Optimal Control of Plasticity Systems

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Optimal Control of Plasticity Systems

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Contents

Abstract

The thesis is concerned with an optimal control problem with an evolution variational inequality (EVI), involving a maximal monotone operator, as a constraint. This abstract setting can be applied to various cases of elasto plasticity.

It is shown that elasto and homogenized plasticity, elasto plasticity with an inertia term and also perfect plasticity can be transformed into a certain EVI. Such an EVI is analyzed in the context of optimal control. Then optimal control problems for each of the mentioned plasticity systems are considered, where the findings in the abstract case are either directly applied (elasto and homogenized plasticity and partly elasto plasticity with an inertia term) or at least partially used (perfect plasticity).

In each case, the existence of a global solution to the corresponding optimal control problem is shown. The state equations, and thus the control problems, are then regularized and results regarding approximation of global minimizers by global minimizers of the regularized problems are proved. For the optimal control problem, constrained by the abstract EVI, first and second order optimality conditions are derived, whereas only first order conditions are investigated for optimal control problems governed by plasticity systems.

A certain difficulty arises in the case of perfect plasticity due to the non-uniqueness of the displacement and the fact that it is only of bounded deformation. This is the main reason for restricting the optimal control problem to the stress as the only state variable when it comes to optimality conditions. Moreover, for this case numerical experiments are presented. The finite element toolbox *FEniCS* was used to solve the involved partial differential equations.

Part I Introduction

The thesis at hand is concerned with the optimal control of four different plasticity systems, namely

(P.i) small-strain quasistatic elasto plasticity with linear kinematic hardening, for short *elasto plasticity*,

(P.ii) small-strain quasistatic homogenized plasticity with linear kinematic hardening, for short *homogenized plasticity*,

(P.iii) small-strain elasto plasticity with an inertia term and linear kinematic hardening, for short *plasticity with inertia* and

(P.iv) small-strain quasistatic elasto perfect plasticity, for short *perfect plasticity*.

We will always use the short term for these systems in what follows. Moreover, we will simply say hardening instead of linear kinematic hardening.

In all four systems a certain equation, respectively inclusion, is present, the so called *flow rule*. All systems have in common, that they can be transformed into an *evolution variational inequality (EVI)* due to the flow rule. In the case of elasto, homogenized and perfect plasticity, this EVI is essentially the flow rule and the solution to it the *plastic strain*. In these three cases, the *stress* depends linearly on the *displacement* and the plastic strain. Therefore the systems can be reduced such that the stress is not present anymore. Then, the *balance of momentum* can be solved and the displacement be obtained if the plastic strain is known. Hence, the displacement and the balance of momentum can be replaced with a solution operator. Inserting this solution operator in the flow rule yields the mentioned EVI. In the case of homogenized plasticity, these steps can also be performed, but one has to take the variables representing the micro structure into account. This idea of the transformation into an EVI originated in [45], at least to the author's knowledge. The case of plasticity with inertia has to be treated differently, the whole system has to be rewritten such that it is equivalent to an EVI. An analog method was also proposed in [45], however, therein the equations were transformed into a second order EVI .

The arising EVI reads as follows:

$$\dot{z} \in A(R\ell - Qz), \qquad z(0) = z_0. \tag{EVI}$$

Therein, A is a *maximal monotone operator*, R and Q are linear and continuous operators and z_0 is the initial condition. In the case of elasto, homogenized and perfect plasticity, the maximal monotone operator might be the subdifferential of an indicator function representing the *von-Mises flow rule*. In the case of plasticity with inertia it will be a certain maximal monotone operator which is in fact not a subdifferential. In the case of elasto, homogenized and perfect plasticity, the operator $\boldsymbol{R}$ is a solution operator of the equations of *linear elasticity*. It maps possible external forces and a potential Dirichlet displacement, both contained in the variable ℓ, into the space of the plastic strain z. The operator $\boldsymbol{Q}$ consists also of a solution operator, but also of the *elasticity tensor* and the *hardening parameter* (except in the case of perfect plasticity, where the hardening parameter is zero). The case of plasticity with inertia is again different. The operators $\boldsymbol{R}$ and $\boldsymbol{Q}$ are simpler, due to the more complex maximal monotone operator, and z is not only (a transformed version of) the plastic strain but contains also the displacement and the velocity of the displacement.

Of course, the description above of the transformation into an EVI and the description of this EVI has only the purpose to give a rough introduction to the idea on which this thesis is mostly based on. Throughout the thesis, we will give rigorous definitions of the mentioned plasticity systems, lay out the transformation into an EVI in detail and present an in-depth analysis of this EVI.

Organization of the Thesis

The thesis is organized as follows: Besides the abstract at the beginning and the conclusion and outlook, the appendix and several lists at the end, it consists of five parts, each part containing two or more chapters. All chapters except Chapter 1 are divided further into two or more sections.

In the current part we present the notation and standing assumptions in Chapter 1 and proceed then with an introduction into elasto plasticity in Chapter 2. Therein, we will apply the above described idea to transform the system of elasto plasticity into an EVI of the form of (EVI).

Part II is devoted to the analysis and optimal control of (EVI). At first we will introduce maximal monotone operators with a closer look at subdifferentials. After this, we will analyse EVIs. We prove the existence of solutions and convergence results in two different settings, in the first one we use a boundedness assumption on the maximal monotone operator. In the second setting, we drop this assumption, but use instead more regular loads, that is, loads which are $\boldsymbol{H}^2$ in time. The first setting is useful for elasto, homogenized and perfect plasticity, the second is tailored for plasticity with inertia. With the existence and convergence results at hand, we finally consider an optimal control problem with an EVI as a constraint. Besides an existence and approximation result, first and second order optimality conditions are proved.

In Part III we apply the abstract results obtained in Part II to the case of elasto and homogenized plasticity. In the case of elasto plasticity, we can use the results from Chapter 2 to consider straightway an optimal control problem. This problem fits exactly into the setting considered in Part II, therefore the results are directly applicable and first order optimality conditions in the form of a KKT system are provided. In Chapter 7 the equations of homogenized plasticity are transformed into an EVI. This chapter is kept rather short, we focus only on the transformation, an application of the general theory developed in Part II is possible and would be similar as the one presented for elasto plasticity. For this reason and the fact that such an application was provided in [71], Part III ends after the transformation of homogenized plasticity into an EVI.

Plasticity with inertia is the topic of Part IV. In contrast to elasto and homogenized plasticity, the findings concerned with optimal control in Part II cannot be applied directly and the transformation into an EVI is different. Nonetheless, the results concerning the existence and convergence of EVIs in the setting with more regular loads, provided in Part II, can be used. This part concludes with first order optimality conditions again in the form of a KKT system.

Finally, in Part V the case of perfect plasticity is handled. The equations are identical to the ones of elasto plasticity, except that the hardening parameter is set to zero. Therefore the equations of perfect plasticity can still be transformed into an EVI of the form of (EVI). As described above, the operator Q consists of a solution operator, the elasticity tensor and, in the case of elasto plasticity, the hardening parameter. As we will see in Chapter 2, the solution operator together with the elasticity tensor is positive, but only the presence of the hardening parameter makes the operator Q coercive . This coercivity will be vital in Part II, hence, most of the results developed in this part are not applicable for perfect plasticity. Moreover, it is also known that the equations of perfect plasticity in general only admits solutions where the displacement is of *bounded deformation* and also not unique. Due to these facts, the existence and approximation of solutions have to be investigated in detail before we can turn to an optimal control problem. The approximation of global minimizers by global minimizers of regularized optimal control problems is, due to the lesser regularity and non-uniqueness of the displacement, all but standard. This is why we then reduce the problem under consideration to the stress, that is, we drop the displacement as a state. For this reduced problem optimality conditions are provided and we finally present numerical experiments.

Comparison with the Literature

As can be inferred from the description above, the objective of the thesis at hand is the study of optimal control problems constrained by the four mentioned plasticity systems. Since these plasticity systems are composed of partial differential equations (PDEs) and variational inequalities (VIs), the optimal control of these systems falls under the topic of optimal control of PDEs and VIs. Such problems are investigated in the literature, let us refer to [100, 54, 23, 1, 16, 17, 94, 79] and the references therein. We also note that the plasticity systems we are considering are *rate-independent*, for a servey of rate-independent systems see [76]. Optimal control problems with rate-independent systems are, e.g., also investigated in [81, 75, 82, 35]. However, when it comes to optimal control problems governed by quasi-static plasticity the literature becomes rather scarce. In the static case there are various articles, for example, [52, 70], but in the quasi-static case, we are only aware of [104, 105, 107, 108].

Beside these references, the articles [71, 72, 73] also deal with optimal control of homogenized and perfect plasticity and serve as a basis for this thesis. In particular, the content of Part II and Chapter 7 is based on [71] and Part V on [72, 73]. Note that some of the substance of these papers can be found one to one in this thesis. Nonetheless, optimal control of plasticity with inertia presented in Part IV, which had also an impact on the content of Part II, is completely new and not published yet. Moreover, the treatment of the existence and approximation of solutions to perfect plasticity in Chapter 11 contains external forces in contrast to the findings in [72, 73], which is a delicate issue due to the so called *safe-load condition* (cf. Definition 11.13). Besides this, the approximation of optimal controls shown in Section 12.1 differs from the one presented in [73], see Remark 12.2. Moreover, the papers [71, 72, 73] give also a good overview of related literature, which again serve as a foundation for our comparison.

Note also that the case of elasto plasticity presented in Chapter 6 was already dealt with in [104, 105, 107, 108]. However, our approach is completely different and elasto plasticity is only one case which fits in the abstract setting considered in Part II.

Throughout the thesis, we compare some of our findings in more detail with the above and further references.

Some Comments on Assumptions

Before we can dive into the mathematical content of our work, we have to elaborate on one speciality regarding our assumptions. At first, we introduce the standing assumptions for the whole thesis in the upcoming chapter. These assumptions are tacitly assumed throughout the thesis without mentioning them every time. Furthermore, we collect all assumptions needed in a certain part right at the beginning of that part. We agree upon the following:

The Assumption Agreement. *In the assumption collection at the beginning of a part, each item is denoted with a roman or arabic number. The roman number refers to a part and the arabic number to a chapter or section. The part, chapter or section associated with an item by its number is the scope in which the assumption is valid. An assumption may be strengthened by a further assumption or weakened inside a certain result.*

Let us give an example how to get all assumptions for a specific result. Consider for example Theorem 5.21. Clearly, at first we have to look at the statement of a result, in case of Theorem 5.21 we have the assumption that a $\delta > 0$ exists such that (5.34) is fulfilled. Next, we look for the assumption collection at the beginning of Part II, namely Assumption II. Since Theorem 5.21 is contained in Section 5.3, Assumptions ⟨5.3.i⟩ to ⟨5.3.iv⟩ are necessary. Notice that Assumption ⟨5.3.i⟩ also includes Assumptions ⟨5.2.i⟩ to ⟨5.2.v⟩. Furthermore, since Section 5.3 belongs to Chapter 5, also Assumptions ⟨5.i⟩ to ⟨5.iv⟩ are required for Theorem 5.21, where Assumption ⟨5.i⟩ includes Assumption ⟨4.i⟩ and Assumption ⟨4.ii⟩. Finally, Chapter 5 is contained in Part II, hence, Assumption ⟨II.i⟩ and Assumption ⟨II.ii⟩ have to be taken into account. Beside all these assumptions from Assumption II, the standing assumptions in Chapter 1 are additionally necessary. Now we have collected every assumption which is required, such that the statement in Theorem 5.21 holds true.

We have decided to present our assumptions according to The Assumption Agreement due to two reasons. At first, it is always clear how to get all assumptions required for a specific result, hence, we do not have to go through a whole section, chapter or part to check if there is an assumption which has to be taken into account. Secondly, our agreement lifts the burden to state many assumptions inside a specific result, possibly repeating previously made assumptions. Of course, our agreement also has a drawback. Some assumptions made for a specific section, chapter or part of the thesis is not necessary for each finding therein. However, the assumptions are always required for the main results, thus it is reasonable to simply suppose that they hold in the whole section, chapter or part.

Note also that the assumptions in the collections are tacitly assumed throughout the associated section, chapter or part. Nonetheless, at various passages we will recall some assumptions to point out important issues. Moreover, they may contain notation and notions which are only defined later. However, these assumptions will not be used until the mentioned terms were introduced. Thus the reader may skip the assumption collections and come back to it when a new chapter or section starts.

Let us now state the assumption collection for this part.

Assumption I. *We impose the following assumptions according to The Assumption Agreement above.*

⟨*2.i*⟩ *Suppose that* $z_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$ *is given.*

⟨*2.ii*⟩ *Let* $f, g \in H^1(H^{-1}_D(\Omega; \mathbb{R}^d))$ *and* $u_D \in H^1(H^1(\Omega; \mathbb{R}^d))$ *be given.*

Note that Assumption I, and also Assumption III, is quite sparse due to the nature of these parts. In Part II, Part IV and Part V quite the contrary is the case. Moreover, the assumption collections for all other parts appear right at the beginning of the corresponding part, Assumption I is an exception.

We can now proceed to lay out our notation and state the standing assumptions in the upcoming chapter.

Chapter 1 Notation and Standing Assumptions

In this chapter we introduce the notation we use and impose all standing assumptions. As said above, these standing assumptions are valid in the whole thesis and will be assumed tacitly, that is, we do not mention them again except for clarification.

Before we start, let us point to the List of Symbols on Page 159. This list gives a good overview of the most used symbols in this thesis. Let us shortly discuss one feature of our notation. Some of the used symbols are defined multiple times. Consider the structure of the thesis into five parts, a symbol has the same definition in one part but may be redefined in another, with the exception of Part III, where the redefinition may happen in Chapter 6 and Chapter 7. Consider for instance the space $\mathcal{X}$ in the List of Symbols. It is an arbitrary Banach space in Part II, but defined in Chapter 6 and Part V (respectively Chapter 13) differently and also referred to in Chapter 7 and Part IV. This has the following reason: In Part II we consider a general setting with given data (loads) in $\mathcal{X}$. In the four considered cases of plasticity, this general setting can be (partly) applied, where in each case the space, corresponding to $\mathcal{X}$ from the general setting, is different. Thus, it is reasonable to denote all these spaces by $\mathcal{X}$. Such "multi" symbols are tagged with a small $_{\mathrm{m}}$ in the List of Symbols.

Let us now continue to introduce our notation and impose the standing assumptions.

Numbers: We use the usual notation for numbers. By $\mathbb{N}$ we denote the natural numbers without zero and by $\mathbb{R}$ the real numbers. When $a, b \in \mathbb{R}$, then $[a, b] \subset \mathbb{R}$ is the closed interval and $(a, b) \subset \mathbb{R}$ the open interval. Half closed or open intervals are denoted analogously. Furthermore, $c > 0$ and $C > 0$ denote generic constants, that is, they may change their values during a calculation.

General Notation: When X is a normed vector space we denote its norm by $\|\cdot\|_X$. For normed vector spaces X and Y we denote the space of linear and continuous functions on X with values in Y by $\mathcal{L}(X;Y)$. We abbreviate $\mathcal{L}(X) := \mathcal{L}(X;X)$. The dual space of X is denoted by $X^* = \mathcal{L}(X;\mathbb{R})$. The inner product of a Hilber space H is denoted by $(\cdot,\cdot)_H$. We may leave out the index X or H of norms and inner products when it is clear from the context. Given a coercive and symmetric operator $G \in \mathcal{L}(H)$ on the Hilbert space H, we denote its largest coercivity constant by γ_G (which equals $\inf_{\|h\|_H=1}(Gh,h)_H$), i.e., $(Gh,h)_H \geq \gamma_G\|h\|_H^2$ for all $h \in H$. With this operator, we can define a new scalar product by $H \times H \ni (h_1,h_2) \mapsto \big(Gh_1,h_2\big)_H \in \mathbb{R}$, which induces an equivalent norm. We denote the Hilbert space equipped with this scalar product by H_G, that is, $\big(h_1,h_2\big)_{H_G} := \big(Gh_1,h_2\big)_H$ for all $h_1,h_2 \in H$. Moreover, we use common terms of functional analysis, e.g. such as weak convergence. We refer to [109, 13, 33, 111, 112, 113, 110, 6, 61, 86].

Matrices: For $d \in \mathbb{N}$, the $d \times d$-dimensional matrices are denoted by $\mathbb{R}^{d\times d}$. Given a matrix $\tau \in \mathbb{R}^{d\times d}$, the transpose is denoted by $\tau^\top$ and we define its deviatoric (i.e., trace-free) part as

$$\tau^D := \tau - \frac{1}{d}\operatorname{tr}(\tau)I,$$

where I is the identity matrix. We use the same notation for matrix-valued functions. The Frobenius norm is denoted by $|A|^2 := \sum_{i,j=1}^d A_{i,j}^2$ for $A \in \mathbb{R}^{d\times d}$ and for the associated scalar product, we write $A : B := \sum_{i,j=1}^{d} A_{i,j} B_{i,j}$, for all $A, B \in \mathbb{R}^{d\times d}$. By $\mathbb{R}_s^{d\times d} \subset \mathbb{R}^{d\times d}$, we denote the space of symmetric matrices, that is, $A \in \mathbb{R}_s^{d\times d}$ if and only if $A_{i,j} = A_{j,i}$ for all $i, j \in \{1, ..., d\}$ (so that $A = A^\top$).

Tensors: Fourth-order tensors, that is, linear and continuous mappings on $\mathbb{R}^{d\times d}$, are denoted by blackboard symbols. We denote the adjoint of a tensor $\mathbb{E} \in \mathcal{L}(\mathbb{R}^{d\times d})$ by $\mathbb{E}^\top$ (that is, we adopt the notation from matrices instead of using the $*$-symbol). The elasticity tensor and hardening parameter are denoted by $\mathbb{C} : \mathbb{R}_s^{d\times d} \to \mathbb{R}_s^{d\times d}$ and $\mathbb{B} : \mathbb{R}_s^{d\times d} \to \mathbb{R}_s^{d\times d}$, respectively. They are both linear and continuous, $\mathbb{C}, \mathbb{B} \in \mathcal{L}(\mathbb{R}_s^{d\times d})$. Moreover, both are symmetric and coercive, where symmetry, for instance for $\mathbb{C}$, means

$$\mathbb{C}\sigma : \tau = \sigma : \mathbb{C}\tau$$

for all $\sigma, \tau \in \mathbb{R}_s^{d\times d}$. In addition, we set $\mathbb{A} := \mathbb{C}^{-1}$ and note that $(\mathbb{A}\tau, \tau)_{\mathbb{R}_s^{d\times d}} \geq \gamma_{\mathbb{C}}/\|\mathbb{C}\|^2 \|\tau\|^2_{\mathbb{R}_s^{d\times d}}$ for all $\tau \in \mathbb{R}_s^{d\times d}$ holds, i.e., $\mathbb{A}$ is coercive with coercivity constant $\gamma_{\mathbb{A}} \geq \gamma_{\mathbb{C}}/\|\mathbb{C}\|^2$. Let us note that $\mathbb{C}$ and $\mathbb{B}$ could also have a spatial dependency, however, to keep the discussion concise, we restrict ourselves to this setting.

Domain: The domain Ω is an open and connected subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$ is the dimension. It is bounded by a Lipschitz boundary Γ, also denoted by $\partial\Omega$. The boundary consists of two disjoint measurable parts, the *Neumann boundary* Γ_N and the *Dirichlet boundary* Γ_D, such that $\Gamma = \Gamma_N \cup \Gamma_D$. While Γ_N is a relatively open subset, Γ_D is a relatively closed subset of Γ with positive boundary measure. The setting $\Gamma_D = \Gamma$ and $\Gamma_N = \emptyset$ would be possible. In addition, the set $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [46]. The outward unit normal vector on the boundary of Ω is denoted by $\nu : \partial\Omega \to \mathbb{R}^d$. Let us already mention that we will not directly use the precise properties of the domain. They are actually mostly used to solve (usually linear) elasticity, see e.g. Theorem 2.5, respectly Corollary 2.6, and in Corollary 8.9. As described above, mostly with the help of Corollary 2.6, we can exchange the balance of momentum with a solution operator which allows the transformation into an EVI.

Lebesgue Spaces: Let $k \in \mathbb{N}$, $p \in [1, \infty]$ with conjugate exponent p', that is, $1/p + 1/p' = 1$, where $1/\infty := 0$, and X be a finite dimensional space. Throughout the thesis, by $L^p(\Omega; X)$ we denote Lebesgue spaces and by $L^p(\Gamma_N; X)$ Lebesgue spaces on the Neumann boundary with values in X. For $u \in L^1(\Omega; X)$, we abbreviate $\int_\Omega u := \int_\Omega u(x)dx$, that is, we do not write "dx" when the argument of the function is not present. Let us refer to [85, 37] for the notion of Lebesgue spaces.

Sobolev Spaces: Sobolev spaces which are k-times (weakly) differentiable are denoted by $W^{k,p}(\Omega; X)$. The set $W_D^{1,p}(\Omega; X)$ is the subspace of $W^{1,p}(\Omega; X)$ which contains all functions whose traces are zero on Γ_D. The dual space of $W_D^{1,p'}(\Omega; X)$, for $p \in (1, \infty)$, is denoted by $W_D^{-1,p}(\Omega; X)$. We use the usual abbreviations $H^k(\Omega; X) := W^{k,2}(\Omega; X)$ and $H_D^1(\Omega; X) := W_D^{1,2}(\Omega; X)$. We only refer to [2] for further information about Sobolev spaces.

Continuous Functions on Ω: The set of continuous functions on Ω is denoted by $C(\Omega)$. The subset of functions which can be extended to $C(\overline{\Omega})$ and are zero on $\partial\Omega$ are denoted by $C_0(\Omega)$. The set $C_c(\Omega)$ contains continuous functions with compact support, $C^k(\Omega)$ continuous functions which are k-times differentiable, $C^\infty(\Omega)$ continuous functions which belong to $C^k(\Omega)$ for every $k \in \mathbb{N}$ and $C_c^\infty(\Omega)$ is the intersection between $C_c(\Omega)$ and $C^\infty(\Omega)$. The set $C(\Omega; X)$ contains continuous functions into the finite dimensional space X, for the other sets of continuous functions we use an analog notation.

Bochner(-Sobolev) Spaces: Let now X be an arbitrary Banch space, possibly with infinite dimension. By $T > 0$ we denote the end time of the considered time horizon $[0, T]$. For $t > 0$ we denote the space of square-integrable Bochner functions on the time interval $[0, t]$ by $L^2(0, t; X)$ and abbreviate $L^2(X) := L^2(0, T; X)$. We use this abbreviation with the exception of the space $L^2(0,T;\mathbb{R})$, where our abbreviation might be confusing. The space of Bochner-Sobolev functions is denoted by $H^1(0,t;X)$ and we also abbreviate $H^1(X) := H^1(0,T;X)$. We use further the rather unusual notation $H^1_0(X) := \{v \in H^1(X) : v(0) = 0\}$ and $H^1_{00}(X) := \{v \in H^1(X) : v(0) = v(T) = 0\}$. For the notion and properties of Bocher and Bochner-Sobolev spaces we refer to [104, Chapter 3.1], [31], [41, Kapitel IV, §1], [109, Chapter V, 5] and [86, Chapter 2.1].

Continuous Functions on $[0,T]$: Similarly, the space of continuous functions is denoted by $C(0,t;X)$ and we write $C(X) := C(0,T;X)$.

Extension of Linear Operators: For two Banach spaces X, Y, when $G \in \mathcal{L}(X;Y)$ is a linear and continuous operator, we can define an operator in $\mathcal{L}(L^2(X);L^2(Y))$ by $G(u)(t) := G(u(t))$ for all $u \in L^2(X)$ and for almost all $t \in [0,T]$, we denote this operator also by G, that is, $G \in \mathcal{L}(L^2(X);L^2(Y))$, and analog for Bochner-Sobolev spaces and the space of continuous functions on $[0,T]$, i.e., $G \in \mathcal{L}(H^1(X);H^1(Y))$ and $G \in \mathcal{L}(C(X);C(Y))$, respectively. We do the same in the case of Lebesgue spaces. For instance, given the elasticity tensor $\mathbb{C} \in \mathcal{L}(\mathbb{R}^{d\times d}_s)$, we have $\mathbb{C} \in \mathcal{L}(L^p(\Omega;\mathbb{R}^{d\times d}_s))$ by $(\mathbb{C}v)(x) := \mathbb{C}v(x)$ for all $v \in L^p(\Omega;\mathbb{R}^{d\times d}_s)$.

We moreover note that some new notation is presented throughout the thesis at appropriate places.

Let us now start with the introduction of elasto plasticity.

Chapter 2 A First Glance at Elasto Plasticity

We start by considering the equations of elasto plasticity, see Part I Item (P.i). The equations can be stated in a *formal* and strong formulation as follows:

$$-\nabla \cdot \sigma = f \qquad \text{in } \Omega, \tag{2.1a}$$
$$\nu \cdot \sigma = g \qquad \text{on } \Gamma_N, \tag{2.1b}$$
$$u = u_D \qquad \text{on } \Gamma_D \tag{2.1c}$$
$$\sigma = \mathbb{C}(\nabla^s u - z) \qquad \text{in } \Omega, \tag{2.1d}$$
$$\dot{z} \in \partial I_{\mathcal{K}(\Omega)}(\sigma - \mathbb{B}z), \quad z(0) = z_0 \qquad \text{in } \Omega. \tag{2.1e}$$

Herein the unknown variables are the displacement $u : [0,T] \times \Omega \to \mathbb{R}^d$, the stress $\sigma : [0,T] \times \Omega \to \mathbb{R}^{d\times d}_s$ and $z : [0,T] \times \Omega \to \mathbb{R}^{d\times d}_s$, an internal variable representing the plastic strain. The given variables are the body or volume force $f : [0,T] \times \Omega \to \mathbb{R}^d$, the force on the Neumann boundary $g : [0,T] \times \Gamma_N \to \mathbb{R}^d$ and the prescribed displacement on the Dirichlet boundary $u_D : [0,T] \times \Gamma_D \to \mathbb{R}^d$. Moreover, the elasticity tensor $\mathbb{C} : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ and hardening parameter $\mathbb{B} : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ are given (see Chapter 1). The *(Von-Mises) flow rule* is stated by using the multivalued mapping $\partial I_{\mathcal{K}(\Omega)} : \mathbb{R}^{d\times d}_s \to 2^{\mathbb{R}^{d\times d}_s}$, given in Definition 2.2. We will give a more detailed physical interpretation after the definition of a solution below. The presentation of elasto plasticity in (2.1) is only one possibility, it can also be found in Han and Reddy [49, Chapter 7], Wachsmuth [104, Chapter 2.2] or Schweizer [87, Chapter 27.3]. Another option, for example, is to consider the dual problem as in Han and Reddy [49, Chapter 8].

We also note that elasto plasticity is a certain type of hysteresis, see e.g. [102, 64, 19, 63]. Since we are only interested in plasticity, we do not have a closer look at the concept of hysteresis.

The reason to consider (2.1) is the following: It is our goal to transform (2.1) into an EVI of the form of (EVI). To this end, we use (2.1d) and insert σ into the other equations. Then we solve, for a known plastic strain z, (2.1a) to (2.1c). Using the obtained solution operator, we have then reduced (2.1) to (2.1e), which is then equivalent to (EVI).

This is essentially the same procedure as in GRÖGER [45, Section 4] or ALBER [3, Chapter 4] and therefore not new. However, since we use this transformation, respectively the idea behind it, throughout the thesis (see for example Chapter 6, Chapter 7, Proposition 11.11 or Proposition 12.17), it is fundamental for us and we thus present it in detail.

Let us finally refer to [49, 104, 80, 67, 47] for the notion of plasticity, elasticity and continuum mechanics in general.

2.1 Definition of a Solution

At first we give the definition of a solution to (2.1). To this end, we formulate the definition of (2.1a) and (2.1b) in a weak sense and afterwards consider the flow rule (2.1e).

Definition 2.1 (Balance of momentum). *Let $p \in [1, \infty]$ and define the* symmetrized gradient

$$\nabla^s : W^{1,p}(\Omega; \mathbb{R}^d) \to L^p(\Omega; \mathbb{R}^{d\times d}_s), \qquad \nabla^s v := \frac{1}{2}(\nabla v + (\nabla v)^\top)$$

for all $v \in W^{1,p}(\Omega; \mathbb{R}^d)$ and the divergence operator

$$\operatorname{div} : L^p(\Omega; \mathbb{R}^{d\times d}_s) \to W^{-1,p}_D(\Omega; \mathbb{R}^d), \qquad \langle \operatorname{div} \tau, \xi\rangle := -\langle \tau, \nabla^s \xi\rangle_{L^p(\Omega;\mathbb{R}^{d\times d}_s)}$$

for all $\tau \in L^p(\Omega; \mathbb{R}^{d\times d}_s)$ and $\xi \in W^{1,p'}_D(\Omega; \mathbb{R}^d)$, which is the adjoint of the symmetrized gradient restricted to $W^{-1,p}_D(\Omega; \mathbb{R}^d)$. We do not incorporate the exponent p into the notation of div*, it will always be clear from the context.*

We say that a stress $\tau \in L^p(\Omega; \mathbb{R}^{d\times d}_s)$ fulfills the balance of momentum *or* equilibrium condition *with respect to $L \in W^{-1,p}_D(\Omega; \mathbb{R}^d)$, when*

$$-\operatorname{div} \tau = L. \tag{2.2}$$

Note that when $L \in L^p(\Omega; \mathbb{R}^d)$ or $L \in L^p(\Gamma_N; \mathbb{R}^d)$, we identify L with $W^{1,p'}_D(\Omega; \mathbb{R}^d) \ni \varphi \mapsto \langle L, \varphi\rangle_{L^p(\Omega;\mathbb{R}^d)}$ or $W^{1,p'}_D(\Omega; \mathbb{R}^d) \ni \varphi \mapsto \langle L, \varphi\rangle_{L^p(\Gamma_N;\mathbb{R}^d)}$, respectively, so that we can also write (2.2).

Let us shortly and *formally* explain the connection between the div operator and (2.1a) and (2.1b). Let us simply assume that σ is of class $C^1(\Omega; \mathbb{R}^{d\times d}_s)$ and fulfills (2.1a) and (2.1b). After testing (2.1a) with a smooth test function and integrating over Ω, we can apply the divergence theorem to see that (2.2) with $L = f + g$ holds. On the other hand, one can analog obtain (2.1a) and (2.1b) from (2.2) with the reversed argumentation. We emphasize that the forces g on the Neumann boundary are incorporated in the div operator, see (2.3a) and (2.4). We come back to this connection when we have a closer look at the physical interpretation below.

To give a definition for (2.1e), we now describe the operator $\partial I_{\mathcal{K}(\Omega)}$ in

Definition 2.2 (Von-Mises flow rule). *Let the* uniaxial yield stress*, or just the* yield stress*, $\gamma > 0$ be given and abbreviate*

$$K := \{\tau \in \mathbb{R}^{d\times d}_s : |\tau^D| \le \gamma\}$$

and

$$\mathcal{K}(\Omega) := \{\tau \in L^2(\Omega; \mathbb{R}^{d\times d}_s) : \tau(x) \in K \text{ f.a.a. } x \in \Omega\}.$$

We say to both sets the set of admissible stresses*. The multivalued operator $\partial I_{\mathcal{K}(\Omega)} : L^2(\Omega; \mathbb{R}^{d\times d}_s) \to 2^{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$ is defined by*

$$\partial I_{\mathcal{K}(\Omega)}(\sigma) := \{\tau \in L^2(\Omega; \mathbb{R}^{d\times d}_s) : 0 \ge (\tau(x), v - \sigma(x))_{\mathbb{R}^{d\times d}} \ \forall v \in K, \text{ f.a.a. } x \in \Omega\}$$

when $\sigma \in \mathcal{K}(\Omega)$ *and by*

$$I_{\mathcal{K}(\Omega)}(\sigma) := \emptyset$$

when $\sigma \notin \mathcal{K}(\Omega)$.

The above defined operator $\partial I_{\mathcal{K}(\Omega)}$ is in fact the subdifferential of the indicator functional of the convex set $\mathcal{K}(\Omega)$, as we will see in Section 3.2. However, for the purpose of this chapter, it is sufficient to simply define $\partial I_{\mathcal{K}(\Omega)}$ as above (in fact, $\partial I_{\mathcal{K}(\Omega)}$ could be exchanged with any other multivalued operator mapping from $L^2(\Omega;\mathbb{R}^{d\times d}_s)$ to $2^{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$, the concrete form is unimportant for Theorem 2.9).

In what follows, recall that $z_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$ is given in Assumption ⟨2.i⟩ and $f, g \in H^1(H^{-1}(\Omega;\mathbb{R}^d))$ and $u_D \in H^1(H^1(\Omega;\mathbb{R}^d))$ in Assumption ⟨2.ii⟩.

Definition 2.3 (Solution to an EVI). *Let* $\sigma \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$. *Then* $z \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *is called solution of* (2.1e), *if*

$$\dot{z}(t) \in \partial I_{\mathcal{K}(\Omega)}(\sigma(t) - \mathbb{B}z(t))$$

holds for almost all $t \in [0,T]$, *and* $z(0) = z_0$. *Note that, due to the continuous embedding* $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)) \hookrightarrow C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *(see [104, Theorem 3.1.41]), the evaluation of* z *in* 0 *is well defined.*

Clearly, the definition of a solution to (2.1e) is quite intuitive, therefore in the rest of this thesis the definitions of EVIs similar to (2.1e) will be analog. Thus, there is no need to give definitions, for example, for (2.17), (EVI_q), (EVI_s) or (4.4).

Now we are in a position to give the definition of a solution to (2.1) in

Definition 2.4 (Solution to elasto plasticity). *The functions* $u \in H^1(H^1(\Omega;\mathbb{R}^d))$, $\sigma \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *and* $z \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *are called solution of* (2.1) *if*

$$-\operatorname{div}\sigma = f + g, \tag{2.3a}$$

$$u - u_D \in H^1(H^1_D(\Omega;\mathbb{R}^d)), \tag{2.3b}$$

$$\sigma = \mathbb{C}(\nabla^s u - z), \tag{2.3c}$$

$$\dot{z} \in \partial I_{\mathcal{K}(\Omega)}(\sigma - \mathbb{B}z), \qquad z(0) = z_0. \tag{2.3d}$$

Later in Section 6.1 the forces f and g will belong to the spaces $L^2(\Omega;\mathbb{R}^d)$ and $L^2(\Gamma_N;\mathbb{R}^d)$, respectively, then (2.3a) is, according to Definition 2.1, equivalent to

$$(\sigma, \nabla^s\zeta)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} = (f,\zeta)_{L^2(\Omega;\mathbb{R}^d)} + (g,\zeta)_{L^2(\Gamma_N;\mathbb{R}^d)} \qquad \forall \zeta \in H^1_D(\Omega;\mathbb{R}^d). \tag{2.4}$$

Due to this reason, we have used f and g in (2.3a) instead of some $L = f + g \in H^1(H^{-1}_D(\Omega;\mathbb{R}^d))$.

Physical Interpretation

The equations (2.1a) and (2.1b), respectively (2.3a), are called *balance of momentum*. The internal forces, that is the stress σ, must be in equilibrium with the external forces, that is the volume force f and the force g on the Neumann boundary. To see how this "equilibrium" can be interpreted, we assume that σ is sufficient regular and use, as already described after Definition 2.1, the divergence theorem in (2.4) to obtain (2.1a) and (2.1b). Taking now an (sufficient regular) arbitrary subset V of Ω, integrating (2.1a) over V and using the divergence theorem once again we get $\int_{\partial V} \nu \cdot \sigma + \int_V f = 0$. Setting $\Gamma_{V,N} := \partial V \cap \Gamma_N$ and $\Gamma_{V,\sigma} := \partial V \setminus \Gamma_{V,N}$ we finally obtain

$$\int_{\Gamma_{V,\sigma}} \nu \cdot \sigma + \int_V f + \int_{\Gamma_{V,N}} g = 0. \tag{2.5}$$

Equation (2.5) can be read as "the sum of all forces on the cutout V equals zero", which is Newton's second law of motion for an not accelerated body, see Figure 1 for an illustration of the described situation. Therein the whole body Ω with the indicated cutout V is depicted on the left side and the cutout V on the right side. Here the internal forces $\nu \cdot \sigma$ must be applied on the new boundary $\Gamma_{V,\sigma}$ of the cutout such that the deformation is kept. We have thus derived (2.5) for all subsets $V \subset \Omega$ from (2.4). Vice versa, when (2.5) holds for all subsets $V \subset \Omega$, we obtain (2.4) with the reversed argumentation. Note that in general our body will be accelerated, in this case the second time derivative of the displacement multiplied by the density of the body has to be added in (2.1a), which would carry over to (2.5). However, the classical argumentation to derive (2.1a), is that the acceleration is small and can thus be neglected. This is also the reason why (2.1) is called *quasistatic* (see Item (P.i)). Nonetheless, in Part IV we will also consider the case where the acceleration is present.

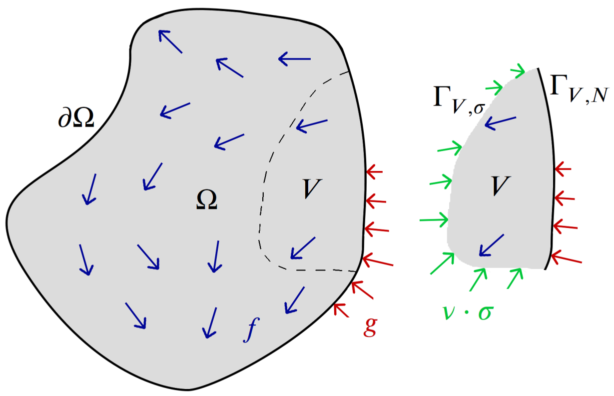


Figure 1: Depiction of the body Ω and the cutout V.

Equation (2.1c), respectively (2.3b), simply represents the fact that the displacement u should equal the prescribed Dirichlet displacement on the Dirichlet boundary. An interpretation is, that the body Ω is "being held" on the Dirichlet boundary.

The split of the symmetrized gradient in (2.3c) can be rewritten as

$$\nabla^s u = e + z, \tag{2.6}$$

where

$$e := \mathbb{C}^{-1}\sigma = \mathbb{A}\sigma. \tag{2.7}$$

The unknown variable e is called *elastic strain*, so that (2.6) represents the assumption that the overall strain $\nabla^s u$ can be split by addition into an elastic part e and a plastic part z. Now (2.7), respectively $\sigma = \mathbb{C}e$, is *Hooke's law*, the internal force, that is the stress σ, is linearly dependent on the (elastic) strain e (whereas the plastic strain z is not).

Finally, the *flow rule* (2.3d) can be interpreted as follows: As long as σ stays in the interior of $\mathcal{K}(\Omega)$, the plastic strain z remains equal to z_0 (typically set to zero in applications, see also Chapter 13). When, due to the external forces and the Dirichlet displacement, the stress reaches the boundary of $\mathcal{K}(\Omega)$, also called *yield surface*, then the plastic strain flows in the direction normal to the yield surface and the stress can take values outside the set of admissible stresses, only $\sigma - \mathbb{B}z$ has to be admissible.

Clearly, there are many more facets in elasto plasticity and its modeling, for a more thorough survey, we again refer, for example, to HAN AND REDDY[49].

2.2 Transformation into an EVI

After we have given the notion of a solution in Definition 2.4, we can now turn to the transformation into an EVI of the type (EVI). As described at the beginning of this chapter, we will replace (2.3a) and (2.3b) with a solution operator. In this chapter we only need to obtain a solution in $H^1(\Omega;\mathbb{R}^d)$, however, since we need to solve it in $W^{1,p}(\Omega;\mathbb{R}^d)$ in Section 6.1, Corollary 8.9 and Proposition 12.17, we present the result already here. This finding was proven in HERZOG ET AL.[50] but we will use it in detail in Corollary 8.9, so we recite it in the following

Theorem 2.5 (Nonlinear elasticity). *Let $b : \Omega \times \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ be a function such that* HERZOG ET AL. *[50, Assumption 1.5 (2)] holds, that is,*

$$b(\cdot, 0) \in L^\infty(\Omega; \mathbb{R}^{d\times d}_s), \tag{2.8}$$

$$b(\cdot, \tau) \text{ is measurable }, \tag{2.9}$$

$$(b(x,\tau) - b(x,\overline{\tau})) : (\tau - \overline{\tau}) \geq m|\tau - \overline{\tau}|^2, \tag{2.10}$$

$$|b(x,\tau) - b(x,\overline{\tau})| \leq M|\tau - \overline{\tau}| \tag{2.11}$$

for almost all $x \in \Omega$ and all $\tau, \overline{\tau} \in \mathbb{R}^{d\times d}_s$ with constants $0 < m \leq M$.

Then there exists $\overline{p} \in (2,\infty)$ such that for every $p \in [2,\overline{p}]$ and $L \in W^{-1,p}_D(\Omega;\mathbb{R}^d)$ there exists a unique $u \in W^{1,p}_D(\Omega;\mathbb{R}^d)$ such that

$$-\operatorname{div} b(\cdot, \nabla^s u(\cdot)) = L.$$

Moreover, there exists a constant C such that

$$\|u_1 - u_2\|_{W^{1,p}_D(\Omega;\mathbb{R}^d)} \leq C\|L_1 - L_2\|_{W^{-1,p}_D(\Omega;\mathbb{R}^d)}$$

holds for all $L_1, L_2 \in W^{-1,p}_D(\Omega;\mathbb{R}^d)$, where u_1 and u_2 are the corresponding solutions.

Proof. The claim follows from HERZOG ET AL. [50, Theorem 1.1 and Remark 1.3]. Note that [50, Assumption 1.5 (1)] holds according to Chapter 1. □

Corollary 2.6 (Linear elasticity). *Let $\overline{p}$ be from Theorem 2.5. Then for all $p \in [\overline{p}', \overline{p}]$, $L \in W^{-1,p}_D(\Omega;\mathbb{R}^d)$ and $u_D \in W^{1,p}(\Omega;\mathbb{R}^d)$, there exists a unique $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that*

$$-\operatorname{div} \mathbb{C}\nabla^s u = L, \tag{2.12a}$$

$$u - u_D \in W^{1,p}_D(\Omega;\mathbb{R}^d). \tag{2.12b}$$

We define the solution operator

$$\mathcal{T} : W^{-1,p}_D(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega;\mathbb{R}^d) \to W^{1,p}(\Omega;\mathbb{R}^d) \qquad (L, u_D) \mapsto u.$$

Furthermore, $\mathcal{T}$ is linear and continuous and we denote it with the same symbol for different p.

Proof. Obviously, the mapping

$$b : \Omega \times \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s \qquad (x,\tau) \mapsto \mathbb{C}\tau$$

fulfills (2.8) and (2.9). Since $\mathbb{C}$ is coercive, (2.10) is fulfilled with $m = \gamma_{\mathbb{C}}$ and (2.11) holds with $M = \|\mathbb{C}\|$. Thus, in the case $p \in [2,\overline{p}]$, we can apply Theorem 2.5 to obtain the unique existence of $v \in W^{1,p}_D(\Omega;\mathbb{R}^d)$ such that

$$-\operatorname{div} \mathbb{C}\nabla^s v = L + \operatorname{div} \mathbb{C}\nabla^s u_D.$$

That the same is true for $p \in [\overline{p}', 2)$ follows from the fact that the adjoint operator of an invertible linear and continuous operator is also invertible. Thus we obtain the assertion with $u := v + u_D$. □

Now, by eliminating σ in (2.1), we obtain the following equivalent equations:

$$\begin{aligned} -\operatorname{div}(\mathbb{C}\nabla^s u - \mathbb{C}z) &= f + g, \\ u - u_D &\in H^1(H^1_D(\Omega;\mathbb{R}^d)), \\ \dot{z} &\in \partial I_{\mathcal{K}(\Omega)}(\mathbb{C}\nabla^s u - (\mathbb{C}+\mathbb{B})z), \qquad z(0) = z_0. \end{aligned} \tag{2.13}$$

We can divide the first equation and inclusion into

$$\begin{aligned} -\operatorname{div}\mathbb{C}\nabla^s u_\ell &= f + g, \\ u_\ell - u_D &\in H^1(H^1_D(\Omega;\mathbb{R}^d)) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} -\operatorname{div}\mathbb{C}\nabla^s u_z &= -\operatorname{div}\mathbb{C}z, \\ u_z &\in H^1(H^1_D(\Omega;\mathbb{R}^d)), \end{aligned} \tag{2.15}$$

to see that, when $z \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ is given, $u \in H^1(H^1_D(\Omega;\mathbb{R}^d))$ is a solution of the first equation and inclusion in (2.13) if and only if $u_\ell \in H^1(H^1_D(\Omega;\mathbb{R}^d))$ is a solution of (2.14) and $u_z \in H^1(H^1_D(\Omega;\mathbb{R}^d))$ is a solution of (2.15) and we have $u = u_\ell + u_z$ (which also follows from the linearity of $\mathcal{T}$). Thanks to Corollary 2.6, we can eliminate the first equation and inclusion in (2.13) and describe the split $u = u_\ell + u_z$ by using the solution operator $\mathcal{T}$, that is,

$$u = \mathcal{T}(f + g - \operatorname{div}\mathbb{C}z, u_D) = \mathcal{T}(f+g, u_D) + \mathcal{T}(-\operatorname{div}\mathbb{C}z, 0) = u_\ell + u_z,$$

and obtain the equivalent inclusion

$$\dot{z} \in \partial I_{\mathcal{K}(\Omega)}(\mathbb{C}\nabla^s\mathcal{T}(f+g,u_D) + \mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z, 0) - (\mathbb{C}+\mathbb{B})z), \qquad z(0) = z_0 \tag{2.16}$$

Let us shorten the notation with the operators in the following

Definition 2.7 (Operators R and Q for elasto plasticity)**.** *We define the operators*

$$R : H^{-1}_D(\Omega;\mathbb{R}^d) \times H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{R}^{d\times d}_s), \qquad (L, u_D) \mapsto \mathbb{C}\nabla^s\mathcal{T}(L, u_D)$$

and

$$Q : L^2(\Omega;\mathbb{R}^{d\times d}_s) \to L^2(\Omega;\mathbb{R}^{d\times d}_s), \qquad z \mapsto (\mathbb{C}+\mathbb{B})z - \mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z, 0).$$

These operators are, according to Corollary 2.6, linear and continuous.

With these operators, (2.16) is equivalent to

$$\dot{z} \in \partial I_{\mathcal{K}(\Omega)}(R\ell - Qz), \qquad z(0) = z_0 \tag{2.17}$$

with $\ell := (f + g, u_D)$. Before we summarize our findings, let us prove that the operator Q is symmetric and coercive , which is crucial, as we will see in Part II.

Lemma 2.8 (Symmetry and coercivity of Q)**.** *The operator*

$$\mathbb{C} - \mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}\cdot, 0) = Q - \mathbb{B} \in \mathcal{L}(L^2(\Omega;\mathbb{R}^{d\times d}_s))$$

is positive and Q is symmetric and coercive.

Proof. At first we address the symmetry of Q. Since $\mathbb{C}$ and $\mathbb{B}$ are symmetric, it remains to prove the symmetry of $\mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}\cdot,0)$,

$$\begin{aligned}\big(\mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z_1,0),z_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} &= \big(\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z_1,0),\mathbb{C}z_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\\ &= \langle\mathcal{T}(-\operatorname{div}\mathbb{C}z_1,0),-\operatorname{div}\mathbb{C}z_2\rangle\\ &= \langle\mathcal{T}(-\operatorname{div}\mathbb{C}z_1,0),-\operatorname{div}\mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z_2,0)\rangle\\ &= \big(\mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z_1,0),\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z_2,0)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\end{aligned}$$

for all $z_1,z_2\in L^2(\Omega;\mathbb{R}^{d\times d}_s)$.

Let us now come to the positivity of $\mathbb{C}-\mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}\cdot,0)$. We have

$$\operatorname{div}\mathbb{C}(z-\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z,0)) = \operatorname{div}\mathbb{C}z-\operatorname{div}\mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z,0)=0,$$

hence

$$\begin{aligned}(\mathbb{C}z-\mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z,0),z)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} &= (\mathbb{C}(z-\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z,0),z)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\\ &= (\mathbb{C}(z-\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z,0),z-\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z,0))_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\\ &= \|z-\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}z,0)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{C}}}\geq 0\end{aligned}$$

for all $z\in L^2(\Omega;\mathbb{R}^{d\times d}_s)$. The coercivity of Q follows now from its definition. □

Let us finally collect our results in the following

Theorem 2.9 (Transformation of elasto plasticity into an EVI)**.** *The tuple* $(u,\sigma,z)\in H^1(H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^{d\times d}_s)\times L^2(\Omega;\mathbb{R}^{d\times d}_s))$*, is a solution of* (2.1)*, if and only if* $z\in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)))$ *is a solution of* (2.17)*, with* $\ell := (f+g,u_D)\in H^1(H^{-1}_D(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^d))$*,* $u=\mathcal{T}(f+g,u_D)+\mathcal{T}(-\operatorname{div}\mathbb{C}z,0)$ *and* $\sigma=\mathbb{C}(\nabla^s u-z)$*. Moreover, the operator* Q*, involved in* (2.17) *and given in Definition 2.7, is symmetric and coercive .*

Proof. The assertion follows from the definition of R and Q and Lemma 2.8. □

Having seen that the equations of elasto plasticity can be reduced to an EVI, it is now natural to consider this EVI in a general and abstract setting and derive optimality conditions for an optimal control problem constraint by such an EVI. However, let us emphasize that, as already noted in the beginning of Part I, an optimal control problem with elasto plasticity as a constraint was already analyzed in WACHSMUTH [104, Chapter 5] and is therefore not new. Nonetheless, the analysis in the upcoming part can also be applied (partly) for homogenized plasticity, plasticity with inertia and perfect plasticity, where all three applications were, to the author's current knowledge, until now not considered in the context of optimal control (except in [71, 72, 73]). Furthermore, we will also generalize the EVI by exchanging the operator $\partial I_{\mathcal{K}(\Omega)}$ with a general, maximal monotone operator. Since in all our applications we will always choose a subdifferential of an indicator function, this generalization seems at first only to be of theoretical interest. However, as it turns out in Part IV, we will actually need this generalization, see Remark 8.16.

Part II Evolution Variational Inequalities (EVIs)

As agreed upon at the beginning of Part I, we start by collecting all assumptions needed in this part.

Assumption II. *We impose the following assumptions according to The Assumption Agreement in the beginning of Part I.*

⟨II.i⟩ *By $\mathcal{H}$ we denote a Hilbert space and by $\mathcal{X}$ a Banach space.*

⟨II.ii⟩ *The multivalued operator $A : \mathcal{H} \mapsto 2^{\mathcal{H}}$ is maximal monotone (see Definition 3.1), if not otherwise said.*

⟨4.i⟩ *The operators $R : \mathcal{X} \to \mathcal{H}$ and $Q : \mathcal{H} \to \mathcal{H}$ are linear and continuous. Furthermore, Q is self-adjoint and coercive .*

⟨4.ii⟩ *The initial condition z_0 belongs to $\mathcal{H}$.*

⟨4.iii⟩ *The initial condition q_0 belongs to the domain of A, that is, $q_0 \in D(A)$, see Definition 3.1.*

⟨5.i⟩ *Suppose that Assumption ⟨4.i⟩ and Assumption ⟨4.ii⟩ hold.*

⟨5.ii⟩ *The maximal monotone operator $A : \mathcal{H} \mapsto 2^{\mathcal{H}}$ has the boundedness property (see Definition 3.5).*

⟨5.iii⟩ *The set M is a nonempty and closed subset of $D(A)$.*

⟨5.iv⟩ *There exists a space $\mathcal{W}$ such that $\mathcal{H}$ is a subspace of $\mathcal{W}$ and the injection $\mathcal{H} \hookrightarrow \mathcal{W}$ is continuous. Furthermore, the space $\mathcal{X}_c$ is reflexive and compactly embedded into $\mathcal{X}$. By*

$$J : H^1(\mathcal{W}) \times H^1(\mathcal{X}_c) \to \mathbb{R}$$

we denote the objective function.

⟨5.1.i⟩ *The objective function consists of two parts,*

$$J : H^1(\mathcal{W}) \times H^1(\mathcal{X}_c) \to \mathbb{R}, \qquad (z, \ell) \mapsto \Psi(z, \ell) + \Phi(\ell),$$

where $\Psi : H^1(\mathcal{W}) \times H^1(\mathcal{X}_c) \to \mathbb{R}$ and $\Phi : H^1(\mathcal{X}_c) \to \mathbb{R}$ are both weakly lower semicontinuous. Moreover, Ψ is bounded from below and continuous in the first argument, while Φ is coercive.

⟨5.1.ii⟩ *Let* $\{A_n\}_{n\in\mathbb{N}}$ *be a sequence of Lipschitz continuous operators from* $\mathcal{H}$ *to* $\mathcal{H}$ *such that, together with a sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$*, which converges towards zero, it holds*

$$\frac{1}{\lambda_n}\exp\Big(\frac{T\|Q\|_{L(\mathcal{H};\mathcal{H})}}{\lambda_n}\Big)\sup_{h\in\mathcal{H}}\|A_n(h)-A_{\lambda_n}(h)\|_{\mathcal{H}}\to 0, \tag{2.18}$$

i.e., the requirements in Lemma 4.17 are fulfilled.

⟨5.2.i⟩ *Additional to* $\mathcal{W}$*,* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are Banach spaces, such that the embeddings* $\mathcal{Y}\hookrightarrow\mathcal{Z}\hookrightarrow\mathcal{H}\hookrightarrow\mathcal{W}$ *are continuous. We also suppose that* $\mathcal{Y}$ *is dense in* $\mathcal{Z}$*,* $\mathcal{Z}$ *dense in* $\mathcal{H}$ *and* $\mathcal{H}$ *dense in* $\mathcal{W}$*. Furthermore, we assume that* $\mathcal{H}$ *is separable.*

⟨5.2.ii⟩ *The operator* Q *is extendable to an element of* $\mathcal{L}(\mathcal{Y})$ *and* $\mathcal{L}(\mathcal{Z})$*, and* R *to an element of* $\mathcal{L}(\mathcal{X};\mathcal{Y})$*. We denote all operators again by* Q *and* R*, respectively.*

⟨5.2.iii⟩ *Let* $z_0\in\mathcal{Y}$ *such that* $-Qz_0\in D(A)$*. The set* M *in the definition of the set of admissible controls is given by the singleton* $M=\{-Qz_0\}$ *such that*

$$\mathcal{A}_{\mathcal{L}} := \mathcal{A}_{\mathcal{L}}\big(z_0;\{-Qz_0\}\big)=\big\{\ell\in H^1(\mathcal{X}):\ \ell(0)\in\ker R\big\}.$$

⟨5.2.iv⟩ *The operator* $A_s:\mathcal{Y}\to\mathcal{Y}$ *is Lipschitz continuous and Fréchet differentiable from* $\mathcal{Y}$ *to* $\mathcal{Z}$*. Moreover, the extension of* $A_s'(y)$ *to elements of* $\mathcal{L}(\mathcal{Z})$ *and* $\mathcal{L}(\mathcal{H})$*, respectively, are denoted by the same symbol. There exists a constant* C *such that these extensions satisfy* $\|A_s'(y)z\|_{\mathcal{Z}}\le C\|z\|_{\mathcal{Z}}$ *and* $\|A_s'(y)h\|_{\mathcal{H}}\le C\|h\|_{\mathcal{H}}$ *for all* $y\in\mathcal{Y}$*,* $z\in\mathcal{Z}$*, and* $h\in\mathcal{H}$*.*

⟨5.2.v⟩ *The objective function* $J:\ H^1(\mathcal{W})\times H^1(\mathcal{X}_c)\to\mathbb{R}$ *is Fréchet differentiable.*

⟨5.3.i⟩ *We suppose that Assumptions ⟨5.2.i⟩ to ⟨5.2.v⟩ hold.*

⟨5.3.ii⟩ *The Fréchet-derivative* A_s' *is Lipschitz continuous from* $\mathcal{Y}$ *to* $\mathcal{L}(\mathcal{Z})$*. Moreover, for every* $y\in\mathcal{Y}$*, the extension of* $A_s'(y)$ *to an element of* $\mathcal{L}(\mathcal{W})$ *is Lipschitz continuous from* $\mathcal{Y}$ *to* $\mathcal{L}(\mathcal{W})$*. Furthermore, there is a constant* $C>0$ *such that* $\|A_s'(y)w\|_{\mathcal{W}}\le C\|w\|_{\mathcal{W}}$ *holds for all* $y\in\mathcal{Y}$ *and all* $w\in\mathcal{W}$*.*

⟨5.3.iii⟩ A_s' *is Fréchet-differentiable from* $\mathcal{Y}$ *to* $\mathcal{L}(\mathcal{Z};\mathcal{W})$*. For all* $y\in\mathcal{Y}$*, the extension of* $A_s''(y)$ *to an element of* $\mathcal{L}(\mathcal{Z};\mathcal{L}(\mathcal{Z};\mathcal{W}))$ *is such that the mapping* $y\mapsto A_s''(y)$ *is continuous in these spaces. Moreover, there exists a constant* C *such that* $\|A_s''(y)[z_1,z_2]\|_{\mathcal{W}}\le C\|z_1\|_{\mathcal{Z}}\|z_2\|_{\mathcal{Z}}$ *for all* $y\in\mathcal{Y}$ *and all* $z_1,z_2\in\mathcal{Z}$*.*

⟨5.3.iv⟩ $J:\ H^1(\mathcal{W})\times H^1(\mathcal{X}_c)\to\mathbb{R}$ *is twice continuously Fréchet differentiable.*

Let us shortly comment on Assumption II. The assumptions made for Chapter 5 are for the optimal control problem and we will discuss them later. Aside from these assumptions, there are only assumptions for the whole Part II and for Chapter 4, which are all in connection with (EVI) considered in Chapter 4. When we apply our results in Chapter 6, the space $\mathcal{H}$ will be $L^2(\Omega;\mathbb{R}_s^{d\times d})$ and $\mathcal{X}$ a negative Sobolev space. We will further choose A to be the subdifferential of an indicator function and R and Q as in Definition 2.7, see Chapter 2.

The present part is concerned with an optimal control problem constraint by an abstract evolution variational inequality (EVI). We mention that most of the presented results are either known from the literature (that is, the results we build upon,) or were presented in MEINLSCHIDT ET AL.[71]. At first we will deal with *maximal monotone operators*, and in particular with *subdifferentials* of an indicator function, in Chapter 3. Then, in Chapter 4 we investigate the abstract EVI and the convergence properties of the corresponding solution operator. It is noteworthy that we do not prove the existence of a solution in $H^1(\mathcal{H})$ for an arbitrary maximal monotone operator and loads in $H^1(\mathcal{X})$ (see also BREZIS [15, Chapter II], here only the existence of a *weak* solution in $C(\mathcal{H})$ was proved). We will either need that the maximal monotone operator satisfies the *boundedness property* (see Definition 3.5) or that the loads belong to $H^2(\mathcal{X})$. Where the first criterion is fulfilled by subdifferentials of an indicator function so that the results can be applied for elasto and homogenized plasticity, we will see in Part IV that there the second criterion holds naturally. Finally, in Chapter 5 we tackle an optimal control problem constraint by the previously analyzed EVI.

As said above and elaborated on in Part I, the foundation of the content in this part is given in [71]. We moreover will compare our results in particular in the beginning of each chapter.

Chapter 3 Maximal Monotone Operators

In this chapter we deal with maximal monotone operators and subdifferentials.

Let us note that there exists many more examples of maximal monotone operators and applications as given in this thesis, we refer for example to ZEIDLER [113, Chapter 32]. Moreover, this chapter gives only a rudimentary introduction to the topic, for a more detailed presentation of maximal monotone operators and subdifferentials we refer to the literature about monotone operators and convex analysis, cf. [15, 113, 110, 90, 10, 103, 84, 34].

We also note that this chapter only deals with maximal monotone operators which are independent of time. There exists also analysis for EVIs where the maximal monotone operator is dependent on time, see for example [10, 77, Chapter III §4] or [77, Chapter 6 §4]. Another topic which is strongly related to maximal monotone operators are *semigroups*, which can be found in most of the references mentioned above. Since we do not need them in what follows, we also do not investigate them.

We first give the definition of a maximal monotone operator and present some basic properties and then consider a special operator, the subdifferential. Since this chapter serves only as a reference for the results we need in the following, all presented findings can be found in the literature. Thus, readers who are already familiar with maximal monotone operators and the subdifferential may skip this chapter and consult it only when needed later.

3.1 Definition and Properties

We start by giving the definition of a maximal monotone operator. A maximal monotone operator is a multivalued and a monotone operator which cannot be extended. That is, one can think of a maximal monotone operator as a monotone operator which does not have any "holes", see the explanation after the following definition.

Definition 3.1 (Maximal monotone operator)**.** *A multivalued operator*

$$A : \mathcal{H} \to 2^{\mathcal{H}}$$

is called monotone *when*

$$\big(a_1 - a_2, h_1 - h_2\big)_{\mathcal{H}} \geq 0$$

for all $h_1, h_2 \in \mathcal{H}$ and all $a_1 \in A(h_1)$, $a_2 \in A(h_2)$.

The operator A is called maximal monotone *when it is monotone and has no proper extension, that is, when for two elements $h_1, a_1 \in \mathcal{H}$ the inequality*

$$\big(a_1 - a_2, h_1 - h_2\big)_{\mathcal{H}} \geq 0$$

holds for all $h_2 \in \mathcal{H}$ and $a_2 \in A(h_2)$, then $a_1 \in A(h_1)$.

Morevoer, the set

$$D(A) := \{h \in \mathcal{H} : A(h) \neq \emptyset\}$$

is called effective domain *or just* domain *of A and*

$$R(A) := \cup_{h \in H} A(h)$$

is called the range *of A.*

Let us shortly have a look at Figure 2. Therein, we see on the left a monotone operator (mapping from $\mathbb{R}$ to $2^{\mathbb{R}}$) which is not maximal monotone. Its extension to a maximal monotone operator is depicted on the right (note that this is the subdifferential of the indicator functional of the interval $[-1, 1]$, cf. Definition 3.11 and (3.4) below).

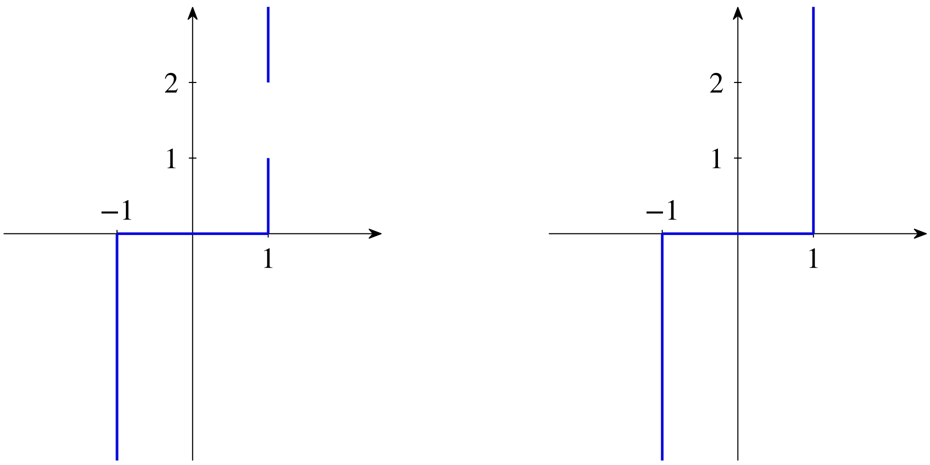


Figure 2: Example of a monotone (left) and a maximal monotone (right) operator.

Recall that A is always a maximal monotone operator in what follows, except when otherwise said, as stated in Assumption ⟨II.ii⟩.

In the following we will for convenience often write $A(h_1)$ instead of a_1 and then "for all $a_1 \in A(h_1)$", that is, for example, the definition of monotonicity would then read

$$\left(A(h_1) - A(h_2), h_1 - h_2\right)_{\mathcal{H}} \geq 0$$

for all $h_1, h_2 \in \mathcal{H}$.

As the example in Figure 2 indicates, $A(h)$ is, for all $h \in \mathcal{H}$, closed and convex. To see this let $h, a \in \mathcal{H}$ and $a_n \in A(h)$ for all $n \in \mathbb{N}$ such that $a_n \to a$. Since A is monotone, we have

$$\left(a_n - A(\tilde{h}), h - \tilde{h}\right)_{\mathcal{H}} \geq 0$$

for all $n \in \mathbb{N}$ and all $\tilde{h} \in \mathcal{H}$, hence, letting $n \to \infty$, we see that the same is true for a, which implies, thanks to maximal monotonicity of A, $a \in A(h)$. Let now $h \in \mathcal{H}$ and $a, b \in A(h)$. Then

$$\begin{aligned}&\left((\lambda a + (1-\lambda)b) - A(\tilde{h}), h - \tilde{h}\right)_{\mathcal{H}} \\ &\quad = \lambda \left(a - A(\tilde{h}), h - \tilde{h}\right)_{\mathcal{H}} + (1-\lambda)\left(b - A(\tilde{h}), h - \tilde{h}\right)_{\mathcal{H}} \geq 0\end{aligned}$$

for all $\lambda \in [0, 1]$ and all $\tilde{h} \in \mathcal{H}$, which shows $(\lambda a + (1-\lambda)b) \in A(h)$.

Thanks to this observation and the fact that the projection onto closed and convex sets in Hilbert spaces is well defined, we may make the following

Definition 3.2 (Selection operator). *We define the following mapping, which takes the element with the smallest norm of the set $A(h)$,*

$$A^0 : D(A) \to \mathcal{H}, \qquad h \mapsto \underset{v \in A(h)}{\arg\min} \|v\|_{\mathcal{H}}, \tag{3.1}$$

that is, $A^0(h)$ is the projection of 0 onto $A(h)$, $A^0(h) = \pi_{A(h)}(0)$, for all $h \in \mathcal{H}$.

In Chapter 4 we will often transform the occurring EVI so that the operator QA appears. Thus, it is convenient to give this operator a new name. Recall that the space $\mathcal{H}_{Q^{-1}}$ is the set $\mathcal{H}$ equipped with the scalar product $(\cdot,\cdot)_{\mathcal{H}_{Q^{-1}}} = \left(Q^{-1}\cdot,\cdot\right)_{\mathcal{H}}$, as defined in Chapter 1.

Definition 3.3 (Transformed maximal monotone operator). *We define the mapping*

$$A_Q : \mathcal{H}_{Q^{-1}} \to 2^{\mathcal{H}_{Q^{-1}}}, \qquad h \mapsto QA(h),$$

where $QA(h) = \{g \in \mathcal{H} : \exists \tilde{g} \in A(h) \text{ with } g = Q\tilde{g}\}$ for all $h \in \mathcal{H}$.

Lemma 3.4 (A_Q is maximal monotone). *The mapping $A_Q : \mathcal{H}_{Q^{-1}} \to 2^{\mathcal{H}_{Q^{-1}}}$ is a maximal monotone operator and*

(i) $\sqrt{\gamma_{Q^{-1}}}\|h\|_{\mathcal{H}} \le \|h\|_{\mathcal{H}_{Q^{-1}}} \le \sqrt{\|Q^{-1}\|}\|h\|_{\mathcal{H}}$ *and*

(ii) $\frac{\sqrt{\gamma_{Q^{-1}}}}{\|Q^{-1}\|_{\mathcal{H}}}\|A^0(h)\|_{\mathcal{H}} \le \|A_Q^0(h)\|_{\mathcal{H}_{Q^{-1}}} \le \sqrt{\|Q^{-1}\|_{\mathcal{H}}}\|Q\|_{\mathcal{H}}\|A^0(h)\|_{\mathcal{H}}$

hold for all $h \in \mathcal{H}$.

Proof. The monotonicity of A_Q follows from

$$\left(A_Q(h_1) - A_Q(h_2), h_1 - h_2\right)_{Q^{-1}} = \left(A(h_1) - A(h_2), h_1 - h_2\right) \ge 0.$$

To prove the maximal monotonicity of A_Q let $h_1, a_1 \in \mathcal{H}$ such that

$$\left(a_1 - A_Q(h_2), h_1 - h_2\right)_{\mathcal{H}_{Q^{-1}}} \ge 0$$

for all $h_2 \in \mathcal{H}$. Setting $b_1 := Q^{-1}a_1$ we get

$$\left(b_1 - A(h_2), h_1 - h_2\right)_{\mathcal{H}} \ge 0$$

for all $h_2 \in \mathcal{H}$, hence, $b_1 \in A(h_1)$, which implies $a_1 \in A_Q(h_1)$ as desired.

The first inequality in Item (i) follows from the coercivity of Q^{-1} and the second inequality from the continuity of Q^{-1}.

From the definition of A^0 we obtain $\|A^0(h)\|_{\mathcal{H}} \le \|v\|_{\mathcal{H}}$ for all $v \in A(h)$, choosing $v = Q^{-1}A_Q^0(h)$ and applying the first inequality in (i) we derive

$$\|A^0(h)\|_{\mathcal{H}} \le \|Q^{-1}\|_{\mathcal{H}}\|A_Q^0(h)\|_{\mathcal{H}} \le \frac{\|Q^{-1}\|_{\mathcal{H}}}{\sqrt{\gamma_{Q^{-1}}}}\|A_Q^0(h)\|_{\mathcal{H}_{Q^{-1}}},$$

so that the first inequality in Item (ii) holds. To verify the second inequality in Item (ii) we can argue analoguously to get

$$\|A_Q^0(h)\|_{\mathcal{H}_{Q^{-1}}} \le \|QA^0(h)\|_{\mathcal{H}_{Q^{-1}}} \le \sqrt{\|Q^{-1}\|}\|QA^0(h)\|_{\mathcal{H}} \le \sqrt{\|Q^{-1}\|}\|Q\|\|A^0(h)\|_{\mathcal{H}},$$

which completes the proof. □

As already noted at the beginning of this part, we will need either an assumption on A or more regular loads to prove the existence of a solution to an EVI . This additional assumption is the *boundedness property* given in

Definition 3.5 (Boundedness property)**.** *We say that A has the* boundedness property *when the domain $D(A)$ is closed and A^0 is bounded on bounded subsets of $D(A)$.*

In Chapter 5 we derive optimality conditions for an optimal control problem with an EVI as a constraint. A typically approach is to regularize the optimal control problem, respectively the constraint, prove convergence of global minimizers of the regularized problems towards a global minimizer of the original problem to justify the "replacement" of the original problem with the regularized one, and finally to derive optimality conditions for the regularized problem. This is also the way we take in Chapter 5. For this purpose, we regularize the maximal monotone operator A, the classical regularization is

The Yosida Approximation

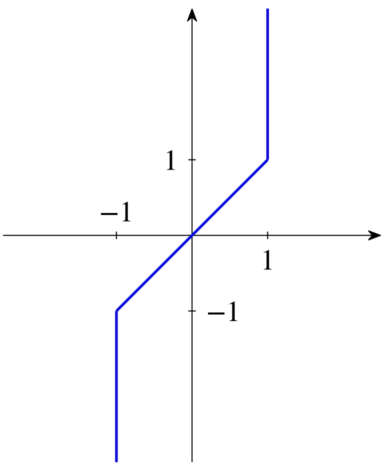
To define the Yosida approximation we need the so called *resolvent* of a multivalued operator. For every multivalued operator the resolvent can be defined itself as a multivalued operator.

Definition 3.6 (Resolvent of a multivalued operator)**.** *Let $\lambda > 0$. The* resolvent $R_\lambda : \mathcal{H} \to 2^{\mathcal{H}}$ *of a multivalued operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is defined by*

$$R_\lambda := (I + \lambda A)^{-1},$$

that is, $r \in R_\lambda(h)$ if and only if $h \in r + \lambda A(r)$. Note that $R_\lambda(h) \subset D(A)$ holds for all $h \in \mathcal{H}$.

For an illustration of the resolvent we consider Figure 3. On the left we see the mapping $(I + \lambda A) : \mathbb{R} \to 2^{\mathbb{R}}$, where A is the maximal monotone operator from Figure 2. On the right the resolvent is shown. Note that both mappings do not depend on λ since $\lambda A = A$ in this case.



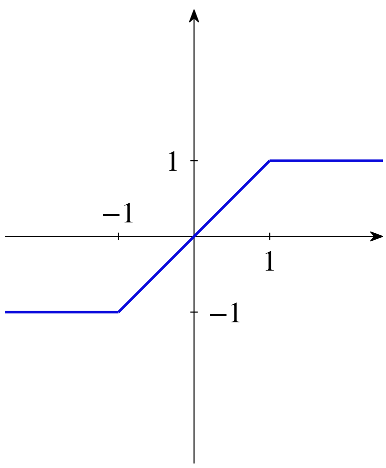


Figure 3: Example of $(I + \lambda A)$ (left) and the resolvent (right) of A, where A is the maximal monotone operator from Figure 2.

The resolvent depicted in Figure 3 is a single valued mapping on the whole space (the real numbers in this case). Moreover, its range is contained in the domain of the corresponding maximal monotone operator. This holds in fact for every maximal monotone operator as we will now see in

Proposition 3.7 (Equivalence of maximal monotonicity)**.** *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued operator. Then the following properties are equivalent:*

(i) A is maximal monotone.

(ii) A is monotone and $R(I + A) = H$.

(iii) For all $\lambda > 0$, the resolvent R_λ maps $\mathcal{H}$ into $D(A)$, that is, the set $R_\lambda(h)$ consists of one element for all $h \in \mathcal{H}$. Additionally, R_λ is nonexpansive.

Proof. This is the statement in ZEIDLER [110, Proposition 55.1 (B)] □

The equivalence between Item (i) and Item (iii) in Proposition 3.7 makes it possible to define the Yosida approximation.

Definition 3.8 (Yosida approximation)**.** *Let $\lambda > 0$. The* Yosida approximation *$A_\lambda : \mathcal{H} \to \mathcal{H}$ of the maximal monotone operator A is defined by*

$$A_\lambda := \frac{1}{\lambda}(I - R_\lambda). \tag{3.2}$$

For ease of notation, in particular in Part V, we abbreviate

$$A_0 := A$$

(not to be confused with A^0 given in Definition 3.2).

Let us continue the example presented in Figure 2 and Figure 3. In Figure 4 the Yosida approximation of the maximal monotone operator from Figure 2 is shown. Clearly, the Yosida approximation seems to be the natural single valued approximation of this maximal monotone operator.

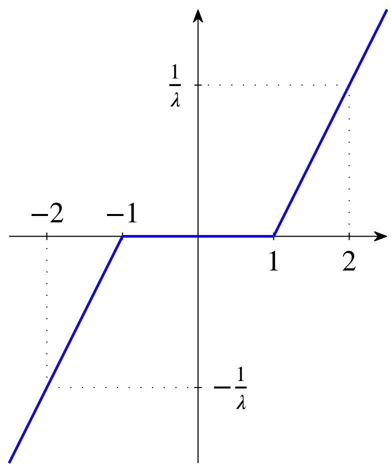


Figure 4: Example of the Yosida approximation of the maximal monotone operator from Figure 2.

Note that the Yosida approximation in Figure 4 has some nice properties, e.g., it is Lipschitz continuous with Lipschitz constant $1/\lambda$ and itself (maximal) monotone. These properties, among others, hold in fact for every maximal monotone operator as stated in

Proposition 3.9 (Properties of the Yosida approximation)**.** *For all $\lambda, \mu > 0$ and $h_1, h_2 \in \mathcal{H}$ the following holds:*

(i) $A_\lambda(h_1) \in A(R_\lambda(h_1))$,

(ii) $\|A_\lambda(h_1) - A_\lambda(h_2)\|_{\mathcal{H}} \leq \frac{1}{\lambda}\|h_1 - h_2\|_{\mathcal{H}}$,

(iii) A_λ *and* R_λ *are maximal monotone.*

(iv) $(A_\lambda)_\mu = A_{\lambda+\mu}$

Proof. Only the maximal monotonicity of R_λ is not stated in ZEIDLER [110, Proposition 55.2], however, by Item (i) we have

$$\big(A_\lambda(h_1) - A_\lambda(h_2), R_\lambda(h_1) - R_\lambda(h_2)\big)_{\mathcal{H}} \geq 0$$

which is equivalent to

$$\|R_\lambda(h_1) - R_\lambda(h_2)\|_{\mathcal{H}}^2 \leq \big(R_\lambda(h_1) - R_\lambda(h_2), h_1 - h_2\big)_{\mathcal{H}}$$

for all $h_1, h_2 \in \mathcal{H}$, hence, R_λ is monotone and continuous, the claim follows now as in ZEIDLER [113, Example 32.4]. □

The Yosida approximation, as the name already says, approximates the corresponding maximal monotone operator in a certain sense, cf. [110, Corollary 55.3]. These approximation properties are used to prove the existence of an EVI , see e.g. [15, Proposition 3.4] or [110, Theorem 55.A]. However, we will only use these existence results so that there is no need to repeat the approximation properties here. Note also that throughout the thesis we use the Yosida approximation to approximate solutions of an EVI, see, e.g. Section 4.2, Section 8.3 or Section 11.2. The approximation method involving the Yosida approximation is also called *vanishing viscosity*.

Let us end this section with a result concerning a "double" Yosida approximation. Due to Proposition 3.9 Item (iv) it might seem at first irrelevant to consider the Yosida approximation twice, however, as we will see in Theorem 4.12, this can actually be utilized, cf. also Remark 4.13.

Lemma 3.10 (Double Yosida inequality)**.** *The inequality*

$$\|(A_\mu)_\lambda(h) - A_\lambda(h)\|_{\mathcal{H}} \leq \Big(\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\sqrt{\frac{\mu}{2\lambda-\mu}}\Big)\|A_\lambda(h)\|_{\mathcal{H}}$$

holds for all $2\lambda > \mu > 0$ *and all* $h \in \mathcal{H}$.

Proof. At first we prove that, for all $2\lambda > \mu > 0$ and $h \in \mathcal{H}$, the following inequality holds

$$\|R_\lambda(h) - R_{\lambda+\mu}(h)\|_{\mathcal{H}} \leq \sqrt{\frac{\mu}{2\lambda-\mu}}\|h - R_\lambda(h)\|_{\mathcal{H}}. \tag{3.3}$$

For this purpose, let $h \in \mathcal{H}$ be arbitrary and set $y_1 := R_\lambda(h)$ and $y_2 := R_{\lambda+\mu}(h)$. Then we have $h \in y_1 + \lambda A(y_1)$, hence, $\frac{h-y_1}{\lambda} \in A(y_1)$ and analogously $\frac{h-y_2}{\lambda+\mu} \in A(y_2)$. The monotonicity of A thus implies

$$\begin{aligned}
0 &\leq \left(\frac{\lambda+\mu}{\lambda}(h-y_1) - (h-y_2), y_1 - y_2\right)_{\mathcal{H}} \\
&= \frac{\mu}{\lambda}\big(h - y_1, y_1 - y_2\big)_{\mathcal{H}} + \big(y_2 - y_1, y_1 - y_2\big)_{\mathcal{H}} \\
&\leq \frac{\mu}{2\lambda}(\|h - y_1\|_{\mathcal{H}}^2 + \|y_1 - y_2\|_{\mathcal{H}}^2) - \|y_1 - y_2\|_{\mathcal{H}}^2 \\
&= \Big(\frac{\mu}{2\lambda} - 1\Big)\|y_1 - y_2\|_{\mathcal{H}}^2 + \frac{\mu}{2\lambda}\|h - y_1\|_{\mathcal{H}}^2,
\end{aligned}$$

hence,

$$\|y_1 - y_2\|_{\mathcal{H}}^2 \le \frac{\mu}{2\lambda - \mu}\|h - y_1\|_{\mathcal{H}}^2,$$

which yields (3.3). With this inequality and Proposition 3.9 Item (iv) at hand, we obtain

$$\begin{aligned}\|(A_\mu)_\lambda(h) - A_\lambda(h)\|_{\mathcal{H}} &= \|\frac{1}{\mu+\lambda}(h - R_{\mu+\lambda}(h)) - \frac{1}{\lambda}(h - R_\lambda(h))\|_{\mathcal{H}} \\ &\le \frac{1}{\mu+\lambda}\|R_\lambda(h) - R_{\lambda+\mu}(h)\|_{\mathcal{H}} + \big(\frac{1}{\lambda} - \frac{1}{\mu+\lambda}\big)\|h - R_\lambda(h)\|_{\mathcal{H}} \\ &\le \Big(\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\sqrt{\frac{\mu}{2\lambda-\mu}}\Big)\|A_\lambda(h)\|_{\mathcal{H}}\end{aligned}$$

which completes the proof. □

3.2 The Subdifferential

The subdifferential is a concept in the theory of convex analysis. It generalizes the concept of a derivative. In fact, one can prove that the subdifferential coincides with the Gâteaux derivative when a function is Gâteaux differentiable. When a function is not differentiable in the classical sense, one can still take affin linear mappings which are "below" of the function, these mappings are called *subgradients* and the set of all subgradients is the *subdifferential*.

Let us note that one can also derive optimality conditions with the help of the subdifferential, however, we do not make use of the subdifferential in this way, we will only use it for the *von-Mises flow rule*, see Definition 2.2.

As mentioned above, we present only the results which we need for our analysis, for a more extensive presentation of the subdifferential and convex analysis in general we refer to [84, 34, 55].

Definition 3.11 (The subdifferential). *Let $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a functional. Then the* subdifferential *at $u \in \mathcal{H}$ is the set*

$$\partial f(u) := \{h \in \mathcal{H} : f(v) \ge f(u) + (h, v - u)_{\mathcal{H}} \ \forall v \in \mathcal{H}\}.$$

Important for the upcoming analysis is the fact that certain subdifferentials are maximal monotone, this is the content of

Proposition 3.12 (Subdifferentials are maximal monotone). *Let $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a convex, lower semicontinuous and proper functional, that is, $f \not\equiv \infty$. Then the multivalued map*

$$\partial f : \mathcal{H} \to 2^{\mathcal{H}}, \qquad h \mapsto \partial f(h)$$

is maximal monotone.

