 0.0006 |
| 0.5 | 0.0 | 0.0202 | -0.0003 | 0.0067 | 0.0096 | 0.0166 | 0.0185 | 0.0191 | 0.0193 | 0.0193 | 0.0034 |
| | 0.3 | 0.0282 | 0.0012 | 0.0107 | 0.0140 | 0.0233 | 0.0259 | 0.0266 | 0.0270 | 0.0271 | 0.0095 |
| | 0.6 | 0.0464 | 0.0079 | 0.0230 | 0.0256 | 0.0385 | 0.0426 | 0.0439 | 0.0444 | 0.0446 | 0.0254 |
| | 0.9 | 0.1310 | 0.0818 | 0.1082 | 0.1051 | 0.1141 | 0.1216 | 0.1249 | 0.1263 | 0.1266 | 0.1087 |
| Panel B: RMSE | | | | | | | | | | | |
| 0.0 | 0.0 | 0.0248 | 0.0345 | 0.0268 | 0.0267 | 0.0256 | 0.0253 | 0.0251 | 0.0250 | 0.0250 | 0.0263 |
| | 0.3 | 0.0345 | 0.0488 | 0.0376 | 0.0374 | 0.0357 | 0.0351 | 0.0349 | 0.0348 | 0.0348 | 0.0368 |
| | 0.6 | 0.0561 | 0.0832 | 0.0629 | 0.0627 | 0.0586 | 0.0573 | 0.0568 | 0.0567 | 0.0566 | 0.0614 |
| | 0.9 | 0.1529 | 0.2513 | 0.1781 | 0.1816 | 0.1670 | 0.1583 | 0.1559 | 0.1553 | 0.1552 | 0.1701 |
| 0.5 | 0.0 | 0.0383 | 0.0318 | 0.0278 | 0.0304 | 0.0364 | 0.0376 | 0.0378 | 0.0379 | 0.0379 | 0.0251 |
| | 0.3 | 0.0532 | 0.0451 | 0.0397 | 0.0429 | 0.0506 | 0.0523 | 0.0525 | 0.0526 | 0.0527 | 0.0373 |
| | 0.6 | 0.0860 | 0.0781 | 0.0695 | 0.0726 | 0.0825 | 0.0847 | 0.0851 | 0.0852 | 0.0853 | 0.0691 |
| | 0.9 | 0.2254 | 0.2668 | 0.2241 | 0.2258 | 0.2256 | 0.2250 | 0.2249 | 0.2251 | 0.2251 | 0.2259 |

where

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix} \sim NID\left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho_3 & \rho_3 \\ \rho_3 & 1 & \rho_3 \\ \rho_3 & \rho_3 & 1 \end{bmatrix} \right).$$

Throughout, we consider the frequency $\omega = \pi/2$ and set the regression parameters to $\delta = 1+2i$, $\beta_1 = 3+4i$ and $\beta_2 = 5+6i$. The values for $\rho_1$ and $\rho_2$ are chosen from the set $\{0, 0.3, 0.6, 0.9\}$ and $\rho_3$ is either zero or $1/2$. The parameter $\rho_1$ controls the distance of the zero of the autoregressive polynomial of $u_t$ to the unit root $e^{i\omega}$. The second parameter, $\rho_2$, controls the amount of intertemporal correlation between the regression errors and the regressor innovations. The third parameter, $\rho_3$, controls the amount of contemporaneous correlation in $\varepsilon_t$. Hence, if either $\rho_2$ or $\rho_3$ is different from zero then there is regressor endogeneity and the OLS estimator of the parameters in (32) gets contaminated by a second order bias. We consider sample sizes $T \in \{50, 100, 200, 500, 1000\}$ and the number of replications is 5000.

Spectral density estimators that are required for the computation of the SFM-OLS estimator are computed via (15) for which we choose the Bartlett, the Parzen and the Quadratic Spectral (QS) kernels. The bandwidth parameter is set to $M = bT$ with $b$ chosen from an equidistant grid on the unit interval. Additionally, we compute the bandwidth according to the data dependent bandwidth selection rule of Gregoir (2006, 2010).[1]

## 6.1. Estimation Performance

We briefly summarize the simulation results for the bias and the root mean squared error (RMSE) of $\mathrm{Re}(\beta_1)$ which are given in Table 3. The SFM-OLS results are very similar for all kernel functions, so we only report the results for the Bartlett kernel. For brevity, only the results for $\rho_2 = 0$ and $T = 100$ are presented.

If $\rho_3 = 0$ (no regressor endogeneity) and if there is no seasonal autocorrelation in the regression errors ($\rho_1 = 0$) then all estimators exhibit a bias similar in magnitude. This bias increases slightly when $\rho_1$ get larger. This effect is, however, much more prominent for the SFM-OLS estimator, especially when a larger bandwidth parameter is used. This effect is amplified if $\rho_3 = 0.5$. In this case the bias of the SIM-OLS estimator is still small in magnitude whereas the bias of the OLS and SFM-OLS estimators get comparatively large. Again, the latter gets larger with increasing bandwidth parameter. This effect is not surprising because in case of regressor endogeneity the limiting distribution of the OLS estimator is contaminated by an additive bias. The SFM-OLS corrects for this bias. However, this bias correction crucially relies on the estimation of $\Omega_\omega$ and $\Delta_\omega$ which might be poor in small samples or when the bandwidth parameter is chosen inappropriately. The advantage of the SIM-OLS is that it does not hinge upon any nonparametric corrections.

The RMSE of SFM-OLS estimator gets higher with increasing $\rho_1$, where the bandwidth parameter does not seem to have any impact on the estimation performance. This changes when comparing the estimation results for different values of $\rho_3$. In this case the RMSE differences become apparent only if the estimation is performed with a large bandwidth parameter. The results for the OLS estimator are comparable to those of the SFM-OLS estimator with large bandwidth. The RMSE of the SIM-OLS estimator is, apart from a few exceptions, consistently higher than the RMSE of the OLS or SFM-OLS estimator. This is because the variance of the SIM-OLS estimator is larger than the variance of the other two estimators, see Vogelsang and Wagner (2014, Proposition 2)[2].

## 6.2. Inference using SIM-OLS

We now provide some simulation results for the hypothesis tests introduced in Sections 3 and 4. Tables 4 and 5 report the empirical sizes of the $t$-test for the null hypothesis $H_0\colon \mathrm{Re}(\beta_1) = 3$ against the two-sided alternative $H_1\colon \mathrm{Re}(\beta_1) \neq 3$ for sample sizes $T = 100$ and $T = 500$, respectively. For the computation of the bandwidth parameter we apply the data dependent bandwidth selection rule of Gregoir (2006, 2010).

When the parameters $\rho_1$, $\rho_2$ and $\rho_3$ are equal to zero all three estimators perform well with OLS showing the best performance. The empirical sizes get more distorted when at least one of the parameters increases. Clearly, the OLS-based textbook $t$-test suffers from the highest size distortions as it does neither correct for the endogeneity bias nor for the autocorrelation in the regression error terms. The empirical sizes of the $t$-tests based upon the SFM-OLS estimator

[1] Gregoir generalizes the bandwidth selection rule of Newey and West (1994), but without considering the QS kernel. However, an extension for this particular kernel function can be derived analogously.

[2] Vogelsang and Wagner show that the variance that corresponds to the $\delta$ and $\beta$ parts of the SIM-OLS estimator is larger (in terms of the Loewner order) than the variance of the SFM-OLS estimator. Although they consider the zero frequency case only it can be shown that their result remains valid for the general frequency case.

**Table 4:** Empirical null rejection probabilities for $H_0\colon \mathrm{Re}(\beta_1) = 3$ against the two-sided alternative, data dependent bandwidth, $T = 100$, 5% level.

| $\rho_1$ | $\rho_2$ | OLS | SFM-OLS | | | SIM-OLS | | |
|---|---|---|---|---|---|---|---|---|
| | | | Bartlett | Parzen | QS | Bartlett | Parzen | QS |
| Panel A: $\rho_3 = 0$ | | | | | | | | |
| 0.0 | 0.0 | 0.0500 | 0.0802 | 0.0964 | 0.0928 | 0.0596 | 0.0674 | 0.0652 |
| | 0.3 | 0.0648 | 0.0778 | 0.0896 | 0.0868 | 0.0486 | 0.0616 | 0.0580 |
| | 0.6 | 0.0814 | 0.0754 | 0.0858 | 0.0814 | 0.0238 | 0.0390 | 0.0378 |
| | 0.9 | 0.0912 | 0.0712 | 0.0820 | 0.0764 | 0.0120 | 0.0224 | 0.0224 |
| 0.3 | 0.0 | 0.1630 | 0.1464 | 0.1290 | 0.1308 | 0.1200 | 0.1024 | 0.1010 |
| | 0.3 | 0.1726 | 0.1488 | 0.1318 | 0.1298 | 0.0998 | 0.0884 | 0.0868 |
| | 0.6 | 0.2042 | 0.1506 | 0.1232 | 0.1224 | 0.0626 | 0.0624 | 0.0596 |
| | 0.9 | 0.2258 | 0.1580 | 0.1216 | 0.1218 | 0.0300 | 0.0344 | 0.0330 |
| 0.6 | 0.0 | 0.3280 | 0.2340 | 0.1974 | 0.2028 | 0.2112 | 0.1704 | 0.1756 |
| | 0.3 | 0.3440 | 0.2392 | 0.1962 | 0.2034 | 0.1964 | 0.1538 | 0.1584 |
| | 0.6 | 0.3720 | 0.2336 | 0.1952 | 0.2048 | 0.1418 | 0.1176 | 0.1212 |
| | 0.9 | 0.3918 | 0.2262 | 0.2006 | 0.2034 | 0.0970 | 0.0868 | 0.0880 |
| 0.9 | 0.0 | 0.5808 | 0.3722 | 0.3350 | 0.3328 | 0.3948 | 0.3374 | 0.3292 |
| | 0.3 | 0.6074 | 0.4078 | 0.3622 | 0.3610 | 0.4076 | 0.3532 | 0.3442 |
| | 0.6 | 0.6318 | 0.4372 | 0.3904 | 0.3898 | 0.4290 | 0.3680 | 0.3586 |
| | 0.9 | 0.6422 | 0.4690 | 0.4210 | 0.4150 | 0.4344 | 0.3766 | 0.3664 |
| Panel B: $\rho_3 = 0.5$ | | | | | | | | |
| 0.0 | 0.0 | 0.1430 | 0.0736 | 0.0946 | 0.0884 | 0.0502 | 0.0546 | 0.0540 |
| | 0.3 | 0.1414 | 0.0774 | 0.0930 | 0.0882 | 0.0294 | 0.0384 | 0.0372 |
| | 0.6 | 0.1378 | 0.0756 | 0.0846 | 0.0802 | 0.0074 | 0.0178 | 0.0174 |
| | 0.9 | 0.1332 | 0.0716 | 0.0822 | 0.0724 | 0.0022 | 0.0078 | 0.0086 |
| 0.3 | 0.0 | 0.2912 | 0.1582 | 0.1384 | 0.1386 | 0.1052 | 0.0834 | 0.0864 |
| | 0.3 | 0.2850 | 0.1666 | 0.1376 | 0.1370 | 0.0728 | 0.0618 | 0.0644 |
| | 0.6 | 0.2800 | 0.1666 | 0.1308 | 0.1322 | 0.0266 | 0.0286 | 0.0290 |
| | 0.9 | 0.2754 | 0.1696 | 0.1274 | 0.1258 | 0.0084 | 0.0128 | 0.0118 |
| 0.6 | 0.0 | 0.4692 | 0.2736 | 0.2280 | 0.2318 | 0.1926 | 0.1486 | 0.1562 |
| | 0.3 | 0.4654 | 0.2816 | 0.2334 | 0.2420 | 0.1586 | 0.1272 | 0.1308 |
| | 0.6 | 0.4596 | 0.2720 | 0.2298 | 0.2356 | 0.0988 | 0.0838 | 0.0874 |
| | 0.9 | 0.4546 | 0.2586 | 0.2256 | 0.2268 | 0.0612 | 0.0554 | 0.0580 |
| 0.9 | 0.0 | 0.7130 | 0.4514 | 0.4224 | 0.4250 | 0.3684 | 0.3408 | 0.3430 |
| | 0.3 | 0.7034 | 0.4640 | 0.4318 | 0.4338 | 0.3716 | 0.3410 | 0.3410 |
| | 0.6 | 0.6982 | 0.4856 | 0.4482 | 0.4468 | 0.3964 | 0.3590 | 0.3570 |
| | 0.9 | 0.6984 | 0.5066 | 0.4696 | 0.4626 | 0.4170 | 0.3716 | 0.3660 |

are considerably higher than those based upon the SIM-OLS estimator. Interestingly, both tests tend to underreject the null hypothesis if $\rho_2$ increases. We observe this in particular for the SIM-OLS based test in small samples. It is also remarkable that the choice of the kernel function affects the results heavily. In particular, when we use the Bartlett kernel the tests perform worse than with the Parzen or QS kernel. However, the SIM-OLS based tests do not react as sensitive to the kernel choice as the SFM-OLS based tests.

