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Synergies, Cooperation and Syndication in Venture Capital Game, Portfolio Optimization with Genetic Algorithms and Asset Auctions:
Essays in Finance

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Table of Contents

List of Figures

List of Tables

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1 Introduction

The fast-growing field of behavioral finance uses insights from different scientific fields to understand the impact of human behavior on an individuals' decision-making ability and the subsequent impact on organizations, investments, and markets. Interdisciplinary studies combine findings from psychology, medicine, mathematics, and physics, to explain what guides our choices, including simple day-to-day tasks, such as what level of effort would we exert in solving a particular problem, to more complex ones, such as why does a venture capitalist opt for a risky project? Why don't bidders in the auction proceed with the bidding? Why do investors confuse between a good company and a good stock? Do start-up founders care less about their product after giving out some share to the investors? Why do people increase the amount of risk they are willing to take based on good or bad performance?

In finance, we study how individuals and organizations acquire and allocate resources, maximizing their utility, while considering the associated risks or exposure. Previously, research analysts mostly relied on psychological influences; however, in recent decades, financial trends have moved away from social sciences and toward natural sciences. According to natural sciences, the universe follows a logical order. Thus, financial decision-making is usually modeled and based on assumptions regarding the behavior of individuals and markets.

This thesis looks at all those scientific phenomena from the both empirical and theoretical side, with empirical trying to strengthen theoretical assumptions or even to expand it like it is shown in chapter 3, where theory proposed in the previous chapter expands with one additional focus - on the venture capital syndication. We tried to fortify theoretical assumptions in chapter 5 with empirical analysis with various broadband auctions, while chapter 4 bases its findings solely on empirical finings developed through computer simulations.

In recent decades, signaling theory is, in combination with empirical studies, widely used to model decision-making strategies of various players in the funding of new ventures. Innovative start-up companies often face difficulties in obtaining finance from traditional sources, such as bank loans or public stock markets. Lately, many startup firms have received seed financing from wealthy individuals like angles or organizations, such as venture capitalists (VCs). However, there is a lack of academic understanding regarding the economic and behavioral factors that motivate not only an entrepreneur's choice of financier but also the level of effort that a financier can expect from the entrepreneur after successful financing. Previous studies have discussed the adverse effects of conflicts between the financier and the entrepreneur on the venture's performance. This battle of interests creates a moral hazard issue, a term often used in economics to describe situations where parties act in their self-interest, regardless of its effect on the other parties involved (Elitzur and Gavious, 2003).

In chapter 2, we propose a two-stage financing model with three players that consider the output elasticities of all parties using the Cobb-Douglas utility function. The entrepreneur can observe the previous track record of potential investors to obtain knowledge about their response later in the game. Financier observes output elasticities of two other players after the first financing round, adapting their strategy for negotiations in the second round of funding. Additionally, we introduce the complementary aspect of the project, showing that synergies between existing and added management influence the three-player game dynamics and, thus, the exit stage value of the project.

Theoretical findings in chapter 2 suggest that a higher complementary coefficient between players on both stages can lead to a higher level of effort from all three players, taking game dynamics away from the moral hazard problem and causing higher exit stage payoffs. Previous track record of the angel and VC and output elasticity of the entrepreneur, combined with the company's shares offered the angel and VC, impact the three-player game dynamic, causing some players to reduce their efforts after specific funding rounds.

This study also provides insight into the signaling aspect of the angel's reputation during the VC funding round. Specifically, our research reveals that the higher company share of the angel would increase the level of the effort of the VC until a point dependent on the output elasticities of the angel and the entrepreneur. However, higher output elasticity of the angel would lead to a decrease in the level of the VC's effort. For practitioners, this finding implies that VCs could see firms with higher angel backing as firms whose founders are less prone to cash-in early and leave. Additionally, VCs perceive that the angels with highly effective mentoring and management would do 'the heavy lifting' instead of their management.

The next section empirically expands on the theoretical proposal with the focus on the venture capital syndication. Our empirical results show that VC syndication increases the average amount of funding offered to entrepreneurs as well as that syndicated ventures have a higher number of funding rounds, resulting in a higher number of possible entry-points provided by those start-ups. At the same time, investors capitalize on less exposure to a specific company's risk, enabling them to fund additional projects and increase their expected returns. In this chapter, we also investigated the influence of academic level of founder and employees on the chance of syndication. Our empirical results suggest that academic titles of higher grade, like PhDs have an influence on the opportunity of syndication, implying that knowledge of entrepreneurs, portrayed by degrees, serves as a signaling factor for venture capitalists and possibly also angels.

Based on the Basel Capital Accords, the Capital Requirements (CR) for market risk exposure of banks are a nonlinear function of Expected Shortfall (ES). In chapter 4, we used ES as a risk measure for the portfolio selection problem, using genetic algorithms (GA) to improve and iterate portfolios. Although GAs are widely used across the literature to improve the portfolio's performance, to our knowledge, this is the first paper that studies ES portfolio optimization using the actual portfolio and various GA techniques. We evaluated the best setting for this problem by combining different evolution schemes. When a crossover was set to single-point crossover, non-uniform mutation outperformed one-point and two-point mutations. Our results suggest that one-point mutation was outperformed heavily by the other two methods in terms of convergence. Conversely, when using a linear crossover, one-point mutation outperformed the other two methods. Next, we tested The Strength Pareto Evolutionary Algorithm 2 (SPEA2) of Zitzler et al. (2002). Our results suggested that a two-point GA that minimized the risk for a given level of expected return slightly outperformed the results of the SPEA2. Compared with the previous industry standard for risk measure—Value-at-Risk, we show that both frontiers differed, especially at the low return side. The converted Value-at-Risk solutions were not evenly distributed along the efficient frontier and even inadequate for some ES values.

Since its establishment, the game theory has been closely tied to the financial decision-making process. Game theory, the part of economics that studies the "rules of the game," provides a framework in which interacting choices of economic agents produce outcomes with respect to the preferences of those agents, where the outcomes in question might have been intended by none of the agents. Thus, to cope with such complications, it is necessary to pay careful attention to the formal details of a specific market, as well as define a new dataset to supplement the traditional analytical theories (Roth 2002).

Despite the long history of auctions dating to ancient Greece, the first formal analysis of auctions began in 1961 and was the basis of the ground-breaking work of Nobel Prize winner William Vickrey. In the decades that followed, economists devoted significant resources to develop a better understanding of sales by auction, from the standpoint of both buyers (bidding strategy) and sellers (auction design). In chapter 5, we use the game theory approach to examine the first-price package auction design for illiquid asset auctions. Our theoretical work suggests that every case that can be presented as a two or three asset game, as well as longer games that can be presented as two and three asset subgames, has a strong equilibrium if the bidders' budgets and utilities for every asset are common knowledge. The empirical part of chapter 5 uses various Federal Communications Commission broadband auctions to illustrate how, depending on the relationship between certain utilities and budgets, does every version of the game shows that every player can create an optimal strategy, which does not require bidding until their utility for every asset, lowering exposure, and preserving them from obtaining unwanted packages. Furthermore, the empirical evidence shows the existence and importance of a strategically placed important asset, but only in those cases where utility on consecutive packages is higher than the sum of utilities of separate assets. This observation is consistent with the results of Bulow et al. (2009), who showed that early bidding focused on not only the largest assets but also the strategically placed assets.

Summarizing, the second and fifth chapter focuses on decision making in different environments, relying on finding equilibrium and determining the factors that change the games' outcome. The third chapter empirically examines scenarios from the second chapter, expanding it into the concept of venture capital syndication. Without a direct connection to the previous chapters, the fourth chapter uses simulations to determine optimal portfolios using Expected shortfall as a risk measure using different genetic algorithms.

2 Cooperation and Synergies in Partnership Games in Venture Capital Funding

Researchers often address venture capital-backed startups as a breakthrough innovation in the business model, novel technology or a combination of both and call it a vital source of innovation and technological development and serve as a significant source of new wealth creation (e.g., Bygrave & Timmons, 1991). A major concern for start-up companies is how to secure financing: There are various types of funding, depending on the stage of start-up development, information on success probability and the number of patents. In the following work, we will present a two-stage financing model that can be adopted in more complex practical applications with multiple stage financing. Although there are more than two types financing of entrepreneurs, like angels, incubators, bank loans, crowdfunding and venture capital (VC), we will, without loss of generality, address entry stage financing as angels and later stage financing as VC.

Angels are wealthy individuals who usually provide seed funds to start-up companies and are not necessarily connected to entrepreneurship (Wong et al. 2009), while in last decades start-up accelerators, grew in popularity as not only primary source of funds but also entrepreneurial nurturing and advisory management. Kim and Wagman (2014) state that accelerators make a small equity investment in a group of start-ups and require a relatively early exit. Y Combinator, the most prominent start-up accelerator in recent years runs its program twice a year, where it invests $11,000 (plus $3000 per founder) in each participating venture and takes on average a 6% equity stake in the start-up.

On the other hand, VCs provide professional support services as well as capitals to entrepreneurs usually at a later stage, when more information of entrepreneur's probability of success is revealed. VCs decide how much money they will invest in the firm and weigh the return they expect concerning the risk involved. In making their decisions, VCs take into account the decisions of the other players (Elitzur and Gavious, 2003). Our data suggest that year 2015 showed the peak of total worldwide funding with over $140 billion of capital invested through VCs, while 2016 showed sharp 24% decline with $127.2 billion invested, reflecting ongoing investor concerns within the VC market.

In order for an entrepreneur to secure financing, she needs to signal enterprise quality firstly to an angel who has no prior beliefs about entrepreneur's quality and secondly, to VCs on the later stage. We assume that all three players aim towards the point where they can 'cash out' on their investment which is presented through both money and effort. The 'cash out' stage is commonly referred to 'exit' stage, which can be an initial public offer (IPO), acquisition and in some cases a private offering. Both VCs and first stage financiers might offer their management in order to navigate the company's success and quality of the signal that will be provided to the next stage financier or during the exit strategy. Previous studies discuss the confrontational effects of conflicts between the angel, VC and the entrepreneur on the performance of the venture. Some state that direct effect of the conflict of interests is a moral hazard problem since all three players act in their self-interest (often regarding their complex portfolios), regardless the outcome on the other parties involved. For example, VC with similar profile enterprise in her portfolio might push one to early and less profitable exit, satisfied with lesser profit, but securing bigger market share for the potently more successful enterprise, regardless of angel's and entrepreneur's stakes.

There are numerous works written explaining why entrepreneur – venture capitalist cooperation is so crucial, and how failing proper establishment thereof can have a significant impact. Knowing how to develop and exploit a new technology is generally specific to a particular entrepreneur, and a venture capitalist is unlikely to locate another entrepreneur with the skills necessary to exploit that same opportunity (Cable and Shine 1997). This can be particularly convoluted when the start-up in question is on the right track to becoming a disruptive technology. Many examples arose in the last decade: in 2008 after two successful rounds of investment Twitter founder Jack Dorsey was pressured by investors to resign as CEO. He reportedly lost his position for leaving work early to enjoy other pursuits such as yoga and fashion design. He was reestablished in 2015 as the current management was unable to generate a profit and expand the customer base. In 2017 Uber founder and CEO Travis Kalanick stepped down under pressure from investors after he was unable to address allegations of a toxic work environment in his company.

Previous studies confirm that entrepreneur exerts less effort than ideal because they rely on the actions of the other players. Similarly, the VC pays less than preferably because they rely on the angel and the entrepreneur to pick up the slack (Elitzur and Gavious, 2003). Other propose a prisoner's dilemma framework in order to provide a useful model for understanding entrepreneur-venture capitalist relationships (Cable and Shine 1997). In the following work, we will consider a cooperation game model applied to classical signaling theory following Spence (1973).

Various models of signaling game between an entrepreneur and different types of investors are presented in growing literature. This study concludes four strands of previous research: The first strand draws conclusions from works on signaling, primarily by Spence (1973), where it is assumed that there are two types of workers, with either high or low level of productivity, which, in our case translates in three types of entrepreneur: unsuccessful and condemned to fail, successful and disruptive. Conti, Thursby, and Thursby (2013) propose a single-stage financing model, in which a technology entrepreneur strategically chooses the number of patents to attract potential investors in a seed investment market. Following Spence (1973), Kim and Wagman (2016) introduce two-stage financing model where the entrepreneur chooses either an offer from angels or that from VCs in the first stage, and the choice by the entrepreneur becomes a signal to the second-stage investors. On the other hand, Hah, Kim and Kwon (2016) investigate two-stage start-up financing wherein nature determines the ability of the technology entrepreneur, and she strategically chooses a costly patent level as a signal to inform her ability to potential investors and consider the cases where agents may have ambiguous beliefs about the entrepreneur's type. Similar to our model, in Elitzur and Gavious (2003), an angel invests in the first stage, and a venture capitalist does at the second stage. They focus on the role of the angel as an advisor of the entrepreneur, neglecting the previous track record of the angel, and later VC. In their model, the angel initially determines the shares of all agents, including her own, and then the entrepreneur and the venture capital only choose effort level and the amount of investment, respectively. Alike, Kim and Wagman (2016) propose a model in which only VCs have an option of reinvestment in the second stage, but it is impossible for angels to participate in the second stage after providing financial support at the first stage. Their model introduces signaling game, but they do not consider the effort taken by the entrepreneur to access seed capital, therefore skipping the partnership game aspect.

The second strand of documents introduces game theory approach to solving start-up paradigm. Cable and Shane (1997) present prisoner’s dilemma to entrepreneur venture capitalist relationships concluding that the probability of cooperative entrepreneur- venture capitalist relationships increases with the generosity shown by one party toward the other and when penalties against non-cooperative behaviour are instituted. Their findings are in line with the preceding literature. Bowden (1994) argued that venture capitalists who demand a smaller portion of the equity in business are more likely to procure entrepreneurs to work with them. Likewise, Hustedde and Pulver stated that "the higher the percentage of equity entrepreneurs are willing to surrender to outside venture capitalists, the more likely they are to be successful in acquiring funding" (1992).

A more analytical approach tries to shed light on two-player, multi-round game. Zacharakis and Meyer (2000) suggest that actuarial models could improve the screening by VCs of prospective investments. Zacharakis and Shepherd (2001) find that VCs are overconfident of their ability to make investment decisions and that this overconfidence negatively affects the accuracy of their decisions. Higashide and Birley (2002) show that disagreements between the VC and the entrepreneur could be beneficial for the performance of the venture. Elitzur and Gavious (2003) note that the body of work before their research dealt explicitly with the relationship between VCs and entrepreneurs but ignored the role of the angel altogether. They added the third player to the mix, showing significantly different game dynamics, proving that the angel in the game does not have the same properties and influences game differently than another VC.

The third strand of literature documents the importance and the development of repeated partnership game theory in venturing. Following Radner, Myerson and Maskin (1986) who observe that in repeated partnership game with imperfect monitoring in which all supergame equilibria with positive discount rates are bounded away from full efficiency uniformly in the discount rate. Abreu, Milgrom and Pearce expose limits of the Folk Theorem showing that when monitoring is imperfect, short periods can make it costly or even impossible to provide sufficient incentive adding that when modelling a given partnership as a repeated game, one ought to consider not only the players' abilities to adjust their actions but also the timing of their information flow. Kandori and Obara (2006) state that repeated games depend not solely on public signals but also on players' actions in the past. Their main finding is that players can sometimes make better use of information by using their own strategies where efficiency in repeated games can be improved. Bonatti and Hörner (2011) provide a mathematical proof that moral hazard distorts not only the amount of effort but also its timing. Free-riding leads not only to a reduction in the effort but also to procrastination, while deadlines might help to migrate moral hazard.

The last strand of documents captures the difference between successful and disruptive venture. For decades both managers and academics have troubles understanding what constitutes disruptive innovation. Christensen (1997) focused primarily on technological innovation and explored how new technologies came to surpass seemingly superior technologies in a market. Later on, Christensen corrected himself broadening definition not only technologies but also products and business models. Markides (2006) differs between two, declaring that, to qualify as an innovation, the new business model must enlarge the existing economic pie, either by attracting new customers into the market or by encouraging existing customers to consume more. On the other side, radical innovations are disruptive to consumers because they introduce products and value propositions that disturb current consumer habits and behaviors in a significant way. Without going to the detailed analysis, we will define disruptive venture by valuation and market share not separating into two categories. [1]

2.1 The model

Let us consider a simple model with three risk neutral players: an entrepreneur, an angel investor, and a venture capitalist. For simplicity, let us assume that the entrepreneur does not invest any sufficient money but brings an initial start-up idea and the primary workforce. The entrepreneur signals potential angels about the type of her idea. The angel receives the signal and provides seed financing with advice and management to the firm. We also assume that the firm will need further investments which are not coming either from entrepreneur or angel. Together they signal potential further investors. Venture capitalist, as a third player in the game, provides further capital and

[1] In a recent survey of the business literature, highly successful disruptive ventures are referred to as “unicorns”. Unicorns are defined as companies with a valuation of over $1B. It is hard to separate technological from business model innovation in every case (Uber is a transport platform with advanced route optimizing the algorithm, but they develop a concept of self-driving vehicles, Xiaomi started as a software company, expanding to hardware business etc.)

additional management to ensure that start-up is on the right track and steer it to the successful exit.[2]

The entrepreneur's risky project generates random asset values π, which is defined on $\Omega = \{unsuccessful, successful, disruptive\}$ and it has the following distribution:

$$\pi(\omega) = \begin{cases} 0\ if\ \omega = unsuccessful \\ A\ if\ \omega = successful \\ B\ if\ \omega = disruptive \end{cases}$$

with the probability of start-up being unsuccessful p_L , the probability of being successful p_H and the probability of being disruptive p_D where $p_D \ll p_H$. Therefore, the expected project value is

$$E_s[\pi] = p_H A + p_D B. \quad (1)$$

The timeline of events consists of four periods $\tau = 0,1,2,3$ with the signaling occurring before every one of the steps 1,2,3. At the time $\tau = 0$ entrepreneur learns the success probability $p_H + p_D$ and that information is available only to her until the end of the game. Let us assume that the project is state-of-the-art and thus has no prior records of success. Therefore, investors might not learn the exact success probability at the time $\tau = 0$ and can only assume the success probability thorough signaling of the entrepreneur. The angel acquires the information needed for a reliable signal about the venture and in the next step after implementation of its advice and strategy in cooperation with the entrepreneur credibly reveals that information to VCs before the next round of funding.[3]

On the first stage of financing, the entrepreneur needs investment K_1 at the time $\tau = 1$. She offers the portion of the company $\alpha \in (0,1)$ in exchange for seed investment and advice. After updating her beliefs of the entrepreneur's type, while developing and advising the project, angel in cooperation with the entrepreneur signals potential VCs. When follow-up investment K_2 is needed

[2] This paper discusses only a subset of startups which need further financing by VC for their continued success. Although the need for VC financing means multiple rounds of investments by different VCs, without loss of generalization, we assume that VCs are presented as one player with one advisory addition.
[3] Depending on the angel's portfolio size (Kim and Wagman. 2014) and availability of human resources (Fulghieri and Sevilir. 2009) this signal can contain more noise than the previous one, but following the literature (e.g., Grossman, 1981; Milgrom, 1981) firms cannot misrepresent information due to potentially severe legal penalties for false reporting. Additionally, angles have significant reputation concerns which strongly discourage them from providing false information.

at $\tau = 2$, VC offers an investment K_2 for $\beta \in (0,1)$ portion of the company. If the deal falls thru and no VCs offer an investment, the project needs to shut down, and revenue is not generated, i.e., $R = 0$. As a result of second stage financing, the angel's share α is diluted and her final share of the company is $\alpha(1-\beta)$. Thus, the entrepreneur's share at $\tau = 2$ is given by:

$$\theta = 1 - (\alpha(1-\beta) + \beta)$$

If the project is successful, it is directed towards the successful exit at the time $\tau = 2$. Realised value of the project becomes V and entrepreneur, angel and VC are paid proportionally to their equity shares.

The sequence of events is summarized in Figure 2.1.

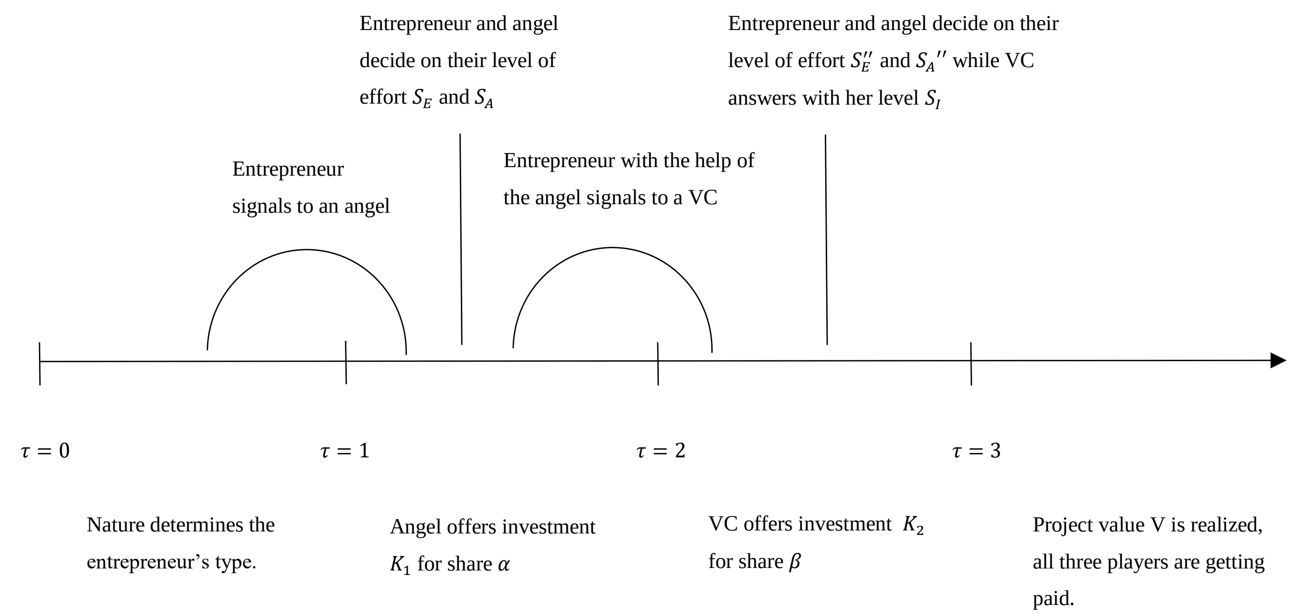

Figure 2.1: Timeline of events

Following Elitzur and Gavious (2003), based on her observation of K_1 and α, entrepreneur decides on her level of effort S_E at $\tau = 1$ which is observable by the entrepreneur only.[4] At the $\tau = 1$ angel also chooses her level of effort S_A. The effort is assumed to be costly for the entrepreneur,

[4] Following classical partnership game theory (Radner et al., 1986) we will later call this level of effort optimal strategy or the best response

and we also assume the classical assumptions that the cost of effort is included in increases in effort.

At $\tau = 2$, after the VC determines how much he will invest in the company, both entrepreneur and angel would adapt their levels of effort to S_E' and S_A' based on their observation of K_1 and β. At the $\tau = 2$ VC also chooses her level of effort S_I. We assume that the expected value of the firm at the cash-out stage, V_2, increases with diminishing marginal returns when the entrepreneur, angel or VC put more effort into the firm, when the angel's investment, K_1, is increased, or when the investment by the VC, K_2, increases.

2.2 Partnership Game and Payoff of Two Agents

Similar to (Radner et al., 1986) we suppose that the partners choose their respective efforts simultaneously and that neither partner can observe the other's effort. This absence of observability introduces an element of "moral hazard" into the situation, and motivates players to believe that the success of the project depends only on disruptiveness of the project and not on individual efforts. The strategy of entrepreneur and angel after $\tau = 1$ are $S_E \in [0,\ s_E]$ and $S_A \in [0,\ s_A]$ where S_E is the strategy for the cost of the effort of entrepreneur and S_A the strategy for is the cost of the effort of the angel, independent from investment K_1.

Let us define the value of the project at the time $\tau = 1$ as:

$$V_1 = \pi b {S_E}^p {S_A}^q \tag{2}$$

Where $b \in \left[0,\ \frac{1}{\frac{1}{2}(s_E + s_A)}\right]$ is synergy or complementary coefficient[5] and

${S_E}^p {S_A}^q$ is Cobb Douglas utility function where p and q are the output elasticities of two players and we assume that $p + q < 1$.[6]

[5] Synergy coefficient represents synergistic, value-creating abilities of partnership. The value created by the partnership may result from more efficient management, economies of scale, improved production techniques, the combination of complementary resources, the redeployment of assets to more profitable uses, the exploitation of market power etc. (Bradley et al.,1988). Unlike in Fairchild (2009), we believe that both entrepreneur and any type of investor believe in their synergy, where failing to produce high output synergy will deflate the value of the project, thus $0 \leq b \leq 1$.