Proof. This was proven in ROCKAFELLAR [83, Theorem A]. □

The von-Mises Flow Rule as a Subdifferential

In Definition 2.2 we have simply defined the multivalued operator $\partial I_{\mathcal{K}(\Omega)}$. By considering the indicator functional

$$I_M : \mathcal{H} \to \{0, \infty\}, \qquad h \mapsto \begin{cases} 0, & h \in M, \\ \infty, & h \notin M \end{cases} \tag{3.4}$$

for a set $M \subset \mathcal{H}$, we see that the operator given in Definition 2.2 coincides in the case $\mathcal{H} = L^2(\Omega; \mathbb{R}_s^{d\times d})$ with the subdifferential of the indicator functional with $M = \mathcal{K}(\Omega)$.

Lemma 3.13 (Subdifferentials fulfill the boundedness property). *Let $\mathcal{K}(\Omega)$ be a nonempty, closed and convex set. Then the subdifferential $\partial I_{\mathcal{K}(\Omega)} : L^2(\Omega; \mathbb{R}_s^{d\times d}) \to 2^{L^2(\Omega;\mathbb{R}_s^{d\times d})}$ fulfills the boundedness property .*

Proof. According to the definition of the subdifferential and the indicator functional, we have

$$\partial I_{\mathcal{K}(\Omega)}(\sigma) = \{\tau \in L^2(\Omega; \mathbb{R}_s^{d\times d}) : I_{\mathcal{K}(\Omega)}(v) \geq I_{\mathcal{K}(\Omega)}(\sigma) + (\tau, v - \sigma)_{\mathcal{H}} \;\; \forall v \in L^2(\Omega; \mathbb{R}_s^{d\times d})\}$$

for all $\sigma \in L^2(\Omega; \mathbb{R}_s^{d\times d})$. Hence, the domain $D(\partial I_{\mathcal{K}(\Omega)}) = \mathcal{K}(\Omega)$ is closed and we have $0 \in \partial I_{\mathcal{K}(\Omega)}(\sigma)$ for all $\sigma \in D(\partial I_{\mathcal{K}(\Omega)})$, thus $\partial I^0_{\mathcal{K}(\Omega)} \equiv 0$ is trivially bounded on bounded subsets of $D(\partial I_{\mathcal{K}(\Omega)}(\sigma))$. □

Let us summarize our findings in the following

Proposition 3.14 (Properties of subdifferentials). *Let $\mathcal{K}(\Omega)$ be a nonempty, closed and convex subset of $L^2(\Omega; \mathbb{R}_s^{d\times d})$. Then the subdifferential $\partial I_{\mathcal{K}(\Omega)} : L^2(\Omega; \mathbb{R}_s^{d\times d}) \to 2^{L^2(\Omega;\mathbb{R}_s^{d\times d})}$ is a maximal monotone operator which fulfills the boundedness property .*

Proof. It is obvious that $I_{\mathcal{K}(\Omega)}$ is convex, lower semicontinuous and proper, hence, $\partial I_{\mathcal{K}(\Omega)}$ is maximal monotone according to Proposition 3.12 and fulfills the boundedness property thanks to Lemma 3.13. □

In view of Theorem 2.9 and Proposition 3.14, we see that the equations of elasto plasticity can be transformed into (2.17), which is an EVI with the maximal monotone operator $\partial I_{\mathcal{K}(\Omega)}$, Chapter 4 and Chapter 5 are concerned with exactly this type of inclusion in a general setting.

Before we continue with this general setting, let us consider the Yosida approximation of $\partial I_{\mathcal{K}(\Omega)}$, respectively $I_{\mathcal{K}(\Omega)}$, where $\mathcal{K}(\Omega)$ is given in Definition 2.2. We denote the Yosida approximation of $\partial I_{\mathcal{K}(\Omega)}$ by ∂I_λ for $\lambda > 0$. In this case the function

$$I_\lambda := \frac{1}{2\lambda} \|I - \pi_{\mathcal{K}(\Omega)}\|^2_{\mathcal{H}} : \mathcal{H} \to \mathbb{R},$$

where $\pi_{\mathcal{K}(\Omega)} : L^2(\Omega; \mathbb{R}_s^{d\times d}) \to L^2(\Omega; \mathbb{R}_s^{d\times d})$ is the projection onto $\mathcal{K}(\Omega)$, is Fréchet differentiable and we have

$$\partial I_\lambda = I'_\lambda = \frac{1}{\lambda}(I - \pi_{\mathcal{K}(\Omega)}), \tag{3.5}$$

here the properties of I_λ and the first equation above follows from Moreau's theorem (see SHOWALTER [90, Chapter IV, Proposition 1.8]) and the second equation from HAN AND REDDY [48, Lemma 8.6]. Note that

$$I_\lambda = \frac{\lambda}{2} \|\partial I_\lambda\|^2_{\mathcal{H}} \tag{3.6}$$

holds. From (3.5) and from the definition of the Yosida approximation (Definition 3.8) we get $R_\lambda = \pi_{\mathcal{K}(\Omega)}$. We mention that this identity can also be obtained by using the definition of the resolvent, one easily checks that, for $y, h \in \mathcal{H}$, $y = R_\lambda(h)$ ($\in D(A)$) holds if and only if $y \in \mathcal{K}(\Omega)$ ($= D(A)$) and

$$0 \geq \left(\frac{h - y}{\lambda}, v - y\right)_{\mathcal{H}}$$

is satisfied for all $v \in \mathcal{K}(\Omega)$. Multiplying this inequality with λ gives exactly the necessary and sufficient first order optimality condition of the minimization problem associated with the projection $\pi_{\mathcal{K}(\Omega)}$, cf. [54, Lemma 1.10] or [42, Satz 2.18]. Note also that, since $\mathcal{K}(\Omega)$ is defined pointwise,

$$\partial I_\lambda(\tau)(x) = \varphi_\lambda(\tau(x)) = \frac{1}{\lambda}(\tau(x) - \pi_K(\tau(x))) \tag{3.7}$$

for all $\tau \in \mathcal{H}$ and almost all $x \in \Omega$, where $\pi_K : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ is the projection onto K and we have abbreviated $\varphi_\lambda := \frac{1}{\lambda}(I - \pi_K) : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$.

In what follows, with a slight abuse of notation, we denote both I'_λ and φ_λ by ∂I_λ.

Until now, the precise structure of K was not important, we only needed $\mathcal{K}(\Omega) = \{\tau \in L^2(\Omega; \mathbb{R}^{d\times d}_s) : \tau(x) \in K \text{ f.a.a. } x \in \Omega\}$. However, when K has in fact the form in Definition 2.2, that is, $K = \{\tau \in \mathbb{R}^{d\times d}_s : |\tau^D| \le \gamma\}$, then a straightforward computation shows that

$$\partial I_\lambda(\tau) = \frac{1}{\lambda} \max\left(0, 1 - \frac{\gamma}{|\tau^D|}\right)\tau^D \tag{3.8}$$

for all $\tau \in \mathbb{R}^{d\times d}_s$. Herein, again with a slight abuse of notation, we denote the Nemyzki operator in $L^\infty(\Omega)$ associated with the pointwise maximum, i.e., $\mathbb{R} \ni r \mapsto \max\{0, r\} \in \mathbb{R}$, by the same symbol. In addition, we set $\max\{0, 1 - \gamma/r\} := 0$, if $r = 0$.

Regularity in Space

The operator ∂I_λ has the property that it maps $H^1(\Omega; \mathbb{R}^{d\times d}_s)$ into $H^1(\Omega; \mathbb{R}^{d\times d}_s)$ and it also preserves two properties associated with the Lipschitz continuity and the monotonicity, this is the content of Lemma 3.15 and Corollary 3.17. This regularity in space will come into play in Section 12.1.

Lemma 3.15 (Space regularity of ∂I_λ). *Let $\lambda > 0$. The operator ∂I_λ maps $H^1(\Omega; \mathbb{R}^{d\times d}_s)$ into $H^1(\Omega; \mathbb{R}^{d\times d}_s)$. For $\tau \in H^1(\Omega; \mathbb{R}^{d\times d}_s)$ the weak derivative of $\sigma := \partial I_\lambda(\tau)$ in direction x_j, $j \in \{1, ..., n\}$, is given by*

$$\partial_j \sigma = \frac{1}{\lambda} \max\left\{0, 1 - \frac{\gamma}{|\tau^D|}\right\}(\partial_j \tau)^D + \frac{1}{\lambda}\chi_M \frac{\gamma}{|\tau^D|^3}(\tau^D : (\partial_j \tau)^D)\tau^D, \tag{3.9}$$

where $M := \{x \in \Omega : |\tau(x)^D| > \gamma\}$.

Proof. We note at first that $\lambda(\partial I_\lambda(\tau))_{i,k}$ is the product of $u := \max\left\{0, 1 - \frac{\gamma}{|\tau^D|}\right\}$ and $\tau^D_{i,k}$ for $i, k \in \{1, ..., d\}$. We have $\tau^D_{i,k} \in H^1(\Omega; \mathbb{R})$, if additionally $u \in H^1(\Omega; \mathbb{R})$ with

$$\partial_j u = \chi_M \frac{\gamma}{|\tau^D|^3}(\tau^D : (\partial_j \tau)^D)$$

for all $j \in \{1, ..., n\}$, then a product rule for Sobolev functions (see for instance [87, Page 57]) gives $u\tau^D_{i,k} \in W^{1,1}(\Omega; \mathbb{R})$ and we obtain (3.9). From (3.9) we can derive further that $\partial I_\lambda(\tau) \in H^1(\Omega; \mathbb{R})$. So it remains to prove the above properties of u.

To this end we select a sequence $\tau_n \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^{d\times d}_s)$ such that $\tau_n \to \tau$ in $H^1(\Omega; \mathbb{R}^{d\times d}_s)$, $\tau_n(x) \to \tau(x)$ and $\partial_j(\tau_n)(x) \to \partial_j(\tau)(x)$ for all $j \in \{1, ..., n\}$ and almost all $x \in \Omega$. Let $s > 0$, we define

$$\max_s : \mathbb{R} \to \mathbb{R} \qquad r \mapsto \begin{cases} \max\{0, r\}, & |r| \ge s, \\ \frac{1}{4s}(r+s)^2, & |r| < s, \end{cases} \tag{3.10}$$

so that $\max_s \in C^1(\mathbb{R}, \mathbb{R})$. For $\phi \in C_c^\infty(\Omega; \mathbb{R})$ we have

$$\int_\Omega \partial_j \phi(x) \max_s\left(1 - \frac{\gamma}{|\tau_n(x)^D|}\right) dx$$
$$= \int_\Omega \phi(x) \max_s'\left(1 - \frac{\gamma}{|\tau_n(x)^D|}\right) \frac{\gamma}{|\tau_n(x)^D|^3}(\tau_n(x)^D : (\partial_j \tau_n(x))^D) dx,$$

considering the limit $n \to \infty$ and then $s \to 0$ and using Lebesgue's dominated convergence theorem yields the assertion. □

Lemma 3.16 (Deviatoric properties). *We have*

$$|\tau^D| \leq |\tau|$$

and

$$\tau^D : \sigma = \tau^D : \sigma^D$$

for all $\tau, \sigma \in \mathbb{R}_s^{d\times d}$.

Proof. We can simply calculate

$$|\tau^D|^2 = |\tau|^2 - 2\frac{\mathrm{tr}(\tau)}{n} I : \tau + |\frac{\mathrm{tr}(\tau)}{n} I|^2 = |\tau|^2 - \frac{\mathrm{tr}(\tau)^2}{n} \leq |\tau|^2$$

and

$$\tau^D : \sigma = \tau^D : \sigma^D + (\tau - \frac{\mathrm{tr}(\tau)}{n} I) : \frac{\mathrm{tr}(\sigma)}{n} I = \tau^D : \sigma^D$$

for all $\tau, \sigma \in \mathbb{R}_s^{d\times d}$. □

Corollary 3.17 (H^1 properties of ∂I_λ). *Let $\lambda > 0$, then*

(i) $\|\partial I_\lambda(\tau)\|_{H^1(\Omega;\mathbb{R}_s^{d\times d})} \leq \frac{3}{\lambda}\|\tau\|_{H^1(\Omega;\mathbb{R}_s^{d\times d})}$ *and*

(ii) $\partial_j(\partial I_\lambda(\tau)) : \partial_j \tau \geq 0$

hold for all $j \in \{1, ..., n\}$ and all $\tau \in H^1(\Omega; \mathbb{R}_s^{d\times d})$.

Proof. Both claims follow easily from Lemma 3.15 using Lemma 3.16. □

The results in Lemma 3.15 and Corollary 3.17 can be used to obtain more regularity of a solution to a particular EVI when the given data is also more regular. We note that these results can be extended to more general Sobolev spaces and settings than the von-Mises flow rule, see [73, Lemma 5.2]. Moreover, the assertion in Lemma 3.15 can also be deduced from the chain rule for Sobolev functions (using the Lipschitz continuity of the Yosida approximation), see [114, Thm 2.1.11]. However, the results presented above are sufficient for us, see also Remark 12.8, and we decided to provide a direct proof for Lemma 3.15.

Let us also mention that ∂I_λ is neither Lipschitz continuous on $H^1(\Omega; \mathbb{R}_s^{d\times d})$ nor is $\partial_j \partial I_\lambda$ monotone on $L^2(\Omega; \mathbb{R}_s^{d\times d})$. This is already the case for the max-function on $H^1(\Omega; \mathbb{R})$, one may choose $\sigma \in H^1(\Omega; \mathbb{R})$ with $\sigma(x) < 0$ for almost all $x \in \Omega$ and $\tau = \sigma + \varepsilon$ such that $\tau(x) > 0$ for almost all $x \in \Omega$. Now, the difference $\max(0, \sigma) - \max(0, \tau)$ may be arbitrary large in $H^1(\Omega; \mathbb{R})$ (due to the derivative) but the difference $\sigma - \tau$ remains small, thus the operator cannot be Lipschitz continuous. One sees with a similar argument that $\partial_j \partial I_\lambda$ is not monotone on $L^2(\Omega; \mathbb{R}_s^{d\times d})$.

This regularity will be needed to construct a *recovery sequence* in Section 12.1, which is used to prove the global optimality of an accumulation point of global minimizers of certain regularized optimal control problems in Theorem 12.9.

Let us now finish this section with the

Smoothing of the von-Mises Flow Rule

Clearly, while ∂I_λ is globally Lipschitz continuous, it is not differentiable due to the non-smoothness of the max function. Since we aim for optimality conditions in Chapter 6, we now smoothen ∂I_λ further to obtain a differentiable mapping. To this end, we define

$$\partial I_{\lambda,s} : \mathcal{H} \to \mathcal{H}, \qquad \tau \mapsto \frac{1}{\lambda} \max{}_s \left(1 - \frac{\gamma}{|\tau^D|}\right)\tau^D, \tag{3.11}$$

where $\max_s$ is given in (3.10) for a smoothing parameter $s \in (0,1)$. One easily checks that $\max_s \in C^1(\mathbb{R})$ with

$$\max{}_s'(r) = \begin{cases} 0, & r \le s, \\ \frac{r+s}{2s}, & |r| < s \\ 1, & r \ge s \end{cases}$$

and that $|\max_s(r) - \max(0,r)| \le \frac{s}{4}$ holds for all $r \in \mathbb{R}$. Moreover, using the fact that $|1 - \gamma/|r|| < s$ if and only if $|r| \in (\gamma/1{+}s, \gamma/1{-}s)$ for $r \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\|\partial I_{\lambda,s}(\tau) - \partial I_\lambda(\tau)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\le \frac{1}{\lambda}\Big(\int_\Omega \Big|\max{}_s\left(1 - \frac{\gamma}{|\tau(x)^D|}\right) - \max\left(0, 1 - \frac{\gamma}{|\tau(x)^D|}\right)\Big|^2 |\tau(x)^D|^2\Big)^{1/2} \\
&\le \frac{|\Omega|\gamma s}{4\lambda(1-s)}
\end{aligned}$$

and in particular

$$\|\partial I_{\lambda,s}(\tau) - \partial I_\lambda(\tau)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \le |\Omega|\gamma \frac{s}{\lambda} \tag{3.12}$$

for $s \in (0, 3/4]$ and all $\tau \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$. This inequality shows that $\partial I_{\lambda,s}$ and ∂I_λ are globally close when s is small (relative to λ). This property will come in Chapter 6 into play when we prove convergence of global minimizers of a smoothed optimization problem.

For later reference (see Example 9.16) we define

$$R_s : \mathcal{H} \to \mathcal{H}, \qquad \tau \mapsto \tau - \max{}_s \left(1 - \frac{\gamma}{|\tau^D|}\right)\tau^D, \tag{3.13}$$

that is, $R_s = I - \lambda\partial I_{\lambda,s}$ or equivalently $\partial I_{\lambda,s} = 1/\lambda(I - R_s)$ (see also the relationship between the Yosida approximation and the resolvent in Definition 3.8). We denote, again with a slight abuse of notation, the pointwise operator, $\mathbb{R}^{d\times d}_s \ni \tau \mapsto \frac{1}{\lambda}\max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\tau^D \in \mathbb{R}^{d\times d}_s$ also by $\partial I_{\lambda,s}$, and do the same for R_s.

Let us collect properties concerning monotonicity and Lipschitz continuity of $\partial I_{\lambda,s}$ and R_s in the following

Lemma 3.18 (Properties of $\partial I_{\lambda,s}$). *For every $\lambda > 0$ and $s \in (0,1)$, the following properties hold:*

(i) The mapping $\partial I_{\lambda,s}$ is maximal monotone.

(ii) The mapping $\partial I_{\lambda,s}$ is Lipschitz continuous with constant $1/\lambda$.

(iii) The mapping R_s is maximal monotone.

(iv) The mapping R_s is Lipschitz continuous with constant 1.

The mappings can be taken from $\mathbb{R}_s^{d\times d}$ to $\mathbb{R}_s^{d\times d}$ or from $L^2(\Omega;\mathbb{R}_s^{d\times d})$ to $L^2(\Omega;\mathbb{R}_s^{d\times d})$.

Proof. It is well known that, since $\max_s$ is continuously differentiable and convex,

$$\max_s(x) - \max_s(y) \geq \max_s(y)'(x-y) \tag{3.14}$$

holds for all $x, y \in \mathbb{R}$.

Let $\tau, \sigma \in \mathbb{R}_s^{d\times d}$, w.l.o.g. we can assume that $|\sigma^D| \geq |\tau^D| > 0$ and

$$\max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right) \geq \max_s\left(1 - \frac{\gamma}{|\tau^D|}\right),$$

then, using (3.14) with $x = 1 - \frac{\gamma}{|\tau^D|}$ and $y = 1 - \frac{\gamma}{|\sigma^D|}$, we get

$$\begin{aligned}
&|\max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\tau^D - \max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right)\sigma^D| \\
&\quad\leq (\max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right) - \max_s\left(1 - \frac{\gamma}{|\tau^D|}\right))\,|\tau^D| + \max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right)|\tau^D - \sigma^D| \\
&\quad\leq \gamma\max_s'\left(1 - \frac{\gamma}{|\sigma^D|}\right)|\tau^D|\,|\frac{1}{|\tau^D|} - \frac{1}{|\sigma^D|}| + \max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right)|\tau^D - \sigma^D|,
\end{aligned}$$

taking into account that

$$\gamma|\tau^D|\,|\frac{1}{|\tau^D|} - \frac{1}{|\sigma^D|}| = \gamma\left|\frac{|\tau^D| - |\sigma^D|}{|\sigma^D|}\right| \leq \frac{\gamma}{|\sigma^D|}|\tau^D - \sigma^D|$$

and using the first inequality in Lemma 3.16, we obtain

$$\begin{aligned}
&|\max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\tau^D - \max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right)\sigma^D| \\
&\quad\leq \left(\max_s'\left(1 - \frac{\gamma}{|\sigma^D|}\right)\frac{\gamma}{|\sigma^D|} + \max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right)\right)|\tau^D - \sigma^D| \\
&\quad\leq \max_s(1)|\tau - \sigma| = |\tau - \sigma|,
\end{aligned}$$

where we used (3.14) again with $x = 1$ and $y = 1 - \frac{\gamma}{|\sigma^D|}$. This proves Item (ii) from which one also immediately deduces Item (iv). We also get

$$\begin{aligned}
&(\boldsymbol{R}_s(\tau) - \boldsymbol{R}_s(\sigma)) : (\tau - \sigma) \\
&\quad= |\tau - \sigma|^2 - \left(\max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\tau^D - \max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right)\sigma^D\right) : (\tau - \sigma) \\
&\quad\geq |\tau - \sigma|^2 - |\tau - \sigma|^2 = 0
\end{aligned}$$

which shows the monotonicity of $\boldsymbol{R}_s$. Let us finally address the monotonicity of $\partial\boldsymbol{I}_{\lambda,s}$. We have

$$\begin{aligned}
&\lambda(\partial\boldsymbol{I}_{\lambda,s}(\sigma) - \partial\boldsymbol{I}_{\lambda,s}(\tau)) : (\sigma - \tau) \\
&\quad= \left(\max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right)\sigma^D - \max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\tau^D\right) : (\sigma^D - \tau^D) \\
&\quad= \left(\max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right) - \max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\right)\sigma^D : (\sigma^D - \tau^D) \\
&\qquad+ \max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)(\sigma^D - \tau^D) : (\sigma^D - \tau^D) \\
&\quad\geq \left(\max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right) - \max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\right)(|\sigma^D|^2 - \tau^D : \sigma^D) \\
&\quad\geq \left(\max_s\left(1 - \frac{\gamma}{|\sigma^D|}\right) - \max_s\left(1 - \frac{\gamma}{|\tau^D|}\right)\right)|\sigma^D|(|\sigma^D| - |\tau^D|) \\
&\quad\geq 0,
\end{aligned}$$

so that $\partial I_{\lambda,s}$ is monotone. The maximal monotonicity of $\partial I_{\lambda,s}$ and R_s follows now as in ZEIDLER [113, Example 32.4]. □

Now we address the desired differentiability of $\partial I_{\lambda,s}$ in

Lemma 3.19 (Differentiability of $\partial I_{\lambda,s}$)**.** *Let $\lambda, s > 0$. The operator $\partial I_{\lambda,s}$ is continuously Fréchet differentiable from $L^{p_1}(\Omega;\mathbb{R}_s^{d\times d})$ to $L^{p_2}(\Omega;\mathbb{R}_s^{d\times d})$ with $1 \le p_2 < p_1 < \infty$ and its directional derivative at $\tau \in L^{p_1}(\Omega;\mathbb{R}_s^{d\times d})$ in direction $h \in L^{p_2}(\Omega;\mathbb{R}_s^{d\times d})$ is given by*

$$\partial I'_{\lambda,s}(\tau)h = \frac{1}{\lambda}\max{}'_s\Big(1-\frac{\gamma}{|\tau^D|}\Big)\frac{\gamma}{|\tau^D|^3}(\tau^D : h^D)\tau^D + \frac{1}{\lambda}\max{}_s\Big(1-\frac{\gamma}{|\tau^D|}\Big)h^D.$$

Moreover, $\partial I'_{\lambda,s}(\tau)$ is extendable to an element of $\mathcal{L}(L^p(\Omega;\mathbb{R}_s^{d\times d}))$ for all $p \in [1,\infty]$, and is, in the case $p = 2$, self-adjoint .

Proof. This can be proven as in HERZOG ET AL. [51, Proposition 2.11] by using the general result in GOLDBERG ET AL. [44, Theorem 7]. □

Remark 3.20 ($\partial I_{\lambda,s}$ is twice differentiable)**.** *The operator $\partial I_{\lambda,s}$ is also twice continuously Fréchet differentiable from $L^{p_1}(\Omega;\mathbb{R}_s^{d\times d})$ to $L^{p_3}(\Omega;\mathbb{R}_s^{d\times d})$ when $1 \le p_3 < p_1/2$ and $\max_s$ is more regular, this was shown in [71, Lemma 7.26] (respectively [106] and [44, Theorem 9]), therein also the second derivative of $\partial I_{\lambda,s}$ is given. An example of the more regular $\max_s$ is given in [71, Example 7.21]. This operator satisfies the Assumptions ⟨5.3.i⟩ to ⟨5.3.iv⟩, as was shown in [71, Corollary 7.27] (in the case of homogenized plasticity, that is, $\partial I_{\lambda,s}$ was considered as a mapping on $L^{p_1}(\Omega \times Y;\mathbb{R}_s^{d\times d})$, where $Y = [0,1]^d$, but it clearly also holds in our case). Since we will not apply the second order optimality conditions, developed in Section 5.3, in detail in one of the plasticity cases, this remark is sufficient for us, see also the end of Chapter 6.*

Having collected all results about maximal monotone operators and subdifferentials which we will need, we can finally turn to EVIs .

Chapter 4 Analysis of EVIs

After collecting important properties of maximal monotone operators in the last chapter, we can start the analysis of a generalized version of the EVI obtained in Theorem 2.9, that is, we consider

$$\dot z \in A(R\ell - Qz), \qquad z(0) = z_0, \tag{EVI}$$

where ℓ is the given input and z the desired solution. In the following we will say *load* to ℓ, this is based on the application given in Chapter 2 where ℓ contains the exterior forces and the Dirichlet displacement, cf. Theorem 2.9. Recall Assumption ⟨II.ii⟩, Assumption ⟨4.i⟩ and Assumption ⟨4.ii⟩ where the properties of A, R, Q and z_0 are given.

To the author's knowledge, there exists no analysis for this EVI in the literature, however, as we will see in Lemma 4.3, (EVI) can be equivalently transformed into (EVI$_q$) and there exists an extensive analysis in the literature for EVIs of this form. We have already given some references at the beginning of Chapter 3 which can also be consulted regarding EVIs, but our main references are [15, 113, 110], which are sufficient for our analysis of (EVI). Nonetheless, we cannot draw all needed results for Chapter 5 from them. Where the existence of a solution and a priori estimates (Theorem 4.5 and Theorem 4.7) can be somewhat easily deduced from the results in [15, 110], the desired convergence results of the solution operator, presented in Section 4.2, will require a more carefull inspection.

4.1 Definition and Existence of a Solution

Despite that this section carries the word "definition" we do not give a definition of a solution to (EVI) since it is analog to Definition 2.3, as we have agreed upon right after Definition 2.3.

Before we transform (EVI) into an EVI which form occurs in the literature, let us shortly consider the space of loads. It is obvious that it is not sufficient to require only $\ell \in L^\infty(\mathcal{X})$, due to the possible jumps in time and the regularity $z \in H^1(\mathcal{H}) \subset C(\mathcal{H})$ it is immediately clear that there cannot exist a solution in general. Therefore the load has to be at least continuous, since in the context of optimal control, we are interested in loads which are contained in a Hilbert space, it is natural to consider loads in $H^1(\mathcal{X})$, which embeds continuously into $C(\mathcal{X})$, see [104, Theorem 3.1.41]. The regularity of the load after the announced transformation is then also in accordance with the desired regularity in the literature.

One important property is the uniqueness of a solution, this can be shown easily for an EVI as we now see in

Lemma 4.1 (Uniqueness of a solution). *Let $\ell \in H^1(\mathcal{X})$. Assume that $z_1, z_2 \in H^1(\mathcal{H})$ are two solutions of* (EVI). *Then $z_1 = z_2$.*

Proof. We observe that

$$\Big(\dot{z}_1(t) - \dot{z}_1(t), Q(z_1(t) - z_2(t))\Big)_{\mathcal{H}} = \Big(\dot{z}_1 - \dot{z}_2, Qz_1(t) - R\ell(t) - Qz_2(t) + R\ell(t)\Big)_{\mathcal{H}} \le 0$$

holds for almost all $t \in [0, T]$, where we have used the monotonicity of A. Integrating this inequality and using the coercivity of Q gives the desired result. □

Transformation

For the following transformation it is convenient to introduce some abbreviations.

Definition 4.2 (z to q mapping). *We define*

$$\mathfrak{Q} : \mathcal{H} \times \mathcal{X} \to \mathcal{H}, \qquad (z, \ell) \mapsto R\ell - Qz$$

and

$$\mathfrak{Z} : \mathcal{H} \times \mathcal{X} \to \mathcal{H}, \qquad (q, \ell) \mapsto Q^{-1}(R\ell - q)$$

such that $\mathfrak{Q}(\cdot, \ell)^{-1} = \mathfrak{Z}(\cdot, \ell)$ for a fixed $\ell \in \mathcal{X}$.

We are now in the position to present the transformation. We note that this transformation was also used in GRÖGER [45, Theorem 4.1]. Recall that q_0 is given in Assumption ⟨4.iii⟩.

Lemma 4.3 (Transformation of an EVI). *Let $\ell \in H^1(\mathcal{X})$. Then $z \in H^1(\mathcal{H})$ is the solution of* (EVI) *if and only if $q \in H^1(\mathcal{H})$ is the solution of*

$$\dot{q} + A_Q(q) \ni R\dot{\ell}, \quad q(0) = q_0 \qquad (\mathrm{EVI}_q)$$

with $q_0 = \mathfrak{Q}(z_0, \ell(0))$, and we have $z = \mathfrak{Z}(q, \ell)$ or, equivalently, $q = \mathfrak{Q}(z, \ell)$, where the maximal monotone operator A_Q is given in Definition 3.3 (see also Lemma 3.4).

Proof. This follows immediately from the definitions of $\mathfrak{Q}$ and $\mathfrak{Z}$. □

Existence of a Solution and A Priori Estimates

To prove the existence of a solution to (EVI) it is obvious that the value at $t = 0$ of the load has to be in compliance with the initial condition. This is captured in

Definition 4.4 (Admissible loads). *For $z_0 \in \mathcal{H}$ and $M \subset D(A)$, we define the set*

$$\mathcal{A}_{\mathcal{L}}(z_0, M) := \{\ell \in H^1(\mathcal{X}) : R\ell(0) - Qz_0 \in M\}$$

of admissible loads.

Now we can prove the existence of a solution, at first under the boundedness property and then with H^2 loads. Clearly, since the subdifferential of an indicator function fulfills the boundedness property (Proposition 3.14) and the loads ℓ in Theorem 2.9 are H^1 in time, the first result is sufficient for the application to elasto plasticity (and also to homogenized plasticity as we will see in Chapter 7). However, for the application of plasticity with an inertia term in Part IV we will need the second result.

Theorem 4.5 (Existence of a solution under the boundedness property). *We assume that A has the boundedness property (see Definition 3.5).*

Let $\ell \in \mathcal{A}_{\mathcal{L}}(z_0, D(A))$. Then there exists a unique solution $z \in H^1(\mathcal{H})$ of (EVI). *Furthermore, there exists a constant C, independent of z_0 and ℓ, such that*

$$\|z\|_{C(\mathcal{H})} \leq C\big(1 + \|z_0\|_{\mathcal{H}} + \|\ell\|_{C(\mathcal{X})} + \|\dot{\ell}\|_{L^1(\mathcal{X})}\big), \tag{4.1}$$

$$\|\dot{z}\|_{L^2(\mathcal{H})} \leq C\big(\|\dot{\ell}\|_{L^2(\mathcal{X})} + \sup_{\tau\in[0,T]} \|A^0(R\ell(\tau) - Qz(\tau))\|_{\mathcal{H}}\big), \tag{4.2}$$

where A^0 is given in Definition 3.2.

Proof. Since A has the boundedness property (thus A_Q has it also, according to Lemma 3.4) and $R\ell(0) - Qz_0 \in D(A)$, we can apply Brezis [15, Proposition 3.4] to obtain a unique solution $q \in H^1(\mathcal{H})$ of (EVI_q). Thanks to Lemma 4.3, this gives also the existence of a unique solution of (EVI).

To verify the estimate in (4.1), we employ Brezis [15, Lemme 3.1], which gives

$$\|q(t) - \tilde{q}(t)\|_{\mathcal{H}_{Q^{-1}}} \leq \|R\ell(0) - Qz_0 - a\|_{\mathcal{H}_{Q^{-1}}} + \int_0^t \|R\dot{\ell}(\tau)\|_{\mathcal{H}_{Q^{-1}}} d\tau$$

for all $t \in [0, T]$, where $\tilde{q}$ is the unique solution of

$$\dot{\tilde{q}} + A_Q(\tilde{q}) \ni 0, \qquad q(0) = a$$

with an arbitrary element $a \in D(A)$. The inequality

$$\|z(t)\|_{\mathcal{H}} = \|\mathfrak{Z}(q(t), \ell(t))\|_{\mathcal{H}} \leq \|Q^{-1}\|\big(\|R\|\|\ell(t)\|_{\mathcal{X}} + \|q(t)\|_{\mathcal{H}}\big)$$

together with

$$\|q(t)\|_{\mathcal{H}} \leq \frac{1}{\sqrt{\gamma_{Q^{-1}}}}\|q(t) - \tilde{q}(t)\|_{\mathcal{H}_{Q^{-1}}} + \|\tilde{q}(t)\|_{\mathcal{H}},$$

where we have used Lemma 3.4, gives (4.1).

To prove (4.2), we deduce from BREZIS [15, Proposition 3.4] and the associated proof that

$$\|q(t) - q(s)\|_{\mathcal{H}_{Q^{-1}}} \leq \int_s^t \|R\dot{\ell}(\tau)\|_{\mathcal{H}_{Q^{-1}}} d\tau + \sup_{\tau\in[0,T]} \|A_Q^0(q(\tau))\|_{\mathcal{H}_{Q^{-1}}}(t-s).$$

Dividing this inequality by $(t-s)$ and letting $t \to s$ yields

$$\|\dot{q}(s)\|_{\mathcal{H}_{Q^{-1}}} \leq \|R\dot{\ell}(s)\|_{\mathcal{H}_{Q^{-1}}} + \sup_{\tau\in[0,T]} \|A_Q^0(q(\tau))\|_{\mathcal{H}_{Q^{-1}}}$$

for almost all $s \in [0,T]$. Taking once again Lemma 3.4 and Lemma 4.3 into account, we get (4.2). □

Remark 4.6 (Weakend boundedness property)**.** *In order to prove Theorem 4.5, it is sufficient to require that A^0 is bounded on compact subsets (in addition to the closedness of $D(A)$), cf.* BREZIS *[15, Proposition 3.4]. However, the boundedness on bounded sets of A^0 is needed to prove Theorem 4.11 below, therefore we have required it right away.*

Theorem 4.7 (Existence of a solution with H^2 loads)**.** *Let $\ell \in H^2(\mathcal{X})$. Then there exists a unique solution $q \in H^1(\mathcal{H})$ of* (EVI$_q$)*. Furthermore, there exists a constant C such that*

$$\|\dot{q}\|_{L^2(\mathcal{H})} \leq C\big(\|A^0(q_0)\|_{\mathcal{H}} + \|\dot{\ell}\|_{H^1(\mathcal{X})}\big) \tag{4.3}$$

holds, where the constant C is independent of q_0 and ℓ.

Proof. Since $\ell \in H^2(\mathcal{X})$ we have $R\dot{\ell} \in H^1(\mathcal{H})$, therefore we can apply ZEIDLER [110, Theorem 55.A] to obtain the existence of a unique solution $q \in H^1(\mathcal{H})$ to (EVI$_q$).

To verify (4.3), we consider the solution $q_\lambda \in C^1(\mathcal{H})$ of

$$\dot{q}_\lambda + A_{Q,\lambda}(q_\lambda) = R\dot{\ell}, \quad q_\lambda(0) = q_0.$$

That this equation obtains a unique solution follows from Theorem A.7. According to the proof of ZEIDLER [110, Theorem 55.A], the inequality

$$\|\dot{q}_\lambda\|_{L^2(\mathcal{H}_{Q^{-1}})} \leq C\big(\|A_Q^0(q_0)\|_{\mathcal{H}_{Q^{-1}}} + \|\dot{\ell}\|_{H^1(\mathcal{X})}\big)$$

holds for all $\lambda > 0$. Since $q_\lambda \rightharpoonup q$ in $H^1(\mathcal{H}_{Q^{-1}})$ (cf. again ZEIDLER [110, Theorem 55.A]), we can let $\lambda \searrow 0$ and use Lemma 3.4 to get (4.3). □

Note that the solution q in Theorem 4.7 has the regularity $\dot{q} \in L^\infty(\mathcal{H})$ according to ZEIDLER [110, Corollary 55.4], however, we do not need this regularity in the following.

4.2 Convergence of Solutions

In the context of optimal control it is necessary to obtain results concerning the sensitivity of solutions to a state equation with respect to controls. This becomes already clear in the proof of the existence of optimal solutions in Theorem 5.2, where we use the direct method of the calculus of variations.

This section is devoted to such a sensitivity analysis, we start by providing a result regarding the Yosida approximation but for fixed data. This will come in handy in Lemma 4.9 because this essentially enables a "replacement" of the maximal monotone operator with the Yosida approximation.

We emphasize that in both Proposition 4.8 and Lemma 4.9 we simply assume that there exists a solution to (EVI). These findings can then be applied in the case of the boundedness property, where the existence is provided by Theorem 4.5, but also in the case of H^2 loads, where the existence is guaranteed by Theorem 4.7.

Proposition 4.8 (Convergence of the Yosida approximation for fixed data)**.** *Let $\ell \in H^1(\mathcal{X})$ and assume that $z \in H^1(\mathcal{H})$ is the solution of* (EVI)*. Moreover, let $z_\lambda \in H^1(\mathcal{H})$ be the solution of*

$$\dot{z}_\lambda = A_\lambda(R\ell - Qz_\lambda), \qquad z_\lambda(0) = z_0 \tag{4.4}$$

for all $\lambda > 0$. Then $z_\lambda \to z$ in $H^1(\mathcal{H})$ as $\lambda \searrow 0$ and the following inequality holds

$$\|z_\lambda - z\|^2_{C(\mathcal{H})} + \frac{\lambda}{\gamma_Q}\|\dot{z}_\lambda\|^2_{L^2(\mathcal{H})} + \frac{\lambda}{\gamma_Q}\|\dot{z}_\lambda - \dot{z}\|^2_{L^2(\mathcal{H})} \leq \frac{\lambda}{\gamma_Q}\|\dot{z}\|^2_{L^2(\mathcal{H})}. \tag{4.5}$$

Proof. The proof in principle follows the lines of BREZIS [15, Proposition 3.11], but our assumptions and assertions are different.

First of all, since $z \mapsto A_\lambda(R\ell - Qz)$ is Lipschitz continuous by Proposition 3.9 Item (ii), the existence of a unique solution of (4.4) follows from Theorem A.7. Moreover, Proposition 3.9 Item (i) and the definition of A_λ (see Definition 3.8) give

$$\begin{aligned}
\frac{d}{dt}\left(Q(z_\lambda(t) - z(t)), z_\lambda(t) - z(t)\right)_{\mathcal{H}} &= 2\left(\dot{z}_\lambda(t) - \dot{z}(t), Q(z_\lambda(t) - z(t))\right)_{\mathcal{H}} \\
&= -2\left(\dot{z}_\lambda(t) - \dot{z}(t), R_\lambda\big[R\ell(t) - Qz_\lambda(t)\big] - \big[R\ell(t) - Qz(t)\big]\right)_{\mathcal{H}} \\
&\quad - 2\left(\dot{z}_\lambda(t) - \dot{z}(t), R\ell(t) - Qz_\lambda(t) - R_\lambda\big[R\ell(t) - Qz_\lambda(t)\big]\right)_{\mathcal{H}} \\
&\leq -2\lambda\left(\dot{z}_\lambda(t) - \dot{z}(t), \dot{z}_\lambda(t)\right)_{\mathcal{H}} \\
&= \lambda\Big(\|\dot{z}(t)\|^2_{\mathcal{H}} - \|\dot{z}_\lambda(t)\|^2_{\mathcal{H}} - \|\dot{z}_\lambda(t) - \dot{z}(t)\|^2_{\mathcal{H}}\Big).
\end{aligned}$$

By integrating this inequality and using the coercivity of Q, we obtain the desired inequality.

In order to prove the strong convergence of z_λ to z in $H^1(\mathcal{H})$, we note that $z_\lambda \to z$ in $C(\mathcal{H})$ and $\|\dot{z}_\lambda\|_{L^2(\mathcal{H})} \leq \|\dot{z}\|_{L^2(\mathcal{H})}$ follow from the gained inequality. Hence, $z_\lambda \rightharpoonup z$ in $H^1(\mathcal{H})$ and the desired strong convergence follows from Lemma A.3. □

Let us already say that in the following lemma the operators A_n will be later simply set to A or to the yosida approximation for $\lambda_n > 0$, that is, $A_n = A_{\lambda_n}$, see Remark 4.10. Moreover, this lemma is the key to prove convergence results under the boundedness property in Theorem 4.11 and Theorem 4.12, and also with H^2 loads in Theorem 4.14.

Lemma 4.9 (Key result for convergence properties)**.** *Let $\ell \in H^1(\mathcal{X})$ and assume that $z \in H^1(\mathcal{H})$ is the solution of* (EVI)*. Moreover, let $\{z_{n,0}\}_{n\in\mathbb{N}} \subset \mathcal{H}$ and $\{\ell_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{X})$ be sequences such that $z_{n,0} \to z_0$ in $\mathcal{H}$ and $\ell_n \to \ell$ in $L^1(\mathcal{X})$. Assume further that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of maximal monotone operators such that for every $\lambda > 0$ the convergence*

$$A_{n,\lambda}(h) \to A_\lambda(h) \tag{4.6}$$

holds for all $h \in (R\ell - Qz_\lambda)([0,T])$, as $n \to \infty$, where z_λ is the solution of (4.4) *and $A_{n,\lambda}$ denotes the Yosida approximation of A_n. Furthermore, if the sequence $\{z_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{H})$ satisfies*

$$\dot{z}_n \in A_n(R\ell_n - Qz_n), \qquad z_n(0) = z_{n,0} \tag{4.7}$$

and the derivatives $\dot{z}_n$ are bounded in $L^2(\mathcal{H})$, then $z_n \rightharpoonup z$ in $H^1(\mathcal{H})$ and $z_n \to z$ in $C(\mathcal{H})$.

Before we continue with the proof, let us shortly depict the idea of it in Figure 5. Therein, z_λ and $z_{n,\lambda}$ are the solutions of (4.8) and (4.9), respectively. As we can see, choosing at first a small λ and afterwards a large n yields essentially the assertion (see also (4.10)).

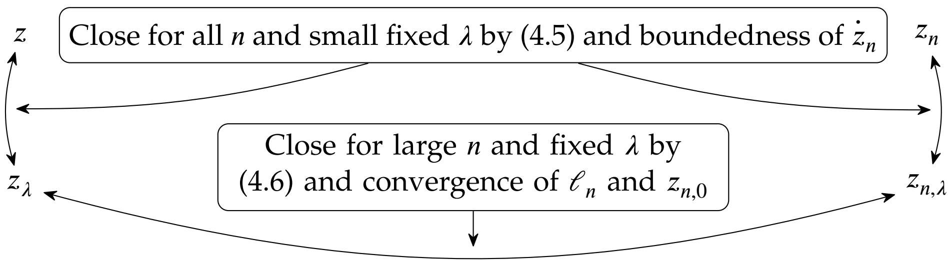


Figure 5: Idea of the proof of Lemma 4.9.

Proof of Lemma 4.9. Let $\lambda > 0$ be fixed, but arbitrary and define $z_\lambda, z_{n,\lambda} \in H^1(\mathcal{H})$ as solutions of

$$\dot{z}_\lambda = A_\lambda(R\ell_n - Qz_\lambda), \qquad z_\lambda(0) = z_0, \tag{4.8}$$

and

$$\dot{z}_{n,\lambda} = A_{n,\lambda}(R\ell_n - Qz_{n,\lambda}), \qquad z_{n,\lambda}(0) = z_{n,0}, \tag{4.9}$$

respectively. In both cases, existence and uniqueness follows from Theorem A.7. Owing to Proposition 3.9 Item (ii), we obtain

$$\begin{aligned}
\|\dot{z}_\lambda(t) - \dot{z}_{n,\lambda}(t)\|_{\mathcal{H}} &\leq \|A_\lambda(R\ell(t) - Qz_\lambda(t)) - A_{n,\lambda}(R\ell(t) - Qz_\lambda(t))\|_{\mathcal{H}} \\
&\quad + \|A_{n,\lambda}(R\ell(t) - Qz_\lambda(t)) - A_{n,\lambda}(R\ell_n(t) - Qz_{n,\lambda}(t))\|_{\mathcal{H}} \\
&\leq \|A_\lambda(R\ell(t) - Qz_\lambda(t)) - A_{n,\lambda}(R\ell(t) - Qz_\lambda(t))\|_{\mathcal{H}} \\
&\quad + \frac{\|Q\|_{L(\mathcal{H};\mathcal{H})}}{\lambda}\|z_\lambda(t) - z_{n,\lambda}(t)\|_{\mathcal{H}} + \frac{\|R\|_{L(\mathcal{X};\mathcal{H})}}{\lambda}\|\ell(t) - \ell_n(t)\|_{\mathcal{X}},
\end{aligned}$$

and therefore, Lemma A.8 Item (i) (with $u := z_\lambda - z_{n,\lambda}$ and $\alpha := \|A_\lambda(R\ell - Qz_\lambda) - A_{n,\lambda}(R\ell - Qz_\lambda)\|_{\mathcal{H}} - \frac{\|R\|_{L(\mathcal{X};\mathcal{H})}}{\lambda}\|\ell - \ell_n\|_{\mathcal{X}}$) implies

$$\begin{aligned}
\|z_\lambda - z_{n,\lambda}\|_{C(\mathcal{H})} \leq C(\lambda)\Big(&\|z_0 - z_{n,0}\|_{\mathcal{H}} + \|\ell - \ell_n\|_{L^1(\mathcal{X})} \\
&+ \|A_\lambda(R\ell - Qz_\lambda) - A_{n,\lambda}(R\ell - Qz_\lambda)\|_{L^1(\mathcal{H})}\Big).
\end{aligned}$$

The operators $A_{n,\lambda}$ are uniformly Lipschitz continuous with Lipschitz constant λ^{-1}. Thus, thanks to (4.6), we can apply Lemma A.1 with $\mathcal{M} := (R\ell - Qz_\lambda)[0,T]$, $\mathcal{N} := \mathcal{H}$, $G_n := A_{n,\lambda}$ and $G := A_\lambda$. Together with the assumptions on ℓ_n and $z_{n,0}$ this gives that the right side of the inequality above converges to zero as $n \to \infty$. Using this, Proposition 4.8, and (4.5) (with $A = A$ but also with $A = A_n$), we conclude

$$\begin{aligned}
\limsup_{n\to\infty} \|z - z_n\|_{C(\mathcal{H})} &\leq \|z - z_\lambda\|_{C(\mathcal{H})} + \limsup_{n\to\infty} \|z_{n,\lambda} - z_n\|_{C(\mathcal{H})} \\
&\leq \sqrt{\frac{\lambda}{\gamma_Q}}\big(\|\dot{z}\|_{L^2(\mathcal{H})} + \sup_{n\in\mathbb{N}} \|\dot{z}_n\|_{L^2(\mathcal{H})}\big).
\end{aligned} \tag{4.10}$$

Now, since λ was arbitrary, (4.10) holds for every $\lambda > 0$. Therefore, as $\dot{z}_n$ is bounded in $L^2(\mathcal{H})$ by assumption, we obtain $z_n \to z$ in $C(\mathcal{H})$. Moreover, again due to the boundedness assumption on $\dot{z}_n$, there is a weakly converging subsequence in $H^1(\mathcal{H})$. Due to $z_n \to z$ in $C(\mathcal{H})$, the weak limit is unique and hence, the whole sequence z_n converges weakly to z in $H^1(\mathcal{H})$. □

Remark 4.10 (Choice of A_n)**.** *In what follows, we will either choose $A_n := A$ or $A_n := A_{\lambda_n}$, where $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ converges towards zero. In the case $A_n := A_{\lambda_n}$, the requirement* (4.6) *is fulfilled thanks to Lemma 3.10.*

Convergence under the Boundedness Property

In Section 4.1 we have proven the existence of a solution to (EVI) once with the boundedness property and once with H^2 loads. Therefore, we also need at least one of these assumptions to prove convergence of solutions, we start with two results under the boundedness property.

Theorem 4.11 (Convergence under the boundedness property)**.** *We assume that A has the boundedness property (see Definition 3.5).*

Let $\ell \in \mathcal{A}_{\mathcal{L}}(z_0, D(A))$ and $z \in H^1(\mathcal{H})$ the solution of (EVI) *(whose existence is guaranteed by Theorem 4.5). Let $\{\ell_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_{\mathcal{L}}(z_{n,0}, D(A))$ be a sequence such that $z_{n,0} \to z_0$ in $\mathcal{H}$, $\ell_n \rightharpoonup \ell$ in $H^1(\mathcal{X})$ and $\ell_n \to \ell$ in $L^1(\mathcal{X})$. Moreover, denote the solution of*

$$\dot{z}_n \in A(R\ell_n - Qz_n), \qquad z_n(0) = z_{n,0}$$

by $z_n \in H^1(\mathcal{H})$ (whose existence is again guaranteed by Theorem 4.5). Then $z_n \rightharpoonup z$ in $H^1(\mathcal{H})$ and $z_n \to z$ in $C(\mathcal{H})$.

If additionally $\ell_n \to \ell$ in $H^1(\mathcal{X})$, A is a subdifferential of a convex, lower semicontinuous and proper functional (see Section 3.2), that is, $A = \partial\phi$, and $\phi(R\ell_n(0) - Qz_{n,0}) \to \phi(R\ell(0) - Qz_0)$, then $z_n \to z$ in $H^1(\mathcal{H})$.

Proof. Thanks to Theorem 4.5, to be more precise (4.1), $\{z_n\}$ is bounded in $C([0,T];\mathcal{H})$. Since A has the boundedness property , A^0 is bounded on bounded sets, hence, (4.2) then gives that $\{\dot{z}_n\}$ is bounded in $L^2(\mathcal{H})$. Therefore, we can apply Lemma 4.9 with $A_n := A$ for all $n \in \mathbb{N}$ to obtain $z_n \rightharpoonup z$ in $H^1(\mathcal{H})$ and $z_n \to z$ in $C(\mathcal{H})$.

If the additional requirements hold, we can follow the lines of GRÖGER [45, Theorem 4.2 step 3)] to get

$$\begin{aligned}
\limsup_{n\to\infty} \|\dot{z}_n\|^2_{L^2(\mathcal{H}_Q)} &= \limsup_{n\to\infty} \int_0^T \left(Q\dot{z}_n, \dot{z}_n\right)_{\mathcal{H}} dt \\
&= \limsup_{n\to\infty} -\left(R\dot{\ell}_n - Q\dot{z}_n, \dot{z}_n\right)_{L^2(\mathcal{H})} + \left(R\dot{\ell}, \dot{z}\right)_{L^2(\mathcal{H})} \\
&= \limsup_{n\to\infty} \phi(R\ell_n(0) - Qz_{n,0}) - \phi(R\ell_n(T) - Qz_n(T)) + \left(R\dot{\ell}, \dot{z}\right)_{L^2(\mathcal{H})} \\
&\leq \phi(R\ell(0) - Qz_0) - \phi(R\ell(T) - Qz(T)) + \left(R\dot{\ell}, \dot{z}\right)_{L^2(\mathcal{H})} \\
&= \int_0^T \left(Q\dot{z}, \dot{z}\right)_{\mathcal{H}} dt = \|\dot{z}\|^2_{L^2(\mathcal{H}_Q)}
\end{aligned}$$

where the third and fifth equation follows from [15, Lemme 3.3]. Hence, the strong convergence $z_n \to z$ in $H^1(\mathcal{H}_Q)$ (and thus in $H^1(\mathcal{H})$) follows from Lemma A.3 with $H = L^2(\mathcal{H})$ and $x_n = \dot{z}_n$. □

Theorem 4.12 (Convergence of the Yosida approximation under the boundedness property)**.** *The assertion in Theorem 4.11 holds true when z_n is, for every $n \in \mathbb{N}$, the solution of*

$$\dot{z}_n = A_{\lambda_n}(R\ell_n - Qz_n), \qquad z_n(0) = z_{n,0}, \tag{4.11}$$

where $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ is a sequence converging to zero.

Proof. According to Lemma 3.10, the sequence of maximal monotone operators $A_n := A_{\lambda_n}$ fulfills (4.6) so that it only remains to prove that $\dot{z}_n$ is bounded in $L^2(\mathcal{H})$ to apply again Lemma 4.9. To this end, let $v_n \in H^1(\mathcal{H})$ be the solution of

$$\dot{v}_n \in A(R\ell_n - Qv_n), \qquad v_n(0) = z_{n,0},$$

whose existence is guaranteed by Theorem 4.5 (note that $\ell_n \in \mathcal{A}_{\mathcal{L}}(z_{n,0}, D(A))$ by assumption). Thanks to Theorem 4.11, it holds $v_n \rightharpoonup z$ in $H^1(\mathcal{H})$. From Proposition 4.8, it follows $\|\dot{z}_n\|_{L^2(\mathcal{H})} \leq \|\dot{v}_n\|_{L^2(\mathcal{H})}$ and consequently, $\dot{z}_n$ is bounded in $L^2(\mathcal{H})$. Thus, Lemma 4.9 yields $z_n \rightharpoonup z$ in $H^1(\mathcal{H})$ and $z_n \to z$ in $C(\mathcal{H})$, as claimed.

If the additional requirements hold, then Theorem 4.11 implies $v_n \to z$ in $H^1(\mathcal{H})$ so that Lemma A.3 gives the strong convergence $z_n \to z$ in $H^1(\mathcal{H})$ because of $\|\dot{z}_n\|_{L^2(\mathcal{H})} \leq \|\dot{v}_n\|_{L^2(\mathcal{H})}$ as seen above. □

Remark 4.13 (Double Yosida approximation)**.** *The assertions of Theorem 4.11 and Theorem 4.12 are remarkable due to the following: As a first approach to prove the (strong) convergence of the states in $H^1(\mathcal{H})$, one is tempted to follow the lines of the proofs of* BREZIS *[15, Lemme 3.1] and Proposition 4.8, respectively. This would however require the strong convergence of the* derivatives *of the given loads, which we want to avoid in order to enable less regular controls. The detour via the Yosida approximation in Lemma 4.9 (see also Figure 5), which most notably implies that in the case of Theorem 4.12 a "double" Yosida approximation is used, allows to overcome this issue. The same is also true for Theorem 4.14 when q_n is the solution of* (4.13)*. This method was also used in* BREZIS *[15, Theoreme 3.16], however, this result is not usable for us, since it is concerned with* weak solutions *of EVIs .*

Convergence with H^2 Loads

We continue with two convergence results with H^2 loads. In the first one we treat both solutions for the original problem and for the regularized problems, the arguments are similar to the ones used in Theorem 4.11 and Theorem 4.12, but, of course, we use Theorem 4.7 instead of Theorem 4.5. Note that Theorem 4.14 and also Corollary 4.15 is formulated in the q-variable, this is (almost) exactly the formulation we need in Theorem 8.17 and Proposition 8.18 when treating plasticity with inertia.

Theorem 4.14 (Convergence with H^2 loads)**.** *Let $\ell \in H^2(\mathcal{X})$ and $q \in H^1(\mathcal{H})$ the solution of* (EVI_q) *(whose existence is guaranteed by Theorem 4.7). Let $\{q_{n,0}\}_{n\in\mathbb{N}} \subset D(A)$ and $\{\ell_n\}_{n\in\mathbb{N}} \subset H^2(\mathcal{X})$ be sequences such that $q_{n,0} \to q$ in $\mathcal{H}$, $A^0(q_{n,0})$ is bounded in $\mathcal{H}$, $\ell_n \rightharpoonup \ell$ in $H^2(\mathcal{X})$, $\ell_n \to \ell$ in $L^1(\mathcal{X})$ and $\ell_n(0) \to \ell(0)$ in $\mathcal{X}$. Moreover, by $q_n \in H^1(\mathcal{H})$ we denote either the solution of*

$$\dot{q}_n + A_Q(q_n) \ni R\dot{\ell}_n, \qquad q_n(0) = q_{n,0} \tag{4.12}$$

(whose existence is again guaranteed by Theorem 4.7) or the solution of

$$\dot{q}_n + A_{\lambda_n,Q}(q_n) = R\dot{\ell}_n, \qquad q_n(0) = q_{n,0}, \tag{4.13}$$

where we have abbreviated $A_{\lambda_n,Q} := (A_{\lambda_n})_Q = QA_{\lambda_n}$. Then $q_n \rightharpoonup q$ in $H^1(\mathcal{H})$ and $\mathfrak{Z}(q_n, \ell_n) \to \mathfrak{Z}(q, \ell)$ in $C(\mathcal{H})$.

Proof. At first we assume that q_n is the solution of (4.12) and employ Theorem 4.7, to be more precise, (4.3), to obtain the boundedness of q_n in $H^1(\mathcal{H})$. According to Lemma 4.3, $z_n := \mathfrak{Z}(q_n, \ell_n) \in H^1(\mathcal{H})$ is the solution of

$$\dot{z}_n \in A(R\ell_n - Qz_n), \qquad z_n(0) = z_{n,0},$$

with $z_{n,0} := \mathfrak{Z}(q_{n,0}, \ell_n(0)) = Q^{-1}(R\ell_n(0) - q_{n,0})$. We can now apply Lemma 4.9 to derive $z_n \rightharpoonup \mathfrak{Z}(q, \ell)$ in $H^1(\mathcal{H})$ and $z_n \to \mathfrak{Z}(q, \ell)$ in $C(\mathcal{H})$.

Let us now assume that q_n is the solution of (4.13). As above, thanks to Lemma 4.3, $z_n := \mathfrak{Z}(q_n, \ell_n) \in H^1(\mathcal{H})$ is the solution of

$$\dot{z}_n = A_{\lambda_n}(R\ell_n - Qz_n), \qquad z_n(0) = z_{n,0},$$

with $z_{n,0} := \mathfrak{Z}(q_{n,0}, \ell_n(0)) = Q^{-1}(R\ell_n(0) - q_{n,0})$. Similar as in the proof of Theorem 4.12, we consider the solution of

$$\dot{v}_n \in A(R\ell_n - Qv_n), \quad v_n(0) = z_{n,0},$$

then, thanks to the first part of this proof, v_n is bounded in $H^1(\mathcal{H})$. Using Proposition 4.8, we obtain the boundedness of $\dot{z}_n$ in $L^2(\mathcal{H})$, hence, Lemma 4.9 gives the desired result (with $A_n := A_{\lambda_n}$, which is a possible choice since it fulfills (4.6), according to Lemma 3.10). □

The assumption that the loads are contained in $H^2(\mathcal{X})$ enables us to consider a general, maximal monotone operator, this is exactly what we need in Part IV. Unfortunately, in order to prove the strong convergence of solutions to the regularized equations in Theorem 4.12, we needed that A is a subdifferential, but this will not be the case in Part IV, see Remark 8.16. However, the strong convergence of solutions by *fixed* loads is sufficient in Theorem 9.3 (respectively Theorem 5.4) and this can be directly obtained from Proposition 4.8 as shown in

Corollary 4.15 (Convergence of the Yosida approximation with fixed H^2 loads)**.** *Let $\ell \in H^2(\mathcal{X})$ and $q \in H^1(\mathcal{H})$ the solution of* (EVI$_q$) *(whose existence is guaranteed by Theorem 4.7). Furthermore, let $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence converging towards zero and $q_n \in H^1(\mathcal{H})$ the solution of*

$$\dot{q}_n + A_{\lambda_n,Q}(q_n) = R\dot{\ell}, \quad q_n(0) = q_0,$$

where we have again abbreviated $A_{\lambda_n,Q} := (A_{\lambda_n})_Q = QA_{\lambda_n}$. Then $q_n \to q$ in $H^1(\mathcal{H})$.

Proof. Note that the existence and uniqueness of q_n follows from Theorem A.7. Applying Lemma 4.3 with $A = A_{\lambda_n}$, we see that $z_n := \mathfrak{Z}(q_n, \ell) \in H^1(\mathcal{H})$ is the solution of

$$\dot{z}_n = A_{\lambda_n}(R\ell - Qz_n), \quad z_n(0) = z_0,$$

with $z_0 := \mathfrak{Z}(q_0, \ell(0)) = Q^{-1}(R\ell(0) - q_0)$. We can now apply Proposition 4.8 to obtain $z_n \to z$, therefore $q_n = \mathfrak{Q}(z_n, \ell) \to \mathfrak{Q}(z, \ell) = q$ in $H^1(\mathcal{H})$. □

Remark 4.16 (Extension to general Bochner-Sobolev spaces)**.** *It is to be noted that most of the above results can also be shown in more general Bochner-Sobolev spaces, that is, when loads are contained in $W^{1,r}(\mathcal{X})$ and states in $W^{1,r}(\mathcal{X})$ for some $r \in [1,\infty)$. However, since a Hilbert space setting is advantageous when it comes to the derivation of optimality conditions, we focus on the case $r = 2$.*

We end this chapter with a result concerning a further smoothing of the Yosida approximation. Later we apply this result with $A_n = \partial I_{\lambda_n,s_n}$ with suitable sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$, see Section 6.1. As will become clear in the next chapter, we use this further smoothing to derive optimality conditions.

Lemma 4.17 (Convergence of the regularized Yosida approximation)**.** *Consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, which converges to zero, and a sequence of Lipschitz continuous operators $A_n : \mathcal{H} \to \mathcal{H}$, $n \in \mathbb{N}$, such that*

$$\frac{1}{\lambda_n} \exp\Big(\frac{T\|Q\|_{L(\mathcal{H};\mathcal{H})}}{\lambda_n}\Big) \sup_{h\in\mathcal{H}} \|A_n(h) - A_{\lambda_n}(h)\|_{\mathcal{H}} \to 0. \tag{4.14}$$

Let moreover $\{\ell_n\}_{n\in\mathbb{N}} \subset C(\mathcal{X})$ be given and denote by $z_n, z_{\lambda_n} \in C^1(\mathcal{H})$ the solutions of

$$\begin{aligned} \dot{z}_n &= A_n(R\ell_n - Qz_n), \quad z_n(0) = z_0, \\ \dot{z}_{\lambda_n} &= A_{\lambda_n}(R\ell_n - Qz_{\lambda_n}), \quad z_{\lambda_n}(0) = z_0. \end{aligned}$$

Then $\|z_n - z_{\lambda_n}\|_{C^1(\mathcal{H})} \to 0$.

Proof. Again, thanks to the Lipschitz continuity of A_n and A_{λ_n}, the existence and uniqueness of z_n and $z_{n,\lambda}$ follows from Theorem A.7. Moreover, the continuity of ℓ_n carries over to the continuity of $\dot{z}_n$ and $\dot{z}_{n,\lambda}$. Let us abbreviate $c_n := \sup_{h\in\mathcal{H}} \|A_n(h) - A_{\lambda_n}(h)\|_{\mathcal{H}}$. Then, according to the definition of z_n and z_{λ_n}, we find

$$\|\dot{z}_n(t) - \dot{z}_{\lambda_n}(t)\|_{\mathcal{H}} \leq c_n + \frac{\|Q\|_{L(\mathcal{H};\mathcal{H})}}{\lambda_n}\|z_n(t) - z_{\lambda_n}(t)\|_{\mathcal{H}}$$

for all $t \in [0, T]$, so that Lemma A.8 Item (ii) yields

$$\|\dot{z}_n(t) - \dot{z}_{\lambda_n}(t)\|_{\mathcal{H}} \leq \frac{\|Q\|_{L(\mathcal{H};\mathcal{H})}}{\lambda_n}\Big(T \exp\Big(\frac{\|Q\|_{L(\mathcal{H};\mathcal{H})}}{\lambda_n}T\Big) + 1\Big)c_n$$

for all $t \in [0, T]$, which completes the proof. □

Chapter 5 Optimal Control

The last chapter was devoted to the state equation, the evolution variational inequality (EVI), of the optimal control problem which will be considered in this chapter. We have collected all results we need for the investigation of such an optimal control problem, which is the purpose of this chapter.

The optimal control of this EVI is the core of the thesis at hand. As we have seen in Chapter 2 and will see in Chapter 7, the optimal control of this EVI has a direct application in elasto and homogenized plasticity. Moreover, it is also related to the problem considered in Part IV for plasticity with inertia and some of the results in this chapter can also be used in the case of perfect plasticity in Part V.

We note that we require A to have the boundedness property, see Assumption ⟨5.ii⟩. We do this since in all applications we will choose A to be the subdifferential of an indicator function, which has the boundedness property (cf. Proposition 3.14, note also that the arising operator $\mathcal{A}$ in Part IV does not have it, the results in Chapter 4 about H^2 loads are tailored to this case). However, it is also possible to require loads to be H^2 in time and to drop the boundedness property of A. The analysis in Section 5.1 would not much be affected and in Section 5.2 and Section 5.3 the boundedness property plays no role (Assumption ⟨5.ii⟩ could also only be required in Section 5.1 but we decided to suppose it in the whole chapter to make clear that we consider optimal control with respect to the boundedness property).

At first we will prove the existence of optimal controls and then a result regarding the approximation of optimal controls with optimal controls of a smoothed problem, this is the content of Section 5.1. With this result at hand, it is reasonable to "replace" the original problem with the regularized one in Section 5.2 and provide optimality conditions only for the regularized problem. Second-order sufficient conditions are finally addressed in Section 5.3.

As said in the beginning of this part, its content, and thus the content of this chapter, is based on [71]. Let us follow the comparison with the literature therein to put our work into perspective. To this end, we assume that A is in fact the convex subdifferential of a proper, convex, and lower semicontinuous functional ϕ. Then, by convex duality (cf. [84, 34]), the state equation is equivalent to

$$0 \in \partial\phi^*(\dot{z}) + \mathcal{E}'(z), \qquad z(0) = z_0, \tag{5.1}$$

where $\mathcal{E}$ is the quadratic energy functional given by

$$\mathcal{E}(z) := \tfrac{1}{2}(Qz, z)_{\mathcal{H}} - \langle R\ell, z\rangle.$$

The literature on optimization problems with (5.1) as a constraint is rather scarce. The first investigation on optimal control of equations of type (5.1) is probably the sweeping process, see [78], which is a special case of (5.1). Optimal control problems of this type with a certain function ϕ and $R = Q = I$ are examined in [16, 17, 18, 1, 24, 25, 26, 8], where the underlying Hilbert space is mostly finite dimensional, the infinite dimensional case is investigated in [94, 43]. As already elaborated in Part I, optimal control problems governed by quasi-static elastoplasticity with linear kinematic hardening and the von Mises yield condition are treated in [105, 107, 108]. All mentioned problems can be seen as special cases of the problem we are considering. Our analysis is therefore a generalization of existing results on optimal control for non-smooth evolution problems, but can also be applied to problems that were not treated in the literature so far. We also mention that optimal control problems governed by (5.1) with a non-convex energy are investigated in [81, 82], however, these problems are not covered by our analysis.

5.1 Existence and Approximation of Optimal Controls

Unsurprisingly, the optimal control problem under consideration reads

$$\left\{\begin{array}{ll} \min & J(z,\ell) = \Psi(z,\ell) + \Phi(\ell), \\ \text{s.t.} & \dot{z} \in A(R\ell - Qz), \qquad z(0) = z_0, \\ & (z,\ell) \in H^1(\mathcal{H}) \times \big(H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}(z_0; M)\big). \end{array}\right. \tag{5.2}$$

Herein, the properties of the objective function and the maximal monotone operator A are given in Assumptions ⟨5.ii⟩ to ⟨5.iv⟩ and Assumption ⟨5.1.i⟩. Note that we require loads in $H^1(\mathcal{X}_c)$ and not only in $H^1(\mathcal{X})$. In Theorem 5.2 and Theorem 5.4 we will use the compact embedding $\mathcal{X}_c \overset{c}{\hookrightarrow} \mathcal{X}$ to obtain the desired strong convergence of loads in $L^1(\mathcal{X})$, required in Theorem 4.11 and Theorem 4.12, for an otherwise only bounded sequence.

Existence of Optimal Controls

Clearly, the first step is to prove the existence of optimal controls. Before we do so, let us define the solution operator of (EVI), which will become later in handy when we prove the convergence of global minimizers.

Definition 5.1 (Solution operator of (EVI))**.** *Due to Theorem 4.5, there exists for every $\ell \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}(z_0; M)$ a solution $z \in H^1(\mathcal{H})$ of the state equation* (EVI)*. Consequently, we may define the solution operator*

$$\mathcal{S}:\ H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}(z_0; M) \ni \ell \to z \in H^1(\mathcal{H}).$$

This operator will be frequently called control-to-state map.

With the definition above, problem (5.2) is equivalent to the *reduced problem*:

$$(5.2) \quad \Longleftrightarrow \quad \left\{\begin{array}{ll} \min & J(\mathcal{S}(\ell), \ell), \\ \text{s.t.} & \ell \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}(z_0; M). \end{array}\right.$$

Theorem 5.2 (Existence of a global solution)**.** *There exists a global solution of* (5.2).

Proof. Based on Theorem 4.11, the proof follows the standard direct method of the calculus of variations. First of all, since Ψ is bounded from below and Φ is coercive, every infimal sequence of controls is bounded in $H^1(\mathcal{X}_c)$ and thus admits a weakly converging subsequence (recall that we assumed $\mathcal{X}_c$ to be reflexive in Assumption ⟨5.iv⟩). Due to the compact embedding of $\mathcal{X}_c$ in $\mathcal{X}$, this sequence converges strongly in $C(\mathcal{X})$ so that the weak limit belongs to $\mathcal{A}_{\mathcal{L}}(z_0; M)$, due the closeness of M. Moreover, thanks to the weak convergence in $H^1(\mathcal{X})$ and the strong convergence in $C(\mathcal{X})$, Theorem 4.11 gives weak convergence of the associated states in $H^1(\mathcal{H})$ and shows that the limit is admissible. The weak lower semicontinuity of Ψ and Φ together with $\mathcal{H} \hookrightarrow \mathcal{W}$ then implies the optimality of the weak limit. □

It is obvious that in view of the nonlinear state equation, one cannot expect the optimal solution to be unique. Note that, since $D(A)$ is closed by Assumption ⟨5.ii⟩, the choice $M = D(A)$ is feasible.

Convergence of Global Minimizers

In Section 5.2 and Section 5.3 we will derive optimality conditions for a regularized optimal control problem, that global minimizers of these regularized problems are related the global minimizers of the original problem is the topic in the following. The regularized problems read as follows

$$\left\{\begin{aligned} &\min \quad J(z,\ell), \\ &\text{s.t.} \quad \dot z = A_n(R\ell - Qz), \qquad z(0) = z_0, \\ &\qquad\ (z,\ell) \in H^1(\mathcal{H}) \times \big(H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}(z_0; M)\big), \end{aligned}\right. \tag{5.3}$$

that is, the operator A is exchanged by A_n, where the properties of A_n are given in Assumption ⟨5.1.ii⟩. Since A_n is Lipschitz continuous, the equation

$$\dot z = A_n(R\ell - Qz), \qquad z(0) = z_0$$

admits a unique solution for every $\ell \in L^2(\mathcal{X})$, according to Theorem A.7. Similar to Definition 5.1, we denote the associated solution operator by

$$\mathcal{S}_n : L^2(\mathcal{X}) \to H^1(\mathcal{H}). \tag{5.4}$$

Moreover, the solution operator associated with the Yosida approximation, i.e., the solution operator of $\dot z = A_{\lambda_n}(R\ell - Qz)$, $z(0) = z_0$, is denoted by

$$\mathcal{S}_{\lambda_n} : L^2(\mathcal{X}) \to H^1(\mathcal{H}). \tag{5.5}$$

Proposition 5.3 (Existence of optimal solutions of the regularized problems)**.** *Let $n \in \mathbb{N}$, then there exists a global solution of* (5.3)*.*

Proof. Let $\ell_1, \ell_2 \in L^2(\mathcal{X})$ be arbitrary and define $z_i := \mathcal{S}_n(\ell_i)$, $i = 1, 2$. Then, due to the Lipschitz continuity of A_n, we have for almost all $t \in [0, T]$

$$\begin{aligned} \|\dot z_1(t) - \dot z_2(t)\|_{\mathcal{H}} &= \|A_n(R\ell_1(t) - Qz_1(t)) - A_n(R\ell_2(t) - Qz_2(t))\|_{\mathcal{H}} \\ &\le c\big(\|l_1(t) - l_2(t)\|_{\mathcal{X}} + \|z_1(t) - z_2(t)\|_{\mathcal{H}}\big), \end{aligned} \tag{5.6}$$

which yields, according to Lemma A.8 Item (iii), the Lipschitz continuity of $\mathcal{S}_n$. Using this together with the fact that $\mathcal{X}_c$ is compactly embedded into $\mathcal{X}$, one can argue as in the proof of Theorem 5.2 to obtain the existence of a global solution of (5.3) for all $n \in \mathbb{N}$. □

Theorem 5.4 (Weak approximation of global minimizers). *Let $\{\overline{\ell}_n\}_{n\in\mathbb{N}}$ be a sequence of globally optimal controls of* (5.3). *Then there exists a weak accumulation point and every weak accumulation point is a global solution of* (5.2).

Proof. Due to $M \subset D(A)$, Proposition 4.8 gives $\mathcal{S}_{\lambda_n}(\overline{\ell}_1) \to \mathcal{S}(\overline{\ell}_1)$ in $H^1(\mathcal{H})$ so that Lemma 4.17 yields $\mathcal{S}_n(\overline{\ell}_1) \to \mathcal{S}(\overline{\ell}_1)$ in $H^1(\mathcal{H})$ and thus

$$\limsup_{n\to\infty} \Psi(\mathcal{S}_n(\overline{\ell}_n), \overline{\ell}_n) + \Phi(\overline{\ell}_n) = \limsup_{n\to\infty} J(\mathcal{S}_n(\overline{\ell}_n), \overline{\ell}_n) \le \limsup_{n\to\infty} J(\mathcal{S}_n(\overline{\ell}_1), \overline{\ell}_1) = J(\mathcal{S}(\overline{\ell}_1), \overline{\ell}_1).$$

Hence, by virtue of the boundedness of Ψ from below and the coercivity of Φ, $\{\overline{\ell}_n\}$ is bounded and therefore admits a weak accumulation point in $H^1(\mathcal{X}_c)$.

Let us now assume that a given subsequence of $\{\overline{\ell}_n\}_{n\in\mathbb{N}}$, denoted by the same symbol for simplicity, converges weakly to $\tilde{\ell}$ in $H^1(\mathcal{X}_c)$. Since $\mathcal{X}_c$ is compactly embedded in $\mathcal{X}$, we obtain $\overline{\ell}_n \to \tilde{\ell}$ in $C(\mathcal{X})$ and consequently, $\tilde{\ell} \in \mathcal{A}_{\mathcal{L}}(z_0; M)$. In addition, the strong convergence in $C(\mathcal{X})$ in combination with Theorem 4.12 and Lemma 4.17 yields weak convergence of the states, i.e., $\mathcal{S}_n(\overline{\ell}_n) \rightharpoonup \mathcal{S}(\tilde{\ell})$ in $H^1(\mathcal{H})$ and thus also in $H^1(\mathcal{W})$. Now, let $\overline{\ell}$ be a global solution of (5.2). We can again use Proposition 4.8 and Lemma 4.17 to obtain $\mathcal{S}_n(\overline{\ell}) \to \mathcal{S}(\overline{\ell})$ in $H^1(\mathcal{H})$. This, together with the weak lower semicontinuity of Ψ and Φ, and the continuity of Ψ in the first argument, implies

$$\begin{aligned} J(\mathcal{S}(\tilde{\ell}), \tilde{\ell}) &= \Psi(\mathcal{S}(\tilde{\ell}), \tilde{\ell}) + \Phi(\tilde{\ell}) \\ &\le \liminf_{n\to\infty} \Psi(\mathcal{S}_n(\overline{\ell}_n), \overline{\ell}_n) + \Phi(\overline{\ell}_n) \\ &\le \limsup_{n\to\infty} J(\mathcal{S}_n(\overline{\ell}_n), \overline{\ell}_n) \le \limsup_{n\to\infty} J(\mathcal{S}_n(\overline{\ell}), \overline{\ell}) = J(\mathcal{S}(\overline{\ell}), \overline{\ell}), \end{aligned} \tag{5.7}$$

giving in turn the optimality of the weak limit. □

Corollary 5.5 (Strong approximation of global minimizers). *Suppose that $\Phi : H^1(\mathcal{X}_c) \to \mathbb{R}$ is such that, if a sequence $\{\ell_n\}_{n\in\mathbb{N}}$ satisfies $\ell_n \rightharpoonup \ell$ in $H^1(\mathcal{X}_c)$ and $\Phi(\ell_n) \to \Phi(\ell)$, then $\ell_n \to \ell$ in $H^1(\mathcal{X}_c)$. Then every weak accumulation point of a sequence of globally optimal controls of* (5.3) *is also a strong one.*

Moreover, if in addition, at least one of the following holds

(i) A is a subdifferential of a convex, lower semicontinuous and proper functional ϕ (see Section 3.2), that is $A = \partial\phi$, and ϕ is continuous on M or

(ii) $\Psi : H^1(\mathcal{W}) \times H^1(\mathcal{X}_c) \to \mathbb{R}$ is such that, if sequences $\{z_n\}_{n\in\mathbb{N}}$ and $\{\ell_n\}_{n\in\mathbb{N}}$ satisfy $z_n \rightharpoonup z$ in $H^1(\mathcal{H})$ and $\ell_n \to \ell$ in $H^1(\mathcal{X}_c)$ and $\Psi(z_n, \ell_n) \to \Psi(z, \ell)$, then $z_n \to z$ in $H^1(\mathcal{H})$,

then the associated sequence of states also converges strongly in $H^1(\mathcal{H})$.

Proof. Consider an arbitrary accumulation point $\tilde{\ell}$ of a sequence of global minimizers of (5.3), i.e., $\overline{\ell}_n \rightharpoonup \tilde{\ell}$ in $H^1(\mathcal{X}_c)$. From the previous proof, we know that then (5.7) holds, giving in turn

$$\Psi(\mathcal{S}_n(\overline{\ell}_n), \overline{\ell}_n) + \Phi(\overline{\ell}_n) \to \Psi(\mathcal{S}(\tilde{\ell}), \tilde{\ell}) + \Phi(\tilde{\ell}).$$

Since $\mathcal{S}_n(\overline{\ell}_n) \rightharpoonup \mathcal{S}(\tilde{\ell})$, as seen in the previous proof, and both, Ψ and Φ, are weakly lower semicontinuous by assumption, this implies $\Phi(\overline{\ell}_n) \to \Phi(\tilde{\ell})$ and $\Psi(\mathcal{S}_n(\overline{\ell}_n), \overline{\ell}_n) \to \Psi(\mathcal{S}(\tilde{\ell}), \tilde{\ell})$. The hypothesis on Φ thus yields $\overline{\ell}_n \to \tilde{\ell}$ in $H^1(\mathcal{X}_c)$ so that $\tilde{\ell}$ is indeed a strong accumulation point as claimed.

Due to $\mathcal{X}_c \hookrightarrow \mathcal{X}$, the strong convergence carries over to $H^1(\mathcal{X})$ and therefore, we deduce from Theorem 4.12 that $\mathcal{S}_{\lambda_n}(\overline{\ell}_n) \to \mathcal{S}(\tilde{\ell})$ in $H^1(\mathcal{H})$, provided that A is a subdifferential and

$\phi(R\overline{\ell}_n(0) - Qz_0) \to \phi(R\tilde{\ell}(0) - Qz_0)$ holds. If the additional requirements on Ψ are fulfilled, we also obtain the strong convergence $\mathcal{S}_{\lambda_n}(\overline{\ell}_n) \to \mathcal{S}(\tilde{\ell})$ in $H^1(\mathcal{H})$, since we already showed $\overline{\ell}_n \to \tilde{\ell}$ in $H^1(\mathcal{X}_c)$. Thus, in both cases, Lemma 4.17 gives $\mathcal{S}_n(\overline{\ell}_n) \to \mathcal{S}(\tilde{\ell})$ in $H^1(\mathcal{H})$, which is the second assertion. □

Remark 5.6 (Approximation of local minimizers)**.** *By standard localization arguments, the above convergence analysis can be adapted to approximate local minimizers. Following the lines of, for instance, [20], one can show that, under the assumptions of Corollary 5.5, every strict local minimum of* (5.2) *can be approximated by a sequence of local minima of* (5.3)*. A local minimizer $\overline{\ell}$ of* (5.2)*, which is not necessarily strict, can be approximated by replacing the objective in* (5.3) *by $\overline{J}(z,l) := J(z,l) + \|\ell - \overline{\ell}\|^2_{H^1(\mathcal{X}_c)}$, which is of course only of theoretical interest, cf. e.g. [9]. Since these results and their proofs are standard, we omitted them.*

5.2 First-Order Optimality Conditions

Due to the results from the last section, it is reasonable to "exchange" (5.2) with (5.3). To this end, we consider

$$\begin{cases} \min & J(z,\ell), \\ \text{s.t.} & \dot{z} = A_s(R\ell - Qz), \quad z(0) = z_0, \\ & (z,\ell) \in H^1(\mathcal{H}) \times \big(H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}\big), \end{cases} \tag{5.8}$$

where A_s is supposed to be equal to A_n for one $n \in \mathbb{N}$, however, to derive optimality conditions for (5.8), this is not important, we only need Assumption ⟨5.2.iv⟩.

Let us shortly comment on the Assumptions ⟨5.2.i⟩ to ⟨5.2.v⟩. At first it is well known that a norm gap is often indispensable to ensure Fréchet-differentiability, see Lemma 3.19 and in particular [44, Section 3.1]. Since we will use $\partial I_{\lambda,s}$ (see (3.11)) in our applications, this will be the case therein, see Section 6.2, Section 9.2 (cf. Example 9.16) and Section 12.3, which is the reason for considering the space $\mathcal{Y}$ in context of the Fréchet-differentiability of A_s. Moreover, since we are interested in deriving second order optimality conditions, we will need a second norm gap (cf. Remark 3.20), which is the purpose of the space $\mathcal{Z}$. Clearly, then we also need to solve the regularized equation in these spaces so that Assumption ⟨5.2.ii⟩ becomes necessary. The additional requirements in Assumption ⟨5.2.iv⟩ for A_s are technical details, which we will use for the (twice) differentiability in Theorem 5.9 and Proposition 5.17. Finally, in Assumption ⟨5.2.iii⟩ we reduce the set of admissible loads to $\mathcal{A}_{\mathcal{L}}$, that is, when we assume for a moment that R is injective (which is the case in our applications, see Definition 2.7, Definition 7.5, Definition 8.5 and Proposition 12.17), we consider loads which are zero at the initial time $t = 0$. Thanks to this reduction the set of admissible controls is a linear subspace of $H^1(\mathcal{X}_c)$ and thus itself a Hilbert space. We note that one could allow for additional control constraints in our analysis such as, for example, box constraints over the whole time interval or vanishing final loading, i.e., $\ell(T) = 0$, which is certainly meaningful for many applications. However, since the differentiability of the control-to-state map is the essential issue in the derivation of optimality conditions and additional control constraints can be incorporated by standard argument, we choose to omit them to keep the discussion more concise.