Next we investigate the performance of the fixed-$b$ version of the $t$-type test and demonstrate its superiority over conventional tests. For the calculation of $\hat{\sigma}^2_{\omega,u\cdot v}$ and $\tilde{\sigma}^2_{\omega,u\cdot v}$ we set the bandwidth parameter to $M = bT$ with 50 equidistantly distributed values on the unit interval

**Table 5:** Empirical null rejection probabilities for $H_0\colon \mathrm{Re}(\beta_1) = 3$ against the two-sided alternative, data dependent bandwidth, $T = 500$, 5% level.

| $\rho_1$ | $\rho_2$ | OLS | SFM-OLS | | | SIM-OLS | | |
|---|---|---|---|---|---|---|---|---|
| | | | Bartlett | Parzen | QS | Bartlett | Parzen | QS |
| Panel A: $\rho_3 = 0$ | | | | | | | | |
| 0.0 | 0.0 | 0.0490 | 0.0560 | 0.0640 | 0.0608 | 0.0546 | 0.0572 | 0.0562 |
| | 0.3 | 0.0684 | 0.0522 | 0.0598 | 0.0592 | 0.0466 | 0.0538 | 0.0536 |
| | 0.6 | 0.0884 | 0.0456 | 0.0556 | 0.0524 | 0.0266 | 0.0456 | 0.0436 |
| | 0.9 | 0.1026 | 0.0354 | 0.0418 | 0.0414 | 0.0126 | 0.0322 | 0.0326 |
| 0.3 | 0.0 | 0.1530 | 0.0926 | 0.0784 | 0.0778 | 0.0924 | 0.0704 | 0.0716 |
| | 0.3 | 0.1774 | 0.0940 | 0.0738 | 0.0740 | 0.0742 | 0.0638 | 0.0636 |
| | 0.6 | 0.2146 | 0.0862 | 0.0648 | 0.0656 | 0.0448 | 0.0486 | 0.0498 |
| | 0.9 | 0.2338 | 0.0822 | 0.0540 | 0.0538 | 0.0230 | 0.0322 | 0.0328 |
| 0.6 | 0.0 | 0.3382 | 0.1562 | 0.1130 | 0.1204 | 0.1464 | 0.1046 | 0.1118 |
| | 0.3 | 0.3552 | 0.1440 | 0.1112 | 0.1126 | 0.1206 | 0.0930 | 0.0966 |
| | 0.6 | 0.3880 | 0.1274 | 0.1008 | 0.0984 | 0.0724 | 0.0636 | 0.0664 |
| | 0.9 | 0.4124 | 0.0980 | 0.0850 | 0.0860 | 0.0390 | 0.0412 | 0.0418 |
| 0.9 | 0.0 | 0.6504 | 0.2938 | 0.2084 | 0.2094 | 0.2874 | 0.1918 | 0.1998 |
| | 0.3 | 0.6586 | 0.2920 | 0.2098 | 0.2064 | 0.2578 | 0.1760 | 0.1836 |
| | 0.6 | 0.6836 | 0.2530 | 0.2008 | 0.2096 | 0.1634 | 0.1248 | 0.1406 |
| | 0.9 | 0.7044 | 0.2210 | 0.1926 | 0.2086 | 0.0950 | 0.0890 | 0.1012 |
| Panel B: $\rho_3 = 0.5$ | | | | | | | | |
| 0.0 | 0.0 | 0.1464 | 0.0618 | 0.0680 | 0.0668 | 0.0550 | 0.0560 | 0.0568 |
| | 0.3 | 0.1470 | 0.0550 | 0.0640 | 0.0634 | 0.0368 | 0.0460 | 0.0468 |
| | 0.6 | 0.1468 | 0.0414 | 0.0478 | 0.0478 | 0.0096 | 0.0298 | 0.0302 |
| | 0.9 | 0.1444 | 0.0326 | 0.0358 | 0.0340 | 0.0026 | 0.0132 | 0.0138 |
| 0.3 | 0.0 | 0.2912 | 0.1052 | 0.0844 | 0.0826 | 0.0894 | 0.0688 | 0.0708 |
| | 0.3 | 0.2930 | 0.0992 | 0.0774 | 0.0784 | 0.0562 | 0.0526 | 0.0536 |
| | 0.6 | 0.2940 | 0.0904 | 0.0628 | 0.0630 | 0.0186 | 0.0288 | 0.0304 |
| | 0.9 | 0.2892 | 0.0820 | 0.0532 | 0.0510 | 0.0052 | 0.0106 | 0.0120 |
| 0.6 | 0.0 | 0.4672 | 0.1844 | 0.1352 | 0.1344 | 0.1340 | 0.0978 | 0.1008 |
| | 0.3 | 0.4740 | 0.1770 | 0.1270 | 0.1244 | 0.0920 | 0.0708 | 0.0710 |
| | 0.6 | 0.4718 | 0.1508 | 0.1100 | 0.1080 | 0.0394 | 0.0362 | 0.0378 |
| | 0.9 | 0.4724 | 0.1172 | 0.0926 | 0.0884 | 0.0116 | 0.0126 | 0.0116 |
| 0.9 | 0.0 | 0.7330 | 0.2676 | 0.2208 | 0.2624 | 0.1600 | 0.1350 | 0.1780 |
| | 0.3 | 0.7284 | 0.3192 | 0.2394 | 0.2574 | 0.1666 | 0.1226 | 0.1450 |
| | 0.6 | 0.7344 | 0.3794 | 0.2696 | 0.2462 | 0.1390 | 0.0948 | 0.0938 |
| | 0.9 | 0.7318 | 0.2792 | 0.2242 | 0.2398 | 0.0650 | 0.0548 | 0.0606 |

for $b$. Figure 1 illustrates the impact that the bandwidth has on the empirical null rejection probabilities of the $t$-type tests. We only display results where we use the Bartlett kernel since there are only small qualitative differences when using the Parzen or the QS kernel[3]. SFM-OLS and SIM-OLS based tests that rely on the conventional estimator $\hat{\sigma}^2_{\omega,u\cdot v}$ are labeled by SFM-OLS and SIM-OLS (O), respectively. In both cases the tests are carried out using standard normal critical values. Test statistics that include $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$ are labeled by SIM-OLS (N) and SIM-

[3]Although the results are qualitatively similar we have to mention that the tests based upon the Bartlett kernel suffer from the highest size-distortions. This is not surprising as we have already observed this phenomenon for the conventional tests in Tables 4 and 5. The results for the Parzen and QS kernel are available upon request.

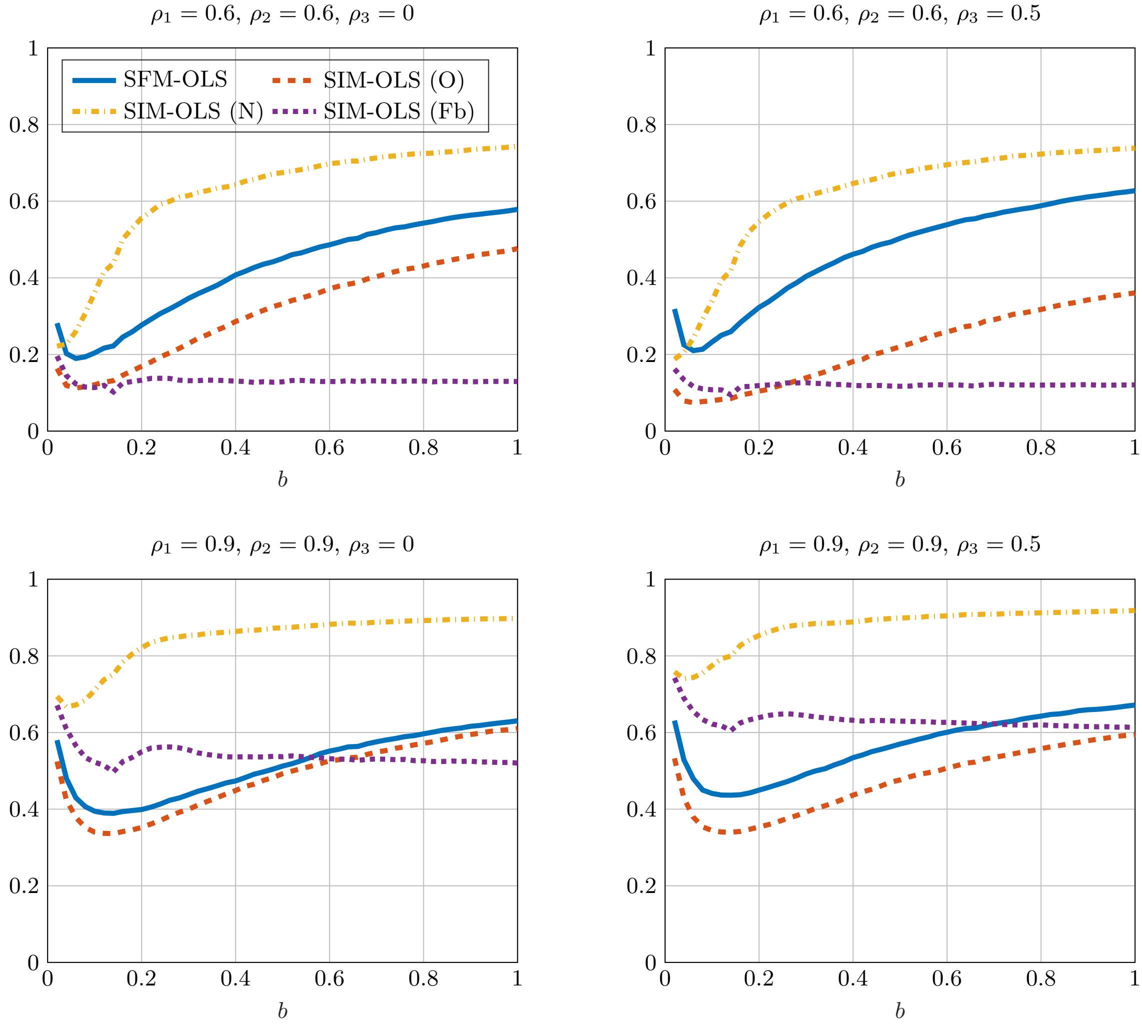


**Figure 1:** Empirical null rejection probabilities, $t$-test for $\mathrm{Re}(\beta_1)$, $T = 100$, Bartlett kernel.

OLS (Fb), respectively. The former is carried out using standard normal critical values and is included only as benchmark whereas the latter uses fixed-$b$ critical values from Table 1.

The empirical null rejection probabilities of the SFM-OLS based tests are always higher than those of SIM-OLS (Fb), especially if $\rho_3 = 0.5$. As expected, SIM-OLS (N) tests exhibit the highest size distortions since these are carried out using wrong critical values. The SIM-OLS (Fb) tests perform quite decently if $\rho_1 = \rho_2 = 0.6$. Nevertheless, the empirical sizes are around 13% for nearly all values of $b$ (except for very small values where they are slightly higher). If $\rho_1 = \rho_2 = 0.9$ the all tests fail miserably in holding the significance level. However, the value of $b$ impacts the empirical sizes of the SIM-OLS (fb) tests only marginally.

Figure 2 illustrates how the size distortions of the SIM-OLS (fb) tests vanish asymptotically. If $T = 200$ and $\rho_1 = \rho_2 = 0.6$ the empirical sizes drop below 10% for all values of $b$ and they approach the nominal 5% level for $T = 500$. If $\rho_1 = \rho_2 = 0.9$ the test statistics struggle to reach the nominal significance level for moderate sample sizes but evidently approach 5% when $T = 1000$.

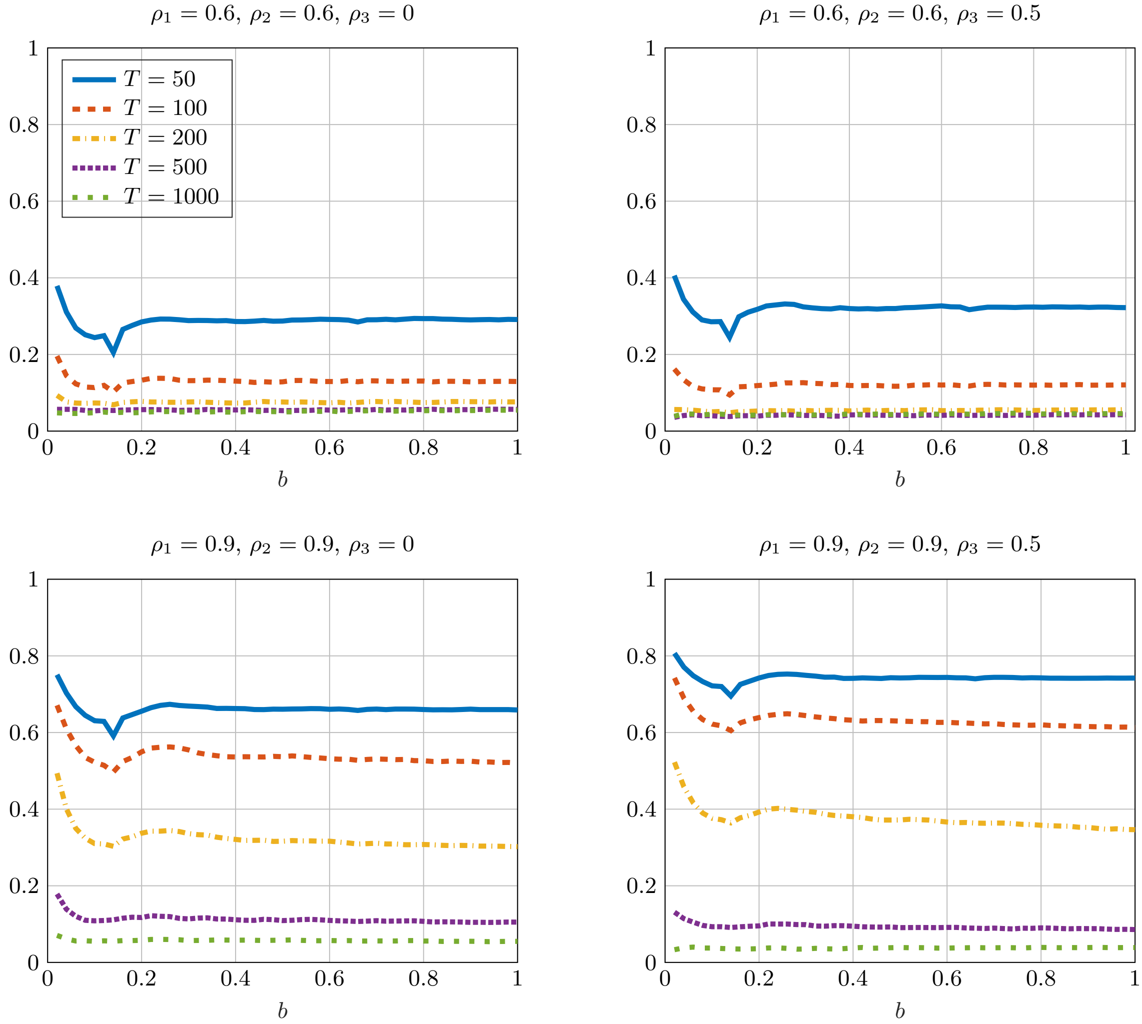


**Figure 2:** Empirical null rejection probabilities, fixed-$b$ $t$-type test for $\mathrm{Re}(\beta_1)$, Bartlett kernel.