[6] An increasing body of literature discusses the interpretation of output elasticities of Cobb–Douglas production function. If $p + q = 1$ then inputs and output show a decreasing return to scale relationship. If $p + q < 1$ then

The utility function of the entrepreneur between $\tau = 1$ and $\tau = 2$, dependent of her strategy and angel's strategy, denoted as $U_E{}'(S_E, S_A)$ is her prior to $\tau = 2$ evaluation share, times the evaluation value of the company at $\tau = 2$, less his cost of effort, S_E, given by:

$$U_E'(S_E, S_A) = (1 - \alpha)\pi b {S_E}^p {S_A}^q - S_E \quad (3)$$

Equivalently the utility function of the angel between, $\tau = 1$ and $\tau = 2$, dependent of her strategy and entrepreneur's strategy is given by:

$$U_A'(S_E, S_A) = \alpha\pi b {S_E}^p {S_A}^q - S_A \quad (4)$$

The first-order condition of the entrepreneur's utility function with respect to her strategy yields:

$$\frac{\partial U_E{}'(S_E, S_A)}{\partial S_E} = (1 - \alpha)\pi p b {S_E}^{p-1} {S_A}^q - 1$$

when $\frac{\partial U_E{}'(S_E, S_A)}{\partial S_E} = 0$

Entrepreneur, wanting to maximize her utility, leads to:

$$\widehat{S_E} = (\pi(1 - \alpha)bp)^{\frac{1}{1-p}} {S_A}^{\frac{q}{1-p}} \quad (5)$$

Similarly, angel's utility function with respect to her strategy yields:

$$\frac{\partial U_A{}'(S_E, S_A)}{\partial S_A} = \alpha\pi b q {S_E}^p {S_A}^{q-1} - 1 = 0$$

Leading to:

$$\widehat{S_A} = (\pi\alpha b q)^{\frac{1}{1-q}} {S_E}^{\frac{p}{1-q}} \quad (6)$$

Type 1 Nash Equilibrium is achieved when:

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inputs and output show a constant return to scale relationship. Finally, if $p + q > 1$ then inputs and output show an increasing return to relationship (Pendharkar et al., 2008). Others state that empirical success is an illusion and that, as a rule, the two exponents, p and q, have values incompatible with the distributive shares that the Cobb-Douglas claims to explain (Labini, 1995). The numerical solution is a specific case of entrepreneurial effort and wealth in privately held firms shows that $p = q = 0.4$. Yet, Pendharkar et al.(2008) empirical findings for the Cobb–Douglas production function properties of software development effort show $p + q > 1$ proving that the production function exhibits increasing returns to scale economy.

$$S_E = \widehat{S_E} \text{ and } S_A = \widehat{S_A}$$

which implies:

$$\widehat{S_E} = (\pi(1-\alpha)bp)^{\frac{1}{1-p}}(\pi\alpha bq)^{\frac{q}{(1-p)(1-q)}}\widehat{S_E}^{\frac{pq}{(1-p)(1-q)}}$$

Leading to:

$$\widehat{S_E} = \pi^{\frac{1}{-p-q+1}}(bp^{1-q}q^q(1-\alpha)^{1-q}\alpha^q)^{\frac{1}{-p-q+1}} \quad (7)$$

Similarly, angel's optimal strategy is:

$$\widehat{S_A} = (\pi\alpha bq)^{\frac{1}{1-q}}(\pi b(1-\alpha)p^{1-q}q^q\alpha^q)^{\frac{p}{(1-q-p)(1-q)}} \quad (8)$$

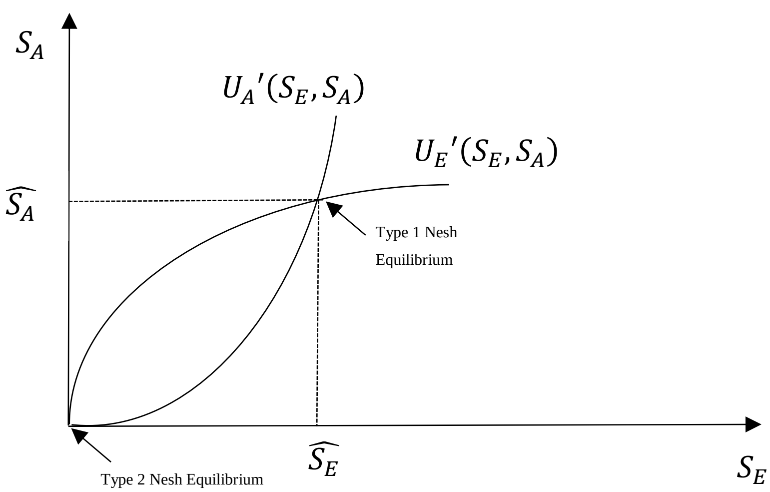


Figure 2.2: The two equilibrium types

Best response functions, as well as two types of Nesh equilibrium are given in Figure 2.2.

Let us formalize the discussion above with the following proposition:

Proposition 2.1: Only two types of equilibria can exist in the game:

$$1)\ S_E = S_A = 0$$

$$2)\ S_E = \widehat{S_E} \text{ and } S_A = \widehat{S_A}$$

Proof: Trivially, if $S_A = 0$ following (5) the entrepreneur's best response is $S_E = 0$

If $S_A \neq 0$ since

$$\frac{\partial^2 U_E{'}(S_E,S_A)}{\partial^2 S_E} < 0 \text{ and } \frac{\partial U_A{'}(S_E,S_A)}{\partial S_A} < 0$$

both $U_E'(S_E, S_A)$ and $U_A'(S_E, S_A)$ are concave functions, following Theorem 3 in Rosen (1965) there exists a normalized equilibrium point to a concave n-person game for every specified $S_A > 0$.

Following strict concavity and Theorem 4 in Rosen (1965) there is a unique normalized equilibrium point for every $\widehat{S_E} \in (0, S_E)$ and $\widehat{S_A} \in (0, S_A)$ and it is achieved when

$$U_A'(S_E, S_A) = U_E'(S_E, S_A) \ \blacksquare$$

Proposition 2.2: Given that $\alpha > 0$ and $S_E \neq 0$, $S_A \neq 0$ increase in the complementary coefficient b will increase optimal strategies of both players $\widehat{S_E}$ and $\widehat{S_A}$ (Figure 2.3).

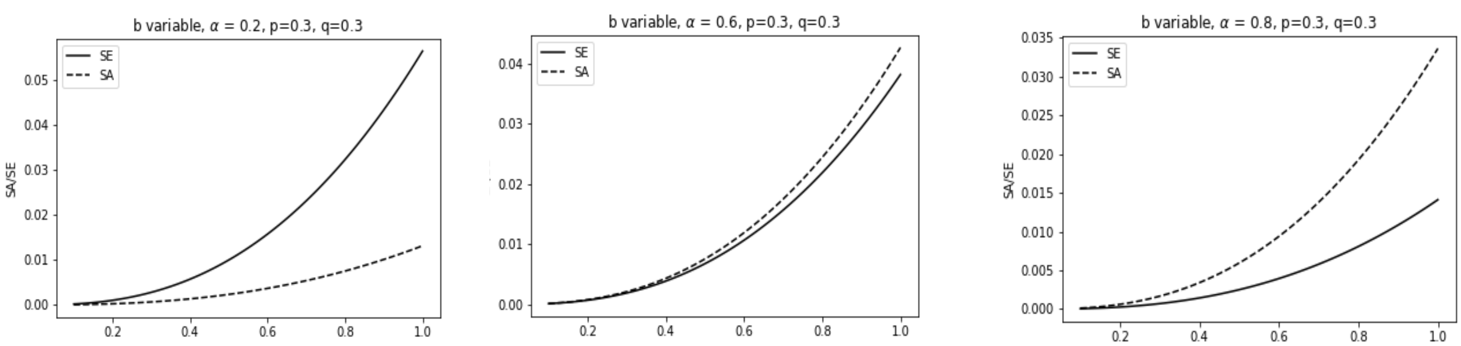


Figure 2.3: Player's strategies dependence on complementary coefficient b. Although in all three cases optimal strategies increase with the increase of the complementary coefficient, their relationship depends on the share of the company α

Proof: Following (7) and (8) we can infer that the variable b is on the power of $\frac{1}{1-(p+q)}$ and $\frac{p}{(1-q-p)(1-q)}$ respectively, which are both greater than zero, since $p+q<0$, hense $\widehat{S_E}$ and $\widehat{S_A}$ increase when b increases. ■

Proposition 2.3: Given that $\alpha > 0$ and $S_E \neq 0$, $S_A \neq 0$ increase in the portion of the company α proposed from entrepreneur to the angel will increase the optimal strategy of entrepreneur $\widehat{S_E}$ if $\alpha \in (0, \alpha_E)$ and decrease if $\alpha \in (\alpha_E, 1)$ and decrease the best response of the angel $\widehat{S_A}$ if $\alpha \in (0, \alpha_A)$ and decrease if $\alpha \in (\alpha_A, 1)$, where $\alpha_E \leq \alpha_A$ and $\alpha_E = q$ and

$$\alpha_A = \frac{pq-(p+q)+1}{pq-q+1}$$

In special case, $\alpha_E = \alpha_A$ when

$$q = \frac{-p+\sqrt{-p(3p-4)}+2}{2-2p}$$

Player's strategies dependence of the share offered to the angel in combination with different elasticity coefficients are presented in Figure 2.4.

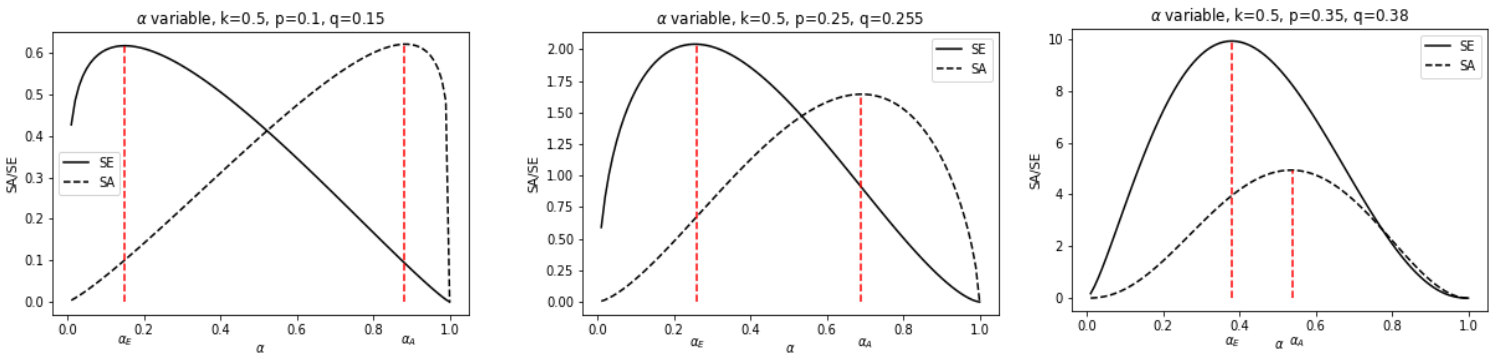


Figure 2.4: Player's strategies dependence of the share offered to the angel

Proof: Following (7) and taking the derivative of $\widehat{S_E}$ with the respect of α yields:

$$\frac{\partial \widehat{S_E}}{\partial \alpha} = \frac{1}{-p-q+1} \pi^{\frac{1}{-p-q+1}} (bq^q(1-\alpha)^{1-q}\alpha^q p^{1-q})^{\frac{1}{-p-q+1}-1}$$

$$(bp^{1-q}q^{q+1}(1-\alpha)^{1-q}\alpha^{q-1} - b(1-q)q^q(1-\alpha)^{-q}\alpha^q p^{1-q})$$

Maximum of the function is achieved when alternative form of the derivative is equal to zero:

$$\frac{(q-\alpha)\pi^{\frac{1}{-p-q+1}}(bq^q(1-\alpha)^{1-q}\alpha^q p^{1-q})^{\frac{1}{-p-q+1}}}{\alpha(1-\alpha)(p+q-1)} = 0$$

yields $\alpha_E = q$

Following (8) and taking the derivative of $\widehat{S_A}$ with the respect of α yields:

$$\frac{\partial \widehat{S_A}}{\partial \alpha} = \frac{b\, p\pi \left((1-\alpha)\, b\, q^q \pi\, \alpha^q\, p^{1-q}\right)^{\frac{p}{(1-q)(-p-q+1)}} (\alpha\, b\, p\, \pi)^{\frac{1}{1-q}-1}}{1-q}$$
$$+ \frac{1}{(1-q)(-p-q+1)} p\left((1-\alpha)\, b\, q^q \pi\, \alpha^q\, p^{1-q}\right)^{\frac{p}{(1-q)(-p-q+1)}-1}$$
$$\cdot \left((1-\alpha)\, b\, q^{q+1}\, \pi\, \alpha^{q-1}\, p^{1-q} - b\, q^q\, \pi\, \alpha^q\, p^{1-q}\right)(\alpha\, b\, p\, z)^{\frac{1}{1-q}}$$

Maximum of the function is achieved when alternative form of the derivative is equal to zero:

$$\frac{(\alpha\, p\, q - \alpha\, q + \alpha - p\, q + p + q - 1)(\alpha\, b\, p\, \pi)^{\frac{1}{1-q}}\left((1-\alpha)\, b\, q^q\, z\, \alpha^q\, p^{1-q}\right)^{\frac{p}{(1-q)(-p-q+1)}}}{(\alpha-1)\, \alpha\, (q-1)(p+q-1)} = 0$$

Yields:

$$\alpha_A = \frac{p\, q - p - q + 1}{p\, q - q + 1}.$$

Equalizing maximums of the functions $\widehat{S_E}$ and $\widehat{S_A}$ gives:

$$q = \frac{p\, q - p - q + 1}{p\, q - q + 1}$$

Knowing that $p < 1$, $p \neq 0$ and $q < 1$ we get:

$$q = \frac{-p+\sqrt{-p(3p-4)}+2}{2-2p} \quad \blacksquare$$

2.3 Partnership Game and Payoff of Three Agents

On the later stage, after the project is successfully initiated, the entrepreneur and angel signal the VCs in order to secure further funding. Similarly, to the seed investment market, VCs offer capital in return for the equity shares and entrepreneur chooses a VC who offers the lowest share. The strategy of entrepreneur, angel and VC after $\tau = 2$ are $S_E{}'' \in [0,\ s_E]$, $S_A{}'' \in [0,\ s_A]$ and $S_I \in [0,\ s_I]$ where $S_E{}'', S_A{}''$ and $S_I{}''$ are the strategies for the cost of effort of entrepreneur, angel and VC, respectively. Similar to the case with two agents we value of the project is:

$$V_2 = \pi b S_E{}''^p S_A{}''^q S_I{}''^r \tag{9}$$

The utilities of agents, depending on strategies of other players are following:

$$U_E''(S_E{}'', S_A{}'', S_I{}'') = \big(1 - (\alpha(1-\beta) + \beta)\big)\pi b S_E{}''^p S_A{}''^q S_I{}''^r - S_E{}'' \tag{10}$$

$$U_A''(S_E{}'', S_A{}'', S_I{}'') = \alpha(1-\beta)\pi b S_E{}''^p S_A{}''^q S_I{}''^r \ - S_A{}'' \tag{11}$$

$$U_I''(S_E{}'', S_A{}'', S_I{}'') = \beta\pi b S_E{}''^p S_A{}''^q S_I{}''^r \ - S_I{}'' \tag{12}$$

The first-order condition of the player's utility function with respect to her strategy yields:

$$\frac{\partial U_E''(S_E{}'', S_A{}'', S_I{}'')}{\partial S_E{}''} = p\pi b S_A{}''^q S_I{}''^r S_E{}''^{p-1}\big(1 - \beta - \alpha(1-\beta)\big) - 1 = 0$$

$$\frac{\partial U_A''(S_E{}'', S_A{}'', S_I{}'')}{\partial S_A{}''} = q\pi b\alpha(1-\beta) S_E{}''^p S_I{}''^r S_A{}''^{q-1} - 1 = 0$$

$$\frac{\partial U_I''(S_E{}'', S_A{}'', S_I{}'')}{\partial S_I{}''} = r\pi b\beta S_E{}''^p S_I{}''^{r-1} S_A{}''^q - 1 = 0$$

Players, wanting to maximize their utility, lead to their best responses:

$$\widehat{S_E{}''} = \big(\pi b p S_A{}''^q S_I{}''^r\big)^{\frac{1}{1-p}}((1-\alpha)(1-\beta))^{\frac{1}{1-p}} \tag{13}$$

$$\widehat{S_A{}''} = \big[\pi b q\alpha(1-\beta) S_E{}''^p S_I{}''^r\big]^{\frac{1}{1-q}} \tag{14}$$

$$\widehat{S_I}'' = \left[r\pi b\beta {S_E''}^{p} {S_A''}^{q}\right]^{\frac{1}{1-r}} \tag{15}$$

Proposition 2.4: Only two types of equilibria can exist in the game:

1) $\widehat{S_E}'' = \widehat{S_A}'' = \widehat{S_I}'' = 0$

2) $S_E'' = \widehat{S_E}''$, $S_A'' = \widehat{S_A}''$ and $S_I'' = \widehat{S_I}''$

Proof: Solving equations (13) – (15) for $\widehat{S_E}''$, $\widehat{S_A}''$ and $\widehat{S_I}''$ leads to:

$$ln\widehat{S_E}'' = -\frac{-\left(-n - \frac{jr}{1-q}\right)\left(-\frac{q}{1-p} - \frac{qr}{(1-p)(1-r)}\right) + \left(-m - \frac{jr}{1-p}\right)\left(1 - \frac{qr}{(1-q)(1-r)}\right)}{1 - \frac{pq}{(1-p)(1-q)} - \frac{pr}{(1-p)(1-r)} - \frac{qr}{(1-q)(1-r)} - \frac{2pqr}{(1-p)(1-q)(1-r)}} \tag{16}$$

$$ln\widehat{S_A}'' = -\frac{n + mp - np - mp^2 - nq + npq + jr - nr + nqr - jr^2}{-1 + p + q + r} \tag{17}$$

$$ln\widehat{S_I}'' = -\frac{j - jp + mp - mp^2 - jq + nq - nq^2 - jr + jpr + jqr}{-1 + p + q + r} \tag{18}$$

Where

$$m = \frac{1}{1-p}\ln\big(\pi bp(1-\alpha)(1-\beta)\big),$$

$$n = \frac{1}{1-q}\,ln\big(\pi bq\alpha(1-\beta)\big),$$

$$j = \frac{1}{1-r}\ln(r\pi b\beta)$$

Trivially, if $S_I'' = 0$ following (13) the entrepreneur's best response is $S_E'' = 0$. If entrepreneur's choses $S_E'' = 0$ following (14), angel's best response is $S_A'' = 0$.

$$\text{If } S_I'' \neq 0 \text{ since } \frac{\partial^2 U_E''(S_E,S_A,S_I)}{\partial^2 S_E''} < 0\,,\ \frac{\partial U_A''(S_E,S_A,S_I)}{\partial S_A''} < 0 \text{ and } \frac{\partial U_I''(S_E,S_A,S_I)}{\partial S_I''} < 0 \tag{19}$$

$U_E''(S_E, S_A, S_I)$, $U_A''(S_E, S_A, S_I)$ and $U_I''(S_E, S_A, S_I)$ are concave functions, following Theorem 3 in Rosen (1965) there exists a normalized equilibrium point to a concave n-person game for every specified $S_I'' > 0$.

Following strict concavity and Theorem 4 in Rosen (1965) there is a unique normalized equilibrium point for every $\widehat{S_E}'' \in (0, S_E'')$, $\widehat{S_A}'' \in (0, S_A'')$ and $\widehat{S_I}'' \in (0, S_I'')$ and it is achieved when

$$U_E''(S_E, S_A, S_I) = U_A''(S_E, S_A, S_I) = U_I''(S_E, S_A, S_I) \blacksquare$$

Proposition 2.5: Given that $\alpha > 0, \beta > 0$ and $S_E'' \neq 0$, $S_A'' \neq 0$, $S_I'' \neq 0$ increase in complementary coefficient b will increase optimal strategies of all players $\widehat{S_E}''$, $\widehat{S_A}''$ and $\widehat{S_I}''$ (Figure 2.5).

Proof: Following (16) – (18) increase in b will increase m, n and j , which, after short calculation leads to increase in $ln\widehat{S_E}''$, $ln\widehat{S_A}''$and $ln\widehat{S_I}''$, hence growth of $\widehat{S_E}''$, $\widehat{S_A}''$ and $\widehat{S_I}''$. ■

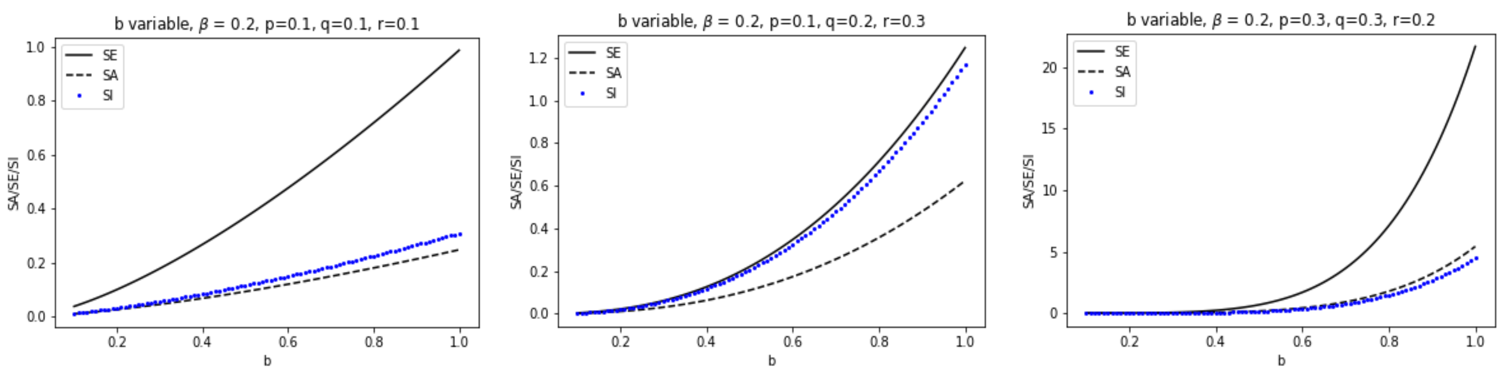


Figure 2.5: Three players' optimal response dependence on complementary coefficient b in combination with players' elasticity coefficients.

Different scenarios or players' strategies changes with the increase of b, with different combinations of Cobb-Douglas elasticity coefficients, are presented in Figure 2.5. Having lower but equal elasticity coefficients for all three players' results in quicker separation of players 'efforts. Entrepreneur, in this case, chooses the highest effort levels and has the incentive to increase it the most as cooperation between players rises. As intuitions suggest, the angel is the one who has the incentive only slightly to increase his efforts, and that strategy will follow in different elasticity coefficient scenarios.

In a case where elasticity coefficient of VC is greater than both entrepreneur and angel, his optimal strategy is to put his efforts above all players and increase efforts much faster with an increase of cooperation factor than in the case where Cobb-Douglas elasticity coefficients were the same for all players.

Since angel's portion of the company gets deflated, although with VC funding his investment gets a higher probability of success, his believes of project value do not get changes, intuition suggests that his efforts will not dramatically increase with the increase of cooperation between players, as shown in Figure 2.5. Even when his elasticity coefficient is highest of all players, his efforts remain low.