Differentiability of the Regularized Control-to-State Mapping

Theorem A.7 immediately implies that the state equation in (5.8), i.e.,

$$\dot{z} = A_s(R\ell - Qz), \quad z(0) = z_0, \tag{EVI$_s$}$$

admits a unique solution $z \in H^1(\mathcal{Y})$ for every right hand side $\ell \in L^2(\mathcal{X})$, provided that $z_0 \in \mathcal{Y}$. Therefore, similar to above, we can define the associated solution operator

$$\mathcal{S}_s : L^2(\mathcal{X}) \to H^1(\mathcal{Y}) \tag{5.9}$$

(for fixed $z_0 \in \mathcal{Y}$). We will frequently consider this operator with different domains, e.g. $H^1(\mathcal{X})$, and ranges, in particular $H^1(\mathcal{Z})$. With a little abuse of notation, these operators are denoted by the same symbol.

Lemma 5.7 (Lipschitz continuity of $\mathcal{S}_s$)**.** *The solution operator $\mathcal{S}_s$ is globally Lipschitz continuous from $L^2(\mathcal{X})$ to $H^1(\mathcal{Y})$.*

Proof. This can be proven completely analogously to the Lipschitz continuity of $\mathcal{S}_n$ from $L^2(\mathcal{X})$ to $H^1(\mathcal{H})$ in the proof of Proposition 5.3. □

Lemma 5.8 (Existence of the derivative of $\mathcal{S}_s$)**.** *Let $y \in L^2(\mathcal{Y})$ and $w \in L^2(\mathcal{Z})$ be given. Then there exists a unique solution $\eta \in H^1(\mathcal{Z})$ of*

$$\dot{\eta} = A_s'(y)(w - Q\eta), \qquad \eta(0) = 0. \tag{5.10}$$

Proof. Let us define

$$B : [0,T] \times \mathcal{Z} \to \mathcal{Z}, \qquad (t,\eta) \mapsto A_s'(y(t))(w(t) - Q\eta)$$

so that (5.10) becomes $\dot{\eta}(t) = B(t,\eta(t))$ a.e. in $[0,T]$, $\eta(0) = 0$. Now, given $\eta \in L^2(\mathcal{Z})$, $[0,T] \ni t \mapsto B(t,\eta(t)) \in \mathcal{Z}$ is Bochner measurable, which can be seen as follows: At first, it is Bochner measurable as a pointwise limit of Bochner measurable functions when the direction $w - Q\eta$ is an element of $L^2(\mathcal{Y})$, thanks to the differentiability of A_s. When only $w - Q\eta \in L^2(\mathcal{Z})$, then we can approximate with a sequence contained in $L^2(\mathcal{Y})$ (note that $L^2(\mathcal{Y})$ is dense in $L^2(\mathcal{Z})$ due to Assumption ⟨5.2.i⟩). The measurability follows then from the fact that the function is a pointwise limit of measurable functions.

Furthermore, Assumption ⟨5.2.iv⟩ implies for almost all $t \in [0,T]$ and all $\eta_1, \eta_2 \in \mathcal{Z}$ that $\|B(t,\eta_1) - B(t,\eta_2)\|_{\mathcal{Z}} \leq C\,\|\eta_1 - \eta_2\|_{\mathcal{Z}}$. Therefore, we can apply once again Theorem A.7, which gives the assertion. □

With the above result at hand we can proceed with the differentiability of $\mathcal{S}_s$. The prove of the differentiability is rather technical, but also uses essentially only Assumption ⟨5.2.iv⟩, Gronwall's inequality and Lebesgues dominated convergence theorem.

Theorem 5.9 (Fréchet-differentiability of $\mathcal{S}_s$)**.** *The solution operator $\mathcal{S}_s$ is Fréchet differentiable from $H^1(\mathcal{X})$ to $H^1(\mathcal{Z})$. Its directional derivative at $\ell \in H^1(\mathcal{X})$ in direction $h \in H^1(\mathcal{X})$ is given by the unique solution of*

$$\dot{\eta} = A_s'(R\ell - Qz)(Rh - Q\eta), \qquad \eta(0) = 0, \tag{5.11}$$

where $z := \mathcal{S}_s(\ell) \in H^1(\mathcal{Y})$. Moreover, there exists a constant C such that $\|\mathcal{S}_s'(\ell)h\|_{H^1(\mathcal{Z})} \leq C\|h\|_{L^2(\mathcal{X})}$ holds for all $\ell, h \in H^1(\mathcal{X})$.

Proof. Let $\ell, h \in H^1(\mathcal{X})$ be arbitrary and abbreviate $z_h := \mathcal{S}_s(\ell + h)$. Thanks to Lemma 5.8, there exists a unique solution $\eta \in H^1(\mathcal{Z})$ of (5.11). Clearly, the solution operator of (5.11) is linear with respect to h. Moreover, Assumption ⟨5.2.iv⟩ implies for almost all $t \in [0,T]$ that

$$\|\dot{\eta}(t)\|_{\mathcal{Z}} \leq C\big(\|h(t)\|_{\mathcal{X}} + \|\eta(t)\|_{\mathcal{Z}}\big),$$

so that Lemma A.8 Item (iii) gives $\|\eta\|_{H^1(\mathcal{Z})} \leq C\|h\|_{L^2(\mathcal{X})}$, i.e., the continuity of the solution operator of (5.11). This also proves the asserted inequality (after having proved that $\eta = \mathcal{S}_s'(\ell)h$, which we do next).

It remains to verify the remainder term property. For this purpose, let us denote the remainder term of A_s by r_1, i.e.,

$$A_s(y+\zeta) = A_s(y) + A_s'(y)\zeta + r_1(y;\zeta) \quad \text{with} \quad \frac{\|r_1(y;\zeta)\|_{\mathcal{Z}}}{\|\zeta\|_{\mathcal{Y}}} \to 0 \text{ as } \zeta \to 0 \text{ in } \mathcal{Y}.$$

Moreover, we abbreviate

$$y := R\ell - Qz \in H^1(\mathcal{Y}) \quad \text{and} \quad \zeta := Rh - Q(z_h - z) \in H^1(\mathcal{Y}).$$

Then, in view of the definition of z, z_h, and η (as solution of (5.11)), we find for almost all $t \in [0,T]$

$$\begin{aligned}
&\|\dot{z}_h(t) - \dot{z}(t) - \dot{\eta}(t)\|_{\mathcal{Z}} \\
&\quad = \|A_s(y(t)+\zeta(t)) - A_s(y(t)) - A_s'(y(t))(\zeta(t) + Q(z_h(t) - z(t) - \eta(t)))\|_{\mathcal{Z}} \\
&\quad \leq \|A_s'(y(t))Q(z_h(t) - z(t) - \eta(t))\|_{\mathcal{Z}} + \|r_1(y(t);\zeta(t))\|_{\mathcal{Z}}.
\end{aligned}$$

Hence, Assumption ⟨5.2.iv⟩ and Lemma A.8 Item (iii) yield

$$\|z_h - z - \eta\|_{H^1(\mathcal{Z})} \leq C\|r_1(y;\zeta)\|_{L^2(\mathcal{Z})} \tag{5.12}$$

(note that $r_1(y;\zeta) \in L^2(\mathcal{Z})$ by its definition as remainder term). Furthermore, thanks to Lemma 5.7 and the definition of ζ, we obtain

$$\|\zeta\|_{H^1(\mathcal{Y})} \leq C\|h\|_{H^1(\mathcal{X})} \tag{5.13}$$

such that $h \to 0$ in $H^1(\mathcal{X})$ implies $\zeta \to 0$ in $H^1(\mathcal{Y})$. The continuous embedding $H^1(\mathcal{Y}) \hookrightarrow C(\mathcal{Y})$ and the remainder term property of r_1 thus give for almost all $t \in [0,T]$ that

$$\frac{\|r_1(y(t);\zeta(t))\|_{\mathcal{Z}}}{\|h\|_{H^1(\mathcal{X})}} \leq C\frac{\|r_1(y(t);\zeta(t))\|_{\mathcal{Z}}}{\|\zeta(t)\|_{\mathcal{Y}}}\frac{\|\zeta\|_{H^1(\mathcal{Y})}}{\|h\|_{H^1(\mathcal{X})}} \to 0 \tag{5.14}$$

as $h \to 0$ in $H^1(\mathcal{X})$. Moreover, the Lipschitz continuity of $A_s : \mathcal{Y} \to \mathcal{Y}$ together with Assumption ⟨5.2.iv⟩, $\mathcal{Y} \hookrightarrow \mathcal{Z}$, and (5.13) yield for almost all $t \in [0,T]$ that

$$\frac{\|r_1(y(t);\zeta(t))\|_{\mathcal{Z}}}{\|h\|_{H^1(\mathcal{X})}} = \frac{\|(A_s(y+\zeta) - A_s(y) - A_s'(y)\zeta)(t)\|_{\mathcal{Z}}}{\|h\|_{H^1(\mathcal{X})}} \leq C\frac{\|\zeta(t)\|_{\mathcal{Y}}}{\|h\|_{H^1(\mathcal{X})}} \leq C.$$

In combination with (5.14) and Lebesgue's dominated convergence theorem, this yields

$$\frac{\|r_1(y;\zeta)\|_{L^2(\mathcal{Z})}}{\|h\|_{H^1(\mathcal{X})}} \to 0$$

as $h \to 0$ in $H^1(\mathcal{X})$, which, in view of (5.12) finishes the proof. □

One might wonder why we did not employ the implicit function theorem to show the differentiability of $\mathcal{S}_s$. The reason is that $H : z \mapsto \dot{z} - A_s(R\ell - Qz)$ is Fréchet-differentiable from $H^1(\mathcal{Y})$ to $L^2(\mathcal{Z})$, but the derivative $H'(z)$ is not continuously invertible in these spaces, cf. Lemma 5.8. On the other hand, due to the differentiability properties of A_s arising from the norm gap (see the discussion at the beginning of this section), H is not differentiable from $H^1(\mathcal{Y})$ to $L^2(\mathcal{Y})$, which would be the right spaces for the existence result from Lemma 5.8. The same observation for a more abstract setting was already made in [106].

Adjoint Equation

A typical approach to make the purely primal necessary optimality conditions (see Lemma 5.10 below) more accessible is to use the so called *adjoint state*. We refer for instance to TRÖLTZSCH [100, Section 1.4.4, 2.13.3] or HINZE ET AL. [54, Section 1.6.2].

Regarding the purely primal necessary optimality conditions, the chain rule immediately gives that the reduced objective function defined by

$$F: H^1(\mathcal{X}_c) \to \mathbb{R}, \qquad \ell \mapsto J(\mathcal{S}_s(\ell), \ell) \tag{5.15}$$

is Fréchet-differentiable, too. Thus, by standard arguments, one derives the following

Lemma 5.10 (Purely primal necessary optimality conditions). *If a control $\overline{\ell} \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$ with associated state $\overline{z} = \mathcal{S}_s(\overline{\ell})$ is locally optimal for* (5.8)*, then*

$$F'(\overline{\ell})h = J_z'(\overline{z}, \overline{\ell})\mathcal{S}_s'(\overline{\ell})h + J_\ell'(\overline{z}, \overline{\ell})h = 0, \tag{5.16}$$

for all $h \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$.

Next, to reformulate (5.16) in terms of a KKT system, we introduce an *adjoint equation*, which formally reads

$$\dot{\varphi} = QA_s'(y)^*\varphi + v, \qquad \varphi(T) = 0. \tag{5.17}$$

Depending on the regularity of the right hand side v, we define different notions of solutions:

Definition 5.11 (Weak solution of an ODE). *Let $y \in L^2(\mathcal{Y})$ and $v \in H^1(\mathcal{H})^*$ be given. A function $\varphi \in L^2(\mathcal{H})$ is called* weak solution *of* (5.17)*, if*

$$-\left(\varphi, \dot{\eta}\right)_{L^2(\mathcal{H})} = \left(\varphi, A_s'(y)Q\eta\right)_{L^2(\mathcal{H})} + v(\eta) \tag{5.18}$$

holds for all $\eta \in H^1(\mathcal{H})$ with $\eta(0) = 0$. Note that the measurability of $A_s'(y)Q\eta$ can be shown as in Lemma 5.8.

If v takes the form

$$v(\eta) = \left(v_1, \eta\right)_{L^2(\mathcal{H})} + \left(v_2, \eta(T)\right)_{\mathcal{H}} \tag{5.19}$$

with some $v_1 \in L^2(\mathcal{H})$ and $v_2 \in \mathcal{H}$, then we call $\varphi \in H^1(\mathcal{H})$ strong solution *of* (5.17)*, if, for almost all $t \in [0, T]$,*

$$\dot{\varphi}(t) = \left(QA_s'(y)^*\varphi\right)(t) + v_1(t), \qquad \varphi(T) = -v_2. \tag{5.20}$$

Before formulating the KKT system in Theorem 5.13 with the help of the adjoint state as the solution of the adjoint equation, we first prove the existence of a solution of the adjoint equation in

Lemma 5.12 (Existence of weak solutions). *Let $y \in L^2(\mathcal{Y})$ and $v \in H^1(\mathcal{H})^*$. Then there is a unique weak solution of* (5.17)*, which is given by $\varphi := -v \circ \mathcal{S}_y \in L^2(\mathcal{H})^* = L^2(\mathcal{H})$, where $\mathcal{S}_y : L^2(\mathcal{H}) \to H^1(\mathcal{H})$ is the solution operator of*

$$\dot{\eta} = -A_s'(y)Q\eta + w, \qquad \eta(0) = 0, \tag{5.21}$$

that is, $\mathcal{S}_y(w) = \eta$.

Moreover, if v is of the form (5.19)*, then there exists a unique strong solution of* (5.17)*, and the weak and the strong solution coincide.*

Proof. At first note that the existence of a solution of (5.21) can be proven exactly as in Lemma 5.8. Let $\eta \in H^1(\mathcal{H})$ with $\eta(0) = 0$ be arbitrary and define $w := \dot{\eta} + A_s'(y)Q\eta \in L^2(\mathcal{H})$, hence, $\eta = \mathcal{S}_y(w)$. By the definition of w and φ, it follows that

$$\Big(\dot{\eta} + A_s'(y)Q\eta, \varphi\Big)_{L^2(\mathcal{H})} = (\varphi, w)_{L^2(\mathcal{H})} = -v(\mathcal{S}_y(w)) = -v(\eta),$$

i.e., (5.18) holds. Since η was arbitrary, we see that φ is a weak solution of (5.17).

To prove uniqueness, let $\tilde{\varphi} \in L^2(\mathcal{H})$ be another weak solution. Then, we choose an arbitrary $w \in L^2(\mathcal{H})$ and set $\eta := \mathcal{S}_y(w)$ to see that

$$(\varphi, w)_{L^2(\mathcal{H})} = -v(\eta) = \Big(\tilde{\varphi}, \dot{\eta} + A_s'(y)Q\eta\Big)_{L^2(\mathcal{H})} = (\tilde{\varphi}, w)_{L^2(\mathcal{H})},$$

and therefore $\varphi = \tilde{\varphi}$.

Now we turn to the strong solution and suppose that v is given as in (5.19). Existence and uniqueness of a strong solution follows from Theorem A.7, as we will elaborate in the following. Let us consider the affine-linear operator

$$B : [0,T] \times \mathcal{H} \to \mathcal{H}, \qquad B(t,\varphi) = QA_s'(y(t))^*\varphi + v_1(t).$$

Since $\mathcal{H}$ is separable by Assumption ⟨5.2.i⟩ (this is the only time we need the separability of $\mathcal{H}$), we can apply [41, Chap. IV, Thm. 1.4] to obtain that, for every $\varphi \in L^2(\mathcal{H})$, the mapping $[0,T] \mapsto B(t,\varphi(t))$ is Bochner measurable (note that the measurability of $[0,T] \ni t \mapsto A_s'(y(t))Qh \in \mathcal{H}$ for $h \in \mathcal{H}$ follows as in Lemma 5.8). Moreover, since $\|A_s'(y)^*\|_{L(\mathcal{H};\mathcal{H})} = \|A_s'(y)\|_{L(\mathcal{H};\mathcal{H})}$, Assumption ⟨5.2.iv⟩ yields that B is also Lipschitz continuous w.r.t. the second variable for almost all $t \in [0,T]$. Therefore, we can apply Theorem A.7 to establish the existence of a unique strong solution.

Finally, if we test (5.20) with an arbitrary $\eta \in H^1(\mathcal{H})$ with $\eta(0) = 0$ and integrate by parts, then we see that every strong solution is also a weak solution. Since the latter one is unique, as seen above, we deduce that weak and strong solution coincide. □

Theorem 5.13 (KKT-Conditions for (5.8))**.** *Let $\overline{\ell} \in H^1(\mathcal{X}_c) \cap \mathcal{A}_\mathcal{L}$ be a locally optimal control for* (5.8) *with associated state $\overline{z} = \mathcal{S}_s(\overline{\ell})$. Then there exists a unique* adjoint state $\varphi \in L^2(\mathcal{H})$ *such that the following optimality system is fulfilled*

$$\dot{\overline{z}} = A_s(R\overline{\ell} - Q\overline{z}), \qquad \overline{z}(0) = z_0, \tag{5.22a}$$

$$-\Big(\varphi, \dot{\eta}\Big)_{L^2(\mathcal{H})} = \Big(\varphi, A_s'(R\overline{\ell} - Q\overline{z})Q\eta\Big)_{L^2(\mathcal{H})} + J_z'(\overline{z}, \overline{\ell})\eta \quad \forall \eta \in H^1(\mathcal{H}) : \eta(0) = 0 \tag{5.22b}$$

$$\Big(\varphi, A_s'(R\overline{\ell} - Q\overline{z})Rh\Big)_{L^2(\mathcal{H})} = J_\ell'(\overline{z}, \overline{\ell})h \qquad \forall h \in H^1(\mathcal{X}_c) \cap \mathcal{A}_\mathcal{L}. \tag{5.22c}$$

If J enjoys extra regularity, namely

$$J(z,\ell) = \Psi_1(z,\ell) + \Psi_2(z(T),\ell(T)) + \Phi(\ell) \tag{5.23}$$

with two Fréchet-differentiable functionals $\Psi_1 : L^2(\mathcal{H}) \times H^1(\mathcal{X}_c) \to \mathbb{R}$ and $\Psi_2 : \mathcal{H} \times \mathcal{X}_c \to \mathbb{R}$, then $\varphi \in H^1(\mathcal{H})$ is a strong solution of

$$\begin{aligned} \dot{\varphi}(t) &= \big(QA_s'(R\overline{\ell} - Q\overline{z})^*\varphi\big)(t) + \Psi_{1,z}'(\overline{z}, \overline{\ell}), \\ \varphi(T) &= -\Psi_{2,z}'(\overline{z}(T), \overline{\ell}(T)). \end{aligned} \tag{5.24}$$

Proof. Since $J_z'(\overline{z}, \overline{\ell}) \in H^1(\mathcal{W})^* \hookrightarrow H^1(\mathcal{H})^*$, Lemma 5.12 gives the existence of a unique solution of (5.22b). Now, let $h \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$ be arbitrary and define $\eta := \mathcal{S}_s'(\overline{\ell})h \in H^1(\mathcal{Z}) \subset H^1(\mathcal{H})$. The weak form of the adjoint equation then implies

$$\Big(\varphi, A_s'(R\overline{\ell} - Q\overline{z})Rh\Big)_{L^2(\mathcal{H})} = \Big(\varphi, \dot{\eta} + A_s'(R\overline{\ell} - Q\overline{z})Q\eta\Big)_{L^2(\mathcal{H})} = -J_z'(\mathcal{S}_s(\overline{\ell}), \overline{\ell})\eta. \tag{5.25}$$

This together with Lemma 5.10 shows that $(\overline{z}, \overline{\ell}, \varphi)$ fulfills the optimality system (5.22). If J is of the form in (5.23), then Lemma 5.12 implies that the weak solution of the adjoint equation is in fact a strong solution and solves (5.24). □

As the following corollary shows, the existence of an adjoint state is not only necessary for (5.16), but also sufficient.

Corollary 5.14 (KKT equivalence)**.** *The function $\overline{\ell} \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$ with associated state $\overline{z} = \mathcal{S}_s(\overline{\ell})$ fulfills* (5.16) *if and only if there exists an adjoint state $\varphi \in L^2(\mathcal{H})$ such that $(\overline{z}, \overline{\ell}, \varphi)$ satisfies the optimality system* (5.22).

Proof. The proof of Theorem 5.13 already shows that (5.16) implies the optimality system in (5.22).

To prove the reverse implication, assume that $(\overline{z}, \overline{\ell}, \varphi)$ fulfills the optimality system (5.22). Then choose an arbitrary $h \in H^1(\mathcal{H})$, define $\eta := \mathcal{S}_s'(\overline{\ell})h$, and use the fact that φ is the weak solution of (5.22b) to obtain (5.25). This together with (5.22c) finally give (5.16). □

Example 5.15 (Specified objective function)**.** *Under suitable additional assumptions, it is possible to further simplify the gradient equation* (5.22c)*. For this purpose assume that R is injective (so that $\mathcal{A}_{\mathcal{L}} = \{\ell \in H^1(0,T,\mathcal{X}) : \ell(0) = 0\}$), $\mathcal{X}_c$ is a Hilbert space, and*

$$J(z,\ell) = \Psi_1(z,\ell) + \Psi_2(z(T), \ell(T)) + \frac{\alpha}{2}\|\dot{\ell}\|^2_{L^2(\mathcal{X}_c)}, \tag{5.26}$$

where $\Psi_1 : H^1(\mathcal{W}) \times L^2(0,T;\mathcal{X}_c) \to \mathbb{R}$ and $\Psi_2 : \mathcal{W} \times \mathcal{X}_c \to \mathbb{R}$ are Fréchet-differentiable and $\alpha > 0$. This type of objective will also appear in the application problem in Chapter 6. Then (5.22c) *becomes*

$$\begin{aligned} \alpha(\partial_t\overline{\ell}, \partial_t h)_{L^2(\mathcal{X}_c)} - \int_0^T \langle R^* A_s'(R\overline{\ell} - Q\overline{z})^*\varphi, h\rangle_{\mathcal{X}^*,\mathcal{X}} dt& \\ + \int_0^T \Psi_{1,\ell}'(\overline{z}, \overline{\ell})h\,dt + \Psi_{2,\ell}'(\overline{z}(T), \overline{\ell}(T))h(T) = 0& \\ \forall h \in H^1(\mathcal{X}_c) \text{ with } h(0) = 0,& \end{aligned} \tag{5.27}$$

where we identified $\partial_\ell\Psi_1(\overline{z}, \overline{\ell}) \in (L^2(\mathcal{X}_c))^* = L^2(\mathcal{X}_c)$ (note that $\mathcal{X}_c$ as a Hilbert space satisfies the Radon-Nikodým-property). Since $\mathcal{X}_c \hookrightarrow \mathcal{X}$, we may identify $R^*A_s'(R\overline{\ell} - Q\overline{z})^*\varphi$ with an element of $L^2(\mathcal{X}_c)$, too, which we denote by the same symbol. Then, if we choose $h(t) = \psi(t)\,\xi$ with $\psi \in C_c^\infty[0,T]$ and $\xi \in \mathcal{X}_c$ arbitrary, we obtain

$$\Big(-\int_0^T \Big[\alpha\partial_t\psi\partial_t\overline{\ell} + R^*A_s'(R\overline{\ell} - Q\overline{z})^*\varphi\psi - \Psi_{1,\ell}'(\overline{z}, \overline{\ell})\psi\Big]dt, \xi\Big)_{\mathcal{X}_c} = 0.$$

Now, since $\xi \in \mathcal{X}_c$ was arbitrary, we find that the second distributional time derivative of $\overline{\ell}$ is a regular distribution in $L^2(\mathcal{X}_c)$, i.e., $\overline{\ell} \in H^2(0,T;\mathcal{X}_c)$, satisfying for almost all $t \in [0,T]$

$$\alpha\partial_t^2\overline{\ell}(t) + R^*A_s'\big(R\overline{\ell}(t) - Q\overline{z}(t)\big)^*\varphi(t) = \Psi_{1,\ell}'(\overline{z}, \overline{\ell})(t) \quad \text{in } \mathcal{X}_c. \tag{5.28}$$

Since $\mathcal{X}_c$ is supposed to be a Hilbert space, we can apply integration by parts to (5.27). *Together with* $\overline{\ell} \in \mathcal{A}_{\mathcal{L}} = \{\ell \in H^1(\mathcal{X}) : \ell(0) = 0\}$ *and* (5.28), *this implies the following boundary conditions:*

$$\overline{\ell}(0) = 0, \quad \alpha \partial_t \overline{\ell}(T) = -\Psi'_{2,\ell}(\overline{z}(T), \overline{\ell}(T)), \tag{5.29}$$

where we again identified $\partial_\ell \Psi_2(\overline{z}(T), \overline{\ell}(T)) \in \mathcal{X}_c^*$ with its Riesz representative. In summary, we have thus seen that the gradient equation in (5.22c) *becomes an operator boundary value problem in $\mathcal{X}_c$, namely* (5.28)–(5.29).

5.3 Second-Order Sufficient Conditions

Having derived first order optimality conditions in Theorem 5.13, we want to go further and strive for second order sufficient conditions. The approach in this section is analog to the one in the last section, we first tend to the Lipschitz continuity and the differentiability of $\mathcal{S}_s'$ and then turn to the desired optimality conditions.

Lemma 5.16 (Lipschitz continuity of $\mathcal{S}_s'$)**.** *The derivative $\mathcal{S}_s'$ is Lipschitz continuous from $H^1(\mathcal{X})$ to $L(H^1(\mathcal{X}); H^1(\mathcal{Z}))$.*

Proof. Let $\ell_1, \ell_2, h \in H^1(\mathcal{X})$ be arbitrary and abbreviate

$$z_i := \mathcal{S}_s(\ell_i), \quad \eta_i := \mathcal{S}_s'(\ell_i)h, \quad \text{and} \quad y_i := R\ell_i - Qz_i, \quad i = 1, 2.$$

Using the first Lipschitz-assumption in Assumption ⟨5.3.ii⟩, we deduce for almost all $t \in [0, T]$ that

$$\begin{aligned}
&\|\dot\eta_1(t) - \dot\eta_2(t)\|_{\mathcal{Z}} \\
&\quad = \|A_s'(y_1(t))(Rh(t) - Q\eta_1(t)) - A_s'(y_2(t))(Rh(t) - Q\eta_2(t))\|_{\mathcal{Z}} \\
&\quad = \|\big(A_s'(y_1(t)) - A_s'(y_2(t))\big)(Rh(t) - Q\eta_1(t)) + A_s'(y_2(t))Q(\eta_1(t) - \eta_2(t))\|_{\mathcal{Z}} \\
&\quad \le C\big(\|y_1(t) - y_2(t)\|_Y \|Rh(t) - Q\eta_1(t)\|_{\mathcal{Z}} + \|\eta_1(t) - \eta_2(t)\|_{\mathcal{Z}}\big).
\end{aligned}$$

Lemma A.8 together with the definition of y_1 and y_2 yields

$$\begin{aligned}
\|\eta_1 - \eta_2\|_{H^1(\mathcal{Z})} &\le C\|R(\ell_1 - \ell_2) - Q(z_1 - z_2)\|_{L^2(\mathcal{Y})}\|Rh - Q\eta_1\|_{H^1(\mathcal{Z})} \\
&\le C\|\ell_1 - \ell_2\|_{L^2(\mathcal{X})}\|h\|_{H^1(\mathcal{X})},
\end{aligned}$$

where we used the continuous embedding $H^1(\mathcal{Z}) \hookrightarrow C(\mathcal{Z})$ (see [104, Theorem 3.1.41]), Lemma 5.7 and the estimate in Theorem 5.9. □

The proof of the following proposition is analog to the one of Theorem 5.9. As it was already the case for Theorem 5.9, the proof is rather technical and uses only fundamental mathematical concepts such as Gronwall's inequality and Lebesgues dominated convergence theorem.

Proposition 5.17 (Second derivative of the solution operator)**.** *The solution operator $\mathcal{S}_s : H^1(\mathcal{X}) \to H^1(\mathcal{W})$ is twice Fréchet differentiable. Given $\ell, h_1, h_2 \in H^1(\mathcal{X})$, its second derivative $\mathcal{S}_s''(\ell)[h_1, h_2] \in H^1(\mathcal{W})$ is given by the unique solution of*

$$\dot\xi = A_s''(R\ell - Qz)[Rh_1 - Q\eta_1, Rh_2 - Q\eta_2] - A_s'(R\ell - Qz)Q\xi, \quad \xi(0) = 0, \tag{5.30}$$

where $z := \mathcal{S}_s(\ell) \in H^1(\mathcal{Y})$ and $\eta_i := \mathcal{S}_s'(\ell)h_i \in H^1(\mathcal{Z})$, $i = 1, 2$.

Moreover, there exists a constant C such that

$$\|\mathcal{S}_s''(\ell)[h_1, h_2]\|_{H^1(\mathcal{W})} \le C\|h_1\|_{H^1(\mathcal{X})}\|h_2\|_{H^1(\mathcal{X})} \tag{5.31}$$

for all $\ell, h_1, h_2 \in H^1(\mathcal{X})$.

Proof. Let $\ell, h_1, h_2 \in H^1(\mathcal{X})$ be arbitrary and define $z := \mathcal{S}_s(\ell)$, $z_1 := \mathcal{S}_s(\ell + h_1)$, $\eta_i := \mathcal{S}_s'(\ell)h_i \in H^1(\mathcal{Z})$, $i \in \{1,2\}$, and $\eta_{1,2} := \mathcal{S}_s'(\ell + h_1)h_2$.

We first address the existence of solutions to (5.30). We argue similarly to Lemma 5.8 and set

$$w\colon [0,T] \to \mathcal{W}, \qquad t \mapsto A_s''(R\ell(t) - Qz(t))[Rh_1(t) - Q\eta_1(t), Rh_2(t) - Q\eta_2(t)].$$

From the estimate in Assumption ⟨5.3.iii⟩ it follows that

$$\|w(t)\|_{\mathcal{W}} \le C\|Rh_1(t) - Q\eta_1(t)\|_{\mathcal{Z}}\|Rh_2(t) - Q\eta_2(t)\|_{\mathcal{Z}},$$

and, since w is Bochner measurable, which can be shown as in Lemma 5.8, we obtain $w \in L^2(\mathcal{W})$. Since $A_s'(y)$ is assumed to be bounded in $\mathcal{W}$ by Assumption ⟨5.3.ii⟩, we can now follow the proof of Lemma 5.8 (with $\mathcal{W}$ instead of $\mathcal{Z}$) to deduce the existence of a unique solution $\xi \in H^1(\mathcal{W})$ of (5.30). The (bi-)linearity of the associated solution operator w.r.t. h_1 and h_2 is straightforward to see. For its continuity, we calculate

$$\|\dot{\xi}(t)\|_{\mathcal{W}} \le C\|Rh_1(t) - Q\eta_1(t)\|_{\mathcal{Z}}\|Rh_2(t) - Q\eta_2(t)\|_{\mathcal{Z}} + C\|\xi(t)\|_{\mathcal{W}}$$

so that Lemma A.8, the continuous embedding $H^1(\mathcal{X}) \hookrightarrow C(\mathcal{X})$ (see [104, Theorem 3.1.41]) and the estimate in Theorem 5.9 give

$$\begin{aligned}
\|\xi\|_{H^1(\mathcal{W})} &\le C\| \, \|Rh_1 - Q\eta_1\|_{\mathcal{Z}}\|Rh_2 - Q\eta_2\|_{\mathcal{Z}} \, \|_{L^2(0,T;\mathbb{R})} \\
&\le C\|Rh_1 - Q\eta_1\|_{H^1(\mathcal{Z})}\|Rh_2 - Q\eta_2\|_{H^1(\mathcal{Z})} \\
&\le C\|h_1\|_{H^1(\mathcal{X})}\|h_2\|_{H^1(\mathcal{X})}.
\end{aligned}$$

This shows also (5.31) (after having proved that $\xi = \mathcal{S}_s''(\ell)[h_1, h_2]$).

It only remains to prove the remainder term property. To this end, we define

$$y := R\ell - Qz, \qquad \zeta := Rh_1 - Q(z_1 - z).$$

Then, the equations for $\eta_{1,2}$, η_2, and ξ lead to

$$\begin{aligned}
\dot{\eta}_{1,2} - \dot{\eta}_2 - \dot{\xi} &= A_s'(y+\zeta)(Rh_2 - Q\eta_{1,2}) - A_s'(y)(Rh_2 - Q\eta_2) \\
&\quad - A_s''(y)[Rh_1 - Q\eta_1, Rh_2 - Q\eta_2] + A_s'(y)Q\xi \\
&= \big(A_s'(y+\zeta) - A_s'(y)\big)(Rh_2 - Q\eta_{1,2}) \\
&\quad - A_s''(y)[Rh_1 - Q\eta_1, Rh_2 - Q\eta_2] - A_s'(y)Q(\eta_{1,2} - \eta_2 - \xi) \\
&= A_s''(y)[\zeta, Rh_2 - Q\eta_{1,2}] + r_2(y;\zeta)(Rh_2 - Q\eta_{1,2}) \\
&\quad - A_s''(y)[Rh_1 - Q\eta_1, Rh_2 - Q\eta_2] - A_s'(y)Q(\eta_{1,2} - \eta_2 - \xi) \\
&= A_s''(y)[\zeta, Q(\eta_2 - \eta_{1,2})] - A_s''(y)[Q(z_1 - z - \eta_1), Rh_2 - Q\eta_2] \\
&\quad + r_2(y;\zeta)(Rh_2 - Q\eta_{1,2}) - A_s'(y)Q(\eta_{1,2} - \eta_2 - \xi),
\end{aligned}$$

where $r_2(y;\zeta) := A_s'(y+\zeta) - A_s'(y) - A_s''(y)\zeta \in L^2(L(\mathcal{Z};\mathcal{W}))$ denotes the corresponding remainder term. The estimate in Assumption ⟨5.3.iii⟩ thus implies

$$\begin{aligned}
&\|\dot{\eta}_{1,2}(t) - \dot{\eta}_2(t) - \dot{\xi}(t)\|_{\mathcal{W}} \\
&\le C\big(\|\zeta(t))\|_{\mathcal{Z}}\|\eta_2(t) - \eta_{1,2}(t)\|_{\mathcal{Z}} + \|z_1(t) - z(t) - \eta_1(t)\|_{\mathcal{Z}}\|Rh_2(t) - Q\eta_2(t)\|_{\mathcal{Z}} \\
&\qquad + \|r_2(y(t),\zeta(t))\|_{L(\mathcal{Z};\mathcal{W})}\|Rh_2(t) - Q\eta_{1,2}(t)\|_{\mathcal{Z}} + \|\eta_{1,2}(t) - \eta_2(t) - \xi(t)\|_{\mathcal{W}}\big)
\end{aligned}$$

for almost all $t \in [0,T]$, such that Lemma A.8 Item (iii) yields

$$
\begin{aligned}
&\|\eta_{1,2} - \eta_2 - \xi\|_{H^1(\mathcal{W})} \\
&\leq C\big(\|Rh_1 - Q(z_1 - z)\|_{L^\infty(0,T;\mathcal{Z})}\|\eta_2 - \eta_{1,2}\|_{L^2(\mathcal{Z})} \\
&\qquad + \|z_1 - z - \eta_1\|_{L^\infty(0,T;\mathcal{Z})}\|Rh_2 - Q\eta_2\|_{L^2(\mathcal{Z})} \\
&\qquad + \|r_2(y;\zeta)\|_{L^2(L(\mathcal{Z};\mathcal{W}))}\|Rh_2 - Q\eta_{1,2}\|_{H^1(\mathcal{Z})}\big) \\
&\leq C\|h_2\|_{H^1(\mathcal{X})}\big(\|h_1\|^2_{H^1(\mathcal{X})} + \|z_1 - z - \eta_1\|_{H^1(\mathcal{Z})} + \|r_2(y;\zeta)\|_{L^2(L(\mathcal{Z};\mathcal{W}))}\big),
\end{aligned}
$$

where we used Lemma 5.7, Lemma 5.16 and the estimate in Theorem 5.9. Denoting the solution operator of (5.30) already by $\mathcal{S}_s''(\ell)[h_1,h_2]$, we have thus shown

$$
\begin{aligned}
&\|\mathcal{S}_s'(\ell + h_1) - \mathcal{S}_s'(\ell) - \mathcal{S}_s''(\ell)h_1\|_{L(H^1(\mathcal{X});H^1(\mathcal{W}))} \\
&\qquad \leq C\big(\|h_1\|^2_{H^1(\mathcal{X})} + \|\mathcal{S}_s(\ell + h_1) - \mathcal{S}_s(\ell) - \mathcal{S}_s'(\ell)h_1\|_{H^1(\mathcal{Z})} \\
&\qquad\qquad + \|r_2(y;\zeta)\|_{L^2(L(\mathcal{Z};\mathcal{W}))}\big).
\end{aligned}
$$

Therefore, thanks to the Fréchet-differentiability of $\mathcal{S}_s : H^1(\mathcal{X}) \to H^1(\mathcal{Z})$, it only remains to show that

$$
\frac{\|r_2(y;\zeta)\|_{L^2(L(\mathcal{Z};\mathcal{W}))}}{\|h_1\|_{H^1(\mathcal{X})}} \to 0, \tag{5.32}
$$

as $0 \neq h_1 \to 0$ in $H^1(\mathcal{X})$. To this end, we note that the embedding $H^1(\mathcal{Y}) \hookrightarrow C(\mathcal{Y})$ and Lemma 5.7 yield for all $t \in [0,T]$

$$
\frac{\|\zeta(t)\|_{\mathcal{Y}}}{\|h_1\|_{H^1(\mathcal{X})}} \leq C\frac{\|\zeta\|_{H^1(\mathcal{Y})}}{\|h_1\|_{H^1(\mathcal{X})}} = C\frac{\|Rh_1 - Q(z_1 - z)\|_{H^1(\mathcal{Y})}}{\|h_1\|_{H^1(\mathcal{X})}} \leq C \tag{5.33}
$$

Hence, thanks to the Fréchet-differentiability of $A_s' : \mathcal{Y} \to L(\mathcal{Z};\mathcal{W})$, we have for almost all $t \in [0,T]$

$$
\frac{\|r_2(y;\zeta)(t)\|_{L(\mathcal{Z};\mathcal{W})}}{\|h_1\|_{H^1(\mathcal{X})}} \leq C\frac{\|r_2(y;\zeta)(t)\|_{L(\mathcal{Z};\mathcal{W})}}{\|\zeta(t)\|_{\mathcal{Y}}} \to 0
$$

as $0 \neq h_1 \to 0$ in $H^1(\mathcal{X})$. Furthermore, using the Lipschitz continuity of $A_s' : \mathcal{Y} \to L(\mathcal{Z};\mathcal{Z})$, the estimate for A_s'' in Assumption ⟨5.3.iii⟩ and again (5.33), we deduce

$$
\begin{aligned}
&\frac{\|r_2(y;\zeta)(t)\|_{L(\mathcal{Z};\mathcal{W})}}{\|h_1\|_{H^1(\mathcal{X})}} \\
&\quad = \frac{\|A_s'(y(t) + \zeta(t)) - A_s'(y(t)) - A_s''(y(t))\zeta(t)\|_{L(\mathcal{Z};\mathcal{W})}}{\|h_1\|_{H^1(\mathcal{X})}} \leq C\frac{\|\zeta(t)\|_{\mathcal{Y}}}{\|h_1\|_{H^1(\mathcal{X})}} \leq C
\end{aligned}
$$

for almost all $t \in [0,T]$. The convergence in (5.32) now follows from Lebesgue's dominated convergence theorem (see [104, Theorem 3.1.29]). □

In Theorem 5.21 we will use a general result from the literature to derive second order optimality conditions, for this result we need the continuity of the second derivative. A first step in this direction provides

Lemma 5.18 (Continuity estimate of $\mathcal{S}_s''$)**.** *There exists a constant C such that*

$$
\begin{aligned}
&\|\mathcal{S}_s''(\ell_1) - \mathcal{S}_s''(\ell_2)\|_{L(H^1(\mathcal{X});L(H^1(\mathcal{X});H^1(\mathcal{W})))} \\
&\leq C\big(\|A_s''(R\ell_1 - Qz_1) - A_s''(R\ell_2 - Qz_2)\|_{L^2(L(\mathcal{Z};L(\mathcal{Z};\mathcal{W})))} + \|\ell_1 - \ell_2\|_{H^1(\mathcal{X})}\big)
\end{aligned}
$$

holds for all $\ell_1, \ell_2 \in H^1(\mathcal{X})$, where $z_i := \mathcal{S}_s(\ell_i)$, $i = 1,2$.

Proof. Let $\ell_1, \ell_2, h_1, h_2 \in H^1(\mathcal{X})$ be arbitrary. We again abbreviate $z_i := \mathcal{S}_s(\ell_i)$, $\eta_{i,j} := \mathcal{S}_s'(\ell_i)h_j$, $\xi_i := \mathcal{S}_s''(\ell_i)[h_1, h_2]$, and $y_i := R\ell_i - Qz_i$ for $i, j \in \{1, 2\}$. By the equation for $\mathcal{S}_s''$, we obtain for almost all $t \in [0, T]$

$$\begin{aligned}\dot{\xi}_1 - \dot{\xi}_2 &= A_s''(y_1)[Rh_1 - Q\eta_{1,1}, Rh_2 - Q\eta_{1,2}] - A_s'(y_1)Q\xi_1 \\ &\quad - A_s''(y_2)[Rh_1 - Q\eta_{2,1}, Rh_2 - Q\eta_{2,2}] - A_s'(y_2)Q\xi_2 \\ &= \big(A_s''(y_1)(Rh_1 - Q\eta_{1,1}) - A_s''(y_2)(Rh_1 - Q\eta_{2,1})\big)(Rh_2 - Q\eta_{1,2}) \\ &\quad + A_s''(y_2)[Rh_1 - Q\eta_{2,1}, Q(\eta_{2,2} - \eta_{1,2})] \\ &\quad + \big(A_s'(y_2) - A_s'(y_1)\big)Q\xi_1 + A_s'(y_2)Q(\xi_2 - \xi_1).\end{aligned}$$

With the help of

$$\begin{aligned}A_s''(y_1)(Rh_1 - Q\eta_{1,1}) &- A_s''(y_2)(Rh_1 - Q\eta_{2,1}) \\ &= \big(A_s''(y_1) - A_s''(y_2)\big)(Rh_1 - Q\eta_{1,1}) + A_s''(y_2)Q(\eta_{2,1} - \eta_{1,1}),\end{aligned}$$

we obtain

$$\begin{aligned}\|\dot{\xi}_1(t) - \dot{\xi}_2(t)\|_{\mathcal{W}} \leq C\Big(&\|A_s''(y_1(t)) - A_s''(y_2(t))\|_{L(\mathcal{Z};L(\mathcal{Z};\mathcal{W}))}\|Rh_1(t) - Q\eta_{1,1}(t)\|_{\mathcal{Z}} \\ &+ \|\eta_{1,1}(t) - \eta_{2,1}(t)\|_{\mathcal{Z}}\Big)\|Rh_2(t) - Q\eta_{1,2}(t)\|_{\mathcal{Z}} \\ &+ C\|Rh_1(t) - Q\eta_{2,1}(t)\|_{\mathcal{Z}}\|\eta_{1,2}(t) - \eta_{2,2}(t)\|_{\mathcal{Z}} \\ &+ C\|y_1(t) - y_2(t)\|_{\mathcal{Y}}\|\xi_1(t)\|_{\mathcal{W}} + C\|\xi_1(t) - \xi_2(t)\|_{\mathcal{W}}.\end{aligned}$$

Therefore we can use once again Lemma A.8 Item (iii) and the continuous embedding $H^1(\mathcal{Z}) \hookrightarrow C(\mathcal{Z})$ to arrive at

$$\begin{aligned}\|\xi_1 - \xi_2\|_{H^1(\mathcal{W})} &\leq C\Big[\|Rh_1 - Q\eta_{2,1}\|_{H^1(\mathcal{Z})}\|\eta_{1,2} - \eta_{2,2}\|_{H^1(\mathcal{Z})} + \|y_1 - y_2\|_{H^1(\mathcal{Y})}\|\xi_1\|_{H^1(\mathcal{W})} \\ &\qquad + \Big(\|A_s''(y_1) - A_s''(y_2)\|_{L^2(L(\mathcal{Z};L(\mathcal{Z};\mathcal{W})))}\|Rh_1 - Q\eta_{1,1}\|_{H^1(\mathcal{Z})} \\ &\qquad + \|\eta_{1,1} - \eta_{2,1}\|_{H^1(\mathcal{Z})}\Big)\|Rh_2 - Q\eta_{1,2}\|_{H^1(\mathcal{Z})}\Big] \\ &\leq C\Big(\|A_s''(y_1) - A_s''(y_2)\|_{L^2(L(\mathcal{Z};L(\mathcal{Z};\mathcal{W})))} + \|\ell_1 - \ell_2\|_{H^1(\mathcal{X})}\Big)\|h_1\|_{H^1(\mathcal{X})}\|h_2\|_{H^1(\mathcal{X})},\end{aligned}$$

where we also used the estimate in Theorem 5.9, (5.31), and the Lipschitz continuity of $\mathcal{S}_s'$ according to Lemma 5.16. □

Clearly, if A_s'' were Lipschitz continuous from $\mathcal{Y}$ to $L(\mathcal{Z}; L(\mathcal{Z}; \mathcal{W}))$, then Lemma 5.18 would immediately imply the Lipschitz continuity of $\mathcal{S}_s''$. However, to obtain the continuity of the second derivative, this additional assumption is not necessary as the following theorem shows.

Theorem 5.19 (Second-order continuous Fréchet-differentiability of the solution operator)**.** *The solution operator $\mathcal{S}_s : H^1(\mathcal{X}) \to H^1(\mathcal{W})$ is twice continuously Fréchet-differentiable. Its second derivative at $\ell \in H^1(\mathcal{X})$ in directions $h_1, h_2 \in H^1(\mathcal{X})$ is given by the unique solution of* (5.30).

Proof. Thanks to Proposition 5.17, we only have to show that the operator $\mathcal{S}_s'' : H^1(\mathcal{X}) \to L(H^1(\mathcal{X}); L(H^1(\mathcal{X}); H^1(\mathcal{W})))$ is continuous. For this let $\{\ell_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{X})$ and $\ell \in H^1(\mathcal{X})$ be given such that $\ell_n \to \ell$ in $H^1(\mathcal{X})$ so that in particular $\ell_n \to \ell$ in $C(\mathcal{X})$ (see [104, Theorem 3.1.41]). Then, Lemma 5.7 implies $z_n := \mathcal{S}_s(\ell_n) \to \mathcal{S}_s(\ell) =: z$ in $C(\mathcal{Y})$. With this convergence results at hand, we can apply Lemma A.2 with $M = [0,T]$, $N = \mathcal{Y}$, $G_n = R\ell_n - Qz_n$ and $G = R\ell - Qz$ to see that

$$U := \Big(\bigcup_{n=1}^{\infty}(R\ell_n - Qz_n)([0,T])\Big) \cup \Big((R\ell - Qz)([0,T])\Big)$$

is compact. Therefore, thanks to the continuity assumption in Assumption ⟨5.3.iii⟩, $A_s'' : \mathcal{Y} \to L(\mathcal{Z}; L(\mathcal{Z}; \mathcal{W}))$ is uniformly continuous on U. Consequently, $A_s''(R\ell_n - Qz_n)$ converges to $A_s''(R\ell - Qz)$ in $C(L(\mathcal{Z}; L(\mathcal{Z}; \mathcal{W})))$, which, together with Lemma 5.18, yields the assertion. □

Remark 5.20 (Solutions are more regular). *It is to be noted that the regularized state equation* (EVI_s) *and the equations corresponding to the derivatives of $\mathcal{S}_s$, i.e.,* (5.11) *and* (5.30)*, provide more regular solutions under the hypothesis of Assumption ⟨5.2.iv⟩ and Assumption ⟨5.3.ii⟩ & Assumption ⟨5.3.iii⟩. Indeed, if $\ell, h_1, h_2 \in H^1(\mathcal{X})$, then the solutions of all three equations can be shown to be continuously differentiable in time with values in the respective spaces ($\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{W}$, respectively). Moreover, the time derivatives of z and η are absolutely continuous and the same would hold for ξ, if A_s'' were Lipschitz continuous. However, we did not exploit this additional regularity, since the original unregularized problem* (5.2) *does not provide this property in general.*

Theorem 5.21 (Second-order sufficient optimality conditions for (5.8)). *Let $(\overline{z}, \overline{\ell}, \varphi) \in H^1(\mathcal{Y}) \times (H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}) \times L^2(\mathcal{H})$ be a solution of the optimality system* (5.22)*. Moreover, suppose that there is a $\delta > 0$ such that*

$$F''(\overline{\ell})h^2 \geq \delta \|h\|^2_{H^1(\mathcal{X}_c)} \tag{5.34}$$

for all $h \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$, where F is the reduced objective function from (5.15)*. Then $(\overline{z}, \overline{\ell})$ is locally optimal for* (5.8) *and there exist $\varepsilon > 0$ and $\tau > 0$ such that the following* quadratic growth condition

$$F(\ell) \geq F(\overline{\ell}) + \tau \|\ell - \overline{\ell}\|^2_{H^1(\mathcal{X}_c)} \tag{5.35}$$

holds for all $\ell \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$ with $\|\ell - \overline{\ell}\|_{H^1(\mathcal{X}_c)} \leq \varepsilon$.

Proof. Thanks to the assumptions on J and Theorem 5.19, the chain rule implies that the reduced objective function $F(\cdot) = J(\mathcal{S}_s(\cdot), \cdot) : H^1(\mathcal{X}_c) \to \mathbb{R}$ is twice continuously Fréchet-differentiable and, according to Corollary 5.14, the equation in (5.16) holds for all $h \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$. Since $\mathcal{A}_{\mathcal{L}}$ is a linear subspace, the claim then follows from standard arguments, see e.g. [100, Satz 4.23]. □

Remark 5.22 (Second-order necessary optimality conditions for (5.8)). *If a control $\overline{\ell} \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$ with associated state $\overline{z} = \mathcal{S}_s(\overline{\ell})$ is locally optimal for* (5.8)*, then*

$$F''(\overline{\ell})h^2 \geq 0$$

must hold for all $h \in H^1(\mathcal{X}_c) \cap \mathcal{A}_{\mathcal{L}}$. This follows also from standard arguments, see for instance [100, Satz 4.27].

Corollary 5.23 (Specified form of F''). *Assume in addition to the hypothesis of Assumption ⟨5.3.iii⟩ that $\|A_s''(y)[z_1, z_2]\|_{\mathcal{H}} \leq C\|z_1\|_{\mathcal{Z}}\|z_2\|_{\mathcal{Z}}$ for all $y \in \mathcal{Y}$ and $z_1, z_2 \in \mathcal{Z}$, i.e., the last inequality in Assumption ⟨5.3.iii⟩ holds in $\mathcal{H}$ instead of the (possible) weaker space $\mathcal{W}$. Then it holds for all $\ell, h \in H^1(\mathcal{H})$ that*

$$F''(\ell)h^2 = \Psi''(z, \ell)(\eta, h)^2 + \Phi''(\ell)h^2 - \big(\varphi, A_s''(R\ell - Qz)(Rh - Q\eta)^2\big)_{L^2(\mathcal{H})},$$

where $z = \mathcal{S}_s(\ell)$, $\eta = \mathcal{S}_s'(\ell)h$, and φ solves the adjoint equation in (5.22b).

Proof. Let us again abbreviate $y = R\ell - Qz$. According to the chain rule, the second derivative of the reduced objective is given by

$$\begin{aligned} F''(\ell)h^2 &= J''_{\ell\ell}(z,\ell)h^2 + J''_{zz}(z,\ell)\eta^2 + 2J''_{\ell z}(z,\ell)[h,\eta] + J'_z(z,\ell)\xi \\ &= \Psi''(z,\ell)(\eta,h)^2 + \Phi''(\ell)h^2 + J'_z(z,\ell)\xi \end{aligned}$$

with $z = \mathcal{S}_s(\ell)$, $\eta = \mathcal{S}'_s(\ell)h$, and $\xi = \mathcal{S}''_s(\ell)h^2$. Now, since $A''_s(y)$ is a bilinear form on $\mathcal{H}$ by assumption, we obtain that $\xi \in H^1(\mathcal{H})$. Therefore, we are allowed to test the adjoint equation in (5.22b) (in its weak form) with ξ, which results in

$$J'_z(z,\ell)\xi = -\left(\varphi, \dot{\xi} + A'_s(y)Q\xi\right)_{L^2(\mathcal{H})} = -\left(\varphi, A''_s(y)(Rh - Q\eta)^2\right)_{L^2(\mathcal{H})},$$

where we used the precise form of $\mathcal{S}''_s(\ell)$ in (5.30) for the last identity. □

Remark 5.24 (Coercivity of Q)**.** *For the analysis of the original equation* (EVI) *the coercivity of Q was vital (see Section 4.1), however, we never used it in Section 5.2 and this section, so that all findings in these two sections still hold when we drop the coercivity of Q.*

With the result concerning the second order sufficient optimality conditions in Theorem 5.21 we arrived at the end of this part of the thesis. It was devoted to the analysis and optimal control of (EVI) with the main results given in Theorem 5.2 (existence of global minimizers), Theorem 5.4 & Corollary 5.5 (approximation of global minimizers), Theorem 5.13 (first order optimality conditions) and Theorem 5.21 (second order optimality conditions). As been said in the introduction to Chapter 5, this part and in particular Chapter 5 plays a central role in the thesis at hand. In what follows, we will use the mentioned and more results from this part to analyze the optimal control of elasto and homogenized plasticity, plasticity with inertia and perfect plasticity. To apply the analysis from this part to these different plasticity systems, it was held abstract and general. This stays in contrast to the following parts, where we consider mostly specific spaces and operators.

We continue with elasto plasticity followed by the similar (with respect to the transformation into an EVI) homogenized plasticity.

Part III Elasto and Homogenized Plasticity

In Chapter 6 we use the observations from Chapter 2 and apply the findings from Part II. In Chapter 7 we transform the system of *homogenized plasticity* into an EVI as we did in Chapter 2 for elasto plasticity. In both cases, the necessary assumptions and requirements are presented throughout the corresponding chapter, therefore no assumptions are needed in the assumption collection agreed upon in The Assumption Agreement for this part.

Assumption III. *All assumptions are presented throughout Chapter 6 and Chapter 7, respectively.*

Both cases, the elasto plasticity in Chapter 6 and homogenized plasticity in Chapter 7, fit perfectly into the general setting analyzed in Part II, which is the reason for presenting it before plasticity with inertia and perfect plasticity. Due to this fact, there is only little analysis required in this part, so it as rather short in comparison to the last one. We start with the continuation of Chapter 2.

Chapter 6 Elasto Plasticity

Since we have already presented the equations of elasto plasticity in Section 2.1 and transformed them into an EVI in Section 2.2, we can continue straightaway with the optimal control problem.

6.1 Optimal Control Problem

The concrete optimization problem we consider reads as follows

$$
\left\{
\begin{aligned}
\min \quad & J(z, f, g, u_D) = \frac{1}{2}\|z(T) - z_d\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{\alpha}{2}\|(\dot{f}, \dot{g}, \dot{u}_D)\|^2_{L^2(L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_D;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d))}, \\
\text{s.t.} \quad & -\operatorname{div}\sigma = f + g \\
& u - u_D \in H^1_D(\Omega;\mathbb{R}^{d\times d}_s) \\
& \sigma = \mathbb{C}(\nabla^s u - z) \\
& \dot{z} \in \partial I_{\mathcal{K}(\Omega)}(\sigma - \mathbb{B}z), \qquad z(0) = 0 \\
& (u, \sigma, z) \in H^1(H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)) \\
& (f, g, u_D) \in H^1_0(L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_D;\mathbb{R}^d) \times H^2(\Omega;\mathbb{R}^d)),
\end{aligned}
\right.
\tag{6.1}
$$

where $z_d \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$ is a desired state and $\alpha > 0$ is the Tikhonov parameter.

Let us shortly comment on our choice of this optimization problem. At first it is to be noted that we have chosen the plastic strain z as the state variable. Clearly, one might want to control the stress or the displacement, however, these can be easily added as states in (6.1), one only has to be a bit more careful when it comes to the transformation for example in the proof of Theorem 6.3. We have only chosen the plastic strain as the only state variable because then the results from Part II can be applied more directly, which keeps the discussion more focused on the essentials. We note that an optimal control problem with the displacement and the (two scale) stress of homogenized plasticity as states was considered in MEINLSCHMIDT ET AL. [71, Section 7.3]. Secondly, we have chosen $H_0^1(H^2(\Omega;\mathbb{R}^d))$ as the control space for the Dirichlet displacement to obtain a compact embedding into $L^1(H^1(\Omega;\mathbb{R}^d))$, as is necessary to apply some results from Part II, see Assumption ⟨5.iv⟩. This might be undesirable, for example for numerical experiments, since then the $H^2(\Omega;\mathbb{R}^d)$-norm appears in the objective function. This can be avoided by adding a linear and continuous mapping which is compact from some space into $H^1(\Omega;\mathbb{R}^d)$, this mapping, for example, can be created by solving the equations of elasticity, then the control u_D can be exchanged by controls which also belong to a Lebesgue space. This is exactly what we will do in Part V. We have simply chosen directly the Dirichlet displacement in $H^2(\Omega;\mathbb{R}^d)$ only for simplicity so that we can, once again, keep the discussion more concise. Moreover, for the same reasoning, we set the initial condition to zero.

Now we define most of the necessary operators, spaces and parameters from Part II to apply the results therein. We set

$$
\begin{aligned}
&\mathcal{X} := W_D^{-1,p_1}(\Omega;\mathbb{R}^d)\times W^{1,p_1}(\Omega;\mathbb{R}^d), && \mathcal{X}_c := L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_N;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d),\\
&\mathcal{H} := \mathcal{W} := L^2(\Omega;\mathbb{R}^{d\times d}_s),\\
&\Psi(z,f,g,u_D) := \|z(T)-z_d\|_{\mathcal{H}}, && \Phi(f,g,u_D) := \frac{\alpha}{2}\|(\dot f,\dot g,\dot u_D)\|_{L^2(\mathcal{X}_c)}\\
&A := \partial I_{\mathcal{K}(\Omega)}, && M := \{0\},\\
&A_n := \partial I_{\lambda_n,s_n}, \quad z_0 := 0,
\end{aligned}
$$

with $p_1\in(2,\overline{p}]$, where $\overline{p}$ is from Theorem 2.5, and $2 > dp_1/(d+p_1)$ (such that $L^2(\Omega;\mathbb{R}^d)$ is compactly embedded into $W_D^{-1,p_1}(\Omega;\mathbb{R}^d)$) and $2 > (d-1)p_1/d$ (such that $L^2(\Gamma_N;\mathbb{R}^d)$ is compactly embedded into $W_D^{-1,p_1}(\Omega;\mathbb{R}^d)$), so that $\mathcal{X}_c$ is compactly embedded into $\mathcal{X}$ via the canonical embedding

$$
\mathcal{X}_c \ni (f,g,u_D)\mapsto ((f,\cdot)_{L^2(\Omega;\mathbb{R}^d)} + (g,\cdot)_{L^2(\Gamma_N;\mathbb{R}^d)}, u_D)\in\mathcal{X}.
$$

Moreover, the set $\mathcal{K}(\Omega)$ is given in Definition 2.2 and the mapping $\partial I_{\lambda_n,s_n}$ in (3.11), where $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$ are sequences such that

$$
\frac{1}{\lambda_n}\exp\Big(\frac{T\|Q\|}{\lambda_n}\Big)|\Omega|\gamma\frac{s_n}{\lambda_n}\to 0,
$$

as $n\to\infty$. With this choice and in view of (3.12), the requirement in Lemma 4.17 and thus Assumption ⟨5.1.ii⟩ holds.

Let us also recall the operators R and Q from Definition 2.7. Note that there R is defined on $H_D^{-1}(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^d)$, thanks to Corollary 2.6 we can also define the same operator on $\mathcal{X}$, with a slight abuse of notation we denote both operators by R. We also denote the same mapping defined on $\mathcal{X}_c$ by R. Clearly, Assumption ⟨4.i⟩ holds according to Corollary 2.6 and Lemma 2.8.

Moreover, Assumption ⟨II.ii⟩ and Assumption ⟨5.ii⟩ is fulfilled due to Proposition 3.14, and it is easy to verify Assumptions ⟨5.iii⟩ to ⟨5.iv⟩ and Assumption ⟨5.1.i⟩.

Now, since the optimization problems (5.2) and (6.1) are equivalent according to Theorem 2.9, with $\ell = (f, g, u_D)$, we are in a position to apply the results given in Section 5.1. To this end, we also consider the regularized problems

$$\left\{\begin{aligned} \min \quad & J(z, f, g, u_D) = \frac{1}{2}\|z(T) - z_d\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ & \qquad + \frac{\alpha}{2}\|(\dot{f}, \dot{g}, \dot{u}_D)\|^2_{L^2(L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_N;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d))}, \\ \text{s.t.} \quad & -\operatorname{div}\sigma = f + g \\ & u - u_D \in H^1_D(\Omega;\mathbb{R}^{d\times d}_s) \\ & \sigma = \mathbb{C}(\nabla^s u - z) \\ & \dot{z} = \partial I_{\lambda_n, s_n}(\sigma - \mathbb{B}z), \qquad z(0) = 0 \\ & (u, \sigma, z) \in H^1(H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)) \\ & (f, g, u_D) \in H^1_0(L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_N;\mathbb{R}^d) \times H^2(\Omega;\mathbb{R}^d)), \end{aligned}\right. \tag{6.2}$$

which are also equivalent to (5.3).

6.2 Application of the General Theory

We start with the application of the findings in Section 5.1.

Theorem 6.1 (Existence and convergence of global minimizers for elasto plasticity)**.** *There exists a global minimizer of* (6.1) *and of* (6.2) *for every $n \in \mathbb{N}$.*

Moreover, if $(z_n, f_n, g_n, u_{D,n})$ is a global minimizer of (6.2) *for every $n \in \mathbb{N}$, then every weak accumulation point is a strong accumulation point and a global minimizer of* (6.1)*. Futhermore, there exists an accumulation point.*

Proof. The assertions follow from Theorem 5.2, Proposition 5.3, Theorem 5.4 and Corollary 5.5. Note that the requirement on $\Phi = \alpha\|\cdot\|_{H^1(\mathcal{X}_c)}$ is fulfilled since $\mathcal{X}_c = L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_D;\mathbb{R}^d) \times H^2(\Omega;\mathbb{R}^d)$ is a Hilbert space. □

Let us now turn to the results in Section 5.2, in particular Theorem 5.13. At first we have to verify Assumptions ⟨5.2.i⟩ to ⟨5.2.v⟩. To this end, we define

$$\mathcal{Y} := L^{p_1}(\Omega;\mathbb{R}^{d\times d}_s), \qquad \mathcal{Z} := L^{p_2}(\Omega;\mathbb{R}^{d\times d}_s), \qquad A_s := \partial I_{\lambda,s}$$

with $2 < p_2 < p_1$ (recall the requirements on p_1 from above) and $\lambda > 0$, $s \in (0,1)$. Clearly, Assumption ⟨5.2.i⟩, Assumption ⟨5.2.iii⟩ and Assumption ⟨5.2.v⟩ are fulfilled and Corollary 2.6 shows that Assumption ⟨5.2.ii⟩ also holds. Thanks to Lemma 3.18, A_s is Lipschitz continuous and Lemma 3.19 shows that it is also Fréchet differentiable from $\mathcal{Y}$ to $\mathcal{Z}$. The other requirements in Assumption ⟨5.2.iv⟩ can be easily verified, hence, Assumption ⟨5.2.iv⟩ is also fulfilled.

Now we consider

$$\left\{\begin{aligned} \min \quad & J(z,f,g,u_D) = \frac{1}{2}\|z(T)-z_d\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ & \qquad + \frac{\alpha}{2}\|(\dot{f},\dot{g},\dot{u}_D)\|^2_{L^2(L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_N;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d))}, \\ \text{s.t.} \quad & -\operatorname{div}\sigma = f+g \\ & u-u_D \in H^1_D(\Omega;\mathbb{R}^{d\times d}_s) \\ & \sigma = \mathbb{C}(\nabla^s u - z) \\ & \dot{z} = \partial I_{\lambda,s}(\sigma - \mathbb{B}z), \quad z(0)=0 \\ & (u,\sigma,z) \in H^1(H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^{d\times d}_s)\times L^2(\Omega;\mathbb{R}^{d\times d}_s)) \\ & (f,g,u_D) \in H^1_0(L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_N;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d)), \end{aligned}\right. \tag{6.3}$$

which is again the same problem as (5.8). Before we can apply Theorem 5.13 we have to determine the adjoint operator of R in

Lemma 6.2 (Adjoint operator of R in the case of elasto plasticity). *Let $\tau \in \mathcal{H}$, $f \in H^1_D(\Omega;\mathbb{R}^d)$ the solution of*

$$-\operatorname{div}\mathbb{C}\nabla^s f = -\operatorname{div}\mathbb{C}\tau$$

(that is, $f = \mathcal{T}(-\operatorname{div}\mathbb{C}\tau, 0)$), $g \in L^2(\Gamma_N,\mathbb{R}^d)$ defined by $g := f|_{\Gamma_N}$ and $u_D \in H^2(\Omega;\mathbb{R}^{d\times d}_s)$ the solution of

$$\big(u_D, v\big)_{H^2(\Omega;\mathbb{R}^d)} = (\mathbb{C}(\tau - \nabla^s f), \nabla^s v)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

for all $v \in H^2(\Omega;\mathbb{R}^d)$ (which existence is guaranteed by the Riesz representation theorem). Then we have $R^*\tau = (f,g,u_D)$ (where R^* is the adjoint operator of $R \in \mathcal{L}(\mathcal{X}_c,\mathcal{H})$).

Proof. Let $(\xi_f,\xi_g,\xi_{u_D}) \in \mathcal{X}_c$ be arbitrary and $v := \mathcal{T}(\xi_f+\xi_g,\xi_{u_D}) \in H^1(\Omega;\mathbb{R}^d)$, according to the definition of R we have $R(\xi_f,\xi_g,\xi_{u_D}) = \mathbb{C}\nabla^s v$. Using the fact that $\mathcal{T}$ is a solution operator, we get

$$\begin{aligned} \big(R(\xi_f,\xi_g,\xi_{u_D}),\tau\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} &= \big(\nabla^s(v-\xi_{u_D}),\mathbb{C}\tau\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \big(\nabla^s\xi_{u_D},\mathbb{C}\tau\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ &= \big(\mathbb{C}\nabla^s f,\nabla^s(v-\xi_{u_D})\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \big(\nabla^s\xi_{u_D},\mathbb{C}\tau\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ &= (\nabla^s f,\mathbb{C}\nabla^s v)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \big(\nabla^s\xi_{u_D},\mathbb{C}(\tau-\nabla^s f)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ &= \big(f,\xi_f\big)_{L^2(\Omega;\mathbb{R}^d)} + \big(g,\xi_g\big)_{L^2(\Gamma_N;\mathbb{R}^d)} + \big(u_D,\xi_{u_D}\big)_{H^2(\Omega;\mathbb{R}^d)} \\ &= \big((\xi_f,\xi_g,\xi_{u_D}),(f,g,u_D)\big)_{\mathcal{X}_c}, \end{aligned}$$

which proves the assertion. □

Theorem 6.3 (KKT-conditions for elasto plasticity). *Let $(\overline{z},\overline{f},\overline{g},\overline{u}_D) \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)\times L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_N;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d))$ be a locally optimal solution for* (6.3). *Then there exists $\overline{u} \in H^1(H^1(\Omega;\mathbb{R}^{d\times d}_s))$, $\overline{\sigma} \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$, a unique adjoint state $(\varphi,\zeta,v_D) \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)\times H^1_D(\Omega;\mathbb{R}^{d\times d}_s)\times H^2(\Omega;\mathbb{R}^d))$ such that the following optimality system is fulfilled*

State equation:

$$
\begin{aligned}
-\operatorname{div}\overline{\sigma} &= \overline{f} + \overline{g} && (6.4a)\\
\overline{u} - \overline{u}_D &\in H^1_D(\Omega;\mathbb{R}^{d\times d}_s) && (6.4b)\\
\overline{\sigma} &= \mathbb{C}(\nabla^s\overline{u} - \overline{z}) && (6.4c)\\
\dot{\overline{z}} &= \partial I_{\lambda,s}(\overline{\sigma} - \mathbb{B}\overline{z}), \qquad \overline{z}(0) = 0 && (6.4d)
\end{aligned}
$$

Adjoint equation :

$$
\begin{aligned}
\dot{\varphi} &= (\mathbb{C} + \mathbb{B})\partial I'_{\lambda,s}(\overline{\sigma} - \mathbb{B}\overline{z})\varphi - \mathbb{C}\nabla^s\zeta && (6.4e)\\
-\operatorname{div}\mathbb{C}\nabla^s\zeta &= -\operatorname{div}\mathbb{C}\partial I'_{\lambda,s}(\overline{\sigma} - \mathbb{B}\overline{z})\varphi && (6.4f)\\
\varphi(T) &= \overline{z}(T) - z_d && (6.4g)
\end{aligned}
$$

Gradient equation:

$$
\begin{aligned}
&\alpha\partial_t^2(\overline{f},\overline{g},\overline{u}_D) + (\zeta,\zeta|_{\Gamma_N}, v_D) = 0 && (6.4h)\\
&\big(v_D(t), v\big)_{H^2(\Omega;\mathbb{R}^d)} = \Big(\mathbb{C}(\partial I'_{\lambda,s}(\overline{\sigma}(t) - \mathbb{B}\overline{z}(t))\varphi(t)) - \nabla^s\zeta(t)), \nabla^s v\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} && (6.4i)\\
&\qquad \forall t\in[0,T], \forall v\in H^2(\Omega;\mathbb{R}^d),\\
&(\overline{f},\overline{g},\overline{u}_D)(0) = \partial_t(\overline{f},\overline{g},\overline{u}_D)(T) = 0. && (6.4j)
\end{aligned}
$$

Proof. Using Theorem 5.13 and Example 5.15 (with $\Psi_1 = 0$ and $\Psi_2 = \Psi$), we see that if $(\overline{z},\overline{f},\overline{g},\overline{u}_D) \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)\times L^2(\Omega;\mathbb{R}^d)\times L^2(\Gamma_N;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d))$ is a locally optimal solution for (6.3), then there exists an unique adjoint state $\varphi \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$ such that

$$
\begin{aligned}
\dot{\overline{z}} &= \partial I_{\lambda,s}(R\overline{\ell} - Q\overline{z}), \qquad \overline{z}(0) = z_0,\\
\dot{\varphi} &= Q\partial I'_{\lambda,s}(R\overline{\ell} - Q\overline{z})^*\varphi,\\
\varphi(T) &= -\Psi'_z(\overline{z}(T),\overline{\ell}(T)).\\
\alpha\partial_t^2\overline{\ell} + R^*\partial I'_{\lambda,s}(R\overline{\ell} - Q\overline{z})^*\varphi &= 0\\
\overline{\ell}(0) = 0, &\qquad \alpha\partial_t\overline{\ell}(T) = -\Psi'_\ell(\overline{z}(T),\overline{\ell}(T))
\end{aligned}
$$

holds with $\overline{\ell} = (\overline{f},\overline{g},\overline{u}_D)$. Taking the concrete form of the objective function and the operators R and Q into account, using Lemma 3.19 (to see that $\partial I'_{\lambda,s}(\tau)$ is self-adjoint for all $\tau \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$) and applying Lemma 6.2, one obtians the desired optimality system. □

Let us now look at the results in Section 5.3. To shorten the discussion, we will not apply the results therein in detail, but only comment on them. Clearly, we can also apply Theorem 5.21 and Corollary 5.23 when we verify Assumptions ⟨5.3.ii⟩ to ⟨5.3.iv⟩. Here it is notable that a difficulty arises due to the second derivative of $\partial I_{\lambda,s}$. As was discussed in Remark 3.20, $\partial I_{\lambda,s}$ is from $L^{p_1}(\Omega;\mathbb{R}^{d\times d}_s)$ to $L^{p_3}(\Omega;\mathbb{R}^{d\times d}_s)$ twice differentiable only when $1 \le p_3 < p_1/2$ (and, of course, $\max_s$ is sufficient regular). Taking into account that, when we choose $\mathcal{Y} = L^{p_1}(\Omega;\mathbb{R}^{d\times d}_s)$ and $\mathcal{W} = L^{p_3}(\Omega;\mathbb{R}^{d\times d}_s)$, the operators R and Q also have to be well defined on $\mathcal{Y}$ and $\mathcal{W}$, Corollary 2.6 gives $\overline{p}' \le p_3 < p_1/2 \le \overline{p}/2$ which results in

$$
1 = \frac{1}{\overline{p}} + \frac{1}{\overline{p}'} > \frac{3}{\overline{p}}
$$

so that

$$\overline{p} > 3.$$

Since Corollary 2.6 ensures only $\overline{p} > 2$ and one cannot expect $\overline{p}$ to be significantly larger than 2, see e.g. [36, 89, 74], the requirement $\overline{p} > 3$ is quite restrictive. Furthermore, when $\overline{p}$ is not greater than 4, the necessity $2p_3 < p_1$ gives that $p_3 < 2$ so that we cannot choose $\mathcal{W} = \mathcal{H}$ and the concrete objective function from this chapter is no possible choice. However, we note that it is possible to use an objective function involving the displacement with a $L^2(\Omega; \mathbb{R}^d)$-norm, cf. [71, Remark 7.20] (but one still requires $\overline{p} > 3$). Nonetheless, when we want to use the objective from this chapter, we need $\overline{p} > 4$.

We also mention that in Meyer et al. [71, Section 7.4], in particular in Remark 7.28 therein, the case of homogenized plasticity was analyzed (the topic of the next chapter) with analog results for $\overline{p}$.

We summarize the discussion and the results given in this chapter. The results presented in Section 5.1 and Section 5.2 can be applied in the case of elasto plasticity leading to Theorem 6.1 and Theorem 6.3. The results from Section 5.3 can also be applied, however, $\overline{p}$ from Corollary 2.6 then needs to satisfy $\overline{p} > 3$ or, depending on the desired objective function, even $\overline{p} > 4$, which is quite restrictive.

Chapter 7 Homogenized Plasticity

In this chapter, we show that a system of equations that arises as homogenization limit in elasto plasticity, which was derived in Schweizer [88, Theorem 2.2], can also be transformed into an EVI of the type (EVI). It describes the evolution of plastic deformation in a material with periodic microstructure. We emphasize that this chapter contains only the transformation into an EVI, we will not consider an optimal control problem to shorten the thesis and since the analysis would be similar to the one in Chapter 6. We also elaborate on this in more detail at the end of this chapter.

In [88], the so called *two-scale convergence* was used to derive the homogenized system. Since we are only interested in applying our results from Part II to this limit system, we do not elaborate on two-scale convergence, but only refer to [4], which also gives more insight into homogenization. We also note that in [88] the obtained system contains the inertia term, that is the second time derivative of the displacement. In contrast, we are considering the quasistatic case by neglecting this term.

7.1 Definition and Notation

As usual, at first we present the equations formally in its strong form, namely

$$\begin{aligned}
-\nabla_x \cdot \pi\Sigma &= f && \text{in } \Omega, && (7.1a)\\
-\nabla_y \cdot \Sigma &= 0 && \text{in } \Omega \times Y, && (7.1b)\\
\nu \cdot \pi\Sigma &= g && \text{on } \Gamma_N, && (7.1c)\\
u &= 0 && \text{on } \Gamma_D, && (7.1d)\\
\Sigma &= \mathbb{C}(\nabla_x^s u + \nabla_y^s v - Bz) && \text{in } \Omega \times Y, && (7.1e)\\
\dot{z} &\in A(B^\top\Sigma - \mathbb{B}z), \quad z(0) = z_0 && \text{in } \Omega \times Y. && (7.1f)
\end{aligned}$$

Herein, $Y = [0,1]^d$ is the unit cell. The mapping $u\colon (0,T)\times\Omega \to \mathbb{R}^d$ is the displacement on the macro level, while $v\colon (0,T)\times\Omega\times Y \to \mathbb{R}^d$ is the displacement reflecting the micro structure. The stress tensor is denoted by $\Sigma\colon (0,T)\times\Omega\times Y \to \mathbb{R}^{d\times d}_s$ and $z\colon (0,T)\times\Omega\times Y \to \mathbb{V}$ is the internal variable describing changes in the material behavior under plastic deformation (such as hardening), where $\mathbb{V}$ is a finite dimensional Banach space. Moreover, $\nabla^s_x := \frac{1}{2}(\nabla_x + \nabla_x^\top)$ is the linearized strain in Ω and ∇^s_y is defined analogously. The elasticity tensor and the hardening parameter satisfy $\mathbb{C} \in L^\infty(\Omega\times Y; \mathcal{L}(\mathbb{R}^{d\times d}_s))$ and $\mathbb{B} \in L^\infty(\Omega\times Y; \mathcal{L}(\mathbb{V}))$ and are symmetric and uniformly coercive , i.e., there exist constants $\underline{c} > 0$ and $\underline{b} > 0$ such that

$$
\begin{aligned}
\mathbb{C}(x,y)\sigma : \sigma &\geq \underline{c}\|\sigma\|^2_{\mathbb{R}^{d\times d}} && \forall \sigma \in \mathbb{R}^{d\times d}_s, && \text{f.a.a. } (x,y)\in\Omega\times Y,\\
\mathbb{B}(x,y)\zeta : \zeta &\geq \underline{b}\|\zeta\|^2_{\mathbb{V}} && \forall \zeta \in \mathbb{V}, && \text{f.a.a. } (x,y)\in\Omega\times Y.
\end{aligned}
$$

It is to be noted that this is an exception from the general requirement on $\mathbb{C}$ and $\mathbb{B}$ made in Chapter 1. As we have noted there, in the other cases of plasticity, one could also assume that the elasticity tensor and hardening parameter have a spatial dependency, but we neglected this for convenience. In the case of homogenized plasticity, the spatial dependency is however vital since it represents the micro structure of the material. In addition, $B \in L^\infty(\Omega\times Y; \mathcal{L}(\mathbb{V}; \mathbb{R}^{d\times d}_s))$ is a given linear mapping by which one recovers the plastic strain from the internal variables z. Once again, such a mapping could also be integrated in elasto plasticity in Chapter 2 and Chapter 6 but we omitted this also for the sake of simplicity. The evolution of the internal variables is determined by a maximal monotone operator $A\colon L^2(\Omega\times Y;\mathbb{V}) \to 2^{L^2(\Omega\times Y;\mathbb{V})}$. This maximal monotone mapping can also be chosen, similar as in the case of elasto-plasticity in Definition 2.2, as a subdifferential to represent the case of linear kinematic hardening with the von-Mises flow rule . However, let us simply write A for the sake of convenience. Finally, z_0 is a given initial state and π is the averaging over the unit cell, i.e.,

$$
\pi : \Sigma \mapsto \fint_Y \Sigma(\cdot,y)dy := \frac{1}{|Y|}\int_Y \Sigma(\cdot,y)dy.
$$

For a precise notion of a solution for the homogenized elastoplasticity system in (7.1), we define the function space for the micro displacement:

Definition 7.1 (Space for the micro displacement)**.** *The function space for the micro displacement is defined as*

$$
\begin{aligned}
V_0 := \{ v \in L^2(\Omega\times Y;\mathbb{R}^d) : v(x,\cdot) \in H^1_{per}(Y;\mathbb{R}^d), \nabla^s_y v \in L^2(\Omega\times Y;\mathbb{R}^d)\\
\text{and } \int_Y v(x,y)\,dy = 0 \text{ f.a.a. } x\in\Omega\},
\end{aligned}
$$

where $H^1_{per}(Y;\mathbb{R}^d)$ is the subspace of $H^1(Y;\mathbb{R}^d)$ consisting of Y-periodic functions. With the scalar product

$$
\big(v_1,v_2\big)_{V_0(\Omega\times Y;\mathbb{R}^d)} := \big(v_1,v_2\big)_{L^2(\Omega\times Y;\mathbb{R}^d)} + \Big(\nabla^s_y v_1, \nabla^s_y v_2\Big)_{L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)}
$$

this space becomes a Hilbert space. Note that the scalar product is well defined according to Korn's and Poincaré's inequality for functions with zero mean value.

With this definition at hand, we are now in the position to define our precise notion of solutions to (7.1):

Definition 7.2 (Weak solutions of homogenized plasticity). *Let $f, g \in H^1(H_D^{-1}(\Omega;\mathbb{R}^d))$ and $z_0 \in L^2(\Omega \times Y;\mathbb{V})$. Then we say that a tuple*

$$(u, v, z, \Sigma) \in H^1(H_D^1(\Omega;\mathbb{R}^d)) \times H^1(V_0) \times H^1(L^2(\Omega \times Y;\mathbb{V})) \times H^1(L^2(\Omega \times Y;\mathbb{R}_s^{d\times d}))$$

is a solution of (7.1)*, if, for almost all $t \in [0, T]$, there holds*

$$\big(\pi\Sigma(t), \nabla_x^s\varphi\big)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} = \langle (f+g)(t), \varphi\rangle \quad \forall \varphi \in H_D^{-1}(\Omega;\mathbb{R}^d), \tag{7.2a}$$

$$\Big(\Sigma(t), \nabla_y^s\psi\Big)_{L^2(\Omega\times Y;\mathbb{R}_s^{d\times d})} = 0 \quad \forall \psi \in V_0, \tag{7.2b}$$

$$\Sigma = \mathbb{C}(\pi_r^{-1}\nabla_x^s u + \nabla_y^s v - Bz), \tag{7.2c}$$

$$\dot z \in A(B^\top\Sigma - \mathbb{B}z), \quad z(0) = z_0, \tag{7.2d}$$

where

$$\pi_r^{-1} : L^2(\Omega;\mathbb{R}_s^{d\times d}) \ni \varepsilon \mapsto \Big(\Omega \times Y \ni (x, y) \mapsto \varepsilon(x) \in \mathbb{R}_s^{d\times d}\Big) \in L^2(\Omega \times Y;\mathbb{R}_s^{d\times d}).$$

In the following, we will frequently consider π_r^{-1} in different domains and ranges, for simplicity denoted by the same symbol.