We now turn to the power properties of the tests. Starting from the true value of $\mathrm{Re}(\beta_1) = 3$ we consider under the alternative $\mathrm{Re}(\beta_1) \in (3, 4]$ using 20 values generated on an equidistant grid with mesh 0.05. We focus on the size-corrected power since it is a useful tool for theoretical comparisons as it overcomes potential over- (and under-) rejection problems under the null hypothesis. For brevity, we only report the results for $T = 100$, $\rho_1 = \rho_2 = 0.6$ and $\rho_3 = 0$. Figure 3 displays the power curves, starting at 5% under the null hypothesis, for the $t$-type tests based upon the estimators considered previously. The SFM-OLS and SIM-OLS (O) tests use the data dependent bandwidth rule for spectral density estimation and we display the power curves for the fixed-$b$ versions using $b = 0.02$, $b = 0.3$ and $b = 1$. The left and right panel display the power curves when the corresponding test statistics are calculated using the Bartlett and QS kernel, respectively. Two main observations can be made. First, the power curves of the SFM-OLS and the OLS based tests are fairly indistinguishable whereas the SIM-OLS (O) test exhibits slightly lower size corrected power. For SIM-OLS (Fb) tests we have to distinguish between the chosen kernel function. If the Bartlett kernel is used then the fixed-$b$ tests have

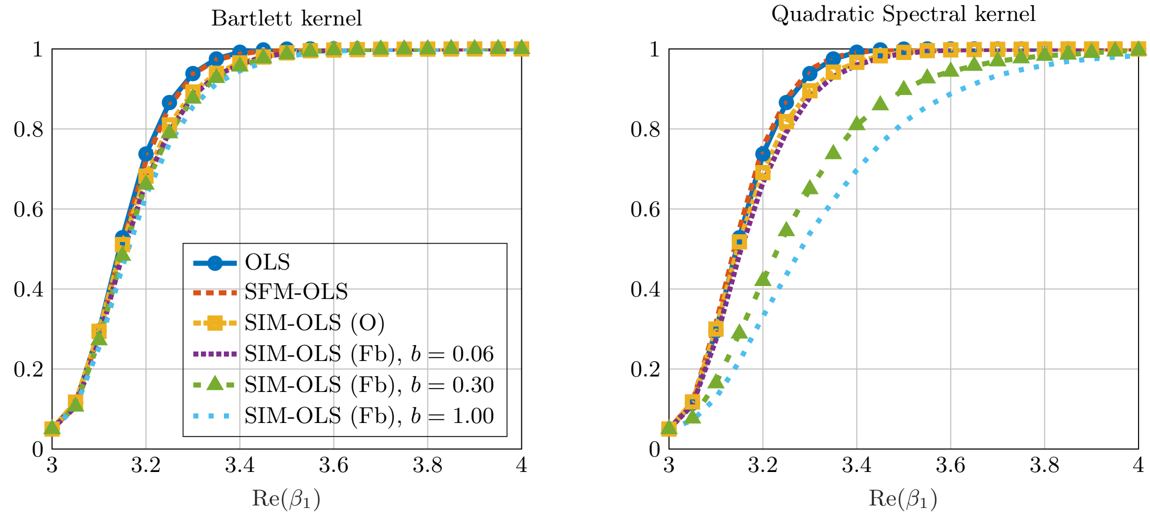


**Figure 3:** Size corrected power of $t$-type tests for $\text{Re}(\beta_1)$, $T = 100$, $\rho_1 = \rho_2 = 0.6$ and $\rho_3 = 0$. Data dependent bandwidth selection rule for spectral density estimation used for SFM-OLS and SIM-OLS (O).

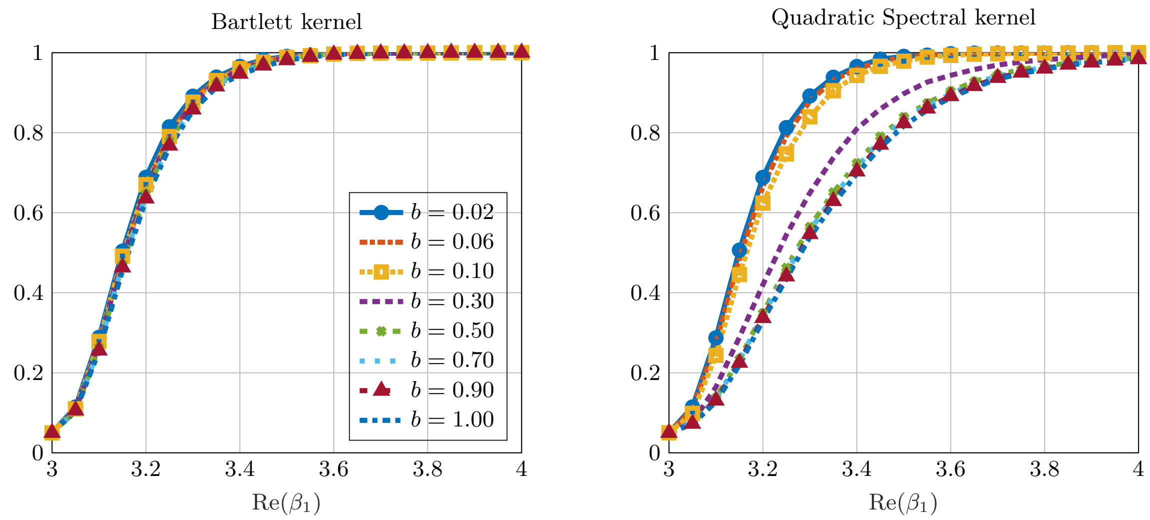


**Figure 4:** Size corrected power of $t$-type tests for $\text{Re}(\beta_1)$, $T = 100$, $\rho_1 = \rho_2 = 0.6$ and $\rho_3 = 0$. Tests are based on the fixed-$b$ version of SIM-OLS for various values of $b$.

almost the same power, irrespective of $b$. On the other hand, if the QS kernel[4] is used the power drops drastically for higher values of $b$. To illustrate this effect Figure 4 displays the power curves for the fixed-$b$ tests for eight different values of $b$. Obviously, there are virtually no power differences when the Bartlett kernel is used but a large power decrease with increasing $b$ when we use the QS kernel. Similar observations regarding the sensitivity of the QS kernel (as well as the Daniell kernel) to the choice of $b$ have already been made by Kiefer and Vogelsang (2005) and Vogelsang and Wagner (2014).

[4]Similar observations are also made for the Parzen kernel.

**Table 6:** Empirical null rejection probabilities of cointegration tests, QS kernel, 5% level.

| | $\rho_1 = \rho_2$ | $M = 4$ | 8 | 12 | 16 | 20 | Auto |
|---|---|---|---|---|---|---|---|
| Panel A: $T = 100$ | | | | | | | |
| $K^G$ | 0.0 | 0.0318 | 0.0256 | 0.0322 | 0.1100 | 0.3656 | 0.0358 |
| | 0.3 | 0.0368 | 0.0236 | 0.0234 | 0.0614 | 0.2562 | 0.0776 |
| | 0.6 | 0.0778 | 0.0248 | 0.0196 | 0.0358 | 0.1376 | 0.1534 |
| | 0.9 | 0.5982 | 0.1372 | 0.0510 | 0.0456 | 0.0996 | 0.1432 |
| $\tilde{K}$ | 0.0 | 0.0898 | 0.3282 | 0.8278 | 0.9630 | 0.9876 | 0.0724 |
| | 0.3 | 0.0888 | 0.1922 | 0.5972 | 0.8606 | 0.9402 | 0.1226 |
| | 0.6 | 0.1270 | 0.1430 | 0.3390 | 0.6318 | 0.7890 | 0.1920 |
| | 0.9 | 0.7348 | 0.3924 | 0.4040 | 0.5684 | 0.7282 | 0.5208 |
| $\tilde{\tilde{K}}$ | 0.0 | 0.1728 | 0.6622 | 0.8330 | 0.8994 | 0.9416 | 0.1044 |
| | 0.3 | 0.0756 | 0.3786 | 0.6368 | 0.7644 | 0.8502 | 0.1128 |
| | 0.6 | 0.0258 | 0.0848 | 0.2544 | 0.4096 | 0.5380 | 0.0726 |
| | 0.9 | 0.0058 | 0.0008 | 0.0046 | 0.0218 | 0.0522 | 0.0408 |
| Panel B: $T = 200$ | | | | | | | |
| $K^G$ | 0.0 | 0.0444 | 0.0392 | 0.0358 | 0.0340 | 0.0370 | 0.0440 |
| | 0.3 | 0.0494 | 0.0344 | 0.0314 | 0.0292 | 0.0286 | 0.0538 |
| | 0.6 | 0.1046 | 0.0380 | 0.0266 | 0.0220 | 0.0192 | 0.1056 |
| | 0.9 | 0.8568 | 0.3332 | 0.1410 | 0.0688 | 0.0416 | 0.1960 |
| $\tilde{K}$ | 0.0 | 0.0682 | 0.0996 | 0.1732 | 0.3492 | 0.6208 | 0.0702 |
| | 0.3 | 0.0742 | 0.0806 | 0.1176 | 0.2020 | 0.3872 | 0.0858 |
| | 0.6 | 0.0962 | 0.0654 | 0.0726 | 0.0966 | 0.1516 | 0.1054 |
| | 0.9 | 0.8550 | 0.4144 | 0.2464 | 0.2028 | 0.2076 | 0.2950 |
| $\tilde{\tilde{K}}$ | 0.0 | 0.0738 | 0.1956 | 0.4916 | 0.6906 | 0.7884 | 0.1004 |
| | 0.3 | 0.0624 | 0.0982 | 0.2844 | 0.4782 | 0.6188 | 0.0934 |
| | 0.6 | 0.0630 | 0.0452 | 0.0898 | 0.1862 | 0.2912 | 0.0792 |
| | 0.9 | 0.0630 | 0.0066 | 0.0038 | 0.0040 | 0.0084 | 0.0052 |

The block-rows $K^G$ report the results from using the test statistic of the SFM-OLS based cointegration test of Gregoir (2010). The block-rows $\tilde{K}$ and $\tilde{\tilde{K}}$ report the results from using the test statistics introduced in section 5 with critical values given in Table 2.

## 6.3. Cointegration Tests

Let us now briefly turn to cointegration testing. Table 6 displays the empirical null rejection probabilities of the cointegration tests introduced in Section 5 where the data is once again simulated according to (32). We compare the SIM-OLS residuals based tests, denoted by $\tilde{K}$ and $\tilde{\tilde{K}}$, with the SFM-OLS residuals based test of Gregoir (2010), labeled by $K^G$. For brevity, we only report the results for $\rho_1 = \rho_2$, $\rho_3 = 0$ and for $\hat{\sigma}^2_{\omega,u\cdot v}$ being computed using the QS kernel. For the latter we use fixed bandwidths as well as the data dependent bandwidth rule of Gregoir (2010).

We first summarize our findings when the data dependent bandwidth rule is used. If $T = 100$ and if the parameters $\rho_1$ and $\rho_2$ are at most 0.3 then the size distortions of the $K^G$ test are close to the nominal 5% level. Otherwise the tests exhibit empirical sizes at around 15%. The $\tilde{K}$ test yields acceptable empirical sizes only for $\rho_1 = \rho_2 = 0$ and overrejects the null hypothesis in all other scenarios. If $\rho_1 = \rho_2 = 0.9$ the empirical null rejection probability even exceeds 50%. On the contrary, the $\tilde{\tilde{K}}$ test has problems to keep the significance level for low values of $\rho_1$ and

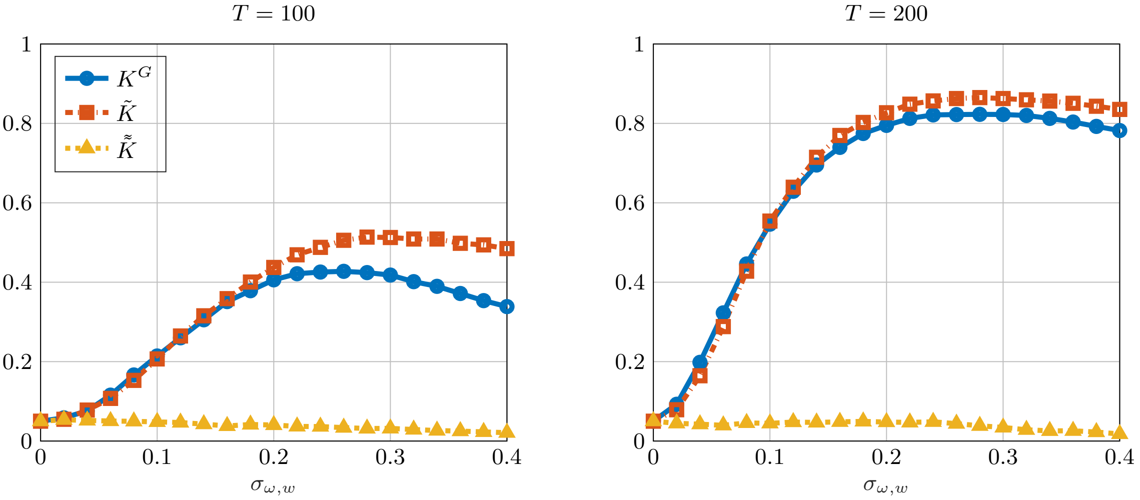


**Figure 5:** Size-corrected power of cointegration tests. Data generating process with $\rho_1 = 0.3$, $\rho_2 = 0.3$ and $\rho_3 = 0$. Spectral density estimate with QS kernel and data dependent bandwidth selection rule.