Proposition 2.6: Given that $\beta > 0$ and $S_E \neq 0$, $S_A \neq 0$, $S_I \neq 0$ increase in the portion of the company β proposed from the entrepreneur to the angel will increase the optimal strategy of entrepreneur $\widehat{S_E}$ and the angel $\widehat{S_A}$ if $\beta \in (0, \beta_E)$ and decrease if $\beta \in (\beta_E, 1)$ and increase the best response of the VC if $\widehat{S_I}$ if $\beta \in (0, \beta_I)$ and decrease if $\beta \in (\beta_I, 1)$, where $\beta_E \leq \beta_A$ and

$\beta_E = r$ and

$$\beta_I = 1 - q - p$$

In special case, $\beta_E = \beta_I$ when

$$p + q + r \rightarrow 1$$

(Figure 2.6)

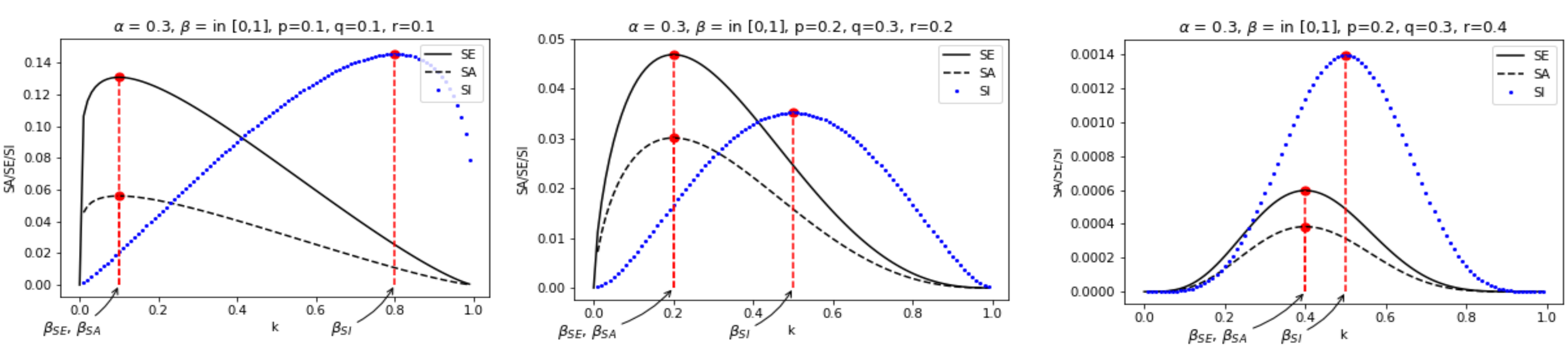


Figure 2.6: Player's best response functions' dependence on the share offered to the VC in combination with different elasticity coefficients. The entrepreneur and the angel always achieve maximum at the same share β

Proof: Following (16) and taking the derivative of $\widehat{S_E''}$ with the respect of β after simplifying yields:

$$\frac{\partial \widehat{S_E''}}{\partial \beta} = \frac{be^{\ln\left(\pi(\beta-r)\left(-\alpha(-1+\beta)q^2+\beta r^2-(-1+\alpha)(-1+\beta)p(-1+q+r)\right)\right)}}{(-1+\beta)\beta(-1+p+q+r)^2} \tag{20}$$

$$\frac{\partial \widehat{S_A''}}{\partial \beta} = \frac{be^{-\ln\left(\pi(-1+\alpha)(-1+\beta)p^2+\alpha(-1+\beta)pq+\alpha(-1+\beta)q(-1+r)+\beta r^2\right)}}{(-1+\beta)\beta(-1+p+q+r)^2} \tag{21}$$

$$\frac{\partial \widehat{S_I''}}{\partial \beta} = \frac{be^{\ln(\pi(-1+\beta+p+q)((-1+\alpha)(-1+\beta)p^2-\alpha(-1+\beta)q^2-\beta pr+\beta(-1+q)r)}}{(-1+\beta)\beta(-1+p+q+r)^2} \tag{22}$$

Maximum of the function is achieved when derivatives is equal to zero leading to $\beta = r$ from both (20) and (21) and $\beta = p + q + r = 1$ from (22).

Equalizing maximums of the functions $\widehat{S_E''}$ and $\widehat{S_I''}$ gives:

$$p + q + r \to 1 \ \blacksquare$$

Elasticity coefficients of all three players have a significant role in forming of the best response of other players. Proposition 2.6 may explain that angel and entrepreneur always have the same incentive when it comes to their optimal strategy with regards to β and they will both evince the maximum of efforts at the certain level of β that depends on r showing that the prior track record of VC – their management skills and investment strategies are essential for negotiations of the funding round at time $\tau = 2$. Figure 2.6 also confirms the smaller mentoring and management role of the angel after successful investment by VC. Angel will evince a smaller amount of effort than entrepreneur after $\tau = 2$, even with significantly higher elasticity coefficient, assuming that $\alpha < 0.5$. Similarly, since β_{SE} for which $\widehat{S_E''}$=$\widehat{S_I''}$ is always higher than β_{AI} for which $\widehat{S_A''}$=$\widehat{S_I''}$, the angel has the motive to lead the negotiations for the round of VC funding. Figure 2.6 also shows that, although $\widehat{S_I''}$ achieves maximum for β that only depends on p and q, having higher r will influence the shape of the curve of $\widehat{S_I''}$ and therefore if and for which β optimal response of VC surpasses efforts of both entrepreneur and angel.

2.4 Angels' reputation and signaling in VC funding

In previous literature, it was discussed that reputation of the angel, as well as the stake of the angel in the company α in the time $\tau = 2$ plays a significant role in signaling the venture capitalist (see Hsu, 2005). Having a higher reputation and better track record as an angel while having a high share of the project can increase VC's efforts after successful funding at $\tau = 2$.

Proposition 2.7: Given that $\alpha > 0$ and $\beta > 0$, $S_E{''} \neq 0$, $S_A{''} \neq 0$, $S_I{''} \neq 0$ increase in angel's share of the company α on $\left(0, \frac{q}{p+q}\right)$ will increase the optimal strategy of VC $\widehat{S_I{''}}$ and decrease on $\alpha \in \left(\frac{q}{p+q}, 1\right)$ (Figure 2.7).

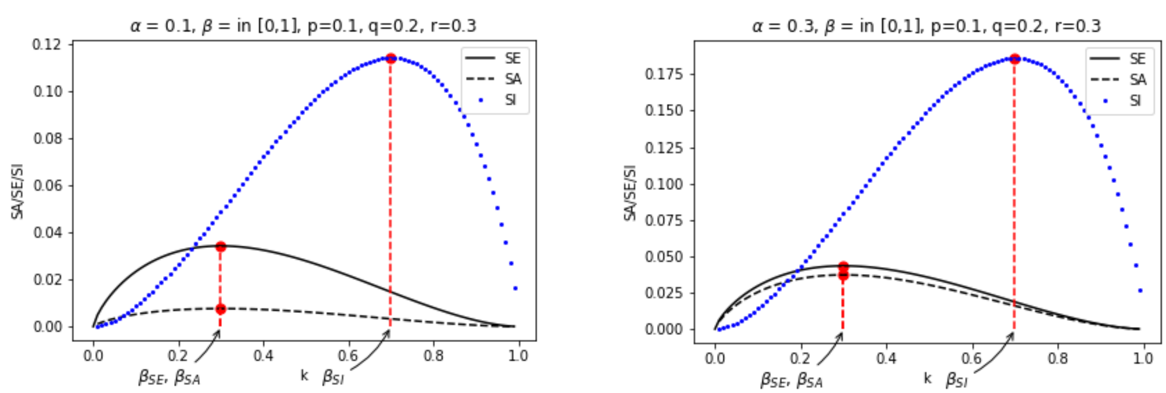


Figure 2.7: Increase in angel's share of the company α will increase optimal strategy of VC after successful funding round

Proof: Following (18) and taking the first derivative of $\widehat{S_I{''}}$ with respect to α yields

$$\frac{\partial \ln \widehat{S_I''}}{\partial \alpha} = \frac{\alpha p - q + \alpha q}{\alpha(\alpha - 1)(1 - p - q - r)}$$

$\frac{\partial \ln \widehat{S_I''}}{\partial \alpha} = 0$ is when $\alpha = \frac{q}{p+q}$. ■

Proposition 2.8: Given that $\alpha > 0$ and $\beta > 0$, $S_E{}'' \neq 0$, $S_A{}'' \neq 0$, $S_I{}'' \neq 0$ higher angel's elasticity coefficient q yields lower optimal strategy of VC $\widehat{S_I{}''}$ (Figure 2.8).

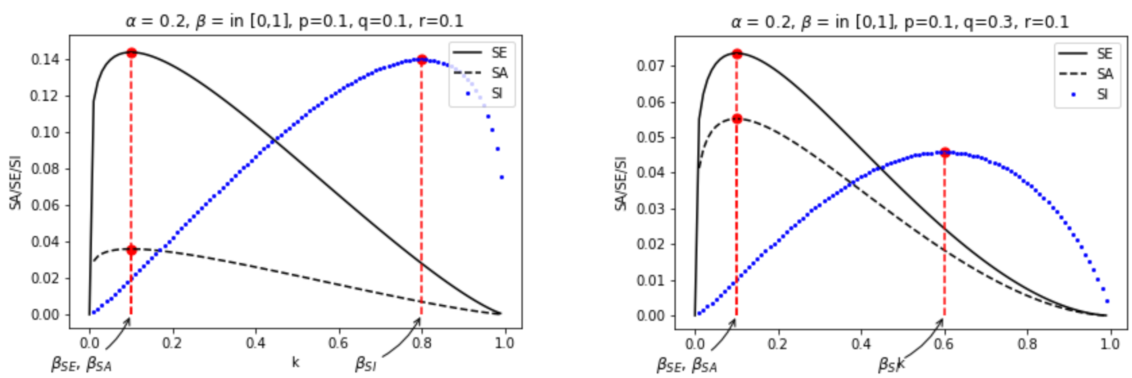


Figure 2.8: Higher angel's elasticity coefficient q will yield lower optimal strategy of VC after successful second funding round

Proof: Following (18) and taking the first derivative of $\widehat{S_I{}''}$ with respect to q yields

$$\frac{\partial \ln \widehat{S_I''}}{\partial q} = -\frac{p + q + r - \ln\big(-\pi\alpha b q(\beta - 1)\big) - r\ln(\pi b\beta r) - p\ln\big(\pi b p(\alpha - 1)(\beta - 1)\big)}{(p + q + r - 1)^2}$$

$$- \frac{p\ln\big(-\pi\alpha b q(\beta - 1)\big) + r\ln\big(-\pi\alpha b q(\beta - 1)\big) - 1}{(p + q + r - 1)^2}$$

Which is negative for all $q \in (0, 1)$. ■

VC observes angle's share of the company and elasticity coefficients of entrepreneur and angel after τ=1. She will evince a higher level of effort if angles share of the company satisfies condition from Proposition 2.7 (Figure 2.7). Angel's trust in the company can be a good signal only until a certain limit, after which VC, if decided to join the game, will evince a lower level of effort, expecting the angel to do "heavy lifting" of mentoring and management. Similar will happen with higher angel's elasticity coefficient (Figure 2.8).

2.5 Concluding Remarks

This study addresses a significant gap in the current academic literature on VCs. Entrepreneurs in early-stage investments may face a decision between multiple offers from angels that might seem identical in value provided. Angles that involve themselves in multiple ventures (like incubators) usually worry about the exit stage and payoff, not taking into account that enthusiasm of founders may drop off significantly due to the lack of synergy with the management added. Our study proposes a two-stage financing model that considers output elasticities of all three players through the Cobb-Douglas utility function. The entrepreneur can observe the previous track record of potential investors to obtain knowledge about her best response later in the game. VC observes output elasticities of two other players after the first financing round, adapting her strategy for negotiations in the second round of funding. Additionally, we introduce the complementary aspect of the project, showing that synergies between existing and added management influence three-player game dynamics and thus the exit stage value of the project.

Our findings suggest that higher complementary coefficient between players on both stages lead to the higher level of effort from all three players, taking game dynamics away from moral hazard problem and causing higher exit stage payoffs. Previous track record of the angel and VC and output elasticity of the entrepreneur, in combination with the share of the company, offered to angel and VC impact the three-player game dynamic causing some players to reduce their efforts after certain funding rounds.

This study also provides insight into the signaling aspect of the angel's reputation during the VC funding round. Specifically, our research reveals that angel's higher company share will increase the level of the effort of the VC until some point that depends on output elasticities of angel and entrepreneur, but at the same time, higher output elasticity of angel leads to decrease of the level of the effort of the VC. This finding for practitioners implies that VCs could see firms with higher angel backing as firms whose founders are less prone to cash in early and leave as well as the angels with highly effective mentoring and management will do ‘the heavy lifting’ instead of VC’s management.

We acknowledge several limitations in this research and recognize opportunities for future studies in this area. First, probabilities of the venture being a specific type are not discussed in this paper. Second, empirical research of previous track record of angels and venture capitalists can give insight on modelling output elasticities on the start-up industry. Lastly, we did not investigate in this chapter is the syndication of investments by several VCs. Syndication has a vital role in that it diversifies the risk of the VCs involved.

3 The Signaling Effect of Venture Capital Syndication on Funding, Value, and Risk Sharing with Angel-backed Start-ups

Venture capital investments are of major importance within the start-up industry. Newly founded companies are often dependent on financial resources from the outside. In the previous chapter, we modelled the dynamics between founders, angels and venture capitalist in the theoretical sense, but also focusing on difficulties that occur in the cooperation between the different players in start-up funding, and therefore solely on egoistic behavior of one or more parties, which are likely to contradict each other. For example, the situation can incentivize entrepreneurs to rely on the investors help and expertise and reduce the effort they make in order to maximize their own output at the cost of their partners. Equally, VCs could make smaller investments and rely on angel investment and entrepreneurial effort in company development. The fear of partners acting in self-interest also causes a moral hazard for the other participant. In our theoretical model the entrepreneur is first to decide on his course of action, which in theory is done by approaching an angel and migrating moral hazard by signaling her commitment to the company.

Afterwards, the angel establishes his investment in the company and sets certain key values that influence the other actors' decision-making. This includes key figures like the amount of capital he invests and the number of shares, demanded in return. These factors are exogenous to the entrepreneur, who is reacting to these inputs decides whether he should work with the angel and how much effort he puts into his company. As previously discussed, this is one of the moments where egoistic behavior can occur. Following this logic, we suggest a point of equilibrium, where all forms of moral hazard are mitigated as much as possible. In this state, all individuals make the ideal decisions that lead to a flourishing company. For this process, the angel is established as a so-called 'Stackelberg' leader. In general, this implies that he is conscious of the other players' reactions towards his chosen actions. It is assumed that he is able to predict the reactions of the entrepreneur and the VC. The leadership role of angels is also the main reason why only entrepreneurs and VCs are suspected of egoistic behavior. While the angel can estimate their actions, she cannot exploit the other players as she is the first to make a move (figure 2.1 from the chapter 2). In the previous chapter, we show that type 1 equilibrium is determined by entrepreneurial effort, angel's effort, venture capital funding and synergy coefficient. Elitzur and Gavious (2003), imply that in general entrepreneurs and VCs are the ones able to utilize moral hazard in order to exploit angels and each other. Their model, however, has limitations which might be compensated by test results. For example, it only accounts for start-ups with the involvement of venture capitalists in general. But additional players increase the chance of moral hazard. Especially inexperienced investors are linked to a high risk in this regard (Casamatta and Haritchabalet, 2007). Beneficial effects that might be provided through the cooperation of multiple independent VCs are also not included. However, syndication in this context might account for a reduction of risk on the side of the investors or add value to the company.

We usually perceive venture capital syndication as beneficial, but the cause of those benefits is debated among researchers. The two most discussed explanations are the value-adding and the risk-reduction hypothesis. The idea of value-adding argues that a higher number of VCs creates additional value for the funded company (Brander et al. 2002). This implies that an investor's worth does not solely consist of his monetary investment but is also affected by other factors. For example, each investor provides his own skillset and the information he has. In the case of angel-backed start-ups, their distinct properties enable VCs and angel investors to supply different kinds of support according to their own individual skillset. Venture capitalists are expected to provide knowledge and experience in financial matters, primarily. Angels, on the other hand, are able to provide first-hand knowledge to the entrepreneur, especially if they originate from the same industry as the entrepreneur or have an entrepreneurial background themselves (Dutta and Folta, 2016). Some research shows that angels add less value than their VC counterpart, but they are chosen more frequently, leading to beliefs that there are factors more important than monetary (Fairchild, 2011).

In contrast to that theory, the concept of risk-reduction reasons that syndication benefits the investors by mitigating factors that could endanger their invested capital. In this case, additional investors provide knowledge and skills which reduce a start-up's risk of failure. The theory basically provides a different interpretation of the same factors that were also included in the value-adding hypothesis. Here, additional investors and knowledge are seen as a way of mitigating and diversifying risk rather than increasing revenue. Lockett and Wright (2003) further distinguish between reducing risk and sharing it. Sharing in this context mostly focusses on diversifying the investor's portfolio and thus reducing its exposure to the inherent risk of every single investment. Syndication in this context enables the venture capitalist to invest in a variety of diverse firms while still being able to guarantee the amount of funding needed by each individual start-up. However, this is regarded as a hard task in the venture capital industry because of the possible necessity of additional future funding and ex-ante information asymmetries. For the aspect of risk reduction, they interpret the knowledge, skills and experience of venture capitalists as non-financial resources. Should a company decide against syndicating its investments, it is deemed useful to specialize in certain competencies.

Another notable theory discusses syndication as a way of improving the venture capitalists' selection process. Investments with a relatively high expected return, would not be syndicated as venture capitalists would not want to share their profits (Brander et al. 2002). This can be seen as a mixture of both value-adding and risk-reduction as it provides the possibility of focusing a VCs attention on the most promising investments.

To summarize, the aforementioned theories have a different view about the benefits of venture capital syndication, but they are not exclusive. We visualized the general ideas in the figure 3.1. Two views argue that the resources of added venture capitalist will reap the benefit in one way or another. One strand of literature argues that it will affect the funded company by increasing its overall valuation. In the other, it reduces the risk of failure associated with the start-up. This supports the idea that both theories might have a certain degree of truth to them and are equally relevant for start-ups. A synergy between them can lead to a higher company value as well as lower risk. Which factor is dominant in a certain situation depends on the observed company. Company's profile, novelty of the product or number of patents can have a significant influence on the perceived risk for the venture capitalists. Higher valuation is a direct improvement in contrast to the investment of a single VC. However, entrepreneurs might also benefit indirectly as individual investors will require fewer shares in order to justify funding a given firm. These shares can then benefit the entrepreneur by giving him more control of the company or by increasing his financial gain in the exit-stage.

Lehman (2006) takes an empirical look at the differences between start-ups funded by single or multiple VCs. The data arrives from a variety of sources and contains a total of 108 VC-backed German companies. Lehmann also implies that the number of venture capitalists grows with increasing risk and that the equity stake of individual VCs is smaller in a syndicated situation. Next, he assumes that the innovative power of a firm is a factor in VC syndication.

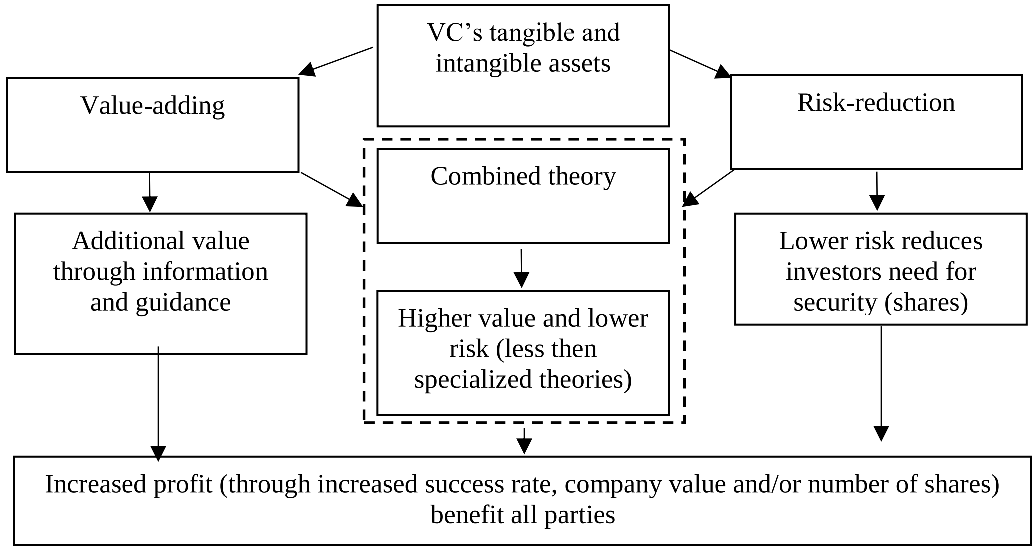


Figure 3.1: Visualization of the value-adding, risk-reduction and combined hypotheses. The theoretical effect of the respective theory is shown isolated. As explained, synergy effects of the theories can be expected. These can lead to a combination of the effects, possibly leading to additional value while the risk is also reduced.

In this case, the numbers of patents owned by the company prior to its initial public offering are supposed to influence the chance of syndication positively. A similar connection is implied for the number of VCs in the company. The findings fail to support the claim that the chance of syndication to occur is linked to either the companies age or the number of intangible assets. On the other hand, the same study suggests a significant connection between the company risk and the number of venture capitalists funding a start-up. Furthermore, syndication is shown not to have a significant influence on the stock market performance of the funded companies. We believe that data limited not only in size but specifically focused on only one country limits the findings of this study. This study as well doesn't consider later economic recessions which highly influenced VC decision-making and it doesn't consider the founder's efforts.

3.1 Data and methodology

We propose dataset acquired from CrunchBase on 17^{th} of October 2019 that includes 672.214 companies from different markets. These companies originate from a total of 208 different countries around the world. More than 238.000 of them are located in the USA. In a first step, the organizations are split up according to their designation: one containing all types of investors and one including all companies which received funding. A short overview shows that 1107 organizations are classified as providers of ‘Angel Investment’ and therefore fall into the category of angels. The second group supplying ‘Venture Capital’ is more than six times the size and contains 7460 different venture capitalists. The second group containing ‘Venture Capital’ is more than six times the size and contains 7460 different venture capitalists. The second group supplying ‘Venture Capital’ is more than six times the size and contains 7460 different venture capitalists. However, it is possible for an organization to participate in the market in a multitude of roles. For example, a VC can decide to supply seed investment and act like an angel in certain situations. These need to be identified as they can be difficult to assign to a specific group. In this case, comparing the data shows that it contains 659 organizations which have been active in both roles, as angels and venture capitalists. However, it is unclear if these investors supplied both kinds of investment to start-ups present in this dataset. These labels are not necessarily linked to the investments presented in the dataset. They could be the result of prior investments.

As already mentioned, the correct classification of angels and venture capitalists is of major importance, because the classification has the power to influence which companies will be added to either test group or which ones will be left out. It also determines which of the sample groups a firm is assigned to. This can happen by linking additional VCs to a company which was formally regarded to be backed by only a single venture capitalist. Certain start-ups could also be excluded from testing entirely. For example, if an angel is wrongfully classified as a venture capitalist, a firm could appear to be exclusively funded by venture capitalists and therefore be falsely excluded from the sample groups.

Next, it is important to find the start-ups which received the funds provided by the now classified investors. Those firms need to fulfil additional criteria to be eligible for testing. First, the start-ups are checked for a clear exit-point at which the investors are assumed to leave, and the company value is known. This results in a large reduction in the number of possible candidates for testing. The exclusion of companies without a clear exit point lowers the number of eligible investees to 86.846 companies.

Comparing the resulting data frame and the ones containing the angels and VC investors demonstrates the surprisingly small number of companies funded by both types of investors. It shows that only 190 of the funded companies have received angel investments. In comparison, the group of venture capital funded companies counts a total of 4212 firms. Next, these two remaining data frames are searched for common entries. The result is a data frame containing a total number of 117 companies which fulfil the imposed limitations. The last important step is to divide these 117 into the two test groups. First the ones with angel investment and a single VC and then those with angel-backing and multiple venture capitalists. The data frame of eligible companies was created by searching those VC-funded companies that were also included in the list of angel-funded firms. Therefore, every company with a single entry in the list has no more than a single VC. These 29 start-ups can directly be added to the first, non-syndicated sample group. The remaining companies are then manually sorted by the number of different VCs[7] and assigned to the fitting group. This leads to a group of 64 different start-ups funded by angels and multiple VCs. The second sample group is slightly smaller and accounts for 52[8] companies which fall into the category of angel-backed start-ups with a single supporting venture capitalist.

Classification of additional investors

Previously mentioned classification of investors that fit into both categories of investors is of high importance. We propose an alternative idea of classifying the entirety of the investors. For every investor we added what kind of funding they provided to the definitions of their business. In order to guarantee a clear distinction, angels are only those investors that provided ‘angel investment’ and ‘seed investment’ and are additionally classified under ‘Angel Investment’ in the existing approach. VCs are by definition excluded from angel investments. Seed investment, however, is accepted as a possibility for venture capitalists for two reasons. One is that research has seen VCs moving more and more into earlier forms of investment (Kim and Wagman, 2016). The other reason lies in the number of companies eligible for testing. It would decrease dramatically if seed investment is regarded as an option unique to angels. Since there is not a lot of start-ups belonging to the both groups we end up with the smaller sample size of 114 companies, showing that both approaches have flaws which led to start-ups being disregarded. This is the main reason why decision is made to continue with the first approach.

[7] Some companies are included in the data frame multiple times even though they only have a single VC. The main reason for this is that some VCs have funded the same company multiple times over several funding rounds. Redundancy of data is also a possibility.

[8] The missing company was dropped even though it fitted the description. This happened because the data did not contain crucial data about it (in this case the amount of provided funding was not mentioned).

The remaining 659 investors are processed separately by using the described alternative approach. Angel investors are limited to angel and seed investment. VCs on the other side are excluded from both investment types to ensure a clear cut. Separating the groups like this leads to a clear distinction for the remaining investors. This results in increased number for both investor groups. This effect transfers to the test groups, which also grow in size. Here, 57 additional companies can be identified as dual-funded and the combined volume of both test groups grows from 117 to 174 individual start-ups. Now 77 angel-backed start-ups with a single supporting VC have been identified for testing, increasing the size of this group by 25 additional companies. The effect on the syndicated group is slightly bigger. The number increases from 64 to 97 firms. These changes include possible transitions of start-ups from the non-syndicated group to the syndicated one because additional investors are identified as VCs. Although one can argue that the rigorous classification demanded from the firms significantly reduces number of companies for testing, the number is significantly bigger and more diverse than previous literature.