7.2 Transformation into an EVI

In what follows, we reduce the system (7.1) to an equation in the internal variable z only and it will turn out that this equation has exactly the form of the general equation (EVI). To this end, we proceed similar to Chapter 2 (that is, we use the idea in GRÖGER [45, Chapter 4]). For this purpose, let us define the following operators:

Definition 7.3 (Symmetrized gradient and divergence operator for homogenized plasticity). *We define the* symmetrized gradient *(for homogenized plasticity)*

$$\nabla_{(x,y)}^s : H_D^1(\Omega;\mathbb{R}^d) \times V_0 \to L^2(\Omega \times Y;\mathbb{R}_s^{d\times d}), \qquad \nabla_{(x,y)}^s(u, v) := \pi_r^{-1}\nabla_x^s u + \nabla_y^s v.$$

For its adjoint, that is, the divergence operator *(for homogenized plasticity), we write*

$$\begin{aligned}
&\operatorname{div}_{(x,y)} : L^2(\Omega \times Y;\mathbb{R}_s^{d\times d}) \to H_D^{-1}(\Omega;\mathbb{R}^d) \times V_0^*,\\
&\langle \operatorname{div}_{(x,y)} \sigma, (\varphi, \psi)\rangle := -\langle \nabla_{(x,y)}^{s\,*}\sigma, (\varphi, \psi)\rangle\\
&\qquad = -\int_{\Omega\times Y} \sigma(x, y) : (\nabla_x^s\varphi(x) + \nabla_y^s\psi(x, y))d(x, y).
\end{aligned}$$

For the replacement of (7.1a) to (7.1e), respectively (7.2a) to (7.2c), with a solution operator, we need the following

Lemma 7.4 (Existence for linear elasticity). *For every $(L_1, L_2) \in H_D^{-1}(\Omega;\mathbb{R}^d) \times V_0^*$, there exists a unique solution $(u, v) \in H_D^1(\Omega;\mathbb{R}^d) \times V_0$ of*

$$-\operatorname{div}_{(x,y)} \mathbb{C}\nabla_{(x,y)}^s(u, v) = (L_1, L_2) \tag{7.3}$$

and there is a constant $C > 0$, independent of L_1 and L_2, such that

$$\|(u, v)\|_{H_D^1(\Omega;\mathbb{R}^d)\times V_0} \le C\big(\|L_1\|_{H_D^{-1}(\Omega;\mathbb{R}^d)} + \|L_2\|_{V_0^*}\big).$$

Proof. The left hand side of (7.3) gives rise to a bilinear form b on the Hilbert space $H^1_D(\Omega;\mathbb{R}^d)\times V_0$:

$$b\big((u,v),(\varphi,\psi)\big) := \Big(\mathbb{C}\nabla^s_{(x,y)}(u,v), \nabla^s_{(x,y)}(\varphi,\psi)\Big)_{L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)}.$$

Clearly, b is bounded. Using Korn's and Poincaré's inequality for functions with zero mean value, we obtain

$$\begin{aligned} b\big(((u,v),(u,v)\big) &\geq \underline{c}\|\nabla^s_{(x,y)}(u,v)\|^2_{L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)} \\ &= \underline{c}|Y|\ \|\nabla^s_x u\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \underline{c}\|\nabla^s_y v\|^2_{L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)} \\ &\geq C\big(\|u\|_{H^1_D(\Omega;\mathbb{R}^d)} + \|v\|_{V_0(\Omega\times Y;\mathbb{R}^d)}\big)^2 \end{aligned}$$

for all $(u,v)\in H^1_D(\Omega;\mathbb{R}^d)\times V_0$, where we additionally used the Y-periodictiy of v in the equation. Hence, b is also coercive so that the claim follows from the Lax-Milgram lemma. □

Now we are in the position to reduce (7.1) to an equation in the variable z only. For this purpose, we need the following

Definition 7.5 (Operators R and Q for homogenized plasticity)**.** *By Lemma 7.4, the solution operator associated with* (7.3) *and denoted by*

$$\mathcal{T} := \big(-\operatorname{div}_{(x,y)}\mathbb{C}\nabla^s_{(x,y)}\big)^{-1} : H^{-1}_D(\Omega;\mathbb{R}^d)\times V_0^* \to H^1_D(\Omega;\mathbb{R}^d)\times V_0,$$

is well defined, linear and bounded. Now, we have everything at hand to define the mappings R and Q from our general equation (EVI) *for the special case of homogenized plasticity:*

$$\begin{aligned} &R:\ H^{-1}_D(\Omega;\mathbb{R}^d)\ni L\mapsto B^\top\mathbb{C}\nabla^s_{(x,y)}\mathcal{T}(L,0)\in L^2(\Omega\times Y;\mathbb{V}),\\ &Q:\ L^2(\Omega\times Y;\mathbb{V})\ni z\mapsto (B^\top\mathbb{C}B+\mathbb{B})z - B^\top\mathbb{C}\nabla^s_{(x,y)}\mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}Bz))\in L^2(\Omega\times Y;\mathbb{V}). \end{aligned}$$

The reason for defining the operators Q and R in the way we did in Definition 7.5 is the following: Owing to Lemma 7.4, given $z\in L^2(\Omega\times Y;\mathbb{V})$, one can solve (7.2a) to (7.2c) so that the tuple $(u,v,\Sigma)\in H^1_D(\Omega;\mathbb{R}^d)\times V_0\times L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)$ is uniquely determined by z. Even more, using the operators from Definition 7.5, we see that the solution of (7.2a) to (7.2c) for given z is

$$\begin{aligned} (u,v) &= \mathcal{T}(f+g-\operatorname{div}_{(x,y)}(\mathbb{C}Bz),0),\\ \Sigma &= \mathbb{C}(\nabla^s_{(x,y)}(u,v) - Bz). \end{aligned}$$

Using the last equation and employing the definition of R and Q in Definition 7.5 then yields

$$\dot{z}\in A(B^\top\Sigma - \mathbb{B}z) = A(R\ell - Qz),\qquad z(0)=z_0, \tag{7.4}$$

with $\ell = f+g$, i.e., exactly an evolution equation of the general form of (EVI). This shows that the system (7.1) of homogenized plasticity fits into the setting analyzed in Part II. Let us consider the operator Q and prove in particular the important coercivity in the following lemma. Note that the proof is analog to Lemma 2.8, however, we present it for the sake of completeness.

Lemma 7.6 (Properties of Q)**.** *The operator Q from Definition 7.5 is self-adjoint and coercive .*

Proof. Since $\mathbb{B}$ is symmetric and coercive, it is sufficient to prove that the operator $B^\top \mathbb{C} B - B^\top \mathbb{C} \nabla^s_{(x,y)} \mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}B\cdot)) = Q - \mathbb{B} \in \mathcal{L}(L^2(\Omega\times Y;\mathbb{V}))$ is symmetric and positive. To prove the symmetry, first observe that $B^\top\mathbb{C}B$ is symmetric by the symmetry of $\mathbb{C}$. The symmetry of $\mathbb{C}$ moreover implies that $\mathcal{T}$, i.e., the solution operator of (7.3), is self-adjoint . Therefore, we have for all $z_1, z_2 \in L^2(\Omega\times Y;\mathbb{V})$ that

$$\begin{aligned}\langle B^\top \mathbb{C} \nabla^s_{(x,y)} \mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}Bz_1)), z_2\rangle_{L^2(\Omega\times Y;\mathbb{V})} &= \langle -\operatorname{div}_{(x,y)}(\mathbb{C}Bz_2), \mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}Bz_1))\rangle \\ &= \langle \mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}Bz_2)), -\operatorname{div}_{(x,y)}(\mathbb{C}Bz_1)\rangle \\ &= \langle z_1, B^\top \mathbb{C}\nabla^s_{(x,y)}\mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}Bz_2))\rangle_{L^2(\Omega\times Y;\mathbb{V})}\end{aligned}$$

so that $B^\top\mathbb{C}\nabla^s_{(x,y)}\mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}B\cdot))$ is also symmetric. To show the positivity of $B^\top\mathbb{C}B - B^\top\mathbb{C}\nabla^s_{(x,y)}\mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}B\cdot))$, let $z \in L^2(\Omega\times Y;\mathbb{V})$ be arbitrary. To shorten the notation, we abbreviate $(u_z, v_z) := \mathcal{T}(-\operatorname{div}_{(x,y)}(\mathbb{C}Bz))$. Then, by testing the equation for (u_z, v_z), i.e., (7.3) with $(L_1, L_2) = -\operatorname{div}_{(x,y)}(\mathbb{C}Bz)$, with $(-u_z, -v_z)$, we arrive at

$$\Big(\mathbb{C}(Bz - \nabla^s_{(x,y)}(u_z, v_z), -\nabla^s_{(x,y)}(u_z, v_z)\Big)_{L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)} = 0.$$

The coercivity of $\mathbb{C}$ therefore implies

$$\begin{aligned}\Big(B^\top\mathbb{C}Bz - B^\top\mathbb{C}\nabla^s_{(x,y)}(u_z, v_z), z\Big)_{L^2(\Omega\times Y;\mathbb{V})} &= \Big(\mathbb{C}(Bz - \nabla^s_{(x,y)}(u_z, v_z), Bz\Big)_{L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)} \\ &= \Big(\mathbb{C}(Bz - \nabla^s_{(x,y)}(u_z, v_z), Bz - \nabla^s_{(x,y)}(u_z, v_z)\Big)_{L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)} \\ &\geq 0.\end{aligned}$$

Since z was arbitrary, this proves the positivity. □

Let us collect our findings in the following

Theorem 7.7 (Transformation of homogenized plasticity into an EVI)**.** *Let $f, g \in H^1(H^{-1}_D(\Omega;\mathbb{R}^d))$ and $z_0 \in L^2(\Omega\times Y;\mathbb{V})$. Then $(u, v, \Sigma, z) \in H^1(H^{-1}_D(\Omega;\mathbb{R}^d)\times V_0 \times L^2(\Omega\times Y;\mathbb{R}^{d\times d}_s)\times L^2(\Omega\times Y;\mathbb{V}))$ is a solution of* (7.1)*, if and only if $z \in H^1(L^2(\Omega\times Y;\mathbb{V})))$ is a solution of* (7.4) *with $\ell = f + g$, $(u, v) = \mathcal{T}(\ell - \operatorname{div}_{(x,y)}(\mathbb{C}Bz), 0)$ and $\Sigma = \mathbb{C}(\nabla^s_{(x,y)}(u, v) - Bz)$. Moreover, the operator Q, involved in* (7.4) *and given in Definition 7.5, is symmetric and coercive .*

One can now proceed as in Chapter 6, that is, apply the general results from Part II to an optimal control problem with (7.1) as a constraint. Since this procedure is similar to the one in Chapter 6, we do not present this and only refer to MEINLSCHMIDT ET AL.[71], therein such an optimal control problem was analyzed. As we already noted at the end of Chapter 6, the same issue regarding a necessary norm gap will arise when using the findings in Section 5.3. Let us additionally mention that one needs an analogon for Corollary 2.6 (when $p \neq 2$) for the case of homogenized plasticity, however, this was also addressed in MEINLSCHMIDT ET AL. [71, Lemma 7.6].

With the presentation of the transformation of homogenized plasticity into an EVI, exactly of the form analyzed in Part II, this part already ends. As we have seen, both elasto and homogenized plasticity can be treated with the analysis given in Part II. In contrast, this will not be the case in the upcoming part, where we will still be able to use results concerning H^2 loads from Chapter 4 but, in particular since the in Part IV arising operator $\mathcal{A}$ does not have the boundedness property, the findings in Chapter 5 will not be directly applicable.

Part IV Plasticity with Inertia

As before and agreed upon in the beginning of Part I, we collect at first all needed assumptions for this part.

Assumption IV. *We impose the following assumptions according to The Assumption Agreement in the beginning of Part I.*

⟨IV.i⟩ *Suppose that* $A : L^2(\Omega; \mathbb{R}^{d\times d}_s) \to 2^{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$ *is a maximal monotone operator. Moreover, for every* $\lambda > 0$ *the resolvent* $\boldsymbol{R}_\lambda$ *can be expressed pointwise, that is, there exists* $\tilde{\boldsymbol{R}}_\lambda : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ *such that*

$$R_\lambda(\tau)(x) = \tilde{R}_\lambda(\tau(x)) \quad \text{f.a.a. } x \in \Omega \text{ and } \forall \tau \in L^2(\Omega; \mathbb{R}^{d\times d}_s). \tag{7.5}$$

With a slight abuse of notation we denote also $\tilde{\boldsymbol{R}}_\lambda$ *by* $\boldsymbol{R}_\lambda$*.*

It is to be noted that this is the case in Section 3.2 for the subdifferential of an indicator function of the pointwise defined set $\mathcal{K}(\Omega)$*, where the resolvent is simply the projection onto this set, see also* (3.7)*.*

⟨IV.ii⟩ *The density of* Ω *is given by* $\rho > 0$*.*

⟨IV.iii⟩ *We choose* $u_0, v_0 \in H^1_D(\Omega; \mathbb{R}^d)$ *and* $z_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$ *and define* $q_0 := \mathbb{C}\nabla^s u_0 - (\mathbb{C} + \mathbb{B})z_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$*. Moreover, we assume that* (u_0, v_0, q_0) *is an element of* $D(\mathcal{A})$*, where* $D(\mathcal{A})$ *is given in Definition 8.5.*

⟨IV.iv⟩ *We set*

$$\mathbb{D} := \mathbb{B}(\mathbb{C} + \mathbb{B})^{-1}\mathbb{C} \quad \text{and} \quad \mathbb{E} := \mathbb{C}(\mathbb{C} + \mathbb{B})^{-1} \tag{7.6}$$

and note that $\mathbb{D}$ *is symmetric and coercive , according to Lemma 8.4.*

⟨8⟩ *Let* $f \in H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ *be given.*

⟨9⟩ *By* $J : L^2(\mathcal{H}) \times \mathfrak{X}_c \to \mathbb{R}$*,* $J(u, v, z, f) := \Psi(u, v, z) + \frac{\alpha}{2}\|f\|^2_{\mathfrak{X}_c}$ *we denote the objective function, where* $\mathcal{H}$ *is given in Definition 8.5 and the control space* $\mathfrak{X}_c$ *is a subspace of* $H^1(L^2(\Omega; \mathbb{R}^d))$*. We assume that* $\Psi : L^2(\mathcal{H}) \to \mathbb{R}$ *is weakly lower semicontinuous, continuous and bounded from below and that the Tikhonov paramenter* α *is a positive constant.*

⟨9.1⟩ *Let the control space* $\mathfrak{X}_c \hookrightarrow H^1(L^2(\Omega; \mathbb{R}^d))$ *be a Hilbert space such that* $\mathcal{F} : H^1(L^2(\Omega; \mathbb{R}^d)) \to H^2(L^2(\Omega; \mathbb{R}^d))$ *with* $\mathcal{F}(f)(t) = \int_0^t f(s)ds$ *(given in Definition 8.14) is compact from* $\mathfrak{X}_c$ *into* $L^1(L^2(\Omega; \mathbb{R}^d))$*.*

⟨9.2.i⟩ *Let* $\boldsymbol{R}_s : \mathbb{R}_s^{d\times d} \to \mathbb{R}_s^{d\times d}$ *be monotone, Lipschitz continuous and Fréchet differentiable.*

⟨9.2.ii⟩ *We fix* $2 < \hat{p} < p < \overline{p}$, *where* $\overline{p}$ *is from Theorem 2.5, such that* $2 - \frac{d}{2} \geq -\frac{d}{r}$.

⟨9.2.iii⟩ *Let* (u_0, v_0, q_0) *be an element of* $\mathcal{Y}_p$, *where* p *is given in Assumption ⟨9.2.ii⟩ and* $\mathcal{Y}_p$ *in Definition 8.5.*

In this part we consider the case of plasticity with inertia, that is, we do not neglect the second time derivative of the displacement, as was done in Chapter 2. That said, the system of equations considered in Chapter 8 is the same as in Chapter 2 except that the inertia term is added to the balance of momentum. Nonetheless, the system cannot be transformed into an EVI as we did in Chapter 2, therein we used the fact that the balance of momentum, after replaceing the stress with the displacement and the plastic strain, can be solved when the plastic strain is known, that is, the balance of momentum could be replaced with a solution operator depending on the plastic strain (and, of course, the external forces and the prescribed displacement). Clearly, due to the inertia term we cannot proceed analog. However, as we will see in Lemma 8.6, one can still transform the system of plasticity with inertia into an EVI. To this end, a new maximal monotone operator is introduced in Definition 8.5. Unfortunately, this operator is not a subdifferential (Remark 8.16) and it can be easily seen that it does not even fulfill the boundedness property (see Definition 3.5). Due to these facts, we cannot directly apply the results given in Chapter 5. Nonetheless, since in Section 5.2 a regularized operator was considered, we can still use the findings therein. To this end, after we have proven the existence of a solution and given an a priori estimate in Section 8.2 and used the convergence results presented in Section 4.2, we regularize the maximal monotone operator in Section 9.2 to derive finally first order optimality conditions in form of a KKT system.

As we elaborated on in the beginning of Part I, the existing results about optimal control of plasticity are rather scarce and, to the knowledge of the author, there are none concerning plasticity with inertia. The findings given in this part are also not yet published.

Chapter 8 State Equation

We begin our investigation with the state equation. At first we give the definition of a solution and then transform the state equation into an EVI with a new (maximal monotone) operator $\mathcal{A}$. In Section 8.2 we prove the existence of a solution by showing that the operator $\mathcal{A}$ is maximal monotone, then we can apply Theorem 4.7.

Let us note that the existence of a solution was already proven in GRÖGER [45, Theorem 5.1] by using essentially the same transformation into an EVI as we will do. However, there it was transformed into a second order EVI and the maximal monotonicity of the (slightly different) operator given therein was proven in another way. In contrast, we consider a first order EVI and will provide the concrete form of the resolvent in Proposition 8.11 (which will be also used later in Section 9.2), the fact that $\mathcal{A}$ is maximal monotone will then follow easily.

The formal strong formulation of the state equation reads

$$
\begin{aligned}
\rho\ddot{u} - \nabla \cdot \sigma &= f && \text{in } \Omega, && (8.1a)\\
\nu \cdot \sigma &= 0 && \text{on } \Gamma_N, && (8.1b)\\
u &= 0 && \text{on } \Gamma_D, && (8.1c)\\
\sigma &= \mathbb{C}(\nabla^s u - z) && \text{in } \Omega, && (8.1d)\\
\dot{z} &\in A(\sigma - \mathbb{B}z) && \text{in } \Omega, && (8.1e)\\
(u, \dot{u}, z)(0) &= (u_0, v_0, z_0) && \text{in } \Omega. && (8.1f)
\end{aligned}
$$

In contrast to elasto plasticity, the second time derivative of the displacement, multiplied with the density ρ, is now present in (8.1a). Note that we have assumed that the density is constant in Ω, see Assumption ⟨IV.ii⟩. It is possible to consider a density which has a spatial dependency (that is, a function from Ω to $(0, \infty]$), one has then in particular to verify that Q, given in Definition 8.5, is well defined, that is, the multiplication of ρ (and also $1/\rho$) with a Sobolev function is again a Sobolev function. However, for simplicity we assume that ρ is constant. The description and physical interpretation of (8.1) are almost identical as in the case of elasto plasticity, see Chapter 2.

Note that we consider only volume forces in (8.1). An adaption of the upcoming analysis to include Neumann boundary forces and prescribed Dirichlet displacements is *not* straightforward, see Remark 8.8.

8.1 Definition and Transformation

Let us begin with the definition of a solution to the state equation (8.1), all necessary tools were already established in Chapter 2.

Definition 8.1 (Solution to plasticity with inertia)**.** *We call $u \in H^1(H^1_D(\Omega; \mathbb{R}^d)) \cap H^2(L^2(\Omega; \mathbb{R}^d))$, $\sigma \in H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ and $z \in H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ solution of* (8.1) *if*

$$
\begin{aligned}
\rho\ddot{u} - \operatorname{div} \mathbb{C}(\nabla^s u - z) &= f,\\
\dot{z} &\in A(\mathbb{C}\nabla^s u - (\mathbb{C} + \mathbb{B})z),\\
(u, \dot{u}, z)(0) &= (u_0, v_0, z_0)
\end{aligned}
\tag{8.2}
$$

holds, where the div *operator is given in Definition 2.1, and we have $\sigma = \mathbb{C}(\nabla^s u - z)$. Since σ can be obtained directly from u and z, we will also call (u, z) a solution of* (8.1) *when it fulfills* (8.2).

Before we can transform the state equation into an EVI we need to reformulate it, to this end we introduce the following

Definition 8.2 (z to q mapping)**.** *We define*

$$
\mathfrak{Q} : H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_s) \to L^2(\Omega; \mathbb{R}^{d\times d}_s), \qquad (u, z) \mapsto \mathbb{C}\nabla^s u - (\mathbb{C} + \mathbb{B})z
$$

and its inverse (for fixed u)

$$
\mathfrak{Z} : H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_s) \to L^2(\Omega; \mathbb{R}^{d\times d}_s), \qquad (u, q) \mapsto (\mathbb{C} + \mathbb{B})^{-1}(\mathbb{C}\nabla^s u - q).
$$

Note that we have defined these operators already in Definition 4.2, thus, with a slight abuse of notation, we have redefined them above. However, these operators are strongly related, to see this let us consider for a moment the case of elasto plasticity from Chapter 2. Due to (2.1d) and Theorem 2.9 we see that $R\ell - Qz = \mathbb{C}\nabla^s u - (\mathbb{C} + \mathbb{B})z$ holds, where R and Q are given in Definition 2.7. This justifies our notation for $\mathfrak{Q}$ and we have defined $\mathfrak{Z}$ again such that $\mathfrak{Q}(u,\cdot)^{-1} = \mathfrak{Z}(u,\cdot)$ holds for a fixed $u \in H^1(\Omega;\mathbb{R}^d)$.

For the following lemma recall the definition of $\mathbb{D}$ and $\mathbb{E}$ given in Assumption ⟨IV.iv⟩, that is, $\mathbb{D} = \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}$ and $\mathbb{E} = \mathbb{C}(\mathbb{C}+\mathbb{B})^{-1}$.

Lemma 8.3 (Transformation of z to q)**.** *We consider*

$$\begin{aligned} \rho\ddot{u} - \operatorname{div}(\mathbb{D}\nabla^s u + \mathbb{E}q) &= f, \\ (\mathbb{C}+\mathbb{B})^{-1}\dot{q} + A(q) - \mathbb{E}^\top\nabla^s\dot{u} &\ni 0, \\ (u,\dot{u},q)(0) = (u_0,v_0,q_0) &= (u_0,v_0,\mathfrak{Q}(u_0,z_0)) \end{aligned} \tag{8.3}$$

for functions $u \in H^1(H^1_D(\Omega;\mathbb{R}^d)) \cap H^2(L^2(\Omega;\mathbb{R}^d))$, $q \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$. Recall that $\mathbb{E}^\top$ is the adjoint of $\mathbb{E}$. Then the following holds:

When (u,z) is a solution of (8.1)*, then $(u,q) = (u,\mathfrak{Q}(u,z))$ solves* (8.3)*. Vice versa, when (u,q) solves* (8.3)*, then $(u,z) = (u,\mathfrak{Z}(u,q))$ is a solution of* (8.1)*.*

Proof. Both implications can be immediately obtained by using the definition of $\mathfrak{Q}$ and $\mathfrak{Z}$ and inserting z in (8.2) and q in (8.3), respectively (note that $\mathbb{C} - \mathbb{C}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C} = (I - \mathbb{C}(\mathbb{C}+\mathbb{B})^{-1})\mathbb{C} = \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C} = \mathbb{D}$). □

Lemma 8.4 (Properties of $\mathbb{D}$)**.** *The tensor $\mathbb{D} = \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}$ is symmetric and coercive .*

Proof. Let $\tau \in \mathbb{R}^{d\times d}_s$ be arbitrary. Following GRÖGER [45, Lemma 4.2] we have

$$\begin{aligned} \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C} &= \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}(\mathbb{C}+\mathbb{B}-\mathbb{B}) \\ &= \mathbb{B} - \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{B} \\ &= \mathbb{B} - (\mathbb{C}+\mathbb{B}-\mathbb{C})(\mathbb{C}+\mathbb{B})^{-1}\mathbb{B} \\ &= \mathbb{C}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{B}, \end{aligned}$$

hence,

$$\begin{aligned} \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau : \tau &= \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau : \Big((\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau + (\mathbb{C}+\mathbb{B})^{-1}\mathbb{B}\tau\Big) \\ &= \mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau : (\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau + \mathbb{C}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{B}\tau : (\mathbb{C}+\mathbb{B})^{-1}\mathbb{B}\tau \\ &\geq \gamma_{\mathbb{B}}\|(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau\|^2_{\mathbb{R}^{d\times d}_s} + \gamma_{\mathbb{C}}\|(\mathbb{C}+\mathbb{B})^{-1}\mathbb{B}\tau\|^2_{\mathbb{R}^{d\times d}_s}. \end{aligned}$$

Furthermore, we have

$$\|\tau\|_{\mathbb{R}^{d\times d}_s} = \|\mathbb{C}^{-1}(\mathbb{C}+\mathbb{B})(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau\|_{\mathbb{R}^{d\times d}_s} \leq \|\mathbb{C}^{-1}(\mathbb{C}+\mathbb{B})\|\,\|(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau\|_{\mathbb{R}^{d\times d}_s}$$

and analog

$$\|\tau\|_{\mathbb{R}^{d\times d}_s} \leq \|\mathbb{B}^{-1}(\mathbb{C}+\mathbb{B})\|\,\|(\mathbb{C}+\mathbb{B})^{-1}\mathbb{B}\tau\|_{\mathbb{R}^{d\times d}_s},$$

so that we finally arrive at

$$\mathbb{B}(\mathbb{C}+\mathbb{B})^{-1}\mathbb{C}\tau : \tau \geq \Big(\frac{\gamma_{\mathbb{B}}}{\|\mathbb{C}^{-1}(\mathbb{C}+\mathbb{B})\|^2} + \frac{\gamma_{\mathbb{C}}}{\|\mathbb{B}^{-1}(\mathbb{C}+\mathbb{B})\|^2}\Big)\|\tau\|^2_{\mathbb{R}^{d\times d}_s},$$

which completes the proof. □

We are now in the position to introduce the EVI, respectively the operator $\mathcal{A}$.

Definition 8.5 (The operator $\mathcal{A}$). *For $p \in [1, \infty]$ we set*

$$\mathcal{Y}_p := W^{1,p}_D(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathbb{R}^{d\times d}_s) \quad \text{and} \quad \mathcal{H} := \mathcal{Y}_2$$

The scalar product on $\mathcal{H}$ is defined by

$$\big((u_1, v_1, q_1), (u_2, v_2, q_2)\big)_{\mathcal{H}} := \big(\mathbb{D}\nabla^s u_1, \nabla^s u_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \big(v_1, v_2\big)_{L^2(\Omega;\mathbb{R}^d)} + \big(q_1, q_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}.$$

We define

$$\mathcal{A} : D(\mathcal{A}) \to 2^{\mathcal{H}}, \qquad (u, v, q) \mapsto \begin{pmatrix} -v \\ -\operatorname{div}(\mathbb{D}\nabla^s u + \mathbb{E}q) \\ A(q) - \mathbb{E}^\top \nabla^s v \end{pmatrix}$$

with the domain

$$D(\mathcal{A}) := \{(u, v, q) \in H^1_D(\Omega; \mathbb{R}^d) \times H^1_D(\Omega; \mathbb{R}^d) \times D(A) : \operatorname{div}(\mathbb{D}\nabla^s u + \mathbb{E}q) \in L^2(\Omega; \mathbb{R}^d)\}.$$

Moreover, we set

$$R : L^2(\Omega; \mathbb{R}^d) \to \mathcal{Y}_\infty, \qquad f \mapsto (0, f, 0)$$

and

$$Q := (I, (1/\rho)I, \mathbb{C} + \mathbb{B}).$$

Lemma 8.6 (Transformation into an EVI). *The tuple (u, q) solves* (8.3) *if and only if $(u, v, q) = (u, \dot{u}, q) \in H^1(\mathcal{H})$ is a solution of*

$$Q^{-1}(\dot{u}, \dot{v}, \dot{q}) + \mathcal{A}(u, v, q) \ni Rf, \qquad (u, v, q)(0) = (u_0, v_0, q_0). \tag{8.4}$$

Proof. This follows immediately from the definition of $\mathcal{A}$. □

We emphasize that this is essentially the same EVI as (EVI_q) but with f instead of $\dot{\ell}$, that is, we gain regularity in time so that plasticity with inertia fits into the setting of Part II with H^2-*loads*.

Remark 8.7 (Consequences of the transformation). *We note that this transformation has some consequences for the optimal control problem and its regularization discussed in Chapter 9. A first approach to regularize* (8.1) *would be to simply regularize the operator A, as we did in the case of elasto plasticity. However, our approach is different, due to the transformation into an EVI we can regularize the operator $\mathcal{A}$, this is our method in Section 8.3 and Chapter 9. We also mention that the fact that $v = \dot{u}$ will be lost after the regularization (cf. Corollary 9.4 and Definition 9.6) and that we will transform our objective function in Definition 9.1, so that we obtain an optimal control problem with respect to the state* (u, v, q) *in* (9.2)*. The KKT-conditions given in Theorem 9.13 below are then also formulated for this transformed problem.*

Remark 8.8 (Neumann surface forces and Dirichlet displacement). *Let us shortly discuss some issues with possible surface forces and Dirichlet displacements. Regarding surface forces, they are currently equal to zero and contained in the domain $D(\mathcal{A})$ by the requirement* $\operatorname{div}(\mathbb{D}\nabla^s u + \mathbb{E}q) \in L^2(\Omega; \mathbb{R}^d)$ *(see Definition 2.1 and the explanation thereafter). Allowing now surface forces which are time dependent, the domain, and thus $\mathcal{A}$ itself, would also depend on the time.*

An approach for Dirichlet displacements would be to exchange the displacement with a "new" displacement minus the Dirichlet displacement, then one could still define the domain $D(\mathcal{A})$ as a subset of $H^1_D(\Omega;\mathbb{R}^d) \times H^1_D(\Omega;\mathbb{R}^d) \times D(A)$. However, this would again make the domain and the operator itself time dependent (the Dirichlet displacement would occur also in the operator).

In both cases one could still show that the arising operator is maximal monotone for a fixed time, but for different points in time the monotonicity would be perturbed by the time dependent functions. Having now a closer look at Theorem 4.7, respectively ZEIDLER *[110, Theorem 55.A], we see that a comparison of two different points in time is used to derive a priori estimates. Following this proof, the time depend functions would occur and a straightforward adaption is not possible.*

At this juncture, let us also elaborate on the underlying spaces of the operator $\mathcal{A}$. One might try to exchange $L^2(\mathbb{R};\Omega)$ with a negative Sobolev space in the definition of $\mathcal{H}$ to allow surface forces. However, with this definition of $\mathcal{H}$, for instance, the proof of Lemma 8.12 (which is used to show the monotonicity of $\mathcal{A}$) would not be valid anymore. Thus, our choice of $\mathcal{H}$ seems reasonable.

8.2 Existence of a Solution

We prove now the existence of a solution to (8.1) by using Theorem 4.7, thus we need to show that $\mathcal{A}$ is maximal monotone. Since the monotonicity of $\mathcal{A}$ can be easily obtained (cf. Lemma 8.12), it remains to prove that the resolvent exists (cf. the proof of Proposition 8.13). For this it is sufficient to show the existence of a solution to (8.9) in the case $p = 2$. However, since the existence and Lipschitz continuity for $p > 2$ is needed to derive optimality conditions in Section 9.2, we already provide the following corollary for later needed results.

Corollary 8.9 (Extended nonlinear elasticity). *Let $\lambda > 0$ and $p \in [2, \overline{p}]$, where $\overline{p}$ is from Theorem 2.5, with $2 - \frac{d}{2} \geq -\frac{d}{p}$. We assume that there exist $m, M, D \in \mathbb{R}$, $D \geq 0 < m \leq M$, such that the family of functions $\{b_\sigma : \Omega \times \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s\}_{\sigma\in\mathbb{R}^{d\times d}_s}$ has the following properties:*

$$b_0(\cdot, 0) \in L^\infty(\Omega;\mathbb{R}^{d\times d}_s), \tag{8.5}$$

$$b_\sigma(\cdot,\tau) \text{ is measurable}, \tag{8.6}$$

$$(b_\sigma(x,\tau) - b_{\overline{\sigma}}(x,\overline{\tau})) : (\tau - \overline{\tau}) + D(|\sigma - \overline{\sigma}| + |\tau - \overline{\tau}|)|\sigma - \overline{\sigma}| \geq m|\tau - \overline{\tau}|^2, \tag{8.7}$$

$$|b_\sigma(x,\tau) - b_{\overline{\sigma}}(x,\overline{\tau})| \leq M\Big(|\tau - \overline{\tau}| + |\sigma - \overline{\sigma}|\Big) \tag{8.8}$$

for almost all $x \in \Omega$ and all $\sigma, \overline{\sigma}, \tau, \overline{\tau} \in \mathbb{R}^{d\times d}_s$.

Then for every $\varphi \in L^p(\Omega;\mathbb{R}^{d\times d}_s)$ and $L \in W^{-1,p}_D(\Omega;\mathbb{R}^d)$ there exists a unique solution $u \in W^{1,p}_D(\Omega;\mathbb{R}^d)$ of

$$-\operatorname{div} b_\varphi(\cdot, \nabla^s u) + \frac{u}{\lambda^2} = L.$$

Moreover, there exists a constant C such that the inequality

$$\|u_1 - u_2\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \leq C\Big(\|\varphi_1 - \varphi_2\|_{L^p(\Omega;\mathbb{R}^{d\times d}_s)} + \|L_1 - L_2\|_{W^{-1,p}_D(\Omega;\mathbb{R}^d)}\Big)$$

holds for all $\varphi_1, \varphi_2 \in L^p(\Omega;\mathbb{R}^{d\times d}_s)$ and $L_1, L_2 \in W^{-1,p}_D(\Omega;\mathbb{R}^d)$, where u_1 and u_2 are the solutions with respect to (φ_1, L_1) and (φ_2, L_2).

Proof. Note that $b_\sigma(\cdot,\tau) \in L^p(\Omega;\mathbb{R}^{d\times d}_s)$ holds for all $\tau, \sigma \in L^p(\Omega;\mathbb{R}^{d\times d}_s)$ (and in fact for all $p \in [1,\infty]$), which follows from (8.5), (8.6) and (8.8) (taking into account that a pointwise limit of measurable functions is also measurable, see [104, Corollary 3.1.5]).

Let us at first consider the case $p = 2$. Then the existence of a solution follows from the Browder-Minty theorem, Korn's inequality and the Poincaré inequality. In order to verify the inequality, let $\varphi_1, \varphi_2 \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$, $L_1, L_2 \in H^{-1}(\Omega; \mathbb{R}^d)$ and $u_1, u_2 \in H^1_D(\Omega; \mathbb{R}^d)$ the corresponding solutions. Then we obtain

$$\begin{aligned}\langle L_1 - L_2, u_1 - u_2\rangle &= \big(b_{\varphi_1(\cdot)}(\cdot, \nabla^s u_1(\cdot)) - b_{\varphi_2(\cdot)}(\cdot, \nabla^s u_2(\cdot)), \nabla^s(u_1 - u_2)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left\|\frac{u_1 - u_2}{\lambda}\right\|^2_{L^2(\Omega;\mathbb{R}^d)}\\ &\geq m\|\nabla^s(u_1 - u_2)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - D\|\varphi_1 - \varphi_2\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\\ &\quad - D\int_\Omega |\nabla^s(u_1 - u_2)|\,|\varphi_1 - \varphi_2| + \frac{1}{\lambda^2}\|u_1 - u_2\|^2_{L^2(\Omega;\mathbb{R}^d)},\end{aligned}$$

hence, the asserted inequality is fulfilled.

For the general case let now $\varphi \in L^\infty(\Omega; \mathbb{R}^{d\times d}_s)$ and $L \in W^{-1,p}_D(\Omega; \mathbb{R}^d)$, we define $b : \Omega \times \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ by

$$b(x, \tau) := b_{\varphi(x)}(x, \tau)$$

and $L_u \in W^{-1,p}_D(\Omega; \mathbb{R}^d)$ by

$$\langle L_u, v\rangle := \langle L, v\rangle - \frac{1}{\lambda^2}\,(u, v)_{L^2(\Omega;\mathbb{R}^d)}\,,$$

where $u \in H^1_D(\Omega; \mathbb{R}^d) \hookrightarrow L^q(\Omega; \mathbb{R}^d)$, with $1 - \frac{d}{2} = -\frac{d}{q}$ when $d > 2$ and $q = \infty$ otherwise, is the solution in the case $p = 2$ and $v \in W^{1,p'}(\Omega; \mathbb{R}^d) \hookrightarrow L^{q'}(\Omega; \mathbb{R}^d)$ (note that $1 - \frac{d}{p'} + \frac{d}{q'} = 1 + \frac{d}{p} - \frac{d}{q} = 2 + \frac{d}{p} - \frac{d}{2} \geq 0$ when $d > 2$ and $1 - \frac{d}{p'} + \frac{d}{q'} = 1 - \frac{d}{p'} + d \geq 1 \geq 0$ otherwise). We can now apply Theorem 2.5 (here we need $\varphi \in L^\infty(\Omega; \mathbb{R}^{d\times d}_s)$ to satisfy (2.8)) to obtain $\overline{u} \in W^{1,p}_D(\Omega; \mathbb{R}^d)$ such that

$$\big(b(\cdot, \nabla^s\overline{u}), \nabla^s v\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} = \langle L_u, v\rangle,$$

that is,

$$\big(b_\varphi(\cdot, \nabla^s\overline{u}), \nabla^s v\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{1}{\lambda^2}\,(u, v)_{L^2(\Omega;\mathbb{R}^d)} = \langle L, v\rangle,$$

holds for all $v \in W^{1,p'}_D(\Omega; \mathbb{R}^d)$, we get in particular $u = \overline{u} \in W^{1,p}_D(\Omega; \mathbb{R}^d)$ since u is the unique solution of the equation above for all $v \in H^1_D(\Omega; \mathbb{R}^d)$.

To prove the asserted inequality let $\varphi_1, \varphi_2 \in L^\infty(\Omega; \mathbb{R}^{d\times d}_s)$, $L_1, L_2 \in W^{-1,p}_D(\Omega; \mathbb{R}^d)$ and $u_1, u_2 \in W^{1,p}(\Omega; \mathbb{R}^d)$ the corresponding solutions and define L_{u_1}, L_{u_2} as before. Considering the proof of Herzog et al. [50, Theorem 1.1] (Gröger [46, Theorem 1], respectively) one can see that there exists a constant $c > 0$, depending only on p, m and M (thus not on $L_1, L_2, \varphi_1, \varphi_2$), such that

$$\|u_1 - u_2\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \leq c\|A_1(u_2) - A_2(u_2) - L_{u_1} + L_{u_2}\|_{W^{-1,p}_D(\Omega;\mathbb{R}^d)},$$

where $A_i : W^{1,p}(\Omega; \mathbb{R}^d) \to W^{-1,p}_D(\Omega; \mathbb{R}^d)$ is defined by

$$\langle A_i(v_1), v_2\rangle := \Big(b_{\varphi_i}(\cdot, \nabla^s v_1), \nabla^s v_2\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

for all $v_1 \in W^{1,p}(\Omega;\mathbb{R}^d)$, $v_2 \in W^{1,p'}(\Omega;\mathbb{R}^d)$ and for $i \in \{1,2\}$. We finally obtain

$$\begin{aligned}\|u_1 - u_2\|_{W^{1,p}(\Omega;\mathbb{R}^d)} &\le c\Big(\|A_1(u_2) - A_2(u_2)\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} + \|L_{u_1} - L_{u_2}\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)}\Big) \\ &\le c\Big(M\|\varphi_1 - \varphi_2\|_{L^p(\Omega;\mathbb{R}^{d\times d}_s)} + \|L_1 - L_2\|_{W_D^{-1,p}(\Omega;\mathbb{R}^d)} \\ &\qquad + \frac{C}{\lambda^2}\|u_1 - u_2\|_{H^1(\Omega;\mathbb{R}^d)}\Big),\end{aligned}$$

where we have used again the embeddings $H^1(\Omega;\mathbb{R}^d) \hookrightarrow L^q(\Omega;\mathbb{R}^d)$ and $W^{1,p'}(\Omega;\mathbb{R}^d) \hookrightarrow L^{q'}(\Omega;\mathbb{R}^d)$. Taking into account that the assertion is already proven in the case $p = 2$, we see that the desired inequality holds.

One can now obtain the result for all $\varphi_1, \varphi_2 \in L^p(\Omega;\mathbb{R}^{d\times d}_s)$ by an approximation (using the just proven inequality to see that the corresponding sequence u_n is a Cauchy sequence). □

The operator R_0 in the following proposition will later be the resolvent, or a smoothed version of the resolvent, of A and should not be confused with R from Definition 8.5.

Proposition 8.10 (Solution operator $\mathcal{T}_{R_0}$). *Let $\lambda > 0$ and $p \ge 2$ as in Corollary 8.9 and $h = (h_1, h_2, h_3) \in \mathcal{Y}_p$. Moreover, let $R_0 : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ be Lipschitz continuous and monotone. Then there exists a unique solution $u \in W^{1,p}_D(\Omega;\mathbb{R}^d)$ of*

$$-\operatorname{div}(\mathbb{D}\nabla^s u + \mathbb{E}R_0(\mathbb{E}^\top\nabla^s(u - h_1) + h_3)) = \frac{h_2}{\lambda} + \frac{h_1 - u}{\lambda^2}. \tag{8.9}$$

We denote the solution operator of this equation by $\mathcal{T}_{R_0} : \mathcal{Y}_p \to W^{1,p}_D(\Omega;\mathbb{R}^d)$, that is, $\mathcal{T}_{R_0}(h) = u$. Furthermore, $\mathcal{T}_{R_0}$ is Lipschitz continuous. Note that the dependency of $\mathcal{T}_{R_0}$ on λ and p will always be clear from the context.

Proof. For all $\sigma \in \mathbb{R}^{d\times d}_s$ we define $b_\sigma : \Omega \times \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ by

$$b_\sigma(x,\tau) := \mathbb{D}\tau + \mathbb{E}R_0(\mathbb{E}^\top\tau + \sigma),$$

then the assertion follows from Corollary 8.9 (with $\varphi := -\mathbb{E}^\top\nabla^s h_1 + h_3$ for a given $h \in \mathcal{Y}_p$), let us only prove that (8.7) is fulfilled, the other requirements can be easily checked. To this end let $\sigma, \overline{\sigma}, \tau, \overline{\tau} \in \mathbb{R}^{d\times d}_s$, then

$$\begin{aligned}&(b_\sigma(x,\tau) - b_{\overline{\sigma}}(x,\overline{\tau})) : (\tau - \overline{\tau}) \\ &\quad\ge \gamma_{\mathbb{D}}|\tau - \overline{\tau}|^2 + \Big(R_0(\mathbb{E}^\top\tau + \sigma) - R_0(\mathbb{E}^\top\overline{\tau} + \overline{\sigma})\Big) : \Big(\mathbb{E}^\top(\tau - \overline{\tau}) + (\sigma - \overline{\sigma})\Big) \\ &\qquad - \Big(R_0(\mathbb{E}^\top\tau + \sigma) - R_0(\mathbb{E}^\top\overline{\tau} + \overline{\sigma})\Big) : (\sigma - \overline{\sigma}) \\ &\quad\ge \gamma_{\mathbb{D}}|\tau - \overline{\tau}|^2 - L_{R_0}|\sigma - \overline{\sigma}|^2 - L_{R_0}\|\mathbb{E}\|\,|\tau - \overline{\tau}|\,|\sigma - \overline{\sigma}|\end{aligned}$$

holds, where L_{R_0} is the Lipschitz constant of R_0. □

Note that $R_\lambda : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ (see Assumption ⟨IV.i⟩) fulfills the requirements in Proposition 8.10 since $R_\lambda : L^2(\Omega;\mathbb{R}^{d\times d}_s) \to L^2(\Omega;\mathbb{R}^{d\times d}_s)$ is Lipschitz continuous and also monotone (Proposition 3.7 Item (iii) and Proposition 3.9 Item (iii)) and due to (7.5) these properties carry over to $R_\lambda : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$.

Let us also mention that R_0 in Proposition 8.10 does not have to be monotone, the inequality

$$(R_0(a) - R_0(b)) : (a - b) \ge -\varepsilon|a - b|^2$$

for $a, b \in \mathbb{R}^{d\times d}_s$ with $\varepsilon < \gamma_{\mathbb{D}}/\|\mathbb{E}^\top\|^2$ would be sufficient.

We can now prove the existence of the resolvent of $\mathcal{A}$, from which we can then derive the maximal monotonicity of $\mathcal{A}$ in Proposition 8.13 below.

Proposition 8.11 (Existence of the resolvent of $\mathcal{A}$). *For every $\lambda > 0$ and $h = (h_1, h_2, h_3) \in \mathcal{H}$, the tuple*

$$\begin{pmatrix} u \\ v \\ q \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{R_\lambda}(h) \\ \frac{1}{\lambda}(\mathcal{T}_{R_\lambda}(h) - h_1) \\ R_\lambda(\mathbb{E}^\top \nabla^s(\mathcal{T}_{R_\lambda}(h) - h_1) + h_3) \end{pmatrix}$$

is contained in $D(\mathcal{A})$ and the unique solution of $(u, v, q) + \lambda\mathcal{A}(u, v, q) \ni h$.

Proof. Using the definition of $\mathcal{T}_{R_\lambda}$ we get

$$-\lambda \operatorname{div}(\mathbb{D}\nabla^s u + \mathbb{E}q) = h_2 - v,$$

which is the second row in $(u, v, q) + \lambda\mathcal{A}(u, v, q) \ni h$ and we also get $(u, v, q) \in D(\mathcal{A})$ (note that $rg(R_\lambda) \subset D(A)$). That the first and last row in $(u, v, q) + \lambda\mathcal{A}(u, v, q) \ni h$ is also fulfilled follows immediately from the definitions of u, v and q.

Furthermore, when (u, v, q) is a solution of $(u, v, q) + \lambda\mathcal{A}(u, v, q) \ni h$, then one verifies analog that (u, v, q) must have the claimed form, therefore the uniqueness follows from the uniqueness of a solution to (8.9). □

Lemma 8.12 (Monotonicity of $\mathcal{A}$). *The equation*

$$\big(\mathcal{A}(u_1, v_1, q_1) - \mathcal{A}(u_2, v_2, q_2), (u_1, v_1, q_1) - (u_2, v_2, q_2)\big)_{\mathcal{H}} = \big(A(q_1) - A(q_2), q_1 - q_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

holds for all $(u_1, v_1, q_1), (u_2, v_2, q_2) \in D(\mathcal{A})$.

Proof. Using the definition of $\mathcal{A}$ and the scalar product in $\mathcal{H}$ we obtain

$$\begin{aligned}
&\big(\mathcal{A}(u_1, v_1, q_1), (u_1, v_1, q_1) - (u_2, v_2, q_2)\big)_{\mathcal{H}} \\
&\quad = -\big(\mathbb{D}\nabla^s v_1, \nabla^s(u_1 - u_2)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - \big(\operatorname{div}(\mathbb{D}\nabla^s u_1 + \mathbb{E}q_1), v_1 - v_2\big)_{L^2(\Omega;\mathbb{R}^d)} \\
&\qquad\quad + \big(A(q_1) - \mathbb{E}^\top\nabla^s v_1, q_1 - q_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\quad = \big(\mathbb{D}\nabla^s v_1, \nabla^s u_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - \big(\mathbb{D}\nabla^s u_1, \nabla^s v_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - \big(\mathbb{E}^\top\nabla^s v_2, q_1\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\qquad\quad + \big(\mathbb{E}^\top\nabla^s v_1, q_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \big(A(q_1), q_1 - q_2\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)},
\end{aligned}$$

evaluating now $\big(\mathcal{A}(u_2, v_2, q_2), (u_1, v_1, q_1) - (u_2, v_2, q_2)\big)_{\mathcal{H}}$ and taking the difference yields the assertion. □

Proposition 8.13 ($\mathcal{A}$ is maximal monotone). *The operator $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone.*

Proof. The monotonicity of $\mathcal{A}$ follows immediately from Lemma 8.12 and the monotonicity of A.

To prove that $\mathcal{A}$ is maximal monotone, it is, according to Proposition 3.7, sufficient that $R(I + \mathcal{A}) = \mathcal{H}$, that is, we have to show that for every $(h_1, h_2, h_3) \in \mathcal{H}$ there exists $(u, v, q) \in D(\mathcal{A})$ such that $(u, v, q) + \mathcal{A}(u, v, q) \ni (h_1, h_2, h_3)$. This follows from Proposition 8.11 with $\lambda = 1$. □

In what follows it is convenient to give the integration operator a name.

Definition 8.14 (Integration operator). *We define $\mathcal{F} : H^1(L^2(\Omega; \mathbb{R}^d)) \to H^2(L^2(\Omega; \mathbb{R}^d))$ by $(\mathcal{F}f)(t) := \int_0^t f(s)ds$ for all $f \in H^1(L^2(\Omega; \mathbb{R}^d))$. Moreover, we abbreviate $\mathcal{F}_\rho := \mathcal{F}/\rho$. As usual, we denote the operators with different inverse images and ranges with the same symbol, for instance $\mathcal{F} : L^2(L^2(\Omega; \mathbb{R}^d)) \to H^1(L^2(\Omega; \mathbb{R}^d))$.*

Theorem 8.15 (Existence of a solution to the state equation). *There exists a unique solution $(u, v, q) \in H^1(\mathcal{H})$ of* (8.4). *Moreover, the inequality*

$$\|(\dot{u}, \dot{v}, \dot{q})\|_{L^2(\mathcal{H})} \leq C(1 + \|f\|_{H^1(L^2(\Omega;\mathbb{R}^d))})$$

holds, where the constant C does not depend on f.

Proof. The tuple (u, v, q) is a solution of (8.4) if and only if $w \in H^1(\mathcal{H})$ solves

$$\dot{w} + \mathcal{A}_Q(w) \ni R\dot{F}, \qquad w(0) = (u_0, v_0, q_0),$$

where $F := \mathcal{F}_\rho f \in H^2(L^2(\Omega; \mathbb{R}^d))$, and $w = (u, v, q)$ (recall the notation of $\mathcal{A}_Q$ given in Definition 3.3 and note that $QR = (0, \rho\, \cdot, 0)$). The assertion follows now from Theorem 4.7. □

Remark 8.16 ($\mathcal{A}$ is not a subdifferential). *Let us show that the maximal monotone operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is not a subdifferential, that is, there exists no proper, convex and lower semicontinuous function $\Phi : \mathcal{H} \to (-\infty, \infty]$ such that*

$$\mathcal{A}(u, v, q) = \partial\Phi(u, v, q) = \{(c, d, e) \in \mathcal{H} : \Phi(\hat{u}, \hat{v}, \hat{q}) \geq \Phi(u, v, q) + ((c, d, e), (\hat{u} - u, \hat{v} - v, \hat{q} - q))_{\mathcal{H}} \ \forall(\hat{u}, \hat{v}, \hat{q}) \in \mathcal{H}\}$$

holds for all $(u, v, q) \in \mathcal{H}$. In fact, there exists even not any function $\Phi : \mathcal{H} \to (-\infty, \infty]$ such that the equation above holds, which can be seen as follows:

Let us assume that such a Φ exists and recall that $(u_0, v_0, q_0) \in D(\mathcal{A})$ according to Assumption ⟨IV.iii⟩. Then, using Lemma 8.12 with $(u_1, v_1, q_1) = (u + u_0, v, q_0)$ and $(u_2, v_2, q_2) = (u_0, 0, q_0)$,

$$\begin{aligned}\Phi(u_0, 0, q_0) &\geq \Phi(u + u_0, v, q_0) - \big(\mathcal{A}(u + u_0, v, q_0), (u, v, 0)\big)_{\mathcal{H}} \\ &= \Phi(u + u_0, v, q_0) - \big(\mathcal{A}(u_0, 0, q_0), (u, v, 0)\big)_{\mathcal{H}} \\ &\geq \Phi(u_0, 0, q_0) + \big(\mathcal{A}(u_0, 0, q_0), (u, v, 0)\big)_{\mathcal{H}} - \big(\mathcal{A}(u_0, 0, q_0), (u, v, 0)\big)_{\mathcal{H}} \\ &= \Phi(u_0, 0, q_0)\end{aligned}$$

holds for all (u, v) such that $(u + u_0, v, q_0) \in D(\mathcal{A})$, hence,

$$\begin{aligned}\big(\mathcal{A}(u_0, 0, q_0), (\hat{u} - u, \hat{v} - v, 0)\big)_{\mathcal{H}} &= \Phi(\hat{u} + u_0, \hat{v}, q_0) - \Phi(u + u_0, v, q_0) \\ &\geq \big(\mathcal{A}(u + u_0, v, q_0), (\hat{u} - u, \hat{v} - v, 0)\big)_{\mathcal{H}}\end{aligned}$$

which gives

$$0 \geq (\mathbb{D}\nabla^s u, \nabla^s \hat{v})_{\mathcal{H}} - (\mathbb{D}\nabla^s \hat{u}, \nabla^s v)_{\mathcal{H}}$$

for all $(\hat{u}, \hat{v}), (u, v)$ such that $(u + u_0, v, q_0), (\hat{u} + u_0, \hat{v}, q_0) \in D(\mathcal{A})$. Choosing now an arbitrary $u \in C_c^\infty(\Omega; \mathbb{R}^d)$, $u \neq 0$, $\hat{v} = u$ and $\hat{u} = v = 0$, we obtain the desired contradiction.

In light of Remark 8.16, the case of plasticity with inertia essentially differs from elasto and homogenized plasticity in two aspects. First, we gain more regularity in time as explained after Lemma 8.6. Second, we lose the fact that $\mathcal{A}$ has the boundedness property (which is easily seen by its definition) and it is not a subdifferential. This is exactly the second case we have considered in Chapter 4. It is also to be noted that Remark 8.16 is independent of the operator A.

8.3 Regularization and Convergence Results

As explained above, the case of plasticity of inertia fits into the case of EVIs with H^2-loads considered in Part II (as was already the case for Theorem 8.15) so that we can apply Theorem 4.14 and Corollary 4.15.

Theorem 8.17 (Weak convergence of the state). *Let $\{f_n\}_{n\in\mathbb{N}} \subset H^1(L^2(\Omega;\mathbb{R}^d))$ such that $f_n \rightharpoonup f$ in $H^1(L^2(\Omega;\mathbb{R}^d))$ and $\mathcal{F}f_n \to \mathcal{F}f$ in $L^1(L^2(\Omega;\mathbb{R}^d))$. Moreover, let $(u,v,q) \in H^1(\mathcal{H})$ be the solution of (8.4) and $(u_n,v_n,q_n) \in H^1(\mathcal{H})$, for every $n \in \mathbb{N}$, the solution of*

$$Q^{-1}(\dot{u}_n,\dot{v}_n,\dot{q}_n) + \mathcal{A}(u_n,v_n,q_n) \ni Rf_n, \qquad (u_n,v_n,q_n)(0) = (u_0,v_0,q_0)$$

or

$$Q^{-1}(\dot{u}_n,\dot{v}_n,\dot{q}_n) + \mathcal{A}_{\lambda_n}(u_n,v_n,q_n) = Rf_n, \qquad (u_n,v_n,q_n)(0) = (u_0,v_0,q_0),$$

where $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\lambda_n \searrow 0$.

Then $(u_n,v_n,q_n) \rightharpoonup (u,v,q)$ in $H^1(\mathcal{H})$ and $(u_n,v_n,q_n) \to (u,v,q)$ in $C(H^1(\Omega;\mathbb{R}^d))\times L^1(L^2(\Omega;\mathbb{R}^d))\times C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$. If additionally $\mathcal{F}f_n \to \mathcal{F}f$ in $C(L^2(\Omega;\mathbb{R}^d))$, then $v_n \to v$ in $C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$.

Proof. We can apply a transformation analog to the one in Theorem 8.15, the assertion follows then from Theorem 4.14 (where q therein takes the form (u,v,q) and q_n the form (u_n,v_n,q_n)). Note that the convergence $\mathfrak{Z}(q_n,\ell_n) \to \mathfrak{Z}(q,\ell)$ in Theorem 4.14 then means $Q^{-1}(R\mathcal{F}f_n - (u_n,v_n,q_n)) \to Q^{-1}(R\mathcal{F}f - (u,v,q))$ in $C(\mathcal{H})$, so that the convergence $(u_n,v_n,q_n) \to (u,v,q)$ in $C(H^1(\Omega;\mathbb{R}^d)) \times L^1(L^2(\Omega;\mathbb{R}^d)) \times C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ follows from the fact that the range of R is a subset of $\{0\} \times L^2(\Omega;\mathbb{R}^d) \times \{0\}$. □

Proposition 8.18 (Strong convergence for fixed forces). *Let $(u,v,q) \in H^1(\mathcal{H})$ be the solution of (8.4) and $(u_n,v_n,q_n) \in H^1(\mathcal{H})$, for every $n \in \mathbb{N}$, the solution of*

$$Q^{-1}(\dot{u}_n,\dot{v}_n,\dot{q}_n) + \mathcal{A}_{\lambda_n}(u_n,v_n,q_n) = Rf, \qquad (u_n,v_n,q_n)(0) = (u_0,v_0,q_0).$$

where $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\lambda_n \searrow 0$.

Then $(u_n,v_n,q_n) \to (u,v,q)$ in $H^1(\mathcal{H})$.

Proof. Again, we apply a transformation analog to the one in Theorem 8.15, then the assertion follows from Corollary 4.15 (where q therein takes the form (u,v,q) and q_n the form (u_n,v_n,q_n)). □

With these results we have collected everything we need to analyze an optimal control problem with plasticity with inertia as a constraint in the upcoming chapter.

Chapter 9 Optimal Control

The procedure in this chapter is similar to the one in Chapter 5. Since the smoothed operator $\mathcal{A}_s$, given in Definition 9.6, possesses the required properties for A_s in Section 5.2, we can apply the finding concerned with the differentiability of the solution operator associated with the EVI therein (the boundedness property of A is not necessary in Section 5.2, see the beginning of Chapter 5). Before we give the details in Section 9.2, we tend to the existence and approximation of optimal controls.

9.1 Existence and Approximation of Optimal Controls

We consider the following optimal control problem:

$$\begin{cases} \min & J(u,\dot{u},z,f) = \Psi(u,\dot{u},z) + \frac{\alpha}{2}\|f\|^2_{\mathfrak{X}_c}, \\ \text{s.t.} & \rho\ddot{u} - \operatorname{div}\mathbb{C}(\nabla^s u - z) = f, \\ & \dot{z} \in A(\mathbb{C}\nabla^s u - (\mathbb{C}+\mathbb{B})z), \\ & (u,\dot{u},z)(0) = (u_0,v_0,z_0), \\ & u \in H^1(H^1_D(\Omega;\mathbb{R}^d)) \cap H^2(L^2(\Omega;\mathbb{R}^d)), \\ & z \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)), \\ & f \in \mathfrak{X}_c. \end{cases} \tag{9.1}$$

Note that the stress is not present in (9.1) but can be easily integrated in Ψ due to $\sigma = \mathbb{C}(\nabla^s u - z)$. Moreover, we require in Assumption ⟨9⟩ that Ψ is defined on $L^2(\mathcal{H})$ and not on $H^1(\mathcal{H})$, which excludes for example evaluations at certain points in time. Analog to Section 5.2 we could also consider an objective function on $H^1(\mathcal{H})$, then we would only obtain a (possible) weak solution ϕ of the adjoint state in Theorem 9.13 as in Theorem 5.13. We decided to define Ψ on $L^2(\mathcal{H})$ only for simplicity and to keep the discussion concise.

Since we have transformed our state equation (8.1) into (8.3) by introducing the new variable q, it is reasonable to do the same with the optimal control problem. To this end, we need the following

Definition 9.1 (Transformed objective function)**.** *We define*

$$\Psi_z : L^2(\mathcal{H}) \to \mathbb{R}, \qquad (u,v,q) \mapsto \Psi(u,v,\mathfrak{Z}(u,q))$$

and the transformed objective function

$$J_z : L^2(\mathcal{H}) \times \mathfrak{X}_c \to \mathbb{R}, \qquad (u,v,q,f) \mapsto \Psi_z(u,v,q) + \frac{\alpha}{2}\|f\|^2_{\mathfrak{X}_c}$$

Using the definition above and the transformation of the state equation into (8.4), we obtain the equivalence of (9.1) and

$$\begin{cases} \min & J_z(u,v,q,f) = \Psi_z(u,v,q) + \frac{\alpha}{2}\|f\|^2_{\mathfrak{X}_c}, \\ \text{s.t.} & Q^{-1}(\dot{u},\dot{v},\dot{q}) + \mathcal{A}(u,v,q) \ni Rf, \quad (u,v,q)(0) = (u_0,v_0,q_0), \\ & (u,v,q) \in H^1(H^1_D(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)), \\ & f \in \mathfrak{X}_c. \end{cases} \tag{9.2}$$

Let us shortly interrupt the discussion and give two examples for the control space $\mathfrak{X}_c$.

Example 9.2 (Control space)**.** *In order to satisfy Assumption ⟨9.1⟩, we can use the lemma of Lions-Aubin (cf.* SHOWALTER *[90, III. Proposition 1.3]) and for instance choose* $\mathfrak{X}_c = H^1(L^2(\Omega;\mathbb{R}^d)) \cap L^2(H^1(\Omega;\mathbb{R}^d))$ *or* $\mathfrak{X}_c = \{f \in H^1(L^2(\Omega;\mathbb{R}^d)) : \mathcal{F}f \in L^2(H^1(\Omega;\mathbb{R}^d))\}$ *with corresponding norms.*

Let us now select a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ such that $\lambda_n \searrow 0$. We consider the regularized optimization problem

$$\begin{cases} \min & J_z(u,v,q,f) = \Psi_z(u,v,q) + \frac{\alpha}{2}\|f\|^2_{\mathfrak{X}_c}, \\ \text{s.t.} & Q^{-1}(\dot{u},\dot{v},\dot{q}) + \mathcal{A}_{\lambda_n}(u,v,q) = Rf, \quad (u,v,q)(0) = (u_0,v_0,q_0) \\ & (u,v,q) \in H^1(H^1_D(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)), \\ & f \in \mathfrak{X}_c. \end{cases} \tag{9.3}$$

Theorem 9.3 (Existence and approximation of optimal solutions)**.** *There exists a global solution of* (9.2) *(and thus of* (9.1)*) and of* (9.3) *for every* $n \in \mathbb{N}$*.*

Moreover, let $(\overline{u}_n, \overline{v}_n, \overline{q}_n, \overline{f}_n)_{n\in\mathbb{N}}$ *be a sequence of global solution of* (9.3)*. Then there exists a weak accumulation point* $(\overline{u}, \overline{v}, \overline{q}, \overline{f})$ *and every weak accumulation point is a global solution of* (9.2)*. The subsequence of states which converges weakly towards* $(\overline{u}, \overline{v}, \overline{q})$ *in* $H^1(\mathcal{H})$ *converges also strongly in* $C(H^1(\Omega;\mathbb{R}^d)) \times L^1(L^2(\Omega;\mathbb{R}^d)) \times C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *and, when* $\mathcal{F}$ *is compact from* $\mathfrak{X}_c$ *into* $C(L^2(\Omega;\mathbb{R}^d))$*, then the subsequence of* v_n *converges also strongly in* $C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$*. Moreover, the subsequence of controls converges strongly to* $\overline{f}$ *in* $\mathfrak{X}_c$*.*

Proof. The existence of a global solution to (9.2) follows from the standard direct method of the calculus of variations using Theorem 8.17 and Assumption ⟨9.1⟩, the proof is analog to the proof of Theorem 5.2. The existence of a global solution to (9.3) follows easily using the Lipschitz continuity of $\mathcal{A}_{\lambda_n}$ (which implies the Lipschitz continuity of the corresponding solution operator), see Proposition 5.3.

The convergence result can also be obtained by standard arguments using again Theorem 8.17 and Proposition 8.18, the proof is analog to the one from Theorem 5.4 and Corollary 5.5. Note that the strong convergence of the states in $C(H^1(\Omega;\mathbb{R}^d)) \times L^1(L^2(\Omega;\mathbb{R}^d)) \times C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$, and also of v_n in $C(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ when $\mathcal{F}$ is compact from $\mathfrak{X}_c$ into $C(L^2(\Omega;\mathbb{R}^d))$, follows directly from Theorem 8.17. □

Note that a strong convergence result of the states (in $H^1(\mathcal{H})$) is not provided in the theorem above. In Corollary 5.5 we were able to prove the strong convergence either when the associated maximal monotone operator is a subdifferential, which is here not the case (Remark 8.16), or when it can be deduced from the weak convergence and the convergence of the evaluations of Ψ. Since we supposed that Ψ is defined on $L^2(\mathcal{H})$, this cannot be the case. However, as elaborated on at the beginning of this section, it is possible for instance to consider a different Ψ defined on $H^1(\mathcal{H})$ such that this property holds.

Having dealt with the existence and approximation of optimal solutions we turn to the optimality condition for a further smoothed problem.

9.2 Optimality Conditions

In order to derive first order optimality conditions we smoothen at first the optimal control problem further. Then we prove the differentiability of the smoothed solution operator and can after that finally present our main result in this part, the KKT conditions for the smoothed optimization problem.

Smoothed Optimization Problem

Thanks to Proposition 8.11, we can give the precise form of the resolvent and Yosida approximation of $\mathcal{A}$ in the following

Corollary 9.4 (Precise form of the resolvent)**.** *Let* $\lambda > 0$ *and denote the resolvent of* $\mathcal{A}$ *by* $\mathcal{R}_\lambda$*. Then*

$$\mathcal{R}_\lambda(h) = \begin{pmatrix} \mathcal{T}_{R_\lambda}(h) \\ \frac{1}{\lambda}\mathcal{T}_{R_\lambda}(h) - \frac{h_1}{\lambda} \\ R_\lambda(\mathbb{E}^\top \nabla^s(\mathcal{T}_{R_\lambda}(h) - h_1) + h_3) \end{pmatrix}$$

so that

$$\mathcal{A}_\lambda(h) = \frac{1}{\lambda}\begin{pmatrix} h_1 - \mathcal{T}_{R_\lambda}(h) \\ h_2 - \frac{1}{\lambda}\mathcal{T}_{R_\lambda}(h) + \frac{h_1}{\lambda} \\ h_3 - R_\lambda(\mathbb{E}^\top \nabla^s(\mathcal{T}_{R_\lambda}(h) - h_1) + h_3) \end{pmatrix}$$

for every $h = (h_1, h_2, h_3) \in \mathcal{H}$.

The Yosida approximation $\mathcal{A}_\lambda$ is in view of Proposition 8.10 Lipschitz continuous from $\mathcal{Y}_p$ to $\mathcal{Y}_p$, where p is given in Assumption ⟨9.2.ii⟩. Therefore the state equation in (9.3) admits a solution in $\mathcal{Y}_p$ (note that R maps into $\mathcal{Y}_\infty$). However, since this regularity is not present in (9.1), we did not use it. In contrast, the same is true for the smoothed Yosida approximation, which is given below in Definition 9.6 (see Definition 9.10), but here this additional regularity will be used to prove the differentiability of the smoothed solution operator in Proposition 9.11.

In order to smoothen the Yosida approximation, respectively the resolvent, of $\mathcal{A}$, we smoothen the resolvent of A and then define the smoothed resolvent for $\mathcal{A}$ analog to $\mathcal{R}_\lambda$. We denote this smoothed resolvent of A by $R_s : \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ (which indicates that the resolvent of A can be expressed pointwise), from the properties given in Assumption ⟨9.2.i⟩ one can easily derive the following inequalities, which will be useful when proving the differentiability of $\mathcal{T}_{R_s}$ in Lemma 9.8 below.

Lemma 9.5 (Properties of R_s')**.** *There exists a constant C such that $|R_s'(\sigma)\tau| \le C|\tau|$ and $0 \le R_s'(\sigma)\tau : \tau$ holds for all $\sigma, \tau \in \mathbb{R}^{d\times d}_s$. Moreover, the same is true for $R_s'(\cdot)^*$.*

Proof. Let $\sigma, \tau \in \mathbb{R}^{d\times d}_s$ be arbitrary. The Lipschitz continuity and Fréchet differentiability of R_s gives

$$\Big|\frac{r(t\tau)}{t} + R_s'(\sigma)\tau\Big| = \frac{|R_s(\sigma + t\tau) - R_s(\sigma)|}{t} \le L|\tau|$$

for all $t \in \mathbb{R} \setminus \{0\}$, where r is the remainder term of R_s. The limit $t \to 0$ yields the first assertion.

The second claim follows using the monotonicity,

$$0 \le \frac{R_s(\sigma + t\tau) - R_s(\sigma)}{t} : \tau \to R_s'(\sigma)\tau : \tau$$

as $0 \ne t \to 0$.

Now, by definition we have $R_s'(\sigma)\tau : \eta = \tau : R_s'(\sigma)^*\eta$ for all $\sigma, \tau, \eta \in \mathbb{R}^{d\times d}_s$, so that the second assertion also holds for $R_s'(\cdot)^*$. Choosing in particular $\tau = R_s(\sigma)^*\eta$ we get

$$|R_s'(\sigma)^*\eta|^2 = |R_s'(\sigma)R_s'(\sigma)^*\eta : \eta| \le C|R_s'(\sigma)^*\eta|\,|\eta|,$$

which yields the first assertion for $R_s'(\cdot)^*$. □

Definition 9.6 (Smoothed resolvent)**.** *Let $\lambda_s \in (0, \infty)$. We define*

$$\mathcal{R}_s : \mathcal{Y}_p \to \mathcal{Y}_p, \quad h = (h_1, h_2, h_3) \mapsto \begin{pmatrix} \mathcal{T}_{R_s}(h) \\ \frac{1}{\lambda_s}\mathcal{T}_{R_s}(h) - \frac{h_1}{\lambda_s} \\ R_s(\mathbb{E}^\top \nabla^s(\mathcal{T}_{R_s}(h) - h_1) + h_3) \end{pmatrix}$$

and $\mathcal{A}_s := \frac{1}{\lambda_s}(I - \mathcal{R}_s)$ (see Assumption ⟨9.2.ii⟩ for p). According to Proposition 8.10 and Assumption ⟨9.2.i⟩, $\mathcal{R}_s$ and $\mathcal{A}_s$ are well defined and Lipschitz continuous. As usual, with a slight abuse of notation, we denote operators for different p with the same symbol.

Let us now consider the smoothed optimization problem

$$\begin{cases} \min & J_z(u,v,q,f) = \Psi_z(u,v,q) + \frac{\alpha}{2}\|f\|^2_{\mathfrak{X}_c}, \\ \text{s.t.} & Q^{-1}(\dot{u},\dot{v},\dot{q}) + \mathcal{A}_s(u,v,q) = Rf, \quad (u,v,q)(0) = (u_0,v_0,q_0) \\ & (u,v,q) \in H^1(H^1_D(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^{d\times d}_s)), \\ & f \in \mathfrak{X}_c. \end{cases} \tag{9.4}$$

Analog to Theorem 9.3 one can analogously prove that there exists a global solution of (9.4).

As we did in Theorem 5.4, when $\mathcal{A}_s$ and $\mathcal{A}_{\lambda_s}$ are globally "close together", one can prove a result analog to the convergence result in Theorem 9.3 with a sequence $(\overline{u}_s,\overline{v}_s,\overline{q}_s,\overline{f}_s)_{s>0}$ of global solutions to (9.4) when $\sup_{h\in\mathcal{H}}\|\mathcal{A}_{\lambda_s}(h)-\mathcal{A}_s(h)\|_{\mathcal{H}}$ tends fast enough to zero relative to λ_s. The following lemma shows that this is the case when the same is true for $\frac{1}{\lambda_s}\sup_{\tau\in L^2(\Omega;\mathbb{R}^{d\times d}_s)}\|A_{\lambda_s}(\tau)-A_s(\tau)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$ with $A_s = \frac{1}{\lambda_s}(I-R_s)$, which holds in the case of the von-Mises flow rule for suitable sequences $\{\lambda_s\}_{\lambda_s>0}$ and $\{s\}_{s>0}$, cf. (3.12). Note also that Lemma 4.17 was used in Theorem 5.4, so that it was in particular required that $\sup_{\tau\in L^2(\Omega;\mathbb{R}^{d\times d}_s)}\|A_{\lambda_s}(\tau)-A_s(\tau)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$ tends faster to zero than $\exp(\frac{1}{\lambda_s})$, thus the additional factor $\frac{1}{\lambda_s}$ does not play a big role.

Lemma 9.7 (Convergence of the smoothed resolvent)**.** *The inequality*

$$\|\mathcal{A}_{\lambda_s}(h)-\mathcal{A}_s(h)\|_{\mathcal{H}} \le C\sqrt{1+\frac{1}{\lambda_s^2}}\sup_{\tau\in L^2(\Omega;\mathbb{R}^{d\times d}_s)}\|A_{\lambda_s}(\tau)-A_s(\tau)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

holds for all $h\in L^2(\Omega;\mathbb{R}^{d\times d}_s)$, where $A_s := \frac{1}{\lambda_s}(I-R_s)$ and the constant does only depend on $\mathbb{C}$ and $\mathbb{B}$, $C = C(\mathbb{C},\mathbb{B})$.

Proof. Let us abbreviate

$$M := \sup_{\tau\in L^2(\Omega;\mathbb{R}^{d\times d}_s)}\|R_{\lambda_s}(\tau)-R_s(\tau)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}.$$

Due to the definitions of $\mathcal{A}_s$ and A_s we only have to prove that

$$\|\mathcal{R}_{\lambda_s}(h)-\mathcal{R}_s(h)\|_{\mathcal{H}} \le C\sqrt{1+\frac{1}{\lambda_s^2}}M \tag{9.5}$$

holds for all $h\in\mathcal{H}$. To this end let $h\in\mathcal{H}$ be arbitrary and abbreviate $u := \mathcal{T}_{R_{\lambda_s}}(h), u_s := \mathcal{T}_{R_s}(h)\in H^1_D(\Omega;\mathbb{R}^d)$, hence, u is the solution of (8.9) with respect to R_{λ_s} and u_s with respect to R_s, testing both equations with $u-u_s$ and subtracting the second from the first, we get

$$\begin{aligned} &\|\nabla^s(u-u_s)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{D}}} + \left\|\frac{u-u_s}{\lambda_s}\right\|^2_{L^2(\Omega;\mathbb{R}^d)} \\ &= -\Big(\mathbb{E}(R_{\lambda_s}(w)-R_s(w_s)),\nabla^s(u-u_s)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ &= -\Big((R_{\lambda_s}(w_s)-R_s(w_s)),w-w_s\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - \Big((R_{\lambda_s}(w)-R_{\lambda_s}(w_s)),w-w_s\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ &\le -\Big(\mathbb{D}^{-1}\mathbb{E}(R_{\lambda_s}(w_s)-R_s(w_s)),\nabla^s(u-u_s)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{D}}} \\ &\le \frac{1}{2}\|\mathbb{D}^{-1}\mathbb{E}(R_{\lambda_s}(w)-R_s(w_s))\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{D}}} + \frac{1}{2}\|\nabla^s(u-u_s)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{D}}} \\ &\le \frac{\|\mathbb{E}^\top\mathbb{D}^{-1}\mathbb{E}\|}{2}M^2 + \frac{1}{2}\|\nabla^s(u-u_s)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{D}}} \end{aligned}$$

with $w := \mathbb{E}^\top \nabla^s(u - h_1) + h_3)$ and $w_s := \mathbb{E}^\top \nabla^s(u_s - h_1) + h_3)$, where we used in particular the monotonicity of R_{λ_s}. Thus we obtain

$$\|\nabla^s(u - u_s)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{D}}} + \left\|\frac{u - u_s}{\lambda_s}\right\|^2_{L^2(\Omega;\mathbb{R}^d)} \le CM^2. \tag{9.6}$$

We get further

$$\begin{aligned}\|R_{\lambda_s}(w) - R_s(w_s)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} &\le \|R_{\lambda_s}(w) - R_{\lambda_s}(w_s)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \|R_{\lambda_s}(w_s) - R_s(w_s)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ &\le \frac{C}{\lambda_s} M + M\end{aligned}$$

where we have used (9.6). We arrive at

$$\|\mathcal{R}_{\lambda_s}(h) - \mathcal{R}_s(h)\|^2_{\mathcal{H}} \le CM^2 + \frac{C}{\lambda_s^2} M^2$$

which implies (9.5). □

KKT-Conditions

To establish KKT conditions we first need to prove the Fréchet differentiability of the smoothed solution operator of the constraint in (9.4). To this end, we need two norm gaps in Lemma 9.8 and Proposition 9.9, recall that the corresponding coefficients are fixed in Assumption ⟨9.2.ii⟩.

Lemma 9.8 (Fréchet differentiability of $\mathcal{T}_{R_s}$). *The operator $\mathcal{T}_{R_s}$ is from $\mathcal{Y}_p$ into $W^{1,\hat{p}}_D(\Omega;\mathbb{R}^d)$ Fréchet differentiable and, for $h, g \in \mathcal{Y}_p$, $\eta := \mathcal{T}'_{R_s}(h)g$ is of class $W^{1,p}_D(\Omega;\mathbb{R}^d)$ and the unique solution of*

$$-\operatorname{div}(\mathbb{D}\nabla^s\eta + \mathbb{E}R'_s(\mathbb{E}^\top\nabla^s(u - h_1) + h_3)(\mathbb{E}^\top\nabla^s(\eta - g_1) + g_3))) = \frac{g_2}{\lambda_s} + \frac{g_1 - \eta}{\lambda_s^2}, \tag{9.7}$$

where $u := \mathcal{T}_{R_s}(h)$.

Moreover, there exists a constant C such that the extension of $\mathcal{T}'_{R_s}(h)$ to an element of $L(\mathcal{H}; H^1_D(\Omega;\mathbb{R}^d))$ fulfills $\|\mathcal{T}'_{R_s}(h)g\|_{H^1_D(\Omega;\mathbb{R}^d)} \le C\|g\|_{\mathcal{H}}$ for all $h \in \mathcal{Y}_p$ and $g \in \mathcal{H}$.

Proof. Let $h, g \in \mathcal{Y}_p$. At first we prove that (9.7) has a unique solution $\eta \in W^{1,p}_D(\Omega;\mathbb{R}^d)$ with respect to h and g. For $\sigma \in \mathbb{R}^{d\times d}_s$ we define $b_\sigma : \Omega \times \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s$ by

$$b_\sigma(x,\tau) := \mathbb{D}\tau + \mathbb{E}R'_s(\mathbb{E}^\top\nabla^s(u(x) - h_1(x)) + h_3(x)))(\mathbb{E}^\top\tau + \sigma)$$

for almost all $x \in \Omega$ and all $\tau \in \mathbb{R}^{d\times d}_s$. The existence of η follows now from Corollary 8.9 (with $\varphi := -\mathbb{E}^\top\nabla^s g_1 + g_3$), when we have verified the requirements on b_σ therein. Moreover, Corollary 8.9 also shows that the solution operator of (9.7) is continuous with respect to $g \in \mathcal{Y}_p$ (clearly, it is also linear).

Clearly, $b_0(x, 0) = 0 \in L^\infty(\Omega;\mathbb{R}^{d\times d}_s)$ and $b_\sigma(\cdot,\tau)$ is measurable as a pointwise limit of measurable functions (see [104, Corollary 3.1.5]), for all $\tau, \sigma \in \mathbb{R}^{d\times d}_s$. Moreover, we have

$$\begin{aligned}(b_\sigma(x,\tau) &- b_{\overline{\sigma}}(x,\overline{\tau})) : (\tau - \overline{\tau}) \\ &\ge \gamma_{\mathbb{D}}|\tau - \overline{\tau}|^2 + R'_s(w(x))(\mathbb{E}^\top(\tau - \overline{\tau}) + (\sigma - \overline{\sigma})) : \mathbb{E}^\top(\tau - \overline{\tau}) \\ &\ge \gamma_{\mathbb{D}}|\tau - \overline{\tau}|^2 - C|\sigma - \overline{\sigma}|\,|\tau - \overline{\tau}|,\end{aligned}$$

with $w := \mathbb{E}^\top \nabla^s(u-h)+h_3)$, and

$$|b_\sigma(x,\tau)-b_{\overline{\sigma}}(x,\overline{\tau})| \le C\Big(|\tau-\overline{\tau}|+|\sigma-\overline{\sigma}|\Big)$$

for all $\sigma,\overline{\sigma},\tau,\overline{\tau}\in\mathbb{R}^{d\times d}_s$ and almost all $x\in\Omega$, where we have used Lemma 9.5 in both estimations. Therefore (8.5) to (8.8) are fulfilled.