$\rho_2$ whereas the empirical sizes are close to 5% for higher parameter values. In case of $T = 200$ the empirical null rejection probabilities of the $\tilde{K}$ test improve for all parameter constellations. However, for $\rho_1 = \rho_2 = 0.9$ the empirical size is sill very high with 29.5%. The empirical sizes of the $\tilde{\tilde{K}}$ test do not improve with the higher sample size except in case $\rho_1 = \rho_2 = 0.9$, where the test strongly underrejects. The performance results of the $K^G$ test are mixed. If $\rho_1$ and $\rho_2$ are at most 0.6 the test still performs extremely well. However, when the parameters are equal to 0.9 the empirical size jumps to 19.6% (compared to 14.3% when $T = 100$), an observation that is unexpected when the sample size increases.

If we use fixed bandwidths the picture gets cloudy, especially for the small sample size $T = 100$. For example, if $M = 4$ the $K^G$ test improves (compared to the automatic bandwidth selection case) for values of $\rho_1$ and $\rho_2$ up to 0.6 but breaks down to an empirical null rejection probability of around 60% if the parameters are set to 0.9. This is not surprising as the spectral density requires a higher parameter the closer the roots of the autoregressive polynomial approach the unit circle. Therefore it is also not surprising that the empirical null rejection probabilities of the $K^G$ test drop closely to the nominal level if the bandwidth is set to $M = 12$ or $M = 16$. On the other hand, if $M = 16$ and $\rho_1 = \rho_2 = 0$ the empirical size rises up to 11%. The $\tilde{K}$ test performs very badly for $M \geq 8$, even if $\rho_1$ is high. If $M = 4$ the test exhibits only small size distortions for small values of $\rho_1$ and $\rho_2$ but fails miserably if these parameters are set to 0.9. The $\tilde{\tilde{K}}$ test shows once again some strange behavior. If $\rho_1$ and $\rho_2$ are equal to zero this test is massively overrejecting the null hypothesis whereas it heavily underrejects the null hypothesis if the parameters get large. If $T = 200$ we observe that the $K^G$ test is close to 5% (but slightly below) if $\rho_1$ and $\rho_2$ are zero or 0.3, irrespective of the choice of $M$. It still overrejects the null hypothesis when the parameters are large and the bandwidth is chosen too small. The empirical sizes of the $\tilde{K}$ test also improve for the larger sample size but this test still suffers from severe size distortions if large bandwidths are used. The performance of the $\tilde{\tilde{K}}$ test is quite decent for $M = 4$. If however, larger bandwidths are used the test drastically overrejects (underrejects) the null hypothesis if the parameters $\rho_1$ and $\rho_2$ are small (large).

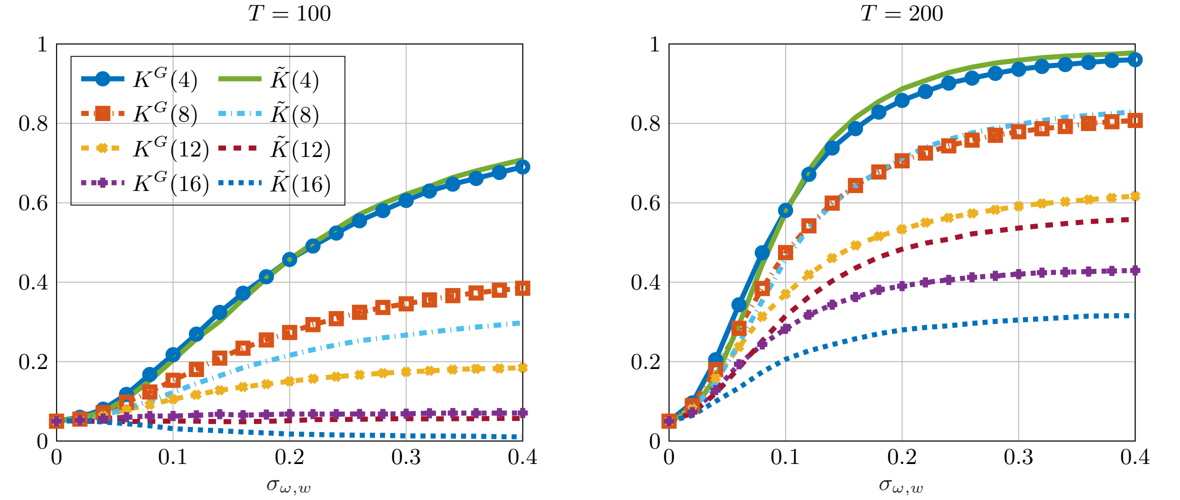


**Figure 6:** Size-corrected power of cointegration tests. Data generating process with $\rho_1 = 0.3$, $\rho_2 = 0.3$ and $\rho_3 = 0$. Spectral density estimate with QS kernel and various fixed bandwidths (in parenthesis).

We close this section by investigating the size corrected power of the cointegration tests. Therefore, we add $\sigma_{\omega,w} z_t$ to the regression error $u_t$, where we generate $z_t$ according to (26) with $z_0 = 0$ and i.i.d. standard normally distributed $w_t$, which are independent of $u_t$ and $v_t$. We vary $\sigma_{\omega,w}$ on an equidistant grid of 21 values on $[0, 0.4]$, where $\sigma_{\omega,w} = 0$ corresponds to the null hypothesis of seasonal cointegration. For brevity, we restrict our report to the case $\rho_1 = \rho_2 = 0.3$.

We start by depicting our observations when the bandwidths are chosen according to the data dependent bandwidth rule. Figure 5 displays the power curves of the three considered tests for $T = 100$ and $T = 200$, repsectively. There are two main findings here. First, the $\tilde{\tilde{K}}$ test seems to have no power at all. Second, the power curves of the $K^G$ and $\tilde{K}$ tests initially rise with increasing $\sigma_{\omega,w}$ and reach a local maximum at about $\sigma_{\omega,w} = 0.2$, from where they begin to fall. The reason for this strange behaviour is the data dependent bandwidth rule which is designed to deliver appropriate bandwidths under the null hypothesis. Under the alternative hypothesis, however, it is not clear if the tests are still consistent when the data dependent bandwidth rule is used. Similar observations have been made by Xiao and Phillips (2002). If we use fixed bandwidths the picture gets more familiar. Figure 6 displays the power curves for the $K^G$ and the $\tilde{K}$ tests[5] for four different bandwidths. The power of both tests increases with increasing sample size, regardless of the chosen bandwidth. Furthermore, the power curves of the $K^G$ and $\tilde{K}$ tests are nearly equal for bandwidths $M = 4$ and $M = 8$. For larger bandwidths, however, the $K^G$ test has clearly more power than the $\tilde{K}$ test.

## 7. Conclusion

This paper presents the integrated modified OLS estimator for seasonally cointegrating regressions at an arbitrary frequency. In contrast to the SFM-OLS estimator of Gregoir (2010)

[5]We do not present the power curves for the $\tilde{\tilde{K}}$ test since they are similar in shape to the power curve in Figure 5. If $T = 1000$ (or higher) the test has higher power and the power curves increase slowly. Corresponding simulation results are available upon request.

the SIM-OLS estimator does not require any tuning parameters to have an zero-mean mixed (complex) Gaussian limiting distribution. This limiting distribution forms the basis for parameter inference. However, consistent spectral density estimation is still necessary for test construction.

Typically, spectral densities are estimated by kernel estimators that require an appropriate choice of a kernel function and a bandwidth parameter. Since the latter in particular has a major impact on the outcome of hypothesis tests we also introduce fixed-$b$ tests. To construct fixed-$b$ test statistics we first have to augment the SIM-OLS regression with an additionally constructed regressor. The residuals resulting from this augmented regression are incorporated into the construction of the test statistics in a second step. These fixed-$b$ test statistics have an asymptotically nuisance parameter free limiting distribution which depends on the choice of $b$ but can be tabulated and used for $t$- or Wald-type tests.

Additionally, we also introduce a KPSS-type cointegration test. We show that tests based upon the residuals of both the standard and the augmented fixed-$b$ regressions are consistent. However, estimation of the spectral density has to be performed by classical methods as it turns out that the fixed-$b$ versions are not consistent.

The theoretical results are complemented by a simulation study where we compare the performance of the SIM-OLS estimator and test statistics based upon it with OLS and SFM-OLS. Similar to the zero-frequency case the bias of the SIM-OLS estimator is usually smaller than the bias of the OLS and the SFM-OLS estimators whereas its RMSE is slightly larger than the RMSE of the OLS and SFM-OLS estimators. In terms of empirical sizes tests based upon the SIM-OLS estimator outperform tests upon SFM-OLS especially if there is a high amount of contemporal regressor endogeneity. The fixed-$b$ versions of the tests clearly exhibit the smallest size distortions with only a minor power loss. Overall, the simulation results for the estimation performance and the hypothesis tests support the findings in Vogelsang and Wagner (2014) for the zero-frequency case. However, the simulation results for the cointegration tests indicate that SIM-OLS residual based tests perform poorer than tests based on SFM-OLS residuals both in terms of finite sample sizes as well as in terms of size corrected power. Therefore, we do not recommend to use these tests in applied work.

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## A. Proofs

*Proof of Theorem 1.* The proof is a direct generalization of the proof of Theorem 2 of Vogelsang and Wagner (2014). Let $\theta_0 = [\delta', \beta', 0]$. We consider the centered and scaled OLS estimator of regression (10),

$$\begin{aligned} A_{SIM}^{-1}(\tilde{\theta} - \theta_0) &= A_{SIM}^{-1}(S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}})^{-1}(S_\omega^{\tilde{x}*} S_\omega^{u}) \\ &= (T^{-2} A_{SIM} S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}} A_{SIM})^{-1}(T^{-2} A_{SIM} S_\omega^{\tilde{x}*} S_\omega^{u}). \end{aligned} \tag{34}$$

We investigate the limits of both terms of the right hand side of (34) separately. The functional central limit theorem (7) in conjunction with Assumption 1 and the continuous mapping theorem yields

$$\begin{bmatrix} T^{-1} G_D^{-1} \sum_{t=1}^{[rT]} e^{i\omega t} d_t \\ T^{-3/2} \sum_{t=1}^{[rT]} e^{i\omega t} x_t \\ T^{-1/2} e^{i\omega [rT]} x_{[rT]} \end{bmatrix} \Rightarrow \begin{bmatrix} \int_0^r f(s)\, ds \\ \Omega_{\omega,vv}^{1/2} \tau_\omega \int_0^r W_v(s)\, ds \\ \Omega_{\omega,vv}^{1/2} \tau_\omega W_v(r) \end{bmatrix} = \tau_\omega \Pi_\omega g(r).$$

Since $e^{-ir} = \overline{e^{ir}}$ and $e^{-ir} e^{ir} = 1$, exemplary one block entry of the matrix $T^{-2} A_{SIM} S_\omega^{\tilde{x}*} S_\omega^{\tilde{x}} A_{SIM}$ is given by

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^{T} T^{-3/2} \overline{S_{\omega,t_t}^{\tilde{x}}} T^{-1/2} \tilde{x}_t' &= \frac{1}{T} \sum_{t=1}^{T} \frac{e^{i\omega t}}{T^{3/2}} \overline{S_{\omega,t_t}^{\tilde{x}}} \frac{e^{-i\omega t}}{T^{1/2}} \tilde{x}_t' \\ &= \frac{1}{T} \sum_{t=1}^{T} \overline{\frac{e^{-i\omega t}}{T^{3/2}} S_{\omega,t}^{\tilde{x}}} \frac{e^{i\omega t}}{T^{1/2}} \tilde{x}_t' \\ &\Rightarrow \tau_\omega^2 \int \int_0^r \overline{B_v(s)}\, ds B_v(r)'\, dr \\ &= \tau_\omega^2 \overline{\Omega}_{\omega,vv}^{1/2} \int \int_0^r \overline{W_v(s)}\, ds W_v(r)'\, dr\, \Omega_{\omega,vv}^{1/2\prime}. \end{aligned}$$

Consequently, we obtain from the continuous mapping theorem

$$(T^{-2}A_{SIM}S_\omega^{\tilde{x}*}S_\omega^{\tilde{x}}A_{SIM})^{-1} \Rightarrow \left(\tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}g(r)'\,dr\,\Pi_\omega'\right)^{-1}.$$