The process of data procession as it was explained so far is visualized in figure 3.2.

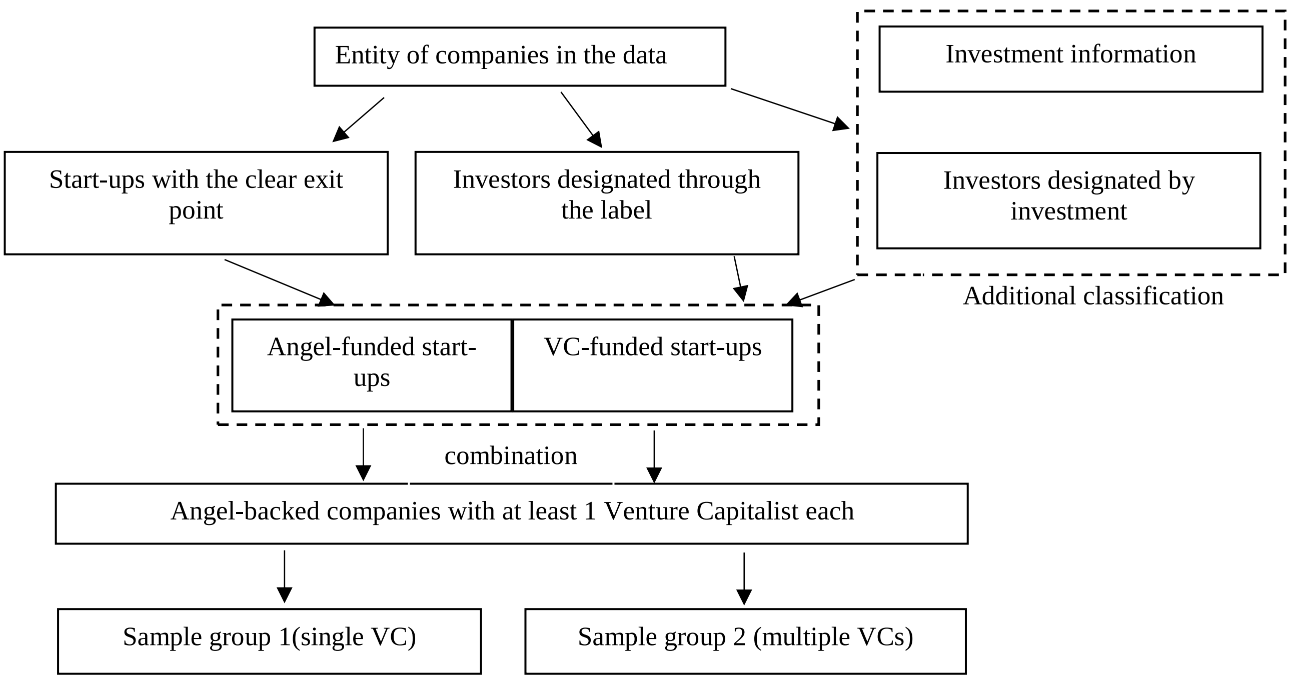


Figure 3.2: Data processing visualized

Here, the outlined section in the upper right represents the additional actions undertaken in this subchapter to account for the investors which are classified as both business angels and venture capitalists. It is important to remember that the chosen classification for the remaining investors is not our first choice of differentiation. Investments with an unknown type are still excluded.

Indirectly represented values

Some value adding and risk sharing characteristics of the founders and financiers that have high impact on venture capital syndication, but are not directly available in the provided data, therefore it is not possible to access them directly, but they are scattered across multiple different data frames. Additional interest of this research are company's age and the number of academic titles in the company.

We define company's age as the period of the time between a start-up's founding day and its exit-stage. This can either be the day of acquisition or the day when it is first offered publicly. A shortened or a lengthened time span from entry to exit is an important variable in the decision making of all involved parties and especially of the investors. Lacking exposure in the base data, the age of the individual start-ups needs to be calculated with aspects given in different data frames.

Since syndication can have a variety of influences on the different groups of companies, they are separated for the next step. This is done by utilizing the type of exit-point they provide as the distinguishing feature. For acquired companies, the calculation revolves around computing the difference between the date of acquisition and the founding day. For firms whose exit stage was marked by an IPO, this is handled similarly with the date of IPO as the final exit-point. The calculation results in the exact company age in days for each represented firm, which was rounded to the closest full day for the mean values and standard deviations.

Finally, the number of degrees per company is provided by simply counting how often each start-ups identification code is mentioned in the resulting data frame. These counts are then added to the corresponding sample group's information and expanded by including 0 as value for all start-ups without mention.

3.2 Descriptive statistics and comparative foundation

Before detailing empirical results, it is can be helpful to take a look at the means of eligible key values because it supports identification of promising test subjects. Therefore, different stages of the data selection process are chosen. This also helps to depict the influence of the imposed limitations and the benefits presented by syndication. As shown in table 3.1 the average funding provided to the start-ups increases with each step of the selection process leading to 103,23 million USD for companies the dual-funded group. This noteworthy increase can be explained by the fact that unfiltered data contains many companies without available information on funding volume. It also includes the failed start-ups, which in this case means they fased out without reaching one of the defined exit stages. The ones we chose for testing, however, are guaranteed to be at least partially successful by the definition of the limitations as reaching the exit-stage is seen as market success.

Table 3.1: mean key values of companies in different stages of data selection. The mean was only calculated for companies with available data for the concerning issue. Firms with missing information were excluded in the respective calculation. are presented in million US Dollar.

Variable	All organizations	IPO + acquired	dual funded
Number of companies	667 221	86 723	174
average funding received	25.06	67,68	103.23
funding rounds	0.3361	0,4679	4.4655
employee count < 251	386 017	31 905	136
employee count 251 - 5000	46 413	11 712	29
employee count > 5000	11 632	5 008	2

The increases in other categories were also mostly expected. Just like the average funding volume, the average number of funding rounds also increases. Since the tested observations are made ex-post, it seems logical for start-ups with a higher investor count to also have more possible entry-points. However, in the context of funding rounds, the relatively small difference between the means of the entirety and the start-ups that reached their exit-stage is unexpected, posing a question that for successful exit number of funding rounds doesn’t necessarily play a big role. The larger number for dual-funded companies, however, was expected, because of the higher minimum number of investors for every company, since every entry in this group has at least two independent investors.

In our dataset information concerning employment figures is provided in buckets without the nominal numbers. The values shown in table 3.1 therefore only depict the available data presented in broader and combined buckets. They can be seen as representatives for small, medium and large companies. Results show that dual-funded companies have relatively low employee counts. While more than 5.7% of firms with a clearly defined exit-stage had 5000 or more employees, that value is 1.15% for dual funded start-ups, used in later testing. This is in the contrast with the previous research. It might hint at a peculiarity of VC syndication, where many of the companies in the sample group were acquired before they could grow their employee numbers to the sizes of their non-funded competitors. The other reason might lie in the theory presented in the previous chapter, where high synergy coefficient leads to less exerted effort needed for success of the project and therefore less workforce, especially in the fields where angel of VC can offer their expertise. It is also possible that the number of employees hired by start-ups was limited in order to reduce cost and make the company more appealing to new investors. The same logic applies for a cost reduction demanded by existing investors.

The literature has the opposing views on the impact of additional venture capitalist on the key characteristics of the start-up. Despite the ideas of the value-adding hypothesis, it is possible that they have a negative effect on company's value. Shortened or lengthened company age, meaning the time span before an acquisition or IPO takes place can have a big impact on the momentum of the company, and therefore it's value. Investors could use their power to influence this characteristic. Alternatively, they can also serve as a signaling factor for possible acquirers. We separated angel-backed start-ups into groups of with either syndicated or non-syndicated VC investments for comparison (table 3.2).

Comparing the two groups exposes a few obvious differences. An increase in the average investors per company can be observed for the syndicated group. A similar effect applies to the average number of funding rounds a start-up offers. Both of these increases were expected by the definition of the sample groups and the underlying theory, especially due to the increased minimal number of investors per company.

Table 3.2: Mean value comparison between test groups. Monetary values are depicted in US Dollar and age presented in days.

Variable	Single VC mean	Syndicated mean	Difference
funding received	43.418.905	149.950.770	106.531.865
funding rounds	3,74	5,04	1,30
investor count	4,84	5,56	0,71
acquisition price	329.807.757	359.593.333	29.785.576
value at IPO	360.772.642	3.384.538.182	3.023.765.540
acquisition age	2.234	2.654	420
IPO age	3.828	3.295	-533
number of degrees	0,40	0,84	0,43

Interesting, however, are the changes for the monetary values; the mean funding value is more than tripled, when compared to the non-syndicated sample group. This increase is of a larger scale than the one observed for the investor count. It implies that syndication might not only add more investors to a start-up but also serves as an incentive to supply higher amounts of investment in the process. This corresponds to the effect Lehmann (2006) observed in the context of shares held by investors under syndication.

Perhaps the most captivating finding presents itself when comparing the mean market values of the companies at the respective exit-stages. Within the non-syndicated sample group, the values for companies of either exit-stage are quite similar. The test group containing syndicated investments, however, shows a different outcome for this aspect. Here the acquisition price shows an increase of about 9% compared to the first sample group. This was expected in the context of the proposed theory. However, the increase seems rather small and its significance is therefore questionable. In general, the increase in funding volume noted in the table 3.2 would be a reason to expect an increase of equal size for the mean of the selling value. Reasons for this change need to be fathomed in the later chapters, should it prove to be significant.

However, the value at IPO shows more extreme changes since it is nearly ten times larger for the syndicated group. Significance in this context could imply a massive advantage brought by venture capital syndication. This would apply for start-ups that strive towards a listing at a stock exchange. Those findings combined can signify a pendency of benefits of VC syndication depending on long-term entrepreneurial decisions. For example, founders with the aim of selling their company with a profit might want to receive funding by a smaller number of investors because additional investors not necessarily increase the selling price and would demand additional shares. On the other side, entrepreneurs that seek to go public could profit immensely from allowing additional venture capitalists into their firm as they have the prospect of significantly increasing their company's performance. At the same time, this big discrepancy could influence the decision-making in stakeholders whether to go in the direction of acquisition or to file for IPO if such opportunity is presented.

3.3 Venture capital syndication: Value, funding rounds and human capital

After gaining an insight into the data and assessing the position of companies without dual funding, we propose hypothesis that through empirical testing can shine more light on venture capital syndication. It is instinctive to imply that a significant connection between VC syndication and an increase in values for key characteristics does not exist. Therefore, the tested hypothesis (H0a) claims that syndication does not affect the key characteristics of angel-backed start-ups. Specifically, for the two-sided t-test[9], it claims that the value of u_1 is equal to u_2. The alternative hypothesis (H1a) claims that u_2 is significantly different from u_1. This means that differences in both directions are tested and if H0 is rejected, a significant effect in either direction is plausible. The direction of this proposed effect can be found by looking at the means.

	Proposition	Explanation
Null hypothesis (H0a)	$u_1 = u_2$	*VC syndication has no significant impact on the key value of angel-backed start-ups*
Alternative hypothesis (H1a)	$u_1 \neq u_2$	*VC syndication has a significant impact on the key value of angel-backed start-ups*

[9] A one-sided test would imply that one value is at least equal if not higher than the other. For example, this can mean $u_1 \geq u_2$. Testing this does not alter the methodology and would lead to the same test statistic and degrees of freedom as the two-sided approach. Therefore, at a later stage it might be useful to look at the results from this perspective in addition.

The hypotheses are formulated relatively open because each value will be analyzed separately and independently from the others. As seen by the previous comparison, the size of the discrepancy between the two sample groups can vary between the characteristics. The formulation enables the use of a fitting test for each specific value. Keeping in mind the different results Lehmann's tests yielded for different determinants of entrepreneurial success, it is clear that testing each of the variables should be done with as little interference from the others as possible. Therefore, the potential exists that for some key values H0 will not be rejected, while it will be for others. The same applies to the kind of impact made in the sample groups. Depending on the regarded aspect, syndication can either have an increasing or decreasing effect.

This leads to assumptions regarding signaling of the companies with the syndicated investments and what impact does that information have on number of funding rounds:

Null hypothesis (H0b)

Companies with syndicated investments do not feature a significantly larger number of funding rounds compared to their non-syndicated counterparts.

Alternative hypothesis (H1b)

Companies with syndicated investments feature a significantly larger number of funding rounds compared to their non-syndicated counterparts.

The last argument is based on the additional value in form of effort and knowledge founders provide in the partnership. Our goal is to see can level of education, specifically number of PhD titles among founders and their employees have a significant impact on venture capital syndication. Since venture capital firms provide mostly managerial advice and information, one would assume that the syndication of investment would increase with the human capital signaled from the company.

Null hypothesis (H0c)

Syndicated companies on average don't have significantly more high-grade academic titles among their employees.

Alternative hypothesis (H1c)

Syndicated companies on average don't have significantly more high-grade academic titles among their employees.

3.4 Testing and results

For the test of choice, we opted for a two-sided t-test because it is possible for syndication to affect the key characteristics in either direction. As shown in the previous subchapters, increases and decreases of means between the two sample groups are not uniform and also vary in size. This implies that a test needs to verify whether the resulting values are significantly different from each other. Afterwards, a one-sided t-test could be used in order to test whether the syndicated value is significantly different in a single direction. The two-sided t-test also has its limitations. For a regular two-sided t-test homogeneity of variance is presumed for the samples (Eckstein, 2014), which given dataset cannot guarantee. This is why we chose Welch-test, as it enables the investigation of data without presuming heterogeneity of variance. This is better suited to test the given sample groups and evaluate the significance of the differences identified earlier.

Beforehand, we need to define the individual variables essential to execute the Welch-test. As the subject of the test, we need the means of the chosen key characteristics the basis of the calculations. In addition, the sample size, the number of samples in the respective group, is required for test statistic and the degrees of freedom. The same applies for each characteristic's standard deviation. Finally, it is necessary to calculate the test statistic and the degrees of freedom in order to make a comparison possible.

The Welch test statistic (t) is calculated by utilizing the means of the individual variable (x_1 and x_2), their respective standard deviation (s_1 and s_2) and the sizes of the sample groups (n_1 and n_2). The mean value for each sample is calculated as $\overline{x_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$ and the associated standard deviation is calculated as ${s_1}^2 = \frac{1}{n_1 - 1}\sum(x_{i1} - \overline{x_\iota})^2$. Having those two variables set up, we calculate t as follows[10]:

[10] Calculation of the Welch test statistic (t) based on Derrick et al. (2016)

$$t = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}}}$$

With the degrees of freedom (df) being dependent on the value's samples size (n) and standard deviation (s) in the following relationship:

$$df = \frac{[\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}]}{\frac{[\frac{{s_1}^2}{n_1}]^2}{n_1 - 1} + \frac{[\frac{{s_2}^2}{n_2}]^2}{n_2 - 1}}$$

After determining the test statistics and the degrees of freedom connected to them, a decision about the hypothesis (H0) can be made for each individual characteristic[11]. For a two-sided t-test, the possibility of rejection is linked to the operator of the t value. The null hypothesis can be rejected if the absolute amount of the t-value is larger than the absolute amount of the comparative statistic at a given confidence level.

There are two main factors that drive the choice of characteristics used in testing. The first one concerns the observed mean values. If they differ substantially between the two sample groups then the chance of finding a significant impact is high and the value is included. The second part of the test group is made up of values that are of special interest in the context of venture capital investments and need to be tested. They are added to the pool of tested characteristics, even if a first look at the observed values does not promise significance due to small differences. Using the explained methodology, the values of interest are then calculated for the start-ups contained in the sample groups. Since the means, as shown previously, have already been computed for most values, the next step is to calculate the associated standard deviations. With those factors prepared, we calculate Welch test statistic. In the following step, it will then be compared to the comparative values provided in the table of the t-distribution. Additionally, results illustrate the dimensions of the associated standard deviation which has even larger values than those of the associated means. In the context of these large values, the values of the t-statistic were expected to be of larger size before the test's execution. The results are summarized in table 3.3.

[11] In this context rejecting H0 means showing that a significant connection between venture capital syndication and an increase in value for the respective key value cannot be denied. Not rejecting it indicates that such a connection cannot be found in the given dataset.

Table 3.3: Welch test statistic of the sample groups and comparative values for all 174 firms before the IPO. Syndicated refers to those firms, which are financed by two or more venture capitalists. The comparison of mean values shows a large difference in value which was already observed before.

Variable	Sample size		Mean		Standard Deviation		t-value
	Single VC	Syndicated	Single VC	Syndicated	Single VC	Syndicated	
Funding received	75	96	43.418.905	149.950.770	70.703.993	528.374.085	1.95322737
Funding rounds	77	97	3.74	5.04	2.39	2.71	3.36183392
Investor count	77	97	4.84	5.56	4.19	3.67	1.17699279
Acquisition price	21	29	329.807.757	359.593.333	613.802.055	415.583.480	0.19268251
Value at IPO	8	19	360.772.642	3.384.538.182	190.688.339	7.608.564.979	1.73100557
Acquisition age in days	67	80	2234	2654	1473	1472	1.72228353
IPO age in days	12	25	3828	3295	1959	1221	-0.86527754
Number of degrees	77	97	0.4026	0.8351	1.0670	1.4265	2.28673897

Surprisingly, the calculated test statistics are of moderate sizes and do not deviate massively from the comparative statistics. Comparing those two statistics shows which differences are significant and which ones are not. It also determines the degree of significance.

For a such comparison we need the degrees of freedom for each of the chosen characteristics which are presented in table 3.4. The calculated values range from 15 up to 171 df. Considering the dimensions of mean values and the measured standard deviations, some characteristics were expected to feature even more degrees of freedom. They still are of reasonably expected size and are still included within the list of comparative statistics, although some values are estimated[12].

Table 3.4: Degrees of freedom (df) calculated for the respective values of table 3.3

Variable	Degrees of freedom	Comparative statistic at confidence level		
		1%	5%	10%
Funding received	99	2,626	1,984	1,66
Funding rounds	170	2,609	1,976	1,655
Investor count	151	2,609	1,976	1,655
Acquisition price	32	2,738	2,037	1,694
Value at IPO	18	2,878	2,101	1,734
Acquisition age in days	140	2,609	1,976	1,655
IPO age in days	15	2,947	2,131	1,753
Number of degrees	171	2,609	1,976	1,655

The associated values of the comparative statistic are shown for confidence levels ranging from 1% to 10%. We need them in order to specify the reliability of the given results. As explained earlier, the presented values can be applied to conduct one-sided and two-sided tests alike. The application of either test is dependent on the situation, but in general, a two-sided approach is used. It might be helpful to consider lower levels of confidence if test statistic and comparative statistic for a key value are only differing by a small margin. This, however, should only be done in rare cases as lower confidence

[12] The difference in comparative statistics are diminishing for higher numbers of df and estimations do not change the outcome of tests.

levels have diminishing significance in testing the hypothesis. As such, finding a significant impact of venture capital syndication on angel-backed start-ups on a 1% level is much more promising than finding it on a level of 10%. This is also illustrated by the difference in value between the 1% and the 10% level of the comparative statistics.

A comparison of the test statistics depicted in table 3.3 with the comparative statistics of table 3.4 shows the significance of the tested key characteristics. The hypothesis (H0) can be rejected if the t-value is bigger than the comparative statistic at a certain confidence level. This is the case for some of the tested values. The funding received by the angel-backed start-ups increased significantly on a confidence level of just below 5%, but well above 10%, leading us to conclude that additional investors also bring more funds than a single VC would. Syndication, therefore, does not only lead to a division of funds among a variety of partners but also leads to an overall increase in funding. This bears similarity to the behavior of company shares observed by Lehmann, where it is concluded that the number of shares for an individual investor decreased with syndication but was increasing overall. Companies with syndicated investments are also proven to feature a larger number of funding rounds compared to their non-syndicated counterparts. This is confirmed at a 1% confidence level, leading to the rejection of hypothesis H0b, leading to conclusion that more entry points for investors are provided by companies with syndicated investments. With a calculated test statistic of 2.29, the average number of PhDs. in the companies is significant at the 5% level of confidence for a two-sided test. This shows that syndicated companies on average have more high-grade academic titles among their employees. Furthermore, the larger standard deviation supports the claim of larger extreme values. This is confirmed by investigating the underlying data. There, it can be seen that the maximum number of degrees within a single company within the non-syndicated group is 6. For start-ups with syndicated investment, the maximum value is 8 and the number of companies with academic titles in their ranks is also increased.

The results concerning the age of the acquired companies don’t show strong significance. With a t-value barely larger than the comparative statistic, it is significantly different at a 10% confidence level. It implies that non-syndicated companies are on average acquired at a younger age than syndicated ones and vice versa. This was suggested by the difference between the means of about 420 days. In contrast, the age at IPO suggested the opposite connection. Here, the mean values suggest that syndication shortens the timespan between founding and the initial public offering. However, for this difference, the test shows no significance at any displayed level. Therefore, this effect cannot be proven and the null hypothesis is not rejected for this characteristic.

Similar results follow corresponding company valuations. The acquisition prices, as suggested by the small difference between the mean values, do not significantly differ from each other. The company valuations at IPO, however, misses significance at the 10% level at a relatively small margin. With a test statistic of 1,731 and a comparative statistic of 1,734, it is significant at a level of just below 10%. This margin is small enough to consider the character as significant in the context of this test. Especially since there were comparably few entries within the sample groups of this particular characteristic. Combined, the syndicated and non-syndicated groups measure only 27 entries. With 8 samples the latter is considerably smaller than the syndicated one. This lack of entries can cause distortion on many possible levels. As such, it is possible that a larger size could have led to more moderate means and standard deviations. Those, in consequence, would affect the test statistic and the degrees of freedom for these key characteristics and could have led to a different result.

3.5 Implications on the previous theoretical research

The abovementioned findings have the power to influence angel-backed start-ups in a variety of ways. With these results, many new questions and opportunities for future research arise. First, it is shown that companies with syndicated venture capital investments have a significantly higher funding volume and at the same time also offer more funding rounds for investors. It would be interesting in this context to compare the number of shares owned by the individual partners between the sample groups. Findings in this field can shine a further light on moral-hazard game, as well as synergy and cooperation game presented in the previous chapter. Unfortunately, the number of shares demanded by the individual investors data is limited and therefore cannot be subject of the performed tests.

On the one hand, the increased investment volume implies an increase in the number of shares demanded by angels and venture capitalists. As they provide more money for the firm it is likely that they request a larger number of shares as security for their investment, as shown by previous research. On the contrary, the significantly higher number of funding rounds could counteract this influence. The reason for this lies in the time span over which the investments are made. Additional funding rounds prolong the funding process. Generally, in later stages of the start-up's development, the risk for investors is decreasing while the company value increases (Lehmann, 2006). That is why VCs are likely to demand fewer shares for an investment of equal size if it happens at a later point in time. These two effects are therefore working against each other. An additional test with fitting data is necessary, in order to find out if one of them is predominant. In this context, the profit of a founder is dependent on the company's value over time and especially at his personal exit-point and the number of shares he holds at that moment. Therefore, an entrepreneur might decide against additional venture capitalists when the decrease in shares outweighs the possible benefit of additional value provided by said VC, which aligns with the theory proposed in the previous chapter. Investors follow a similar thought process, as their profit also depends on the company value and the associated worth of their shares. They do not want to reduce profits by diluting shares.

Number of PhD titles among the founders and employees, which might also signal entrepreneur's effort is significantly higher for the companies with syndicated investments, which could be valuable information for start-ups looking for investments. It indicates an increased interest of venture capitalists related to the number of academic titles. A possible reason for this connection is the smaller perceived risk of failure associated with knowledgeable entrepreneurs. In this scenario, an academic title serves as a signaling factor in order to assure investors of an entrepreneur's credibility and knowledge. Therefore, founders seeking additional investments could increase their chances of gaining venture capitalists' attention by employing board members with fitting degrees. It is also possible that venture capitalists see entrepreneur's knowledge as increase in synergy coefficient which, according to theory presented in the previous chapter, could potentially lead to higher valuations and attract even more funding.

In this work we made a strong distinction between types of exit. Differentiating between IPOs and acquisitions of start-ups was necessary because the different exit-stages are likely to be subject to different influences, which might influence players to react in a different manner. A significant effect of syndication on the final value of the start-up companies was found only for IPO-exit values. On average the chosen start-ups were found to have a nearly ten times higher value for their first offering at a stock exchange than those presented in the non-syndicated group. Still, the test shows that significance in this matter is missed really closely. This fact is amplified by the mentioned small difference between test and comparative statistic and the large associated standard deviations. Therefore, conducting additional tests with larger sample groups seems promising in this regard. Erasing doubt about the significant effect of syndication on the selling price would offer entrepreneurs the possibility to take this option if an IPO is a goal they aim for. The use for investors is similar: knowing additional partners can change the company's worth by significantly increasing or lowering the potential final value of the company will make syndication decision easier for venture capitalists and angels.