Considering now the equations for $u_g := \mathcal{T}_{R_s}(h+g)$ and $u := \mathcal{T}_{R_s}(h)$, we see that

$$\begin{aligned}-\operatorname{div}(\mathbb{D}\nabla^s(u_g-u-\eta))+\frac{u_g-u-\eta}{\lambda_s^2} &= \operatorname{div}(\mathbb{E}(R_s(\mu+v_g)-R_s(\mu)-R_s'(\mu)v_g))\\ &\quad+\operatorname{div}(\mathbb{E}R_s'(\mu)((\mathbb{E}^\top\nabla^s(u_g-u-\eta))),\end{aligned}$$

where

$$\begin{aligned}\mu &:= \mathbb{E}^\top\nabla^s(u-h_1)+h_3),\\ v_g &:= \mathbb{E}^\top\nabla^s(u_g-u-g_1)+g_3)\in L^p(\Omega;\mathbb{R}^{d\times d}_s),\end{aligned}$$

hence,

$$-\operatorname{div}(\mathbb{D}\nabla^s(u_g-u-\eta)-\mathbb{E}R_s'(\mu)((\mathbb{E}^\top\nabla^s(u_g-u-\eta)))+\frac{u_g-u-\eta}{\lambda_s^2}=\operatorname{div}\mathbb{E}r_\mu(v_g),$$

where $r_\mu(v_g)$ is the remainder term of R_s at μ in direction v_g. Applying Corollary 8.9 (Theorem 2.5 is in fact sufficient at this point) once again with

$$b_\sigma(x,\tau) := \mathbb{D}\tau+\mathbb{E}R_s'(\mu(x))\mathbb{E}^\top\tau$$

(and $p=\hat{p}$) we obtain

$$\frac{\|u_g-u-\eta\|_{W^{1,\hat{p}}(\Omega;\mathbb{R}^d)}}{\|g\|_{\mathcal{Y}_p}} \le C\frac{\|r_\mu(v_g)\|_{L^{\hat{p}}(\Omega;\mathbb{R}^{d\times d}_s)}}{\|g\|_{\mathcal{Y}_p}} \le C\frac{\|r_\mu(v_g)\|_{L^{\hat{p}}(\Omega;\mathbb{R}^{d\times d}_s)}}{\|v_g\|_{L^p(\Omega;\mathbb{R}^{d\times d}_s)}}\to 0,$$

as $g\to 0$ in $\mathcal{Y}_p$, where we also used the Lipschitz continuity of $\mathcal{T}_{R_s}$ and the fact that $R_s : L^p(\Omega;\mathbb{R}^{d\times d}_s)\to L^{\hat{p}}(\Omega;\mathbb{R}^{d\times d}_s)$ is Fréchet differentiable (cf. GOLDBERG [44, Theorem 7]).

That the extension of $\mathcal{T}_{R_s}'(h)$ to an element of $L(\mathcal{H};H^1_D(\Omega;\mathbb{R}^d))$ fulfills the asserted inequality, can be proven as above (one can simply test (9.7) with $\eta\in H^1_D(\Omega;\mathbb{R}^d)$ and use Lemma 9.5). □

Proposition 9.9 (Fréchet differentiability of $\mathcal{A}_s$)**.** *The mapping $\mathcal{R}_s$ is from $\mathcal{Y}_p$ to $\mathcal{H}$ Fréchet differentiable and there exists a constant C such that the extension of $\mathcal{R}_s'(h)\in\mathcal{L}(\mathcal{Y}_p,\mathcal{H})$ to an element of $\mathcal{L}(\mathcal{H})$ fulfills $\|\mathcal{R}_s'(h)g\|_\mathcal{H}\le C\|g\|_\mathcal{H}$ for all $h\in\mathcal{Y}_p$ and $g\in\mathcal{H}$.*

For $h\in\mathcal{Y}_p$ and $g\in\mathcal{H}$ we have

$$\mathcal{R}_s'(h)g=\begin{pmatrix}\mathcal{T}_{R_s}'(h)g\\ \frac{1}{\lambda_s}\mathcal{T}_{R_s}'(h)g-\frac{g_1}{\lambda_s}\\ R_s'(\mathbb{E}^\top\nabla^s(\mathcal{T}_{R_s}(h)-h_1)+h_3)(\mathbb{E}^\top\nabla^s(\mathcal{T}_{R_s}'(h)g-g_1)+g_3)\end{pmatrix}$$

The same is true for $\mathcal{A}_s=\frac{1}{\lambda_s}(I-\mathcal{R}_s)$ with $\mathcal{A}_s'(h)g=\frac{1}{\lambda_s}(g-\mathcal{R}_s'(h)g)$ for all $h\in\mathcal{Y}_p$ and $g\in\mathcal{H}$.

Proof. The assertion follows from Lemma 9.8, Lemma 9.5 for the estimate of $(\mathcal{R}_s'(h)g)_3$, the fact that $R_s : L^{\hat{p}}(\Omega;\mathbb{R}^{d\times d}_s)\to L^2(\Omega;\mathbb{R}^{d\times d}_s)$ is Fréchet differentiable (cf. GOLDBERG [44, Theorem 7]) and the chain rule. □

Now, we can use Theorem 5.9 to derive the differentiability of the solution operator of the constraint in (9.4) from the differentiability of $\mathcal{A}_s$. To this end, we first introduce the solution operator in

Definition 9.10 (Smoothed solution operator). *We denote the solution operator of*

$$Q^{-1}(\dot{u}, \dot{v}, \dot{q}) + \mathcal{A}_s(u, v, q) = Rf, \quad (u, v, q)(0) = (u_0, v_0, q_0) \tag{9.8}$$

by $\mathcal{S}_s : L^2(L^2(\Omega; \mathbb{R}^d)) \to H^1(\mathcal{Y}_p)$, that is, $\mathcal{S}_s(f) = (u, v, q)$, which existence follows from Theorem A.7 since $\mathcal{A}_s$ is Lipschitz continuous according to Definition 9.6. Here we use the improved regularity of (u_0, v_0, q_0), see Assumption ⟨9.2.iii⟩.

Proposition 9.11 (Fréchet differentiability of the smoothed solution operator). *The solution operator $\mathcal{S}_s : L^2(L^2(\Omega; \mathbb{R}^d)) \to H^1(\mathcal{Y}_p)$ is Lipschitz continuous, $\mathcal{S}_s : H^1(L^2(\Omega; \mathbb{R}^d)) \to H^1(\mathcal{H})$ is Fréchet differentiable and, for $f, g \in H^1(L^2(\Omega; \mathbb{R}^d))$, $\eta := \mathcal{S}_s'(f)g \in H^1(\mathcal{H})$ is the unique solution of*

$$Q^{-1}\dot{\eta} + \mathcal{A}_s'(w)\eta = Rg, \quad \eta(0) = 0, \tag{9.9}$$

where $w := \mathcal{S}_s(f)$. Moreover, there exists a constant C, such that $\|\mathcal{S}_s'(f)g\|_{H^1(\mathcal{H})} \leq C\|g\|_{L^2(L^2(\Omega;\mathbb{R}^d))}$ holds for all $f, g \in H^1(L^2(\Omega; \mathbb{R}^d))$.

Proof. Our goal is to use Theorem 5.9, to this end we first consider a transformed equation. We set $p_0 := Q^{-1}(RF(0) - (u_0, v_0, q_0)) = -Q^{-1}(u_0, v_0, q_0)$ and denote the solution operator of

$$\dot{p} = \mathcal{A}_s(RF - Qp), \quad p(0) = p_0 \tag{9.10}$$

by $\tilde{\mathcal{S}}_s : L^2(L^2(\Omega; \mathbb{R}^d)) \to H^1(\mathcal{Y}_p)$, that is, $\tilde{\mathcal{S}}_s(F) = p$. We can now transform (9.8) into

$$(\dot{u}, \dot{v}, \dot{q}) + (\mathcal{A}_s)_Q(u, v, q) = R\dot{F}, \quad (u, v, q)(0) = (u_0, v_0, q_0), \tag{9.11}$$

where $F := \mathcal{F}_\rho f \in H^2(L^2(\Omega; \mathbb{R}^d))$ (note that $QR = (0, \rho\ \cdot, 0)$). The equivalence between (9.10) and (9.11) can be shown as in Lemma 4.3, therefore $\mathcal{S}_s(f) = R\mathcal{F}_\rho f - Q\tilde{\mathcal{S}}_s(\mathcal{F}_\rho f)$ for all $f \in L^2(L^2(\Omega; \mathbb{R}^d))$.

We can now apply Lemma 5.7 and Theorem 5.9 (with $\mathcal{X} = L^2(\Omega; \mathbb{R}^d)$, $\mathcal{Y} = \mathcal{Y}_p$, $\mathcal{Z} = \mathcal{H}$, $z = p$ and $z_0 = p_0$), note that Assumptions ⟨5.2.i⟩ to ⟨5.2.iv⟩ are fulfilled. In particular, Assumption ⟨5.2.iv⟩ is satisfied thanks to Proposition 9.9 and Assumption ⟨5.2.iii⟩ holds according to Assumption ⟨9.2.iii⟩ (since $p_0 = -Q^{-1}(u_0, v_0, q_0)$ we acutally have $-Qp_0 = (u_0, v_0, q_0) \in D(\mathcal{A})$ by Assumption ⟨IV.iii⟩, which is the second requirement in Assumption ⟨5.2.iii⟩ (the set $\mathcal{A}_\mathcal{L}$ therein is here not important), however, this is not necessary for Lemma 5.7 and Theorem 5.9). Moreover, Assumption ⟨5.2.v⟩ is clearly uninportant and it can be easily seen that the requirements made in Assumption II for the whole Part II and Chapter 5 are either fulfilled or also not necessary. Thus the solution operator $\tilde{\mathcal{S}}_s : L^2(L^2(\Omega; \mathbb{R}^d)) \to H^1(\mathcal{Y}_p)$ is Lipschitz continuous and $\tilde{\mathcal{S}}_s : H^1(L^2(\Omega; \mathbb{R}^d)) \to H^1(\mathcal{H})$ is Fréchet differentiable, hence, the desired Lipschitz continuity and Fréchet differentiability also hold for $\mathcal{S}_s$. Furthermore, the asserted inequality holds and we have $\eta = R\mathcal{F}_\rho g - Q\tilde{\eta}$, where $\eta := \mathcal{S}_s'(f)g$ and $\tilde{\eta} := \tilde{\mathcal{S}}_s'(\mathcal{F}_\rho f)\mathcal{F}_\rho g$. Theorem 5.9 also shows that $\tilde{\eta}$ is the unique solution of

$$\partial_t\tilde{\eta} = \mathcal{A}_s'(R\mathcal{F}_\rho f - Qp)(R\mathcal{F}_\rho g - Q\tilde{\eta}), \quad \tilde{\eta}(0) = 0,$$

where $p := \tilde{\mathcal{S}}_s(\mathcal{F}_\rho f)$. Taking into account that $\tilde{\eta} = R\mathcal{F}g - Q^{-1}\eta$ and $\partial_t\mathcal{F}g = g$, we see that η is the solution of (9.9). □

Remark 9.12 (Control space). *As seen in the proposition above, the smoothed solution operator defined on $H^1(L^2(\Omega;\mathbb{R}^d))$ is Fréchet differentiable. The norm gaps, which arise from the exponents in Assumption ⟨9.2.ii⟩, are only needed for the differentiability of $\mathcal{T}_{R_s}$ but not in the control space. Unfortunately, we still require the compactness property imposed on $\mathfrak{X}_c$ in Assumption ⟨9.1⟩ to use the convergence results in Section 8.3 so that the findings in Section 9.1 hold true. However, we can avoid taking a subspace of $H^1(L^{\tilde{p}}(\Omega;\mathbb{R}^d))$, for a certain $\tilde{p} > 2$, as the control space.*

Let us now consider the following reduced optimization problem

$$\min_{f\in\mathfrak{X}_c} F_z(f), \tag{9.12}$$

where the reduced objective function $F_z : \mathfrak{X}_c \to \mathbb{R}$ is defined by $F_z(f) := J_z(\mathcal{S}_s(f), f)$. Clearly, (9.12) and (9.4) are equivalent.

We can finally present the main result of Part IV.

Theorem 9.13 (KKT-conditions for (9.12)). *Let $\overline{f} \in \mathfrak{X}_c$ and abbreviate $(\overline{u},\overline{v},\overline{q}) := \mathcal{S}_s(\overline{f}) \in H^1(\mathcal{Y}_p)$ and $\overline{w} := \mathcal{T}_{R_s}(\overline{u},\overline{v},\overline{q}) \in H^1(W^{1,p}_D(\Omega;\mathbb{R}^{d\times d}_s))$. Then the variational equation*

$$F_z'(\overline{f})g = \Psi_z'(\mathcal{S}_s(\overline{f}))\mathcal{S}_s'(\overline{f})g + \alpha\left(\overline{f},g\right)_{\mathfrak{X}_c} = 0 \tag{9.13}$$

holds for all $g \in \mathfrak{X}_c$ if and only if there exists an unique adjoint state $(\varphi,\eta^*) = (\varphi_1,\varphi_2,\varphi_3,\eta^*) \in H^1(\mathcal{H}\times H^1_D(\Omega;\mathbb{R}^d))$ such that the following optimality system is satisfied:

State equation:

$$\begin{pmatrix}\dot{\overline{u}}\\ \dot{\overline{v}}\\ \dot{\overline{q}}\end{pmatrix} = \frac{1}{\lambda_s}\begin{pmatrix}\overline{w}-\overline{u}\\ (\overline{w}-\overline{u})/(\rho\lambda_s) - \overline{v}/\rho\\ (\mathbb{C}+\mathbb{B})(\overline{p}-\overline{q})\end{pmatrix} + \begin{pmatrix}0\\ \overline{f}/\rho\\ 0\end{pmatrix} \tag{9.14a}$$

$$-\operatorname{div}(\mathbb{D}\nabla^s\overline{w} + \mathbb{E}\overline{p}) = \overline{v}/\lambda_s + (\overline{w}-\overline{u})/\lambda_s \tag{9.14b}$$

$$\overline{p} = R_s(\mathbb{E}\nabla^s(\overline{w}-\overline{u}) + \overline{q}) \tag{9.14c}$$

$$(\overline{u},\overline{v},\overline{q})(0) = (u_0,v_0,q_0) \tag{9.14d}$$

Adjoint equation :

$$\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\\ \dot{\varphi}_3\end{pmatrix} = \frac{1}{\lambda_s}\begin{pmatrix}\eta^*-\varphi_1\\ (\eta^*-\varphi_1)/(\rho\lambda_s) - \varphi_2/\rho\\ (\mathbb{C}+\mathbb{B})(r^*-\varphi_3)\end{pmatrix} - Q\Psi_z'(\overline{u},\overline{v},\overline{q}) \tag{9.14e}$$

$$-\operatorname{div}(\mathbb{D}\nabla^s\eta^* + \mathbb{E}r^*) = \varphi_2/\lambda_s + (\eta^*-\varphi_1)/\lambda_s \tag{9.14f}$$

$$r^* = R_s'(\mathbb{E}^\top\nabla^s(\overline{w}-\overline{u}) + \overline{q})^*(\mathbb{E}^\top\nabla^s(\eta^*-\varphi_1) + \varphi_3) \tag{9.14g}$$

$$(\varphi_1,\varphi_2,\varphi_3)(T) = 0 \tag{9.14h}$$

Gradient equation:

$$\left(\varphi_2, g\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} = \alpha\left(\overline{f},g\right)_{\mathfrak{X}_c} \qquad \forall g \in \mathfrak{X}_c. \tag{9.14i}$$

In particular, if $\overline{f}$ is locally optimal for (9.12)*, then there exists a unique adjoint state $(\varphi,\eta^*) \in H^1(\mathcal{H}\times H^1_D(\Omega;\mathbb{R}^d))$ such that* (9.14) *is fulfilled.*

Proof. At first we proof that the assertion holds when we exchange (9.14) with

$$
\begin{aligned}
Q^{-1}(\dot{\overline{u}}, \dot{\overline{v}}, \dot{\overline{q}}) + \mathcal{A}_s(\overline{u}, \overline{v}, \overline{q}) &= R\overline{f}, \qquad (\overline{u}, \overline{v}, \overline{q})(0) = (u_0, v_0, q_0),\\
Q^{-1}\dot{\varphi} + \mathcal{A}_s'(\overline{u}, \overline{v}, \overline{q})^*\varphi &= -\Psi_z'(\overline{u}, \overline{v}, \overline{q}), \qquad \varphi(T) = 0,\\
\left(\varphi_2, g\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} &= \alpha\left(\overline{f}, g\right)_{\mathfrak{X}_c} \qquad \forall g \in \mathfrak{X}_c.
\end{aligned}
\tag{9.15}
$$

To this end, let φ be the solution of the second equation in (9.15) (which unique existence follows as in Lemma 5.12) and $\eta := \mathcal{S}_s'(\overline{f})g \in H^1(\mathcal{H})$ for an arbitrary $g \in \mathfrak{X}_c$, then

$$
\begin{aligned}
\left(\varphi_2, g\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} = (\varphi, Rg)_{L^2(\mathcal{H})} &= \left(\varphi, Q^{-1}\dot{\eta}\right)_{L^2(\mathcal{H})} + \left(\varphi, \mathcal{A}_s'(\overline{u}, \overline{v}, \overline{q})\eta\right)_{L^2(\mathcal{H})}\\
&= \left(Q^{-1}\dot{\varphi}, \eta\right)_{L^2(\mathcal{H})} + \left(\mathcal{A}_s'(\overline{u}, \overline{v}, \overline{q})^*\varphi, \eta\right)_{L^2(\mathcal{H})}\\
&= -\left(\Psi_z'(\mathcal{S}_s(\overline{f})), \eta\right)_{L^2(\mathcal{H})}
\end{aligned}
$$

holds for all $g \in \mathfrak{X}_c$, which implies the equivalence between (9.13) and the last equation in (9.15). Moreover, it is well known that if $\overline{f}$ is locally optimal for (9.4), then (9.13) must hold (see also Lemma 5.10).

Let us now prove the equivalence between (9.15) and (9.14). We choose $h, \xi \in \mathcal{H}$ and denote by $\eta^* \in H_D^1(\Omega; \mathbb{R}_s^{d\times d})$ the solution of

$$
-\operatorname{div}(\mathbb{D}\nabla^s\eta^* + \mathbb{E}R_s'(\mathbb{E}^\top\nabla^s(\mathcal{T}_{R_s}(h) - h_1) + h_3)^*(\mathbb{E}^\top\nabla^s(\eta^* - \xi_1) + \xi_3))) = \frac{\xi_2}{\lambda_s} + \frac{\xi_1 - \eta^*}{\lambda_s^2} \tag{9.16}
$$

for all $\phi \in H_D^1(\Omega; \mathbb{R}^d)$ (the existence of η^* follows as in Lemma 9.8, note that the inequalities in Lemma 9.5 hold also for the adjoint operator). Then

$$
\mathcal{R}_s'(h)^*\xi = \begin{pmatrix} \eta^* \\ \frac{1}{\lambda_s}\eta^* - \frac{\xi_1}{\lambda_s} \\ R_s'(\mathbb{E}^\top\nabla^s(\mathcal{T}_{R_s}(h) - h_1) + h_3)^*(\mathbb{E}^\top\nabla^s(\eta^* - \xi_1) + \xi_3) \end{pmatrix}
$$

holds, which can be seen as follows: Let $g \in \mathcal{H}$ and abbreviate

$$
\begin{aligned}
&\eta := \mathcal{T}_{R_s}'(h)g, \qquad \eta_v := \frac{\eta - g_1}{\lambda_s}, \qquad \eta_q := R_s'(\mathbb{E}^\top\nabla^s(\mathcal{T}_{R_s}(h) - h_1) + h_3)(\mathbb{E}^\top\nabla^s(\eta - g_1) + g_3),\\
&\eta_v^* := \frac{\eta^* - \xi_1}{\lambda_s}, \qquad \eta_q^* := R_s'(\mathbb{E}^\top\nabla^s(\mathcal{T}_{R_s}(h) - h_1) + h_3)^*(\mathbb{E}^\top\nabla^s(\eta^* - \xi_1) + \xi_3).
\end{aligned}
$$

Testing (9.7) with $\phi = \xi_1 - \eta^*$ gives

$$
\begin{aligned}
&\left(\mathbb{D}\nabla^s\eta, \nabla^s(\xi_1 - \eta^*)\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} + \left(\eta_v - g_2, \eta_v^*\right)_{L^2(\Omega;\mathbb{R}^d)}\\
&\quad = \left(\mathbb{E}\eta_q, \nabla^s(\eta^* - \xi_1)\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})}\\
&\quad = \left(\mathbb{E}^\top\nabla^s(\eta - g_1) + g_3, \eta_q^*\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} - \left(\eta_q, \xi_3\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})},
\end{aligned}
$$

and testing (9.16) with $\phi = \eta - g_1$ yields

$$
\left(\mathbb{D}\nabla^s\eta^*, \nabla^s(\eta - g_1)\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} + \left(\xi_2 - \eta_v^*, \eta_v\right)_{L^2(\Omega;\mathbb{R}^d)} = \left(\mathbb{E}\eta_q^*, \nabla^s(g_1 - \eta)\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})},
$$

thus, adding both equations together, we arrive at

$$\begin{aligned}\left(\mathbb{D}\nabla^s\eta, \nabla^s\xi_1\right)_{H^1(\Omega;\mathbb{R}^d)} + \left(\eta_v, \xi_2\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left(\eta_q, \xi_3\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ = \left(\mathbb{D}\nabla^s\eta^*, \nabla^s g_1\right)_{H^1(\Omega;\mathbb{R}^d)} + \left(\eta_v^*, g_2\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left(\eta_q^*, g_3\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)},\end{aligned}$$

which is equivalent to

$$\left(\mathcal{R}_s'(h)g, \xi\right)_{\mathcal{H}} = \left(g, \mathcal{R}_s'(h)^*\xi\right)_{\mathcal{H}}.$$

Now one only has to use the definitions of $\mathcal{A}_s$ and R to obtain the equivalence between (9.14) and (9.15). □

Let us end this part with examples about a concrete objective function, the gradient equation in Theorem 9.13 regarding a concrete control space and finally a realization of the maximal monotone operator A (which will be of course the von-Mises flow rule considered in Section 3.2).

Example 9.14 (Concrete objective function)**.** *Let us consider a tracking type objective function, that is,*

$$\Psi(u, v, z) = \frac{1}{2}\|(u, v, z) - (u_d, v_d, z_d)\|^2_{L^2(\mathcal{H})}$$

with a desired state $(u_d, v_d, z_d) \in L^2(\mathcal{H})$. Then

$$\Psi_z(u, v, q) = \frac{1}{2}\|(u, v, (\mathbb{C} + \mathbb{B})^{-1}(\mathbb{C}\nabla^s u - q)) - (u_d, v_d, z_d)\|^2_{L^2(\mathcal{H})}$$

and

$$\Psi_z'(u, v, q) = \begin{pmatrix} \hat{u} \\ v - v_d \\ (\mathbb{C} + \mathbb{B})^{-1}(\mathbb{C}\nabla^s u - q)) - z_d, \end{pmatrix}$$

where $\hat{u}$ is such that $-\operatorname{div}(\mathbb{D}\nabla^s(\hat{u} - u + u_d) - ((\mathbb{C} + \mathbb{B})^{-1}(\mathbb{C}\nabla^s u - q) - z_d)) = 0$, hence, in this example the adjoint equation in (9.14) *has to be completed by this equation. Note that when one uses a finite element approach to solve* (9.14) *numerically, then one can eliminate this additional equation after multiplying* (9.14e) *with a test function, that is, taking the $\mathcal{H}$-scalarproduct. When the $\mathcal{H}$-scalarproduct of $Q\Psi_z'(u, v, q)$ and a test function (η_1, η_2, η_3) is evaluated, the term $\left(\mathbb{D}\nabla^s\hat{u}, \eta_1\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$ arises, then one can use the additional equation to eliminate $\hat{u}$ (respectively the equation).*

Example 9.15 (Concrete control space)**.** *Let us consider the space*

$$\mathfrak{X}_c := \{f \in H^1(L^2(\Omega; \mathbb{R}^d)) \cap L^2(H^1(\Omega; \mathbb{R}^d)) : f(0) = f(T) = 0\}$$

with the scalar product

$$(f, g)_{\mathfrak{X}_c} = \left(\dot{f}, \dot{g}\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} + (\nabla f, \nabla g)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}))},$$

see Example 9.2. The Gradient equation in (9.14) *then becomes*

$$\alpha\left(\dot{f}, \dot{g}\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} + \alpha\,(\nabla f, \nabla g)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}))} = \left(\varphi_2, g\right)_{L^2(L^2(\Omega;\mathbb{R}^d))}$$

for all $g \in \mathfrak{X}_c$, which is the weak formulation of

$$\ddot{f} + \Delta f = -\frac{\varphi_2}{\alpha}.$$

Example 9.16 (Maximal monotone operator A). *Also in the case of plasticity with inertia it is reasonable to consider the case of the von-Mises flow rule introduced in Section 3.2. Let us shortly emphasize that the concrete choice of A has no influence on the facts that $\mathcal{A}$ does neither fulfill the boundedness property, nor is a subdifferential, as we have seen in Remark 8.16. To consider the case of the von-Mises flow rule, we set $A = \partial I_{\mathcal{K}(\Omega)}$, where $\partial I_{\mathcal{K}(\Omega)}$ is given in Definition 2.2 respectively Section 3.2, and R_s is given by* (3.13). *Then Assumption $\langle IV.i \rangle$ holds according to* (3.7) *and* (3.2). *Assumption $\langle 9.2.i \rangle$ is also fulfilled due to Lemma 3.18 Item (iii) & Item (iv) and Lemma 3.19. Note furthermore, that, as we already explained before Lemma 9.7,* (3.12) *shows that Lemma 9.7 can be used to satisfy* (4.14).

With the presentation of the first order optimality conditions in the form of a KKT system in Theorem 9.13 this part ends. We have seen that the abstract theory in Part II can also be applied in the case of plasticity with inertia. This application was not as straightforward as in the case of elasto and homogenized plasticity in Part III due to the lack of the boundedness property of the arising maximal monotone operator $\mathcal{A}$, which is moreover not a subdifferential. However, we could still apply the results in Chapter 4 for H^2-loads and the differentiability results provided in Section 5.2. In the next part we even move further away from Part II, the findings in Chapter 4 will not be applicable at all.

Let us finally say that we decided to omit the application of the results about second order sufficient conditions given Section 5.3 to keep the discussion once again more concise.

Part V Perfect Plasticity

Also in this last part and agreed upon at the beginning of Part I, we collect at first all needed assumptions for this part.

Assumption V. *We impose the following assumptions according to The Assumption Agreement in the beginning of Part I.*

⟨*V.i*⟩ *Let* $K \subset \mathbb{R}^{d\times d}_s$ *be convex and closed such that* $0 \in K^\circ$. *We set*

$$\mathcal{K}(\Omega) := \{\tau \in L^2(\Omega; \mathbb{R}^{d\times d}_s) \,:\, \tau(x) \in K \textit{ f.a.a. } x \in \Omega\}$$

as in Definition 2.2. The precise structure of K *is otherwise not important until Chapter 13, see Assumption* ⟨*13.ii*⟩.

⟨*V.ii*⟩ *We assume that the boundary of* Ω *is of class* C^1 *and that* $d \geq 2$. *This assumption is not necessary in Section 12.2, Section 12.3 and Chapter 13, see Remark 10.3 and also Section 12.2.*

⟨*11.i*⟩ *We assume that* $u_0 \in H^1(\Omega; \mathbb{R}^d)$, $\sigma_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$.

⟨*11.ii*⟩ *Let* $f \in H^1(L^d(\Omega; \mathbb{R}^d))$ *and* $u_D \in H^1(H^1(\Omega; \mathbb{R}^d))$ *such that* $-\operatorname{div}\sigma_0 = f(0)$ *and* $u_0 - u_D(0) \in H^1_D(\Omega; \mathbb{R}^d)$ *holds.*

⟨*11.1*⟩ *We require*

$$\begin{aligned}
&f_n \rightharpoonup f && \text{in } H^1(L^d(\Omega; \mathbb{R}^d))\\
&f_n \to f && \text{in } L^2(L^d(\Omega; \mathbb{R}^d))\\
&u_{D,n} \rightharpoonup u_D && \text{in } H^1(H^1(\Omega; \mathbb{R}^d))\\
&u_{D,n} \to u_D && \text{in } L^2(H^1(\Omega; \mathbb{R}^d))\\
&u_{D,n}(T) \to u_D(T) && \text{in } H^1(\Omega; \mathbb{R}^d),
\end{aligned}$$

where $\{f_n\}_{n\in\mathbb{N}}$ *and* $\{u_{D,n}\}_{n\in\mathbb{N}}$ *are sequences in* $H^1(L^d(\Omega; \mathbb{R}^d))$ *and* $H^1(H^1(\Omega; \mathbb{R}^d))$, *respectively.*

⟨*11.2.i*⟩ *Let* $(\varepsilon, \lambda) \in \mathbb{R}^2 \setminus \{0\}$, $\varepsilon, \lambda \geq 0$, *such that the following holds*

$$\sigma_0 - \varepsilon\mathbb{B}(\nabla^s u_0 - \mathbb{A}\sigma_0) \in \mathcal{K}(\Omega). \tag{9.17}$$

⟨*11.2.ii*⟩ *Let* $\mathcal{S}_\mathcal{L} \subset H^1(L^d(\Omega; \mathbb{R}^d))$ *be a set which fulfills a global safe-load condition with* $M_{\mathcal{S}_\mathcal{L}}$ *and* $\delta_{\mathcal{S}_\mathcal{L}} > 0$ *(see Definition 11.13). Suppose that* $f \in \mathcal{S}_\mathcal{L}$ *with corresponding* $\rho \in W^{1,\infty}(L^\infty(\Omega; \mathbb{R}^{d\times d}_s))$ *from Definition 11.13.*

⟨11.3.i⟩ *Let* $\sigma_0 \in \mathcal{K}(\Omega)$.

⟨11.3.ii⟩ *We again suppose that Assumption ⟨11.2.ii⟩ and Assumption ⟨11.1⟩ holds and that* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\mathcal{L}}$. *Furthermore, let* $-\operatorname{div}\sigma_0 = f_n(0)$ *and* $u_0 - u_{D,n}(0) \in H^1_D(\Omega;\mathbb{R}^d)$ *for all* $n \in \mathbb{N}$.

⟨11.3.iii⟩ *Let* $\{(\varepsilon_n, \lambda_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2 \setminus 0$ *be a sequence such that* $(\varepsilon_n, \lambda_n) \to 0$ *and*

$$\sigma_0 - \varepsilon_n \mathbb{B}(\nabla^s u_0 - \mathbb{A}\sigma_0)) \in \mathcal{K}(\Omega)$$

for all $n \in \mathbb{N}$.

⟨11.3.iv⟩ *We abbreviate* $u_n := u_{\varepsilon_n,\lambda_n}$, $\sigma_n := \sigma_{\varepsilon_n,\lambda_n}$ *and* $z_n := z_{\varepsilon_n,\lambda_n}$, *where* $(u_{\varepsilon_n,\lambda_n}, \sigma_{\varepsilon_n,\lambda_n}, z_{\varepsilon_n,\lambda_n})$ *is the solution of* (11.9) *with* $u_D = u_{D,n}$, $f = f_n$ *and* $g = 0$.

⟨12.i⟩ *The initial condition* u_0 *belongs to* $H^1(\Omega;\mathbb{R}^d)$ *and* σ_0 *to* $\mathcal{K}(\Omega)$ *and fulfills* $-\operatorname{div}\sigma_0 = 0$.

⟨12.ii⟩ *The space* $\mathcal{X}_c$ *is a Hilbert space and the control space is given by* $H^1_{00}(\mathcal{X}_c) = \{\mathfrak{l} \in H^1(\mathcal{X}_c) : \mathfrak{l}(0) = \mathfrak{l}(T) = 0\}$.

The space $\mathcal{X}_c$ *is compactly embedded into a Banach space* $\mathcal{X}$. *The elements* $\mathfrak{l}$ *in* $\mathcal{X}_c$, $\mathcal{X}$ *and* $H^1_{00}(\mathcal{X}_c)$ *are called* pseudo forces .

⟨12.iii⟩ *The function* $\Psi : H^1(H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)) \times H^1_{00}(\mathcal{X}_c)$ *is weakly lower semicontinuous, continuous and bounded from below. The objective function* $J : H^1(H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)) \times H^1_{00}(\mathcal{X}_c)$ *is given in* (12.1), *where* $\alpha > 0$ *is the Tikhonov parameter.*

⟨12.iv⟩ *The operator* $\mathcal{G} : \mathcal{X} \to H^1(\Omega;\mathbb{R}^d)$ *maps pseudo forces to Dirichlet displacements and is linear and continuous. As usual, we denote* $\mathcal{G}$ *restricted to* $\mathcal{X}_c$ *with the same symbol.*

⟨12.v⟩ *The offset* $\mathfrak{a} \in H^1(H^1(\Omega;\mathbb{R}^d))$ *fulfills* $u_0 - \mathfrak{a}(0) \in H^1_D(\Omega;\mathbb{R}^d)$.

⟨12.1.i⟩ *Let the initial condition* σ_0 *be of class* $H^1(\Omega;\mathbb{R}^{d\times d}_s)$ *and* $U \in L^2(H^1(\Omega;\mathbb{R}^{d\times d}_s))$.

⟨12.1.ii⟩ *Let* $\theta \in (0,1)$ *and* $\delta \in (0,\frac{1}{2})$ *be given. The definition of the regularized objective function* $J_\lambda : H^1(H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)) \times H^1_{00}(\mathcal{X}_c) \times (L^2(H^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d)) \cap H^1_0(H^{-1}(\Omega;\mathbb{R}^d))) \to \mathbb{R}$ *is given in* (12.2).

⟨12.2⟩ *With a slight abuse of notation, we denote the objective function* $J : H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)) \times H^1_{00}(\mathcal{X}_c)$, *defined in* (12.12), *again by* J, *where we also defined* $\Psi : H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)) \to \mathbb{R}$ *by* $\Psi(\sigma) := 1/2\|\sigma(T) - \sigma_d\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$, *where* $\sigma_d \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$ *is the desired stress field.*

⟨12.3.i⟩ *We again suppose that Assumption ⟨12.2⟩ is satisfied and choose* $\varepsilon \geq 0$, $\lambda, s > 0$. *Moreover, the pseudo force to Dirichlet boundary displacements mapping* $\mathcal{G}$ *maps* $\mathcal{X}$ *linear and continuous into* $W^{1,p}(\Omega;\mathbb{R}^d)$ *and we have* $\mathfrak{a} \in H^1(W^{1,p}(\Omega;\mathbb{R}^d))$ *for one* $p \in (2,\overline{p}]$, *where* $\overline{p}$ *is from Theorem 2.5.*

⟨12.3.ii⟩ *The initial stress fulfills the regularity* $\sigma_0 \in L^p(\Omega;\mathbb{R}^{d\times d}_s)$, *where* p *is from Assumption ⟨12.3.i⟩, and we have* $u_0 = \mathfrak{a}(0) \in W^{1,p}(\Omega;\mathbb{R}^d)$.

⟨13.i⟩ *The boundary* $\partial\Omega$ *can be split into a* pseudo Dirichlet boundary Λ_D *and a* pseudo Neumann boundary Λ_N. *As for* Γ_D *and* Γ_N, *we require that* Λ_N *is relatively open in* $\partial\Omega$, *while* Λ_D *is relatively closed and has positive boundary measure. Moreover, we also assume that* $\Omega \cup \Lambda_N$ *is regular in the sense of Gröger, cf. [46] (these assumptions are again needed to solve linear elasticity, see also Chapter 1). Furthermore, we require that* $\Gamma_D \subset \Lambda_N$ *and that* Γ_D

and Λ_D have positive distance to each other, i.e.,

$$\operatorname{dist}(\Gamma_D, \Lambda_D) = \inf_{x \in \Lambda_D, \xi \in \Gamma_D} |x - \xi| > 0. \tag{9.18}$$

⟨13.ii⟩ *We choose $\gamma > 0$ (specified in Section 13.3) and set $K := \{\tau \in \mathbb{R}_s^{d \times d} : |\tau^D| \le \gamma\}$ as in Definition 2.2.*

This last part is simultaneously also the most extensive one. This has several reasons.

At first we have to introduce the space of *bounded deformation*, denoted by $BD(\Omega)$, since in the case of perfect plasticity the existence of a displacement can only be shown in this space and not in $H^1(\Omega; \mathbb{R}^d)$. The space of bounded deformation does not have the *Radon-Nikodým property* (see [31, Definition III.1.3] or [29, Appendix D] for the notion of this property and Remark A.9 for the fact that the space of bounded deformation lacks it) and is thus also not reflexive, which follows from the Dunford-Pettis theorem, see [29, Page 516]. Although it can be shown that $BD(\Omega)$ is the dual of a normed space (see [69, 97]), $L^2(BD(\Omega))$ is not the dual space of the square integrable functions with values in the primal space of $BD(\Omega)$ due to the lack of the Radon-Nikodým property (cf. [31, Theorem IV.1.1]). The dual space of the square integrable functions with values in the primal space of $BD(\Omega)$ are only the *weakly measurable* functions (cf. Edwards [33, p. 8.14.1]), also called *scalarwise measurable* functions (which is however a different concept in Edwards [33, p. 8.14.1])), with values in $BD(\Omega)$, denoted by $L^2_w(BD(\Omega))$ (this notation is adopted from Suquet [95]), see Edwards [33, 8.20.3 Theorem] or Tulcea [101, Chapter VII section 5]. Due to these facts we need to consider the space $L^2_w(BD(\Omega))$ to use, for instance results about weakly convergent subsequences. As the reader may not be familiar with the concept of weakly measurable functions, we introduce the space for the displacement from the ground up in Chapter 10 instead of referring only to the abstract results for example given in [33, 101].

A second cause of the length of this part is the fact that most of the results in Part II are not applicable. As we have seen in Lemma 2.8, the in Definition 2.7 introduced operator Q is only coercive when hardening is present. Since hardening is absent in the case of perfect plasticity (which is actually the only difference between elasto and perfect plasticity), the lack of the coercivity of the operator Q prevents us from using the majority of the findings in Part II. This is the reason why we have to go into details when we tend to the definition, regularization and existence of solutions to perfect plasticity in Chapter 11.

Finally, after considering an optimal control problem in Chapter 12 we also present numerical experiments in Chapter 13, which stretches this part further.

Similar as in the case of homogenized plasticity and plasticity with inertia considered in Chapter 7 and Part IV, respectively, except Meyer et al. [72, 73] there are no results about optimal control of perfect plasticity present in the literature, at least to the knowledge of the author. The present part follows [72, 73] but also differs in several aspects, see e.g. Remark 12.2. Concerning the (time dependent) equations of perfect plasticity itself, there are many results available in the literature, see e.g. [57, 60, 95, 27, 28, 11, 40, 92]. We will comment on and use some of the findings in these references throughout this part.

The formal strong formulation reads as follows:

$$-\nabla \cdot \sigma = f \qquad \text{in } \Omega, \tag{9.19a}$$
$$\nu \cdot \sigma = 0 \qquad \text{on } \Gamma_N, \tag{9.19b}$$
$$u = u_D \qquad \text{on } \Gamma_D, \tag{9.19c}$$
$$\sigma = \mathbb{C}(\nabla^s u - z) \qquad \text{in } \Omega, \tag{9.19d}$$
$$\dot{z} \in \partial I_{\mathcal{K}(\Omega)}(\sigma) \qquad \text{in } \Omega \tag{9.19e}$$
$$(u, \sigma)(0) = (u_0, \sigma_0) \qquad \text{in } \Omega. \tag{9.19f}$$

Herein, we do not consider Neumann boundary forces, see also Remark 11.20. We emphasize that the difference between (9.19) and (2.1) lies only in the absence of hardening, that is, the hardening parameter $\mathbb{B}$ is simply set to zero in (9.19). Therefore the physical interpretation for perfect plasticity is analog to the given one in Chapter 2 for elasto plasticity except that the flow rule (9.19e) prohibits the stress to go beyond the yield surface, that is, the stress must stay inside the set $\mathcal{K}(\Omega)$.

Chapter 10 Displacement Space

This chapter is devoted to the introduction of the control space for the displacement. At first we recall the definition and some properties of the space of bounded deformation from the literature in Section 10.1. After that we can develop the space of velocity fields from which we can then derive the space for the displacement in Section 10.2 and Section 10.3, respectively.

Note that the space of bounded deformation is well studied in the literature, the content of Section 10.1 can be found in [85, 37] (measures) and [98] (the space of bounded deformation). Moreover, the space of velocity fields $\mathcal{V}(\Omega)$ developed in Section 10.2 is in fact the space $\{v \in L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d)) \,:\, \nabla^s v \in L^2_w(M(\Omega;\mathbb{R}^{d\times d}_s))\}$, where $L^2_w(M(\Omega;\mathbb{R}^{d\times d}_s)$ (the notation is again adopted from SUQUET [95]) is the space of *weakly measurable* functions with values in $M(\Omega;\mathbb{R}^{d\times d}_s)$, which is the set of regular real Borel measures (see the next section below). We already commented on the space of weakly measurable functions at the beginning of this part, one sees with analog arguments that $L^2_w(M(\Omega;\mathbb{R}^{d\times d}_s))$ is the dual space of $L^2(C_0(\Omega;\mathbb{R}^{d\times d}_s))$. Due to these observations the reader may skip Section 10.1 and / or Section 10.2 if she is already familiar with the mentioned concepts. Moreover, let us note that our development of the space of displacement fields is similar to the development of Bochner spaces, see for instance [87, Kapitel 10.1] (see also Remark 10.9 below), and also borrows some techniques from [33, 101].

10.1 The Space of Bounded Deformation

The space of bounded deformation is the subspace of $L^1(\Omega;\mathbb{R}^d)$ where the symmetrized gradient (which exists as a distribution) is a (regular real Borel) measure. Thus we first introduce (regular real Borel) measures before we can tend to the space of bounded deformation.

Regular real Borel Measures

For the introduction of regular real borel measures we follow [85, 37], see in particular [85, 1.18 Definition, 2.15 Definition & Chapter 6] and [37, Kapitel VIII]. This introduction is intentionally kept as short as possible.

Let us denote the (smallest) σ-Algebra containing all open sets in Ω by $\mathcal{B}(\Omega)$. The elements of $\mathcal{B}(\Omega)$ are called *Borel sets* of Ω (later we will also consider Borel sets of a time interval). A system of Borel sets $\{B_i\}_{i\in\mathbb{N}}$ is called *partition* of $B \in \mathcal{B}(\Omega)$ if $\bigcup_{i=1}^{\infty} B_i = B$ and $B_i \cap B_j = \emptyset$ for all $i, j \in \mathbb{N}$, $i \neq j$. A function $\mu : \mathcal{B}(\Omega) \to \mathbb{R}$ is now called *real Borel Measure* if

$$\mu(B) = \sum_{i=1}^{\infty} \mu(B_i)$$

for all $B \in \mathcal{B}(\Omega)$ and all partitions $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega)$ of B. To define the norm of a real Borel measure we need the *total variation* $|\mu| : \mathcal{B}(\Omega) \to [0, \infty)$ of a real Borel measure μ, which is defined by

$$|\mu|(B) := \sup\Big\{ \sum_{i=1}^{\infty} |\mu(B_i)| : \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } B \Big\}.$$

Note that $|\mu(B)| \leq |\mu|(B) \leq |\mu|(\Omega)$ holds for all $B \in \mathcal{B}(\Omega)$. Moreover, notice that it is a priori not clear that $|\mu|(\Omega) < \infty$, however, that this is actually the case is proven in [85, 6.4 Theorem]. It is evident that the set of all real Borel measures is a vector space and a Banach space with the norm

$$\|\mu\| := |\mu|(\Omega).$$

Now we need to consider a special subspace of the real Borel measures, the set of *regular real Borel measures*, denoted by $M(\Omega)$, where a Borel measure is called *regular* if

$$\begin{aligned} |\mu(B)| &= \sup\{|\mu(K)| : K \subset B, K \text{ is compact}\} && \text{(inner regular)} \\ &= \inf\{|\mu(U)| : B \subset U, U \text{ is open}\} && \text{(outer regular)} \end{aligned}$$

for all $B \in \mathcal{B}(\Omega)$.

As can be found in [85, 6.19 Theorem] or [37, 2.23 Satz], $M(\Omega)$ can be identified with the dual space of the space of all continuous functions "vanishing at infinity", denoted by $C_0(\Omega)$. That is, a function $f : \Omega \to \mathbb{R}$ belongs to $C_0(\Omega)$ if and only if

$$\forall \varepsilon > 0\ \exists K \subset \Omega,\ \text{compact}, : |f(x)| \leq \varepsilon\ \forall x \in \Omega \setminus K. \tag{10.1}$$

Since Ω is open and bounded, the set $C_0(\Omega)$ can be identified with $\{f \in C(\overline{\Omega}) : f|_{\partial\Omega} = 0\}$. Moreover, $C_0(\Omega)$ equipped with the supremum norm is a separable Banach space, a closed subspace of $C(\overline{\Omega})$ and the closure of $C_c(\Omega)$ (the continuous functions which have a compact support). Let us formulate the duality between $C_0(\Omega)$ and $M(\Omega)$ in the following

Theorem 10.1 ($M(\Omega)$ is the dual space of $C_0(\Omega)$)**.** *The mapping*

$$\Phi : M(\Omega) \to C_0(\Omega)^*, \mu \mapsto \Big[C_0(\Omega) \ni f \mapsto \int_\Omega f\, d\mu\Big]$$

is an isometry, that is, Φ is bijective, linear and

$$\|\Phi(\mu)\|_{C_0(\Omega)^*} = \|\mu\|_{M(\Omega)}$$

holds for all $\mu \in M(\Omega)$.

The term $\int_\Omega f\,d\mu$ in the theorem above is the integral of f with respect to μ, for more details we refer to [85, 37, Chapter 1, Kapitel IV].

For simplicity, we will say that $M(\Omega)$ *is* the dual space of $C_0(\Omega)$, $M(\Omega) = C_0(\Omega)^*$. Moreover, we will also say simply *measure* instead of regular real Borel measure. In the following, when $\mu \in M(\Omega)$ and $f \in C_0(\Omega)$, $\langle \mu, f\rangle = \int_\Omega f\,d\mu$ denotes the dual pairing between $M(\Omega)$ and $C_0(\Omega)$. Further we define

$$M(\Omega; \mathbb{R}^{d\times d}_s) := \{\mu : \mathcal{B}(\Omega) \to \mathbb{R}^{d\times d}_s : (\mu)_{i,j} \in M(\Omega)\},$$

which is the dual space of $C_0(\Omega; \mathbb{R}^{d\times d}_s)$. The norm on this is space is simply

$$\|\mu\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} := \sum_{i,j=1}^{d} \|(\mu)_{i,j}\|_{M(\Omega)}$$

for $\mu \in M(\Omega; \mathbb{R}^{d\times d}_s)$. Again, for simplicity we also denote by

$$\langle \mu, f\rangle = \int_\Omega f\,d\mu = \sum_{i,j=1}^{d} \int_\Omega f_{i,j}\,d\mu_{i,j},$$

where $\mu \in M(\Omega; \mathbb{R}^{d\times d}_s)$ and $f \in C_0(\Omega; \mathbb{R}^{d\times d}_s)$, the dual pairing between $M(\Omega; \mathbb{R}^{d\times d}_s)$ and $C_0(\Omega; \mathbb{R}^{d\times d}_s)$.

We mention that there are more duality results similar to the one given in Theorem 10.1, cf. e.g. [37, Kapitel VIII §2], however, we only need the one in Theorem 10.1.

With the definition of measures at hand, we can now introduce

The Space of Bounded Deformation and its Properties

In the following our main resource is TEMAM [98, Chapter II.2 & II.3]. All mentioned properties can be found therein.

At first, the *space of bounded deformation* is defined as

$$BD(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^d) : (\nabla^s u)_{i,j} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \in M(\Omega)\} \tag{10.2}$$

$$= \{u \in L^1(\Omega; \mathbb{R}^d) : \nabla^s u \in M(\Omega; \mathbb{R}^{d\times d}_s)\}, \tag{10.3}$$

where the partial derivatives are understood as distributions. That is, for all $u \in BD(\Omega)$ and all $i, j \in \{1, ..., d\}$ there exists a measure $\mu_{i,j} \in M(\Omega)$ such that

$$\int_\Omega \varphi d\mu_{i,j} = \int_\Omega \varphi d(\nabla^s u)_{i,j} = \frac{1}{2}\int_\Omega \varphi d(\partial_j u_i + \partial_i u_j) = -\frac{1}{2}\int_\Omega u_i \partial_j \varphi + u_j \partial_i \varphi \tag{10.4}$$

for all $\varphi \in C^\infty_c(\Omega)$, where we set $\frac{1}{2}(\partial_j u_i + \partial_i u_j) := (\nabla^s u)_{i,j} := \mu_{i,j}$. We emphasize that $\partial_j u_i$ does not has to be a measure, but only the sum $\partial_j u_i + \partial_i u_j$ (which is the difference to the space of *bounded variation*, see below).

With the norm

$$\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + \|\nabla^s u\|_{M(\Omega;\mathbb{R}^{d\times d}_s)},$$

the space of bounded deformation becomes a Banach space. Moreover, we say that a sequence $\{u_n\}_{n\in\mathbb{N}} \subset BD(\Omega)$ converges weakly* to $u \in BD(\Omega)$, $u_n \rightharpoonup^* u$, when

$$u_n \to u \quad \text{in } L^1(\Omega; \mathbb{R}^d) \text{ and}$$
$$\nabla^s u_n \rightharpoonup^* \nabla^s u \quad \text{in } M(\Omega; \mathbb{R}^{d\times d}_s).$$

Let us additionally mention that when one requires in (10.2) that the full gradient is a measure, then one obtains the space of *bounded variation* (to be more precise, one obtains $BV(\Omega)^d$, where $BV(\Omega)$ is the space of bounded deformation which contains single valued functions). We refer e.g. to [7] for more information about the space of bounded variation. Note that it is known that these two spaces are in fact different (cf. TEMAM [98, page 145]).

We collect the for our analysis important properties of $BD(\Omega)$ in the following

Theorem 10.2 (Properties of $BD(\Omega)$)**.** *We collect some properties of $BD(\Omega)$:*

(i) There exists a continuous, surjective and linear operator $\gamma_0 : BD(\Omega) \to L^1(\partial\Omega; \mathbb{R}^d)$ such that

$$\gamma_0(u) = u|_{\partial\Omega} \qquad \forall u \in BD(\Omega) \cap C(\overline{\Omega}; \mathbb{R}^d).$$

This operator is not *weakly* continuous, that is, $u_n \rightharpoonup^* u$ does not imply $\gamma_0(u_n) \rightharpoonup \gamma_0(u)$. Moreover, for all $i, j \in \{1, ..., d\}$ and all $\varphi \in C(\overline{\Omega})$ we have the generalised* Green's formular

$$\frac{1}{2}\int_\Omega u_i \partial_j \varphi + u_j \partial_i \varphi + \int_\Omega \varphi \, \mathrm{d}(\nabla^s u)_{i,j} = \int_{\partial\Omega} \varphi \frac{1}{2}(\gamma_0(u)_i \nu_j + \gamma_0(u)_j \nu_i),$$

where ν is the outward normal unit vector on $\partial\Omega$. In the following we will simply write u instead of $\gamma_0(u)$.

(ii) The space $BD(\Omega)$ is continuously embedded into $L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d)$, $BD(\Omega) \hookrightarrow L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d)$. It is compactly embedded into $L^p(\Omega; \mathbb{R}^d)$, $BD(\Omega) \overset{c}{\hookrightarrow} L^p(\Omega; \mathbb{R}^d)$, for all $p \in [1, \frac{d}{d-1})$.

(iii) There exists a constant C such that

$$\|u\|_{BD(\Omega)} \le C\Big[\int_{\Gamma_D} |u| + \|\nabla^s u\|_{M(\Omega;\mathbb{R}_s^{d\times d})}\Big]$$

for all $u \in BD(\Omega)$.

Remark 10.3 (Regularity of the boundary of Ω)**.** *Assumption $\langle V.ii\rangle$ is only needed for the results in Theorem 10.2 and thus mainly in the construction of the space of velocity and displacement fields in Section 10.2 and Section 10.3, respectively. This means in particular that Assumption $\langle V.ii\rangle$ can be dropped when we restrict ourself to stress fields in Section 12.2.*

In light of Theorem 10.2 Item (ii), a bounded sequence in $BD(\Omega)$ has a subsequence which converges strongly to a limit u in $L^1(\Omega; \mathbb{R}^d)$. This subsequence further admits another subsequence such that the symmetric gradients converge weakly* to a limit w in $M(\Omega; \mathbb{R}_s^{d\times d})$. By taking the limit in (10.4), one easily verifies that $u \in BD(\Omega)$ with $\nabla^s u = w$, hence, every bounded sequence in $BD(\Omega)$ admits a weakly* convergent subsequence.

This concludes our introduction to the space of bounded deformation. We continue with the space of velocity fields.

10.2 The Space of Velocity Fields

As we elaborated in the beginning of this part and chapter, the space of velocity fields is the subspace of $L^2(L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d))$ such that the symmetric gradient is weakly measurable with values in $M(\Omega; \mathbb{R}_s^{d\times d})$. We specify this in

Definition and Lemma 10.4 (Space of velocity fields)**.** *We define the* space of velocity fields $\mathcal{V}$ *as the linear subspace of $L^2(L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d))$ such that $v \in \mathcal{V}$ if and only if* *(i)* Pointwise regularity. *We have* $v(t) \in BD(\Omega)$ *for almost all* $t \in [0,T]$.

(ii) Weakly measurability . *The symmetric gradient* $\nabla^s v$ *is* weakly measurable, *that is, the mapping* $[0,T] \ni t \mapsto \langle \nabla^s v(t), g(t)\rangle \in \mathbb{R}$ *is an element of* $L^1(0,T;\mathbb{R})$ *for every* $g \in L^2(C_0(\Omega;\mathbb{R}^{d\times d}_s))$, *where* $\langle\cdot,\cdot\rangle$ *denotes the dual pairing between* $M(\Omega;\mathbb{R}^{d\times d}_s)$ *and* $C_0(\Omega;\mathbb{R}^{d\times d}_s)$.

(iii) Finite integrability of the norm. *The mapping* $[0,T] \ni t \mapsto \|\nabla^s v(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \in \mathbb{R}$ *is an element of* $L^2(0,T;\mathbb{R})$.

Moreover, the norm

$$\|\cdot\|_{\mathcal{V}} : \mathcal{V} \to \mathbb{R}, \qquad v \mapsto \Big(\int_0^T \|v(t)\|^2_{BD(\Omega)} dt\Big)^{\frac{1}{2}}$$

is well defined.

Proof. Clearly, if $v, w \in \mathcal{V}$ and $\alpha \in \mathbb{R}$, the element $(\alpha v + w) \in L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$ fulfills Item (i) and Item (ii). In order to see that Item (iii) is fulfilled, we select a dense and countable subset $\{g_k\}_{k\in\mathbb{N}}$ of $C_0(\Omega;\mathbb{R}^{d\times d}_s)$, such that $g_k \neq 0$ for all $k \in \mathbb{N}$, and note that

$$\|\nabla^s(\alpha v + w)(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} = \sup_{k\in\mathbb{N}} \Big(\alpha\langle \nabla^s v(t), g_k/\|g_k\|\rangle + \langle \nabla^s w(t), g_k/\|g_k\|\rangle\Big)$$

for almost all $t \in [0,T]$, hence, as a pointwise supremum of countable measurable functions, $\|\nabla^s(\alpha v+w)(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)}$ is measurable (cf. [85, 1.14 Theorem]). Moreover, the triangle inequality gives a majorant.

Since $\|\cdot\|_{BD(\Omega)} = \|\cdot\|_{L^1(\Omega;\mathbb{R}^d)} + \|\cdot\|_{M(\Omega;\mathbb{R}^{d\times d}_s)}$, the norm $\|\cdot\|_{\mathcal{V}}$ is well defined. □

Note that we have chosen the space $L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d)$ in the definition and Lemma above due to Theorem 10.2 Item (ii). Moreover, similar as in the proof of Definition and Lemma 10.4, Item (ii) therein gives the measurability of $\|v(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)}$ for $v \in \mathcal{V}$, therefore the actual requirement in Item (iii) is only the finiteness of the integral (thus the name of Item (iii)).

Having defined the space of velocity fields, we continue with the definition of the weak convergence in this space and a result about weak compactness.

Definition 10.5 (Weak* convergence in $\mathcal{V}$). *Let* $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ *and* $v \in \mathcal{V}$. *We say that* v_n *converges weakly* towards* v *in* $\mathcal{V}$, *and write* $v_n \rightharpoonup^* v$ *in* $\mathcal{V}$, *if and only if* $v_n \rightharpoonup v$ *in* $L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$ *and*

$$\int_0^T \langle \nabla^s v_n(t), g(t)\rangle dt \to \int_0^T \langle \nabla^s v(t), g(t)\rangle dt \tag{10.5}$$

for all $g \in L^2(C_0(\Omega;\mathbb{R}^{d\times d}_s))$, *as* $n \to \infty$.

Proposition 10.6 (Weak compactness of $\mathcal{V}$). *Let* $C > 0$ *be a constant and* $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ *a sequence such that*

$$\|v_n\|_{\mathcal{V}} \leq C$$

for all $n \in \mathbb{N}$.

Then there exists a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ *and a weak* limit* $v \in B^{\mathcal{V}}_C(0)$ *such that* $v_{n_k} \rightharpoonup^* v$ *in* $\mathcal{V}$, *as* $k \to \infty$.

Proof. In the following we will select several subsequences of $\{v_n\}_{n\in\mathbb{N}}$ and denote them all with the same symbol.

Step (i). Convergence of the symmetric gradient. Let $S = \{g_k\}_{k\in\mathbb{N}}$ be a dense set in $C_0(\Omega;\mathbb{R}^{d\times d}_s)$, such that $g_k \neq 0$ for all $k \in \mathbb{N}$. Since $\|\nabla^s v_n(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)}$ is bounded in $L^2(0,T;\mathbb{R})$, we can select a subsequence and a weak limit $z \in L^2(0,T;\mathbb{R})$, such that $\|\nabla^s v_n(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \rightharpoonup z$ in $L^2(0,T;\mathbb{R})$. Moreover, the mappings

$$z_n^k : [0,T] \to \mathbb{R}, \qquad t \mapsto \langle \nabla^s v_n(t), g_k \rangle$$

are bounded in $L^2(0,T;\mathbb{R})$ with respect to n for every $k \in \mathbb{N}$. By a diagonal method we can extract a subsequence and weak limits $z^k \in L^2(0,T;\mathbb{R})$ such that $z_n^k \rightharpoonup z^k$ in $L^2(0,T;\mathbb{R})$, as $n \to \infty$, for all $k \in \mathbb{N}$.

For almost all $t \in [0,T]$ and all $G \in \operatorname{span}(S)$, $G = \sum_{i=1}^m \alpha_i g_i$, we define

$$(V(t))(G) := \sum_{i=1}^m \alpha_i z^i(t) \in \mathbb{R}.$$

Choosing an arbitrary Borel set $B \in \mathcal{B}([0,T])$ and $G \in \operatorname{span}(S)$ we obtain

$$\int_B (V(t))(G)dt = \sum_{i=1}^m \alpha_i \int_B z^i(t)dt = \sum_{i=1}^m \alpha_i \lim_{n\to\infty} \int_B z_n^i(t)dt = \lim_{n\to\infty} \int_B \langle \nabla^s v_n(t), G\rangle dt,$$

so that

$$\int_B (V(t))(G)dt \leq \lim_{n\to\infty} \int_B \|\nabla^s v_n(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \|G\|_{C_0(\Omega;\mathbb{R}^{d\times d}_s)} dt = \int_B z(t) \|G\|_{C_0(\Omega;\mathbb{R}^{d\times d}_s)} dt,$$

hence,

$$|(V(t))(G)| \leq z(t)\|G\|_{C_0(\Omega;\mathbb{R}^{d\times d}_s)} \tag{10.6}$$

for almost all $t \in [0,T]$ and all $G \in \operatorname{span}(S)$.

Since $V(t)$ is linear by definition, inequality (10.6) proves that $V(t) \in \mathcal{L}(\operatorname{span}(S);\mathbb{R})$ for almost all $t \in [0,T]$. Furthermore, because S, and therefore $\operatorname{span}(S)$, is dense in $C_0(\Omega;\mathbb{R}^{d\times d}_s)$, we can extend $V(t)$ uniquely to an element of $C_0(\Omega;\mathbb{R}^{d\times d}_s)^* = M(\Omega;\mathbb{R}^{d\times d}_s)$, using (10.6) again we obtain

$$\|V(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \leq z(t) \tag{10.7}$$

for almost all $t \in [0,T]$. Moreover, since

$$\|V(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} = \sup_{k\in\mathbb{N}} \frac{\langle V(t), g_k\rangle}{\|g_k\|_{C_0(\Omega;\mathbb{R}^{d\times d}_s)}} = \sup_{k\in\mathbb{N}} z^k(t),$$

$\|V(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)}$ is measurable as the pointwise supremum of countable measurable functions (cf. [85, 1.14 Theorem]), thus $\|V(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \in L^2(0,T;\mathbb{R})$.

Let us select a finite partition $\{B_i\}_{i\in\{1,\dots,m\}} \subset \mathcal{B}([0,T])$ of $[0,T]$, elements $g_i \in S$, $i \in \{1,\dots,m\}$, and a simple function

$$g : [0,T] \to C_0(\Omega;\mathbb{R}^{d\times d}_s), \qquad t \mapsto \sum_{i=1}^m \chi_{B_i}(t) g_i,$$

where χ_{B_i} is the characteristic function of B_i. Then

$$\int_0^T \langle V(t), g(t)\rangle dt = \sum_{i=1}^m \int_{B_i} z^i(t)dt = \lim_{n\to\infty} \int_0^T \langle \nabla^s v_n(t), g(t)\rangle dt. \tag{10.8}$$

Let now $f \in L^2(C_0(\Omega; \mathbb{R}^{d\times d}_s))$ be arbitrary. Because S is dense in $C_0(\Omega; \mathbb{R}^{d\times d}_s)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions with values in S and $f_n \to f$ in $L^2(C_0(\Omega; \mathbb{R}^{d\times d}_s))$ and $f_n(t) \to f(t)$ for almost all $t \in [0, T]$. We have for almost all $t \in [0, T]$

$$\langle V(t), f_n(t)\rangle \to \langle V(t), f(t)\rangle,$$

and

$$|\langle V(t), f_n(t)\rangle| \le \frac{1}{2}\Big[\|V(t)\|^2_{M(\Omega;\mathbb{R}^{d\times d}_s)} + \|f_n(t)\|^2_{C_0(\Omega;\mathbb{R}^{d\times d}_s)}\Big],$$

hence, Lebesgue's dominated convergence theorem (cf. [104, Theorem 3.1.29]) gives $\langle V(\cdot), f_n(\cdot)\rangle \to \langle V(\cdot), f(\cdot)\rangle$ in $L^1(0, T; \mathbb{R})$, as $n \to \infty$. Now, for $\varepsilon > 0$ we select $k \in \mathbb{N}$ such that, using (10.8), we get

$$\begin{aligned}\limsup_{n\to\infty} \Big| \int_0^T \langle V(t), f(t)\rangle dt - \int_0^T \langle \nabla^s v_n(t), f(t)\rangle dt \Big| \\ \le \varepsilon + \limsup_{n\to\infty} \Big| \int_0^T \langle V(t), f_k(t)\rangle dt - \int_0^T \langle \nabla^s v_n(t), f_k(t)\rangle dt \Big| \\ + \limsup_{n\to\infty} \Big| \int_0^T \langle \nabla^s v_n(t)), f_k(t) - f(t)\rangle dt \Big| \le \varepsilon + C\varepsilon,\end{aligned}$$

hence,

$$\int_0^T \langle \nabla^s v_n(t), f(t)\rangle dt \to \int_0^T \langle V(t), f(t)\rangle dt$$

for all $f \in L^2(C_0(\Omega; \mathbb{R}^{d\times d}_s))$, as $n \to \infty$.

Step (ii). Convergence of the full function. Since $BD(\Omega)$ is continuously embedded in $L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d)$ (Theorem 10.2 Item (ii)), v_n is bounded in $L^2(L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d))$ and we can select another subsequence and a weak limit $v \in L^2(L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d))$ such that $v_n \rightharpoonup v$ in $L^2(L^{\frac{d}{d-1}}(\Omega; \mathbb{R}^d))$. For every $\varphi \in C^\infty_c(\Omega; \mathbb{R})$, every Borel set $B \in \mathcal{B}([0, T])$ and all $i, j \in \{1, ..., n\}$ we have

$$\begin{aligned}&\frac{1}{2}\int_B \int_\Omega v_i(t, x)\partial_j \varphi(x) + v_j(t, x)\partial_i \varphi(x) dx dt \\ &\qquad = \lim_{n\to\infty} \frac{1}{2}\int_B \int_\Omega v_{i,n}(t, x)\partial_j \varphi(x) + v_{j,n}(t, x)\partial_i \varphi(x) dx dt \\ &\qquad = \lim_{n\to\infty} -\int_B \int_\Omega \varphi d(\nabla^s v_n(t))_{i,j} dt = -\int_B \int_\Omega \varphi d(V(t))_{i,j} dt,\end{aligned}$$

therefore $V(t) = \nabla^s v(t)$ for almost all $t \in [0, T]$.

Step (iii). Boundedness of the limit. To prove that $\|v\|_{\mathcal{V}} \le C$ we note that $\|v_n(\cdot)\|_{L^1(\Omega;\mathbb{R}^d)}$ is bounded in $L^2(0, T; \mathbb{R})$, thus, for another subsequence, we obtain $\tilde{z} \in L^2(0, T; \mathbb{R})$ such that $\|v_n(\cdot)\|_{L^1(\Omega;\mathbb{R}^d)} \rightharpoonup \tilde{z}$ in $L^2(0, T; \mathbb{R})$. For an arbitrary Borel set $B \in \mathcal{B}([0, T])$ and $\varphi \in L^\infty(\Omega; \mathbb{R}^d)$ we have

$$\begin{aligned}\int_B \int_\Omega v(t, x) \cdot \varphi(x) dx dt &= \lim_{n\to\infty} \int_B \int_\Omega v_n(t, x) \cdot \varphi(x) dx dt \\ &\le \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^d)} \lim_{n\to\infty} \int_B \|v_n(t)\|_{L^1(\Omega;\mathbb{R}^d)} dt = \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^d)} \int_B \tilde{z}(t) dt,\end{aligned}$$

thus

$$\int_{\Omega} v(t,x) \cdot \varphi(x) dx \leq \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^d)} \tilde{z}(t)$$

for almost all $t \in [0,T]$. Choosing in particular $\varphi_i(x) = \chi_{\{x\in\Omega | v_i(t,x)\geq 0\}} - \chi_{\{x\in\Omega | v_i(t,x)<0\}}$ we get $\|v(t)\|_{L^1(\Omega;\mathbb{R}^d)} \leq \tilde{z}(t)$ for almost all $t \in [0,T]$.

Since $(\|v_n(\cdot)\|_{L^1(\Omega;\mathbb{R}^d)} + \|\nabla^s v_n(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)})$ converges weakly towards $(\tilde{z}+z)$ in $L^2(0,T;\mathbb{R})$ and

$$\| \, \|v_n(\cdot)\|_{L^1(\Omega;\mathbb{R}^d)} + \|\nabla^s v_n(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \|_{L^2(0,T;\mathbb{R})} = \|v_n\|_{\mathcal{V}} \leq C$$

we obtain

$$\|v\|_{\mathcal{V}} \leq \|\tilde{z}+z\|_{L^2(0,T;\mathbb{R})} \leq C,$$

where we also have used (10.7), hence, the proof is complete. □

We shortly give some comments on the proof of the proposition above. At first, the gradients of elements of $\mathcal{V}$ are weakly measurable, that is, they belong to $L_w(M(\Omega;\mathbb{R}^{d\times d}_s))$ (we do not use this notation in our development, it was used in SUQUET [95] and we only adopt it here to compare our results with the literature). According to Theorem 10.1 and [33, 8.20.3 Theorem], $L^2_w(M(\Omega;\mathbb{R}^{d\times d}_s))$ is the dual space of $L^2(C_0(\Omega;\mathbb{R}^{d\times d}_s))$, the convergence in (10.5) is then just the weak* convergence in $L^2_w(M(\Omega;\mathbb{R}^{d\times d}_s))$. With this result one obtains the existence of a weak* limit of symmetric gradients from a bounded sequence in $\mathcal{V}$, which we have proven in step (i) of the proof of Proposition 10.6. However, as already said in the beginning of this part, we decided to provide the full proofs without citing results from [33] (or [101]) about weakly measurability.

Let us finally show that the space of velocity fields is complete, for this task the following lemma is useful.

Lemma 10.7 (Limit of the symmetrized gradient). *Let $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$, $v \in \mathcal{V}$ and $V : [0,T] \to M(\Omega;\mathbb{R}^{d\times d}_s)$ such that $v_n \rightharpoonup^* v$ in $\mathcal{V}$ and $\|\nabla^s v_n(\cdot) - V(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \to 0$ in $L^1(0,T;\mathbb{R})$.*

Then $\nabla^s v = V$.

Proof. We have for all $g \in L^\infty(C_0(\Omega;\mathbb{R}^{d\times d}_s))$

$$\begin{aligned}\int_0^T \langle \nabla^s v(t) - V(t), g(t)\rangle dt &= \lim_{n\to\infty} \int_0^T \langle \nabla^s v_n(t) - V(t), g(t)\rangle dt \\ &\leq \lim_{n\to\infty} \int_0^T \|\nabla^s v_n(t) - V(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} dt \|g\|_{L^\infty(C_0(\Omega;\mathbb{R}^{d\times d}_s))} = 0,\end{aligned}$$

thus

$$\int_0^T \langle \nabla^s v(t) - V(t), g(t)\rangle dt = 0.$$

Choosing in particular $g = G\chi_B$ for an arbitrary Borel set $B \in \mathcal{B}([0,T])$ and $G \in C_0(\Omega;\mathbb{R}^{d\times d}_s)$ we get

$$\langle \nabla^s v(t) - V(t), G\rangle = 0$$

hence, $\|\nabla^s v(t) - V(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} = 0$, for almost all $t \in [0,T]$. □

Proposition 10.8 (Banach space). *The space $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is complete.*

Proof. Let $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ be a Cauchy sequence. Since $BD(\Omega)$ is continuously embedded in $L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d)$, v_n is a Cauchy sequence in $L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$ and there exists a strong limit $v \in L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$. Because v_n is also bounded in $\mathcal{V}$, Proposition 10.6 gives $v \in \mathcal{V}$ and $v_n \rightharpoonup^* v$ in $\mathcal{V}$ (for the whole sequence since v is already determined).

We select now a subsequence such that

$$\int_0^T \|\nabla^s v_m(t) - \nabla^s v_n(t)\|^2_{M(\Omega;\mathbb{R}^{d\times d}_s)} dt \le \frac{1}{2^n}$$

for all $n \in \mathbb{N}$ and $m \ge n$. The function

$$U_n : [0,T] \to \mathbb{R}, \qquad t \mapsto \sum_{i=1}^n \|\nabla^s v_{i+1}(t) - \nabla^s v_i(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)}$$

is then bounded in $L^2(0,T;\mathbb{R})$ (due to the triangle inequality) and pointwise monotonically increasing, hence, according to Beppo Levi's monotone convergence theorem (cf. [37, Kapitel IV Satz 2.7]), there exists a limit $U \in L^2(0,T;\mathbb{R})$ such that $U_n \to U$ in $L^2(0,T;\mathbb{R})$ and $U_n(t) \to U(t)$ for almost all $t \in [0,T]$. Due to

$$\|\nabla^s v_n(t) - \nabla^s v_m(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \le |U_{n-1}(t) - U_{m-1}(t)| \tag{10.9}$$

for all $n, m \in \mathbb{N}$, $n \ge m$, the sequence $\nabla^s v_n(t)$ is a Cauchy sequence and has therefore a limit $V(t)$ in $M(\Omega;\mathbb{R}^{d\times d}_s)$ for almost all $t \in [0,T]$. Considering the limit $m \to \infty$ in (10.9) we obtain

$$\|\nabla^s v_n(t) - V(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \le |U_n(t) - U(t)|,$$

and Lebesgue's dominated convergence theorem gives $\|\nabla^s v_n(\cdot) - V(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \in L^2(0,T;\mathbb{R})$ for all $n \in \mathbb{N}$. Yet another application of Lebesgue's dominated convergence theorem yields $\|\nabla^s v_n(\cdot) - V(\cdot)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} \to 0$ in $L^2(0,T;\mathbb{R})$, thus Lemma 10.7 gives $\nabla^s v = V$. □

Remark 10.9 (Completeness of Bochner spaces)**.** *The proof of Proposition 10.8 is similar to the proof of the completeness of Bochner spaces, see for instance [87, Theorem 10.4].*

With the completeness result, we have collected all in the following needed results about the space of velocity fields. In the next section we will integrate functions from $\mathcal{V}$ in time to obtain the space of *displacement fields*.

10.3 The Space of Displacement Fields

When a function v belongs to $\mathcal{V}$, then, by definition, it also belongs to $L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^{d\times d}_s))$. Therefore we can simply (Bochner-)integrate this function to obtain a function in $H^1(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$. This is the method we use to define the space of displacement fields. Before we formulate this in Definition and Lemma 10.11, we show that these functions are *absolutely continuous* with values in $BD(\Omega)$.

Lemma 10.10 (Absolutely continuous)**.** *Let $v \in \mathcal{V}$ and $u_0 \in BD(\Omega)$.*

Then the function $u \in H^1(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$ defined by

$$u(t) := u_0 + \int_0^t v(s)ds \tag{10.10}$$

for almost all $t \in [0,T]$, is an element of $AC(BD(\Omega))$, that is, $u : [0,T] \to BD(\Omega)$ is absolutely continuous.

Proof. We define the function $U : [0,T] \to M(\Omega;\mathbb{R}^{d\times d}_s)$ by

$$\langle U(t), g\rangle := \langle \nabla^s u_0, g\rangle + \int_0^t \langle \nabla^s v(s), g\rangle ds$$

for all $g \in C_0(\Omega;\mathbb{R}^{d\times d}_s)$. Then

$$\|U(t)-U(s)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} = \sup_{g\in C_0(\Omega;\mathbb{R}^{d\times d}_s),\|g\|\le 1} \int_s^t \langle \nabla^s v(\tau), g\rangle d\tau \le \int_s^t \|\nabla^s v(t)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)} d\tau,$$

hence, $U \in AC(M(\Omega;\mathbb{R}^{d\times d}_s))$. Furthermore, for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}\frac{1}{2}\int_\Omega u_i(t)\partial_j\varphi + u_j(t)\partial_i\varphi &= \frac{1}{2}\int_\Omega u_{0,i}\partial_j\varphi + u_{0,j}\partial_i\varphi + \frac{1}{2}\int_\Omega\int_0^t v_i(s)\partial_j\varphi + v_j(s)\partial_i\varphi ds\\ &= -\int_\Omega \varphi d(\nabla^s u_0)_{i,j} - \int_0^t\int_\Omega \varphi d(\nabla^s v(t))_{i,j} ds\\ &= -\int_\Omega \varphi d(U(t))_{i,j},\end{aligned}$$

therefore $u(t) \in BD(\Omega)$ and $\nabla^s u(t) = U(t)$ for almost all $t\in[0,T]$. Since

$$\|u(t)-u(s)\|_{BD(\Omega)} = \|u(t)-u(s)\|_{L^1(\Omega;\mathbb{R}^d)} + \|U(t)-U(s)\|_{M(\Omega;\mathbb{R}^{d\times d}_s)},$$

we get $u \in AC(0,T;BD(\Omega))$. □

In the following we actually do not need the fact that a function defined by (10.10) is absolutely continuous with values in $BD(\Omega)$. However, in DAL MASO ET AL. [27, Theorem 5.2] it was shown that solutions to perfect plasticity are absolutely continuous in time (therein a different notion of a solution was used, as we will discuss in the beginning of the next chapter), Lemma 10.10 gives the same regularity.

Definition and Lemma 10.11 (Space of displacement fields)**.** *We define the* space of displacement fields

$$\mathcal{U} := \{u \in H^1(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d)) : u(0)\in BD(\Omega) \text{ and } \dot{u}\in\mathcal{V}\}.$$

The space of displacement fields is a linear subspace of $H^1(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))\cap AC(0,T;BD(\Omega))$ *and becomes with the norm*

$$\|\cdot\|_{\mathcal{U}} : \mathcal{U}\to\mathbb{R}, \qquad u\mapsto \|u(0)\|_{BD(\Omega)} + \|\dot{u}\|_{\mathcal{V}}$$

a Banach space. Moreover, if $C>0$ *is a constant and* $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{U}$ *a sequence such that*

$$\|u_n\|_{\mathcal{U}} \le C$$

for all $n\in\mathbb{N}$*, then there exists a subsequence* $\{u_{n_k}\}_{k\in\mathbb{N}}$ *and a weak* limit* $u\in\mathcal{U}$ *such that* $u_{n_k}\rightharpoonup^* u$ *in* $\mathcal{U}$*, as* $k\to\infty$*, where the weak convergence in* $\mathcal{U}$ *is defined as follows:*

$w_n \rightharpoonup^* w$ in $\mathcal{U}$ if and only if $\dot{w}_n\rightharpoonup^*\dot{w}$ in $\mathcal{V}$ and $w_n(0)\rightharpoonup^* w(0)$ in $BD(\Omega)$.

Furthermore, if it is known that $u_n(0)\to u(0)$ in $BD(\Omega)$, then $u\in B_C^{\mathcal{U}}(0)$.

Proof. Since $\mathcal{V}$ is a linear subspace of $L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$ and $\mathcal{U} \subset AC(0,T;BD(\Omega))$, $\mathcal{U}$ is a linear subspace of $H^1(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d)) \cap AC(0,T;BD(\Omega))$. That $\|\cdot\|_{\mathcal{U}}$ is a norm on $\mathcal{U}$ is obvious. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ is a Cauchy sequence in $\mathcal{U}$, $\dot{u}_n$ is a Cauchy sequence in $\mathcal{V}$ and $u_n(0)$ in $BD(\Omega)$, hence, there exists $v \in \mathcal{V}$ and $u_0 \in BD(\Omega)$ such that $\dot{u}_n \to v$ in $\mathcal{V}$ and $u_n(0) \to u_0$ in $BD(\Omega)$, then the function defined by (10.10) for almost all $t \in [0,T]$, is the limit of u_n in $\mathcal{U}$.

If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ is bounded by the constant $C > 0$, then $\dot{u}_n$ is bounded in $\mathcal{V}$ and $u_n(0)$ in $BD(\Omega)$, thus there exists a subsequence, $v \in \mathcal{V}$ and $u_0 \in BD(\Omega)$ such that $\dot{u}_n \rightharpoonup^* v$ in $\mathcal{V}$, $u_n(0) \rightharpoonup^* u_0$ in $BD(\Omega)$. Then we can define again the function $u \in \mathcal{U}$ by (10.10) for almost all $t \in [0,T]$ and obtain $u_n \rightharpoonup^* u$ in $\mathcal{U}$.

Let us now assume that $u_n(0) \to u_0 = u(0)$ in $BD(\Omega)$. Then there exists for every $\varepsilon > 0$ a number $n_0 \in \mathbb{N}$ such that

$$\|\dot{u}_n\|_{\mathcal{V}} \leq C - \|u(0)\|_{BD(\Omega)} + \varepsilon$$

for all $n \in \mathbb{N}$, $n \geq n_0$, hence, Proposition 10.6 gives

$$\|\dot{u}\|_{\mathcal{V}} \leq C - \|u(0)\|_{BD(\Omega)} + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we obtain $u \in B_C^{\mathcal{U}}(0)$. □

Corollary 10.12 (Weakly lower* semicontinuity)**.** *Let $u_0 \in BD(\Omega)$ be given. Then the $\mathcal{U}$-norm is weakly lower* semicontinuous on $M := \{u \in \mathcal{U} : u(0) = u_0\}$.*

Proof. Let u_n be a sequence in M such that $u_n \rightharpoonup^* u$ for one $u \in M$. We have to prove that

$$\|u\|_{\mathcal{U}} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{U}}. \tag{10.11}$$

To this end we consider a subsequence u_{n_k} such that

$$\lim_{k\to\infty} \|u_{n_k}\|_{\mathcal{U}} = \liminf_{n\to\infty} \|u_n\|_{\mathcal{U}}.$$

Let now $\varepsilon > 0$ be arbitrary and select $k_0 \in \mathbb{N}$ such that

$$\|u_{n_k}\|_{\mathcal{U}} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{U}} + \varepsilon$$

for all $k \geq k_0$. Since $u_{n_k}(0) = u_0$ holds for all $k \in \mathbb{N}$, Definition and Lemma 10.11 gives

$$\|u\|_{\mathcal{U}} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{U}} + \varepsilon,$$

which yields, thanks to the arbitrariness of ε, (10.11). □

With the corollary above, we have completed the introduction of the space of displacement fields and thus of the control space (for displacements). In the next chapter we can address the equations of perfect plasticity, different solution concepts and the existence of a solution.

Chapter 11 Definition, Regularization and Existence

One of the first (mathematical) works concerning (time dependent) perfect plasticity are [57, 60]. Therein, the flow rule $\nabla^s \dot{u} - \mathbb{A}\dot{\sigma} \in \partial I_{\mathcal{K}(\Omega)}(\sigma)$ (obtained from (9.19e) by using (9.19d)) was weakened by using integration by parts to remove the derivatives from the displacement. Then the existence of displacements in $H^1(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$ for $d = 2, 3$ was proven (and of stresses in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)))$. In the early eighties this work was extended by SUQUET [95], he used essentially the same technique to obtain a weak formulation of the flow rule and proved existence of the velocity of a displacement in $L^2_w(BD(\Omega))$ (see the beginning of this part, the beginning of Chapter 10 and the comment after Proposition 10.6 for the space $L^2_w(BD(\Omega))$). In both works the Yosida approximation was used (this method is also known as *vanishing viscosity*) to prove the existence.

After the work of Suquet the mathematical contributions to perfect plasticity subsided until the paper of DAL MASO ET AL. [27] in 2006. Therein, perfect plasticity was analyzed in the context of *quasistatic evolutions*, also called *energetic solutions* of *rate-independent systems* (see, e.g., [76]). However, as was proven in [27], their notion of a solution is equivalent to the one given in SUQUET [95, 1.4 Formulations. Résultats] (cf. [27, Theorem 6.1 and Remark 6.3]).

After that, several other works concerning perfect plasticity were published, e.g. [28, 11, 40, 92]. The introduction in [40] gives also a good overview of the history of perfect plasticity.

For the optimal control of perfect plasticity, we will follow SUQUET [95] by using the definition given therein and using also the Yosida approximation to regularize the equations of perfect plasticity in Section 11.2. Although the existence of a solution was proven in this paper, we tend to this topic in Section 11.3 in particular to extend this result with varying data, to add hardening and to show the strong convergence of the stresses of regularized solutions. Before we do this, we give the definition of a solution and provide some auxiliary results in the next section.