The second term in (34) is treated similarly. By the functional central limit theorem (7) and the continuous mapping theorem we obtain

$$\begin{bmatrix} T^{-1}G_D^{-1}\sum_{t=1}^{[rT]} e^{-i\omega t}\overline{d}_t T^{-1/2}\sum_{t=1}^{[rT]} e^{i\omega t}u_t \\ T^{-3/2}\sum_{t=1}^{[rT]} e^{-i\omega t}\overline{x}_t T^{-1/2}\sum_{t=1}^{[rT]} e^{i\omega t}u_t \\ T^{-1/2}e^{-i\omega[rT]}\overline{x}_{[rT]}T^{-1/2}\sum_{t=1}^{[rT]} e^{i\omega t}u_t \end{bmatrix} \Rightarrow \tau_\omega\overline{\Pi}_\omega\overline{g(r)}B_u(r).$$

Consequently, we have

$$\begin{aligned} T^{-2}A_{SIM}S_\omega^{\tilde{x}*}S_\omega^u &\Rightarrow \tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}B_u(r)\,dr \\ &= \tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}B_{u\cdot v}(r)\,dr\; + \tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}B_v(r)'\,dr\,\gamma' \\ &= \tau_\omega^2\sigma_{\omega,u\cdot v}\overline{\Pi}_\omega \int \overline{g(r)}W_{u\cdot v}(r)\,dr\; + \tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}W_v(r)'\,dr\,\lambda_{\omega,uv}' \\ &= \tau_\omega^2\sigma_{\omega,u\cdot v}\overline{\Pi}_\omega \int \left[\overline{G(1)} - \overline{G(r)}\right]\,dW_{u\cdot v}(r) + \tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}W_v(r)'\,dr\,\lambda_{\omega,uv}', \end{aligned}$$

where the last equation follows from integration by parts. Since $W(r)$ is the last block entry of $g(r)$ we can rewrite the last term as follows:

$$\left(\tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}g(r)'\,dr\,\Pi_\omega'\right)^{-1}\tau_\omega^2\overline{\Pi}_\omega \int \overline{g(r)}W_v(r)'\,dr\,\lambda_{\omega,uv}' = (\Pi_\omega')^{-1}\begin{bmatrix} 0_{q\times 1} \\ 0_{k\times 1} \\ \lambda_{\omega,uv}' \end{bmatrix}.$$

Using simple matrix algebra we conclude that

$$(\Pi_\omega')^{-1}\begin{bmatrix} 0_{q\times 1} \\ 0_{k\times 1} \\ \lambda_{\omega,uv}' \end{bmatrix} = \begin{bmatrix} 0_{q\times 1} \\ 0_{k\times 1} \\ \overline{\Omega}_{\omega,vv}^{-1}\overline{\Omega}_{\omega,vu} \end{bmatrix},$$

which shows the claim. □

*Proof of Theorem 2.* We prove, conditional upon $B_v$, the convergence of the estimator $\hat{V}_{SIM}$ to $V_{SIM}$. From Theorem 1 we deduce that

$$T^{-2}A_{SIM}S_\omega^{\tilde{x}*}S_\omega^{\tilde{x}}A_{SIM} \Rightarrow \overline{\Pi}_\omega \int \overline{g(s)}g(s)'\,ds\,\Pi_\omega'.$$

Thus, we only have to show convergence of the inner term. From

$$T^{-1/2}e^{i\omega[rT]}A_{SIM}S_{\omega,[rT]}^{\tilde{x}} \Rightarrow g(r)$$

we deduce, iteratively applying the continuous mapping theorem, that

$$
\begin{aligned}
T^{-3/2} A_{SIM} c_{[rT]} &= T^{-3/2} A_{SIM} e^{i\omega T} S_{\omega,T}^{S_{\omega}^{\tilde{x}}} - T^{-3/2} e^{i\omega[rT]} S_{\omega,[rT]}^{S_{\omega}^{\tilde{x}}} \\
&\Rightarrow \Pi_{\omega} \int_0^1 g(s)\, ds - \Pi_{\omega} \int_0^r g(s)\, ds \\
&= \Pi_{\omega}(G(1) - G(r)),
\end{aligned}
$$

implying

$$
T^{-4} A_{SIM} C^* C A_{SIM} \Rightarrow \int \left[\overline{G(1)} - \overline{G(s)}\right] [G(1) - G(s)]'\, ds.
$$

Since the convergence is only conditional upon $B_v$, we also have convergence in probability. Hence, together with $\hat{\sigma}^2_{\omega,u\cdot v} \Rightarrow \sigma^2_{\omega,u\cdot v}$, Slutsky's Theorem yields $\hat{V}_{SIM} \Rightarrow V_{SIM}$.

The limiting distribution of $\hat{W}$ is now obtained using standard arguments. It holds that

$$
\begin{aligned}
\hat{W} &= \tau_{\omega}^2 (R\tilde{\theta} - r)^* \left[R A_{SIM} \hat{V}_{SIM} A_{SIM} R^*\right]^{-1} (R\tilde{\theta} - r) \\
&= \tau_{\omega}^2 \left[R(\tilde{\theta} - \theta)\right]^* \left[R A_{SIM} \hat{V}_{SIM} A_{SIM} R^*\right]^{-1} \left[R(\tilde{\theta} - \theta)\right] \\
&= \tau_{\omega}^2 \left[A_R^{-1} R A_{SIM} A_{SIM}^{-1} (\tilde{\theta} - \theta)\right]^* \left[A_R^{-1} R A_{SIM} \hat{V}_{SIM} A_{SIM} R^* (A_R^{-1})'\right]^{-1} \\
&\quad \times \left[A_R^{-1} R A_{SIM} A_{SIM}^{-1} (\tilde{\theta} - \theta)\right].
\end{aligned}
$$

By condition (13) and using Theorem 1, we conclude that, conditional upon $B_v$,

$$
A_R^{-1} R A_{SIM} A_{SIM}^{-1} (\tilde{\theta} - \theta) \Rightarrow \mathcal{CN}(0, R_A V_{SIM} R_A^*).
$$

and

$$
A_R^{-1} R A_{SIM} \hat{V}_{SIM} A_{SIM} R^* (A_R^{-1})' \Rightarrow R_A V_{SIM} R_A^*.
$$

Hence, by the properties of the complex normal distribution, we obtain $\hat{W} \Rightarrow \chi^2_m$. □

*Proof of Lemma 1.* Let $\tilde{S}^u_{\omega,t}$ denote the OLS residuals from (10), i.e.,

$$
\tilde{S}^u_{\omega,t} = S^y_{\omega,t} - S^{\xi\prime}_{\omega,t} \tilde{\theta} = S^u_{\omega,t} - S^{\Xi\prime}_{\omega,t} (\tilde{\theta} - \theta_a)
$$

with $S^{\Xi}_{\omega,t} = [S^{d\prime}_{\omega,t}, S^{x\prime}_{\omega,t}, x_t']'$ and $\theta_a = [\delta', \beta', 0]'$. Note that for some series $\Xi_t$ we have the following identity

$$
\nabla_{\omega} S^{\Xi}_{\omega,t} = e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} \Xi_j - e^{-i\omega} e^{-i\omega(t-1)} \sum_{j=1}^{t-1} e^{i\omega j} \Xi_j = e^{-i\omega t} e^{i\omega t} \Xi_t = \Xi_t.
$$

Hence, using

$$
B_u(r) = \sigma_{\omega,u\cdot v} W_{u\cdot v}(r) + \lambda_{\omega,uv} W_v(r) = \sigma_{\omega,u\cdot v} W_{u\cdot v}(r) + \Omega_{\omega,uv} \left(\Omega_{\omega,vv}^{-1/2}\right)^* W_v(r)
$$

we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\nabla_\omega \tilde{S}^u_{\omega,t} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}u_t - \frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\Xi_t' A_{SIM}A_{SIM}^{-1}(\tilde{\theta}-\theta_a) \\
&\Rightarrow \tau_\omega B_u(r) - \tau_\omega g(r)'\Pi_\omega' \Big\{\sigma_{\omega,u\cdot v}(\Pi_\omega')^{-1} \\
&\quad \times \left(\int \overline{g(s)}g(s)'\,ds\right)^{-1}\int\left[\overline{G(1)}-\overline{G(s)}\right]\,dW_{u\cdot v}(s) + \tilde{\gamma}\Big\} \\
&= \tau_\omega\sigma_{\omega,u\cdot v}\tilde{P}(r) + \tau_\omega\Omega_{\omega,uv}\left(\Omega_{\omega,vv}^{-1/2}\right)^* W_v(r) - \tau_\omega g(r)'\Pi_\omega'\tilde{\gamma},
\end{aligned}
$$

where

$$
\tilde{\gamma} = \begin{bmatrix} 0_{(q+k)\times 1} \\ \overline{\Omega}_{\omega,vv}^{-1}\overline{\Omega}_{\omega,vu}\end{bmatrix}.
$$

Since

$$
\begin{aligned}
g(r)'\Pi_\omega'\tilde{\gamma} &= g(r)'\begin{bmatrix} \tau_\omega^{-1}I_{q\times q} & 0_{q\times k} & 0_{q\times k} \\ 0_{k\times q} & \Omega_{\omega,vv}^{1/2\prime}0_{k\times k} & \\ 0_{k\times q} & 0_{k\times k} & \Omega_{\omega,vv}^{1/2\prime}\end{bmatrix}\begin{bmatrix} 0_{q\times 1} \\ 0_{k\times 1} \\ \overline{\Omega}_{\omega,vv}^{-1}\overline{\Omega}_{\omega,vu}\end{bmatrix} \\
&= W_v(r)'\Omega_{\omega,vv}^{1/2\prime}\overline{\Omega}_{\omega,vv}^{-1}\overline{\Omega}_{\omega,vu} \\
&= \Omega_{\omega,uv}\left(\Omega_{\omega,vv}^{-1}\right)^*\Omega_{\omega,vv}^{1/2}W_v(r) \\
&= \Omega_{\omega,uv}\Omega_{\omega,vv}^{-1}\Omega_{\omega,vv}^{1/2}W_v(r) \\
&= \Omega_{\omega,vu}\left(\Omega_{\omega,vv}^{-1/2}\right)^* W_v(r),
\end{aligned}
$$

the statement of the lemma follows. □

*Proof of Proposition 1.* The proof is in line with the proof of Lemma 1 in Vogelsang and Wagner (2014). First, we establish an invariance principle for the regressor $z_t = [z_t^{S^d\prime}, z_t^{S^x\prime}, z_t^{x\prime}]'$. It holds that

$$
\begin{aligned}
\frac{e^{i\omega[rT]}}{T^3}G_D^{-1}z_{[rT]}^{S^d} &= \frac{[rT]}{T^3}G_D^{-1}\sum_{t=1}^{T}e^{i\omega t}S_{\omega,t}^d - \frac{1}{T^3}G_D^{-1}\sum_{t=1}^{[rT]}\sum_{j=1}^{t}e^{i\omega j}S_{\omega,j}^d \\
&= \frac{[rT]}{T}\frac{1}{T}\sum_{t=1}^{T}\frac{1}{T}G_D^{-1}\sum_{j=1}^{t}e^{i\omega j}d_j - \frac{1}{T}\sum_{t=1}^{[rT]}\frac{1}{T}\sum_{j=1}^{t}\frac{1}{T}G_D^{-1}\sum_{l=1}^{j}e^{i\omega l}d_l \\
&\Rightarrow r\int_0^1\int_0^s f(u)\,du\,ds - \int_0^r\int_0^s\int_0^u f(w)\,dw\,du\,ds.
\end{aligned}
$$

Next, we obtain

$$
\begin{aligned}
\frac{e^{i\omega[rT]}}{T^{7/2}} z^{S^x}_{[rT]} &= \frac{[rT]}{T^{7/2}} \sum_{t=1}^{T} e^{i\omega t} S^x_{\omega,t} - \frac{1}{T^{7/2}} \sum_{t=1}^{[rT]} \sum_{j=1}^{t} e^{i\omega j} S^x_{\omega,j} \\
&= \frac{[rT]}{T} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{T} \sum_{j=1}^{t} \frac{1}{\sqrt{T}} \sum_{l=1}^{j} e^{i\omega l} v_l - \frac{1}{T} \sum_{t=1}^{[rT]} \frac{1}{T} \sum_{j=1}^{t} \frac{1}{T} \sum_{l=1}^{j} \frac{1}{\sqrt{T}} \sum_{w=1}^{l} e^{i\omega w} v_w \\
&\Rightarrow \tau_\omega r \int_0^1 \int_0^1 B_v(u)\, du\, ds - \tau_\omega \int_0^r \int_0^s \int_0^u B_v(w)\, dw\, du\, ds \\
&= \tau_\omega r \Omega^{1/2}_{\omega,vv} \int_0^1 \int_0^1 W_v(u)\, du\, ds - \tau_\omega \Omega^{1/2}_{\omega,vv} \int_0^r \int_0^s \int_0^u W_v(w)\, dw\, du\, ds
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\frac{e^{i\omega[rT]}}{T^{5/2}} z_{[rT]} &= \frac{[rT]}{T^{5/2}} \sum_{t=1}^{T} e^{i\omega t} x_t - \frac{1}{T^{5/2}} \sum_{t=1}^{[rT]} \sum_{j=1}^{t} e^{i\omega j} x_j \\
&= \frac{[rT]}{T} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{\sqrt{T}} \sum_{j=1}^{t} e^{i\omega j} v_t - \frac{1}{T} \sum_{t=1}^{[rT]} \frac{1}{T} \sum_{j=1}^{t} \frac{1}{\sqrt{T}} \sum_{l=1}^{j} e^{i\omega l} v_l \\
&\Rightarrow r \tau_\omega \Omega^{1/2}_{\omega,vv} \int_0^1 W_v(s)\, ds - \tau_\omega \Omega^{1/2}_{\omega,vv} \int_0^r \int_0^s W_v(u)\, du\, ds.
\end{aligned}
$$