Closely related to the topic of company value is the expected age at which the start-ups reach their respective exit stages. The company age of start-ups that were publicly offered is found to be lower under syndication. The difference in the time spans accounts for about 533 days, which is just short of one and a half year. This sparks interest in the context of the value-adding hypothesis from the previous chapter, where additional effort and funds provided by the investors are supposed to improve a company's performance. So, adding more investors, ceteris paribus, should help companies to achieve their goals faster, implying a shortened timespan from entry-stage to exit-stage. However, even though the mean values seem to support the opposite, the findings do not prove statistically significant. This characteristic might also be included in the context of a possible retest. However, because of the smaller calculated test statistic and the larger sample groups, this result is more reliable than the previous one and changes are not as likely.

For the acquisition age of start-ups, the result yielded by the test is the polar opposite to those concerning IPOs. A significant change is observed at a confidence level of 10%. It is proven that syndication significantly lengthens the average time a start-up needs from its founding to its acquisition. This directly leads to the question about the reasons behind this, in the context of the value-adding hypothesis, unexpected change. A possibility is that investors tend to have higher reservations against acquisitions and use their power to delay them. This would be reasonable, as selling at a later point in time is likely to increase value, and therefore their profit. Less likely seems the idea that entrepreneurs, trying to improve their companies by syndication, are also more reserved against selling it. The most logical explanation is provided by the idea of risk reduction. An increased company age leads to more investment opportunities and allows further investors to enter the company. Therefore, it is more likely that a higher company age leads to more syndication rather than vice versa.

Implication on Equilibrium in Partnership Games

As explained, significant effects for the tested key values harbor the possibility to influence different fields of research. A large impact can be found in an extension of our previous theoretical chapter, where it can empirically confirm the effect of synergic effect of syndication. Our empirical results verified multiple parts of the proclaimed theory. For example, the significantly higher value at IPO or the increased company age at acquisition. At least, for start-ups aiming for an IPO, this implies an increase in the start-up's value without the need to change the entrepreneur's effort function. For acquired companies, the entrepreneurs' benefits might be indirect as they most likely originate from fewer shares given to investors due to a reduction of risk. The assumption of stagnating entrepreneurial effort originates from the fact that the additional funds are provided by new investors and not by the ones which already support the company. Therefore, this increase in funding volume is independent of the entrepreneurial effort the existing venture capitalists expect from the founder they support. Each investor is more likely to have his own expectations in this regard. A new type 1 equilibrium would be the logical consequence. While it would feature higher overall investment, the exposure of every single investor would be decreased, as suggested by Lehmann (2006). Therefore, their individual investments and risks are lowered. This change is shown in figure 3.3.

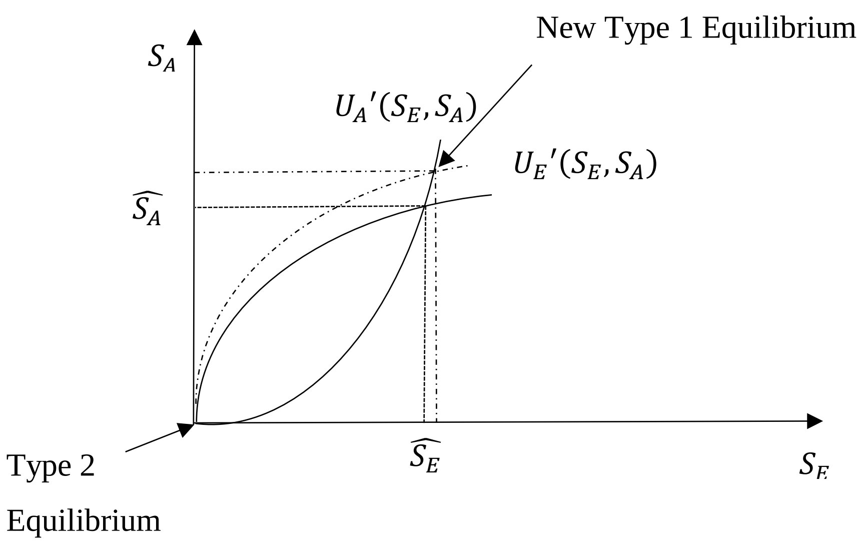


Figure 3.3: Update of figure 2.2 featuring the possible changes of type 1 equilibrium in partnership game through VC syndication

It shows higher ideal values for the investments made by the syndicate of VCs ($\widehat{S_A}$) and the entrepreneur's effort ($\widehat{S_E}$). While for S_E a rather large change can be observed, the changes in S_A seem marginal instead. This fits in with the above-mentioned discussion. In this case, the new equilibrium does not demand a major increase in effort from the founder, because the added value is created by knowledge and skill of the involved VCs as well as their reduced risk. Additional funding is provided by new investors and does not increase expectations. However, as described in chapter 2.1 the risk of moral hazard is likely to rise with the number of VCs, especially if they are unexperienced with syndication (Casamatta and Haritchabalet, 2007).

The trade-off from an entrepreneurial point of view lies in balancing the personal gains and losses of the respective founder. Gains in this context are archived through an increase in company value and consequently higher prices for the shares held by each partner. Counteracting that is the diminishing number of shares an entrepreneur owns under syndication. His shares are diluted with the entry of additional venture capitalists while he improves the future prospects of the company. The results of this trade-off are dependent on the individual entrepreneur as their influence differs according to the goal he sets out for the company. In this sense, an entrepreneur who mainly focusses on the financial gain will likely optimize his expected future pay-out by weighing his shares in the company and their possible future value under syndication. Another type of founder might want to maximize the expected future value of his company with less regard to his own financial profits. This would correspond to goals like ensuring the long-term survival of the start-up or maximizing the company's growth. Reaching those targets might be more important to such an entrepreneur than his raw financial gain. Therefore, such an entrepreneur might give up more shares to increase the likelihood of additional investments occurring from new sources.

The theory in the previous chapter discusses the effects of moral hazard in the relationship of entrepreneurs, angel and venture capitalists. It should be noted that this expansion can also be subject to these effects. While overall the VC's behavior stays as depicted in Figure 3.3, the effect of the effort S_E exerted by the entrepreneur is also subject to change. Depending on the individual founder this influence could manifest in two ways. First, an increase in total investment volume can be an incentive to adjust the effort in an equal manner, resulting in a relocation of the $U_A'(S_E, S_A)$ curve. Since more investment, in theory, causes more effort it would be situated further to the right in the graph. Therefore, an even more favorable type 1 Equilibrium would be created resulting from a new intersection caused by increased values from both $U_E'(S_E, S_A)$ and $U_A'(S_E, S_A)$. Secondly, entrepreneurs can come to the opposite conclusion. Reducing their effort as additional funds are seen as a guarantee of success or slacking off because they expect the investors to compensate for their shortcomings. In this scenario changes in funding volume would lead to a steeper curve of $U_A'(S_E, S_A)$ and with that the intersection would be situated closer to the Type 2 Equilibrium than the first option. Both possible outcomes are presented in Figure 3.4.

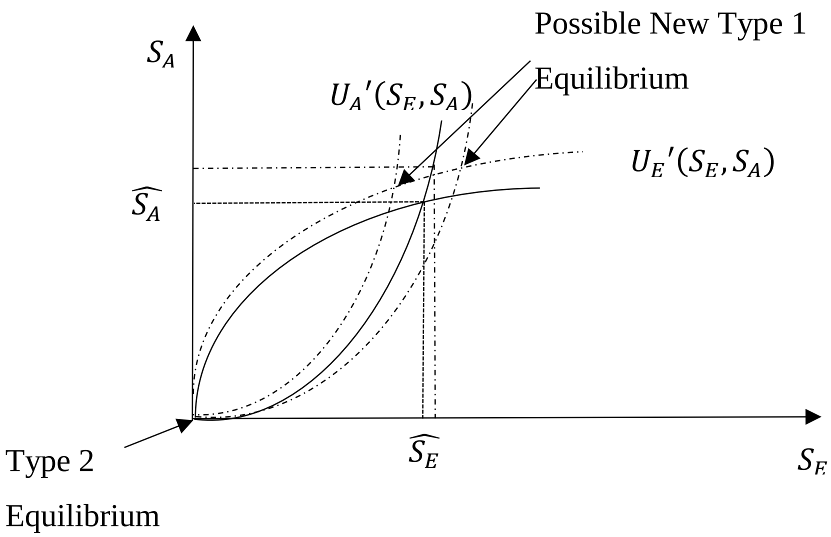


Figure 3.4: Possible impacts of changed entrepreneurial behavior on type 1 equilibrium in partnership game through VC syndication

3.6 Conclusion

In this chapter we analyzed the possible effects of venture capital syndication on angel-funded start-ups. With an initial size of over 600.000 companies, the dataset was filtered according to the limitations of the underlying theories, until 174 eligible companies were used in testing. Our results show that VC syndication increases the average amount of funding offered to entrepreneurs. This presents multiple advantages to the different involved parties. Additional funds accelerate growth and help newly funded firms to survive difficult phases in their development. Especially in earlier phases, this eases the problems of money shortage which is a serious problem for start-ups. Spreading these additional funds among multiple investors also enables entrepreneurs to cooperate with a variety of different companies and individuals. At the same time, investors capitalize on less exposure to a specific company's risk. This enables them to fund additional projects and increase their expected returns. Additionally, smaller investments enable them to further diversify their portfolio and reduce their risk even more.

Our results also suggest that syndicated ventures have higher number of funding rounds, which results in a higher number of possible entry-points provided by those start-ups. However, it is not clear if syndication leads to more funding rounds or vice versa as both options are theoretically possible. This can be a subject of further research.

Academic titles of higher grade, like PhDs. also have an influence on the chance of syndication. While it is possible that syndicated start-ups are urged by investors to hire employees and board members with fitting degrees, it is more likely that knowledge of entrepreneurs, portrayed by degrees, serves as a signaling factor for venture capitalists and possibly also angels. Entrepreneurs can utilize this information to increase their chances of successful future funding. The significantly higher exit valuation of start-ups shows that syndication does not only reduce risk but also adds value to the companies. The proof, however, is limited to this group of companies, as the results for their counterparts that opt for acquisition type of exit do not support the same claim. Further research should shed more life on this type of phenomena.

Company's age on the other side showed a change opposed to the one of company value. Acquired companies were significantly older. This implies a reduced risk for investors based on additional available information as well as the signaling of competitors. In combination with the Lehmann's findings, this suggests a trade-off made between risk and value, depending on the type of exit-stage.

4 Expected Shortfall static portfolio optimization using genetic algorithm

Financial institutions have to allocate so-called ‘economic capital’ in order to guarantee solvency to their clients and counterparties. Although, until recently, Value-at-Risk (VaR) was the go-to industry standard for allocating capital at risk, The Basel IV Capital Accord advocates a shift from VaR to an Expected Shortfall (ES) (or conditional value-at-risk or tail value-at-risk) as measure of risk under stress. This should not come as a surprise, since academics and practitioners state that ES has been characterized as the smallest coherent and law-invariant risk measure to dominate VaR.

There are numerous reasons for this. For a given threshold probability *p*, the VaR is defined so that, with probability *p*, the loss will be smaller than VaR. This definition only gives the minimum loss one can rationally expect, but does not state anything about the typical value of that loss (which can be measured by the Expected Shortfall). The literature also recognizes that the VaR measure is not coherent, because it lacks the mandatory properties of subadditivity and convexity (meaning that summing VaR’s of individual portfolios will not necessarily produce an upper bound for the VaR of the combined portfolios). Therefore, this contradicts the main principle of diversification in finance (Artzner et al, 1999; Tasche, 2002).

Acerbi and Tasche (2001) redefined ES, making it similar to conditional VaR (proposed by Rockafellar and Uryasev, 2000). The ES risk measure is closely related to VaR. For continuous distributions, ES is defined as the conditional expected loss under the condition that it exceeds VaR. Nevertheless, for general distributions (including discrete distributions), ES is defined as the weighted average of VaR and losses strictly exceeding VaR. Since numerical experiments suggest that the minimization of ES also leads to almost optimal solutions in VaR (because VaR, by definition, never exceeds ES); this means that portfolios with low ES must have low VaR as well.

Additionally, when the return-loss distribution is normal, these two measures are equivalent, i.e. they provide the same optimal portfolio. On the other hand, for very skewed distributions, ES and VaR risk optimal portfolios may be quite different (Rockafellar and Uryasev, 2000).

The fundamental principle of diversification was introduced by Markowitz's (1952) mean–variance theory (MV), where an investor attempts to maximize expected portfolio-return for a given amount of portfolio risk or minimize portfolio risk for a given level of expected return. This theory was heavily criticized for certain unrealistic assumptions. In response, researchers combined some real-world constraints (such as bounds on holdings, minimum transaction lots and sector capitalization constraints) to the portfolio model, giving birth to whole new filed of optimization problems.

Beside these single-objective formations there exist a lot of evolutionary algorithms that maintain the original multi-objective structure. These algorithms are called Multi-Objective Evolutionary Algorithms (MOEAs) which have been of increasing interest over the last decades (Anagnostopoulos and Mamanis, 2011). Skolpadungket et al. (2007) did a research on different MOEAS, including Vector Evaluated Genetic Algorithm (VEGA), Fuzzy VEGA, Multi-Objective Genetic Algorithm, Strength Pareto Evolutionary Algorithm 2nd (SPEA2) and Non-Dominated Sorting Genetic Algorithm 2nd (NSGA2). They performed tests on the original model and added cardinality constraints, floor constraints and round-lot constraints and concluded that the SPEA2 performs best. Most of these mentioned articles used Markowitz' original mean-variance risk measure in combination with some additional constraints. But latest developments show an increasing interest in different risk measures. Pflug (2000) presented a comparison of ES and VaR as risk measure for portfolio selection. He pointed out that the ES model can be written as a linear model. Therefore, it is convex and every local minimum is a global minimum. In contrast to that the VaR model is non-convex and therefore much harder to solve.

Genetic algorithms (GAs) have become the method of choice for optimization problems that are too complex to be solved using deterministic techniques. GAs are well suited to multiobjective optimization problems (MOP), as they are inspired by biological processes which are inherently multiobjective. Thanks to Multiobjective Evolutionary Algorithms (MOEAs) techniques, the classical portfolio model can be extended to handle two or more conflicting objectives subject to various realistic constraints (Metaxiotis and Liagkouras, 2012.)

The idea of using techniques based on the simulation of the mechanism of natural selection to solve problems can be traced as far back as the 1930s. At the same time, portfolio optimization problems grow in complexity since they frequently include a lot of additional constraints. This study draws conclusions from three strands of previous research. The first strand concludes from works which attempt to use GA in portfolio optimization. Arnone et al. (1993) considered a bi-objective, mean return-risk, unconstrained portfolio optimization problem regarding downside, variance-based risk measures. They transformed a bi-objective problem into a single-objective problem using a trade-off coefficient. Chang et al. (2009) presented a GA approach to mean return-risk portfolio optimization problems with the cardinality constraint. Similar to Arnone et al. (1993), the authors applied bi-objective to a single-objective problem transformation by using the trade-off coefficient with semi-variance, mean absolute deviation, and variance with skewness as risk measures. Ranković et al. (2014) opt for static portfolio allocation based on historical Value at Risk (VaR) by using GA. They present two different single-objective and a multiobjective techniques for generating mean–VaR efficient frontiers with good risk/reward characteristics of solution portfolios, with a trade-off between the ability to control the diversity of solutions and computation time.

The second strand of literature documents the problem of portfolio optimization under the Expected Shortfall as the risk measure. The concept of Expected Shortfall is gaining popularity in fields well outside the realm of financial analysis. For example, ES as a measure of risk has recently grown popularity in breast cancer therapy (Chan, Mahmoudzadeh, & Purdie, 2014), scheduling (Quan, He, & He, 2014; Sarin, Sherali, & Liao, 2014), and machine learning (Takeda, 2009; Takeda & Kanamori, 2009, 2014; Wang, Dang, & Wang, 2015). In the finance literature, Tasche (2002) provides a setup, documenting a general method on how to attribute Expected Shortfall risk contributions to portfolio components. Ciliberti et al. (2006) discuss the feasibility of portfolio optimization under Expected Shortfall, remarking that using empirical distributions of returns is not well defined when the ratio N/T of assets to data points is larger than a particular critical value. This value depends on the threshold probability of the risk measure in a continuous way. The lower the value of that probability, the larger the length of the time series needed for the portfolio optimization.

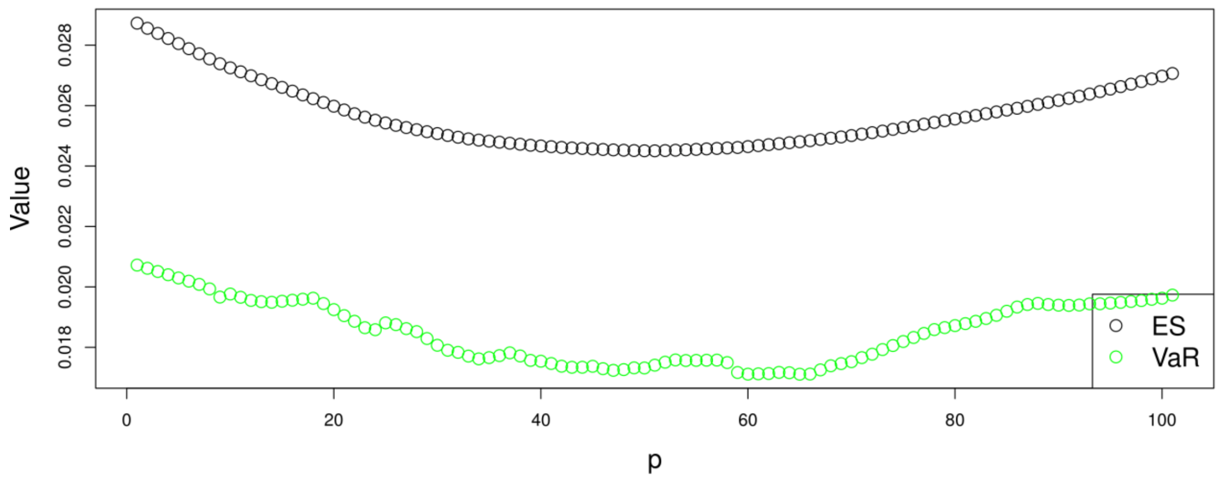


Figure 4.1: VaR and ES of portfolios of 2 assets, Münchner Rückversicherungs-Gesellschaft AG and Beiersdorf AG, showing the existence of local minima for the historical VaR. The proportion of the portfolio invested in Münchner Rück. is (100-p)% the remaining proportion is invested in Beiersdorf AG. Results were obtained for 100 levels of p.

Rockafellar and Uryasev (2000) presented a portfolio selection model that considered ES as measure of risk. Their model computed VaR and optimized ES simultaneously. Krokhmal et al. (2002) adapted and modified this approach. They included transaction costs, cardinality constraints, liquidity constraints and bounds on positions. The resulting model was solved with linear programming and conducted for S&P 100 stocks. Quaranta and Zaffaroni (2008) instead used robust optimization to solve the proposed model and concluded good results, but also emphasized that these results were heavily affected by their particular problem set.

Dallagnol et al. (2009) implemented historical VaR as risk measure and applied two different single-objective techniques. To find portfolios with minimum VaR given a certain level of expected return they used GAs and Particle Swarm Optimization (PSO) and concluded that PSO is in fact faster than GAs in terms of number of iterations and total running time but also more sensible to the initial population. Caccioli et al. (2015) produce contour maps that allow one to quantitatively determine the length of the time series required by the optimization for a given number of different assets in the portfolio, at a given confidence level, and a given level of relative estimation error. They state that the necessary sample sizes invariably turn out to be unrealistically large for any reasonable choice of the number of assets and the confidence level.

The third strand of literature documents the different choices researchers make for GAs to answer complex portfolio optimization problems. Lin and Liu (2008) present a study about portfolio optimization based on the Markowitz (1952), model with the minimum transaction lots constraint. The authors generated mean-variance efficient frontiers by minimizing risk for given return levels using single-objective GA. Corne, Knowles, and Oates (2000) show that SPEA performed outstandingly in comparison to other multiobjective evolutionary algorithms and has therefore been a point of reference in various recent investigations. Zitzler et al. (2002) propose an improved version called SPEA2, which incorporates (unlike to its predecessor) a fine-grained fitness assignment strategy, a density-estimation technique, and an enhanced archive truncation method. Anagnostopoulos and Mamanis (2011) also examine the mean-variance, mean-ES, and mean-VaR optimization problem with quantity, cardinality and class constraints. They showed that NSGA-II, SPEA2 and PESA performed efficiently, and that their performance was independent of the risk-measure used.

Changetal et al. (2009) investigated the performance of GAs for the portfolio optimization problem with different risk measures and cardinality constraint. The risk measures used were mean-variance, semi-variance, mean absolute deviation and variance with skewness. The authors pointed out that GAs are advantageous over other search methods because they are very flexible and obtain good solutions to difficult problems very easily. Krokhmal et al. (2011) gave an overview over different risk measures including, amongst others, Expected Regret, VaR and ES. Ranković et al. (2014) solved the portfolio selection problem with VaR risk measure with three different GAs, two single-objective and one multi-objective one. The first GA considered the formulation with a trade-off parameter, the second minimized VaR with a desired level of return and the third was SPEA2. They showed good risk return characteristics of solution portfolios and emphasized a trade-off between controlling the computational time and diversity of solutions.

This work takes up the ideas of Ranković et al. (2014) but with ES instead of VaR. Furthermore, it is investigated which operators for GAs fit best for the presented model.

Another focus of this chapter is on the effectiveness of different GA techniques for static portfolio optimization when return and percentage ES are set as optimization objectives. We employed a standard single-objective technique and SPEA2 method as a fully multiobjective technique to derive mean return - ES efficient frontier. Additionally, to illustrate effectiveness of our approach, we compared our results with results obtained from the same technique using the mean return-historical VaR efficient frontier proposed by Ranković et al. (2014). To our knowledge, this is the first paper that studies ES portfolio optimization using the actual portfolio and various genetic algorithm techniques.

4.1 Portfolio Selection and Risk Measurement

Beginning of this chapter is dedicated to the general introduction of portfolio selection. Besides, we provide two different amendments and an introduction to risk measurement is provided, including a general definition of risk. After that, we introduce two different risk measurement, the Value at Risk (VaR) and the Expected Shortfall (ES). We compare Pro's and Con's of both measurements, and at the end, a short example clarifies the differences of both measurements.

4.1.1 Portfolio Selection

In the financial world a portfolio describes a compilation of investments, like stocks, bonds and mutual funds (Branke et al., 2009,).

The process of selecting portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio (Markowitz, 1952).

The aim of portfolio selection is to find a portfolio with maximum return and minimum underlying risk. The foundation of modern portfolio theory was provided by Markowitz (1952). The following subsection will give a short recap of the Markowitz Model for portfolio selection.

Markowitz Model:

Before Markowitz introduced his model for selecting the optimal portfolio, researchers believed that investors should maximize discounted returns in favor of optimizing a portfolio. Markowitz (1952) claimed, that the foregoing rule never implies that there is a diversified portfolio which is preferable to all non-diversified portfolios. Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim. The foregoing rule fails to imply diversification no matter how the anticipated returns are formed; whether the same or different discount rates are used for different securities; no matter how these discount rates are decided upon or how they vary over time. The hypothesis implies that the investor places all his funds in the security with the greatest discounted value. If two or more securities have the same value, then any of these or any combination of these is as good as any other.

He came up with the idea that investors should reflect upon return and risk at the same time and that they should chose different investment alternatives depending on their return-risk trade off. To measure risk and return Markowitz introduced expected return and standard deviation, respectively variance (Kolm et al., 2014,). The expected, per period return of a portfolio is given by:

$$r(x_1, \dots, x_N) = E\left[\sum_{j=1}^{N} R_j x_j\right] = \sum_{j=1}^{N} \underbrace{E[R_j]}_{r_j} x_j$$

The decision variable x_i represents the fraction of asset i held in the portfolio. These variables are often referred to as weights. r_i denotes the expected return of asset i. In this chapter log returns are used. In differ to the return the variance of a weighted sum is not the sum of the different variances. To compute the variance the covariance σ_{ij} of R_i and R_j is needed:

σ_{ij} = E[$(R_i - E[R_i])(R_j - E[R_j])$]

The variance of a portfolio is then computed by:

$$V(R) = \sum_{i=1}^{N} x_i{}^2 V(R_i) + 2\sum_{i=1}^{N}\sum_{j>i}^{N} x_i x_{ij} \sigma_{ij} = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_{ij} \sigma_{ij}$$

4.1.2 Portfolio optimization model variations

Following Markowitz (1952), multi-objective optimization model aims to simultaneously maximize the portfolio return $\mu_p(x)$ (Eq. 3.2) and minimize the portfolio risk $\sigma_p^2(x)$ (Eq. 3.1).