11.1 Weak Formulation of the Flow Rule

At first we give the definition of a solution to (9.19). The main idea in this definition is, as already said above, to weaken the flow rule (9.19e) by removing the derivatives of the displacement (and also to drop the boundary conditions (9.19c)). That the original flow rule can be recovered under certain assumptions will be shown in Lemma and Definition 11.2. Recall Assumption ⟨11.i⟩ and Assumption ⟨11.ii⟩.

Definition 11.1 ((Weak) solution of perfect plasticity)**.** *The tuple* $(u, \sigma) \in \mathcal{U} \times H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *is called a* weak solution *or just* solution *of* (9.19)*, if the following conditions hold:*

(i) The initial condition $\sigma(0) = \sigma_0$ *is fulfilled and we have*

$$-\operatorname{div}\sigma(t) = f(t), \qquad \sigma(t) \in \mathcal{K}(\Omega)$$

for all $t \in [0, T]$*, where the* div *operator was introduced in Definition 2.1 and the set* $\mathcal{K}(\Omega)$ *in Assumption ⟨V.i⟩.*

(ii) The initial condition $u(0) = u_0$ *is fulfilled and the* weak flow rule inequality

$$\begin{gathered}\Big(\mathbb{A}\dot\sigma(t) - \nabla^s\dot u_D(t), \tau - \sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \Big(\dot u(t) - \dot u_D(t), \operatorname{div}(\tau - \sigma(t))\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \geq 0 \\ \forall \tau \in \mathcal{K}(\Omega) \text{ with } \operatorname{div}\tau \in L^d(\Omega;\mathbb{R}^d) \text{ and f.a.a. } t \in [0, T]\end{gathered} \tag{11.1}$$

is satisfied.

Moreover, $\sigma \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *is called* reduced solution *of* (9.19) *if Item (i) and the* reduced weak flow rule inequality

$$\begin{gathered}\Big(\mathbb{A}\dot\sigma(t) - \nabla^s\dot u_D(t), \tau - \sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \geq 0 \\ \forall \tau \in \mathcal{K}(\Omega) \text{ with } -\operatorname{div}\tau = f(t) \text{ and f.a.a. } t \in [0, T]\end{gathered} \tag{11.2}$$

holds.

Note that the definitions above correspond to JOHNSON [57, Plasticity Problem I & II] and the definition given in SUQUET [95, 1.4 Formulations. Résultats].

It follows immediately from the definition, that if (u, σ) is a solution, then σ is a reduced solution of (9.19). As hinted above, a solution to (9.19) is equivalent to a *strong solution* when the displacement is more regular and the Dirichlet boundary conditions are satisfied:

Lemma and Definition 11.2 (Equivalence between a solution and a strong solution)**.** *A tuple* $(u, \sigma) \in H^1(H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_s))$ *is a* strong solution *of* (9.19)*, that is,*

$$\begin{aligned} -\operatorname{div}\sigma &= f, \\ \nabla^s \dot{u} - \mathbb{A}\dot{\sigma} &\in \partial I_{\mathcal{K}(\Omega)}(\sigma), \\ (u - u_D)(t) &\in H^1_D(\Omega; \mathbb{R}^d)\ \forall t \in [0, T], \\ (u, \sigma)(0) &= (u_0, \sigma_0) \end{aligned} \tag{11.3}$$

holds, if and only if it is a (weak) solution of (9.19) *and* $(u - u_D)(t) \in H^1_D(\Omega; \mathbb{R}^d)$ *holds for all* $t \in [0, T]$.

Proof. Let $(u, \sigma) \in H^1(H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_s))$ be a strong solution of (9.19), we only have to prove that (u, σ) fulfills the flow rule inequality (11.1) which follows from the definition of the subdifferential and the fact that $(u - u_D)(t) \in H^1_D(\Omega; \mathbb{R}^d)$ for all $t \in [0, T]$,

$$\begin{aligned} 0 &\geq \Big(\nabla^s \dot{u}(t) - \mathbb{A}\dot{\sigma}(t), \tau - \sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\ &= \Big(\nabla^s \dot{u}_D(t) - \mathbb{A}\dot{\sigma}(t), \tau - \sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - \Big(\dot{u}(t) - \dot{u}_D(t), \operatorname{div}(\tau - \sigma(t))\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \end{aligned}$$

for all $\tau \in \mathcal{K}(\Omega)$ with $\operatorname{div}\tau \in L^d(\Omega; \mathbb{R}^d)$ and for almost all $t \in [0, T]$.

Let us now assume that $(u, \sigma) \in H^1(H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_s))$ is a (weak) solution of (9.19) and that $(u - u_D)(t) \in H^1_D(\Omega; \mathbb{R}^d)$ holds for all $t \in [0, T]$. As above we obtain

$$0 \geq \Big(\nabla^s \dot{u}(t) - \mathbb{A}\dot{\sigma}(t), \tau - \sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

for all $\tau \in \mathcal{K}(\Omega)$ with $\operatorname{div}\tau \in L^d(\Omega; \mathbb{R}^d)$ and for almost all $t \in [0, T]$. When we can show that this inequality holds even for all $\tau \in \mathcal{K}(\Omega)$, the proof is complete. To this end, we note that the inequality holds in particular for all $\tau \in C^1_c(\Omega; \mathbb{R}^{d\times d}_s) \cap \mathcal{K}(\Omega)$, thus, it is sufficient to show that $C^\infty_c(\Omega; \mathbb{R}^{d\times d}_s) \cap \mathcal{K}(\Omega)$ is dense in $\mathcal{K}(\Omega)$.

To prove this, let $\tau \in \mathcal{K}(\Omega)$ and $\varepsilon \in (0, 1)$ be arbitrary. We set $\overline{\tau} := (1 - \varepsilon)\tau$ and select a sequence $\{\tau_n\}_{n\in\mathbb{N}} \in C^\infty_c(\Omega; \mathbb{R}^{d\times d}_s)$ such that $\|\tau_n - \overline{\tau}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \leq \frac{\delta^2\varepsilon^3}{2n}$ for all $n \in \mathbb{N}$, where $\delta > 0$ is chosen such that $B_\delta(0) \subset K$ (cf. Assumption ⟨V.i⟩). We define

$$\begin{aligned} S^c_n &:= \{x \in \Omega : \tau_n(x) \in \partial K \cup (\mathbb{R}^{d\times d}_s \setminus K)\}, \\ S^o_n &:= \{x \in \Omega : \tau_n(x) \in (\mathbb{R}^{d\times d}_s \setminus (1 - \frac{\varepsilon}{2})K)\} \end{aligned}$$

(thus S^c_n is compact (τ_n has a compact support so that S^c_n is a subset of this support) and S^o_n open with $S^c_n \subset S^o_n$) and note that

$$\frac{\tau_n(x)}{1 - \frac{\varepsilon}{2}} = \lambda\tau(x) + (1 - \lambda)\sigma(x),$$

with $\lambda := \frac{(1-\varepsilon)}{(1-\frac{\varepsilon}{2})}$ and $\sigma(x) := \frac{\tau_n(x)-\overline{\tau}(x)}{(1-\lambda)(1-\frac{\varepsilon}{2})}$, is an element of K whenever $\tau(x) \in K$ and $|\tau_n(x) - \overline{\tau}(x)| \leq \delta(1-\lambda)(1-\frac{\varepsilon}{2}) = \frac{\delta\varepsilon}{2}$. Therefore we obtain

$$\|\tau_n - \overline{\tau}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \geq \int_{S^o_n} |\tau_n - \overline{\tau}|^2 \geq \frac{\delta^2\varepsilon^2}{2}|S^o_n|$$

so that

$$|S^c_n| \leq |S^o_n| \leq \frac{\varepsilon}{n}.$$

Due to Lebesgue's dominated convergence theorem, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\|\overline{\tau}\|_{L^2(S^o_N;\mathbb{R}^{d\times d}_s)} \leq \|\tau\|_{L^2(S^o_N;\mathbb{R}^{d\times d}_s)} \leq \varepsilon.$$

We select $v \in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that $v(x) = 1$ for all $x \in \mathbb{R}^d \setminus S^o_N$, $v(x) = 0$ for all $x \in S^c_N$ and $v(x) \in [0,1]$ otherwise. The function $\tau_s := v\tau_N$ then fulfills

$$\begin{aligned}
\|\tau - \tau_s\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} &\leq \|\tau - \overline{\tau}\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \|\overline{\tau} - \tau_N\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \|\tau_N - \tau_s\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\leq \varepsilon\|\tau\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{\sqrt{\delta}\varepsilon}{\sqrt{2N}} + \|\tau_N\|_{L^2(S^o_N;\mathbb{R}^{d\times d}_s)} \\
&\leq \varepsilon\|\tau\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{\sqrt{\delta}\varepsilon}{\sqrt{2N}} + \|\overline{\tau} - \tau_N\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \|\overline{\tau}\|_{L^2(S^o_N;\mathbb{R}^{d\times d}_s)} \\
&\leq \varepsilon(\|\tau\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{2\sqrt{\delta}}{\sqrt{2N}} + 1)
\end{aligned}$$

and also $\tau_s \in \mathcal{K}(\Omega) \cap C^\infty_c(\Omega;\mathbb{R}^{d\times d}_s)$. □

Let us note that the unknown plastic strain z in (9.19) does not occur in Definition 11.1 and we have also excluded it in (11.3), however, it can be obtained from u and σ. Analog to DALMASO ET AL. [27, 2.3 Stress and strain, (2.21) & (2.22)], the plastic strain can be set to

$$z(t) = \nabla^s u(t) - e(t) \qquad \text{in } \Omega \tag{11.4}$$

$$z(t) = ((u_D(t) - u(t)) \circledast \nu)\mathcal{H}^{n-1} \qquad \text{on } \Gamma_D \tag{11.5}$$

for $t \in [0,T]$, where the elastic strain is simply $e = \mathbb{A}\sigma$ and the symmetrized tensor product is defined by $a \circledast b := (ab^\top + a^\top b)/2$ for all $a, b \in \mathbb{R}^d$. Here, (11.5) corresponds to the fact that when the displacement field does not equal the Dirichlet displacement on the boundary (which is possible according to Definition 11.1), then this is due to plastic deformation. For more insight on this definition we refer to the discussion after Corollary 3.5 in [73] and DAL MASO ET AL.[27]. We emphasize that from a mechanical standpoint of view, it makes sense to drop the Dirichlet boundary condition in Definition 11.1. This allows *plastic slips* on the boundary.

Auxiliary Results

We start with the uniqueness of the stress. The prove is straightforward and can be found in JOHNSON [57, Theorem 1] but we provide it for the convenience of the reader.

Lemma 11.3 (Reduced solutions are unique)**.** *Assume that $\sigma_1, \sigma_2 \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ are two reduced solutions of* (9.19).

Then $\sigma_1 = \sigma_2$.

Proof. We have

$$\left(\mathbb{A}\dot{\sigma}_i(t) - \nabla^s \dot{u}_D(t), \tau - \sigma_i(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \geq 0$$

for $i = 1, 2$, for almost all $t \in [0, T]$ and for all $\tau \in \mathcal{K}(\Omega)$ with $-\operatorname{div}\tau = f(t)$. Testing the inequality for $i = 1$ with $\tau = \sigma_2(t)$, the inequality for $i = 2$ with $\tau = \sigma_1(t)$ and adding both inequalities together gives

$$\left(\mathbb{A}(\dot{\sigma}_1(t) - \dot{\sigma}_2(t)), \sigma_1(t) - \sigma_2(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \leq 0$$

for almost all $t \in [0, T]$. Integrating this inequality with respect to time and using the coercivity of $\mathbb{A}$, we get

$$\|\sigma_1(t) - \sigma_2(t)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \leq 0,$$

thus $\sigma_1 = \sigma_2$. □

Remark 11.4 (The displacement is not unique)**.** *In Lemma 11.3 it was easily seen that the stress field of a solution to* (9.19) *is unique. However, the same is not true for the displacement field (and thus not for the plastic strain in* (11.5)*), in* SUQUET *[95, 2.1 Exemples] this was shown for a most trivial example in the one dimensional space, see also [73, Example 3.10].*

The following two statements are devoted to a reformulation of the flow rule inequality for time dependent functions. This reformulation is more convenient when taking the limit in the flow rule inequality, for example in Proposition 11.10.

Lemma 11.5 (Time dependent flow rule inequality)**.** *Let* $(u, \sigma) \in H^1(H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega, \mathbb{R}^{d\times d}_s))$. *Then*

$$\begin{gathered}\left(\mathbb{A}\dot{\sigma} - \nabla^s \dot{u}_D, \tau - \sigma\right)_{L^2(L^2(\Omega,\mathbb{R}^{d\times d}_s))} + \left(\dot{u} - \dot{u}_D, \operatorname{div}(\tau - \sigma)\right)_{L^2(L^2(\Omega,\mathbb{R}^d))} \geq 0 \\ \forall \tau \in L^2(L^2(\Omega, \mathbb{R}^{d\times d}_s)) \text{ with } \operatorname{div}\tau \in L^2(L^d(\Omega; \mathbb{R}^d)) \text{ and } \tau(t) \in \mathcal{K}(\Omega) \text{ f.a.a. } t \in [0, T]\end{gathered} \tag{11.6}$$

holds if and only if (11.1) *holds.*

Proof. Clearly, if (11.1) holds, then also (11.6).

Assume now that (11.6) holds. Let $\tau_0 \in \mathcal{K}(\Omega)$ with $\operatorname{div}\tau_0 \in L^d(\Omega; \mathbb{R}^d)$ be arbitrary and $S \in \mathcal{B}([0, T])$ the Borel set containing all $t \in [0, T]$ with

$$\left(\mathbb{A}\dot{\sigma}(t) - \nabla^s \dot{u}_D(t), \tau_0 - \sigma(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left(\dot{u}(t) - \dot{u}_D(t), \operatorname{div}(\tau_0 - \sigma(t))\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} < 0$$

(for a particular choice of representatives of σ, u_D and u.) When we define $\tau(t) := \tau_0$ for all $t \in S$ and $\tau(t) := \sigma(t)$ for all $t \in [0, T] \setminus S$, we see, thanks to (11.6), that S has measure zero. □

Corollary 11.6 (Time dependent reduced flow rule inequality)**.** *Let* $\sigma \in H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$. *Then*

$$\begin{gathered}\left(\mathbb{A}\dot{\sigma} - \nabla^s \dot{u}_D, \tau - \sigma\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \geq 0 \\ \forall \tau \in L^2(L^2(\Omega; \mathbb{R}^{d\times d}_s)) \text{ with } -\operatorname{div}\tau = f \text{ and } \tau(t) \in \mathcal{K}(\Omega) \text{ f.a.a. } t \in [0, T]\end{gathered} \tag{11.7}$$

holds if and only if (11.2) *holds.*

Proof. The proof is analog to the one from Lemma 11.5. □

In the following lemma, we will put the norm of the stress in relationship to the given data. This enables us to obtain a priori estimates for the stress and it is also important to prove strong convergence in Proposition 11.10 Item (ii) and Theorem 11.19 below (using weak and norm convergence).

Lemma 11.7 (Norm of the stress)**.** *Assume that* $(u,\sigma) \in \mathcal{U} \times H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *is a solution of* (9.19). *Then the equation*

$$\|\dot\sigma(t)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{A}}} = \Big(\nabla^s \dot u_D(t), \dot\sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \Big(\dot u(t) - \dot u_D(t), \dot f(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

holds for almost all $t \in [0,T]$.

Proof. There exists a set $N \subset [0,T]$ with measure zero, such that

$$\lim_{h\to 0} \frac{\sigma(t+h)-\sigma(t)}{h} = \dot\sigma(t), \qquad \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \dot f(t)$$

(cf. [104, Theorem 3.1.40]) and

$$\Big(\mathbb{A}\dot\sigma(t) - \nabla^s \dot u_D(t), \tau - \sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \Big(\dot u(t) - \dot u_D(t), \operatorname{div}(\tau - \sigma(t))\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \geq 0$$

hold for all $t \in (0,T) \setminus N$ and all $\tau \in \mathcal{K}(\Omega)$ with $\operatorname{div}\tau \in L^d(\Omega;\mathbb{R}^d)$. Testing this inequality with $\sigma(t \pm h)$ for a sufficient small $h > 0$, we get

$$\Big(\mathbb{A}\dot\sigma(t) - \nabla^s \dot u_D(t), \sigma(t\pm h) - \sigma(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \geq \Big(\dot u(t) - \dot u_D(t), f(t\pm h) - f(t)\Big)_{L^2(\Omega;\mathbb{R}^d)},$$

dividing by h and letting $h \searrow 0$, we obtain the desired equation for all $t \in (0,T) \setminus N$. □

The result above uses the same method as was used in [104, Lemma 3.2.7 and Theorem 3.2.10] and probably originated in [64, Proposition I.3.9].

Corollary 11.8 (Boundedness of the stress)**.** *Assume that* $(u_n,\sigma_n) \in \mathcal{U} \times H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *are weak solutions of* (9.19) *with respect to* $u_{D,n}$ *and* f_n, *and assume that at least one of the following conditions hold:*

(i) $\dot u_n$ *is bounded in* $L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))$.

(ii) $\dot f_n = 0$ *for all* $n \in \mathbb{N}$.

Then the sequence σ_n *is bounded in* $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$.

Proof. Due to Lemma 11.7 we have

$$\|\dot\sigma_n(t)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{A}}} = \Big(\nabla^s \dot u_{D,n}(t), \dot\sigma_n(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \Big(\dot u_n(t) - \dot u_{D,n}(t), \dot f_n(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

for almost all $t \in [0,T]$ and all $n \in \mathbb{N}$. Integrating this equation with respect to time and using the coercivity of $\mathbb{A}$, we get

$$\begin{aligned}
\|\dot\sigma_n\|^2_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} &\leq \frac{1}{\gamma_{\mathbb{A}}}\Big(\Big(\nabla^s \dot u_{D,n}, \dot\sigma_n\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \Big(\dot u_n - \dot u_{D,n}, \dot f_n\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \Big) \\
&\leq \frac{1}{\gamma_{\mathbb{A}}}\Big(\|\nabla^s \dot u_{D,n}\|_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \|\dot\sigma_n\|_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&\qquad + \|\dot u_n - \dot u_{D,n}\|_{L^2(L^{\frac{d}{d-1}}(\Omega;\mathbb{R}^d))} \|\dot f_n\|_{L^2(L^d(\Omega;\mathbb{R}^d))}\Big),
\end{aligned}$$

this together with $\sigma_n(0) = \sigma_0$ for all $n \in \mathbb{N}$ yields the desired boundedness of σ_n in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$. □

Remark 11.9 (Boundedness of the displacement)**.** *It is also possible to prove the boundedness of the displacement, cf. [73, Lemma 4.3]. Such a result was also proven in* DAL MASO ET AL. *[27, Theorem 5.2]. However, this result would be of little use for us since we do not have an analogon in the regularized case (at least without the* safe-load condition*, see Section 11.2) and we also require that the displacement is of class $H^1(\Omega;\mathbb{R}^{d\times d}_s)$ in Chapter 12 so that we could not use such a boundedness result (in $\mathcal{U}$), see also Remark 12.2.*

In the following proposition we prove some continuity properties which in particular show that weak limits of solutions to (9.19) are also solutions. Note that the convergence properties in Assumption ⟨11.1⟩ are only needed for this result.

Proposition 11.10 (Continuity properties of solutions to perfect plasticity)**.** *Suppose that the tuples $(u_n,\sigma_n) \in \mathcal{U} \times H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ are solutions to (9.19) with respect to $u_{D,n}$ and f_n. Then the following two assertions hold:*

(i) Assume additionally that u_n converges weakly towards $u \in \mathcal{U}$. Then there exists $\sigma \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ such that (u,σ) is a solution of (9.19) with respect to u_D and f and we have the weak convergence $\sigma_n \rightharpoonup \sigma$ in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$.*

(ii) Assume that additionally

$$f \equiv 0 \quad \textit{and} \quad f_n \equiv 0$$

holds for all $n \in \mathbb{N}$. Then there exists a reduced solution $\sigma \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)$ of (9.19) with respect to u_D and f and the weak convergence $\sigma_n \rightharpoonup \sigma$ holds in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$.

Moreover, in both cases, when $f_n \to f$ in $H^1(L^d(\Omega;\mathbb{R}^d))$ and $u_{D,n} \to u_D$ in $H^1(H^1(\Omega;\mathbb{R}^d))$ then $\sigma_n \to \sigma$ in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$.

Proof. In both cases, Corollary 11.8 gives the boundedness of σ_n, thus we can select a subsequence, again denoted by σ_n, which converges to a weak limit σ in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$. Since reduced solutions are unique according to Lemma 11.3, we obtain the convergence of the whole sequence by a standart argument once we have shown that σ is a reduced solution. Hence, in both cases we have the weak convergence $\sigma_n \rightharpoonup \sigma$.

Now, since $f_n \to f$ in $L^2(L^d(\Omega;\mathbb{R}^d))$, we can select a subsequence such that $f_n(t) \to f(t)$ for almost all $t \in [0,T]$, thus, using that $(u_n,\sigma_n) \in \mathcal{U} \times H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ are reduced solutions of (9.19), we have $-\operatorname{div}\sigma_n(t) = f_n(t)$ for all $t \in [0,T]$ and $n \in \mathbb{N}$, passing to the limit $n \to \infty$ we obtain $-\operatorname{div}\sigma(t) = f(t)$. Because $\sigma_n(t) \in \mathcal{K}(\Omega)$ for all $t \in [0,T]$, we get $\sigma(t) \in \mathcal{K}(\Omega)$ since $\mathcal{K}(\Omega)$ is weakly closed and $\sigma_n(t) \rightharpoonup \sigma(t)$, according to Lemma A.4, for all $t \in [0,T]$. That the initial conditions $u(0) = u_0$ (only for Item (i)) and $\sigma(0) = \sigma_0$ are fulfilled follows also from Lemma A.4.

Moreover, for the case of Item (i), Lemma 11.5 gives

$$\left(\mathbb{A}\dot\sigma_n - \nabla^s\dot u_{D,n}, \tau - \sigma_n\right)_{L^2(L^2(\Omega,\mathbb{R}^{d\times d}_s))} + \left(\dot u_n - \dot u_{D,n}, \operatorname{div}(\tau - \sigma_n)\right)_{L^2(L^2(\Omega,\mathbb{R}^d))} \geq 0 \tag{11.8}$$

for all $n \in \mathbb{N}$ and all $\tau \in L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ such that $\operatorname{div}\tau \in L^2(L^d(\Omega;\mathbb{R}^d))$ and $\tau(t) \in \mathcal{K}(\Omega)$ for almost all $t \in [0,T]$. We can apply Lemma A.6 to see that

$$\left(\nabla^s\dot u_{D,n}, \sigma_n\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \to \left(\nabla^s\dot u_D, \sigma\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))},$$

so that Lemma A.5 with

$$\begin{aligned} a_n := &-\left(\mathbb{A}\dot\sigma_n - \nabla^s\dot u_{D,n}, \tau\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - \left(\nabla^s\dot u_{D,n}, \sigma_n\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\ &- \left(\dot u_n - \dot u_{D,n}, \operatorname{div}(\tau - \sigma_n)\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} \end{aligned}$$

yields that (u, σ) is a solution of (9.19) with respect to u_D and f.

To prove that the reduced flow rule inequality (11.2) in case that Item (ii) is fulfilled, one can use Corollary 11.6 instead of Lemma 11.5 and argue as above.

Furthermore, to verify the strong convergence $\sigma_n \to \sigma$ in $H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ under the conditions $f_n \to f$ in $H^1(L^d(\Omega; \mathbb{R}^d))$ and $u_{D,n} \to u_D$ in $H^1(H^1(\Omega; \mathbb{R}^d))$, we can simply apply Lemma 11.7. □

With the continuity properties given in Proposition 11.10 we have collected all auxiliary results and can continue to regularize (9.19) and provide a priori estimates.

11.2 Regularization

As already announced at the beginning of this chapter, we will follow SUQUET [95] to prove the existence of a solution to (9.19). To this end, we first regularize (9.19) and provide a priori estimates. Then in Section 11.3 we can show the existence of a solution. For the regularization, we use besides the Yosida approximation (which was employed by Suquet) also vanishing hardening. This approach was used in [11] (in the context of *quasistatic evolutions*), note that our method is more general since we allow for mixed vanishing viscosity and hardening, see (11.10) below. We further provide a strong convergence result for the stress in Theorem 11.19, which is neither in SUQUET [95] nor in [11] present.

In order to prove the existence of a solution to (9.19), we consider for $(\varepsilon, \lambda) \in \mathbb{R}^2 \setminus \{0\}$, $\varepsilon, \lambda \geq 0$ (see Assumption ⟨11.2.i⟩) the following regularized problem:

$$\begin{aligned}
-\nabla \cdot \sigma_{\varepsilon,\lambda} &= f && \text{in } \Omega, && (11.9a)\\
\nu \cdot \sigma_{\varepsilon,\lambda} &= g && \text{on } \Gamma_N, && (11.9b)\\
u_{\varepsilon,\lambda} &= u_D && \text{on } \Gamma_D, && (11.9c)\\
\sigma_{\varepsilon,\lambda} &= \mathbb{C}(\nabla^s u_{\varepsilon,\lambda} - z_{\varepsilon,\lambda}) && \text{in } \Omega, && (11.9d)\\
\dot{z}_{\varepsilon,\lambda} &\in \partial I_\lambda(\sigma_{\varepsilon,\lambda} - \varepsilon \mathbb{B} z_{\varepsilon,\lambda}) && \text{in } \Omega, && (11.9e)\\
(u_{\varepsilon,\lambda}(0), \sigma_{\varepsilon,\lambda}(0)) &= (u_0, \sigma_0) && \text{in } \Omega. && (11.9f)
\end{aligned}$$

Recall that we identify I_0 with the indicator function of $\mathcal{K}(\Omega)$, $I_0 = I_{\mathcal{K}}(\Omega)$, see Definition 3.8. When $\lambda_n > 0$, the inclusion $a \in \partial I_n(b)$ is simply an equation, $a = \partial I_n(b)$, for $a, b \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$.

We emphasize that the following settings are possible

$$\begin{aligned}
&\lambda > 0, \quad \varepsilon = 0 \quad && \text{(vanishing viscosity)},\\
&\lambda = 0, \quad \varepsilon > 0 \quad && \text{(vanishing hardening)}, \qquad (11.10)\\
&\lambda > 0, \quad \varepsilon > 0 \quad && \text{(mixed vanishing viscosity and hardening)}.
\end{aligned}$$

The definition of a solution to (11.9) is analog to the one given in Definition 2.4. Note that we only need $f, g \in H^1(H^{-1}_D(\Omega; \mathbb{R}^d))$ and not the requirement in Assumption ⟨11.2.ii⟩ (and the regularity supposed in Assumption ⟨11.ii⟩), see also Proposition 11.11 and Corollary 11.12. We will need this less regularity later in Section 12.1 for the regularized problem (12.2).

Now, we use the operator R from Definition 2.7, and define analogously

$$Q_\varepsilon : L^2(\Omega; \mathbb{R}^{d\times d}_s) \to L^2(\Omega; \mathbb{R}^{d\times d}_s), \qquad z \mapsto (\mathbb{C} + \varepsilon \mathbb{B}) z - \mathbb{C} \nabla^s \mathcal{T}(-\operatorname{div} \mathbb{C} z, 0), \tag{11.11}$$

where $\mathcal{T}$ is from Corollary 2.6. As in Theorem 2.9, we can transform (11.9) into (EVI), however, we have to take care of the initial conditions since here they are given for u and σ and not for z. This is the content of the following

Proposition 11.11 (Transformation of regularized perfect plasticity into an EVI)**.** *Let $f, g \in H^1(H_D^{-1}(\Omega;\mathbb{R}^d))$ be given (that is, f does not need the regularity supposed in Assumption ⟨11.2.ii⟩ and Assumption ⟨11.ii⟩) such that $-\operatorname{div}\sigma_0 = (f+g)(0)$.*

Then the tuple $(u_{\varepsilon,\lambda}, \sigma_{\varepsilon,\lambda}, z_{\varepsilon,\lambda}) \in H^1(H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}_s^{d\times d})\times L^2(\Omega;\mathbb{R}_s^{d\times d}))$ is a solution of (11.9) *if and only if $z_{\varepsilon,\lambda}$ is a solution of*

$$\dot{z}_{\varepsilon,\lambda} \in \partial I_\lambda\big(R(f+g,u_D) - Q_\varepsilon z_{\varepsilon,\lambda}\big), \qquad z_{\varepsilon,\lambda}(0) = \nabla^s u_0 - \mathbb{A}\sigma_0 \tag{11.12}$$

and $u_{\varepsilon,\lambda}$ and $\sigma_{\varepsilon,\lambda}$ are defined through $u_{\varepsilon,\lambda} = \mathcal{T}(-\operatorname{div}(\mathbb{C}z_{\varepsilon,\lambda}) + (f+g), u_D)$ and $\sigma_{\varepsilon,\lambda} = \mathbb{C}(\nabla^s u_{\varepsilon,\lambda} - z_{\varepsilon,\lambda})$. Moreover, if $\varepsilon > 0$, then Q_ε is coercive .

Proof. In view of the definition of Q_ε and $\mathcal{T}$, we only have to verify that the initial conditions are fulfilled, cf. Chapter 2. Clearly, if $(u_{\varepsilon,\lambda}, \sigma_{\varepsilon,\lambda}, z_{\varepsilon,\lambda})$ is a solution of (11.9), $z_{\varepsilon,\lambda}(0) = \nabla^s u_0 - \mathbb{A}\sigma_0$ follows immediately from (11.9d).

On the other hand, if $z_{\varepsilon,\lambda}$ is a solution of (11.12), then $\sigma_{\varepsilon,\lambda} = \mathbb{C}(\nabla^s u_{\varepsilon,\lambda} - z_{\varepsilon,\lambda})$, $-\operatorname{div}\sigma_0 = (f+g)(0)$ and $u_0 - u_D(0) \in H_D^1(\Omega;\mathbb{R}^d)$ implies

$$u_{\varepsilon,\lambda}(0) = \mathcal{T}(-\operatorname{div}(\mathbb{C}z_{\varepsilon,\lambda}(0)) + (f+g)(0), u_D(0)) = \mathcal{T}(-\operatorname{div}(\mathbb{C}\nabla^s u_0), u_0)$$

hence, $u_{\varepsilon,\lambda}(0) = u_0$ and $\sigma_{\varepsilon,\lambda}(0) = \mathbb{C}(\nabla^s u_0 - z_{\varepsilon,\lambda}(0)) = \sigma_0$.

The coercivity of Q_ε was proven in Lemma 2.8. □

Corollary 11.12 (Existence of a solution to regularized perfect plasticity)**.** *Let again $f, g \in H^1(H_D^{-1}(\Omega;\mathbb{R}^d))$ be as in Proposition 11.11.*

Then there exists a unique solution $(u_{\varepsilon,\lambda}, \sigma_{\varepsilon,\lambda}, z_{\varepsilon,\lambda}) \in H^1(H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}_s^{d\times d})\times L^2(\Omega;\mathbb{R}_s^{d\times d}))$, of (11.9).

Proof. In view of Proposition 11.11, the assertion follows from Theorem 4.5. Note that, for $\lambda = 0$ (thus $\varepsilon > 0$ and Q_ε is coercive), $\partial I_\lambda = \partial I_{\mathcal{K}}(\Omega)$ fulfills the boundedness property and it is also fulfilled for $\lambda > 0$ since ∂I_λ is Lipschitz continuous according to Proposition 3.9 (however, in this case the assertion also follows from Theorem A.7). Moreover, the requirement $\ell \in \mathcal{A}_{\mathcal{L}}(z_0, D(A))$ in Theorem 4.5 means $R(f(0)+g(0), u_D(0)) - Q_\varepsilon(\nabla^s u_0 - \mathbb{A}\sigma_0) \in \mathcal{K}(\Omega)$, which is fulfilled thanks to

$$\begin{aligned} R(f(0)+g(0), u_D(0)) - Q_\varepsilon(\nabla^s u_0 - \mathbb{A}\sigma_0) &= \mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}(\nabla^s u_0 - \mathbb{A}\sigma_0) + (f+g)(0), u_D(0)) \\ &\qquad - (\mathbb{C}+\varepsilon\mathbb{B})(\nabla^s u_0 - \mathbb{A}\sigma_0) \\ &= \mathbb{C}\nabla^s\mathcal{T}(-\operatorname{div}\mathbb{C}\nabla^s u_0, u_0)) - (\mathbb{C}+\varepsilon\mathbb{B})(\nabla^s u_0 - \mathbb{A}\sigma_0) \\ &= \sigma_0 - \varepsilon\mathbb{B}(\nabla^s u_0 - \mathbb{A}\sigma_0) \end{aligned}$$

and (9.17). □

A Priori Estimates

To prove the existence of a solution in the next section, we provide now a priori estimates. It is well known that the so called *safe-load condition* is necessary for such a priori estimates (cf. SUQUET [95, Hypothése 3] or DAL MASO ET AL. [27]). That this is to be expected can be easily seen, in Definition 11.1 we required that the stress is admissible (an element of $\mathcal{K}(\Omega)$) and fulfills the balance of momentum in every time point. Loosely speaking, due to the balance of momentum, the stress directly corresponds to the external forces (cf. Figure 1) and when they are "very large" it seems reasonable that the stress cannot be admissible anymore. Thus an additional condition is needed.

Since we are interested in the behaviour of solutions of the regularized problem under varying data (additional to varying regularization), we will extend the definition of the safe-load condition to a set of functions in

Definition 11.13 (Global safe-load condition)**.** *We say that a set* $\mathcal{S}_\mathcal{L} \subset H^1(L^d(\Omega;\mathbb{R}^d))$ *fulfills a* global safe-load condition, *if it is not empty, weakly closed and there exist two constants* $M_{\mathcal{S}_\mathcal{L}}$ *and* $\delta_{\mathcal{S}_\mathcal{L}} > 0$ *such that for every* $f \in \mathcal{S}_\mathcal{L}$ *there exists* $\rho \in W^{1,\infty}(L^\infty(\Omega;\mathbb{R}^{d\times d}_s))$ *with*

$$-\operatorname{div}\rho(t) = f(t), \qquad \rho(t) + \tau \in \mathcal{K}(\Omega)$$

for all $t \in [0,T]$ *and all* $\tau \in L^\infty(\Omega;\mathbb{R}^{d\times d}_s)$ *with* $\|\tau\|_{L^\infty(\Omega;\mathbb{R}^{d\times d}_s)} \le \delta_{\mathcal{S}_\mathcal{L}}$*, and*

$$\|\rho\|_{W^{1,\infty}(L^\infty(\Omega;\mathbb{R}^{d\times d}_s))} \le M_{\mathcal{S}_\mathcal{L}}\Big[1 + \|f\|_{H^1(L^d(\Omega;\mathbb{R}^d))}\Big].$$

Remark 11.14 (Space of ρ)**.** *We have chosen the space* $W^{1,\infty}(L^\infty(\Omega;\mathbb{R}^{d\times d}_s))$ *in the Definition above in compliance with [95, Hypothèse 3]. However, it is well known that* $L^\infty(0,T;L^\infty(\Omega))$ *is smaller than* $L^\infty([0,T]\times\Omega)$*. Thus, it might be beneficial to choose a different space, note that the regularity of* ρ *is essentially only used in* (11.16)*. Nonetheless, we decided to follow [95].*

Before we continue with the a priori estimates let us shortly interrupt the discussion and comment on some possibilities for a set which fulfills a global safe-load condition. We will keep these comments rather vague since we will drop volume forces and the safe-load condition later in Chapter 12, see also Remark 12.3 and Remark 12.14.

At first, it is obvious that the set $\{0\}$ fulfills a global safe-load condition. Moreover, one can also simply take a set containing only a finite number of elements such that for each exists a suitable ρ.

Another possibility is to solve the equations of linear elasticity and then define $\rho := \mathbb{C}\nabla^s u$ such that $-\operatorname{div}\rho = f$ holds. Then one needs a regularity result which gives the desired regularity of ρ (from a volume force with a suitable regularity) and also an estimate with respect to the volume force. When the higher order terms of the volume force are now bounded one should get a boundedness of ρ in $W^{1,\infty}(L^\infty(\Omega;\mathbb{R}^{d\times d}_s))$ and the extended admissibility of ρ should also be obtained, at least for a sufficient small norm of the volume force. The regularity result given in [21, Theorem 6.3-6] might be used such that the described method works, however, the obtained set would be a (possibly very small) ball with respect to higher derivates.

One may also choose the set $\mathcal{S}_\mathcal{L}$ so that it contains all volume forces such that there exists ρ which fulfills all requirements, but is even more regular and bounded with respect to this regularity (to be able to select a weakly converging subsequence). When there is then a sequence of volume forces in $\mathcal{S}_\mathcal{L}$ with corresponding ρ_n, we can select a subsequence and a weak limit ρ such that all conditions are fulfilled for this weak limit. With this approach one obtains the weakly closedness of the set. The drawback is that it is not quite clear which functions belong to this set (however, with a regularity result of the form of [21, Theorem 6.3-6] it should be possible to get the desired regularity of ρ from a sufficient regular f, and simply scaling f (and thus ρ) gives the desired boundedness of ρ).

In what follows, the precise structure of $\mathcal{S}_\mathcal{L}$ is not important, we simply assume that it is a set which fulfills a safe-load condition, see Assumption ⟨11.2.ii⟩.

In order to prove a priori estimates, we additonally need the following result.

Lemma 11.15 (Chain rule)**.** *Let* $\lambda \ge 0$ *and* $\tau \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$*. Then*

$$\int_a^b \Big(\xi(t), \dot\tau(t)\Big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\,dt = I_\lambda(\tau(b)) - I_\lambda(\tau(a))$$

holds for all $\xi \in L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *such that* $\xi(t) \in \partial I_\lambda(\tau(t))$ *for almost all* $t \in [0,T]$ *and all* $0 \le a \le b \le T$.

Proof. Note that I_λ is Fréchet differentiable for $\lambda > 0$ (cf. (3.5)), hence,

$$\frac{d}{dt} I_\lambda(\tau(t)) = \left(\partial I_\lambda(\tau(t)), \dot{\tau}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

holds for almost all $t \in [0,T]$ and thus the claim holds in this case. The assertion for $\lambda = 0$ can now be proven by an approximation argument, the full proof can be found in Brezis [15, Lemme 3.3]. □

Lemma 11.16 (A priori estimates). *Let* $(u_{\varepsilon,\lambda}, \sigma_{\varepsilon,\lambda}, z_{\varepsilon,\lambda}) \in H^1(H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^{d\times d}_s)\times L^2(\Omega;\mathbb{R}^{d\times d}_s))$ *be the unique solution of* (11.9).

Then the equation

$$\begin{aligned}\left(\mathbb{A}(\dot{\sigma}_{\varepsilon,\lambda}(t) - \dot{\rho}(t)), \sigma_{\varepsilon,\lambda}(t) - \rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \varepsilon\left(\dot{z}_{\varepsilon,\lambda}(t), \mathbb{B}z_{\varepsilon,\lambda}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\\ + \left(\dot{z}_{\varepsilon,\lambda}(t), \sigma_{\varepsilon,\lambda}(t) - \varepsilon\mathbb{B}z_{\varepsilon,\lambda}(t) - \rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} = \left(\nabla^s\dot{u}_D(t) - \mathbb{A}\dot{\rho}(t), (\sigma_{\varepsilon,\lambda}(t) - \rho(t))\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\end{aligned} \tag{11.13}$$

and the inequalities

$$\begin{aligned}&\frac{\gamma_{\mathbb{A}}}{2}\|\sigma_{\varepsilon,\lambda}(t) - \rho(t)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{\varepsilon\gamma_{\mathbb{B}}}{2}\|z_{\varepsilon,\lambda}(t)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left(\dot{z}_{\varepsilon,\lambda}, \sigma_{\varepsilon,\lambda} - \varepsilon\mathbb{B}z_{\varepsilon,\lambda} - \rho\right)_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))}\\ &\quad\le \frac{\gamma_{\mathbb{A}}}{2}\|\sigma_0 - \rho(0)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{\varepsilon\gamma_{\mathbb{B}}}{2}\|z_0\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\\ &\qquad + \left(\|\nabla^s\dot{u}_D - \mathbb{A}\dot{\rho}\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}, \|\sigma_{\varepsilon,\lambda} - \rho\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s))}\right)_{L^2(0,T;\mathbb{R})},\end{aligned} \tag{11.14}$$

with $z_0 := \nabla^s u_0 - \mathbb{A}\sigma_0$,

$$\|\dot{z}_{\varepsilon,\lambda}(t)\|_{L^1(\Omega;\mathbb{R}^{d\times d}_s)} \le \frac{1}{\delta_{S_{\mathcal{L}}}}\left(\dot{z}_{\varepsilon,\lambda}(t), \sigma_{\varepsilon,\lambda}(t) - \varepsilon\mathbb{B}z_{\varepsilon,\lambda}(t) - \rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \tag{11.15}$$

and

$$\begin{aligned}&\frac{\gamma_{\mathbb{A}}}{2}\|\dot{\sigma}_{\varepsilon,\lambda} - \dot{\rho}\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \varepsilon\gamma_{\mathbb{B}}\|\dot{z}_{\varepsilon,\lambda}\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))} + I_\lambda(\sigma_{\varepsilon,\lambda}(t) - \varepsilon\mathbb{B}z_{\varepsilon,\lambda}(t))\\ &\quad\le \frac{1}{2\gamma_{\mathbb{A}}}\|\nabla^s\dot{u}_D - \mathbb{A}\dot{\rho}\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))}\\ &\qquad + \|\dot{z}_{\varepsilon,\lambda}\|_{L^1(0,t;L^1(\Omega;\mathbb{R}^{d\times d}_s))}\|\dot{\rho}\|_{L^\infty(0,t;L^\infty(\Omega;\mathbb{R}^{d\times d}_s))}\end{aligned} \tag{11.16}$$

hold for almost all $t \in [0,T]$. *Moreover, the inequality*

$$\|\dot{\sigma}_{\varepsilon,\lambda}\|^2_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{A}})} \le \left(\dot{\sigma}_{\varepsilon,\lambda}(t), \nabla^s\dot{u}_D(t)\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \left(\dot{f}(t), \dot{u}_{\varepsilon,\lambda}(t) - \dot{u}_D(t)\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \tag{11.17}$$

holds.

Proof. We use (11.9d), $u_\varepsilon, \lambda - u_D \in H^1(H^1(\Omega;\mathbb{R}^d))$ and the fact that $\operatorname{div}(\sigma_{\varepsilon,\lambda}(t) - \rho(t)) = 0$ to obtain (11.13). Integrating (11.13) with respect to time and using the coercivity of $\mathbb{A}$ and $\mathbb{B}$ gives (11.14).

To prove (11.15) we define $\tau \in L^\infty(\Omega;\mathbb{R}^{d\times d}_s)$ for a fixed $t \in [0,T]$ via

$$(\tau(x))_{i,j} := \left\{ \begin{array}{rl} \delta_{S_\mathcal{L}}, & \text{if } (\dot z_{\varepsilon,\lambda}(t))_{i,j} \ge 0, \\ -\delta_{S_\mathcal{L}}, & \text{if } (\dot z_{\varepsilon,\lambda}(t))_{i,j} < 0, \end{array} \right\},$$

for $i,j \in \{1,...,n\}$. Then we can use (11.9e) and the monotonicity of ∂I_λ to obtain

$$\begin{aligned}
\|\dot z_{\varepsilon,\lambda}(t)\|_{L^1(\Omega;\mathbb{R}^{d\times d}_s)} &= \frac{1}{\delta_{S_\mathcal{L}}}\left(\dot z_{\varepsilon,\lambda}(t),\tau\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&= \frac{1}{\delta_{S_\mathcal{L}}}\Big(\left(\dot z_{\varepsilon,\lambda}(t) - \partial I_\lambda(\rho(t)+\tau)),\rho(t)+\tau-\sigma_{\varepsilon,\lambda}(t)+\varepsilon\mathbb{B}z_{\varepsilon,\lambda}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\qquad + \left(\dot z_{\varepsilon,\lambda}(t),\sigma_{\varepsilon,\lambda}(t)-\varepsilon\mathbb{B}z_{\varepsilon,\lambda}(t)-\rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\Big) \\
&\le \frac{1}{\delta_{S_\mathcal{L}}}\left(\dot z_{\varepsilon,\lambda}(t),\sigma_{\varepsilon,\lambda}(t)-\varepsilon\mathbb{B}z_{\varepsilon,\lambda}(t)-\rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)},
\end{aligned}$$

(using $\rho(t)+\tau \in \mathcal{K}(\Omega)$ and $0 \in \partial I_\lambda(\rho(t)+\tau))$ when $\lambda = 0$) so that (11.15) holds for almost all $t \in [0,T]$.

In order to obtain (11.16) we argue as above to get

$$\begin{aligned}
&\left(\mathbb{A}(\dot\sigma_{\varepsilon,\lambda}(t)-\dot\rho(t),\dot\sigma_{\varepsilon,\lambda}(t)-\dot\rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \varepsilon\left(\dot z_{\varepsilon,\lambda}(t),\mathbb{B}\dot z_{\varepsilon,\lambda}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\quad + \left(\dot z_{\varepsilon,\lambda}(t),\dot\sigma_{\varepsilon,\lambda}(t)-\varepsilon\mathbb{B}\dot z_{\varepsilon,\lambda}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\quad = \left(\nabla^s\dot u_D(t)-\mathbb{A}\dot\rho(t),\dot\sigma_{\varepsilon,\lambda}(t)-\dot\rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left(\dot z_{\varepsilon,\lambda}(t),\dot\rho(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}
\end{aligned}$$

for almost all $t \in [0,T]$. Integrating this equation again with respect to time, using the coercivity of $\mathbb{A}$ and $\mathbb{B}$, applying Lemma 11.15 and using (9.17) yields

$$\begin{aligned}
&\gamma_\mathbb{A}\|\dot\sigma_{\varepsilon,\lambda}-\dot\rho\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \varepsilon\gamma_\mathbb{B}\|\dot z_{\varepsilon,\lambda}\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))} + I_\lambda(\sigma_{\varepsilon,\lambda}(t)-\varepsilon\mathbb{B}z_{\varepsilon,\lambda}(t)) \\
&\quad \le \|\nabla^s\dot u_D - \mathbb{A}\dot\rho\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))}\|\dot\sigma_{\varepsilon,\lambda}-\dot\rho\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&\qquad + \|\dot z_{\varepsilon,\lambda}(t)\|_{L^1(0,t;L^1(\Omega;\mathbb{R}^{d\times d}_s))}\|\dot\rho\|_{L^\infty(0,t;L^\infty(\Omega;\mathbb{R}^{d\times d}_s))},
\end{aligned}$$

this together with

$$\begin{aligned}
&\|\nabla^s\dot u_D - \mathbb{A}\dot\rho\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))}\|\dot\sigma_{\varepsilon,\lambda}-\dot\rho\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&\quad \le \frac{1}{2\gamma_\mathbb{A}}\|\nabla^s\dot u_D - \mathbb{A}\dot\rho\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \frac{\gamma_\mathbb{A}}{2}\|\dot\sigma_{\varepsilon,\lambda}-\dot\rho\|^2_{L^2(0,t;L^2(\Omega;\mathbb{R}^{d\times d}_s))}
\end{aligned}$$

gives (11.16).

To see that (11.17) holds, we use (11.9d) and the coercivity of $\mathbb{B}$ to obtain

$$\begin{aligned}
&\left(\mathbb{A}\dot\sigma_{\varepsilon,\lambda}(t),\dot\sigma_{\varepsilon,\lambda}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left(\dot z_{\varepsilon,\lambda}(t),\dot\sigma_{\varepsilon,\lambda}(t)-\varepsilon\mathbb{B}\dot z_{\varepsilon,\lambda}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\quad \le \left(\dot\sigma_{\varepsilon,\lambda}(t),\nabla^s\dot u_{\varepsilon,\lambda}(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&\quad = \left(\dot\sigma_{\varepsilon,\lambda}(t),\nabla^s\dot u_D(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \left(\dot f(t),\dot u_{\varepsilon,\lambda}-\dot u_D\right)_{L^2(\Omega;\mathbb{R}^d)}.
\end{aligned}$$

Integrating this inequality with respect to time and using the non-negativity of I_λ and again Lemma 11.15 and (9.17) yields (11.17). □

With these a priori estimates at hand, we are in the position to prove the existence of a solution in the upcoming section.

11.3 Existence of a Solution

We prove the existence of a solution in two stages. At first we use the previously provided a priori estimates to obtain boundedness properties in Proposition 11.18. Then in Theorem 11.19 we show the existence and also a strong convergence result of the stresses. For both results, recall the definition of (u_n, σ_n, z_n) given in Assumption ⟨11.3.iv⟩.

In order to prove the admissibility of the limit stress, the following lemma is needed.

Lemma 11.17 (Admissibility by weak convergence)**.** *Let $\lambda_n \subset [0, \infty)$ be a sequence converging towards zero, $w \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$ and $\{w_n\}_{n\in\mathbb{N}} \subset L^2(\Omega; \mathbb{R}^{d\times d}_s)$ such that $w_n \rightharpoonup w$ in $L^2(\Omega; \mathbb{R}^{d\times d}_s)$. Suppose that the sequence $I_{\lambda_n}(w_n)$ is bounded. Then $w \in \mathcal{K}(\Omega)$.*

Proof. Clearly, the mapping $L^2(\Omega; \mathbb{R}^{d\times d}_s) \ni \tau \mapsto \|\tau - \pi_{\mathcal{K}(\Omega)}(\tau)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \in \mathbb{R}$ is convex and continuous (cf. Moreau's theorem, see e.g. SHOWALTER [90, Chapter IV Proposition 1.8]) and thus weakly lower semicontinuous, hence,

$$0 \le \|w - \pi_{\mathcal{K}(\Omega)}(w)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \le \liminf_{n\to\infty} \|w_n - \pi_{\mathcal{K}(\Omega)}(w_n)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} = \liminf_{n\to\infty} 2\lambda_n I_{\lambda_n}(w_n) = 0,$$

which implies $w = \pi_{\mathcal{K}(\Omega)}(w)$. □

Proposition 11.18 (Boundedness of regularization sequences)**.** *The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{U}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ in $H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$. Furthermore, there exists $\sigma \in H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ and a subsequence such that*

$$\sigma_n \rightharpoonup \sigma \qquad \text{in } H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$$

and σ fulfills Item (i) in Definition 11.1.

Proof. Let $\rho_n \in W^{1,\infty}(L^\infty(\Omega; \mathbb{R}^{d\times d}_s))$ be from Definition 11.13, for $n \in \mathbb{N}$, corresponding to f_n. Employing (11.14), with $\varepsilon = \varepsilon_n$, $\lambda = \lambda_n$ and $\rho = \rho_n$, and the monotonicity of ∂I_{λ_n} yields the boundedness of σ_n and $\sqrt{\varepsilon_n} z_n$ in $L^\infty(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ and φ_n, defined by

$$\varphi_n(t) := \left(\dot{z}_n(t), \sigma_n(t) - \varepsilon \mathbb{B} z_n(t) - \rho_n(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)},$$

in $L^1(0, T; \mathbb{R})$. Now, by virtue of (11.15), $\dot{z}_n$ is bounded in $L^1(L^1(\Omega; \mathbb{R}^{d\times d}_s))$ and thus (11.16) yields the boundedness of $\dot{\sigma}_n$ in $L^2(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ and $\sqrt{\varepsilon_n}\dot{z}_n$ in $L^2(L^2(\Omega; \mathbb{R}^{d\times d}_s))$. Due to

$$\left|\varepsilon_n \left(\dot{z}_n(t), \mathbb{B} z_n(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\right| \le \|\sqrt{\varepsilon_n}\dot{z}_n(t)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \|\mathbb{B}\| \, \|\sqrt{\varepsilon_n} z_n(t)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

we get the boundedness of $[0, T] \ni t \mapsto \varepsilon_n \left(\dot{z}_n(t), \mathbb{B} z_n(t)\right)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \in \mathbb{R}$ in $L^2(0, T; \mathbb{R})$, so that (11.13) gives the boundedness of φ_n in $L^2(0, T; \mathbb{R})$ (note that we explicitly use the boundedness of σ_n and $\sqrt{\varepsilon_n} z_n$ in L^∞ in time (which however also follows by the embedding $H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s)) \hookrightarrow C(L^2(\Omega; \mathbb{R}^{d\times d}_s))$)). Using again (11.15) we see that $\dot{z}_n$ is bounded in $L^2(L^1(\Omega; \mathbb{R}^{d\times d}_s))$, hence $\nabla^s \dot{u}_n$ is also bounded in $L^2(L^1(\Omega; \mathbb{R}^{d\times d}_s)) \subset L^2(M(\Omega; \mathbb{R}^{d\times d}_s))$. Thanks to Theorem 10.2 Item (iii) and the boundedness of $\dot{u}_{D,n}$ in $L^2(H^1(\Omega; \mathbb{R}^d))$, we see that $\dot{u}_n$ is bounded in $\mathcal{V}$ which immediately yields the boundedness of u_n in $\mathcal{U}$ due to $u_n(0) = u_0$ for all $n \in N$.

We can now select $\sigma \in H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ such that for a subsequence, again denoted by the same symbol,

$$\sigma_n \rightharpoonup \sigma$$

in $H^1(L^2(\Omega;\mathbb{R}_s^{d\times d}))$ and $\sigma_n(t) \rightharpoonup \sigma(t)$ in $L^2(\Omega;\mathbb{R}_s^{d\times d})$ for all $t \in [0,T]$, as $n \to \infty$ (according to Lemma A.4). Clearly, since $-\operatorname{div}\sigma_n(t) = f_n(t)$ for all $n \in \mathbb{N}$, we also have $-\operatorname{div}\sigma(t) = f(t)$ for all $t \in [0,T]$. Since $\sqrt{\varepsilon_n} z_n$ is bounded in $L^\infty(L^2(\Omega;\mathbb{R}_s^{d\times d}))$, we have

$$\sigma_n(t) - \varepsilon_n \mathbb{B} z_n(t) \rightharpoonup \sigma(t)$$

in $L^2(\Omega;\mathbb{R}_s^{d\times d})$. Moreover, (11.16) gives the boundedness of $I_{\lambda_n}(\sigma_n(t) - \varepsilon_n \mathbb{B} z_n(t))$, so that Lemma 11.17 yields $\sigma(t) \in \mathcal{K}(\Omega)$ for almost all $t \in [0,T]$. □

Theorem 11.19 (Existence and approximation of solutions to perfect plasticity)**.** *We have*

$$\sigma_n \rightharpoonup \sigma \qquad \text{in } H^1(L^2(\Omega;\mathbb{R}_s^{d\times d}))$$

and there exists a subsequence such that

$$u_n \rightharpoonup^* u \qquad \text{in } \mathcal{U},$$

as $n \to \infty$, where (u,σ) is a solution of (9.19)*.*

Furthermore, if $u_{D,n} \to u_D$ in $H^1(H^1(\Omega;\mathbb{R}^d))$ and $f_n \to f$ in $H^1(L^d(\Omega;\mathbb{R}_s^{d\times d}))$, then

$$\sigma_n \to \sigma \qquad \text{in } H^1(L^2(\Omega;\mathbb{R}_s^{d\times d})),$$

as $n \to \infty$.

Proof. According to Proposition 11.18 we can select $u \in \mathcal{U}$, $\sigma \in H^1(L^2(\Omega;\mathbb{R}_s^{d\times d}))$ and subsequences such that

$$u_n \rightharpoonup^* u \quad \text{in } \mathcal{U} \qquad \sigma_n \rightharpoonup \sigma \quad \text{in } H^1(L^2(\Omega;\mathbb{R}_s^{d\times d}))$$

(where the definition of the weak* convergence in $\mathcal{U}$ is given in Definition and Lemma 10.11), and σ fulfills Item (i) in Definition 11.1. Choosing an arbitrary $\tau \in L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))$ with $\operatorname{div}\tau \in L^2(L^d(\Omega;\mathbb{R}_s^{d\times d}))$ and $\tau(t) \in \mathcal{K}(\Omega)$ for almost all $t \in [0,T]$, we obtain from the definition of the subdifferential

$$\begin{aligned}
0 &= \int_0^T I_{\lambda_n}(\tau(t))dt \\
&\geq \int_0^T I_{\lambda_n}(\sigma_n - \varepsilon_n \mathbb{B} z_n(t)) + \left(\dot{z}_n(t), \tau(t) - \sigma_n(t) + \varepsilon_n \mathbb{B} z_n(t)\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} dt \\
&\geq \frac{\varepsilon_n}{2}\left(z_n(T), \mathbb{B} z_n(T)\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} - \frac{\varepsilon_n}{2}\left(z_0, \mathbb{B} z_0\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} + \left(\nabla^s \dot{u}_n - \mathbb{A}\dot{\sigma}_n, \tau - \sigma_n\right)_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \\
&\geq -\frac{\varepsilon_n}{2}\left(z_0, \mathbb{B} z_0\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} - \left(\mathbb{A}\dot{\sigma}_n, \tau - \sigma_n\right)_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \\
&\quad + \left(\nabla^s \dot{u}_{D,n}, \tau - \sigma_n\right)_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} - \left(\dot{u}_n - \dot{u}_{D,n}, \operatorname{div}(\tau - \sigma_n)\right)_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))}
\end{aligned} \tag{11.18}$$

where we have abbreviated $z_0 := \nabla^s u_0 - \mathbb{A}\sigma_0$ and used the positivity of I_{λ_n} and the coercivity of $\mathbb{B}$. Defining

$$\begin{aligned}
a_n &:= -\frac{\varepsilon_n}{2}\left(z_0, \mathbb{B} z_0\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} - \left(\mathbb{A}\dot{\sigma}_n, \tau\right)_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} + \left(\nabla^s \dot{u}_{D,n}, \tau - \sigma_n\right)_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \\
&\qquad - \left(\dot{u}_n - \dot{u}_{D,n}, \operatorname{div}(\tau - \sigma_n)\right)_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))}
\end{aligned}$$

we see that

$$-\left(\mathbb{A}\dot{\sigma}_n, \sigma_n\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \geq a_n \tag{11.19}$$

holds for all $n \in \mathbb{N}$, thus, an argumentation as in the proof of Proposition 11.10 shows that (u, σ) is a solution of (9.19). Due to the uniqueness of a reduced solution (Lemma 11.3), we obtain the convergence $\sigma_n \rightharpoonup \sigma$ in $H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ for the whole sequence by standard arguments.

Let us now assume that $u_{D,n} \to u_D$ in $H^1(H^1(\Omega; \mathbb{R}^d))$ and $f_n \to f$ in $H^1(L^d(\Omega; \mathbb{R}^{d\times d}_s))$. According to (11.17), the inequality

$$\|\dot{\sigma}_n\|_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{A}})} \leq \left(\dot{\sigma}_n, \nabla^s \dot{u}_{D,n}\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \left(\dot{f}_n, \dot{u}_n - \dot{u}_{D,n}\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} \tag{11.20}$$

holds for all $n \in \mathbb{N}$. Thanks to Lemma 11.7, the limit of the right side above is $\|\dot{\sigma}\|_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s)_{\mathbb{A}})}$, hence, Lemma A.3 yields the desired strong convergence. □

Remark 11.20 (Safe-load condition and surface forces)**.** *It is also possible to consider surface forces (Neumann boundary forces) in the definition of a solution to* (9.19)*, cf.* SUQUET *[95, 1.4 Formulations. Résultats]. The global safe-load condition has then to be adapted (analog to* SUQUET *[95, Hypothése 3] or* DAL MASO ET AL. *[27]). However, the proofs of some of the results in Section 11.1, in particular those of Lemma 11.7 and more impartantly Proposition 11.10, cannot be adapted so easily. In Lemma 11.7 it is not possible anymore to test with $\sigma(t \pm h)$ (since the test function has to fulfill the Neumann boundary condition, see the definition in* SUQUET *[95, 1.4 Formulations. Résultats]) and in Proposition 11.10 the test function τ in the inequality* (11.8) *would depend on the varying surface forces and thus on n. A similar problem arises in the proof of Theorem 11.19. In* (11.18) *one can indeed test with a function τ such that $\tau\nu$ equals a given surface force g on the Neumann boundary. Then the term*

$$\left(\dot{u}_n - \dot{u}_{D,n}, g - g_n\right)_{L^2(L^2(\Gamma_N;\mathbb{R}^d))}$$

would occur in the last inequality in (11.18)*, where g_n is the boundary force related to σ_n. Since the trace operator is not weakly* continuous (see Theorem 10.2 Item (i)), it is not clear, even under strong convergence assumptions on g_n, that this term vanishes (except when $g_n \equiv g$). Compare also the definition in* SUQUET *[95, 1.4 Formulations. Résultats], where the surface forces do not appear in the variational inequality, but instead are integrated in the set of test functions.*

Due to these reasons we decided to drop the surface forces altogether in this part.

Another noteworthy matter which should be discussed is the fact that we will drop the volume forces in the upcoming chapter, for the reason see Remark 12.3. Thus, one may ask why we have considered volume forces in the first place. This has two reasons, at first it still possible to prove some results in the context of optimal control, see again Remark 12.3 and also Remark 12.14. Secondly, our fidnings extend the present results in the literature (due to the mixed vanishing viscosity and hardening and the strong convergence of the stress, see also the beginning of this section), thus it seems reasonable to give a more complete presentation with volume forces.

We end this chapter with two direct consequences of Theorem 11.19.

Corollary 11.21 (Existence of a solution to perfect plasticity)**.** *There exists a solution of* (9.19)*.*

Proof. This follows directly from Theorem 11.19, we can simply choose $f_n \equiv f$ and $u_{D,n} \equiv u_D$ for all $n \in \mathbb{N}$. □

Corollary 11.22 (Solution and reduced solution)**.** *The function $\sigma \in H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ is a reduced solution of* (9.19) *if and only if there exists $u \in \mathcal{U}$ such that (u, σ) is a solution of* (9.19)*.*

Proof. If there exists $u \in \mathcal{U}$ such that (u, σ) is a solution of (9.19), then (11.2) follows immediately from (11.1).

Let now σ be a reduced solution. Due to Theorem 11.19 there exists a solution $(u, \hat{\sigma})$ of (9.19). According to the first part of the proof, $\hat{\sigma}$ is a reduced solution of (9.19), Lemma 11.3 then yields $\sigma = \hat{\sigma}$. □

Chapter 12 Optimal Control

This chapter is devoted to an optimal control problem with the equations of perfect plasticity as the constraint. At first we will deal with the existence and approximation of optimal controls in Section 12.1. Instead of presenting then optimality conditions we reduce the optimal control problem to the stress as the only state variable. The reason for this reduction will be explained in Section 12.2 in detail. After this reduction we present optimality conditions in Section 12.3.

The optimization problem under consideration reads as follows:

$$\left\{\begin{aligned} \min \quad & J(u, \sigma, \mathfrak{l}) := \frac{1}{2}\|u - u_d\|^2_{H^1(H^1(\Omega;\mathbb{R}^d))} + \Psi(u, \sigma, \mathfrak{l}) \\ & \qquad + \frac{\alpha}{2}\|\dot{\mathfrak{l}}\|^2_{L^2(\mathcal{X}_c)}, \\ \text{s.t.} \quad & \mathfrak{l} \in H^1_{00}(\mathcal{X}_c) \text{ and } (u, \sigma) \in H^1(H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d \times d}_s)) \\ & \text{is a solution of (9.19) with respect to } (f, u_D) = (0, \mathcal{G}\mathfrak{l} + \mathfrak{a}) \end{aligned}\right. \tag{12.1}$$

Note that this is not an optimal control problem in the classical sense since the displacement is not unique for a given Dirichlet displacement. However, it will be one after dropping the displacement as a state in Section 12.2 and we will also call it an optimal control problem in the next section out of convenience.

Let us comment on the precise choice of this problem, for that recall Assumptions ⟨12.ii⟩ to ⟨12.v⟩. At first, we have chosen a tracking type for the displacement since we have no boundedness result. Note that we have chosen $H^1(H^1(\Omega; \mathbb{R}^d))$ as the control space of the displacement, for the reason see Remark 12.2 below. Moreover, it is possible to prove a boundedness result (in $\mathcal{U}$), see Remark 11.9 and again Remark 12.2. Secondly, it is important to note that we use *pseudo forces* $\mathfrak{l}$ together with an offset $\mathfrak{a}$ and the pseudo force to Dirichlet map $\mathcal{G}$ instead of Dirichlet displacements as the control variable. This direct use of Dirichlet displacements as the control variable may be realized by choosing $\mathcal{X} = H^1(\Omega; \mathbb{R}^d)$, $\mathcal{X}_c = H^2(\Omega; \mathbb{R}^d)$ and $\mathcal{G} = id$. However, in this case the $H^2(\Omega; \mathbb{R}^d)$ would be present in the objective function and we want to avoid this for the numerical experiments in Chapter 13. In Chapter 13 the operator $\mathcal{G}$ will be a solution operator of the linear elasticity equations and the space $\mathcal{X}$ a negative Sobolev space (see Section 13.1), thus the name "pseudo forces". Moreover, the requirement $\mathfrak{l}(0) = \mathfrak{l}(T) = 0$ and the additional offset $\mathfrak{a}$ are motivated by our application in Section 13.3. In this application, we will set the initial boundary displacement to zero and fix it at time $t = T$ (that is, $\mathfrak{a}(T)$ is the fixed boundary displacement at end time). The pseudo forces then only have the ability to alter the boundary displacement during the process, but not at the end (or beginning), see also Section 13.3 for a more detailed description. Clearly, for our analysis the offset $\mathfrak{a}$ and the precise requirements on the pseudo forces are not necessary and a generalization is straightforward.

12.1 Existence and Approximation of Optimal Controls

At first we prove the existence of optimal controls which is very brief due to our convergence results in Proposition 11.10. Next, we tackle the approximation of global minimizers which is more intricate, as we will see.

Existence

We prove the existence of a solution in the next theorem, note that Theorem 11.19 does not guaranty that the admissible set of (12.1) is not empty, due to the requirement $u \in H^1(H^1(\Omega;\mathbb{R}^d))$. Nonetheless, when considering a slightly different problem without the off-set $\mathfrak{a}$ and only $\mathfrak{l} \in H^1(\mathcal{X}_c)$ (with $u_0 - \mathcal{G}\mathfrak{l}(0) \in H^1_D(\Omega;\mathbb{R}^{d\times d}_s)$), then one can simply choose $(u,\sigma,\mathfrak{l}) \equiv (u_0,\sigma_0,\mathfrak{l}_0)$, on condition that there exists $\mathfrak{l}_0 \in H^1(\mathcal{X}_c)$ such that $\mathcal{G}\mathfrak{l}_0 = u_0$. However, as said above, we have chosen (12.1) due to our application in Chapter 13 and therein the admissible set is not empty since we only consider the stress as a state.

Theorem 12.1 (Existence of a global solution)**.** *When the admissable set of* (12.1) *is not empty, then there exists a global solution of* (12.1).

Proof. This follows from the standard direct method of the calculus of variations analog to Theorem 5.2. Note that when $(u_n,\sigma_n,\mathfrak{l}_n)$ is a minimizing sequence, then the boundedness of σ_n follows from Item (i) in Proposition 11.10 (respectively Corollary 11.8), which also shows that the weak limit (of a subsequence) is admissible (the necessary covergences of $u_{D,n} = \mathcal{G}\mathfrak{l}_n + \mathfrak{a}$ follow from the compact embedding $\mathcal{X}_c \overset{c}{\hookrightarrow} \mathcal{X}$). □

Before we continue with the regularization of (12.1) let us give two remarks concerning the regularity of the displacement and possible volume forces.

Remark 12.2 (Displacement in $\mathcal{U}$)**.** *We note that Theorem 12.1 would also hold when the displacement has only the regularity $u \in \mathcal{U}$ and we exchange the $H^1(H^1(\Omega;\mathbb{R}^d))$-norm with the $\mathcal{U}$-norm, Proposition 11.10 Item (i) then still holds. There is only one additional aspect to heed, the displacement u_n of a minimizing sequence then converges only weakly* in the sense of Definition and Lemma 10.11, hence, we need the lower weakly* semicontinuity of the $\mathcal{U}$-norm with respect to this weak* convergence, but this was shown in Corollary 10.12.*

It is also to be noted that the admissible set of (12.1) *is then not empty, Corollary 11.21, respectively Theorem 11.19, gives the existence of a solution for every control $\mathfrak{l}$.*

Moreover, one could even drop the norm of the displacement in the objective function since it is possible to proof the boundedness when the Dirichlet displacements are bounded, see Remark 11.9. This setting was considered in [73].

However, we will make use of the regularity $u \in H^1(H^1(\Omega;\mathbb{R}^d))$ in Theorem 12.9 (see also Remark 12.13) and it is not straightforward to reduce the regularity to $u \in \mathcal{U}$, thus we required $u \in H^1(H^1(\Omega;\mathbb{R}^d))$ already in (12.1)*. Note that in [73] a different approach was used to prove the approximation of global minimizers and with this approach it is possible to consider displacements only in $\mathcal{U}$ (the tracking type for the displacement is still needed). But this comes at the cost of an additional requirement on the stress.*

Remark 12.3 (Volume forces)**.** *It is also possible to consider volume forces in* (12.1)*. Then, in particular, these forces have to belong to a set which fulfills a global safe-load condition so that Corollary 11.21, respectively Theorem 11.19, gives the existence of a solution to* (9.19) *for given volume forces (and Dirichlet displacements). Moreover, the volume forces have to belong to a space which embeds compactly into $L^2(L^d(\Omega;\mathbb{R}^d))$, so that we can employ Proposition 11.10 Item (i) to see that the weak limit of a* *minmizing sequence in Theorem 12.1 is admissible. Thus, Theorem 12.1 would still hold, however, with volume forces the construction of a recovery sequence in the proof of Theorem 12.9 would not work (at least with our method), hence, we excluded them already in* (12.1).

Regularization

Now we want to regularize (12.1) and prove in particular the convergence of global minimizers of the regularized problems to a global minimizer of (12.1), this will be the content of Theorem 12.9. A standard procedure is to prove the existence of a weak accumulation point of such a sequence of global minimizers of the regularized problems, show that this weak limit is admissible for (12.1) and then prove the global optimality by using a so called *recovery sequence*. The construction of this recovery sequence is simple when one has a strong convergence of states by fixed controls, this was the case in Theorem 5.4, therein $(S_n(\overline{\ell}), \overline{\ell})$ was the recovery sequence. In our case the strong convergence of states by fixed controls would be $(u_\lambda, \sigma_\lambda) \to (u, \sigma)$, where $(u_\lambda, \sigma_\lambda)$ is the solution for the smoothed equation (11.9) (with $\varepsilon = 0$) and (u, σ) is a solution of (9.19) for fixed u_D.

However, this is here not the case, in Proposition 11.10 we only obtained a strong convergence for the stress, but not for the displacement, which does not even has to be unique. To overcome this problem and construct a recovery sequence, we will fix $u_\lambda = \overline{u}$ (where $(\overline{u}, \overline{\sigma}, \overline{u}_D)$ is a certain global minimizer of (12.1)) and then calculate σ_λ through the flow rule in (11.9), see (i) in the proof of Theorem 12.9. The problem which then arises is that $-\operatorname{div} \sigma_\lambda = 0$ does not has to be fulfilled, hence, we need to consider a regularized optimization problem where forces are allowed, namely

$$\begin{cases} \min \quad J_\lambda(u_\lambda, \sigma_\lambda, \mathfrak{l}_\lambda, \ell_\lambda) := J(u_\lambda, \sigma_\lambda, \mathfrak{l}_\lambda) + \lambda^{-\theta} \|\ell_\lambda\|^2_{L^2(H_D^{-1}(\Omega;\mathbb{R}^d))} \\ \qquad\qquad\qquad + \|\ell_\lambda\|^2_{L^2(H^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d))} + \|\dot{\ell}_\lambda\|^2_{L^2(H_D^{-1}(\Omega;\mathbb{R}^d))}, \\ \text{s.t.} \quad \mathfrak{l} \in H^1_{00}(\mathcal{X}_c), \\ \qquad \ell_\lambda \in (L^2(H_D^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d)) \cap H^1_0(H_D^{-1}(\Omega;\mathbb{R}^d))), \\ \qquad \text{and } (u_\lambda, \sigma_\lambda, z_\lambda) \in H^1(H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}_s) \times L^2(\Omega;\mathbb{R}^{d\times d}_s)) \\ \qquad \text{is a solution of (11.9) with respect to } (f+g, u_D) = (\ell_\lambda, \mathcal{G}\mathfrak{l}_\lambda + \mathfrak{a}), \end{cases} \tag{12.2}$$

where $\theta \in (0,1)$ and $\delta \in (0, \frac{1}{2})$, see Assumption ⟨12.1.ii⟩. Recall that $H^1_0(H_D^{-1}(\Omega;\mathbb{R}^d))$ is the subspace of $H^1(H_D^{-1}(\Omega;\mathbb{R}^d))$ whose functions are zero at $t = 0$ and that $H^1_{00}(\mathcal{X}_c)$ is the subspace of $H^1(\mathcal{X}_c)$ whose functions are zero at $t = 0$ and $t = T$.

Note that we have decided to drop the hardening in (12.2), that is, we focus only on vanishing viscosity. It is also possible to include vanishing hardening, then Proposition 12.7 (and also Lemma 12.6) needs to be extended. Nevertheless, for the sake of convenience, we omitted the hardening in (12.2). Moreover, since we require $\ell_\lambda(0) = 0$, there exists for every pair $(\mathfrak{l}_\lambda, \ell_\lambda)$ of controls a unique solution $(u_\lambda, \sigma_\lambda, z_\lambda)$ of (11.9) according to Corollary 11.12 (the requirements therein on $\ell_\lambda = f + g$ hold according to Assumption ⟨12.i⟩ and the condition on $\mathcal{G}\mathfrak{l}_\lambda + \mathfrak{a} = u_D$ in Assumption ⟨11.ii⟩ is satisfied due to Assumption ⟨12.v⟩).

Why we have selected this objective function and control space for ℓ_λ will become clear in the proof of Theorem 12.9. As a part of the control space we have chosen $H_D^{-s}(\Omega;\mathbb{R}^d)$ with $s = \frac{1}{2} + \delta \in (\frac{1}{2}, 1)$, which is the dual space of $H_D^s(\Omega;\mathbb{R}^d) = \{u \in H^s(\Omega;\mathbb{R}^d) : u|_{\Gamma_D} = 0\}$ (note that $u \in H^s(\Omega;\mathbb{R}^d)$ has a trace for $s \in (\frac{1}{2}, 1)$, cf. Lemma 12.4 below). The space $H^s(\Omega;\mathbb{R}^d)$, for $s \in (0,1)$, is a *fractional Sobolev space*, in the literature also called *Aronszajn*, *Gagliardo* or

Slobodeckij space. The concrete definition of this space is not relevant for us, only the properties mentioned below. For fractional Sobolev spaces in general and its definition we refer to [2, 99, 12, 96, 30, 14] and the references therein. The properties of fractional Sobolev spaces, which are needed in the following, are collected in

Lemma 12.4 (Properties of fractional Sobolev spaces). *The following holds for* $\frac{1}{2} < s < r < 1$*:*

(i) $H^s(\Omega;\mathbb{R}^d)$ *is a subspace of* $L^2(\Omega;\mathbb{R}^d)$ *and a Hilbert space. Moreover,* $H^1(\Omega;\mathbb{R}^d)$ *is a subspace of* $H^s(\Omega;\mathbb{R}^d)$ *(it is continuously embedded).*

(ii) $u \in H^s(\Omega;\mathbb{R}^d)$ *has a trace in* $H^{s-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d) \subset L^2(\partial\Omega;\mathbb{R}^d)$ *and the trace operator is bounded and linear.*

(iii) $H^r(\Omega;\mathbb{R}^d)$ *is compactly embedded into* $H^s(\Omega;\mathbb{R}^d)$*.*

Proof. The first assertion in Item (i) follows from the definition of $H^s(\Omega;\mathbb{R}^d)$ and the second can be found in HITCHHIKER'S GUIDE [30, Section 3]. The claim in Item (ii) is proven in [32, Theorem 1]. The statement in Item (iii) follows from the fact that $H^s(\Omega;\mathbb{R}^d)$ is compactly embedded into $L^2(\Omega;\mathbb{R}^d)$, see HITCHHIKER'S GUIDE [30, Theorem 7.1], and the inequality given in BREZIS ET AL. [14, Theorem 1] or TRIEBEL [99, Section 1.3.3 Theorem (g) & Section 4.3.1 Theorem 2]. □

With these results at hand, we can prove the existence of a solution to (12.2).

Proposition 12.5 (Existence of a global solution to regularized problems). *For every* $\lambda > 0$ *there exists a global solution of* (12.2)*.*

Proof. This can be proven as in Proposition 5.3 (see also Proposition 11.11), using the lemma of Lions-Aubin (cf. SHOWALTER [90, III. Proposition 1.3]) to see that $(L^2(H_D^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d)) \cap H^1(H_D^{-1}(\Omega;\mathbb{R}^d)))$ is compactly embedded into $L^2(H_D^{-1}(\Omega;\mathbb{R}^d))$ (and by using the compact embedding $\mathcal{X}_c \hookrightarrow \mathcal{X}$ to obtain the required strong convergence of the corresponding Dirichlet diplacements derived from the pseudo loads). □

The fractional Sobolev space in (12.2), (only) needed to prove the existence of a global solution in the proposition above, has repercussions on the construction of the mentioned recovery sequence. The obtained stress in (iii) of the proof of Theorem 12.9 needs to be more regular, so that the derived forces also gain the necessary regularity. Thus, before we can prove the approximation of global minimizers, we need to provide the desired regularity results.

Regularity of the Stress

At first we supply a regularity of the stress when the given input is also more regular, recall Assumption ⟨12.1.i⟩. After this we collect all necessary regularity findings in Proposition 12.7.

Lemma 12.6 (Regularity of the stress). *Let* $\lambda > 0$ *and* $\sigma \in H^1(L^2(\Omega;\mathbb{R}_s^{d\times d}))$ *the solution of*

$$U - \mathbb{A}\dot{\sigma} = \partial I_\lambda(\sigma), \qquad \sigma(0) = \sigma_0, \tag{12.3}$$

where U *is given in Assumption ⟨12.1.i⟩. Then* $\sigma \in H^1(H^1(\Omega;\mathbb{R}_s^{d\times d}))$*.*

Proof. At first we consider the case $U \in C(H^1(\Omega;\mathbb{R}_s^{d\times d}))$.

We apply a time discretization scheme, namely the explicit Euler method. For $N \in \mathbb{N}$ and $n \in \{0, ..., N\}$ we set $d_t^N := \frac{T}{N}$ and $t_n^N := n d_t^N$ such that $0 = t_0^N < t_1^N < ... < t_N^N = T$. Now we can define $\sigma_0^N := \sigma_0 \in H^1(\Omega; \mathbb{R}_s^{d\times d})$ and

$$\sigma_n^N := \sigma_{n-1}^N + d_t^N \mathbb{C}(U(t_{n-1}^N) - \partial I_\lambda(\sigma_{n-1}^N)) \in H^1(\Omega; \mathbb{R}_s^{d\times d})$$

(recall that ∂I_λ maps $H^1(\Omega; \mathbb{R}_s^{d\times d})$ into $H^1(\Omega; \mathbb{R}_s^{d\times d})$, according to Lemma 3.15) such that

$$\mathbb{A}\frac{\sigma_n^N - \sigma_{n-1}^N}{d_t^N} + \partial I_\lambda(\sigma_{n-1}^N) = U(t_{n-1}^N) \tag{12.4}$$

for all $N \in \mathbb{N}$ and $n \in \{1, ..., N\}$. We define the piecewise linear interpolation $\sigma^N \in H^1(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ by

$$\sigma^N(t) := \sigma_{n-1}^N + \frac{t - t_{n-1}^N}{d_t^N}(\sigma_n^N - \sigma_{n-1}^N)$$

and the piecewise constant interpolation $\tilde{\sigma}^N \in L^\infty(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ by $\tilde{\sigma}^N(t) := \sigma_{n-1}^N$ for $t \in [t_{n-1}^N, t_n^N)$. The estimate

$$\begin{aligned}
\|\sigma_n^N\|_{H^1(\Omega;\mathbb{R}_s^{d\times d})} &\le \|\sigma_{n-1}^N\|_{H^1(\Omega;\mathbb{R}_s^{d\times d})} \\
&\quad + d_t^N C(\|\sigma_{n-1}^N\|_{H^1(\Omega;\mathbb{R}_s^{d\times d})} + \|U\|_{C(H^1(\Omega;\mathbb{R}_s^{d\times d}))}) \\
&\le ... \le \|\sigma_0\|_{H^1(\Omega;\mathbb{R}_s^{d\times d})} + d_t^N C\Big(\sum_{i=0}^{n-1} \|\sigma_i^N\|_{H^1(\Omega;\mathbb{R}_s^{d\times d})}\Big) + C\|U\|_{C(H^1(\Omega;\mathbb{R}_s^{d\times d})},
\end{aligned}$$

where we have used (i) in Corollary 3.17, together with the discrete Gronwall lemma (cf. HEYWOOD AND RANNACHER [53, Lemma 5.1 and the following remark]) shows that σ^N is bounded in $C(H^1(\Omega; \mathbb{R}_s^{d\times d}))$. Thus $\dot{\sigma}^N(t) = \frac{\sigma_n^N - \sigma_{n-1}^N}{d_t^N}$, $t \in (t_{n-1}^N, t_n^N)$, is, according to (12.4), bounded in $H^1(\Omega; \mathbb{R}_s^{d\times d})$ and we obtain

$$\int_{t_{n-1}^N}^{t_n^N} \|\dot{\sigma}^N(t)\|^2_{H^1(\Omega;\mathbb{R}_s^{d\times d})} dt = d_t^N \left\| \frac{\sigma_n^N - \sigma_{n-1}^N}{d_t^N} \right\|^2_{H^1(\Omega;\mathbb{R}_s^{d\times d})} \le d_t^N C.$$

Taking the sum over n we see that σ^N in bounded in $H^1(H^1(\Omega; \mathbb{R}_s^{d\times d}))$, hence, there exists a subsequence, again denoted by σ^N, and a limit $\sigma \in H^1(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ such that $\sigma^N \rightharpoonup \sigma$ in $H^1(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ and $\sigma^N \to \sigma$ in $C(L^2(\Omega; \mathbb{R}_s^{d\times d}))$. One also verifies easily that $\tilde{\sigma}^N \to \sigma$ in $L^2(L^2(\Omega; \mathbb{R}_s^{d\times d}))$, therefore

$$\begin{aligned}
&\|\mathbb{A}\dot{\sigma}^N + \partial I_\lambda(\sigma^N) - U\|_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \\
&\le \|\mathbb{A}\dot{\sigma}^N + \partial I_\lambda(\tilde{\sigma}^N) - \tilde{U}^N\|_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \\
&\qquad + \|\partial I_\lambda(\sigma^N) - \partial I_\lambda(\tilde{\sigma}^N)\|_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} + \|\tilde{U}^N - U\|_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \\
&\le \frac{1}{\lambda}\|\sigma^N - \tilde{\sigma}^N\|_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} + \|\tilde{U}^N - U\|_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \to 0
\end{aligned}$$

as $N \to \infty$, where $\tilde{U}^N$ is defined as $\tilde{\sigma}^N$, that is, $\tilde{U}^N$ is the piecewise constant interpolation of U.