Combined, we can write

$$
\begin{aligned}
\frac{e^{i\omega[rT]}}{T^{5/2}} A_{SIM} z_{[rT]} &\Rightarrow \tau_\omega \Pi_\omega r \int_0^1 g(s)\, ds - \tau_\omega \Pi_\omega \int_0^r \int_0^s g(u)\, du\, ds \\
&= \tau_\omega \Pi_\omega r G(1) - \tau_\omega \Pi_\omega \int_0^r G(s)\, ds \\
&= \tau_\omega \Pi_\omega \int_0^r [G(1) - G(s)]\, ds.
\end{aligned}
$$

Consequently, we obtain for the cross-product of the regressors and the error terms that

$$
\frac{e^{-i\omega[rT]}}{T^{5/2}} A_{SIM} \overline{z_{[rT]}} \frac{e^{i\omega[rT]}}{\sqrt{T}} S^{u*}_{\omega,[rT]} \Rightarrow \tau^2_\omega \Pi_\omega \int_0^r \left[\overline{G(1)} - \overline{G(s)}\right]\, ds\, B_u(r).
$$

We combine these results with those of Theorem 1. Let $\theta_a = [\delta', \beta', 0_{1\times k}, 0_{1\times(q+2k)}]'$, then

$$
\begin{aligned}
A_M^{-1}(\tilde{\tilde{\theta}} - \theta_a) &\Rightarrow (\Pi'_{\omega,M})^{-1} \left(\int \overline{h(s)} h(s)'\, ds\right)^{-1} \int \overline{h(s)} B_u(s)\, ds \\
&= \sigma_{\omega,u\cdot v} (\Pi'_{\omega,M})^{-1} \left(\int \overline{h(s)} h(s)'\, ds\right)^{-1} \int \overline{h(s)} W_{u\cdot v}(s)\, ds + \tilde{\tilde{\gamma}} \\
&= \sigma_{\omega,u\cdot v} (\Pi'_{\omega,M})^{-1} \left(\int \overline{h(s)} h(s)'\, ds\right)^{-1} \int \left[\overline{H(1)} - \overline{H(s)}\right]\, dW_{u\cdot v}(s) + \tilde{\tilde{\gamma}},
\end{aligned}
$$

with

$$\tilde{\tilde{\gamma}} = \begin{bmatrix} 0_{(q+k)\times 1} \\ \overline{\Omega}_{\omega,vv}^{-1}\overline{\Omega}_{\omega,vu} \\ 0_{(q+2k)\times 1} \end{bmatrix}.$$

The second equation follows from the first one by the same arguments as used in Theorem 1. The last equation follows in the same spirit as in the proof of Theorem 1 from integration by parts. □

*Proof of Lemma 2.* The proof is based on Proposition 1 and is basically in line with the proof of Lemma 1. Let $\tilde{\tilde{S}}^u_{\omega,t}$ denote the OLS residuals from (22), i.e.,

$$\tilde{\tilde{S}}^u_{\omega,t} = S^y_{\omega,t} - S^{\xi\prime}_{\omega,t}\tilde{\tilde{\theta}} = S^u_{\omega,t} - S^{\Xi\prime}_{\omega,t}(\tilde{\tilde{\theta}} - \theta_a)$$

with $\Xi_t = [S^{d\prime}_{\omega,t}, S^{x\prime}_{\omega,t}, x'_t, z'_t]'$ and $\theta_a$ defined in the proof of Proposition 1. By the definition of $B_u(r)$ we obtain

$$\begin{aligned}
\frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\nabla_\omega \tilde{\tilde{S}}^u_{\omega,t} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}u_t - \frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\Xi'_t A_{SIM}A_{SIM}^{-1}(\tilde{\tilde{\theta}} - \theta_a) \\
&\Rightarrow \tau_\omega B_u(r) - \tau_\omega h(r)'\Pi'_{\omega,M}\Bigg\{\sigma_{\omega,u\cdot v}(\Pi'_{\omega,M})^{-1} \\
&\quad \times \left(\int \overline{h(s)}h(s)'\,ds\right)^{-1}\int\left[\overline{H(1)} - \overline{H(s)}\right]\,dW_{u\cdot v}(s) + \tilde{\tilde{\gamma}}\Bigg\} \\
&= \tau_\omega\sigma_{\omega,u\cdot v}\tilde{\tilde{P}}(r) + \tau_\omega\Omega_{\omega,uv}\left(\Omega_{\omega,vv}^{-1/2}\right)^* W_v(r) - \tau_\omega h(r)'\Pi'_{\omega,M}\tilde{\tilde{\gamma}},
\end{aligned}$$

where $\tilde{\tilde{\gamma}}$ is defined in the proof of Proposition 1. The first part of the Lemma follows since

$$h(r)'\Pi'_{\omega,M}\tilde{\tilde{\gamma}} = h(r)'\begin{bmatrix} \tau_\omega^{-1}I_{q\times q} & 0 & 0 & 0 \\ 0 & \Omega^{1/2\prime}_{\omega,vv} & 0 & 0 \\ 0 & 0 & \Omega^{1/2\prime}_{\omega,vv} & 0 \\ 0 & 0 & 0 & \Pi'_\omega \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \overline{\Omega}_{\omega,vv}^{-1}\overline{\Omega}_{\omega,vu} \\ 0 \end{bmatrix} = \Omega_{\omega,vu}\left(\Omega_{\omega,vv}^{-1/2}\right)^* W_v(r),$$

where the last equation is obtained by the same steps as in the proof of Lemma 1.

It remains to show that, conditional upon $W_v$, the limiting distribution of the SIM-OLS estimator, $\Psi_\omega$, is uncorrelated with $\tilde{\tilde{P}}(r)$. Note that all conditional quantities can be treated as non-stochastic. Hence, it suffices to show that the conditional covariance between

$$\Psi^a_\omega = \int\left[\overline{G(1)} - \overline{G(s)}\right]\,dW_{u\cdot v}(s)$$

and $\tilde{\tilde{P}}(r)$ is equal to zero. It holds that

$$\text{Cov}\left(\tilde{\tilde{P}}(r), \Psi^a_\omega\right) = \text{Cov}\left(W_{u\cdot v}(r), \Psi^a_\omega\right) - \text{Cov}\left(R(r), \Psi^a_\omega\right), \tag{35}$$

with

$$R(r) = h(r)' \left( \int \overline{h(s)} h(s)' \, ds \right)^{-1} \int \left[ \overline{H(1)} - \overline{H(s)} \right] \, dW_{u \cdot v}(s).$$

For the first term in (35) it holds that

$$\operatorname{Cov}\left(W_{u \cdot v}(r), \Psi_\omega^a\right) = \tau_\omega^2 \int_0^r [G(1) - G(s)]' \; ds.$$

Up to the scaling factor $\tau_\omega^2$ this expression corresponds to the transpose of the second block entry of $h(r)$, which we label by $h_2(r)$ for brevity. For the second term in (35), we deduce that

$$\begin{aligned}
\operatorname{Cov}\left(R(r), \Psi_\omega^a\right) &= \tau_\omega^2 h(r)' \left( \int \overline{h(s)} h(s)' \, ds \right)^{-1} \int \left[ \overline{H(1)} - \overline{H(s)} \right] [G(1) - G(s)]' \; ds \\
&= \tau_\omega^2 h(r)' \left( \int \overline{h(s)} h(s)' \, ds \right)^{-1} \int \overline{h(s)} h_2(s) \, ds \\
&= \tau_\omega^2 h(r)' \begin{bmatrix} 0_{q+2k} \\ I_{q+2k} \end{bmatrix} \\
&= \tau_\omega^2 h_2(r)',
\end{aligned}$$

where the second equation follows from

$$\begin{aligned}
\int \left[ \overline{H(1)} - \overline{H(s)} \right] [G(1) - G(s)]' \; ds &= \left[ \overline{H(1)} - \overline{H(r)} \right] h_2(r)' \Big|_{r=0}^{r} + \int \overline{h(s)} h_2(s)' \, ds \\
&= \int \overline{h(s)} h_2(s)' \, ds.
\end{aligned}$$

Hence, both terms on the right hand side of (35) coincide, concluding the proof. □

*Proof of Proposition 2.* The proof is similar to the proof of the main result in Hashimzade and Vogelsang (2007). First we define

$$\begin{aligned}
K_{t,s} &= k\left(\frac{|t-s|}{bT}\right) = k\left(\frac{t-s}{bT}\right) = \tilde{k}\left(\frac{t-s}{T}\right) \\
\nabla^2 K_{t,s} &= (K_{t,s} - K_{t,s+1}) - (K_{t+1,s} - K_{t+1,s+1}).
\end{aligned}$$

It holds that

$$\check{\sigma}^2_{\omega,u\cdot v}(\zeta) = \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} k\left(\frac{|t-s|}{bT}\right) e^{i\omega(t-s)} \zeta_t \zeta_s^* = \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} e^{i\omega t} \zeta_t K_{t,s} e^{-i\omega s} \zeta_s^* = \frac{1}{T} \sum_{t=1}^{T} a_t b_t^*$$

with

$$a_t = \zeta_t e^{i\omega t}, \quad b_t = \sum_{s=1}^{T} K_{t,s} e^{i\omega s} \zeta_s.$$

One can easily show the following summation-by-parts formula for complex valued sequences $a_t$ and $b_t$:

$$\sum_{t=1}^{T} a_t b_t^* = \sum_{t=1}^{T-1}\sum_{s=1}^{t} a_s (b_t - b_{t+1})^* + \sum_{t=1}^{T} a_t b_T^*. \tag{36}$$

Hence, we obtain

$$\begin{aligned}
\sum_{t=1}^{T} a_t b_t^* &= \sum_{t=1}^{T-1}\sum_{s=1}^{t} \zeta_s e^{i\omega s}\left(\sum_{j=1}^{T} K_{t,j} e^{i\omega j}\zeta_j - \sum_{j=1}^{T} K_{t+1,k} e^{i\omega j}\zeta_j\right)^* + \sum_{t=1}^{T}\zeta_j e^{i\omega j}\left(\sum_{s=1}^{T} K_{T,s} e^{i\omega s}\zeta_s\right)^* \\
&= \sum_{t=1}^{T-1}\sum_{s=1}^{t} \zeta_s e^{i\omega s}\left(\sum_{j=1}^{T} (K_{t,j} - K_{t+1,j}) e^{i\omega j}\zeta_j\right)^* + \sum_{t=1}^{T}\zeta_t e^{i\omega t}\left(\sum_{s=1}^{T} K_{T,s} e^{i\omega s}\zeta_s\right)^* \\
&= \sum_{t=1}^{T-1} \Lambda_{\omega,t}^{\zeta} C_1^* + \Lambda_{\omega,T}^{\zeta} C_2^*,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_{\omega,t}^{\zeta} &= \sum_{j=1}^{t} e^{i\omega j}\zeta_j \\
C_1 &= \sum_{j=1}^{T} (K_{t,j} - K_{t+1,k}) e^{i\omega j}\zeta_j \\
C_2 &= \sum_{s=1}^{T} K_{T,s} e^{i\omega s}\zeta_s.
\end{aligned}$$

Using (36) once again we rewrite $C_1$ as follows:

$$\begin{aligned}
C_1 &= \sum_{j=1}^{T-1}\left\{\sum_{s=1}^{j} e^{i\omega s}\zeta_s \left[(K_{t,j} - K_{t+1,j}) - (K_{t,j+1} - K_{t+1,j+1})\right]\right\} \\
&\quad + \sum_{j=1}^{T} e^{i\omega j}\zeta_j (K_{t,T} - K_{t+1,T}) \\
&= \sum_{j=1}^{T-1} \Lambda_{\omega,j}^{\zeta} \nabla^2 K_{t,j} + \Lambda_{\omega,T}^{\zeta}(K_{t,T} - K_{t+1,T}),
\end{aligned}$$

where we used the defined quantities $\Lambda_{\omega,t}^{\zeta}$ and $\nabla^2 K_{t,j}$. Similarly, for $C_2$ we obtain

$$\begin{aligned}
C_2 &= \sum_{s=1}^{T-1}\left[\sum_{j=1}^{s} e^{i\omega j}\zeta_j (K_{T,s} - K_{T,s+1})\right] + \sum_{s=1}^{T} e^{i\omega s}\zeta_s K_{T,T} \\
&= \sum_{s=1}^{T-1} \Lambda_{\omega,s}^{\zeta}(K_{T,s} - K_{T,s+1}) + \Lambda_{\omega,T}^{\zeta},
\end{aligned}$$

where the last equation follows from $K_{T,T} = 1$. Noting that $K_{t,s} = K_{s,t}$, we put everything together and get the following expression for $\check{\sigma}^2_{\omega,u\cdot v}$:

$$
\begin{aligned}
\check{\sigma}^2_{\omega,u\cdot v}(\zeta) = \frac{1}{T}\sum_{t=1}^{T-1}\frac{1}{T}\sum_{j=1}^{T-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,t}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,j}\right)(T^2\nabla^2 K_{t,j}) \\
+ \frac{1}{T}\sum_{t=1}^{T-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,t}\right)(T(K_{t,T} - K_{t+1,T}))\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,T}\right) \\
+ \left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,T}\right)\frac{1}{T}\sum_{s=1}^{T-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta^*}_{\omega,s}\right)(T(K_{s,T} - K_{s+1,T})) \\
+ \left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,T}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,T}\right)
\end{aligned}
\tag{37}
$$