Model 1

$$minimize \ \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} = \sigma_p^2(x) \qquad (3.1)$$

$$maximize \ \sum_{j=1}^{n} r_j x_j = \mu_p(x) \qquad (3.2)$$

$$s.t. \sum_{i=1}^{n} x_j = 1, \qquad (3.3)$$

$$0 \le x_j \le u_j, \qquad j = 1, \dots, n. \quad (3.4)$$

Equation 3.3 ensures that the total sum of the fractions is equal to 1. This constraint is also called budget constraint if x_i is interpreted as a fraction of budget invested in asset j. Equation 3.4 prevents to choose negative fractions and limits the maximum fraction of asset j. It therefore prohibits short selling. Unless otherwise specified μ_j is equal to 1. This is widely known quadratic problem, hence convex, and therefore easy to solve. The model proofed to select efficient portfolios if investors are rational and returns are multivariate normally distributed (Branke et al., 2009).

It is easy to convert this multi-objective model into a single-objective one. This is possible by limiting either the risk or the return of the portfolio and minimizing the risk or maximizing the return respectively. For example, equation 3.2 could be used as a constraint that demands at least a certain level ρ of return. This results in the following model:

Model 2

$$minimize \ \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} = \sigma_p^2(x)$$

$$s.t. \ \sum_{j=1}^{n} r_j x_j \ge \rho$$

$$\sum_{i=1}^{n} x_j = 1,$$

$$0 \leq x_j \leq u_j, \qquad j = 1, \dots, n.$$

It is also possible to combine risk and return in a single-objective function. For this a scaling parameter $\lambda \in [0,1]$ is needed which represents the risk-aversion of an investor. Smaller values of λ are indicative of high-risk aversion. Equations 3.1 and 3.2 then change to the following formulation.

<u>Model 3</u>

$$maximize\ \ \lambda \cdot \mu_{p(x)} - (1-\lambda) \cdot \sigma_p^2$$

$$s.t. \sum_{i=1}^{n} x_j = 1,$$

$$0 \leq x_j \leq u_j, \qquad j = 1, \dots, n.$$

All efficient combinations $(\mu_p(x), \sigma_p^2(x))$ build the so-called Efficient Frontier[13].. Figure 4.2 shows an example for the efficient frontier of all DAX assets. The green points symbolize the efficient portfolios, the black points are inefficient portfolios because they are dominated by the green ones. The red squares represent the different assets.[14]

[13] Efficient means that there does not exist any combination of assets which has the same return but less risk (Rifki and Ono, 2012)

[14] To build the frontier the portfolio.optim function from the tseries package of R was used.

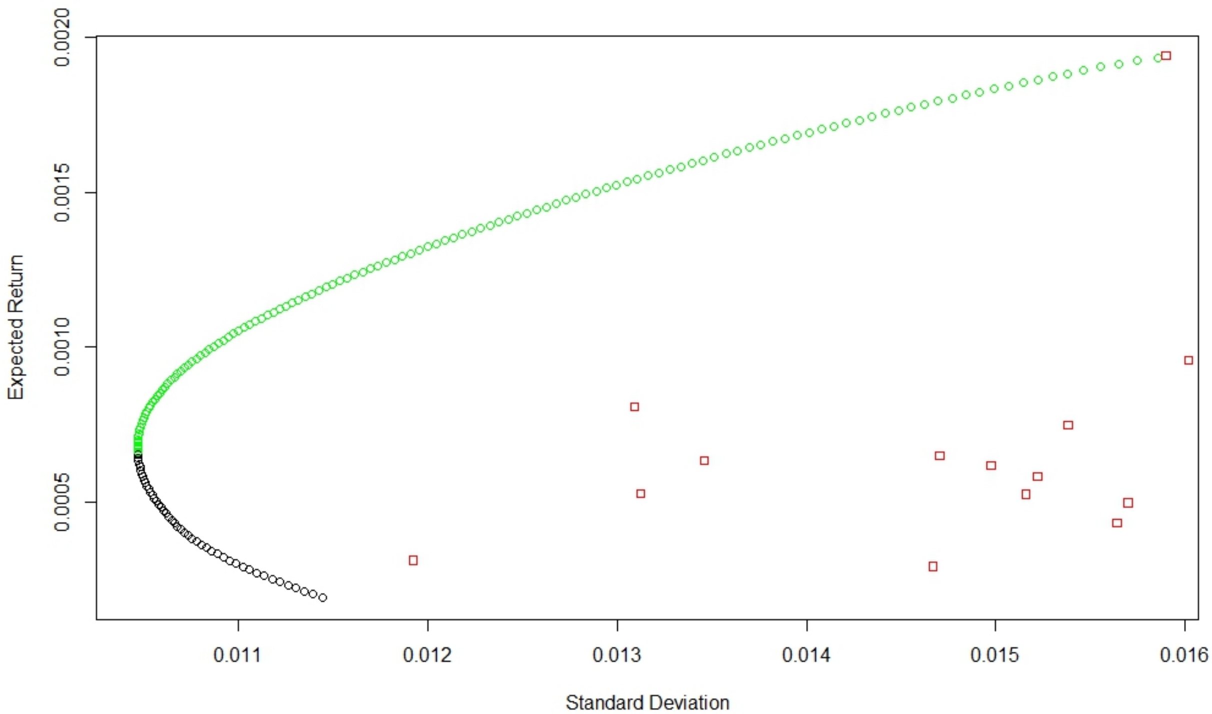


Figure 4.2: Efficient Frontier, DAX. Data from 01.January 2015 – 01. April 2017. For better readability the depicted section was decreased. Therefore not all assets are shown in the figure.

4.1.3 Risk Measurement

There are several ways to measure the risk of an investment. In measuring risk, an investment which is undoubtedly riskier than the other investment must be assigned a higher risk by the chosen method. Therefore, Artzner et al. (1999) introduced the following four axioms for coherent risk measures. A function ρ is a coherent risk measure if it fulfils the following four axioms.

Definition 4.1 (Coherent risk measure)

Let V be a set of real-valued random variables. A function $\rho: V \rightarrow \mathbb{R}$ is coherent if:

$X, Y \in V, X \leq Y \Rightarrow \rho\,(X) \geq \rho\,(Y)$ (Monotonicity)
$X, Y, X + Y \in V \Rightarrow \rho\,(X + Y) \leq \rho\,(X) + \rho\,(Y)$ (Sub-additivity)
$X \in V, h > 0, hX \in V \Rightarrow \rho\,(hX) = h\,\rho(X)$ (Positive homogeneity)
$X \in V, a \in \mathbb{R} \Rightarrow \rho\,(X + a) = \rho\,(X) - a$ (Translation invariance)

Monotonicity ensures that if portfolio Y has greater values in nearly all scenarios than portfolio X, it has to be considered as less risky. Sub-additivity implies that diversification is beneficial. The risk of two portfolios together cannot get worse than the sum of the two separate risks. Positive homogeneity implies that the risk of a portfolio is connected with its size. Increasing size leads to increased risk. Translation invariance describes the principle that adding cash to a portfolio reduces the risk by the same amount.

Taking the standard deviation or variance as a measure of risk conceals risks in itself. Markowitz (1959) pointed out that his initially measure of risk avoids significant losses and profits and that it, therefore, sacrifices too much-expected return since variance considers exceptionally high and meagre returns equally undesirable. He, therefore, introduced a new risk measurement, the semi-variance, also known as semi-deviation, which tends to concentrate on reducing losses an investor worries about under-performance rather than over-performance, semi-deviation is a more appropriate measure of investor's risk than a variance.

Nonetheless, practitioners and academics tend to use variance as a measure of risk for portfolio optimization over decades. As reasons, Markowitz (1991) named costs, convenience and familiarity and therefore focused his research on variance. Since Markowitz was a forerunner in portfolio optimization, other analysts and practitioners followed his lead. Nevertheless, the interest in downside risk is increasing (Estrada, 2015). Two widely known downside risk measurements are VaR and ES, which are introduced hereinafter.

Value-at-Risk

VaR is a standard risk measure for financial risk management and is defined as the possible maximum loss over a given holding period within a fixed confidence level. (Yamai and Yoshiba, 2005). In mathematical terms the VaR is defined as:

Definition 4.2 (Value-at-Risk):

Let $\alpha \in (0,1)$ be a given confidence level and $F_L(x) = P(X \leq x)$ the distribution function of the corresponding loss distribution. For a given loss L the VaR at confidence level α is defined as the smallest number l so that the probability for L to be larger than l is smaller or equal than $(1 - \alpha)$. This definition is illustrated in the figure 4.3.

$VaR_{\alpha} = VaR_{\alpha}(L) = inf\{l \in R: P(L > l) \leq 1 - \alpha\} = \ inf\ \{l \in R: F_L(l) \geq \alpha\}$

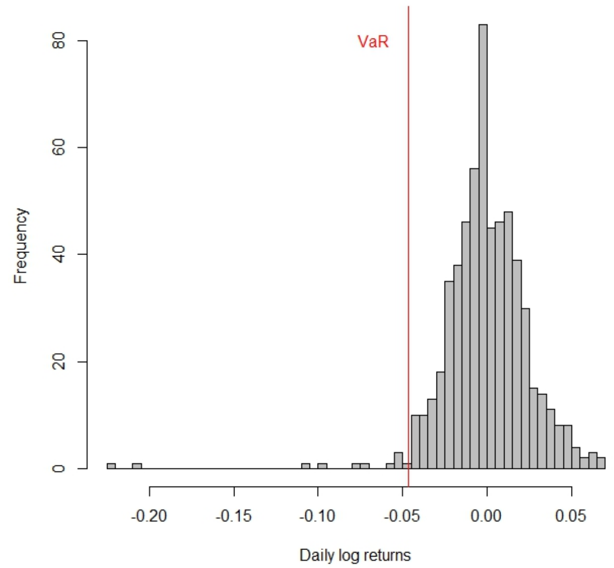


Figure 4.3 Histogram of log returns and 95% VaR, Volkswagen 01. January 2015 – 14. April 2017

There are several advantages when it comes to the use of VaR. Acerbi et al. (2001) emphasized the adaptability of VaR since the measuring unit is always 'lost money' for all financial instruments and pointed out that the VaR takes an estimation of future events into account and returns the risk of a financial instrument as a single value. Both properties would let the VaR be advantageous over the 'Greeks'. However, there are also several disadvantages claimed in the literature. At first, VaR does not regard loss beyond the VaR level, since it only quantifies percentiles of profit-loss distributions. Second, it is, in general, not a coherent measure. It was shown that VaR, as a measure, is not always sub-additive. Sub-additivity mathematically embodies the reduction of risks associated with the concept of diversification.

This for example leads VaR to prohibit diversification in some problem sets (Yamai and Yoshiba, 2005; Artzner et al., 1999). Acerbi et al. (2001) mention that existing literature which show Sub-additivity of VaR under adoption of normally distribution led many practitioners to believe in a general validity. They therefore introduced an example where sub-additivity is violated. Szegö (2002) goes even so far, that in his opinion VaR as measure of risk is not acceptable. To try to measure risk without this (coherent) property is like measuring the distance between two points using a rubber band instead of a ruler (Szegö, 2002).

He also claims that, in the case that VaR is sub-additive, the results obtained by optimizing a portfolio with VaR as risk measure coincide with the Markowitz mean-variance minimized portfolio and therefore makes the use of VaR obsolete since it was introduced to measure risk for weird distributions.

Example 4.1 (VaR not coherent)

Given two different bonds A and B which do not default at the same time. Assuming there exist two default stages for both bonds, namely 70 with a probability of 3% and 90 with a probability of 2%, otherwise the return is set to 100. The following table gives an overview about the possible events.

Event	**A**	**B**	**A+B**	**Prob**
1	70	100	170	3%
2	90	100	190	2%
3	100	70	170	3%
4	100	90	190	2%
5	100	100	200	90%

Assuming that the initial bond value is equal to the expected payoff value the VaR can be computed very easily.

Initial value	98.9	98.9	197.8
VaR	8.9	8.9	27.8

This results in $\rho\,(A+B) > \rho\,(A) + \rho\,(B)$ and therefore harms the Sub-additivity condition of coherent risk measures. Optimizing a portfolio with minimal risk would lead to a one bond solution and would disregard any diversification.

Due to the fact that VaR only refers to the threshold level for losses it is even considered to be one of the reasons for the financial crisis of the years 2007 - 2008 (Lim et al., 2011). Danielsson et al. (2001) point out that VaR regulation can destabilize an economy and induce crashes when they would not otherwise appear.

In consequence of the non-Sub-additivity Artzner et al. (1997) introduced Expected Shortfall, which will be presented in the following.

Expected Shortfall

Over the years, the interest in ES as a measure of risk has grown considerably. Similar to McNeil et al. (2015), we define ES as following:

For a loss L with $E(|L|) < \infty$ and distribution function F_L, the ES at the confidence level $\alpha \in (0,1)$ is defined as

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u\,(F_L)du$$

where $q_u(F_L) = F_\tau(u)$ is the quantile function of F_L. Given the VaR_α, ES_α, can be calculated as

$$ES_\alpha = E[X|X \geq VaR_\alpha(X)]$$

In words, the ES is the expected value of loss exceeding the VaR. Acerbi et al. (2001) mentioned that VaR, as a risk measure, could easily be replaced by ES or any other statistics of the left tail without much effort since the basics of both methods are the same. However, the ES does not come without its disadvantages. One major disadvantage may be the fact that ES is more vulnerable to estimation errors than VaR, and therefore requires a bigger sample size (Yamai and Yoshiba, 2005).

Following the scope of the portfolio optimization, let $f(x,y)$ be the loss of the portfolio, where y is a random vector with the density function $p(y)$ denoting uncertainties of market parameters that affect the loss. The probability of $f(x,y)$ under a threshold value β is $\psi(x,\beta)$. The values of ES for the loss function associated with x and any specified confidence level $\alpha \in (0,1)$ is given by:

$\phi_\alpha(x) = E[f(x,y)\,|f(x,y)\ \geq \beta_\alpha(x)]$

$$= \frac{1}{1-\alpha}\int_{f(x,y)\geq\beta_\alpha(x)} f(x,y)p(y)dy$$

Since this notation is not convenient to use in practice, we reconstruct it as following:

$$\phi_\alpha(x) = \min_{\beta \in R} F_\alpha(x, \beta)$$

where

$$F_\alpha(x,\beta) = \beta + \frac{1}{1-\alpha}\int_{y \in R^m} [f(x,y) - \beta]^+ p(y)\phi$$

and $t^+ = \max\{t, 0\}$

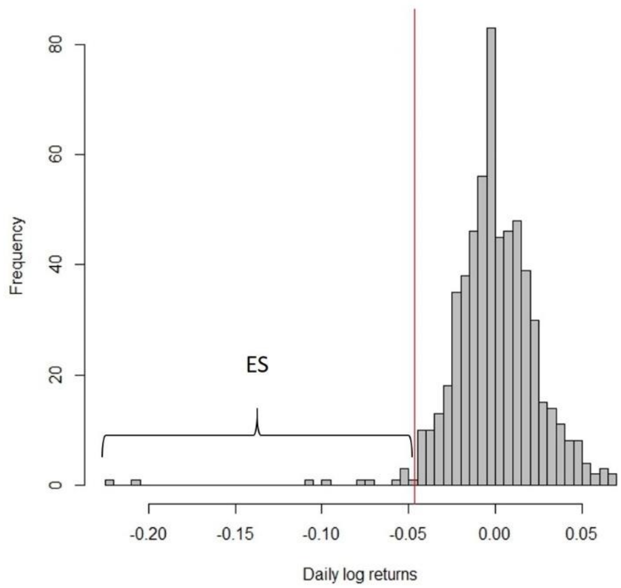


Figure 4.4: Histogram of log returns, 95% VaR and 95% ES, Volkswagen 01. January 2015- 14. April 2017. ES is the expected value of the marked area

Example 4.2 (Differences between VaR and ES)

Considering two investment alternatives A and B with a value of \$100 each. The possible payoff is given in the following table.

Investment A	\$100	Investment B	\$100
Payoff in 96%	\$100	Payoff in 96%	\$100
Payoff in 4%	\$50	Payoff in 4%	\$50
		Payoff in 1%	\$0

Computing the 5% VaR for both investments would result in the same value of \$0 and therefore the VaR approach would not differentiate between both alternatives, whereas computing the 5%

ES would result in a value of -$50 for alternative A and -$62.5 for alternative B. VaR fails to diversify between both alternatives, although alternative B proves to be riskier (based on Acerbi et al. (2001)).

4.2 Genetic Algorithms

GAs are stochastic and iterative algorithms based on the concepts of natural evolution like inheritance, selection, crossover and mutation. The idea is to mimic natural evolution in order to find suitable solutions for complex problems (Mitchell (1998); Holland (1975)).

Table 4.1 gives an overview over some important terminology of biological evolution and its meaning in context of Evolutionary Algorithms.

Table 4.1: Overview of important terminology for Evolutionary Algorithms - taken from Gerdes et al. (2004).

Name	**Biological Evolution**	**Evolutionary Algorithms**
Chromosome	defines attributes of organism	string of numbers or signs, not necessarily fixed length
Gene	single part of chromosome	character, feature,
Allele Population Fitness	characteristic of gene set of organisms survival and reproduction rate of organism	sign, variable value of a sign set of chromosomes value of chromosome
Generation Reproduction	population in time *t* creation of offspring from one or multiple organisms	analogue creation of chromosome from one or multiple chromosomes

GAs start with a random, initial population. What follows is an evaluation step, where each chromosome of the population is rated with the help of a fitness function. Depending on their individual fitness value, every chromosome has a chance of being selected for reproduction. Chromosomes with a higher fitness value have a higher probability of being selected. This represents Darwin's 'Survival of the fittest'. Non-selected chromosomes are discarded. The remaining selected chromosomes provide the basis of the new population that is selected using some strategy (in our case we use a Roulette wheel selection). It follows the alteration step where a new population is built out of the selected chromosomes, and after that, the new population is evaluated again. From this point, the evolutionary cycle starts again (Figure 4.5).

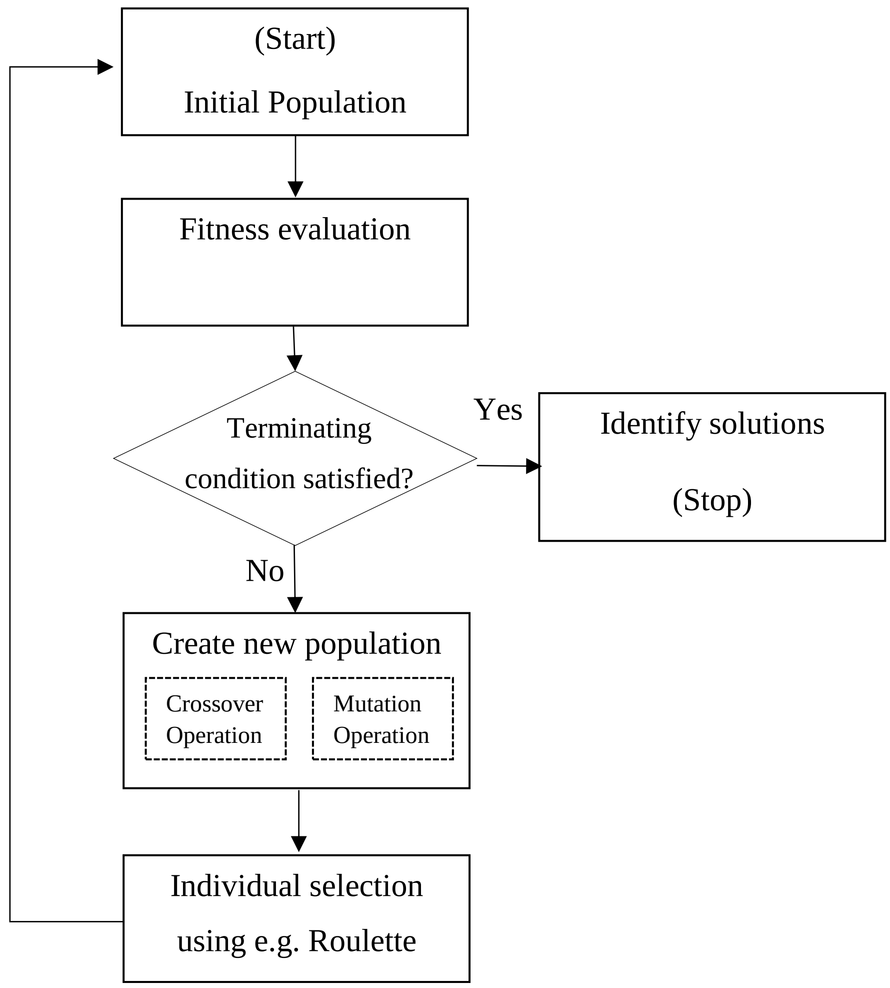


Figure 4.5: Genetic algorithm evolutionary cycle

Due to its robust and efficient nature, GAs are, as a direct search procedure, excellent for solving or approximating optimization problems with, for example, large search spaces, noisy data, multiple optima or non-differentiable objective functions. Because of the stochastic nature of the mutation, selection and crossover, GAs are less likely to get stuck at a local optimum than procedures which use indirect search. Although individuals with a relatively small fitness value have a smaller chance of being selected for recombination, there is still a chance to extend the search into areas which might not be promising at first glance but could still contain valuable solutions. This is also supported by the mutation operator, which ensures diversity by including genetic material in an individual which generally would not be considered. Therefore, it extends the search in different directions. Apart from the positive effects in terms of always finding global optima, GAs also have very good computational properties (Metaxiotis and Liagkouras, 2012). Since the structure is nearly the same for different problem sets, only small changes are necessary to adopt a coded GA to other problems.

4.2.1 Representation of Search Space

Holland (1975) introduced a binary, finite string representation for each individual in combination with a decoding function. Michalewicz (1996) pointed out that the binary representation does perform poorly for multidimensional high-precision problems. They gave the example of 100 variables in the range of [-500, 500] and a required precision of six digits behind the decimal point. This would result in a solution vector of 3000 bits length and would lead to a search space of 10^{1000} possible solutions. They therefore investigated the use of floating point representations which moves the GA closer towards the specific problem space. Additionally, this formulation assures that two solutions which are close to each other in the search space have a similar representation which is not always the case for binary coding and 'allows for an easy and efficient implementation'.

The search space contains all feasible chromosomes (Sivanandam and Deepa, 2007). With respect to the research topic, feasible means that the sum over all alleles of a chromosome is 1 and each allele value is between [0, 1]. Figure 4.6 gives an example for a simple, one-dimensional search space.

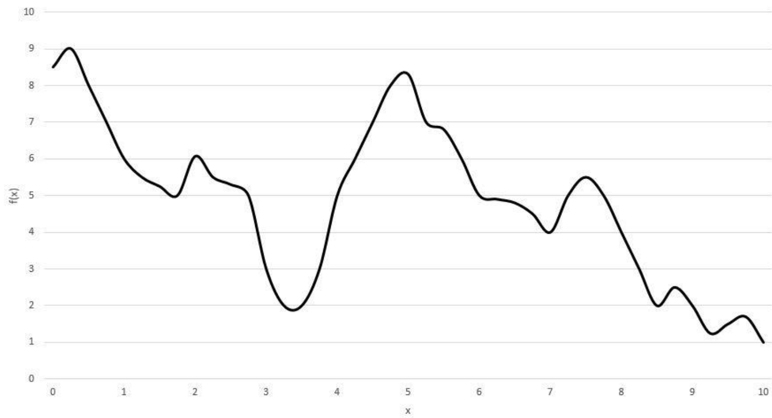


Figure 4.6: Example Search Space - based on Sivanandam and Deepa (2007)

4.2.2 Initialization and Roulette Wheel Selection

The initial first population is significant since it is intended to cover the solution space in the best way possible. Therefore, it is essential to create independent and random individuals. In consequence, each gene within the individual has to be generated randomly and independently. Similar to Ranković et al. (2014), we propose the following implementation:

- Initialize sequence vector $o[i] = rand_{int}[1, N], i = 1, \dots, N,$ where N is the total number of assets and $rand_{int}[\ ]$ is a random function with uniform distribution of integers within the range of $[1, N]$. It our work o is the permutation of indexes of weights in vector x and defines the order of genes' initialization.
- Generate weights in order of o:

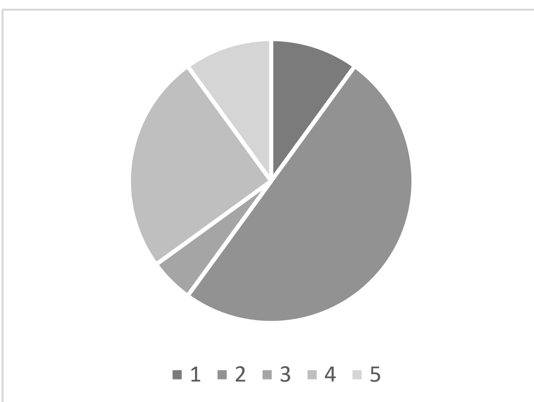
For $i \in \{1, \dots, N-1\}$: $x_{o[i]} = rand(0,1)_{real}(1 - \sum_{j=1}^{i-1} x_{o[j]})$

$x_{o[N]}$= 1- $\sum_{j=1}^{N-1} x_{o[j]}$

After successful initialization, it is necessary to select good individuals for reproduction. There exist plenty of different methods for selection, but we opt for a mixture of the roulette wheel and elitist selection. Roulette wheel selection is based on the chromosome's proportionate fitness. Each chromosome is assigned a slot on a roulette wheel. The slot size depends on the chromosome's fitness in proportion to the entire fitness of all chromosomes. Similar to Burke et al. (2005) we propose a possible implementation for the scheme:

1. Take each of the n chromosomes from the population and evaluate its fitness f_i.
2. Compute the sum over all $f_i, \sum_{i=1}^{n} f_i$. Calculate the slot size p_i for each chromosome i by $p_i = \frac{f_i}{\sum_{i=1}^{n} f_i}$.
3. For each chromosome, evaluate the corresponding cumulative fitness $q_i = \sum_{j=1}^{i} p_j$.
4. Generate $r \in (0,1]$ uniformly.
5. If $r < q_1$ select x_i, else set $i = i + 1$.
6. Repeating step 4 and 5 $2n$ times results in n pairs of parents.