Let now $U \in L^2(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ and $U_n \in C(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ be a sequence such that $U_n \to U$ in $L^2(H^1(\Omega; \mathbb{R}_s^{d\times d}))$. Let $\sigma \in H^1(L^2(\Omega; \mathbb{R}_s^{d\times d}))$ be the solution of (12.3) and $\sigma_n \in H^1(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ for every $n \in \mathbb{N}$ the solution of

$$U_n - \mathbb{A}\dot{\sigma}_n = \partial I_\lambda(\sigma_n), \qquad \sigma_n(0) = \sigma_0. \tag{12.5}$$

Since $\partial I_\lambda : L^2(\Omega; \mathbb{R}_s^{d\times d}) \to L^2(\Omega; \mathbb{R}_s^{d\times d})$ is monotone, one obtains

$$\mathbb{A}(\dot{\sigma}_n(t) - \dot{\sigma}(t)) : (\sigma_n(t) - \sigma(t)) \le (U_n(t) - U(t)) : (\sigma_n(t) - \sigma(t))$$

for almost all $t \in [0,T]$, integrating this inequality with respect to time and using the coercivity of $\mathbb{A}$ shows that $\sigma_n \to \sigma$ in $C(L^2(\Omega; \mathbb{R}_s^{d\times d}))$. Considering now again (12.3) and (12.5) and using the Lipschitz continuity of ∂I_λ (on $L^2(\Omega; \mathbb{R}_s^{d\times d})$) yields the strong convergence in $C^1(L^2(\Omega; \mathbb{R}_s^{d\times d})) \hookrightarrow H^1(L^2(\Omega; \mathbb{R}_s^{d\times d}))$. Using Item (ii) in Corollary 3.17 we get

$$\begin{aligned}\mathbb{A}\partial_j\dot{\sigma}_n(t) : \partial_j\sigma_n(t) &\le \mathbb{A}\partial_j\dot{\sigma}_n(t) : \partial_j\sigma_n(t) + \partial_j(\partial I_\lambda(\sigma_n(t))) : \partial_j\sigma_n(t)\\ &= \partial_j U_n(t) : \partial_j\sigma_n(t)\end{aligned}$$

for almost all $t \in [0,T]$. Once again, integrating this inequality with respect to time and taking the coercivity of $\mathbb{A}$ into account gives the boundedness of σ_n in $C(H^1(\Omega; \mathbb{R}_s^{d\times d}))$. Thanks to Item (i) in Corollary 3.17 and (12.5) we finally obtain the boundedness of σ_n in $H^1(H^1(\Omega; \mathbb{R}_s^{d\times d}))$. □

Proposition 12.7 (Regularity and convergence of the stress). *Let $\sigma_\lambda \in H^1(H^1(\Omega; \mathbb{R}_s^{d\times d}))$ be the solution of*

$$U - \mathbb{A}\dot{\sigma}_\lambda = \partial I_\lambda(\sigma_\lambda), \qquad \sigma_\lambda(0) = \sigma_0 \tag{12.6}$$

for every $\lambda > 0$, and $\sigma \in H^1(L^2(\Omega; \mathbb{R}_s^{d\times d}))$ the solution of

$$U - \mathbb{A}\dot{\sigma} \in \partial I_{\mathcal{K}(\Omega)}(\sigma), \qquad \sigma(0) = \sigma_0. \tag{12.7}$$

Then σ_λ is bounded in $C(H^1(\Omega; \mathbb{R}_s^{d\times d}))$,

$$\begin{aligned}\sigma_\lambda &\to \sigma \quad \text{in } H^1(L^2(\Omega; \mathbb{R}_s^{d\times d})),\\ \sigma_\lambda &\rightharpoonup^* \sigma \quad \text{in } L^\infty(H^1(\Omega; \mathbb{R}_s^{d\times d})),\end{aligned}$$

as $\lambda \searrow 0$, and the inequality

$$\|\sigma_\lambda - \sigma\|^2_{C(L^2(\Omega;\mathbb{R}_s^{d\times d}))} \le \lambda \frac{\|\mathbb{C}\|^2}{\gamma_{\mathbb{C}}} \|U - \mathbb{A}\dot{\sigma}\|^2_{L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))}.$$

holds.

Proof. At first we note that, analog to Lemma 4.3, σ_λ is a solution of (12.6) if and only if $q_\lambda \in H^1(L^2(\Omega; \mathbb{R}_s^{d\times d}))$ is a solution of

$$\dot{q}_\lambda = \partial I_\lambda(\mathbb{C}\mathcal{F}U - \mathbb{C}q_\lambda), \qquad q_\lambda(0) = q_0 := -\mathbb{A}\sigma_0$$

with $q_\lambda = \mathcal{F}U - \mathbb{A}\sigma_\lambda$, respectively $\sigma_\lambda = \mathbb{C}(\mathcal{F}U - q_\lambda)$, and σ is a solution of (12.7) if and only if $q \in H^1(L^2(\Omega; \mathbb{R}_s^{d\times d}))$ is a solution of

$$\dot{q} \in \partial I_{\mathcal{K}(\Omega)}(\mathbb{C}\mathcal{F}U - \mathbb{C}q), \qquad q(0) = q_0,$$

with $q = \mathcal{F}U - \mathbb{A}\sigma$, respectively $\sigma = \mathbb{C}(\mathcal{F}U - q)$, where $\mathcal{F}U \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ is defined by $\mathcal{F}U(t) := \int_0^t U(s)ds$.

The existence of σ, respectively q, follows now from Theorem 4.5 (note that the subdifferential has the boundedness property according to Proposition 3.14) and Proposition 4.8 (with $z = q$, $z_\lambda = q_\lambda$, $z_0 = q_0$, $A = \partial I_{\mathcal{K}(\Omega)}$ and $\mathcal{H} = L^2(\Omega;\mathbb{R}^{d\times d}_s)$) can be employed to get the convergence $\sigma_\lambda \to \sigma$ in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ and the estimate

$$\|\mathbb{A}(\sigma_\lambda - \sigma)\|^2_{C(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \leq \frac{\lambda}{\gamma_{\mathbb{C}}}\|U - \mathbb{A}\dot{\sigma}\|^2_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))}.$$

The desired inequality follows now easily using $\|\sigma_\lambda - \sigma\|_{C(L^2(\Omega;\mathbb{R}^{d\times d}_s))} = \|\mathbb{C}\mathbb{A}(\sigma_\lambda - \sigma)\|_{C(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \leq \|\mathbb{C}\|\,\|\mathbb{A}(\sigma_\lambda - \sigma)\|_{C(L^2(\Omega;\mathbb{R}^{d\times d}_s))}$.

We can now prove the boundedness of σ_λ in $C(H^1(\Omega;\mathbb{R}^{d\times d}_s))$ by applying (ii) in Corollary 3.17 as in the end of the proof of Lemma 12.6. □

Remark 12.8 (Extension of the regularity)**.** *The results in Proposition 12.7 can be extended to more general Sobolev spaces and also to more general settings than the von-Mises flow rule, cf. [73, Lemma 5.2 & 5.3]. These extensions were needed in [73], but since we consider a slightly different optimization problem in* (12.2)*, the results above are sufficient for us.*

Approximation

Having dealt with the regularity of the stress, we are finally in the position to provide the approximation result in the next theorem. Let us emphasize that the method we use to construct a recovery sequence is new and not present in the literature (except in [73]), at least to the knowledge of the author.

Theorem 12.9 (Approximation of global minimizers)**.** *We assume that there exists a global minimizer* $(\overline{u}, \overline{\sigma}, \overline{\mathfrak{l}})$ *of* (12.1) *such that* $\overline{u}$ *has the regularity* $\nabla^s\dot{\overline{u}} \in L^2(H^1(\Omega;\mathbb{R}^{d\times d}_s))$ *and* $(\overline{u} - (\mathcal{G}\overline{\mathfrak{l}} + \mathfrak{a})) \in H^1(H^1_D(\Omega;\mathbb{R}^d))$.

Let $\{\overline{u}_\lambda, \overline{\sigma}_\lambda, \overline{\mathfrak{l}}_\lambda, \overline{\ell}_\lambda\}_{\lambda>0}$ *be a sequence of global minimizers of* (12.2)*. Then every weak accumulation point of* $\{\overline{u}_\lambda, \overline{\sigma}_\lambda, \overline{\mathfrak{l}}_\lambda, \overline{\ell}_\lambda\}_{\lambda>0}$ *is a strong accumulation point which has the form* $(\tilde{u}, \tilde{\sigma}, \tilde{\mathfrak{l}}, 0)$*, that is* $\tilde{\ell} := \lim_{\lambda\searrow 0}\overline{\ell}_\lambda = 0$*, and* $(\tilde{u}, \tilde{\sigma}, \tilde{\mathfrak{l}})$ *is a global minimizer of* (12.1)*. Moreover, there exists an accumulation point.*

Proof. (i) Construction of a recovery sequence. Let $(\overline{u}, \overline{\sigma}, \overline{\mathfrak{l}})$ be a global minimizer of (12.1) such that $\overline{u}$ has the supposed improved regularity. Then we define $\sigma_\lambda \in H^1(H^1(\Omega;\mathbb{R}^{d\times d}_s))$ as the solution of

$$\nabla^s\dot{\overline{u}} - \mathbb{A}\dot{\sigma}_\lambda = \partial I_\lambda(\sigma_\lambda), \qquad \sigma_\lambda(0) = 0$$

(see Lemma 12.6 for the regularity of σ_λ) and $\ell_\lambda \in H^1(H_D^{-\frac{1+\delta}{2}}(\Omega;\mathbb{R}^d))$ by

$$\langle \ell_\lambda(t), \varphi\rangle := \int_\Omega (-\nabla\cdot\sigma_\lambda(t))\cdot\varphi + \int_{\Gamma_N}(\sigma_\lambda(t)\nu)\cdot\varphi \tag{12.8}$$

for all $\varphi \in H_D^{\frac{1+\delta}{2}}(\Omega;\mathbb{R}^d)$ (thanks to Lemma 12.4 Item (ii), φ has a trace in $L^2(\partial\Omega;\mathbb{R}^d)$), that is, $\ell_\lambda = f_\lambda := -\nabla\cdot\sigma_\lambda$ in Ω and $\ell_\lambda = g_\lambda := \sigma_\lambda\nu$ on Γ_N, thus

$$\langle \ell_\lambda(t), \varphi\rangle = \int_\Omega \sigma_\lambda(t) : \nabla^s\varphi$$

for all $\varphi \in H^1_D(\Omega;\mathbb{R}^d)$ and all $t \in [0,T]$, which also shows, thanks to Assumption ⟨12.i⟩, that $\ell_\lambda(0) = 0$ so that ℓ_λ is admissible for (12.2). Thanks to Lemma and Definition 11.2, we can apply Proposition 12.7 with $U = \nabla^s\dot{\overline{u}}$ to get $\sigma_\lambda \to \overline{\sigma}$ in $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$, $\ell_\lambda \to 0$ in $H^1(H^{-1}_D(\Omega;\mathbb{R}^d))$ (since $\operatorname{div}\overline{\sigma} = 0$), $\ell_\lambda \rightharpoonup 0$ in $L^2(H_D^{-\frac{1+\delta}{2}}(\Omega;\mathbb{R}^d))$ and $\|\ell_\lambda\|^2_{C(H^{-1}(\Omega;\mathbb{R}^d))} \leq C\lambda$. The lemma of Lions-Aubin (cf. SHOWALTER [90, III. Proposition 1.3]) then also gives $\ell_\lambda \to 0$ in $L^2(H_D^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d))$ (thanks to Lemma 12.4 Item (iii) and the fact that if an operator is compact, then its adjoint operator is also compact, see [65, 8.2-5 Theorem], $H^{-\frac{1+\delta}{2}}(\Omega;\mathbb{R}^d)$ is compactly embedded into $H^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d)$), hence,

$$J_\lambda(\overline{u},\sigma_\lambda,\overline{\mathfrak{l}},\ell_\lambda) \to J(\overline{u},\overline{\sigma},\overline{\mathfrak{l}})$$

(note that $(\overline{u},\sigma_\lambda,\overline{\mathfrak{l}},\ell_\lambda)$ is admissible for (12.2) by construction).

(ii) Existence of an accumulation point. Since $\{\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda\}_{\lambda>0}$ is a global solution of (12.2), we get

$$J_\lambda(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda) \leq J_\lambda(\overline{u},\sigma_\lambda,\overline{\mathfrak{l}},\ell_\lambda), \tag{12.9}$$

which yields the boundedness of $\{\overline{u}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda\}_{\lambda>0}$ so that the boundedness of $\{\overline{\sigma}_\lambda\}_{\lambda>0}$ follows from (11.17) (note that this inequality still holds when $f_n \in H^1(L^d(\Omega;\mathbb{R}^d))$ is exchanged by $\overline{\ell}_\lambda \in H^1(H^{-1}(\Omega;\mathbb{R}^d))$, see the last part of the proof of Lemma 11.16), hence, there exists a weak accumulation point.

(iii) Admissibility of an accumulation point. Let us assume that a given subsequence of $\{\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda\}_{\lambda>0}$, denoted by the same symbol for simplicity, converges weakly to $(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}},\tilde{\ell})$. Considering again (12.9), we see that $\lambda^{-\theta}\|\overline{\ell}_\lambda\|^2_{L^2(H^{-1}(\Omega;\mathbb{R}^d))}$ is bounded, hence, $\overline{\ell}_\lambda \to \tilde{\ell} = 0$ in $L^2(H^{-1}(\Omega;\mathbb{R}^d))$ and $\operatorname{div}\tilde{\sigma}(t) = 0$ for all $t \in [0,T]$. Due to Lemma A.4, we also obtain $(\tilde{u},\tilde{\sigma})(0) = (u_0,\sigma_0)$. According to the boundedness of $(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda)$, we get the boundedness of $\partial I_\lambda(\overline{\sigma}_\lambda)$ in $L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ by (11.9d) and (11.9e). Thanks to (3.6) we get

$$\|I_\lambda(\overline{\sigma}_\lambda)\|_{L^1(0,T;\mathbb{R})} = \frac{\lambda}{2}\|\partial I_\lambda(\overline{\sigma}_\lambda)\|^2_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \leq C\lambda,$$

hence, $I_\lambda(\overline{\sigma}_\lambda) \to 0$ in $L^1(0,T;\mathbb{R})$ and we can select a subsequence such that $I_\lambda(\overline{\sigma}_\lambda(t)) \to 0$ for almost all $t \in [0,T]$ (cf. [37, 2.7 Korollar]). Because we also have $\overline{\sigma}_\lambda(t) \rightharpoonup \tilde{\sigma}(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d}_s)$ by Lemma A.4, Lemma 11.17 gives $\tilde{\sigma}(t) \in \mathcal{K}(\Omega)$ for all $t \in [0,T]$. To see that $(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}})$ is admissible for (12.1), it remains to show that $(\tilde{u},\tilde{\sigma})$ fulfills (11.6) with $u_D = \mathcal{G}\tilde{\mathfrak{l}} + \mathfrak{a}$. To this end we set $\overline{u}_{D,\lambda} := \mathcal{G}\overline{\mathfrak{l}}_\lambda + \mathfrak{a}$ and use again (11.9d) and (11.9e) to obtain

$$\begin{aligned}
0 = \int_0^T I_\lambda(\tau(t))dt &\overset{(11.9e)}{\geq} \int_0^T I_\lambda(\overline{\sigma}_\lambda(t))dt + \left(\dot{\overline{z}}_\lambda, \tau - \overline{\sigma}_\lambda\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&\overset{(11.9d)}{\geq} \left(\nabla^s\dot{\overline{u}}_\lambda - \mathbb{A}\dot{\overline{\sigma}}_\lambda, \tau - \overline{\sigma}_\lambda\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= \left(\nabla^s\dot{\overline{u}}_{D,\lambda} - \mathbb{A}\dot{\overline{\sigma}}_\lambda, \tau - \overline{\sigma}_\lambda\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} - \left(\dot{\overline{u}}_\lambda - \dot{\overline{u}}_{D,\lambda}, \operatorname{div}\tau\right)_{L^2(L^2(\Omega;\mathbb{R}^d))} \\
&\quad - \int_0^T \langle \overline{\ell}_\lambda(t), \dot{\overline{u}}_\lambda(t) - \dot{\overline{u}}_{D,\lambda}(t)\rangle dt,
\end{aligned} \tag{12.10}$$

for all $\tau \in L^2(L^2(\Omega;\mathbb{R}_s^{d\times d}))$ with $\operatorname{div}\tau \in L^2(L^d(\Omega;\mathbb{R}^d))$ and $\tau(t) \in \mathcal{K}(\Omega)$ for almost all $t \in [0,T]$. We can now argue analog as in the prove of Theorem 11.19 to see that (11.6) is fulfilled, using $\overline{\ell}_\lambda \to 0$ in $L^2(H^{-1}(\Omega;\mathbb{R}^d))$ and the boundedness of $\overline{u}_\lambda$ and $\overline{u}_{D,\lambda}$ in $H^1(H^1(\Omega;\mathbb{R}^d))$.

(iv) Strong accumulation point and global minimizer. The proven convergence properties of the recovery sequence $(\overline{u},\sigma_\lambda,\overline{u}_D,\ell_\lambda)$ give

$$\begin{aligned} J(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}}) &\le \liminf_{\lambda\searrow 0} J(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda) \le \limsup_{\lambda\searrow 0} J(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda) \\ &\le \limsup_{\lambda\searrow 0} J_\lambda(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda) \le \limsup_{\lambda\searrow 0} J_\lambda(\overline{u},\sigma_\lambda,\overline{\mathfrak{l}},\ell_\lambda) = J(\overline{u},\overline{\sigma},\overline{\mathfrak{l}}), \end{aligned} \tag{12.11}$$

which implies that $(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}})$ is a global minimizer of (12.1) and that $J(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda) \to J(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}})$, from which we can deduce that $(\overline{u}_\lambda,\overline{\mathfrak{l}}_\lambda) \to (\tilde{u},\tilde{\mathfrak{l}})$, since we already have the weak convergence. Furthermore, the convergence $\overline{\sigma}_\lambda \to \tilde{\sigma}$ can be proven as in the end of the proof of Theorem 11.19, note that the inequality (11.20) still holds when $f_n \in H^1(L^d(\Omega;\mathbb{R}^d))$ is exchanged by $\overline{\ell}_\lambda \in H^1(H^{-1}(\Omega;\mathbb{R}^d))$ (see again the last part of the proof of Lemma 11.16) and that $\int_0^T \langle \overline{\ell}_\lambda(t), \dot{\overline{u}}_\lambda(t) - \dot{\overline{u}}_{D,\lambda}(t)\rangle dt$ converges to zero as seen in (iii). From (12.11) we can conclude further that

$$\begin{aligned} J(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}}) &= \limsup_{\lambda\searrow 0} J_\lambda(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda) \\ &= J(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}}) + \limsup_{\lambda\searrow 0} \lambda^{-\theta}\|\overline{\ell}_\lambda\|^2_{L^2(H^{-1}(\Omega;\mathbb{R}^d))} + \|\overline{\ell}_\lambda\|^2_{L^2(H^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d))} + \|\dot{\overline{\ell}}_\lambda\|^2_{L^2(H^{-1}(\Omega;\mathbb{R}^d))}, \end{aligned}$$

so that $\overline{\ell}_\lambda \to 0$ in $(L^2(H^{-\frac{1}{2}-\delta}(\Omega;\mathbb{R}^d)) \cap H^1(H^{-1}(\Omega;\mathbb{R}^d)))$ (and $\lambda^{-\frac{\theta}{2}}\overline{\ell}_\lambda \to 0$ in $L^2(H^{-1}(\Omega;\mathbb{R}^d))$). □

Let us conclude this section with some remarks.

Remark 12.10 (Existence of an accumulation point). *Let* $\{\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda\}_{\lambda>0}$ *be the sequence from Theorem 12.9. Note that there might exist* $\lambda > 0$ *such that*

$$J_\lambda(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda) > J_\lambda(u_0,\sigma_0,0,0)$$

since $(u_0,\sigma_0,0,0)$ *is not admissible for* (12.2) *due to the offset* $\mathfrak{a}$. *Fortunately, additionally to use in (iv) of the proof of Theorem 12.9, we could utilize the recovery sequence to obtain the boundedness in* (12.9). *Of course, when considering a slightly different problem, for instance the one described before Theorem 12.1, this problem does not arise.*

Remark 12.11 (Weaker regularity constraint). *The regularity assumption in Theorem 12.9 can be weakened, it is sufficient when there exists a sequence* $(u_n,\sigma_n,\mathfrak{l}_n)$ *of admissible points of* (12.1) *such that* u_n *has the regularity* $\nabla^s \dot{u}_n \in L^2(H^1(\Omega;\mathbb{R}_s^{d\times d}))$, $u_n(t) - \mathcal{G}\mathfrak{l}_n - \mathfrak{a}(t) \in H^1_D(\Omega;\mathbb{R}_s^{d\times d})$ *and* $(u_n,\sigma_n,\mathfrak{l}_n) \to (\overline{u},\overline{\sigma},\overline{\mathfrak{l}})$, *where* $(\overline{u},\overline{\sigma},\overline{\mathfrak{l}}_D)$ *is a global solution of* (12.1). *One can argue as in the proof of Theorem 12.9 but exchange* $(\overline{u},\overline{\sigma},\overline{\mathfrak{l}})$ *with* $(u_n,\sigma_n,\mathfrak{l}_n)$ *and then obtain*

$$\begin{aligned} J(\tilde{u},\tilde{\sigma},\tilde{\mathfrak{l}}) &\le \liminf_{\lambda\searrow 0} J(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda) \\ &\le \limsup_{\lambda\searrow 0} J(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda) \le \limsup_{\lambda\searrow 0} J_\lambda(\overline{u}_\lambda,\overline{\sigma}_\lambda,\overline{\mathfrak{l}}_\lambda,\overline{\ell}_\lambda) \le J(u_n,\sigma_n,\mathfrak{l}_n) \end{aligned}$$

as in (12.11). *After passing to the limit* $n \to \infty$, *we can proceed as before.*

Remark 12.12 (Characteristics of approximable global minimizers). *Since every global minimizer* $(\overline{u}_\lambda, \overline{\sigma}_\lambda, \overline{\mathfrak{l}}_\lambda, \overline{\ell}_\lambda)$ *of* (12.2) *fulfills* $\overline{u}_\lambda - (\mathcal{G}\overline{\mathfrak{l}}_\lambda + \mathfrak{a}) \in H^1(H^1_D(\Omega;\mathbb{R}^d))$*, every accumulation point* $(\tilde{u}, \tilde{\sigma}, \tilde{\mathfrak{l}}, 0)$ *of such a sequence, hence every approximable global minimizer of* (12.1)*, also fulfills* $\tilde{u}_\lambda - (\mathcal{G}\tilde{\mathfrak{l}}_\lambda + \mathfrak{a}) \in H^1(H^1_D(\Omega;\mathbb{R}^d))$.

In light of this observation one could reduce the admissible set of (12.1) *to displacements, which also fulfill the boundary condition. Then, thanks to Lemma and Definition 11.2, it would be equivalent to consider strong solutions of* (9.19) *instead of solutions. Moreover, it is obvious that Proposition 11.10 would still hold and therefore also Theorem 12.1. However, since from a mechanical standpoint of view solutions of* (9.19) *does not need to fulfill the boundary condition (see also the discussion after Lemma and Definition 11.2), we did not include them in* (12.1).

Remark 12.13 (Regularity of the displacement). *Since we have proven in Theorem 11.19 the existence of a solution to* (9.19) *such that* $u \in \mathcal{U}$*, there might not exist a (weak) solution* (u, σ) *with* $u \in H^1(H^1(\Omega;\mathbb{R}^d))$*. We restricted the space of the displacement to* $H^1(H^1(\Omega;\mathbb{R}^d))$ *to obtain the convergence*

$$\int_0^T \langle \overline{\ell}_\lambda(t), \dot{\overline{u}}_\lambda(t) - \dot{\overline{u}}_{D,\lambda}(t)\rangle dt \to 0$$

in (12.10)*. When we would have required only the regularity* $\dot{u} \in \mathcal{V}$ *for the displacement, then the strong convergence* $\overline{\ell}_\lambda \to 0$ *in* $L^2(L^d(\Omega;\mathbb{R}^d))$ *would also yield the convergence above (see Definition 10.5), this could be achieved by adding a suitable norm of* ℓ *to our objective function. However, we would also need to prove this regularity (and the boundedness) of* ℓ_λ*, defined in* (12.8)*, which cannot be concluded from our results in Proposition 12.7.*

Let us note again that in [73] the regularity $u \in \mathcal{U}$ *was sufficient, the mentioned problem does not occur since the considered optimization problem is slightly different (imposing an additional constraint on the stress), cf. Remark 12.2.*

12.2 Reduction to the Stress

In the last section we already neglected volume forces in order to avoid the (global) *safe-load condition*. Where it is possible to consider volume forces for the existence of optimal controls in Section 12.1, they would result in great issues for the approximation of such, cf. Remark 12.3. Despite the omission of volume forces, we needed to go to great lengths to provide an approximation result for global minimizers in Theorem 12.9 by adding volume forces which belong to a fractional Sobolev space. Furthermore, the specific form of the objective function in (12.2) was important in Theorem 12.9. Due to these facts, the study of the optimization problem (12.2) is rather of theoretical interest (but still relevant due to the new idea to construct a recovery sequence).

Moreover, it is also to be noted that, since the displacement field is not unique (see Remark 11.4), one might argue that it does not make much sense to consider the optimal control problem (12.1), at least from an application perspective.

Due to these facts and our intention to investigate and to present numerical experiments for a more applied problem, we will drop the displacement as a state variable in this section so that only the stress field remains as a state. Since, in contrast to the displacement field, the stress is unique, as shown in Lemma 11.3, and we have the strong convergence result from Theorem 11.19 at hand, it is straightforward to prove a result analog to Theorem 12.9.

Since the to the stress reduced optimal control problem is similar to (12.1) (see also (12.12) below), we do not state it. The objective function is simply reduced to $J(\sigma, \mathfrak{l}) := \frac{1}{2}\|\sigma(T) - \sigma_d\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{\alpha}{2}\|\dot{\mathfrak{l}}\|_{L^2(\mathcal{X}_c)}$ (see Assumption ⟨12.2⟩) and for the admissible set we only require that σ is a reduced solution of (9.19). Moreover, let us again mention that Assumption $\langle$V.ii$\rangle$ can be dropped in what follows. This assumption was only needed for Theorem 10.2, see Remark 10.3, and therefore mainly in the construction of the space of displacement fields $\mathcal{U}$. Thus, it is straightforward to see that the results in Chapter 11 concerning the stress are still valid, see also [72].

As mentioned above, it is comparatively simple to prove an existence and an approximation result for optimal controls of the mentioned problem. We start with the

Existence of Optimal Controls

As in the proof of Theorem 12.1, the existence of optimal controls follows from the direct method of the calculus of variations. Here, to obtain the boundedness of the stresses, we make use of Proposition 11.10 Item (ii), which also shows that the limit is admissible. Note, according to Corollary 11.22, that for a reduced solution σ there exists a displacement u, such that (u, σ) is a solution and thus Proposition 11.10 Item (ii) is applicable (however, it is easily seen that the result can be directly proven for reduced solutions).

Approximation of Optimal Controls

The proof of a result, analog to Theorem 12.9 is much simpler than the proof thereof. In contrast to (12.2) one does not need the additional forces ℓ and can simply consider solutions of (11.9) with respect to $(f+g, u_D) = (0, \mathcal{G}\mathfrak{l}+\mathfrak{a})$. Now we can follow the proof of Theorem 12.9, only the construction of a recovery sequence in (i) has to be adapted (note also that the admissibility of an accumulation point in (ii) follows from Theorem 11.19 where $f_n \equiv f \equiv 0$, this result could not be used in the prove of Theorem 12.9 due to the additional forces ℓ which regularity were to weak). However, thanks to the strong convergence result of stresses in Theorem 11.19, one can take simply the solutions with respect to the fixed optimal pseudo force $\bar{\mathfrak{l}}$ and $f \equiv 0$ to obtain the desired recovery sequence.

Let us also refer to Theorem 5.4, where the argumentation was similar and Proposition 4.8 was used to obtain a recovery sequence.

Note also that it is straightforward to add hardening to the state equation as in (11.9).

Remark 12.14 (Volume forces)**.** *In Remark 12.3 we argued that we dropped the volume forces due to the delicate construction of a recovery sequence in Theorem 12.9. As said above, the construction of a recovery sequence after dropping the displacement as a state is much simpler due to the strong convergence in Theorem 11.19 which also holds for (fixed) volume forces when they belong to a set which fulfills a global safe-load condition. Therefore, it is also possible to prove the approximation of global minimizers analog to Theorem 12.9 when volume forces are present. We only have to take care of the boundedness of a minimizing sequence when proving the existence of a global minimizer. Corollary 11.8 is not applicable anymore, but, for example, a tracking type objective function for the stress would be sufficient.*

It would also be an interesting task to derive optimality conditions when volume forces are present (but, loosely speaking, the safe-load condition seems to prevent large volume forces so that the more interesting control is the Dirichlet displacement). Note that the findings in Section 5.2 are not applicable in this case since a set which fulfills a global safe-load condition is not a linear subspace. One would need to extend the results given in Section 5.2 so that sets which fulfill a global safe-load condition are covered, which might not be trivial due to the intricate structure of the global safe-load condition. However, such an extension is beyond the scope of this work.

Remark 12.15 (Approximation by smoothed problems). *As in Section 5.2, we want to smoothen the state equation further by using $\partial I_{\lambda,s}$ (see* (3.11)*) instead of ∂I_λ to obtain a differentiable solution operator so that we can derive optimality conditions. Therefore the approximation result of global minimizers should also hold when we consider such smoothed problems. However, this is the case and can be shown as in Theorem 5.4 by using Lemma 4.17 with $A = \partial I_{\mathcal{K}(\Omega)}$ and $A_n = \partial I_{\lambda_n,s_n}$ for suitable sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$ such that* (4.14) *can be obtained by* (3.12).

This concludes the reduction to the stress and we can bring our attention to the derivation of optimality conditions.

12.3 Optimality Conditions

In the previous section we have reduced the optimization problem (12.1) such that only the stress is present as a state variable. However, we did not state the reduced optimization problem(s) due to the similarity to (12.1) (and (12.2)). Nonetheless, since we investigate the smoothed optimal control problem in more detail in this section, let us do this now for the sake of clarity:

$$\left\{\begin{aligned} &\min && J(\sigma,\mathfrak{l}) := \frac{1}{2}\|\sigma(T)-\sigma_d\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} + \frac{\alpha}{2}\|\dot{\mathfrak{l}}\|^2_{L^2(\mathcal{X}_c)},\\ &\text{s.t.} && -\operatorname{div}\sigma = 0\\ & && u-(\mathcal{G}\mathfrak{l}+\mathfrak{a}) \in H^1(H^1_D(\Omega;\mathbb{R}^{d\times d}_s))\\ & && \sigma = \mathbb{C}(\nabla^s u - z)\\ & && \dot{z} = \partial I_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z),\\ & && (u(0),\sigma(0)) = (\mathfrak{a}(0),\sigma_0)\\ & && u\in H^1(H^1(\Omega;\mathbb{R}^d))\\ & && \sigma, z \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))\\ & && \mathfrak{l}\in H^1_{00}(\mathcal{X}_c), \end{aligned}\right. \tag{12.12}$$

Our main goal in this section is to give a concrete form of the derivative of the *reduced objective function* in Proposition 12.20 by means of the adjoint equation. This derivative will be then used in Chapter 13 to implement a gradient descent method. Thanks to Proposition 12.20, it is then easy to derive optimality conditions in the form of a KKT system in Theorem 12.21.

Let us point to the similarity between (12.12) and (6.3). The difference is that we have now only the Dirichlet displacement, respectively the pseudo forces, as a control variable and the stress instead of the plastic strain as the state. Otherwise, these problems behave similar, even when $\varepsilon = 0$ (see Assumption ⟨12.3.i⟩) due to the Lipschitz continuity of $\partial I_{\lambda,s}$ (cf. Lemma 3.18 Item (ii)). In particular, one could apply the results given in Section 5.2 by using $\sigma = \sigma(z,\mathfrak{l})$ with an analog transformation as in Theorem 2.9. Still, we only use Theorem 5.9 to obtain the differentiability of the solution operator of the state equation and then derive the differentiability of the reduced objective function and the mentioned KKT system from this result. A rigorous application of the results in Section 5.2 by means of the transformation $\sigma = \sigma(z,\mathfrak{l})$ would likely not shorten the argumentation.

Differentiability of the Regularized Solution Operator

At first we introduce the solution operator of the state equation of (12.12) in the following definition. After that, we prove the differentiability in Proposition 12.17 by using the abstract result given in Theorem 5.9 as mentioned above.

Definition 12.16 (Smoothed solution operator)**.** *According to Corollary 11.12, for $\mathfrak{l} \in H^1_{00}(\mathcal{X}_c)$ there exists a unique solution (u, σ, z) of the state equation in* (12.12)*. We denote the with σ associated solution operator by*

$$\mathcal{S}_s : H^1_{00}(\mathcal{X}_c) \to H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s)) \qquad \mathfrak{l} \mapsto \sigma.$$

Of course, this operator depends on ε and λ, but we suppress this dependency to ease notation.

With this solution operator at hand, we can define the *reduced objective function*

$$F : H^1_{00}(\mathcal{X}_c) \to \mathbb{R}, \qquad \mathfrak{l} \mapsto J(\mathcal{S}_s(\mathfrak{l}), \mathfrak{l}).$$

The problem (12.12) is then equivalent to

$$\min_{\mathfrak{l}\in H^1_{00}(\mathcal{X}_c)} \quad F(\mathfrak{l}). \tag{12.13}$$

Proposition 12.17 (Differentiability of the smoothed solution operator)**.** *The solution operator $\mathcal{S}_s$ is Fréchet differentiable from $H^1_{00}(\mathcal{X}_c)$ to $H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$. Its directional derivative at $\mathfrak{l} \in H^1_{00}(\mathcal{X}_c)$ in direction $h \in H^1_{00}(\mathcal{X}_c)$, denoted by $\tau = \mathcal{S}_s'(\ell)h$, is the second component of the unique solution $(v, \tau, \eta) \in H^1(H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_s) \times L^2(\Omega; \mathbb{R}^{d\times d}_s))$ of*

$$\begin{aligned} -\operatorname{div}\tau &= 0, \\ \tau &= \mathbb{C}(\nabla^s v - \eta), \\ \dot{\eta} &= \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)(\tau - \varepsilon\mathbb{B}\eta), \\ v - \mathcal{G}h &\in H^1(H^1_D(\Omega; \mathbb{R}^d)), \\ (v, \tau)(0) &= (0, 0), \end{aligned} \tag{12.14}$$

where (u, σ, z) is the solution of the state equation in (12.12) *associated with $\mathfrak{l}$.*

Proof. As in Proposition 11.11, one can transform the state equation in (12.12) equivalently in

$$\dot{z} = \partial I_{\lambda,s}\big(R(0, \mathcal{G}\mathfrak{l} + \mathfrak{a}) - Q_\varepsilon z\big), \quad z(0) = \nabla^s \mathfrak{a}(0) - \mathbb{A}\sigma_0, \tag{12.15}$$

$$u = \mathcal{T}(-\operatorname{div}(\mathbb{C}z), \mathcal{G}\mathfrak{l} + \mathfrak{a}) \tag{12.16}$$

$$\sigma = \mathbb{C}(\nabla^s u - z). \tag{12.17}$$

The first equation (12.15) has the form of the general equation (EVI) with $R = R(0, \mathcal{G}(\cdot) + \mathfrak{a})$.

At first we consider the solution mapping $\tilde{\mathcal{S}}_z : H^1(W^{1,p}(\Omega; \mathbb{R}^d)) \to H^1(L^p(\Omega; \mathbb{R}^{d\times d}_s))$, where $p > 2$ is given in Assumption ⟨12.3.i⟩, of

$$\dot{z} = \partial I_{\lambda,s}\big(R(0, u_D) - Q_\varepsilon z\big), \quad z(0) = \nabla^s \mathfrak{a}(0) - \mathbb{A}\sigma_0,$$

that is, $\tilde{\mathcal{S}}_z(u_D) = z$ (here we use Assumption ⟨12.3.ii⟩). Note that, completely analog to the discussion in Section 6.2, one verifies that we can apply Theorem 5.9 (the operator Q_ε is not coercive when $\varepsilon = 0$ which was required in Assumption ⟨4.i⟩, however, Theorem 5.9 still holds as discussed in Remark 5.24). Thus, the solution mapping $\tilde{\mathcal{S}}_z$ is Fréchet-differentiable from $H^1(W^{1,p}(\Omega; \mathbb{R}^d))$ to $H^1(L^2(\Omega; \mathbb{R}^{d\times d}_s))$ and its derivative at $u_D \in H^1(W^{1,p}(\Omega; \mathbb{R}^d))$ in direction $h_D \in H^1(W^{1,p}(\Omega; \mathbb{R}^d))$ is the unique solution of

$$\dot{\tilde{\eta}} = \partial I'_{\lambda,s}(R(0, u_D) - Q_\varepsilon \tilde{z})(R(0, h_D) - Q_\varepsilon \tilde{\eta}), \quad \tilde{\eta}(0) = 0,$$

where $\tilde{z} := \tilde{\mathcal{S}}_z(u_D)$.

Now, taking into account that the solution operator of (12.15) is simply the concatenation of $\tilde{\mathcal{S}}_z$ with the affine mapping $H^1_{00}(\mathcal{X}_c) \ni \mathfrak{l} \mapsto \mathcal{G}\mathfrak{l} + \mathfrak{a} \in H^1(W^{1,p}(\Omega;\mathbb{R}^d))$ (here we use Assumption ⟨12.3.i⟩), we see that the solution mapping of (12.15) is Fréchet-differentiable from $H^1_{00}(\mathcal{X}_c)$ to $H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s))$ and its derivative at $\mathfrak{l}$ in direction h is the unique solution of

$$\dot{\eta} = \partial I'_{\lambda,s}(R(0,\mathcal{G}\mathfrak{l} + \mathfrak{a}) - Q_\varepsilon z)(R(0,\mathcal{G}h) - Q_\varepsilon \eta), \qquad \eta(0) = 0,$$

where z is the solution of (12.15). Since all mappings in (12.16) and (12.17) are linear and affine, they are trivially Fréchet-differentiable in their respective spaces and the respective derivatives are given by $v = \mathcal{T}(-\operatorname{div}(\mathbb{C}\eta), \mathcal{G}h)$ and $\tau = \mathbb{C}(\nabla^s v - \eta)$. In view of the definition of $\mathcal{T}$, R and Q_ε, we finally end up with (12.14). □

Adjoint Equation

To give a concrete form of the derivative of the reduced objective function, we need the following adjoint equation. With this adjoint equation at hand, we can provide the desired differentiability in Proposition 12.20.

Definition 12.18 (Adjoint equation). *Let $\mathfrak{l} \in H^1_{00}(\mathcal{X}_c)$ be given and (u,σ,z) the solution of the state equation in* (12.12). *We define the* adjoint state $(v_\varphi, \varphi, v_T) \in H^1(H^1_D(\Omega;\mathbb{R}^d)) \times H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)) \times H^1_D(\Omega;\mathbb{R}^d)$ *as the solution of the* adjoint equation,

$$-\operatorname{div}\mathbb{C}\nabla^s v_\varphi = -\operatorname{div}\mathbb{C}\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi, \tag{12.18a}$$

$$\dot{\varphi} = (\mathbb{C} + \varepsilon\mathbb{B})\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi - \mathbb{C}\nabla^s v_\varphi, \tag{12.18b}$$

$$\varphi(T) = \mathbb{C}(\sigma(T) - \sigma_d - \nabla^s v_T), \tag{12.18c}$$

$$-\operatorname{div}\mathbb{C}\nabla^s v_T = -\operatorname{div}\mathbb{C}(\sigma(T) - \sigma_d), \tag{12.18d}$$

Lemma 12.19 (Existence of an adjoint state). *For every $(\sigma, z) \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s) \times L^2(\Omega;\mathbb{R}^{d\times d}_s))$, there exists a unique adjoint state.*

Proof. Thanks to the definition of Q_ε and $\mathcal{T}$ in Definition 2.7 and Corollary 2.6, the adjoint equation is equivalent to

$$\dot{\varphi} = Q_\varepsilon \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi, \qquad \varphi(T) = \mathbb{C}\big[\sigma(T) - \sigma_d - \nabla^s\mathcal{T}(-\operatorname{div}(\mathbb{C}(\sigma(T) - \sigma_d)), 0)\big].$$

This is an operator equation backward in time, whose existence can be proven by using Theorem A.7. Alternatively, the existence of solutions can be deduced via duality as in Lemma 5.12. □

Proposition 12.20 (Differentiability of the reduced objective function). *The functional F is Fréchet differentiable and*

$$F'(\mathfrak{l})h = \left(\dot{\psi} + \alpha\dot{\mathfrak{l}}, \dot{h}\right)_{L^2(\mathcal{X}_c)}$$

holds for all $\mathfrak{l}, h \in H^1_{00}(\mathcal{X}_c)$, that is, $F'(\mathfrak{l}) = \psi + \alpha\mathfrak{l}$ (Riesz representation), where $\psi \in H^1_{00}(\mathcal{X}_c) \cap H^2(\mathcal{X}_c)$ is given by

$$\psi(t) := \int_0^t \int_0^s L(r)\,dr\,ds - \frac{t}{T}\int_0^T \int_0^s L(r)\,dr\,ds, \tag{12.19}$$

$$L := (\mathbb{C}\nabla^s\mathcal{G})^*(\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi - \nabla^s v_\varphi) \in L^2(\mathcal{X}_c) \tag{12.20}$$

where (u,σ,z) is the solution of the state equation in (12.12) *and $(v_\varphi, \varphi, v_T)$ the adjoint state ($(\mathbb{C}\nabla^s\mathcal{G})^* \in \mathcal{L}(L^2(\Omega;\mathbb{R}^{d\times d}_s);\mathcal{X}_c)$ is the adjoint operator of $\mathbb{C}\nabla^s\mathcal{G} \in \mathcal{L}(\mathcal{X}_c, L^2(\Omega;\mathbb{R}^{d\times d}_s))$).*

Proof. We have $\Psi(\mathfrak{l}) = \frac{1}{2}\|\mathcal{S}_s(\mathfrak{l})(T) - \sigma_d\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$ for all $\mathfrak{l} \in H^1_{00}(\mathcal{X}_c)$. Let $\mathfrak{l}, h \in H^1_{00}(\mathcal{X}_c)$, (u, σ, z) the solution of the state equation in (12.12), (v, τ, η) the solution of (12.14) and $(v_\varphi, \varphi, v_T)$ the adjoint state. Then we have

$$
\begin{aligned}
\Psi'(\mathfrak{l})h &= \big(\sigma(T) - \sigma_d, \tau(T)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&= \big(\sigma(T) - \sigma_d - \nabla^s v_T, \tau(T)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&= \big(\mathbb{C}(\sigma(T) - \sigma_d - \nabla^s v_T), \nabla^s v(T) - \eta(T)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&= \big(\mathbb{C}(\sigma(T) - \sigma_d), \nabla^s v(T)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - \big(\mathbb{C}\nabla^s v_T, \nabla^s v(T)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - (\varphi(T), \eta(T))_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \\
&= -\Big((\varphi(T), \eta(T))_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} - (\varphi(0), \eta(0))_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \Big) \\
&= -\Big(\big(\dot{\varphi}, \eta\big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \big(\varphi, \dot{\eta}\big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \Big) \\
&= -\Big((\mathbb{C} + \varepsilon\mathbb{B})\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi - \mathbb{C}\nabla^s v_\varphi, \eta\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&\quad - \Big(\varphi, \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)(\tau - \varepsilon\mathbb{B}\eta)\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= -\Big((\mathbb{C} + \varepsilon\mathbb{B})\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi - \mathbb{C}\nabla^s v_\varphi, \eta\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&\quad - \Big(\varphi, \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)(\mathbb{C}\nabla^s v - (\mathbb{C} + \varepsilon\mathbb{B})\eta)\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= \big(\mathbb{C}\nabla^s v_\varphi, \eta\big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} - \Big(\mathbb{C}\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi, \nabla^s v\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= \big(\mathbb{C}\nabla^s v_\varphi, \eta\big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} - \Big(\mathbb{C}\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi, \nabla^s(v - \mathcal{G}h)\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&\quad - \Big(\mathbb{C}\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi, \nabla^s\mathcal{G}h\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= \big(\mathbb{C}\nabla^s v_\varphi, \eta - \nabla^s v + \nabla^s\mathcal{G}h\big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} - \Big(\mathbb{C}\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi, \nabla^s\mathcal{G}h\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= -\big(\nabla^s v_\varphi, \tau\big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} + \Big(\mathbb{C}(\nabla^s v_\varphi - \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi), \nabla^s\mathcal{G}h\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= \Big(\nabla^s v_\varphi - \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi, \mathbb{C}\nabla^s\mathcal{G}h\Big)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\
&= \Big((\mathbb{C}\nabla^s\mathcal{G})^*(\nabla^s v_\varphi - \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)^*\varphi), h\Big)_{L^2(\mathcal{X}_c)} \\
&= -(L, h)_{L^2(\mathcal{X}_c)},
\end{aligned}
$$

hence,

$$
\big(\dot{\psi}, \dot{h}\big)_{L^2(\mathcal{X}_c)} = -\big(\ddot{\psi}, h\big)_{L^2(\mathcal{X}_c)} = -(L, h)_{L^2(\mathcal{X}_c)} = \Psi'(\mathfrak{l})h
$$

as claimed. □

Let us finally derive first order optimality conditions in the form of a KKT system by using the finding in the proposition above. We emphasize once again that we do not need the KKT system for the numerical experiments in the upcoming chapter. Therein, we will apply a gradient descent algorithm and thus will only make use of the above provided form of the derivative of the reduced objective function.

Theorem 12.21 (KKT-conditions for (12.13))**.** *Let* $\mathfrak{l} \in H^1_{00}(\mathcal{X}_c)$*,* (u, σ, z) *the solution of the state equation in* (12.12) *and* $(v_\varphi, \varphi, v_T)$ *the adjoint state . Then following assertions are equivalent:*

(i) $F'(\mathfrak{l})h = J'_\sigma(\sigma, \mathfrak{l})S'_s(\mathfrak{l})h + J'_\mathfrak{l}(\sigma, \mathfrak{l})h = 0$ *for all* $h \in H^1_{00}(\mathcal{X}_c)$*,*

(ii) $\mathfrak{l} \in H^2(\mathcal{X}_c)$ *and* $\left(\nabla^s v_\varphi - \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi, \mathbb{C}\nabla^s \mathcal{G}h\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} = \left(\alpha\ddot{\mathfrak{l}}, h\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))}$ *for all* $h \in H^1_{00}(\mathcal{X}_c)$*,*

(iii) $\mathfrak{l} \in H^2(\mathcal{X}_c)$ *and* $\left(\varphi, \partial I'_{\lambda,s}(R(0, \mathcal{G}\mathfrak{l} + \mathfrak{a}) - Q_\varepsilon z)R(0, \mathcal{G}h)\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} = \alpha\left(\dot{\mathfrak{l}}, \dot{h}\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))}$ *for all* $h \in H^1_{00}(\mathcal{X}_c)$*.*

Moreover, if $\mathfrak{l}$ *is a locally optimal control for* (12.13)*, then the assertions above hold.*

Proof. Thanks to Proposition 12.20 we get

$$Item\ (i) \quad \Longleftrightarrow \quad \left(\dot{\psi} + \alpha\dot{\mathfrak{l}}, \dot{h}\right)_{L^2(\mathcal{X}_c)} = 0 \quad \forall h \in H^1_{00}(\mathcal{X}_c), \tag{12.21}$$

where ψ is defined by (12.19). Therefore, analog to Example 5.15, we find that the second distributional time derivative of $\psi + \alpha\mathfrak{l}$ is a regular distribution in $L^2(\mathcal{X}_c)$, i.e., $\psi + \alpha\mathfrak{l} \in H^2(\mathcal{X}_c)$ and thus $\mathfrak{l} \in H^2(\mathcal{X}_c)$ since $\psi \in H^2(\mathcal{X}_c)$. The equivalence of the right side in (12.21) and Item (ii) is now easily obtained through integration by parts and the definition of ψ.

To verify the equivalence between Item (ii) and Item (iii) we note that

$$\begin{aligned} &\left(\nabla^s v_\varphi - \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi, \mathbb{C}\nabla^s \mathcal{G}h\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} \\ &= \left(\nabla^s v_\varphi - \partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi, R(0, \mathcal{G}h)\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))} && \text{(by (12.18a) \& (2.12b))} \\ &= -\left(\partial I'_{\lambda,s}(\sigma - \varepsilon\mathbb{B}z)\varphi, R(0, \mathcal{G}h)\right)_{L^2(L^2(\Omega;\mathbb{R}^{d\times d}_s))}, && \text{(by (2.12a))} \end{aligned}$$

where we have used Definition 2.7 and Corollary 2.6, which (using again integration by parts) yields the assertion since $\sigma - \varepsilon\mathbb{B}z = R(0, \mathcal{G}\mathfrak{l} + \mathfrak{a}) - Q_\varepsilon z$ and $\partial I'_{\lambda,s}(\tau)$ is self-adjoint for all $\tau \in L^2(\Omega; \mathbb{R}^{d\times d}_s)$ (see Lemma 3.19). □

Remark 12.22 (Relationship to Theorem 5.13)**.** *The equation in Theorem 12.21 Item (iii) corresponds to* (5.22c)*.*

Remark 12.23 (Second-order sufficient conditions)**.** *Since the smoothed state equation* (11.9) *is equivalent to* (11.12)*, we could apply the analysis in Section 5.3 to derive second-order sufficient conditions for* (12.13)*. However, as in Chapter 6 and Chapter 9, since we do not want to go beyond the scope of this thesis, we omit this. Let us also refer to the end of Section 6.2, the discussion there applies almost one to one to the scenario considered here.*

Chapter 13 Numerical Experiments

We proceed with the presentation of the numerical experiments. We start with a realization of $\mathcal{G}$ and then describe how we approximate the solution of (12.12). Finally, we choose a test setting and end this thesis by presenting our results for this test setting.

Regarding the literature, we refer to [48, 58, 59, 91] for a survey of numerical analysis concerning plasticity with hardening. Moreover, in [11, Section 5 & 6] the numerical approximation of perfect plasticity is considered in the context of rate-independent systems, see also the introduction in Chapter 11. Numerical approximation of optimal control problems can be found, e.g., in [62, 54, 56]. Regarding optimal control of perfect plasticity, there are no results about numerical approximation available in the literature to the author's knowledge, except the ones given in [72]. These results serve as a basis in what follows, we use in particular the same test setting. However, we extend them with more details.

Note that we only present *experiments*, but do not provide any numerical analysis as in the references above. This is beyond the scope of this thesis.

13.1 A Realization of the Operator $\mathcal{G}$

Recall the definition of the pseudo Dirichlet and Neumann boundary in Assumption ⟨13.i⟩. Analog to $W^{1,p}_D(\Omega;\mathbb{R}^d)$ we define $W^{1,p}_\Lambda(\Omega;\mathbb{R}^d)$ as the subset of functions of $W^{1,p}(\Omega;\mathbb{R}^d)$ which traces are zero on Λ_D. We also denote the dual space by $W^{-1,p'}_\Lambda(\Omega;\mathbb{R}^d)$ and abbreviate $H^1_\Lambda(\Omega;\mathbb{R}^d) := W^{1,2}_\Lambda(\Omega;\mathbb{R}^d)$ and $H^{-1}_\Lambda(\Omega;\mathbb{R}^d) := W^{-1,2}_\Lambda(\Omega;\mathbb{R}^d)$. Clearly, Corollary 2.6 also holds when Γ_D is exchanged with Λ_D. The exponent $\bar{p}$ for Γ_D does not have to be the same as for Λ_D, however, we can assume that they are equal by taking the minimum of both.

Let us now fix $p \in (2, \bar{p}]$ and assume in addition that $2 > {}^{dp}/_{(d+p)}$ and $2 > {}^{(d-1)p}/_{d}$, we define

$$\mathcal{X} := W^{-1,p}_\Lambda(\Omega;\mathbb{R}^d) \quad \text{and} \quad \mathcal{X}_c := L^2(\Omega;\mathbb{R}^d) \times L^2(\Lambda_N;\mathbb{R}^d),$$

so that $\mathcal{X}_c$ is compactly embedded into $\mathcal{X}$ via the canonical embedding $\mathcal{X}_c \ni (f, g) \mapsto (f, \cdot)_{L^2(\Omega;\mathbb{R}^d)} + (g, \cdot)_{L^2(\Lambda_N;\mathbb{R}^d)} \in \mathcal{X}$.

Finally we define $\mathcal{G} : \mathcal{X} \to W^{1,p}_\Lambda(\Omega;\mathbb{R}^d) \subset W^{1,p}(\Omega;\mathbb{R}^d)$ as the solution operator of

$$\big(\mathbb{C}\nabla^s u_D, \nabla^s \phi\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} = \langle \mathfrak{l}, \phi \rangle \quad \forall \phi \in W^{1,p'}_\Lambda(\Omega;\mathbb{R}^d), \tag{13.1}$$

that is, $\mathcal{G}\mathfrak{l} = u_D$. In particular, $\mathcal{G}$ solves *linear elasticity*. Note also that the requirement on $\mathcal{G}$ in Assumption ⟨12.3.i⟩ is satisfied. The following result ensures that our control space is "large enough".

Lemma 13.1 (Control space covers $H^2(\Omega;\mathbb{R}^d)$). *We have*

(i) $\mathcal{T}(0, \mathcal{G}(\mathcal{X})) = \mathcal{T}(0, W^{1,p}(\Omega;\mathbb{R}^d))$ *and*

(ii) $\mathcal{T}(0, \mathcal{G}(\mathcal{X}_c)) \supset \mathcal{T}(0, H^2(\Omega;\mathbb{R}^d))$.

Proof. Due to (9.18), there is a function $\phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that $0 \le \phi \le 1$, $\phi \equiv 1$ on Γ_D and $\phi \equiv 0$ on Λ_D.

Let $u_D \in W^{1,p}(\Omega;\mathbb{R}^d)$ be arbitrary and set $\tilde{u}_D := \phi u_D \in W^{1,p}_\Lambda(\Omega;\mathbb{R}^d) \subset W^{1,p}(\Omega;\mathbb{R}^d)$. It is easy to see that $\mathcal{T}(0, u_D) = \mathcal{T}(0, \tilde{u}_D)$. We define $\mathfrak{l} \in \mathcal{X}$ by $\langle \mathfrak{l}, \phi \rangle := \big(\mathbb{C}\nabla^s \tilde{u}_D, \nabla^s \phi\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$ for all $\phi \in W^{1,p'}_\Lambda(\Omega;\mathbb{R}^d)$, thus, $\mathcal{G}\mathfrak{l} = \tilde{u}_D$ and therefore $\mathcal{T}(0, \mathcal{G}\mathfrak{l}) = \mathcal{T}(0, \tilde{u}_D) = \mathcal{T}(0, u_D)$ which proves $\mathcal{T}(0, \mathcal{G}(\mathcal{X})) \supset \mathcal{T}(0, W^{1,p}(\Omega;\mathbb{R}^d))$. Since we have $\mathcal{G}(\mathcal{X}) \subset W^{1,p}(\Omega;\mathbb{R}^d)$ by definition, the first assertion is proved.

Let now $u_D \in H^2(\Omega;\mathbb{R}^d)$, we define again $\tilde{u}_D := \phi u_D \in H^2(\Omega;\mathbb{R}^d) \cap W^{1,p}_\Lambda(\Omega;\mathbb{R}^d)$ such that $\mathcal{T}(0, u_D) = \mathcal{T}(0, \tilde{u}_D)$ holds. Similar as above we can define $f \in L^2(\Omega;\mathbb{R}^d)$ by $f(x) := -\nabla \cdot \mathbb{C}\nabla^s \tilde{u}_D(x)$ and $g \in L^2(\Lambda_N;\mathbb{R}^d)$ by $g(x) := \mathbb{C}\nabla^s \tilde{u}_D|_{\Lambda_N}(x)$ for almost all $x \in \Omega$. Due to this definition we have $\mathcal{G}(f, g) = \tilde{u}_D$, hence, $\mathcal{T}(0, \mathcal{G}(f, g)) = \mathcal{T}(0, \tilde{u}_D) = \mathcal{T}(0, u_D)$, which proves the second assertion. □

Let us now investigate the adjoint operator of $\mathbb{C}\nabla^s\mathcal{G}$, which is needed in (12.20) to calculate the derivative of the reduced objective function.

Lemma 13.2 (Adjoint operator of $\mathbb{C}\nabla^s\mathcal{G}$). *Let $\tau \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$ be given and $\mathfrak{l} \in H^1_\Lambda(\Omega;\mathbb{R}^d)$ the solution of*

$$(\mathbb{C}\nabla^s\mathfrak{l}, \nabla^s\phi)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} = (\mathbb{C}\tau, \nabla^s\phi)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} \tag{13.2}$$

for all $\phi \in H^1_\Lambda(\Omega;\mathbb{R}^d)$. Then we have $(\mathbb{C}\nabla^s\mathcal{G})^*\tau = (\mathfrak{l}, \mathfrak{l}|_{\Lambda_N}) \in \mathcal{X}_c$.

Proof. Let us abbreviate $(f,g) := (\mathbb{C}\nabla^s\mathcal{G})^*\tau$. We have

$$\big(f, f_\zeta\big)_{L^2(\Omega;\mathbb{R}^d)} + \big(g, g_\zeta\big)_{L^2(\Lambda_N;\mathbb{R}^d)} = \big((f,g),(f_\zeta,g_\zeta)\big)_{\mathcal{X}_c} = \big(\tau, \mathbb{C}\nabla^s\mathcal{G}(f_\zeta,g_\zeta)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}$$

holds for all $(f_\zeta, g_\zeta) \in \mathcal{X}_c$. Using the definition of $\mathcal{G}$ and testing (13.2) with $\phi = \mathcal{G}(f_\zeta, g_\zeta)$ we see that

$$\begin{aligned}\big(\mathfrak{l}, f_\zeta\big)_{L^2(\Omega;\mathbb{R}^d)} + \big(\mathfrak{l}, g_\zeta\big)_{L^2(\Lambda_N;\mathbb{R}^d)} &= \big(\mathbb{C}\nabla^s\mathfrak{l}, \nabla^s\mathcal{G}(f_\zeta,g_\zeta)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)} = \big(\tau, \mathbb{C}\nabla^s\mathcal{G}(f_\zeta,g_\zeta)\big)_{L^2(\Omega;\mathbb{R}^{d\times d}_s)}\\ &= \big(f, f_\zeta\big)_{L^2(\Omega;\mathbb{R}^d)} + \big(g, g_\zeta\big)_{L^2(\Lambda_N;\mathbb{R}^d)}\end{aligned}$$

holds for all $(f_\zeta, g_\zeta) \in \mathcal{X}_c$. □

The lemma above together with Proposition 12.20 shows that the derivative of the smoothed reduced objective function F at $(f,g) = (\mathfrak{l}, \mathfrak{l}|_{\Lambda_N})$ with $\mathfrak{l} \in H^1(H^1_\Lambda(\Omega;\mathbb{R}^d))$ has again the same form. This is in particular convenient for a gradient based optimization method since the two controls f and g are essentially reduced to one control $\mathfrak{l}$. Let us capture this in the following definition and corollary.

Definition 13.3 (Essential control space). *We define the* essential control space *by*

$$H^1_{00}(\mathcal{X}_c)_\mathfrak{l} := \{\tilde{\mathfrak{l}} \in H^1_{00}(\mathcal{X}_c) : \exists \mathfrak{l} \in H^1(H^1_\Lambda(\Omega;\mathbb{R}^d)), \tilde{\mathfrak{l}} = (\mathfrak{l}, \mathfrak{l}|_{\Lambda_N})\}.$$

With a slight abuse of notation, when $\tilde{\mathfrak{l}} \in H^1_{00}(\mathcal{X}_c)_\mathfrak{l}$ and $\mathfrak{l} \in H^1(H^1_\Lambda(\Omega;\mathbb{R}^d))$ is the corresponding function, we also denote $\tilde{\mathfrak{l}}$ by $\mathfrak{l}$, that is, we identify $H^1_{00}(\mathcal{X}_c)_\mathfrak{l}$ with $H^1_{00}(H^1_\Lambda(\Omega;\mathbb{R}^d))$.

Corollary 13.4 (Preservation of the essential control space). *We have*

(i) $(\mathbb{C}\nabla^s\mathcal{G})^*(\{\tau \in H^1(L^2(\Omega;\mathbb{R}^{d\times d}_s)) : \tau(0) = \tau(T) = 0\}) \subset H^1_{00}(\mathcal{X}_c)_\mathfrak{l}$ *and*

(ii) $F'(H^1_{00}(\mathcal{X}_c)_\mathfrak{l}) \subset H^1_{00}(\mathcal{X}_c)_\mathfrak{l}$.

Having clarified the definition of $\mathcal{G}$, we can proceed with the approximation of (12.12), respectively (12.13).

13.2 Approximation

Let us now describe the method we use to approximate solutions of (12.13). We start with the optimization problem and then turn to the state and adjoint equation and the solution operators $\mathcal{G}$ and $\mathcal{G}^*$.

Algorithm 1 Steepest deescent method

1: Choose $\mathfrak{l} \in H^1_{00}(\mathcal{X}_c)_{\mathfrak{l}}$; $\sigma^0 > 0$; $\beta, \gamma_a \in (0,1)$; $0 < TOL \ll 1$ and $m_i \in \mathbb{N}$.
2: **for** $j = 1, ..., m_i$ **do**
3:     **if** $\|F'(\mathfrak{l})\|_{H^1(\mathcal{X}_c)^*} = \|\dot{\psi} + \alpha\dot{\mathfrak{l}}\|_{L^2(\mathcal{X}_c)} < TOL$ **then**
4:         **return** $\mathfrak{l}$
5:     **end if**
6:     $\sigma = \sigma^0$
7:     $s = -F'(\mathfrak{l})$
8:     **while** $F(\mathfrak{l}) - F(\mathfrak{l} + \sigma s) < \gamma_a \sigma \|\dot{s}\|^2_{L^2(\mathcal{X}_c)}$ **do**
9:         $\sigma = \beta\sigma$
10:     **end while**
11:     $\sigma^0 = \sigma/\beta$
12:     $\mathfrak{l} = \mathfrak{l} + \sigma s$.
13: **end for**

Optimization Problem

For the numerical approximation of the solution of (12.13) we use a standard steepest descent method, with an Armijo line search, which is probably one of the most simplest gradient based optimization methods. In the following we use the notation from Proposition 12.20 and Definition 13.3 (see also Corollary 13.4). The method is shown in Algorithm 1 as pseudo code.

Since the steepest descent method and the Armijo line search are well known (see e.g. [62, 42, 93]), there is little need for explanation. Only in Line 11 the update of the initial step size of the Armijo line search is not standard. This update adjusts the initial step size for the next Armijo line search according to the last step size and, loosely speaking, enables the algorithm to find the "correct" step size after some iterations, see also Figure 13.

In order to realize Algorithm 1 we need to evaluate the objective function F and its derivative (respectively the Riesz representative) F'. To this end, we first describe how to solve the equations needed for these evaluations.

State and Adjoint Equation, $\mathcal{G}$ and $\mathcal{G}^*$

Let us refer to the state equation (12.12), the adjoint equation (12.18), and the equations defining $\mathcal{G}$ and its adjoint, (13.1) and (13.2), as *the equations*.

To obtain a time discretized version of the equations we divide the time Interval $[0, 1]$ (we set $T = 1$ as later in Section 13.3) in $n_t \in \mathbb{N}$ parts, each of length $d_t := 1/n_t$. For a discretization of a function $u \in H^1(X)$, where X is a normed space, we choose the point evaluation

$$u_i = u(i/n_t), \qquad i \in \{0, ..., n_t\}$$

and the piecewise affine linear interpolation

$$\tilde{u}(t) := \frac{t - (i-1)d_t}{d_t} u_i + \frac{id_t - t}{d_t} u_{i-1} \tag{13.3}$$

for $t \in [(i-1)d_t, id_t]$. Note that the number of time steps n_t is not integrated in our notation, it will always be clear from the context.

In order to approximate the solutions of the state and adjoint equation, we apply at first an implicit Euler method to obtain a time discretized version of the state and adjoint equation.

For a time discretized version of (13.1) and (13.2) one simply evaluates the functions in id_t for $i \in \{1, ..., n_t\}$. As a solution of the time discretized equations, one obtains $n_t + 1$ elements in the corresponding spaces and can obtain an H^1 function in time by (13.3).

Let us now discuss the discretization in space. To discretize $\Omega = (0,4) \times (0,1)$ (see again Section 13.3) we choose $n_x, n_y \in \mathbb{N}$ and divide Ω in $n_x \cdot n_y$ equal rectangles where n_x is the number of rectangles in x-direction and n_y in y-direction. Each rectangle is then divided into four triangles by the two diagonals. Furthermore, to solve the smoothed state equation we reduce the unknowns to the displacement u and plastic strain z, that is, we eliminate σ by using $\sigma = \mathbb{C}(\nabla^s u - z)$. We are aware that this type of discretization will in general lead to locking effects, but we assume that these can be neglected, as we do not consider "thin" computational domains. We use piecewise linear and continuous elements to approximate the displacement and piecewise constant elements for the stress and plastic strain.

To solve the discretized equations we use the finite element toolbox *FEniCS* (version 2018.1.0), see [39, 5, 66, 68] and [115, 116, 22] for the finite element method in general. For the nonlinear systems of equations we use the in *FEniCS* integrated standard newton solver with a relative and absolute tolerance of 10^{-10}.

Let us now describe the

Calculation of ψ

We depict the implementation of (12.19), that is, we assume that $L \in L^2(\mathcal{X}_c)$ is a given piecewise and affine linear function and integrate this function twice such that the boundary conditions are fulfilled. To this end, we consider piecewise affine linear functions in the following

Lemma 13.5 (Integration of an affine linear function)**.** *Let $n_t \in \mathbb{N}$ and set $d_t := T/n_t$.*

Let $L : [0,T] \to \mathcal{X}_c$ be piecewise affine linear, that is, there exists $L_i \in \mathcal{X}_c$, for $i \in \{0, ..., n_t\}$, with

$$L(t) = \frac{t - (i-1)d_t}{d_t}(L_i - L_{i-1}) + L_{i-1}$$

for $t \in [(i-1)d_t, id_t]$, $i \in \{1, ..., n_t\}$.

Then the following holds:

(i) $\|\dot{L}\|^2_{L^2(\mathcal{X}_c)} = \frac{1}{d_t}\sum_{i=1}^{n_t} \|L_i - L_{i-1}\|^2_{\mathcal{X}_c}$.

(ii) We define $\breve{\psi}(t) := \int_0^t \int_0^s L(x)dxds - \frac{t}{T}\int_0^T \int_0^s L(x)dxds$, *then we have*

$$\begin{aligned}\breve{\psi}(id_t) = \frac{d_t^2}{6}\sum_{j=1}^{i}\sum_{k=1}^{j-1} 3(L_k + L_{k-1}) + L_j + 2L_{j-1} \\ - \frac{id_t^2}{6n_t}\sum_{j=1}^{n_t}\sum_{k=1}^{j-1} 3(L_k + L_{k-1}) + L_j + 2L_{j-1}\end{aligned} \tag{13.4}$$

for all $i \in \{0, ..., n_t\}$.

Proof. We have

$$\|\dot{L}\|^2_{L^2(\mathcal{X}_c)} = \frac{1}{d_t}\sum_{i=1}^{n_t}\frac{1}{d_t}\int_{(i-1)d_t}^{id_t} \|L_i - L_{i-1}\|^2_{\mathcal{X}_c} dt = \frac{1}{d_t}\sum_{i=1}^{n_t}\|L_i - L_{i-1}\|^2_{\mathcal{X}_c},$$

which proves the first assertion.

To prove the second assertion we first calculate for $i \in \{1, ..., n_t\}$ and $s \in [(i-1)d_t, td_t]$

$$\begin{aligned}\int_{(i-1)d_t}^{s} L(x)dx &= \int_{(i-1)d_t}^{s} \frac{x-(i-1)d_t}{d_t}(L_i - L_{i-1}) + L_{i-1}dx \\ &= \frac{(s-(i-1)d_t)^2}{2d_t}(L_i - L_{i-1}) + (s-(i-1)d_t)L_{i-1}\end{aligned}$$

thus

$$\int_{(i-1)d_t}^{id_t} L(x)dx = \frac{d_t}{2}(L_i + L_{i-1}).$$

We have further

$$\begin{aligned}\int_{(i-1)d_t}^{id_t}\int_{(i-1)d_t}^{s} L(x)dxds &= \int_{(i-1)d_t}^{id_t} \frac{(s-(i-1)d_t)^2}{2d_t}(L_i - L_{i-1}) + (s-(i-1)d_t)L_{i-1}ds \\ &= \frac{(id_t-(i-1)d_t)^3}{6d_t}(L_i - L_{i-1}) + \frac{(id_t-(i-1)d_t)^2}{2}L_{i-1} \\ &= \frac{d_t^2}{6}(L_i - L_{i-1}) + \frac{d_t^2}{2}L_{i-1} \\ &= \frac{d_t^2}{6}(L_i + 2L_{i-1})\end{aligned}$$

for $i \in \{1, ..., n_t\}$, therefore

$$\begin{aligned}\int_0^{id_t}\int_0^{s} L(x)dxds &= \sum_{j=1}^{i}\int_{(j-1)d_t}^{jd_t}\Big(\sum_{k=1}^{j-1}\int_{(k-1)d_t}^{kd_t} L(x)dx + \int_{(j-1)d_t}^{s} L(x)dx\Big)ds \\ &= \sum_{j=1}^{i} d_t \sum_{k=1}^{j-1}\frac{d_t}{2}(L_k + L_{k-1}) + \int_{(j-1)d_t}^{jd_t}\int_{(j-1)d_t}^{s} L(x)dxds \\ &= \sum_{j=1}^{i}\frac{d_t^2}{2}\sum_{k=1}^{j-1}(L_k + L_{k-1}) + \frac{d_t^2}{6}(L_j + 2L_{j-1}) \\ &= \frac{d_t^2}{6}\sum_{j=1}^{i}\sum_{k=1}^{j-1} 3(L_k + L_{k-1}) + L_j + 2L_{j-1}\end{aligned}$$

for $i \in \{0, ..., n_t\}$, which yields finally the second assertion. □

Now, let L be given as in Lemma 13.5, we can use (13.4) to calculate the piecewise affine linear function ψ as described in Algorithm 2.

Note that the algorithm only returns the values at id_t. Since L is piecewise affine linear, the actual function $\breve{\psi}$ in Lemma 13.5 is a piecewise cubic function, however, we simply take the piecewise affine linear function which has the same values at id_t.

Let us shortly interrupt the discussion of the implementation to have a closer look at the integration of ψ in (12.19) when L is piecewise affine linear. Instead of integrating L directly as in Lemma 13.5, one could also take the projection of $-L$ into the space of piecewise affine linear functions which fulfill the boundary condition equipped with the $H^1(\mathcal{X}_c)$ norm, that is, solving

$$\left(\dot{\phi}, \dot{h}\right)_{L^2(\mathcal{X}_c)} = -(L, h)_{L^2(\mathcal{X}_c)} \tag{13.5}$$

Algorithm 2 Calculation of ψ

$s = 0$
$\psi_0 = 0$
$\psi_1 = \frac{d_t^2}{6}(L_1 + 2L_0)$
for $i = 2, ..., n_t$ **do**
    $s = s + \frac{d_t^2}{2}(L_{i-1} + L_{i-2})$
    $\psi_i = \psi_{i-1} + s + \frac{d_t^2}{6}(L_i + 2L_{i-1})$
end for
for i = 1, ..., n_t - 1 **do**
    $\psi_i = \psi_i - \frac{i}{n_t}\psi_{n_t}$
end for
$\psi_{n_t} = 0$
return $\psi_0, ..., \psi_{n_t}$

for all piecewise affine linear functions h with $h(0) = h(1) = 0$. Similar to Lemma 13.5 Item (i), one easily verifies that

$$\left(\dot{\phi}, \dot{h}\right)_{L^2(\mathcal{X}_c)} = \frac{1}{d_t} \sum_{i=1}^{n_t} \left(\phi_i - \phi_{i-1}, h_i - h_{i-1}\right)_{\mathcal{X}_c}$$

holds. For the right hand side in (13.5) a straightforward calculation gives

$$-(L, h)_{L^2(\mathcal{X}_c)} = -\sum_{i=1}^{n_t} \frac{d_t^2}{6}\left[2\left(L_i, h_i\right)_{\mathcal{X}_c} + \left(L_i, h_{i-1}\right)_{\mathcal{X}_c} + \left(L_{i-1}, h_i\right)_{\mathcal{X}_c} + 2\left(L_{i-1}, h_{i-1}\right)_{\mathcal{X}_c}\right].$$

Setting now $h_i = 0$ for all $i \in \{1, ..., n_t - 1\} \setminus \{j\}$ for one $j \in \{1, ..., n_t - 1\}$, one arrives at

$$-\phi_{j-1} + 2\phi_j - \phi_{j+1} = -\frac{d_t^2}{6}\left[L_{j-1} + 4L_j + L_{j+1}\right],$$

thus, $(\phi_1, ..., \phi_{n_t-1})$ (recall that $\phi_0 = \phi_{n_t} = 0$) is the solution of

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{n_t-1} \end{bmatrix} = -\frac{d_t^2}{6} \begin{bmatrix} L_0 + 4L_1 + L_2 \\ L_1 + 4L_2 + L_3 \\ L_2 + 4L_3 + L_4 \\ \vdots \\ L_{n_t-1} + 4L_{n_t-1} + L_{n_t} \end{bmatrix}. \tag{13.6}$$

However, as it turns out, the piecewise affine linear function ϕ obtained by (13.6) is the same as the piecewise affine linear function ψ obtained in Algorithm 2 respectively (13.4). To see this, let $\breve{\psi}$ be the function from Lemma 13.5 Item (ii), that is, the piecewise cubic function such that

(13.4) holds. Then we have in particular $\breve{\psi} \in H^2(\mathcal{X}_c)$, hence,

$$\begin{aligned}
-(L,h)_{L^2(\mathcal{X}_c)} &= -\left(\ddot{\breve{\psi}}, h\right)_{L^2(\mathcal{X}_c)} = \left(\dot{\breve{\psi}}, \dot{h}\right)_{L^2(\mathcal{X}_c)} \\
&= \sum_{i=1}^{n_t} \int_{(i-1)d_t}^{id_t} \left(\dot{\breve{\psi}}(t), \frac{h_i - h_{i-1}}{d_t}\right)_{\mathcal{X}_c} dt \\
&= \frac{1}{d_t} \sum_{i=1}^{n_t} \left(\int_{(i-1)d_t}^{id_t} \dot{\breve{\psi}}(t)dt, h_i - h_{i-1}\right)_{\mathcal{X}_c} \\
&= \frac{1}{d_t} \sum_{i=1}^{n_t} \left(\breve{\psi}(id_t) - \breve{\psi}((i-1)d_t), h_i - h_{i-1}\right)_{\mathcal{X}_c} \\
&= \frac{1}{d_t} \sum_{i=1}^{n_t} \left(\psi_i - \psi_{i-1}, h_i - h_{i-1}\right)_{\mathcal{X}_c} = \left(\dot{\psi}, \dot{h}\right)_{L^2(\mathcal{X}_c)}
\end{aligned}$$

for all piecewise affine linear functions h with $h(0) = h(1) = 0$, therefore we obtain $\psi = \phi$.

All in all, one can calculate ψ by (13.4) or by (13.6), we have chosen the first alternative in Algorithm 2.

Let us now continue with the

Evaluation of the Objective Function

Thanks to Corollary 13.4, the control $\mathfrak{l}$ in Algorithm 1 always belongs to the essential control space $H^1_{00}(\mathcal{X}_c)_l$, provided that the initial value does the same. For our experiments we will choose the initial value to be zero, so that this requirement is fulfilled. Thus, using the agreement in Definition 13.3, we have $\mathfrak{l} \in H^1_{00}(H^1_\Lambda(\Omega; \mathbb{R}^2))$.

Using Lemma 13.5 Item (i), we can now evaluate the objective function,

$$\begin{aligned}
F(\mathfrak{l}) &= \frac{1}{2}\|\sigma_{n_t} - \sigma_d\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + \frac{\alpha}{2}\left(\|\dot{\mathfrak{l}}\|^2_{L^2(L^2(\Omega;\mathbb{R}^2))} + \|\dot{\mathfrak{l}}\|^2_{L^2(L^2(\Lambda_N;\mathbb{R}^2))}\right) \\
&= \frac{1}{2}\|\sigma_{n_t} - \sigma_d\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + \frac{\alpha}{2d_t}\left(\sum_{i=1}^{n_t}\|\mathfrak{l}_i - \mathfrak{l}_{i-1}\|^2_{L^2(\Omega;\mathbb{R}^2)} + \sum_{i=1}^{n_t}\|\mathfrak{l}_i - \mathfrak{l}_{i-1}\|^2_{L^2(\Lambda_N;\mathbb{R}^2)}\right),
\end{aligned}$$

where the evaluation of the norms $\|\cdot\|_{L^2(\Omega;\mathbb{R}^2)}$ and $\|\cdot\|_{L^2(\Lambda_N;\mathbb{R}^2)}$ are implemented in *FEniCS* and $\sigma_{n_t} = \sigma(1)$ is obtained by solving the state equation in (12.12) as described above.

Reduced Gradient

Since Algorithm 1 makes use of the gradient of F, we need to implement the gradient according to Proposition 12.20. The procedure is depicted in Algorithm 3 as pseudo code.

Algorithm 3 Computation of the reduced gradient

Require: control function $\mathfrak{l} \in H^1(\mathcal{X}_c)_{\mathfrak{l}}$.

1: Compute the Dirichlet data u_D by solving for all $t \in [0, T]$

$$\left(\mathbb{C}\nabla^s u_D(t), \nabla^s \zeta\right)_{L^2(\Omega;\mathbb{R}_s^{d\times d})} = (\mathfrak{l}(t), \zeta)_{L^2(\Omega;\mathbb{R}^d)} + (\mathfrak{l}(t), \zeta)_{L^2(\Lambda_N;\mathbb{R}^d)}$$

for all $\zeta \in W_\Lambda^{1,p'}(\Omega; \mathbb{R}^d)$.

2: Compute the state (u, σ, z) as solution of the state equation in (12.12) with $\mathcal{G}\mathfrak{l} + \mathfrak{a} = u_D + \mathfrak{a}$ where u_D is from step 1.

3: Solve the adjoint equation in (12.18) with solution $(v_\varphi, \varphi, v_T)$.

4: Compute L as solution of (12.20).

5: Integrate L according to (12.19) to obtain ψ.

6: **return** $\psi + \alpha\mathfrak{l}$ as Riesz representative of $F'(\mathfrak{l})$.

Herein, the equations are solved according to our description above and for the integration of L to obtain ψ we use Algorithm 2.

Remark 13.6 (Mismatch between the discretization of the continuous derivative and the derivative of the discretization)**.** *Let us emphasize, that we have discretized the continuous derivative to obtain ψ from Algorithm 2, this procedure is also called "first optimize, then discretize"-approach. However, the so calculated gradient does not coincide with the gradient of the discretized objective function. Nonetheless, the mismatch does only play a minor role in our numerical experiments, see Figure 11, Table 5 and Table 6.*

13.3 Test Setting

Let us now describe our test setting, we specify the domain Ω and fix the other parameters.

Domain

We set

$$\Omega := (0, 4) \times (0, 1) \subset \mathbb{R}^2$$

with the boundaries

$$\Gamma_D := [\{0\} \cup \{4\}] \times [0, 1], \qquad \Lambda_D := (1, 3) \times [\{0\} \cup \{1\}]$$

and $\Gamma_N := \partial\Omega \setminus \Gamma_D$, $\Lambda_N := \partial\Omega \setminus \Lambda_D$. See Figure 6, therein also the prescribed boundary at $t = 1$ is depicted, the definition of $\mathfrak{a}$ is given below. The length unit of Ω is chosen as [mm].

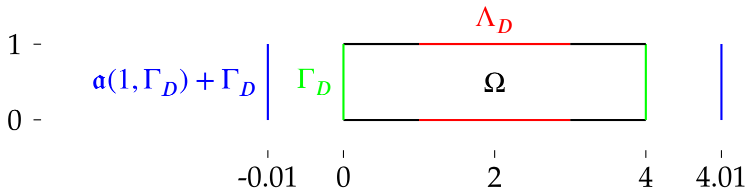


Figure 6: Domain, real (green) and pseudo (red) Dirichlet boundary. The prescribed Dirichlet boundary at $t = 1$ is depicted in blue.