Let $r \in (0, 1]$. By assumption,

$$
\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,[rT]} = \frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\zeta_t \Rightarrow P(r).
$$

Furthermore, if the kernel function is even, twice differentiable with continuous second derivative, it holds that

$$
\lim_{T\to\infty} T(K_{[rT],T} - K_{[rT]+1,T}) = \lim_{T\to\infty}\frac{\tilde{k}\left(\frac{[rT]}{T} - 1\right) - \tilde{k}\left(\frac{[rT]}{T} - 1 + \frac{1}{T}\right)}{\frac{1}{T}} = -\dot{\tilde{k}}(r-1)
$$

and

$$
\begin{aligned}
\lim_{T\to\infty} T^2\nabla^2 K_{[rT],[sT]} &= \lim_{T\to\infty}\left[(K_{[rT],[sT]} - K_{[rT]+1,[sT]}) - (K_{[rT],[sT]+1} - K_{[rT]+1,[sT]+1})\right] \\
&= \lim_{T\to\infty}\frac{1}{T^2}\left[\left\{\tilde{k}\left(\frac{[rT]-[rS]}{T}\right) - \tilde{k}\left(\frac{[rT]-[sT]+1}{T}\right)\right\}\right. \\
&\qquad \left. - \left\{\tilde{k}\left(\frac{[rT]-[sT]-1}{T}\right) - \tilde{k}\left(\frac{[rT]-[sT]}{T}\right)\right\}\right] \\
&= -\ddot{\tilde{k}}(r-s)
\end{aligned}
$$

With this in mind we can rewrite expression (37) as follows

$$
\begin{aligned}
\check{\sigma}^2_{\omega,u\cdot v} = \int\int\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,[rT]}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,[sT]}\right)\left(T^2\nabla^2 K_{[rT],[sT]}\right)\,dr\,ds \\
+ \int\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,[rT]}\right)\left(T(K_{[rT],T} - K_{[rT]+1,T})\right)\,dr\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,T}\right) \\
+ \left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,T}\right)\int\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta^*}_{\omega,[rT]}\right)\left(T(K_{[rT],T} - K_{[rT]+1,T})\right)\,dr \\
+ \left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,T}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,T}\right).
\end{aligned}
$$

The functions $\dot{k}$ and $\dot{\tilde{k}}$ are odd whereas $\ddot{k}$ and $\ddot{\tilde{k}}$ are even. Hence,

$$-\dot{\tilde{k}}(r-1) = -\frac{1}{b}\dot{k}\left(\frac{r-1}{b}\right) = \frac{1}{b}\dot{k}\left(\frac{1-r}{b}\right), \quad \ddot{\tilde{k}}(r) = \frac{1}{b^2}\ddot{k}\left(\frac{r-s}{b}\right).$$

The statement of the theorem now follows directly from the continuous mapping.

If $k$ is the Bartlett kernel we have by definition

$$K_{t,s} = k\left(\frac{t-s}{bT}\right) = \left(1 - \frac{|t-s|}{bT}\right) 1_{\{|t-s|\leq bT\}}.$$

Consequently, it holds that

$$\nabla^2 K_{t,j} = \begin{cases} \frac{2}{bT}, & t = j, \\ -\frac{1}{bT}, & t = j \pm bT, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we can rewrite the first term in (37) as follows:

$$\begin{aligned} &\frac{1}{T}\sum_{t=1}^{T-1}\frac{1}{T}\sum_{j=1}^{T-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,t}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,j}\right)\left(T^2\nabla^2 K_{t,j}\right) \\ &= \frac{2}{bT}\sum_{t=1}^{T-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,t}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,t}\right) \\ &\quad - \frac{1}{bT}\sum_{j=1}^{T-[bT]-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,j+[bT]}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,t}\right) \\ &\quad - \frac{1}{bT}\sum_{t=1}^{T-[bT]-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,t}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,t+[bT]}\right). \end{aligned}$$

By the continuous mapping theorem, these terms converge to the first two summands of $Q(P)$. Using

$$K_{t,T} - K_{t+1,T} = -\frac{1}{bT}1_{\{T-bT\leq t\leq T-1\}},$$

we can rewrite the second term in (37) and obtain

$$\sum_{t=1}^{T-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,t}\right)(K_{t,T}-K_{t+1,T})\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,T}\right) = -\frac{1}{bT}\sum_{t=T-[bT]}^{T-1}\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta}_{\omega,t}\right)\left(\frac{1}{\sqrt{T}}\Lambda^{\zeta*}_{\omega,T}\right).$$

The third term is rewritten analogously. Both terms converge, once again by the continuous mapping theorem, to the third summand of $Q(P)$. This completes the proof. □

*Proof of Proposition 3.* The limiting distributions of $\tilde{\sigma}^2_{\omega,u\cdot v}$ and $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$ follow immediately from Proposition 3 in conjunction with Lemma 1 and Lemma 2, respectively. □

*Proof of Theorem 3.* By the same arguments as in the proof of Theorem 2 we obtain, using the statement of Proposition 3, that

$$\tilde{\tilde{W}} = \tau_\omega^{-2}(R\tilde{\theta} - r)^* \left[ RA_{SIM}\tilde{\tilde{V}}_{SIM}A_{SIM}R^* \right]^{-1} (R\tilde{\theta} - r)$$
$$\Rightarrow \tau_\omega^2 (R_A\Psi_\omega)^* \left[ \tau_\omega^2 Q(\tilde{\tilde{P}}) R_A V_{SIM} R_A^* \right]^{-1} (R_A\Psi_\omega) = \frac{\chi_m^2}{Q(\tilde{\tilde{P}})},$$

where the independence of numerator and denominator can be directly deduced from the independence of $\Psi_\omega$ and $Q(\tilde{\tilde{P}})$. □

*Proof of Theorem 4.* Note that $m_t = u_t$ under the null hypothesis of cointegration. Hence, the limiting distributions of $\tilde{K}$ and $\tilde{\tilde{K}}$ follow from the continuous mapping theorem in conjunction with Lemma 1 and Lemma 2. □

*Proof of Lemma 3.* It holds that

$$\frac{1}{TM_T}\hat{\sigma}_{\omega,u\cdot v}^2 = \frac{1}{TM_T}\hat{\Omega}_{\omega,mm} - \frac{1}{(TM_T)^{1/2}}\hat{\Omega}_{\omega,mv}\hat{\Omega}_{\omega,vv}^{-1}\frac{1}{(TM_T)^{1/2}}\hat{\Omega}_{\omega,vm}, \tag{38}$$

where

$$\hat{\Omega}_\omega = \frac{1}{T}\sum_{t=2}^{T}\sum_{s=2}^{T} k\left(\frac{t-s}{M_T}\right)\hat{\eta}_t\hat{\eta}_s^* = \begin{bmatrix} \hat{\Omega}_{\omega,mm} & \hat{\Omega}_{\omega,mv} \\ \hat{\Omega}_{\omega,vm} & \hat{\Omega}_{\omega,vv} \end{bmatrix},$$

with $\hat{\eta}_t = [\hat{m}_t, \nabla_\omega x_t']'$ with $m_t$ denoting the OLS residuals from (25). We can rewrite $\hat{\Omega}_{\omega,mm}$ as follows:

$$\frac{1}{TM_T}\hat{\Omega}_{\omega,mm} = \frac{1}{TM_T}\left(\hat{\Sigma}_{mm} + \hat{\Lambda}_{\omega,mm} + \hat{\Lambda}_{\omega,mm}^*\right),$$

with

$$\frac{1}{TM_T}\hat{\Sigma}_{mm} = \frac{1}{T^2M_T}\sum_{t=1}^{T}\hat{m}_t\hat{m}_t^*,$$

which converges to zero in probability and

$$\begin{aligned}
\frac{1}{TM_T}\hat{\Lambda}_{\omega,mm} &= \frac{1}{M_T}\sum_{h=1}^{T-1} k\left(\frac{h}{M_T}\right)\frac{1}{T}\sum_{t=1}^{T-h}\frac{e^{i\omega(t+h)}}{\sqrt{T}}\hat{m}_{t+h}\frac{e^{-i\omega t}}{\sqrt{T}}\hat{m}_t^* \\
&= \int_0^{\frac{T-1}{M_T}}\int_0^{1-\frac{[xM_T]}{T}} k\left(\frac{[xM_T]}{M_T}\right)\frac{e^{i\omega([rT]+[xM_T])}}{\sqrt{T}}\hat{m}_{[rT]+[xM_T]}\frac{e^{-i\omega[rT]}}{\sqrt{T}}\hat{m}_{[rT]}^*\,dr\,dx \\
&= \int_0^{\infty}\int_0^{1} F_T(x,r)\frac{e^{i\omega([rT]+[xM_T])}}{\sqrt{T}}\hat{m}_{[rT]+[xM_T]}\frac{e^{-i\omega[rT]}}{\sqrt{T}}\hat{m}_{[rT]}^*\,dr\,dx
\end{aligned} \tag{39}$$

with

$$F_T(x,r) = k\left(\frac{[xM_T]}{M_T}\right)1_{\left[0,\frac{T-1}{M_T}\right)}(x)1_{\left[0,1-\frac{[xM_T]}{T}\right)}(r).$$

If we denote the OLS estimator of $\theta = [\delta', \beta']'$ by $\hat{\theta}$, we derive the following invariance principle:

$$\begin{aligned}
\frac{e^{i\omega[rT]}}{\sqrt{T}}\hat{m}_{[rT]} &= \frac{e^{i\omega[rT]}}{\sqrt{T}}m_{[rT]} - \frac{e^{i\omega[rT]}}{\sqrt{T}}\tilde{x}'_{rT}(\hat{\theta}-\theta) \\
&= \frac{e^{i\omega[rT]}}{\sqrt{T}}m_{[rT]} - \frac{e^{i\omega[rT]}}{\sqrt{T}}\tilde{x}'_{rT}A_{OLS}A_{OLS}^{-1}(\hat{\theta}-\theta) \\
&= \frac{e^{i\omega[rT]}}{\sqrt{T}}m_{[rT]} - \sqrt{T}e^{i\omega[rT]}\tilde{x}'_{rT}A_{OLS}\left(A_{OLS}\sum_{t=1}^{T}\overline{\tilde{x}}_t\tilde{x}'_tA_{OLS}\right)^{-1}\left(\frac{1}{T}A_{OLS}\sum_{t=1}^{T}\overline{\tilde{x}}_t m_t\right) \\
&\Rightarrow B_w(r) - J(r)'\left(\int\overline{J(s)}J(s)'\,ds\right)^{-1}\int\overline{J(s)}B_w(s)\,ds = U(r).
\end{aligned}$$

Since $M_T/T \to 0$ it follows that

$$\frac{e^{i\omega([rT]+[xM_T])}}{\sqrt{T}}\hat{m}_{[rT]+[xM_T]} \Rightarrow U(r)$$

as well. Note that $k([xM_T]/M_T)$ converges uniformly to $k(x)$ and the product of the indicators converges to $1_{[0,1)}(r)$ in the Skorohod topology. Hence, $F_T$ also converges to

$$F(x,r) = k(x)1_{[0,1)}(r)$$

in the Skorohod topology. Therefore, by Billingsley (1968, Theorem 4.1) and the continuous mapping theorem we deduce that

$$\frac{1}{TM_T}\hat{\Lambda}_{\omega,mm} \Rightarrow \int_0^\infty k(x)\,dx\int U(r)U(r)^*\,dr.$$

Since the kernel function is even it follows that

$$\frac{1}{TM_T}\hat{\Omega}_{\omega,mm} \Rightarrow 2\int_0^\infty k(x)\,dx\int U(r)U(r)^*\,dr.$$

It remains to show that $(TM_T)^{-1/2}\hat{\Omega}_{\omega,mv}$ converges to zero in probability. It holds that

$$\begin{aligned}
&\mathbb{E}\left\|\frac{1}{\sqrt{TM_T}}\sum_{h=1}^{T-1}k\left(\frac{h}{M_T}\right)\frac{1}{T}\sum_{t=1}^{T-h}e^{i\omega(t+h)}\hat{m}_{t+h}e^{-i\omega t}v_t^*\right\| \\
&\quad\le \frac{M_T}{\sqrt{T}}\left(\frac{1}{M_T}\sum_{h=1}^{T-1}\left|k\left(\frac{h}{M_T}\right)\right|\right)\max_{1\le h<T}\left(\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T-h}\hat{m}_{t+h}v_t^*\right\|^2\right)^{1/2}.
\end{aligned}$$

From Jansson (2002, Lemma 1) it follows that the second term is bounded. Using similar arguments as in Gregoir (2010, Theorem 2.1) we deduce that the last term is bounded in probability regardless of $h$. Since $M_T/\sqrt{T} \to 0$ as $T \to \infty$ by the whole expression converges to zero which concludes the proof. □

*Proof of Proposition 4.* With $\theta_a$ and $S^{\Xi}_{\omega,t}$ being defined in the proof of Theorem 1 it holds that

$$\frac{e^{i\omega[rT]}}{T^{3/2}}\tilde{S}^m_{\omega,t} = \frac{e^{i\omega[rT]}}{T^{3/2}}S^m_{\omega,t} - \frac{e^{i\omega[rT]}}{T^{3/2}}S^{\Xi\prime}_{\omega,t}(\tilde{\theta}-\theta_a)$$
$$= \frac{1}{T}\sum_{t=1}^{[rT]}\frac{e^{i\omega j}}{\sqrt{T}}m_j - \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}e^{i\omega j}\Xi_j' A_{SIM}A_{SIM}^{-1}(\tilde{\theta}-\theta_a)\frac{1}{T}.$$