Since this is a stochastic procedure, it is possible that the best chromosome is discarded. To prevent this problem, a combination of the roulette wheel and elitist selection is used. Elitist selection copies the best chromosome into the next generation and continues with a different selection scheme for the remaining chromosomes. Figure 4.7 shows an example of a roulette wheel.



Chromosome	**Fitness**	**Percentage**
1	30	10%
2	150	50%
3	15	5%
4	75	25%
5	30	10%
total	300	100%

Figure 4.7 Example of the roulette wheel – similar to Gerdes et al. (2004)

4.2.3 Tournament Selection

The idea of tournament selection is to let individuals compete against each other. The winner of the tournament, the one with the best fitness value, is selected as a parent. If only two individuals participate in the tournament and every individual of a generation has a chance to compete in every tournament, regardless of whether it competed before or not, it is called binary tournament selection with replacement (Nissen, 1997).

4.2.4 Evolution

Crossover

The idea behind crossover is that, due to an exchange of genetic material between individuals, "good" solutions could create "better" ones. Starting from Holland’s (1975) single-point crossover, many different methods of crossover evolved over time. Figure 4.8 shows an example for a single-point crossover.

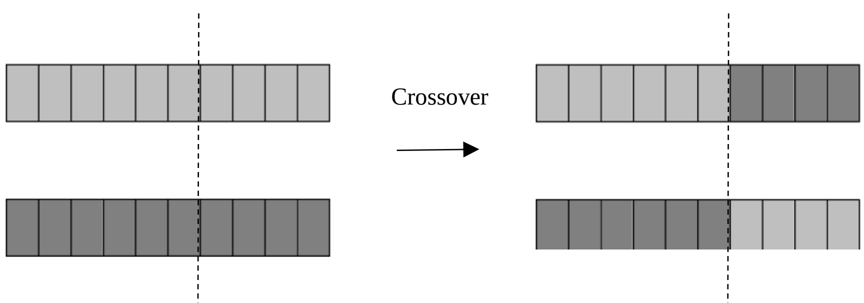


Figure 4.8: Example for single-point crossover

For this procedure, a cross-over point $p \in \{1, \ldots, l-1\}$ is chosen randomly, where l is the number of a chromosome’s genes. After that, all genes lying behind the cross-over point p are swapped between the two selected chromosomes. In figure 4.8 this is done for $p = 6$. This straightforward crossover procedure, unfortunately, comes with some disadvantages. One of them is that genes which are relatively far away from each other have a higher chance of being separated from each other than genes which are relatively close to each other (Eshelman et al., 1989). Wright et al. (1991) mentioned that one-point crossover carries the risk of generating a bad infant solution, despite both parental solutions being relatively good. To counteract this, they introduced linear crossover. As in single-point crossover, two parental individuals, p_1 and p_2, are selected. After that, three offspring are created as follows:

$$c_1 = \frac{1}{2}p_1 + \frac{1}{2}p_2$$

$$c_2 = \frac{3}{2}p_1 - \frac{1}{2}p_2$$

$$c_3 = -\frac{1}{2}p_1 + \frac{3}{2}p_2$$

Each offspring is evaluated, and the best two offspring are transferred to the next generation. They showed that a combination of single-point and linear crossover can outperform single-point crossover. In our work, only the best individual is transferred to the next generation.

Mutation

The mutation is the second most important evolution scheme. It randomly changes the allele values of a chromosome with a probability of p_m and is mainly used to maintain diversity in a population. In contrast to cross-over recombination, the mutation does not necessarily perform changes on all genes of a chromosome and is completed independent of the remaining population. It also prevents some important alleles from going extinct or never even being included in a population (Gerdes et al., 2004). Figure 4.9 shows a simple example where mutation changes different genes of a chromosome.

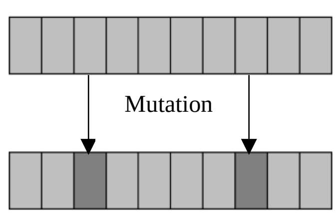


Figure 4.9: One-point mutation

Similar to Michalewicz (1996), for the case of real number implementation of chromosomes, we introduce two groups of mutation- uniform and non-uniform.

uniform: Let $x \in (x_1, \ldots, x_k, \ldots, x_n)$ be a chromosome. Each gene x_k has the same probability of being selected for a mutative process. Let x_k be the only selected gene and $x_k{}'$ its mutated version. Then $x' = (x_1, \ldots, x_k', x_{k+1}, \ldots, x_n)$ is the mutated version of x. The mutation changes the gene value corresponding to a given parameter domain $[l_k, u_k]$. A second possibility is to select two genes of a chromosome and switch a certain amount between both genes. For example, let $x_k = 0.1$, $x_l = 0.5$ and the switched amount be 0.05. Then $x_k' = 0.15$ and $x_l' = 0.45$. Uniform mutations are independent of the underlying generation. In the following, the first mutation scheme will be called one-point mutation and the second scheme two-point mutation.

non-uniform: In opposition to uniform mutations, non-uniform mutations rely on the underlying generation. Let x again be the given chromosome and x_k the selected gene. The variable t defines the generation and assuming z a random variable in $\{0,1\}$. Then:

$$x_k' = \begin{cases} x_k + \Delta(t, u_k - x_k), & if\ z = 0 \\ x_k + \Delta(t, x_k - l_k), & if\ z = 1 \end{cases}$$

Δ is a function which monotonically decreases over time t. Low values of t let the function spread the solutions over the whole search space. With increasing time, the search focuses more and more on the solutions' neighborhood. An example for Δ is $\Delta(t, x) = x(1 - r^{1-\frac{t}{T}})$, where r denotes a random number between [0,1] and T the maximal number of generations. Figure 4.10 shows an example with 100 generations and a starting gene value of 0.6. r is set between [0, 0.1] and fixed at 0.1.

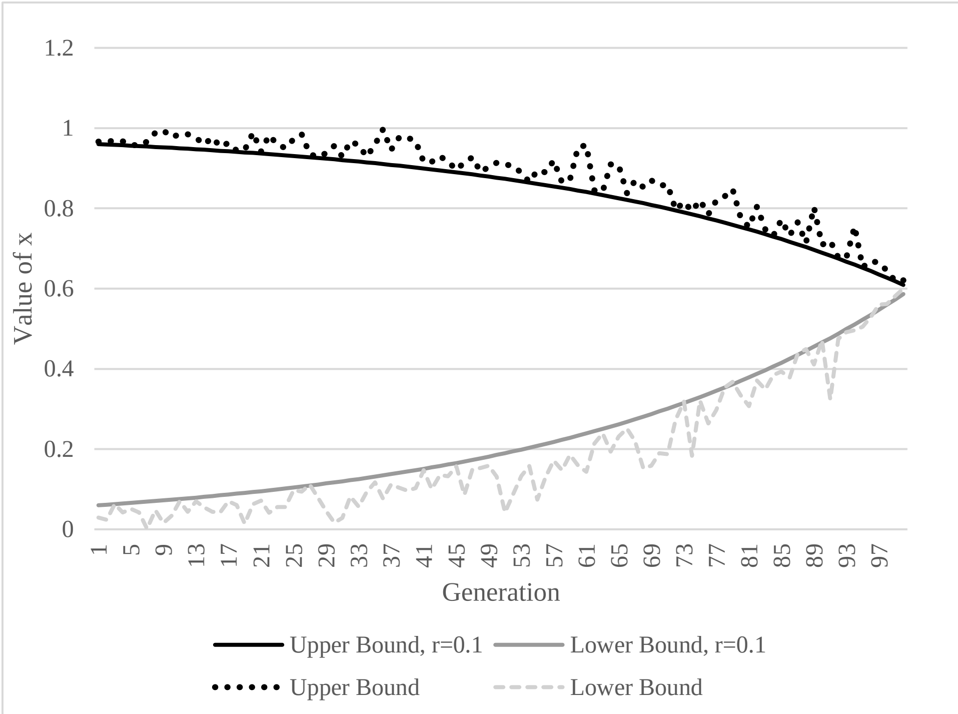


Figure 4.10: Illustration of possible mutation values. Both values are presented, upper and lower bound for r fixed, as well as r taking a random number between [0,0.1]

The possible value of the allele in the first generations is nearly spread over the whole search space. With increasing number of iterations, Δ ensures a smaller variance of the gene value. It should be emphasized that offsprings created with mutation and crossover with a real valued coding are generally not feasible. Without modifications on the mutation and crossover operators, it is possible for genes to be assigned negative values. Therefore, a gene is set to 0 if the assigned value is negative. Additionally, the chromosomes have to be normalized. This is done simply by dividing each gene by the sum of all genes of the chromosome.

The SPEA2 Algorithm

The Strength Pareto Evolutionary Algorithm 2 (SPEA2) of Zitzler et al. (2002) is an algorithm that aims to find or approximate the Pareto-optimal set for multi-objective optimization problems and is a development of the Strength Pareto Evolutionary Algorithm of Zitzler and Thiele (1999). In difference to the other proposed algorithms, the SPEA2 creates a set of non-dominated solutions instead of one single solution. The main loop is given as follows:

Input:

N: population size

$\overline{N}$: archive size

T: maximum number of generations

Output:

A: Non-dominated set.

Step 1 Initialization: Generate an initial population P_0 and create the empty archive $\overline{P_0} = \emptyset$. Set $t = 0$.

Step 2 Fitness assignment: Calculate fitness values of individuals P_t and $\overline{P}_t$.

Step 3 Environmental Selection: Copy all non-dominated individuals in P_t and $\overline{P}_t$ to $\overline{P_{t+1}}$. If size of $\overline{P_{t+1}}$ exceeds $\overline{N}$, reduce $\overline{P_{t+1}}$ by means of the truncation operator, otherwise if size of $\overline{P_{t+1}}$ is less than $\overline{N}$, fill $\overline{P_{t+1}}$ with dominated individuals in P_t and $\overline{P}_t$.

Step 4 Termination: If $t > T$ or another stopping criterion is satisfied, set A to the set of decision vectors represented by the non-dominated individuals in $\overline{P_{t+1}}$. Stop.

Step 5 Mating selection: Perform binary tournament selection with replacement on $\overline{P_{t+1}}$ in order to select the parents.

Step 6 Variation: Apply recombination and mutation operators to the mating pool and set P_{t+1} to the resulting population. Increment generation counter $(t = t + 1)$ and go to step 2.

The fitness assignment is needed to ensure that individuals that are dominated by the same archive members do not have the same fitness value. Therefore, for computing the fitness $F(i)$ of individual i, its strength value $S(i)$, raw fitness $R(i)$ and density $D(i)$ are needed. The strength value is defined as follows:

$$S(i) = |\{j | j \in P_t + \bar{P}_t \wedge i \succ j\}|$$

The strength value represents the number of solutions in the actual population and the current archive that are dominated by solution i. $|\cdot|$ describes the cardinality of a set, $+$ means multi set union and $\succ$ resembles the Pareto-dominance. In the presented optimization problem, solution i dominates solution j if:

$$return(i) \geq return(j) \wedge risk(i) < risk(j) \text{ or } return(i) > return(j) \wedge risk(i) \leq risk(j)$$

With the help of the strength value the raw fitness can be calculated

$$R(i) = \sum\nolimits_{j \in P_t + \bar{P}_t, j > i} S(j)$$

$$\sum_{j \in P_t + \bar{P}_t, j > i} S(j).$$

The raw fitness of individual i depends on its dominators. A value of $R(i) = 0$ corresponds to a non-dominated solution, while a high number of $R(i)$ corresponds to a solution which has many dominators and therefore is less attractive for further investigation. When most individuals are not dominated by each other, the raw fitness fails to diversify between different solutions, since they all have a raw fitness of 0. Therefore, a density information is introduced. For every solution i the distance to every other solution j in the current solution and archive, $| x_i - x_j |$, is computed, sorted in an increasing order and saved in a list. Then, the k'th nearest neighbour, denoted as σ_i^k, is selected. The value of k is set as the square root of the sample size, as suggested by Silverman (1986). The corresponding density to solution i is computed as:

$$D(i) = \frac{1}{\sigma_i^k + 2}.$$

The $+2$ is added to ensure $D(i)$ to be lower than 1 and bigger than 0. The resulting assigned fitness is

$$F(i) = R(i) + D(i)\,.$$

After each solution we assigned a fitness value, and the environmental selection is performed. This step is also referred to as archive update operation. At first, all non-dominated solutions of the current population and archive are copied to the new archive, so

$$\bar{P}_{t+1} = \{\, i \mid i \in P_t + \bar{P}_t \wedge D(i) < 1 \,\}.$$

The number of solutions in $\bar{P}_{t+1}$ is represented by l. There are three different potential cases of the size of l:

1. $N_a = l$, the number of potential archive members is equal the archive size, so nothing has to be done, $\bar{P}_{t+1}$ is the new archive.

2. $N_a > l$, there are not enough non-dominated solutions, so $\bar{P}_{t+1}$ has to be filled up with dominated solutions.

3. $N_a < l$, there are too many potential archive members, so some solutions in $\bar{P}_{t+1}$ have to be deleted.

For case two and case three, the new archive $\bar{P}_{t+1}$ has to be adjusted. In case two, the new archive has to be filled up with dominated solutions. This happens by copying the best-dominated solutions to the new archive. To identify the best solutions, all dominated solutions in $\bar{P}_t$ and P_t are stored in a sorted list with increasing value of F. The first $\bar{N} - l$ individuals are added to $\bar{P}_{t+1}$. Case three is more complicated since it needs to be determined which non dominated solutions are better than other non-dominated solutions. This happens with an iterative archive truncation procedure. The idea behind this procedure is to select solutions which cover a wide range of the search space. In every iteration of the procedure, a solution *i* is removed, which either has the same distances to all other individuals as another individual or is as near to the first k_1 neighbours as another individual, but closer to the k'th nearest neighbour. Formally spoken, i is removed if $i \leq_d j$, $(\forall\, j \in \bar{P}_{t+1})$, with

$$i \leq_d j :\Leftrightarrow \ \forall 0 < k < |\, \bar{P}_{t+1} \,|: \sigma_i^k = \sigma_j^k \ \vee$$
$$\exists\, 0 < |\, \bar{P}_{t+1} \,|: \left[\left(\forall 0 < l < k : \sigma_i^l = \sigma_j^l\right) \wedge \sigma_i^k < \sigma_j^k\right].$$

Figure 4.11 shows an example for $\bar{N} = 5$.

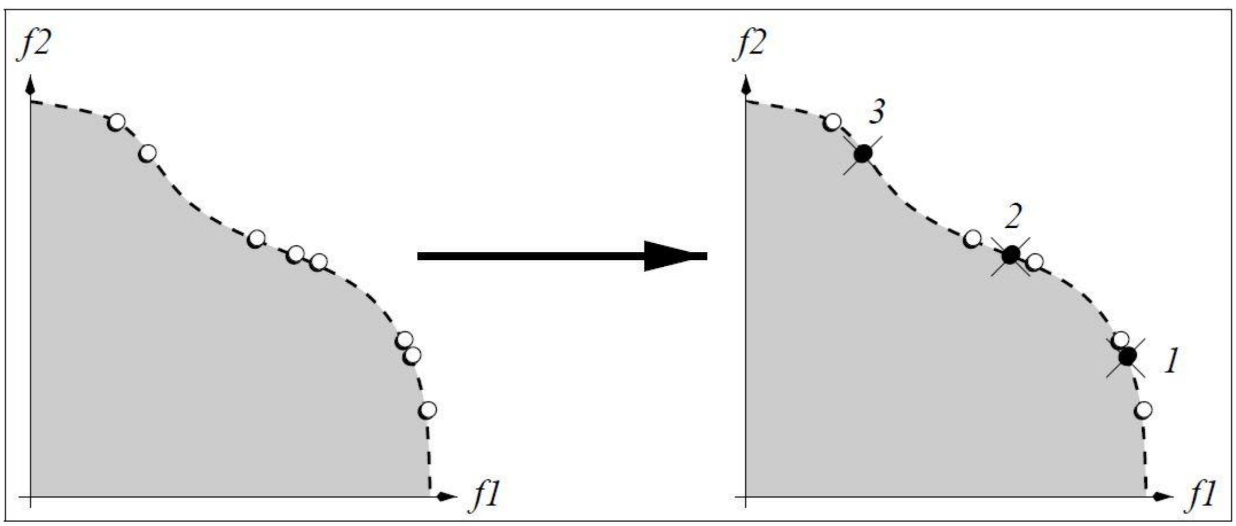


Figure 4.11: Example for the truncation procedure. Left side: potential archive members. Right side: Discarded solutions and removing order – similar to Zitzler et al. (2002)

After filling the archive, the evolutionary steps are performed and parental individuals are selected via binary tournament selection with replacement. Following, single-point crossover and one-point mutation are performed to build the new generation (Zitzler et al., 2002).

4.3 Data and Empirical Results

The following chapter presents the computational results obtained by performing experiments on a historical data set. At first, historical data of DAX from January 2015 to April 2017 is analysed. Based on this, seven different assets are selected for further investigation due to nicely distribution in terms of risk and return (see Figure 4.12).

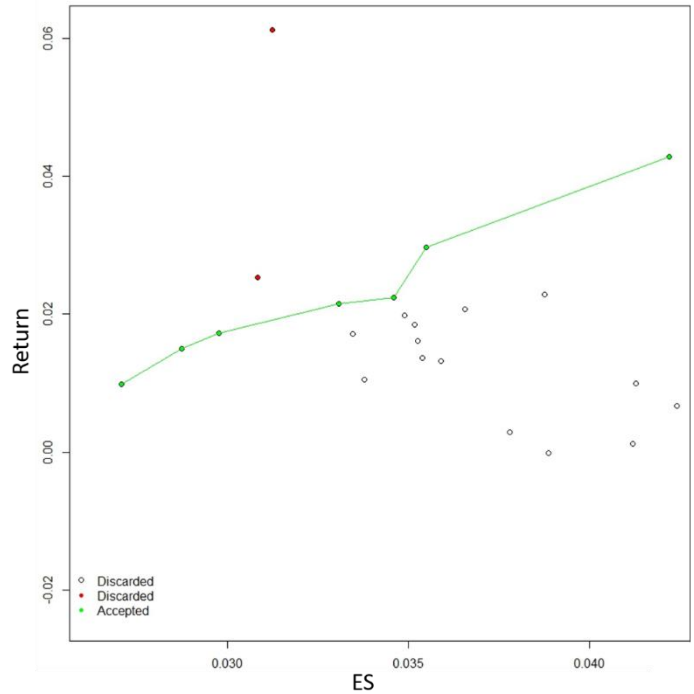


Figure 4.12: Overview of Risk/Return DAX, plotted in ES and daily log return coordinates

Following assets are chosen: Münchner Rückversicherungs-Gesellschaft AG, Beiersdorf AG, Henkel AG & Co. KGaA, Siemens AG, Deutsche Börse AG, Fresenius SE & Co. KGsA and Infineon Technologies AG. Table 4.2 gives an overview over the different return and risk values. Due to computational reasons the daily log returns are scaled to a monthly basis[15], so that the values of risk and return are nearly in the same range. Annualized returns, as it is usual in finance, would increase the difference between the risk and return values too much. As risk component the 5% 1-day historical ES is computed (Table 4.2).

[15] Like annualizing, but for 30 days only.

Table 4.2: Overview of risk and log return

Nr.	Asset	ES	Scaled Daily Log Returns
1	Münchener Rück.	2.71%	0.98%
2	Beiersdorf AG	2.87%	1.50%
3	Henkel AG	2.98%	1.73%
4	Siemens AG & Co. KGaA	3.31%	2.16%
5	Deutsche Boerse AG	3.46%	2.24%
6	Fresenius SE	3.55%	2.97%
7	Infineon Technologies AG	4.22%	4.28%

In the following, three different GA are be presented which use the models presented in chapter 3. GA I uses the formulation of model 3, GA II the formulation of model 2 and GA III the formulation of model 1. Instead of mean-variance the GAs use ES as measure of risk.

5.1 Choice of GA Schemes

To find the optimal settings for the GAs, different combinations of mutation and crossover operations were tested for GA I with trade off parameter λ. The algorithms were programmed in open source program R. The tests were run with an initial population of 50 assets and for 100 iterations each. These settings tend to create good solutions and were also selected due to their required computational time. The initial population was computed before the tests and the population was used for every test run to ensure plausibility. In total, every GA was repeated 100 times. Lambda was set to $\lambda = 0.5$.

5.1.1 Comparison of Mutation Schemes

At first, different mutation operators with a mutation probability of 5% and a mutation value μ of 0.05 were tested. As crossover operation, the single-point crossover (C1) was selected:

- M1: One-point mutation. For each gene is decided independently whether it is mutated or not.
- M2: Non-uniform mutation with $\Delta(t, x) = x \cdot \left(1 - r^{\left(1 - \frac{t}{T}\right)}\right)$.

- M3: Two-point mutation: two different genes are selected and a maximum of μ is swapped between the genes.

The following figures illustrate the results obtained by running the tests for the different settings. Each dot represents the best solution found in iteration *t*. Since the test was run 100 times, for each iteration 100 solutions are plotted.

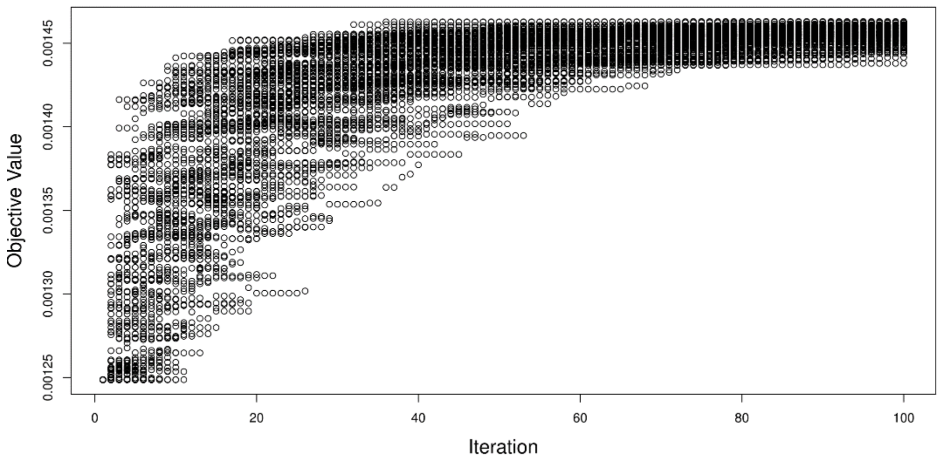


Figure 4.13: Results for M1C1

Obviously, the results for mutation scheme 1 (Figure 4.13) are much more scattered than that is the case for mutation schemes 2 (Figure 4.14) and 3 (Figure 4.15), especially when it comes to the second half of the test phase, therefore it is rejected for further consideration. Mutation schemes 2 and 3 seem to perform equally well. Table 4.3 provides a closer comparison of both implementations:

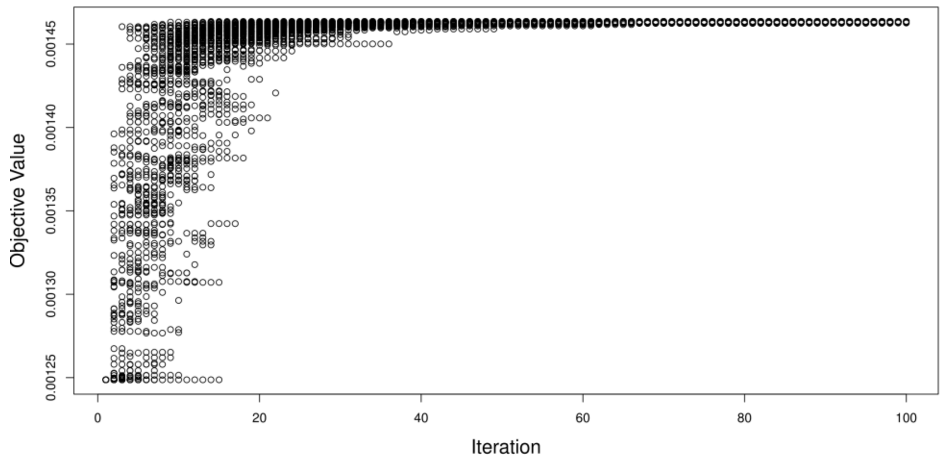


Figure 4.14: Results for M2C1

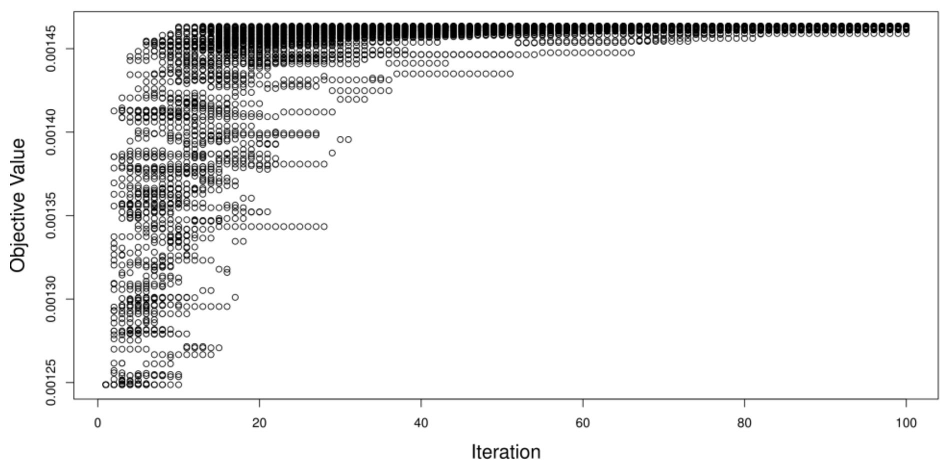


Figure 4.15: Results for M3C1

Mutation scheme 2 outperforms M3 in 77 of 100 cases, has a higher maximal final value, a higher minimal final value and a higher mean final value. Although the differences are very small, it seems that M2 creates better results than M3.