Elasticity Tensor and Hardening Parameter

We choose typical material parameters of steel:

$$E = 210\left[\mathrm{kN}/\mathrm{mm}^2\right] \quad \text{(Young's modulus)}, \tag{13.7}$$

$$\nu = 0.3 \quad \text{(Poisson's ratio)}, \tag{13.8}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \approx 121.1538\left[\mathrm{kN}/\mathrm{mm}^2\right], \tag{13.9}$$

$$\mu = \frac{E}{2+2\nu} \approx 80.7692\left[\mathrm{kN}/\mathrm{mm}^2\right] \quad \text{(Lamé parameters)}, \tag{13.10}$$

$$\gamma = 0.45\left[\mathrm{kN}/\mathrm{mm}^2\right] \quad \text{(uniaxial yield stress)} \tag{13.11}$$

and define the elasticity tensor

$$\mathbb{C}\tau := \lambda \mathrm{tr}(\tau) I + 2\mu\tau$$

for all $\tau \in \mathbb{R}_s^{d\times d}$.

We choose the hardening parameter to be the identity, $\mathbb{B} := I$.

Initial Condition, End Time and Dirichlet Displacement Offset

We set $\sigma_0 = 0$ and $T = 1$. Moreover, let $\mathfrak{a}$ be given by $\mathfrak{a}(t) := t\mathfrak{a}_1$ for $t \in [0,1]$ with $\mathfrak{a}_1(x,y) := ((x-2)/200, 0)$ for $(x,y) \in \Omega$.

Optimization Problem

We set the desired stress to zero, $\sigma_d = 0$, and the Tikhonov parameter α to 10^{-4}. The parameters for the Armijo line search are chosen as typical values: $\sigma^0 = 1$, $\beta = 0.5$ and $\gamma_a = 0.01$. The tolerance TOL in Algorithm 1 is set to $5 * 10^{-4}$ and the maximal numbers of iterations m_i is set to 100.

Regarding this setting our optimization problem can be interpreted as follows. The left and right boundary of the body under consideration Ω is constantly in time pulled apart. The control $\mathfrak{l}$, respectively u_D, can alter this process for $t \in (0,1)$, but at the end (and also the beginning) the control is zero, hence, the position of the Dirichlet boundary at $t = 1$ is predefined. A solution of the optimization problem minimizes the stress at the end of the process and the used pseudo forces, respectively the Dirichlet displacement, where the focus on the minimization is on the stress (Tikhonov paramenter).

In short, a body is pulled apart and we try to minimize the stress at the end of the process by altering the movement of the Dirichlet boundary during the process.

13.4 Results

Let us finally present the numerical results. The parameters

$$\begin{aligned} &\varepsilon, \lambda, s && \text{(smoothing parameters)}, \\ &n_t, n_x, n_y && \text{(discretization parameters)} \end{aligned} \tag{13.12}$$

are not fixed and will vary during our numerical experiments. We will measure different values, in particular the calculated directional derivative in the direction of the anti gradient (that is, our search direction),

$$\nabla_{calc}(\mathfrak{l}_h) := F'(\mathfrak{l}_h)(-F'(\mathfrak{l}_h)) = -\left(F'(\mathfrak{l}_h), F'(\mathfrak{l}_h)\right)_{H^1(\mathcal{X}_c)} = -\|\dot{\psi} + \alpha\dot{\mathfrak{l}}\|^2_{L^2(\mathcal{X}_c)}, \tag{13.13}$$

the real directional derivative (approximated by the difference quotient, thus it is not the real directional derivative but we will call it so),

$$\nabla_{real}(\mathfrak{l}_h, \tau) := \frac{F(\mathfrak{l}_h - \tau\, g_h) - F(\mathfrak{l}_h)}{\tau}, \tag{13.14}$$

the relative error between the calculated and the real directional derivative,

$$\nabla_{err}(\mathfrak{l}_h, \tau) := \left| \frac{\nabla_{calc}(\mathfrak{l}_h) - \nabla_{real}(\mathfrak{l}_h, \tau)}{\nabla_{real}(\mathfrak{l}_h, \tau)} \right|, \tag{13.15}$$

and also the relative distance between the supremum norm of the deviatoric stress and the uniaxial yield stress,

$$\operatorname{dist}_{\mathcal{K}(\Omega)}(\mathfrak{l}_h) := \sup_{(t,x)\in(0,T)\times\Omega} \frac{|\sigma_h^D(t,x)| - \gamma}{\gamma}, \tag{13.16}$$

Herein, σ_h is the discrete stress, which depends on the discrete pseudo force $\mathfrak{l}_h$. Note that these values also depend on the parameters in (13.12).

Before we present the results for the optimal control problem, let us show some tests regarding our calculated gradient.

Gradient Tests

For the gradient tests we have chosen

$$\begin{aligned} &\varepsilon = 0, && \lambda = 1, && s = 10^{-8}, \\ &n_t = 128, && n_x = 64, && n_y = 16 \end{aligned} \tag{13.17}$$

as the standard values. The actual values only differ from these standard values when otherwise said, for example depicted in a table.

Let us at first have a look at Figure 7. In each of the three figures the values of our calculated and the real directional derivative at zero in the direction of the anti gradient, that is, $\nabla_{calc}(0)$ and $\nabla_{real}(0, \tau)$, are depicted with blue triangles and green circles, respectively. Note that the calculated directional derivative does not depend on τ, thus we only see a horizontal line. We have chosen n_t to be 128, 256 and 512 in the upper figure, middle figure and bottom figure, respectively. As we have agreed upon, the other parameters are chosen as in (13.17). What we already see is that $\nabla_{real}(0, \tau)$ and $\nabla_{calc}(0)$ come closer when the number of time steps is increased, at least for moderate step size τ. Furthermore, the behavior of $\nabla_{real}(0, \tau)$ dependent on τ is as expected, it seems to converge to a specific value until the numerical error becomes large for small τ (note that the change for $\tau \in \{10^{-11}, 10^{-10}, ..., 1\}$ is very small which is not visible in the figures).

Let us now describe the Tables 1 to 4. In each table one of the parameters in (13.12), except n_x and n_y, take different values as in (13.17), as depicted in the left columns. There are further shown the values (13.13) to (13.16). Moreover, in the columns α_t, α_ε, α_λ and α_s we see the negative convergence rate of $\nabla_{err}(0, 10^{-8})$, that is,

$$\alpha_x = -\frac{\log\left(\frac{\nabla_{err}(0,10^{-8})(k)}{\nabla_{err}(0,10^{-8})(k-1)}\right)}{\log\left(\frac{x(k)}{x(k-1)}\right)},$$

where $\nabla_{err}(0, 10^{-8})(k)$, respectively $x(k)$, is the corresponding value in k-th row, for $x = n_t, \varepsilon, \lambda, s$. Note that we consider the directional derivative at zero in the direction of the calculated anti gradient, which is the first descent direction of the steepest descent method.

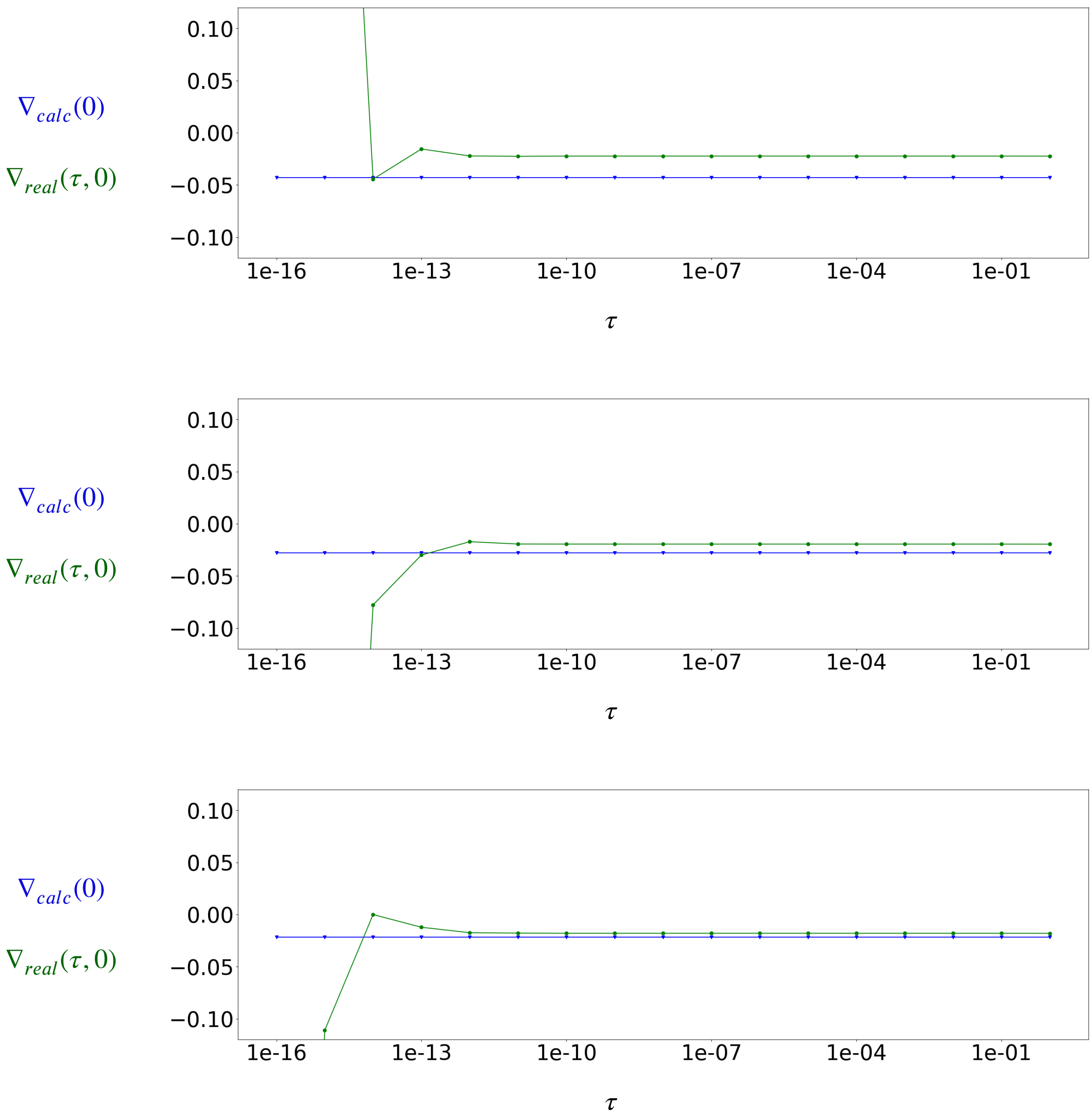


Figure 7: Comparison of the calculated and real directional derivative (in the first step of the steepest descent method).

As we can see in Table 1, when the number of time steps is increased, the relative error between the calculated and real gradient seems to go linearly to zero while the relative distance of the deviatorc stress to the yield surface reaches ≈ 0.05375. This indicates that the main reason for the mismatch between the calculated and real gradient (see Remark 13.6) is the time discretization.

In Table 2 we see the gradient test for different hardening parameters. It is clearly visible that the hardening seems to have almost no effect on the observed values (at least for small hardening parameters). Based on this observation and the goal to approximate solutions without hardening, we have set the hardening parameter to zero for the other experiments.

Table 3 shows the gradient test for different values of the Yosida approximation parameter. Here we can observe that the error between the calculated and the real directional derivative

n_t	$\nabla_{calc}(0)$	$\nabla_{real}(0, 10^{-8})$	$\nabla_{err}(0, 10^{-8})$	α_t	$\text{dist}_{\mathcal{K}(\Omega)}(0)$
4	-17.184163	-0.122446	139.340454	-	0.033011
8	-3.447526	-0.083112	40.480437	1.783317	0.039628
16	-0.773857	-0.053864	13.366766	1.598574	0.044344
32	-0.220765	-0.036898	4.983085	1.423539	0.047053
64	-0.084185	-0.027514	2.059685	1.274615	0.049776
128	-0.042897	-0.022322	0.921712	1.160036	0.051767
256	-0.027937	-0.019484	0.433841	1.087149	0.052683
512	-0.021746	-0.017969	0.210140	1.045816	0.053199
1024	-0.018956	-0.017180	0.103385	1.023329	0.053475
2048	-0.017636	-0.016776	0.051278	1.011619	0.053615
4096	-0.016995	-0.016572	0.025528	1.006230	0.053687
8192	-0.016679	-0.016469	0.012727	1.004155	0.053723
16384	-0.016522	-0.016417	0.006395	0.992925	0.053741
32768	-0.016444	-0.016391	0.003187	1.004852	0.053750

Table 1: Gradient tests with different numbers of time steps.

ε	$\nabla_{calc}(0)$	$\nabla_{real}(0, 10^{-8})$	$\nabla_{err}(0, 10^{-8})$	α_ε	$\text{dist}_{\mathcal{K}(\Omega)}(0)$
1	-0.042569	-0.021908	0.943067	-	0.053682
1e-04	-0.042897	-0.022322	0.921682	-0.002490	0.051767
1e-08	-0.042897	-0.022321	0.921768	1e-05	0.051767
1e-12	-0.042897	-0.022322	0.921709	-7e-06	0.051767
1e-16	-0.042897	-0.022322	0.921715	1e-06	0.051767

Table 2: Gradient tests with different hardening parameters.

seems again to go linearly to zero when λ is increased (which could be caused by the non-smoothness in the case $\lambda = 0$), while the distance of the deviatoric stress to the yield surface increases. The choice of λ then depends on the desired distance of the deviatoric stress to the yield surface, while the error between the calculated and the real directional derivative has to be kept in sight (it could be reduced by increasing the number of time steps).

λ	$\nabla_{calc}(0)$	$\nabla_{real}(0, 10^{-8})$	$\nabla_{err}(0, 10^{-8})$	α_λ	$\text{dist}_{\mathcal{K}(\Omega)}(0)$
0.001	-484.727737	-0.004376	110766.355335	-	0.000308
0.01	-5.392447	-0.004739	1136.799665	1.988724	0.001965
0.1	-0.128744	-0.006565	18.611006	1.785914	0.007489
1	-0.042897	-0.022322	0.921712	1.305175	0.051767
10	-0.189500	-0.174434	0.086372	1.028222	0.278028
100	-0.126870	-0.125766	0.008779	0.992936	1.168526

Table 3: Gradient tests with different Yosida approximation parameters.

Finally, in Table 4 we can observe once again that the smoothing parameter s seems to have only little effect on the observed values, therefore we set $s = 10^{-8}$ for all other experiments.

As said above, we do not show gradient tests for varying values of the space discretization. The reason is that one observes a similar behavior as in the case of ε and s, it seems that they

s	$\nabla_{calc}(0)$	$\nabla_{real}(0, 10^{-8})$	$\nabla_{err}(0, 10^{-8})$	α_s	$\text{dist}_{\mathcal{K}(\Omega)}(0)$
1	-0.024448	-0.019244	0.270470	-	-0.359187
1e-04	-0.042897	-0.022322	0.921709	0.133119	0.051766
1e-08	-0.042897	-0.022322	0.921712	0.0	0.051767
1e-12	-0.042897	-0.022322	0.921714	0.0	0.051767
1e-16	-0.041693	-0.021776	0.914627	-0.000838	0.051767

Table 4: Gradient tests with different smoothing parameters.

have only little effect on the observed values.

All in all, the time step size n_t and the value of the Yosida approximation λ are mainly of interest among the parameters in (13.12). While an increase of the number of time steps has only a positive effect, namely the reduction of the error between the calculated and the real directional derivative, a decrease of λ yields a desired smaller distance of the deviatoric stress to the yield surface but has also the negative effect of an increased error between the calculated and the real directional derivative. Note that ∇_{err} is relatively large, even for a moderate number of time steps and moderate λ, for example, in Table 1 it is almost one hundred percent for 128 time steps (and $\lambda = 1$). However, as it turns out, this error is only in the first few iterations of the steepest descent method so large, in further iterations it is much smaller (see Figure 11, Table 5 and Table 6), which relativizes the impact of this error.

Steepest Descent Method

The parameters are again chosen as in (13.17) if not otherwise said. We will consider in particular only different values for n_t and λ due to the observations of the gradient tests above.

Recall that we have set the tolerance TOL in Algorithm 1 to $5 * 10^{-4}$ and the maximal numbers of iterations m_i to 100.

Let us start with the description of Figures 9 to 13. Therein, the values of λ are chosen as depicted in Figure 8.

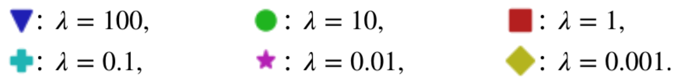


Figure 8: Symbols for different values of λ.

In Figure 9 we see the typical behavior of the steepest descent method. In the first few iterations it makes rapid progress. Then the progress in each iteration decreases until it is quite cumbersome to reduce the value of the objective function further, probably due to the well known slow convergence of the steepest descent method (*zigzagging* effect) see [62, Chapter 3.2.2] or [42, Kapitel 8.2].

In the case of the norm of the gradient (that is, $\|F'(\cdot)\|_{H^1(\mathcal{X}_c)} = \|\partial_t F'(\cdot)\|_{L^2(\mathcal{X}_c)}$) in Figure 10 we see a similar progression. However, in particular in the bottom figure we observe an "up and down" of the norm of the gradient, again indicating the zigzagging effect.

As already mentioned in the description of the gradient tests above, in Figure 11 we see that the error between the calculated and real derivative is in the first few iterations is much larger as in the following iterations, in which it is acceptable. Note that the top figure starts at iteration two.

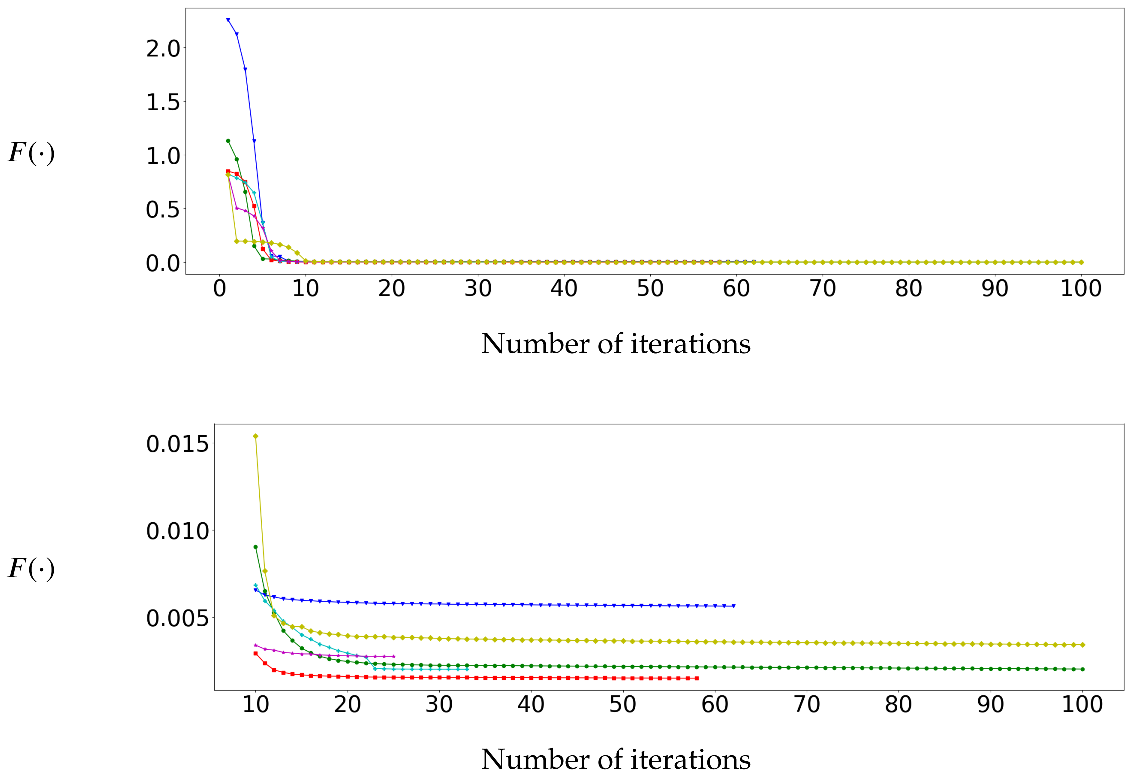


Figure 9: Comparison of the objective evaluation for different values of λ.

In Figure 12 the relative distance between the deviatoric stress and the yield surface is depicted. As was expected, the distance decreases when λ becomes smaller. Furthermore, for every λ we can observe that the distance increases in the first few iterations until the maximum is reached and it then stays almost constant. This can be explained as follows: A possible solution to the optimal control problem is to stretch the domain even further in x-direction and then to press it back so that the prescribed displacement on the Dirichlet boundary at $t = 1$ is fulfilled. Clearly, when this is the case, the maximal distance between the deviatoric stress and the yield surface will be larger as when the pseudo force, and thus the additional Dirichlet displacement, is zero. Hence, during the iterations the domain is stretched more and more until a good displacement was found and then only minor changes occur, which results in the behavior we see in Figure 12. See also Figure 15 and the description thereof below.

Finally, in Figure 13 we see the initial Armijo step size σ_0. Recall that σ_0 is adjusted in each iteration, see Algorithm 1 Line 11. Here we can observe at first a constant increase of σ_0 and then an "up and down" behavior. This is again an indicator of the zigzagging effect.

Let us now have a look at Table 5 and Table 6. These tables are similar to Table 1 and Table 3, except that we now have a column with the number of iterations instead of a column with a negative convergence rate. The calculated and real directional derivative, their error and the distance of the deviatoric stress to the yield surface are evaluated at the final iteration.

Once again, in Table 5 we can observe that the relative error between the calculated and real directional derivative decreases when the number of time steps increases while the relative distance of the deviatoric stress to the yield surface seems to reach ≈ 0.1339. As we have already seen in Figure 11, $\nabla_{err}(\cdot, 10^{-8})$ is much smaller at the end of the iteration as in the beginning, which we observed during the gradient tests.

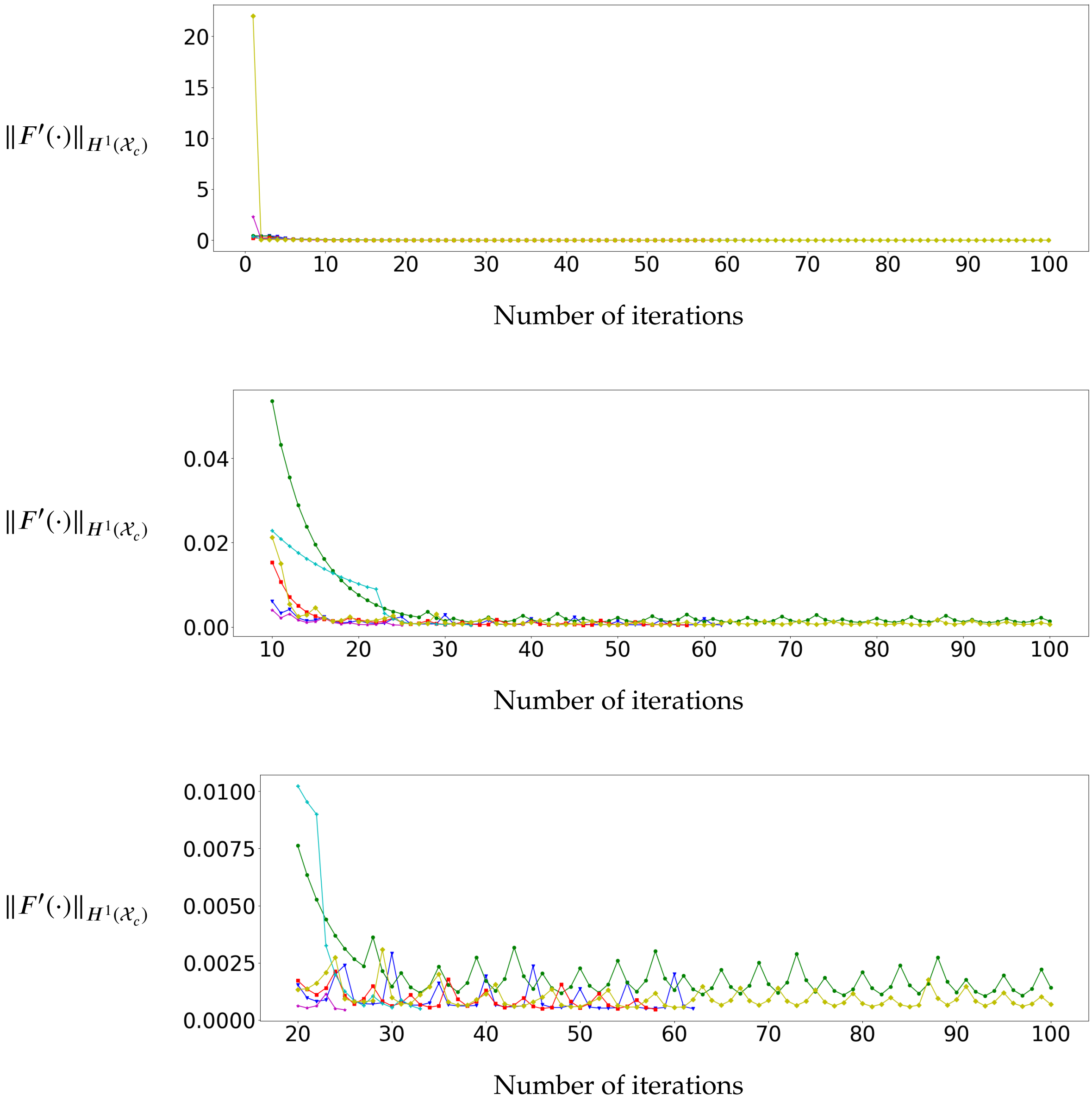
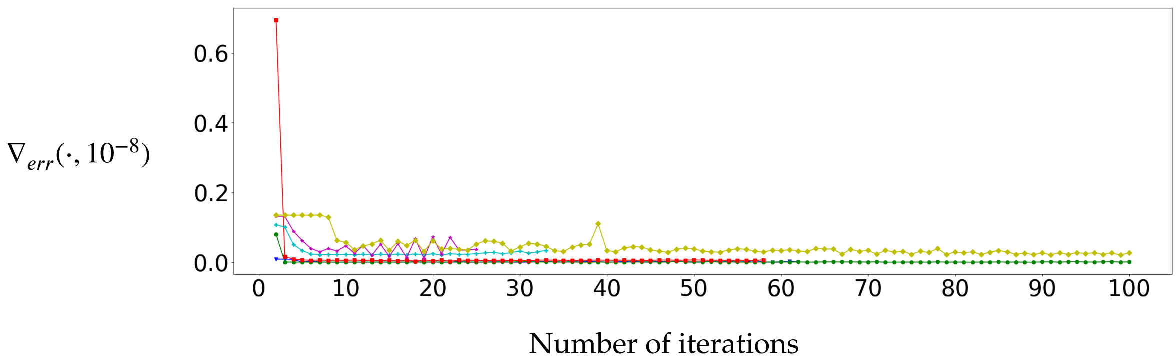


Figure 10: Comparison of the norm of the gradient for different values of λ.

Similar as in Table 3, we observe in Table 6 a decrease of $\nabla_{err}(\cdot, 10^{-8})$ and an increase of $\text{dist}_{\mathcal{K}(\Omega)}(\cdot)$ when λ becomes larger. Regarding the magnitude of $\nabla_{err}(\cdot, 10^{-8})$ in the last iteration, we see the same as described above. Note that $TOL^2 = 2.5 * 10^{-7}$ and $-\nabla_{calc}(\mathfrak{I}) = \|\dot{\psi} + \alpha\dot{\mathfrak{I}}\|^2_{L^2(\mathcal{X}_c)}$ (see (13.13)), in view of Algorithm 1 this explains that the algorithm stopped after 100 iterations in the cases $\lambda = 0.001$ and $\lambda = 10$.

Let us finally reflect upon Figure 15 and Figure 16. The color corresponds to the norm of the stress as depicted in Figure 14. In both figures we see the evolution after optimization. Here we have not stopped when the tolerance TOL or the maximal number of iterations was reached, but simply after 150 iterations. Moreover, the displacement was scaled by a factor 20 for the



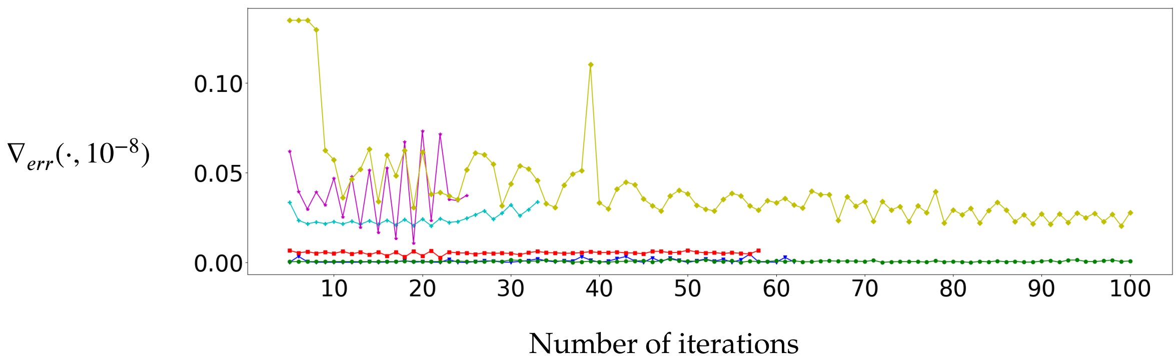


Figure 11: Comparison of the relative error between the calculated and the real directional derivative for different values of λ.

n_t	iteration	$\nabla_{calc}(\cdot)$	$\nabla_{real}(\cdot, 10^{-8})$	$\nabla_{err}(\cdot, 10^{-8})$	$\text{dist}_{\mathcal{K}(\Omega)}(\cdot)$
4	55	-2.4601e-07	-3.1816e-07	0.226817	0.0502
8	51	-2.3590e-07	-2.8903e-07	0.183828	0.0478
16	52	-2.4577e-07	-2.6541e-07	0.074012	0.0497
32	45	-2.4318e-07	-2.5225e-07	0.035941	0.1066
64	77	-2.4627e-07	-2.5056e-07	0.017121	0.1017
128	58	-2.1643e-07	-2.1790e-07	0.006773	0.1365
256	34	-2.4476e-07	-2.4562e-07	0.003481	0.1417
512	48	-2.2542e-07	-2.2541e-07	0.000045	0.1318
1024	43	-1.9258e-07	-1.9225e-07	0.001736	0.1339
2048	41	-2.3150e-07	-2.3165e-07	0.000662	0.1339

Table 5: Comparison of the numerical results for the steepest descent method with different numbers of time steps.

sake of visibility and we have chosen the following parameters:

$$\varepsilon = 0, \qquad \lambda = 1, \qquad s = 10^{-8},$$
$$n_t = 256, \qquad n_x = 128, \qquad n_y = 32$$

In Figure 15 the evolution of the displacement (form of the beam) and the norm of the stress (color) is shown at specific points in time. We observe that until $i = 84$ the norm of the stress

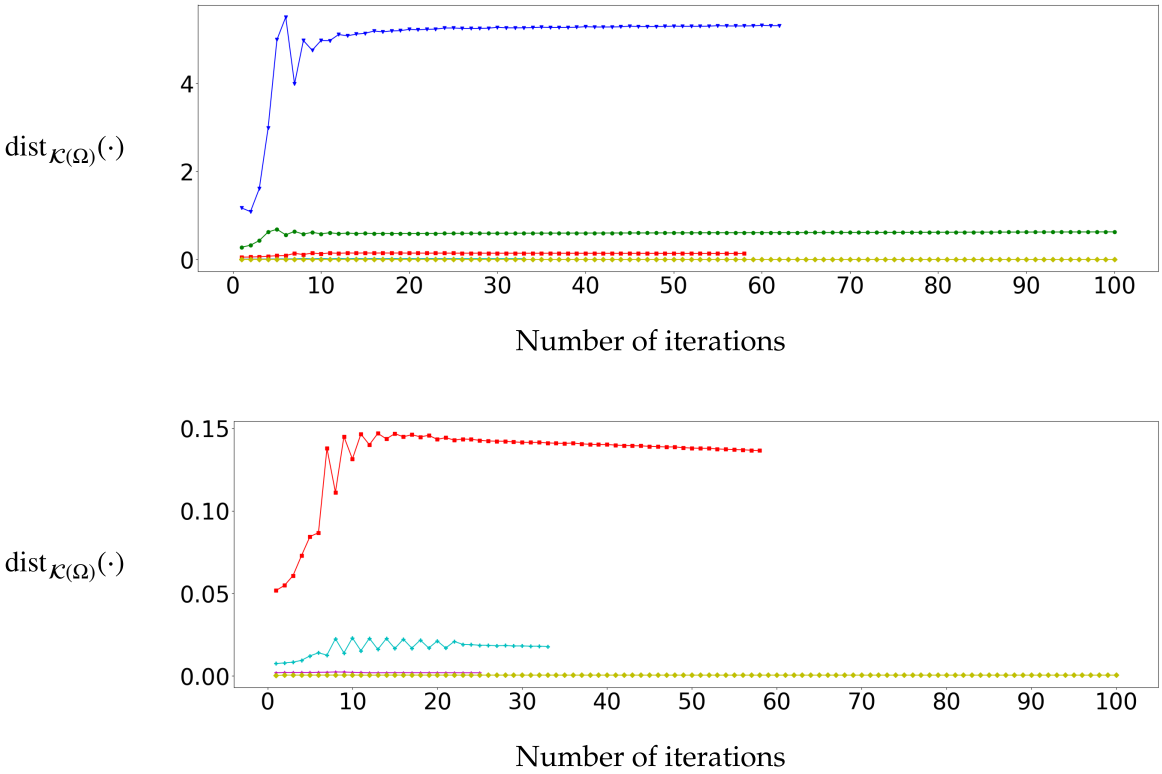


Figure 12: Comparison of relative distance between the supremum norm of the deviatoric stress and the uniaxial yield stress for different values of λ.

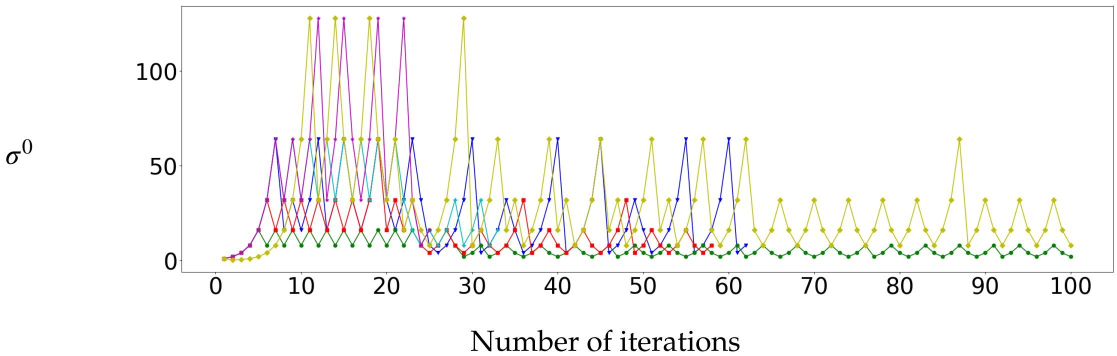


Figure 13: Comparison of the initial Armijo step size for different values of λ.

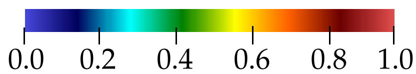


Figure 14: Legend; values in $\left[{}^{\text{kN}}/_{\text{mm}^2}\right]$.

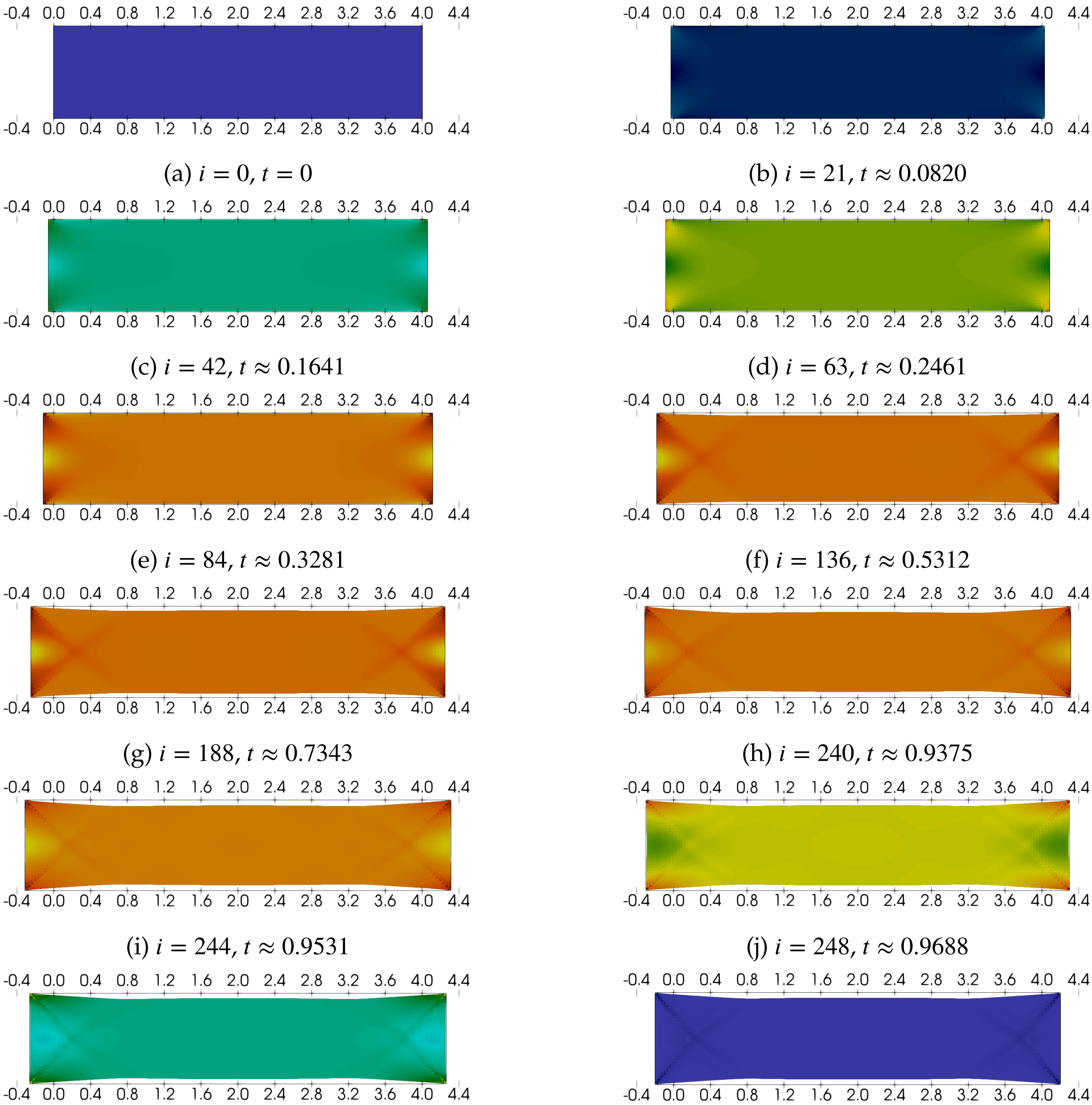
increases constantly in time. Afterwards, between $i = 84$ and $i = 240$, the yield surface is reached and the norm of the stress stays almost constant. Moreover, until $i = 240$ the beam is slowly but constantly pulled apart. From $i = 240$ on, the beam is fast pressed together and the

λ	iteration	$\nabla_{calc}(\cdot)$	$\nabla_{real}(\cdot, 10^{-8})$	$\nabla_{err}(\cdot, 10^{-8})$	$\text{dist}_{\mathcal{K}(\Omega)}(\cdot)$
0.001	100	-4.7174e-07	-4.8520e-07	0.027751	0.00048
0.01	25	-2.0089e-07	-2.0869e-07	0.037369	0.00192
0.1	33	-2.4687e-07	-2.5552e-07	0.033854	0.01781
1	58	-2.1643e-07	-2.1790e-07	0.006773	0.13652
10	100	-2.0106e-06	-2.0122e-06	0.000833	0.62584
100	62	-2.4884e-07	-2.4876e-07	0.000338	5.31148

Table 6: Comparison of the numerical results for the steepest descent method with different values of λ.

norm of the stress shrinks to almost zero as desired.

Figure 16 shows a zoom to the left Dirichlet boundary. We observe that the approximated optimal displacement of the Dirichlet boundary is not constant in vertical direction. Instead, there is a slight curvature of the Dirichlet boundary, i.e., the approximated optimal Dirichlet displacement pulling the beam in horizontal direction slightly varies in vertical direction during the evolution.



(a) $i = 0$, $t = 0$

(b) $i = 21$, $t \approx 0.0820$

(c) $i = 42$, $t \approx 0.1641$

(d) $i = 63$, $t \approx 0.2461$

(e) $i = 84$, $t \approx 0.3281$

(f) $i = 136$, $t \approx 0.5312$

(g) $i = 188$, $t \approx 0.7343$

(h) $i = 240$, $t \approx 0.9375$

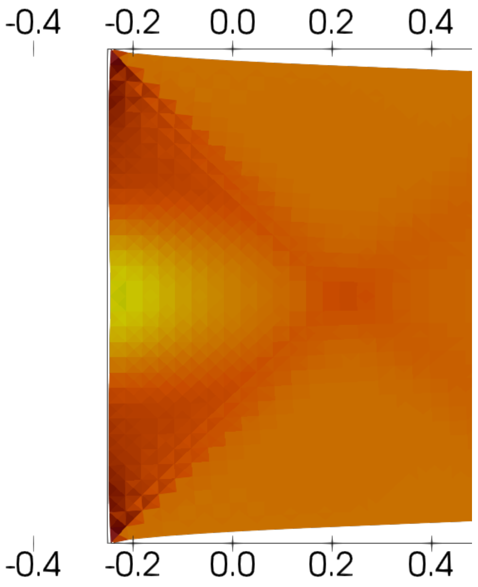
(i) $i = 244$, $t \approx 0.9531$

(j) $i = 248$, $t \approx 0.9688$

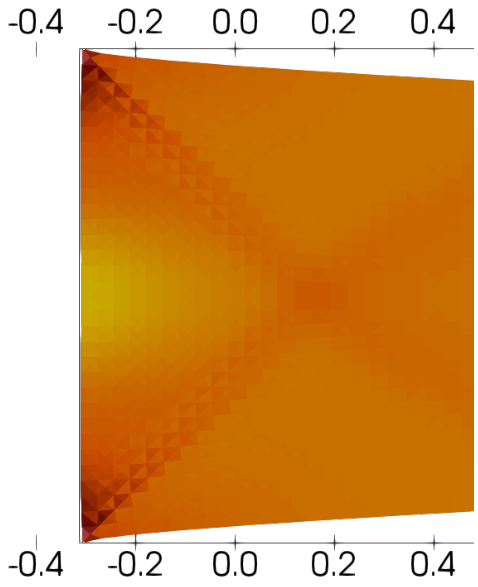
(k) $i = 252$, $t \approx 0.9844$

(l) $i = 256$, $t = 1$

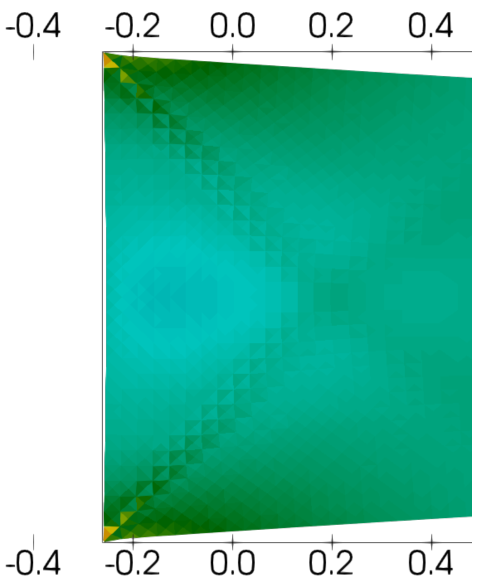
Figure 15: Evolution of $|\sigma(x,t)|$ at the i-th time step.



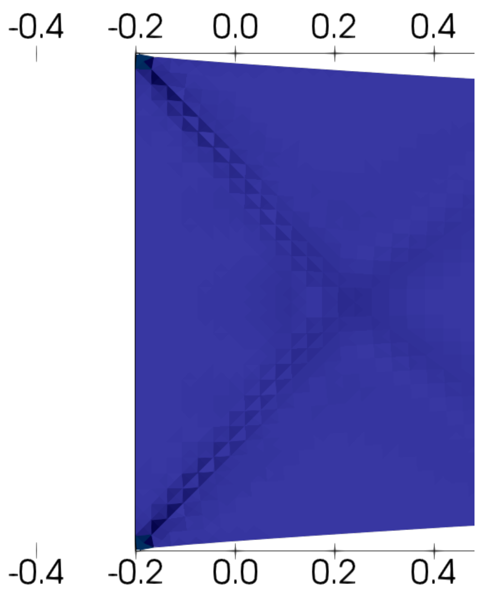
(a) $i = 188$, $t \approx 0.7343$



(b) $i = 240$, $t \approx 0.9375$



(c) $i = 252$, $t \approx 0.9844$



(d) $i = 256$, $t = 1$

Figure 16: Zoom to the left part of the beam from Figure 15.

Conclusion and Outlook

As we have seen in Chapter 2 and in Chapter 7, elasto and homogenized plasticity can both be transformed into an EVI of the form (EVI). Such an EVI was analyzed in an abstract setting in Chapter 4 and an optimal control problem constraint by this EVI was studied in Chapter 5. Therein we have provided first and second order optimality conditions and then applied the general theory up to first order optimality conditions to the case of elasto plasticity in Chapter 6. We have seen that second order sufficient conditions need the critical assumption that the exponent $\overline{p}$ in Theorem 2.5 is greater than 4 (or at least 3), see the end of Chapter 6. An application in the case of homogenized plasticity would also be possible, but this was omitted due to the similarity to elasto plasticity and the fact that this was already addressed in [71].

The findings in Chapter 4 were also used in Part IV. In contrast to elasto and homogenized plasticity, we transformed the system of plasticity with inertia into an EVI with a new maximal monotone operator $\mathcal{A}$. As it turned out, this operator is no subdifferential and also does not satisfy the boundedness property, but we had more regular loads at hand to make use of the results in Chapter 4. This operator had also a huge impact on the first order optimality conditions in the form of a KKT system. The special structure of this system arose from the form of the Yosida approximation in which the solution operator $\mathcal{T}_{R_\lambda}$ ($\mathcal{R}_\lambda$ being the resolvent of $\mathcal{A}$) occurred.

In perfect plasticity the displacement is not unique and only of bounded deformation. Moreover, the lack of hardening immediately implies that the operator Q (from Definition 2.7) is not coercive and thus most of the findings in Part II are not applicable anymore. These are the main reasons for the need for a weaker definition of a solution (essentially a weaker flow rule) and the necessity to prove again the existence and convergence properties of solutions in Chapter 11. We then provided the existence and an approximation result of global minimizers, but without external forces to avoid the safe-load condition. Due to the seen difficulty of this task, we reduced the control problem to the stress as the only state variable, which simplified the control problem thanks to the uniqueness and strong convergence properties of the stress. We thus could finally derive optimality conditions by using the findings in Section 5.2. The last chapter was devoted to numerical experiments.

In short, we have analyzed an abstract EVI in the context of optimal control and applied this to elasto and homogenized plasticity. With some variations, the case of plasticity with inertia could also be handled, but needed further treatment due to the structure of the new maximal monotone operator and its resolvent. To cover perfect plasticity, the equations needed to be analyzed in more detail and several difficulties arose due to the lesser regularity of the displacement and the safe-load condition, which resulted in a stress reduced problem for which we have proved optimality conditions and presented numerical experiments.

Nonetheless, there are still open problems and questions which may be answered in future research. Let us list some of them:

(i) The equations of homogenized plasticity arose in [88] as a (two scale) limit of equations of elasto plasticity where the material has a micro structure. Do optimal solutions to control problems constrained by these equations of elasto plasticity also converge to an optimal solution of a control problem governed by the limit system?

(ii) In [88] the inertia term was not neglected, is it possible to proceed analog to Part IV in the case of homogenized plasticity to handle an inertia term in the context of optimal control? Can convergence results analog to Item (i) be shown?

(iii) We have only considered volume forces in Part IV. As elaborated on in Remark 8.8, a straightforward adoption of the analysis, including surface forces and Dirichlet displacements is not possible since the proposed maximal monotone operator is then time dependent. Is it possible to extend the analysis in Chapter 4 to handle this case?

(iv) In Part V we have neglected surface forces altogether, see Remark 11.20. Could these be incorporated?

(v) In Remark 12.14 we have seen that volume forces could be added as controls in the stress reduced problem such that the approximation of optimal solutions is still possible. Is it then possible to derive optimality conditions?

(vi) In Chapter 13 we have only presented numerical experiments, it might be interesting to prove convergence results.

(vii) In all cases we have only provided optimality conditions for the regularized problem. An often performed procedure is to consider the limit of these optimality conditions with respect to the regularization parameter(s) to derive optimality conditions for the original problem. We have neglected this completely, such an analysis could be worthwhile.

This list of possible subjects of future research concludes the thesis.

Appendix

We collect some auxiliary results for the convenience of the reader. The necessary requirements are given in each result, therefore no assumption collection is needed.

Lemma A.1 (Uniformly convergence by pointwise convergence and Lipschitz estimate)**.** *Let $\mathcal{M}$ be a compact metric space and $\mathcal{N}$ a metric space. Furthermore, let $\{G_n\}_{n\in\mathbb{N}} \subset C(\mathcal{M};\mathcal{N})$, $G \in C(\mathcal{M};\mathcal{N})$ with $G_n(x) \to G(x)$ for all $x \in \mathcal{M}$ and suppose that G_n is uniformly Lipschitz continuous, that is, there exists a constant L such that*

$$d_{\mathcal{N}}(G_n(x), G_n(y)) \leq L d_{\mathcal{M}}(x, y)$$

for all $n \in \mathbb{N}$, $x, y \in \mathcal{M}$.

Then $G_n \to G$ in $C(\mathcal{M};\mathcal{N})$.

Proof. We argue by contradiction. Assume that there exists $\varepsilon > 0$ and a strictly monotonically increasing function $n\colon \mathbb{N} \to \mathbb{N}$, such that for all $k \in \mathbb{N}$ there exists $x_k \in \mathcal{M}$ with

$$\varepsilon \leq d_{\mathcal{N}}(G_{n(k)}(x_k), G(x_k))$$

for all $k \in \mathbb{N}$. Since $\mathcal{M}$ is compact, we can extract a subsequence x_{k_j} of x_k such that $x_{k_j} \to x$ in $\mathcal{M}$, thus

$$\begin{aligned} d_{\mathcal{N}}(G_{n(k_j)}(x_{k_j}), G(x_{k_j})) &\leq d_{\mathcal{N}}(G_{n(k_j)}(x_{k_j}), G_{n(k_j)}(x)) + d_{\mathcal{N}}(G_{n(k_j)}(x), G(x_{k_j})) \\ &\leq L d_{\mathcal{M}}(x_{k_j}, x) + d_{\mathcal{N}}(G_{n(k_j)}(x), G(x_{k_j})) \to 0, \end{aligned}$$

which gives the contradiction. □

Lemma A.2 (Compactness of ranges)**.** *Let $\mathcal{M}$ be a compact metric space and $\mathcal{N}$ a metric space. Furthermore, let $\{G_n\}_{n\in\mathbb{N}} \subset C(\mathcal{M};\mathcal{N})$, $G \in C(\mathcal{M};\mathcal{N})$ with $G_n \to G$ in $C(\mathcal{M};\mathcal{N})$. Define $U_n := G_n(\mathcal{M})$ and $U_0 := G(\mathcal{M})$. Then the set $U := \cup_{n=0}^{\infty} U_n$ is compact.*

Proof. Let $\{y_k\}_{k\in\mathbb{N}} \subset U$. Since a finite union of compact sets is compact, we can assume that there exists a subsequence $\{y_{k_j}\}_{j\in\mathbb{N}}$ and a strictly monotonically increasing function $n\colon \mathbb{N} \to \mathbb{N}$, such that $y_{k_j} \in U_{n(j)}$. Then there exists a sequence $\{x_j\}_{j\in\mathbb{N}} \subset \mathcal{M}$, with $y_{k_j} = G_{n(j)}(x_j)$. Because $\mathcal{M}$ is compact, we can select a subsequence, again denoted by x_j, and a limit $x \in \mathcal{M}$, such that $x_j \to x$, hence,

$$d_{\mathcal{N}}(y_{k_j}, G(x)) \leq d_{\mathcal{N}}(y_{k_j}, G(x_j)) + d_{\mathcal{N}}(G(x_j), G(x)) \to 0,$$

thus the proof is complete. □

The following lemma is a special case of [13, Proposition 3.32], however, for the convenience of the reader we provide the simple proof.

Lemma A.3 (Strong convergence by weak convergence and norm boundedness)**.** *Let H be a Hilbert space, $\{x_n\}_{n\in\mathbb{N}} \subset H$, and $x \in H$. If $x_n \rightharpoonup x$ and $\limsup_{n\to\infty} \|x_n\|_H \leq \|x\|_H$, then $x_n \to x$.*

Proof. This follows immediately from the weakly lower semicontinuity of the norm,

$$\limsup_{n\to\infty} \|x_n\|_H \leq \|x\|_H \leq \liminf_{n\to\infty} \|x_n\|_H \leq \limsup_{n\to\infty} \|x_n\|_H,$$

which gives norm convergence and thus the desired strong convergence. □

Lemma A.4 (Pointwise weak convergence). *Let X be a Banach space, $\{x_n\}_{n\in\mathbb{N}} \subset H^1(X)$ a sequence and $x \in H^1(X)$ such that $x_n \rightharpoonup x$ in $H^1(X)$.*

Then $x_n(t) \rightharpoonup x(t)$ in X for all $t \in [0,T]$.

Proof. According to WACHSMUTH [104, Theorem 3.1.41], the embedding $H^1(X) \hookrightarrow C(X)$ is linear and continuous, thus, for $x^* \in X^*$, $E_{x^*} : H^1(X) \to \mathbb{R}$, $E(x) := \langle x^*, x(t)\rangle$, for a fixed $t \in [0,T]$, is an element of $H^1(X)^*$. □

Lemma A.5 (Inequality preservation by weak convergence). *Let H be a Hilbert space, $\sigma_0 \in H$, $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset H^1(H)$ such that $\tau_n(0) = \sigma_0$ for all $n \in \mathbb{N}$, $a_n \to a$ in $\mathbb{R}$ and $\tau_n \rightharpoonup \tau$ in $H^1(H)$. Moreover, assume that $a_n \le -\left(\mathbb{A}\dot{\tau}_n, \tau_n\right)_{L^2(H)}$ for all $n \in \mathbb{N}$.*

Then $a \le -\left(\mathbb{A}\dot{\tau}, \tau\right)_{L^2(H)}$ holds.

Proof. Using the lower weakly semicontinuity of $\|\cdot\|_{H_\mathbb{A}}$ and Lemma A.4, we deduce

$$\begin{aligned}\liminf_{n\to\infty}\left(\mathbb{A}\dot{\tau}_n, \tau_n\right)_{L^2(H)} &= \frac{1}{2}\liminf_{n\to\infty}\|\tau_n(T)\|^2_{H_\mathbb{A}} - \frac{1}{2}\|\sigma_0\|^2_{H_\mathbb{A}}\\ &\ge \frac{1}{2}\|\tau(T)\|^2_{H_\mathbb{A}} - \frac{1}{2}\|\sigma_0\|^2_{H_\mathbb{A}} = \left(\mathbb{A}\dot{\tau}, \tau\right)_{L^2(H)},\end{aligned}$$

which immediately gives the claim. □

Lemma A.6 (Weak plus weak convergence). *Let H be a Hilbert space, $v, \tau \in H^1(H)$ and $\{v_n\}_{n\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}} \subset H^1(H)$ such that $\tau_n \rightharpoonup \tau$ in $H^1(H)$, $\tau_n(0) \to \tau(0)$, $v_n \to v$ in $L^2(H)$, $v_n(0) \rightharpoonup v(0)$ and $v_n(T) \to v(T)$ in H. Then*

$$\left(\dot{v}_n, \tau_n\right)_{L^2(H)} \to \left(\dot{v}, \tau\right)_{L^2(H)}.$$

Proof. This follows immediately from integration by parts:

$$\begin{aligned}\left(\dot{v}_n, \tau_n\right)_{L^2(H)} &= -\left(v_n, \dot{\tau}_n\right)_{L^2(H)} + \left(v_n(T), \tau_n(T)\right)_H - \left(v_n(0), \tau_n(0)\right)_H\\ &\to -\left(v, \dot{\tau}\right)_{L^2(H)} + (v(T), \tau(T))_H - (v(0), \tau(0))_H = \left(\dot{v}, \tau\right)_{L^2(H)},\end{aligned}$$

where we also used Lemma A.4 to see that $\tau_n(T) \rightharpoonup \tau(T)$ in H. □

In the next theorem we follow the lines of [38, Satz 7.2.5].

Theorem A.7 (Picard-Lindelöf for Banach spaces). *Let X be a Banach space, $r \in [1,\infty)$, $u_0 \in X$, and $B : [0,T]\times X \to X$ a function which is Lipschitz continuous in the second argument, that is, there exists a constant $L > 0$ such that for all $t \in [0,T]$ and all $v, w \in X$*

$$\|B(t,v) - B(t,w)\|_X \le L\|v - w\|_X.$$

Assume further that for an arbitrary $u \in L^r(X)$ the function

$$[0,T] \ni t \mapsto B(t, u(t)) \in X$$

is an element of $L^r(X)$.

Then there exists a unique solution $u \in W^{1,r}(X)$ of

$$\dot{u}(t) = B(t, u(t)), \qquad u(0) = u_0.$$

Proof. Let $t_0 := \min\{\frac{L}{2}, T\}$ and define

$$\mathcal{B} : L^r(0,t_0;X) \to L^r(0,t_0;X), \qquad (\mathcal{B}(u))(t) := u_0 + \int_0^t B(s,u(s))ds.$$

We have for all $u_1, u_2 \in L^r(0,t_0;X)$

$$\begin{aligned}\|\mathcal{B}(u_1) - \mathcal{B}(u_2)\|^r_{L^r(0,t_0;X)} &= \int_0^{t_0} \left\| \int_0^t B(s,u_1(s)) - B(s,u_2(s))ds \right\|_X^r dt \\ &\leq L^r \int_0^{t_0} \|u_1 - u_2\|^r_{L^1(0,t;X)} dt \leq L^r t_0^{1+\frac{r}{r'}} \|u_1 - u_2\|^r_{L^r(0,t_0;X)},\end{aligned}$$

where we used the Lipschitz continuity of B and Hölder's inequality with the conjugate exponent r'. Thus we arrive at

$$\|\mathcal{B}(u_1) - \mathcal{B}(u_2)\|_{L^r(0,t_0;X)} \leq L t_0 \|u_1 - u_2\|_{L^r(0,t_0;X)} \leq \frac{1}{2}\|u_1 - u_2\|_{L^r(0,t_0;X)},$$

hence, Banach fixed-point theorem yields the existence of a unique solution on $[0,t_0]$. Since t_0 does not depend on the initial value, one obtains a unique solution on the whole time interval after applying the above finitely many times with suitable initial values and linear shifts of B. □

Lemma A.8 (Gronwall). *Let X be a Banach space, $r \in [1,\infty)$, $u \in W^{1,r}(X)$ and $\alpha \in L^r(\mathbb{R})$ such that*

$$\|\dot{u}(t)\|_X \leq C\|u(t)\|_X + \alpha(t)$$

for almost all $t \in [0,T]$. Then the following inequalies hold

(i) $\|u\|_{C(X)} \leq \Big(\|u(0)\|_X + \|\alpha\|_{L^1(0,T;\mathbb{R})}\Big)e^{CT}$,

(ii) $\|\dot{u}(t)\|_X \leq C\Big(\|u(0)\|_X + \|\alpha\|_{L^1(0,T;\mathbb{R})}\Big)e^{CT} + \alpha(t)$ *for almost all $t \in [0,T]$ and*

(iii) $\|u\|_{W^{1,r}(X)} \leq (1+C)\Big(\|u(0)\|_X + \|\alpha\|_{L^1(0,T;\mathbb{R})}\Big)e^{CT}T^{\frac{1}{r}} + \|\alpha\|_{L^r(\mathbb{R})}$.

Proof. We have for all $t \in [0,T]$

$$\|u(t)\|_X \leq \|u(0)\|_X + \int_0^t \|\dot{u}(s)\|_X ds \leq \|u(0)\|_X + C\int_0^t \|u(s)\|_X ds + \|\alpha\|_{L^1(0,T;\mathbb{R})},$$

Gronwall's lemma (cf. [15, Lemme A.4]) yields

$$\|u(t)\|_X \leq \Big(\|u(0)\|_X + \|\alpha\|_{L^1(0,T;\mathbb{R})}\Big)e^{CT}$$

so that Item (i) and thus Item (ii) hold. Using these two inequalities, we get

$$\begin{aligned}\|u\|_{W^{1,r}(X)} &\leq \|u\|_{L^r(X)} + \|\dot{u}\|_{L^r(X)} \\ &\leq (1+C)\Big(\|u(0)\|_X + \|\alpha\|_{L^1(0,T;\mathbb{R})}\Big)e^{CT}T^{\frac{1}{r}} + \|\alpha\|_{L^r(\mathbb{R})},\end{aligned}$$

which completes the proof. □

Remark A.9 ($BD(\Omega)$ lacks the Radon-Nikodým property)**.** *The space of bounded deformation $BD(\Omega)$ does not have the Radon-Nikodým property.*

To see this we use two equivalent statements to the Radon-Nikodým property given in [31, VII.6 (3) & (5)] (there are actually over 20 equivalent statements). At first we use the fact that $BD(\Omega)$ has the Radon-Nikodým property if and only if every separable closed linear subspace has it and consider

$$LD(\Omega) := \{u \in BD(\Omega) : \nabla^s u \in L^1(\Omega; \mathbb{R}^{d\times d}_s)\}.$$

Note that this space is treated in TEMAM *[98, Chapter I.1.3] and that it is a closed subspace of $BD(\Omega)$ (cf.* TEMAM *[98, Chapter II.2 Equation 2.31]). Now we use the second equivalent statement from which follows that $LD(\Omega)$ has the Radon-Nikodým property if and only if every absolutely continous function $f : [0,1] \to LD(\Omega)$ is differentiable almost everywhere with $f(b) - f(a) = \int_a^b f'(t)dt$ for any $a, b \in [0,1]$. To contradict this we consider for simplicity the one dimensional case $\Omega = [0,1]$ and define*

$$f : [0,1] \to LD(\Omega), \qquad t \mapsto \left[[0,1] \ni x \mapsto \begin{cases} x, & \text{if } x \le t \\ t, & \text{if } x > t \end{cases} \right]$$

with the derivative

$$\nabla^s(f(t)) = \mathbb{1}_{[0,t]}$$

for all $t \in [0,1]$, that is, $\nabla^s(f(t))$ is one on $[0,t]$ and otherwise zero. One easily verifies that

$$\|f(t+h) - f(t)\|_{L^1([0,1];\mathbb{R})} = \big((1 - (t + h/2)\big)h \le h$$

and

$$\|\nabla^s(f(t+h) - f(t))\|_{L^1([0,1];\mathbb{R})} = h$$

for all $t, h \in [0,1]$ with $t + h \in [0,1]$, so that f is in particular absolutely continuous. If f were differentiable in $t \in [0,1)$, then for the function

$$g : [0, 1-t] \to L^1([0,1];\mathbb{R}), \qquad h \mapsto \frac{\nabla^s(f(t+h) - f(t))}{h}$$

the convergence $g(h) \to G$, as $h \searrow 0$, would hold in $L^1([0,1];\mathbb{R})$ for some $G \in L^1([0,1];\mathbb{R})$. However, we also have

$$\int_0^1 \phi(x)g(h)(x)dx = \frac{1}{h}\int_t^{t+h} \phi(x)dx \to \phi(t),$$

as $h \searrow 0$, for all $\phi \in C([0,1];\mathbb{R})$, which gives the contradiction.

List of Symbols

Numbers

$\mathbb{N}$	Natural numbers without zero
$\mathbb{R}$	Real numbers
∞	Infinite
d	The dimension of the domain Ω, an element of $\mathbb{N}$
E	Young's modulus, defined in (13.7)
ν	Outward unit normal vector (see Page 8), also Poisson's ratio given in (13.8)
λ	Mostly used for the regularization parameter of the Yosida approximation, also used for one of the Lamé parameters, given in (13.9)
μ	Used for the regularization parameter of the Yosida approximation, also used for one of the Lamé parameters, given in (13.10)
γ	Uniaxial yield stress, see Definition 2.2 and (13.11)
ρ	Density, used in Part IV
ε	Used for different purposes, in particular for vanishing hardening in Part V
s	Used as a smoothing parameter, see Section 3.2
p'	Conjugate exponent of $p \in [1, \infty]$, that is, $1/p + 1/p' = 1$, where $1/\infty := 0$
T	End time of the considered time horizon
α	Tikhonov parameter, present in most of the considered optimal control problems
σ^0	Armijo line search parameter, see Algorithm 1
β	Armijo line search parameter, see Algorithm 1
γ_a	Armijo line search parameter, not to be confused with the uniaxial yield stress γ, see Algorithm 1
γ_G	Coercivity constant corresponding to a coercive operator $G \in \mathcal{L}(H)$ on a Hilbert space H, not to be confused with the uniaxial yield stress γ or the Armijo line search parameter γ_a
n_t	Number of time steps in Chapter 13
n_x	Number of rectangles in x-direction in the Discretization in Chapter 13
n_y	Number of rectangles in y-direction in the Discretization in Chapter 13
d_t	Time discretization step length in Chapter 13, equals $1/n_t$

Sets

K	Set of admissible stresses in $\mathbb{R}^{d\times d}_s$, given in Definition 2.2
$\mathcal{K}(\Omega)$	Set of admissible stresses in $L^2(\Omega;\mathbb{R}^{d\times d}_s)$, given in Definition 2.2
$D(A)$	Domain of a maximal monotone operator A
$R(A)$	Range of a maximal monotone operator A
M	Nonenempty and closed subset of $D(A)$, used only in Part II
$\mathcal{A}_{\mathcal{L}}(z_0, M)$	Set of admissible loads, given in Definition 4.4
$\mathcal{A}_{\mathcal{L}}$	Subset of $\mathcal{A}_{\mathcal{L}}(z_0, M)$ which contains only loads which belong to the kernel of R at $t = 0$, given in Assumption ⟨5.2.iii⟩
$\mathcal{S}_{\mathcal{L}}$	A set which fulfills a global safe-load condition, see Definition 11.13 and Assumption ⟨11.2.ii⟩, only used in Chapter 11
$\mathcal{B}(S)$	The (smallest) σ-Algebra containing all open sets in S
$\operatorname{span}(S)$	The set of all (finite) linear combinations of elements in S
$B^X_r(y)$	The ball around y with radius r in a normed space X, that is, the set $\{x \in X : \|x-y\|_X \le r\}$, may be simply written as $B_r(y)$

Domain

Ω	The domain, a subset of $\mathbb{R}^d$. It represents mostly a deformable, continuously distributed body
$\partial\Omega$	Boundary of Ω
Γ	The same as $\partial\Omega$
Γ_D	Dirichlet boundary
Γ_N	Neumann boundary
Λ_D	Pseudo Dirichlet boundary, only used in Chapter 13
Λ_N	Pseudo Neumann boundary, only used in Chapter 13

Spaces

$\mathbb{V}$	Finite dimensional Banach space, used for homogenized plasticity, representing internal variables (such as hardening) in Chapter 7
$L^p(\Omega; X)$	The space of p-integrable Lebesgue functions into a finite dimensional Banach space X
$L^p(\Gamma_N; X)$	The space of p-integrable Lebesgue functions on the Neumann boundary into a finite dimensional Banach space X
$W^{k,p}(\Omega; X)$	The space of p-integrable Sobolev functions, which are k-times differentiable, into a finite dimensional Banach space X
$W^{1,p}_D(\Omega; X)$	Subspace of $W^{1,p}(\Omega; X)$ which contains all functions whose traces vanishes on the Dirchlet boundary
$W^{1,p}_\Lambda(\Omega; X)$	Analog to $W^{1,p}_D(\Omega; X)$ but with vanishing traces on the pseudo Dirichlet boundary, only used in Chapter 13
$W^{-1,p}_D(\Omega; X)$	Dual space of $W^{1,p'}_D(\Omega; X)$
$W^{-1,p}_\Lambda(\Omega; X)$	Dual space of $W^{1,p'}_\Lambda(\Omega; X)$, only used in Chapter 13

$H^k(\Omega; X)$		Abbreviation for $W^{k,2}(\Omega; X)$
$H^1_D(\Omega; X)$		Abbreviation for $W^{1,2}_D(\Omega; X)$
$H^1_\Lambda(\Omega; X)$		Abbreviation for $W^{1,2}_\Lambda(\Omega; X)$, only used in Chapter 13
$H^{-k}(\Omega; X)$		Abbreviation for $W^{-k,2}(\Omega; X)$
$H^{-1}_D(\Omega; X)$		Abbreviation for $W^{-1,2}_D(\Omega; X)$
$H^{-1}_\Lambda(\Omega; X)$		Abbreviation for $W^{-1,2}_\Lambda(\Omega; X)$, only used in Chapter 13
$C(\Omega)$		Continuous functions on Ω
$C_0(\Omega)$		Continuous functions on Ω which are "zero on the boundary" (see (10.1))
$C_c(\Omega)$		Continuous funcitons on Ω which have compact support
$C^k(\Omega)$		Continuous funcitons on Ω which are k-times differentiable ($k \in \mathbb{N}$)
$C^\infty(\Omega)$		Continuous funcitons on Ω which belong to $C^k(\Omega)$ for every $k \in \mathbb{N}$
$C^\infty_c(\Omega)$		Intersection between $C_c(\Omega)$ and $C^\infty(\Omega)$.
$C(\Omega; X)$		Continuous functions on Ω with values in a finite dimensional Banach space X, an analog notation is used for the subsets of $C(\Omega)$ above
$V_0(\Omega \times Y; \mathbb{R}^d)$		Function space for the micro displacement in homogenized plasticity, given in Definition 7.1, used only in Chapter 7
$M(\Omega)$		Set of regular real Borel measures, dual space of $C_0(\Omega)$, only used in Part V
$M(\Omega; X)$		Regular real Borel measures with values in a finite dimensional Banach space X, dual space of $C_0(\Omega; X)$, only used in Part V
$BD(\Omega)$		The space of bounded deformation, only used in Part V
$\mathcal{X}$	m	A Banach space in Part II, set to $W^{-1,p_1}_D(\Omega; \mathbb{R}^d) \times W^{1,p_1}(\Omega; \mathbb{R}^d)$ in Chapter 6 and to $W^{-1,p}_\Lambda(\Omega; \mathbb{R}^d)$ in Chapter 13, a possible choice in Chapter 7 is $H^{-1}_D(\Omega; \mathbb{R}^d)$ and $L^2(\Omega; \mathbb{R}^d)$ in Part IV
$\mathcal{X}_c$	m	A Banach space which is compactly embedded into $\mathcal{X}$ in Part II, set to $L^2(\Omega; \mathbb{R}^d) \times L^2(\Gamma_D; \mathbb{R}^d) \times H^2(\Omega; \mathbb{R}^d)$ in Chapter 6 and to $L^2(\Omega; \mathbb{R}^d) \times L^2(\Lambda_N; \mathbb{R}^d)$ in Chapter 13, a possible choice in Chapter 7 is $L^2(\Omega; \mathbb{R}^d) \times L^2(\Gamma_D; \mathbb{R}^d)$
$\mathcal{Y}$	m	A Banach space in Part II which is continuously embedded into $\mathcal{Z}$, set to $L^{p_1}(\Omega; \mathbb{R}^{d\times d}_s)$ in Chapter 6, denoted by $\mathcal{Y}_p$ in Part IV and the choice in Part V would be $L^p(\Omega; \mathbb{R}^{d\times d}_s)$
$\mathcal{Y}_p$		Set to $W^{1,p}_D(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathbb{R}^{d\times d}_s)$, used only in Part IV
$\mathcal{Z}$	m	A Banach space in Part II which is continuously embedded into $\mathcal{H}$, only needed for second order optimality conditions in Part II, set to $\mathcal{H}$ in all other parts
$\mathcal{H}$	m	A Hilbert space in Part II, set to $L^2(\Omega; \mathbb{R}^{d\times d}_s)$ in Chapter 6 (which would also be the choice in Part V), the obvious choice in Chapter 7 would be $L^2(\Omega \times Y; \mathbb{V})$, and set to $H^1_D(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega; \mathbb{R}^{d\times d}_s)$ in Part IV
$\mathcal{W}$	m	A Banach space in Part II such that $\mathcal{H}$ is continuously embedded into $\mathcal{W}$, only needed for second order optimality conditions in Part II, set to $\mathcal{H}$ in all other parts

$L^2(0,t;X)$	The space of square-integrable Bochner functions on the time intervall $[0,t]$ into a Banach space X
$L^2(X)$	Abbreviation for $L^2(0,T;X)$
$H^1(0,t;X)$	The space of Bochner-Sobolev functions on the time intervall $[0,t]$ into a Banach space X
$H^1(X)$	Abbreviation for $H^1(0,T;X)$
$H^1_0(X)$	Subset of $H^1(X)$ which contains all functions who vanishes at $t=0$
$H^1_{00}(X)$	Subset of $H^1_0(X)$ which contains all functions who vanishes at $t=0$ and $t=T$
$C(0,t;X)$	The space of continuous functions on the time intervall $[0,t]$ into a Banach space X
$C(X)$	Abbreviation for $C(0,T;X)$
$\mathfrak{X}_c$	Subspace of $H^1(L^2(\Omega;\mathbb{R}^d))$, used as a control space in Part IV, later assumed to have a certain compactness into $L^1(L^2(\Omega;\mathbb{R}^d))$, possible choices are given in Example 9.2
$\mathcal{V}$	The space of velocity fields in the case of perfect plasticity in Part V
$\mathcal{U}$	The space of displacement fields in the case of perfect plasticity in Part V
$\mathcal{L}(X;Y)$	The space of linear and continuous operators from a Banach space X into another Banach space Y
$\mathcal{L}(X)$	Abbreviation for $\mathcal{L}(X;X)$

Functions

f	Volume force applied on Ω
g	Neumann boundary force applied on Γ_N
u_D	Dirichlet displacement
u	Displacement of Ω
e	Elastic strain, relationship to σ is $e = \mathbb{A}\sigma$ (Hooke's law), rarely used
σ	Stress, that is, internal forces of Ω
z	Plastic strain
Σ	Stress in the case of homogenized plasticity, used only in Chapter 7
v	Displacement reflecting the microstructure in the case of homogenized plasticity, used only in Chapter 7
u_0	Initial condition for the displacement
σ_0	Initial condition for the stress
z_0	Initial condition for the plastic strain
u_d	Desired displacement, used in (12.1)
σ_d	Desired stress, used in (12.12)
z_d	Desired plastic strain, used in (6.1)
ℓ	Abstract loads \ forces, mostly set to one or more of the functions f, g or u_D
$\mathfrak{l}$	Pseudo forces, used only in Part V to generate Dirichlet displacements

$\mathfrak{a}$		Offset of the Dirichlet displacement, used only in Part V, required only in the application in Chapter 13

Tensors

$\mathbb{C}$	m	Elasticity tensor, given in Chapter 1 for all cases except homogenized plasticity and on Page 63 for homogenized plasticity
$\mathbb{A}$		Inverse of the elasticity tensor (the one for all cases except homogenized plasticity)
$\mathbb{B}$	m	Hardening parameter given in Chapter 1 for all cases except homogenized plasticity and on Page 63 for homogenized plasticity
$\mathbb{D}$		Combination of $\mathbb{C}$ and $\mathbb{B}$, used for plasticity with inertia, defined in (7.6)
$\mathbb{E}$		Combination of $\mathbb{C}$ and $\mathbb{B}$, used for plasticity with inertia, defined in (7.6)

Operators

∇^s		Symmetrized gradient (for elasto plasticity), given in Definition 2.1
∇^s_x		Symmetrized gradient (for homogenized plasticity) with respect to the x-variable, defined on Page 63
∇^s_y		Symmetrized gradient (for homogenized plasticity) with respect to the y-variable, defined on Page 63
$\nabla^s_{(x,y)}$		Symmetrized gradient (for homogenized plasticity) with respect to the x- and y-variable, given in Definition 7.3
div		Divergence operator (for elasto plasticity), given in Definition 2.1
$\mathrm{div}_{(x,y)}$		Divergence operator (for homogenized plasticity) with respect to the x- and y-variable, given in Definition 7.3
max		The maximum of a given function and zero
$\max_s$		Smoothing of max, given in (3.10)
dist_S		The distance to a set S
arg min		The argument which minimizes a further specified function
ker		Kernel of a following operator
$\mathfrak{Q}$	m	Maps z to q for certain given data, given in Definition 4.2 for general EVIs and in Definition 8.2 for plasticity with inertia
$\mathfrak{Z}$	m	Maps q to z for certain given data, given in Definition 4.2 for general EVIs and in Definition 8.2 for plasticity with inertia
$\partial I_{\mathcal{K}(\Omega)}$		Subdifferential of the indicator function of the set $\mathcal{K}(\Omega)$
∂I_λ		Yosida approximation of $\partial I_{\mathcal{K}(\Omega)}$, given in (3.5), see also (3.8)
$\partial I_{\lambda,s}$		Smoothing of ∂I_λ, defined in (3.11)
π_K		Projection onto a convex set $K \subset H$ in a Hilbert space H
A		Maximal monotone operator, set to $\partial I_{\mathcal{K}(\Omega)}$ in most cases
A_n		A sequence of maximal monotone operators, set to the (smoothed) Yosida approximation in most cases
A^0		Projection operator of $A(\cdot)$ onto zero
A_λ		Yosida approximation of a maximal monotone operator A

A_0		Abbreviation for A
A_s		Smoothing of A_λ, when $A = \partial I_{\mathcal{K}(\Omega)}$ then we choose $A_s = \partial I_{\lambda,s}$ where $\partial I_{\lambda,s}$ is given in (3.11)
$\mathcal{A}$		A special Maximal monotone operator, given in Definition 8.5, used only in Part IV
$\mathcal{A}_\lambda$		Yosida approximation of $\mathcal{A}$, specified in Corollary 9.4
$\mathcal{A}_s$		Smoothing of the Yosida approximation of $\mathcal{A}$, given in Definition 9.6
R_λ		Resolvent of a maximal monotone operator A
R_s		Smoothing of R_λ, given in (3.13) in the case of the von-Mises flow rule
$\mathcal{R}_\lambda$		Resolvent of $\mathcal{A}$, specified in Corollary 9.4
$\mathcal{R}_s$		Smoothing of the resolvent of $\mathcal{A}$, given in Definition 9.6
$\mathcal{F}$		Integral operator, given in Definition 8.14, used only in Part IV (and Proposition 12.7)
$\mathcal{F}_\rho$		Equals $\mathcal{F}$ divided by the density ρ, given in Definition 8.14, used only in Part IV
$\mathcal{T}$	m	Solution operator of linear elasticity or a variation, given in the case of elasto (and perfect) plasticity in Corollary 2.6 and in the case of homogenized plasticity in Definition 7.5, for the case of plasticity with inertia see $\mathcal{T}_{R_0}$ below
$\mathcal{T}_{R_0}$		Solution operator of certain nonlinear elasticity with respect to a monotone and Lipschitz continuous operator R_0, given in Proposition 8.10, used only in Part IV
$\mathcal{G}$		Used in Part V as the pseudo force to Dirichlet displacement mapping whereby a realization is given in (13.1)
R	m	Linear and continuous operator from $\mathcal{X}$ to $\mathcal{H}$, not to be confused with the resolvent of a maximal monotone operator, assumed to map into $\mathcal{Y}$ to derive optimality conditions, given in Definition 2.7 for elasto plasticity, in Definition 7.5 for homogenized plasticity, in Definition 8.5 for plasticity with inertia and the choice for perfect plasticity would be the concatenation of R from elasto plasticity and the pseudo force to Dirichlet displacement mapping $\mathcal{G}$ introduced in Assumption $\langle 12.\text{iv}\rangle$
Q	m	Linear and continuous operator from $\mathcal{H}$ to $\mathcal{H}$, assumed to map from $\mathcal{Y}$ into $\mathcal{Y}$ and from $\mathcal{Z}$ into $\mathcal{Z}$ to derive optimality conditions, given in Definition 2.7 for elasto plasticity, in Definition 7.5 for homogenized plasticity and in Definition 8.5 for plasticity with inertia
Q_ε		Linear and continuous mapping from $L^2(\Omega;\mathbb{R}^{d\times d}_s)$ into $L^2(\Omega;\mathbb{R}^{d\times d}_s)$, given in (11.11), used ony in Part V for vanishing viscosity
$\mathcal{S}$		Solution operator of the state equation, given in Definition 5.1 for general EVIs
$\mathcal{S}_n$		Solution operator of an EVI with respect to a maximal monotone operator A_n, defined in (5.4)
$\mathcal{S}_{\lambda_n}$		Solution operator of an EVI with respect to a Yosida approximation A_{λ_n}, defined in (5.5)

$\mathcal{S}_s$	m	Solution operator for a smoothed EVI, defined in (5.9) for general EVIs, in Definition 9.10 for plasticity with inertia and in Definition 12.16 for perfect plasticity

Objective Functions

$\boldsymbol{J}$	m	The objective function, mostly composed of two parts Ψ and Φ, given in Assumption $\langle 5.\text{iv}\rangle$ for general EVIs, in (6.1) for elasto plasticity, in Assumption $\langle 9\rangle$ for plasticity with inertia, in Assumption $\langle 12.\text{iii}\rangle$ for perfect plasticity and in Assumption $\langle 12.2\rangle$ for stress reduced perfect plasticity
Ψ	m	Part of the objective function $\boldsymbol{J}$, depends on the state(s), may also depend on the control(s), for usages see $\boldsymbol{J}$
Φ	m	Part of the objective function $\boldsymbol{J}$, depends only on the control(s), for usages see $\boldsymbol{J}$

Miscellaneous Symbols

$\to$	Used for (strong) convergence
$\rightharpoonup$	Used for weak convergence
$\hookrightarrow$	Continuously embedded
$\overset{c}{\hookrightarrow}$	Compactly embedded

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