Under the alternative hypothesis it holds that

$$\frac{1}{T}A_{SIM}^{-1}(\tilde{\theta}-\theta_a) = \left(A_{SIM}\sum_{t=1}^{T}\overline{S^{\Xi}_{\omega,t}}S^{\Xi\prime}_{\omega,t}A_{SIM}\right)^{-1}\left(\frac{1}{T}A_{SIM}\sum_{t=1}^{T}\overline{S^{\Xi}_{\omega,t}}S^m_{\omega,t}\right)$$
$$= \left(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{\sqrt{T}}A_{SIM}\overline{S^{\Xi}_{\omega,t}}S^{\Xi\prime}_{\omega,t}A_{SIM}\frac{1}{\sqrt{T}}\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}A_{SIM}\frac{1}{\sqrt{T}}\overline{S_{\omega,t}}^{\Xi}S^m_{\omega,t}\frac{1}{T^{3/2}}\right)$$
$$\Rightarrow \left(\int\overline{g(s)}g(s)'\,ds\right)^{-1}\int\overline{g(s)}\int_0^s B_w(u)\,du\,ds.$$

Consequently, we obtain

$$\frac{e^{i\omega[rT]}}{T^{3/2}}\tilde{S}^m_{\omega,t} \Rightarrow \int_0^r B_w(s)\,ds - g(r)'\left(\int\overline{g(s)}g(s)'\,ds\right)^{-1}\int\overline{g(s)}\int_0^s B_w(u)\,du\,ds = \tilde{R}(r)$$

and, by the continuous mapping theorem,

$$\frac{1}{T^4}\sum_{t=1}^{T}\tilde{S}^m_{\omega,t}\tilde{S}^{m*}_{\omega,t} = \frac{1}{T}\sum_{t=1}^{T}\frac{e^{i\omega t}}{T^{3/2}}\tilde{S}^m_{\omega,t}\frac{e^{-i\omega t}}{T^{3/2}}\tilde{S}^{m*}_{\omega,t} \Rightarrow \int\tilde{R}(r)\tilde{R}(r)^*\,dr.$$

Similarly, we derive

$$\frac{1}{T^4}\sum_{t=1}^{T}\tilde{\tilde{S}}^m_{\omega,t}\tilde{\tilde{S}}^{m*}_{\omega,t} \Rightarrow \int\tilde{\tilde{R}}(r)\tilde{\tilde{R}}(r)^*\,dr.$$

The limiting distributions of $\tilde{K}$ and $\tilde{\tilde{K}}$ now follow from Lemma 3. □

*Proof of Theorem 5.* From the proof of Proposition 3 we deduce that $\tilde{\sigma}^2_{\omega,u\cdot v}$ and $\tilde{\tilde{\sigma}}^2_{\omega,u\cdot v}$ are continuous functionals of $\tilde{S}^u_{\omega,[rT]}$ and $\tilde{\tilde{S}}^u_{\omega,[rT]}$, respectively. Since

$$\frac{1}{T^2}\sum_{t=1}^{T}\tilde{S}^u_{\omega,t}\tilde{S}^{u*}_{\omega,t} = \int\frac{e^{i\omega[rT]}}{\sqrt{T}}\tilde{S}^u_{\omega,[rT]}\frac{e^{-i\omega[rT]}}{\sqrt{T}}\tilde{S}^{u*}_{\omega,[rT]}\,dr$$

and

$$\frac{1}{T^2}\sum_{t=1}^{T}\tilde{\tilde{S}}^u_{\omega,t}\tilde{\tilde{S}}^{u*}_{\omega,t} = \int\frac{e^{i\omega[rT]}}{\sqrt{T}}\tilde{\tilde{S}}^u_{\omega,[rT]}\frac{e^{-i\omega[rT]}}{\sqrt{T}}\tilde{\tilde{S}}^{u*}_{\omega,[rT]}\,dr$$

are continuous functionals of these processes, we conclude that $\tilde{K}^b$ and $\tilde{\tilde{K}}^b$ are also continuous functionals. Hence, the theorem follows from Lemma 1 and Lemma 2 in conjunction with the continuous mapping theorem. □

*Proof of Proposition 5.* In order to derive the limiting distributions of the fixed-$b$ statistics under the alternative hypothesis, we only have to verify the claim for the denominator. It holds that

$$\frac{1}{T^2}\tilde{\sigma}^2_{\omega,u\cdot v} = \frac{1}{T^3}\sum_{t=2}^{T}\sum_{s=2}^{T} e^{i\omega(t-s)} k\left(\frac{t-s}{M}\right) \nabla_\omega \tilde{S}^m_{\omega,t} \nabla_\omega \tilde{S}^{m*}_{\omega,s}$$
$$= \sum_{t=2}^{T}\sum_{s=2}^{T} k\left(\frac{t-s}{M}\right) \frac{e^{i\omega t}}{T^{3/2}} \frac{e^{-i\omega s}}{T^{3/2}} \nabla_\omega \tilde{S}^m_{\omega,t} \nabla_\omega \tilde{S}^{m*}_{\omega,s}.$$

Straightforward calculations show that

$$\frac{e^{i\omega[rT]}}{T^{3/2}} \sum_{t=2}^{[rT]} \nabla_\omega \tilde{S}^m_{\omega,t} = \frac{e^{i\omega[rT]}}{T^{3/2}} \tilde{S}^m_{\omega,[rT]} + o_{\mathbb{P}}(1) \Rightarrow \tilde{R}(r).$$

Hence, the proof for $\tilde{K}^b$ follows from Proposition 3. The limiting distribution $\tilde{\tilde{K}}^b$ is proven similarly. □

## B. Complex Normal Distributions

This section contains some definitions and properties of complex normal distributions. We start with the main definition.

**Definition B.1.** Let $X, Y$ be random vectors in $\mathbb{R}^k$, such that $[X', Y']'$ is a $2k$-dimensional normally distributed vector. Then

$$Z := X + iY$$

has a complex normal distribution.

It is well known that the real normal distribution is entirely described by two parameters, the mean vector and the covariance matrix. For the complex normal distribution these quantities are defined by

$$\mu = \mathbb{E}(Z) = \mathbb{E}(X) + i\mathbb{E}(Y)$$
$$\Gamma = \mathbb{E}((Z-\mu)(Z-\mu)^*) = \Gamma_{XX} + \Gamma_{YY} + i(\Gamma_{YX} - \Gamma_{XY}), \tag{40}$$

where

$$\Gamma_{XX} = \mathbb{E}((X-\mathbb{E}(X))(X-\mathbb{E}(X))')$$
$$\Gamma_{YY} = \mathbb{E}((Y-\mathbb{E}(Y))(Y-\mathbb{E}(Y))')$$
$$\Gamma_{XY} = \mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))')$$
$$\Gamma_{YX} = \mathbb{E}((Y-\mathbb{E}(Y))(X-\mathbb{E}(X))')$$

Note that $\Gamma_{XY} = \Gamma'_{YX}$ and $\Gamma = \Gamma^*$. However, the real normal distribution of the $2k$ vector $[X', Y']'$ is not entirely described by $\mu$ and $\Gamma$. Hence, we need an additional matrix, the so-called *relation matrix* or *pseudo-covariance matrix*, defined by

$$C = \mathbb{E}((Z-\mu)(Z-\mu)') = \Gamma_{XX} - \Gamma_{YY} + i(\Gamma_{XY} + \Gamma_{YX}), \tag{41}$$

to completely describe the complex normal distribution. It is often useful to consider complex normal distributions that have an additional property, called circularity or circular symmetry.

**Definition B.2.** Let $Z \in \mathbb{C}^k$ be a zero mean complex normal random vector. The distribution of $Z$ is said to be *circular symmetric* if for all $\phi \in \mathbb{R}$, $e^{i\phi}Z$ has the same distribution as $Z$.

Clearly, every circular symmetric random vector $Z$ must fulfill $\mathbb{E}(Z) = 0$. It is also obvious, that multiplication by $e^{i\phi}$ is not changing the covariance matrix of some random vector $Z$, since

$$\mathbb{E}((e^{i\phi}Z)(e^{i\phi}Z)^*) = \mathbb{E}(e^{i\phi}ZZ^*e^{-i\phi}) = \mathbb{E}(ZZ^*) = \Gamma.$$

For the relation matrix $C$, however, it holds that

$$\mathbb{E}((e^{i\phi}Z)(e^{i\phi}Z)') = e^{2i\phi}\mathbb{E}(ZZ') = e^{2i\phi}C,$$

which is equal to $C$ for all $\phi \in \mathbb{R}$ if and only if $C = 0$. We summarize this in the following proposition.

**Proposition B.1** (Gallager, 2008)**.** *Assume that $Z$ is a complex normal random vector. Then $Z$ is circularly symmetric if and only if $C = 0$. In this case the distribution of $Z$ is entirely determined by $\Gamma$.*

Hence, for a complex normal distribution to be circular symmetric it is sufficient and necessary that the relation matrix is equal to zero. In the following we denote a circularly symmetric complex normal distribution with covariance matrix $\Gamma$ by $\mathcal{CN}(0, \Gamma)$.

With this result we can easily connect the covariance matrix $\Gamma$ of a $\mathcal{CN}(0, \Gamma)$ distributed random vector $Z = X + iY$ and the covariance matrix of $[X', Y']'$, $\Sigma$ say. From (40) and (41) we obtain the following relations:

$$\begin{aligned}
\Gamma_{XX} + \Gamma_{YY} &= \mathrm{Re}(\Gamma), \\
\Gamma_{YX} - \Gamma_{XY} &= \mathrm{Im}(\Gamma), \\
\Gamma_{XX} - \Gamma_{YY} &= 0, \\
\Gamma_{XY} + \Gamma_{YX} &= 0.
\end{aligned}$$

Hence, the covariance matrix of $[X', Y']'$ is given by

$$\Sigma = \frac{1}{2}\begin{bmatrix} \mathrm{Re}(\Gamma) & -\mathrm{Im}(\Gamma) \\ \mathrm{Im}(\Gamma) & \mathrm{Re}(\Gamma) \end{bmatrix}.$$

An important and in fact trivial example of circular symmetric normal random vectors is $W = [W_1, \ldots, W_k]'$, where $W_i = X_i + iY_i$ and all $X_i$ and $Y_i$ are i.i.d. $\mathcal{N}(0, 1/2)$-distributed random variables. Clearly, it holds $\mu = \mathbb{E}(Z) = 0$, $\Gamma = \mathbb{E}(WW^*) = I_k$ and $C = \mathbb{E}(WW') = 0$

and we can use the somewhat familiar notation $W \sim \mathcal{CN}(0, I_k)$. Let $M \in \mathbb{C}^{m\times k}$ denote some nonsingular complex matrix and define $Z = MW$. Then it holds that $\mathbb{E}(Z) = 0$,

$$\Gamma_Z = \mathbb{E}((MW)(MW)^*) = \mathbb{E}(MWW^*M^*) = M\mathbb{E}(WW^*)M^* = MM^*$$

and

$$C_Z = \mathbb{E}((MW)(MW') = \mathbb{E}(MWW'M') = M\mathbb{E}(WW')M' = 0.$$

Hence, $Z \sim \mathcal{CN}(0, MM^*)$. On the other hand, if $Z \sim \mathcal{CN}(0, \Gamma)$ with positive definite covariance matrix $\Gamma$, there exists a matrix $R$ such that $RR^* = \Gamma$. This can be summarized in the following proposition.

**Proposition B.2** (Gallager, 2008)**.** *A necessary and sufficient condition of a random vector $Z$ to have a circular symmetric normal distribution is that $Z = AW$ where $W \sim \mathcal{CN}(0, I)$ and $A$ is an arbitrary complex matrix.*

We deduce an important property from this proposition: Let $Z \sim \mathcal{CN}(0, \Gamma)$ and let $A$ be such that $AA^* = \Gamma$. Then there exists $W \sim \mathcal{CN}(0, I_k)$ and it holds that $W = AZ$. Consequently,

$$2Z^*\Gamma^{-1}Z = 2(AW)^*(AA^*)^{-1}(AW) = 2W^*A^*(A^*)^{-1}A^{-1}AW = 2W^*W \sim \chi^2_{2k},$$

since

$$2W^*W = 2\sum_{l=1}^{k} W_l\overline{W}_l = 2\sum_{l=1}^{k} \mathrm{Re}(W_l)^2 + \mathrm{Im}(W_l)^2 = \sum_{l=1}^{2k} \tilde{W}_l^2,$$

with $\tilde{W}_l$ being i.i.d. standard normally distributed.

At the end of this section we briefly link the theory of complex normal distributions to complex Brownian motions that frequently occurred in the main text as limiting processes of functional central limit theorems of the form

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\eta_t \Rightarrow \frac{1}{\sqrt{2}}B(r) = \frac{1}{\sqrt{2}}\Omega_\omega^{1/2}W(r),$$

with $W(r) = W_{\mathrm{Re}}(r) + iW_{\mathrm{Im}}(r)$, where $W_{\mathrm{Re}}$ and $W_{\mathrm{Im}}$ are independent standard Wiener processes. Since $W_{\mathrm{Re}}(r) \sim \mathcal{N}(0, rI)$ and $W_{\mathrm{Im}}(r) \sim \mathcal{N}(0, rI)$, we obtain $W(r) \sim \mathcal{CN}(0, 2rI)$ and by Proposition B.2,

$$\frac{1}{\sqrt{2}}\Omega_\omega^{1/2}W(r) \sim \mathcal{CN}(0, r\Omega_\omega),$$

since $\Omega_\omega^{1/2}\Omega_\omega^{1/2^*} = \Omega_\omega$.