Table 4.3: Detailed comparison of M2 and M3

	M2	**M3**
Maximal final value	0.001463545	0.001463544
Mean final value	0.001463374	0.001462972
Minimal final value	0.001462589	0.001459038
Better final result	77 times	23 times

4.3.1 Comparison of Crossover Schemes

As next step, we will clarify the choice of the crossover scheme. Therefore, the crossover scheme is changed to linear crossover. As before, M1, M2 and M3 represent the three different mutation methods. For each combination, 100 test runs were performed with a population size of 50, lambda was set to $\lambda = 0.5$ again. Figure 4.16 show the results for the different combinations.

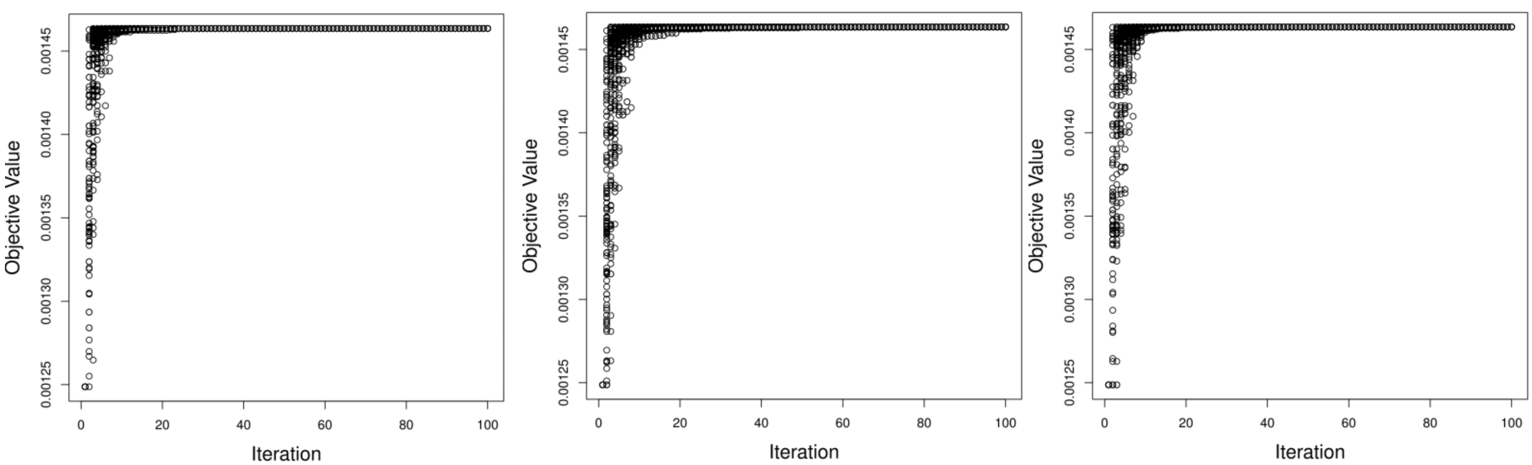


Figure 4.16: Results for M1C2, M2C2, and M3C2 in that order

It is noticeable that this time all three combinations seem to perform very evenly. Especially the fact that all three combinations have a similar final value for each repetition is eye-catching. Table 4.4 compares the three methods in detail.

Table 4.4: Detailed comparison of M1C2, M2C2 and M3C2

	M1C2	**M2C2**	**M3C2**
Maximal final value	0.001464	0.001464	0.001464
Mean final value	0.001464	0.001464	0.001464
Minimal final value	0.001464	0.001463	0.001463
Better final result	64 times	13 times	23 times

The results conclude that the linear crossover scheme outperforms the normal mutation. This is supported by the fact that it results in faster convergence of the algorithm, higher final values, and also less scattered solutions. Figure 4.17 compares the distribution of the final values of the best crossover 1 combination with all crossover 2 combinations.

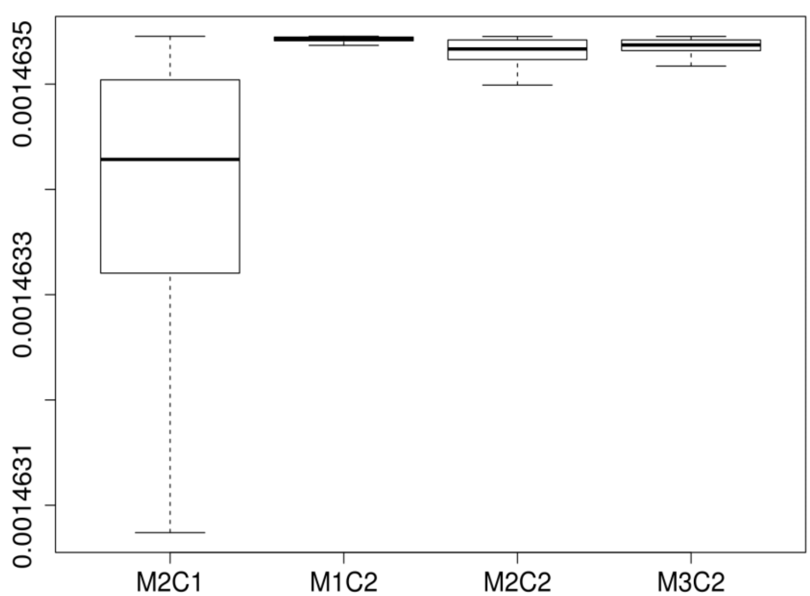


Figure 4.17: Boxplot comparison of final values M2C1, M1C2, M2C2 and M3C2, outliers

Surprisingly, in combination with linear crossover, the normal single-point mutation performs best, whereas in combination with normal crossover it performs worst. All results lead to the assumption, that linear crossover has the major impact on the creation of good solutions.

4.3.2 Computational Results

The next section presents the results obtained by running the three different presented models in terms of distribution and quality of resulting solutions.

Single-Objective Technique GA I - Equidistant Values of Trade-Off Parameter

The following subsection presents the results of solving the portfolio optimization problem for transformed ES model I with the help of the selected GA operators. The GA was performed for 101 equidistant levels of $\lambda \in [0,1]$. From these 101 levels, 21 levels were chosen for further research. The results were plotted in return and risk coordinates (Figure 4.18). Each dot represents the solution of a single run for a given level of lambda; the filled dots represent the 21 selected levels of lambda and the blue squares the underlying seven assets (Table 4.1). The different values of lambda correspond to different weighted importance of risk and return.

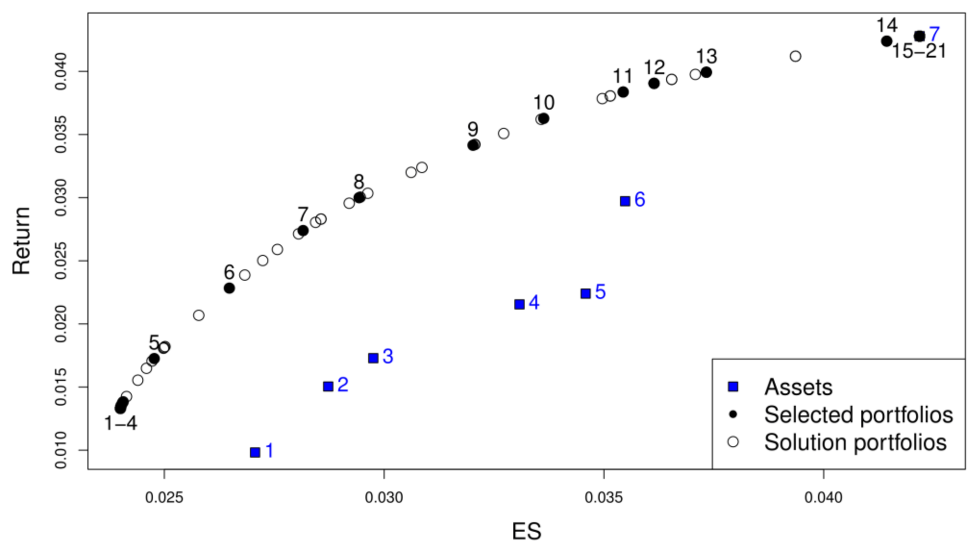


Figure 4.18: Test results for 101 levels of λ

Table 4.5: Results for 21 chosen levels of λ

Lambda	Return	ES	Lambda	Return	ES
0	0.0133	-0.024	0.55	0.03904	-0.03614
0.05	0.01331	-0.024	0.6	0.03993	-0.03733
0.1	0.01347	-0.02401	0.65	0.04239	-0.04144
0.15	0.01383	-0.02407	0.7	0.04279	-0.04218
0.2	0.01725	-0.02477	0.75	0.04279	-0.04218
0.25	0.02283	-0.02648	0.8	0.04279	-0.04218
0.3	0.02739	-0.02816	0.85	0.04279	-0.04218
0.35	0.02999	-0.02943	0.9	0.04279	-0.04218
0.4	0.03415	-0.03202	0.95	0.04279	-0.04218
0.45	0.03627	-0.03363	1	0.04279	-0.04218
0.5	0.03837	-0.03544			

As it can be seen in Figure 4.18, the resulting portfolios have a better return/risk ratio than the single assets. This underlines the importance of diversification. In general, the resulting portfolios are distributed over the whole resulting efficient frontier, but not uniformly, as it can be seen by the distribution of the 21 chosen lambda values (Table 4.5).

As Ranković et al. (2014) pointed out, a high level of diversity along the efficient frontier is crucial in portfolio optimization, since different risk-return portfolios fit for different investors and therefore a wider range of investors can be satisfied.

Single-Objective Technique GA II - Predefined Levels of Return

To improve the diversity of the resulting portfolios along the efficient frontier, the objective function was changed to minimizing the risk for a certain minimal level of desired return. This was simply performed by computing the return of the individual portfolio and setting the objective value to either the corresponding risk if the return generates at least the desired level of return, or 1 if the portfolio does not satisfy the return constraint. Again, 101 levels of return were tested. The different values of return were computed as follows:

$$l_i = r_{min} + (i-1) \cdot \frac{r_{max} - r_{min}}{100}$$

$$r_{min} \coloneqq min\, r_i, r_{max} \coloneqq \max r_i\,, i \in \{1, \dots 7\}$$

Afterwards, 21 different equidistant levels of return were selected. As seen in Table 4.6, the GA II without certain adjustments on the input generation does not necessarily create portfolios for all levels of return. The colored entries emphasize this in the table. For this test, a start population was created which does not contain individuals with a high level of return. Since the introduced penalty function is of a simple nature and does not consider if an individual is very close or very far from being acceptable, there is a possibility that there will never be feasible solutions in an acceptable time frame.

Table 4.6: Results of performing GA II with no modifications

Target Value	Return	ES	Target Value	Return	ES
0.00982	0.01331	-0.024	0.02795	0.02798	-0.02842
0.01147	0.01331	-0.024	0.0296	0.02962	-0.02924
0.01312	0.0133	-0.024	0.03125	0.03127	-0.03021
0.01476	0.01479	-0.02426	0.0329	0.02408	-0.02931
0.01641	0.01643	-0.02459	0.03455	0.0226	-0.02715
0.01806	0.01809	-0.02502	0.03619	0.02702	-0.02856
0.01971	0.01976	-0.02551	0.03784	0.02098	-0.02687
0.02136	0.02142	-0.02605	0.03949	0.02569	-0.02805
0.02301	0.02303	-0.02659	0.04114	0.02633	-0.02862
0.02465	0.02473	-0.02718	0.04279	0.01939	-0.02606
0.0263	0.02632	-0.02779			

Therefore, some modifications to the initial population were performed. At first, the single asset solutions were added to the initial population. This ensures the existence of a solution for every required level of return because the maximal return is given by the maximal return of the single assets. Secondly, the best solution of the run with the next higher level of return was added. On account of that, the iterations were started with the highest level of return and continued with decreasing values.

The results are given in Figure 4.19.

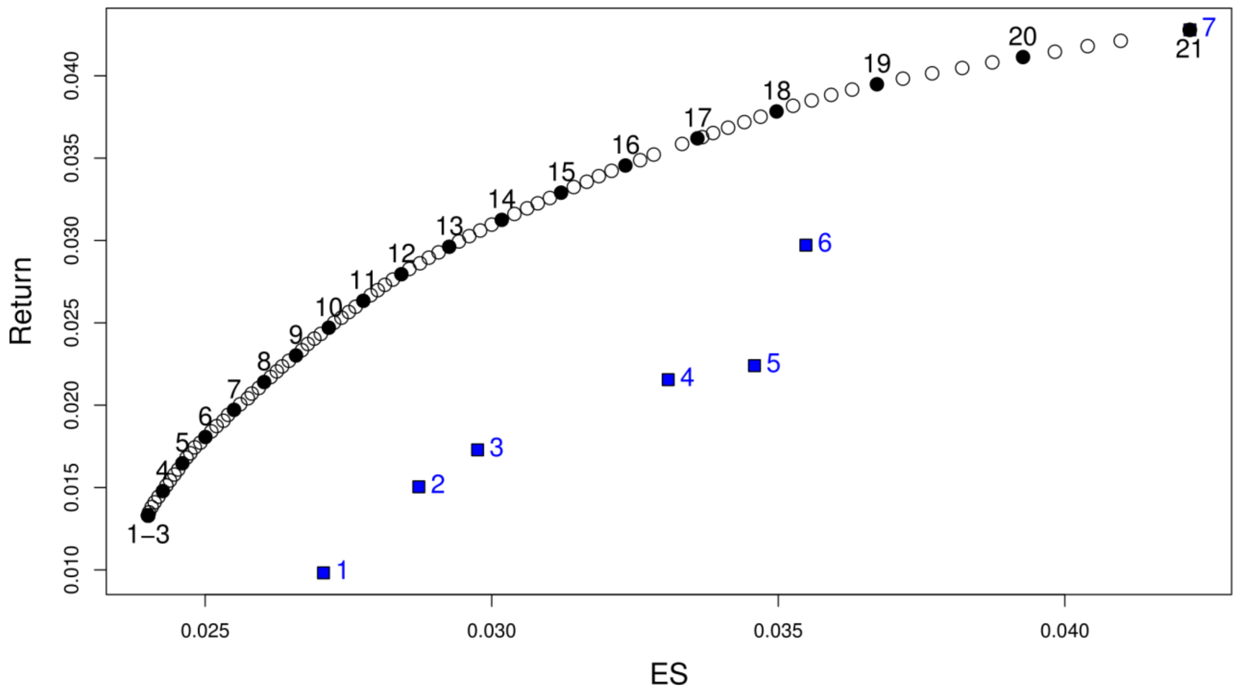


Figure 4.19: Test results modified GA II

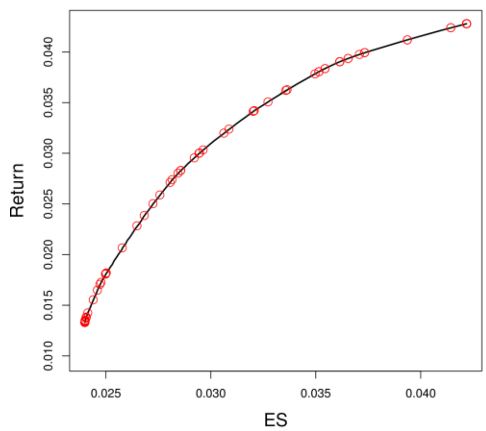
Like GA I the portfolios constructed with GA II have a superior risk/return ratio than the single assets. Compared to the GA I portfolios the portfolios created with GA II are better distributed along the efficient frontier. The results of the 21 chosen levels of return are given in Table 4.7.

Table 4.7: Results for GA II with modification

Target Value	Return	ES	Target Value	Return	ES
0.00982	0.0133	-0.024	0.02795	0.02795	-0.02842
0.01147	0.01331	-0.024	0.0296	0.02962	-0.02926
0.01312	0.0133	-0.024	0.03125	0.03125	-0.03017
0.01476	0.01479	-0.02426	0.0329	0.0329	-0.03121
0.01641	0.01648	-0.0246	0.03455	0.03455	-0.03233
0.01806	0.01807	-0.025	0.03619	0.03619	-0.03359
0.01971	0.01971	-0.0255	0.03784	0.03784	-0.03497
0.02136	0.02141	-0.02603	0.03949	0.03949	-0.03672
0.02301	0.02302	-0.02658	0.04114	0.04114	-0.03927
0.02465	0.0247	-0.02715	0.04279	0.04279	-0.04218
0.0263	0.02634	-0.02776			

It is noticeable that the algorithm finds portfolios which generate returns that are very close to the target value for most desired levels of return, but not for the lowest returns. This is due to the fact, that the algorithm only takes the resulting risk into account. It would not return a portfolio that has lower return and higher risk.

Summarizing both methods, it can be said that the use of GA with equidistant levels of return results in solution portfolios that are well spread along the efficient frontier, which is not the case for the GA with equidistant levels of lambda. But it should be pointed out, that both variations create nearly the same efficient frontier which can be seen in Figure 4.20.



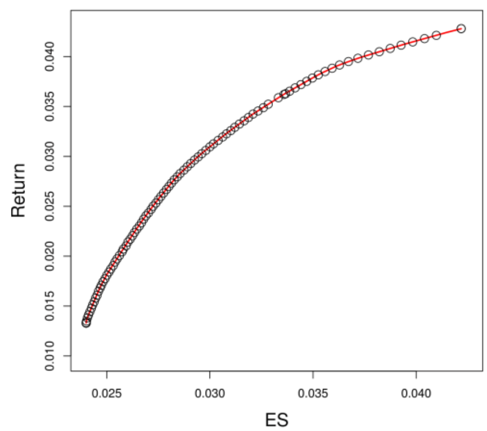


Figure 4.20: Comparison of solutions for GA I and GA II. Black: Solutions for GA II. Red: solutions for GA I. Solutions of GA I are plotted on efficient frontier of GA II and vice versa

Multi-Objective GA III - SPEA2 Algorithm

The next subsection presents the results obtained by using the SPEA2 algorithm, which was introduced in chapter 4.2.4. Two different test runs were performed, both with an initial population size of 500 individuals and 200 iterations. For the first test run the archive size was set to 21 and for the second run it was set to 250.

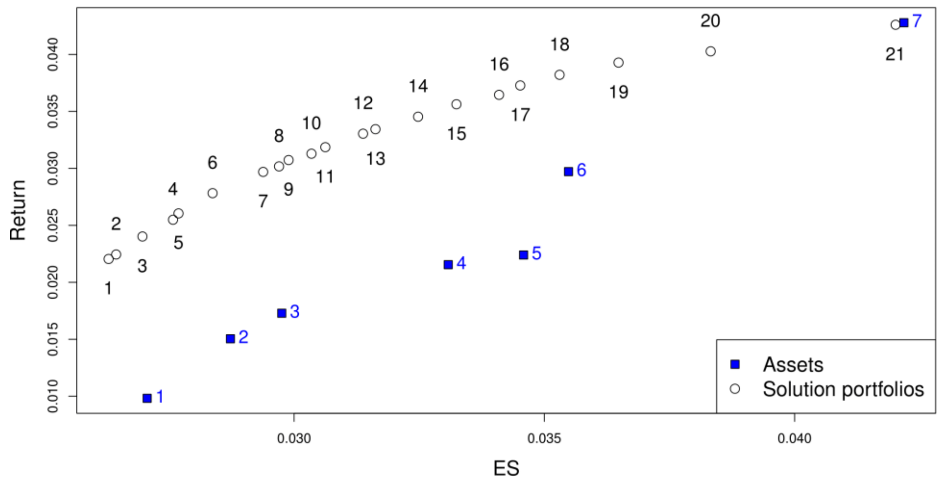


Figure 4.21: Results of SPEA2 for archive size 21

It can be seen in Figure 4.21 that, like GA I and GA II, the SPEA2 algorithm creates portfolios with superior risk and return ratio than the single assets. The solutions are spread across the efficient frontier but tend to lack around the minimum risk achieved by GA I and GA II. In opposite to the results of Ranković et al. (2014) the solutions do not lack around the maximum return. This was insured by including the one assets solutions in the start population. The exact results of the test run are given in Table 4.8.

Table 4.8: Results for SPEA2 with archive size 21

Number	Return	ES	Number	Return	ES
1	0.02205	-0.0263	12	0.03305	-0.03138
2	0.02245	-0.02644	13	0.03344	-0.03163
3	0.02402	-0.02697	14	0.03454	-0.03248
4	0.0255	-0.02758	15	0.03563	-0.03324
5	0.02606	-0.02769	16	0.03645	-0.0341
6	0.02783	-0.02837	17	0.03727	-0.03452
7	0.02968	-0.02938	18	0.0382	-0.0353
8	0.03018	-0.0297	19	0.03928	-0.03648
9	0.03074	-0.02989	20	0.04028	-0.03832
10	0.03129	-0.03035	21	0.04261	-0.04202
11	0.03186	-0.03062			

When analyzing the second run it is recognizable that most solution portfolios are concentrated around the maximum return and around the middle. In comparison to the first run it is noticeable that there exists a solution with lower risk, but as before, solutions seem to lack around the minimum risk (Figure 4.22).

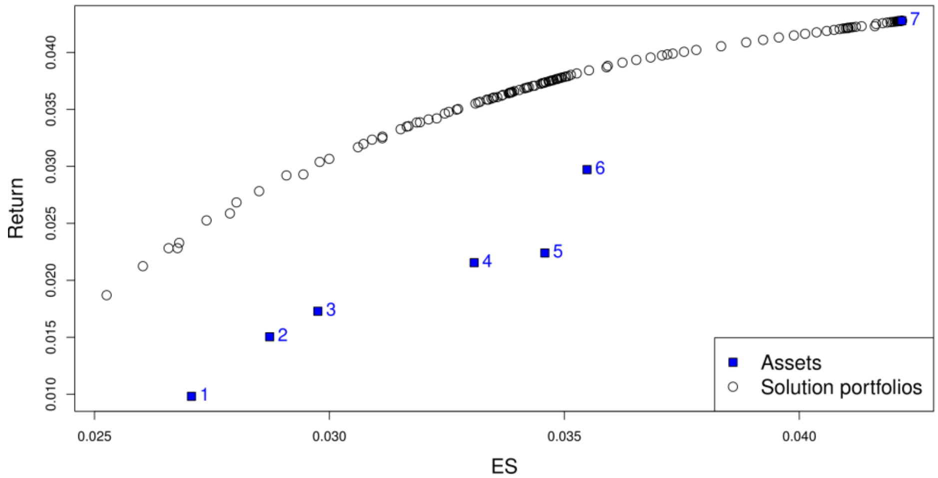


Figure 4.22: Results for SPEA2 with archive size 250

Evaluation and Comparison with VaR

Following Ranković et al. (2014) portfolio optimization techniques should satisfy the following three conditions:

1) find well distributed, efficient portfolios
2) find a portfolio with minimal risk regarding a certain level of return
3) run in an acceptable time

It is obvious that GA I does not satisfy the first condition since the solutions are not well distributed along the efficient frontier. It is also not possible to search for a portfolio with a desired level of return directly. GA II instead satisfies the first two conditions. GA III does only satisfy condition one. It is not possible to make a statement about condition three that has general validity. In this paper GA I and GA II had the same running time and performed much faster than GA III (whose second run had a running time of two days). This is in contradiction to the results of Ranković et al. (2014) where the SPEA2 algorithm run much faster than the normal single objective GA (GA I) with different lambda coefficients. These differences result among other things from the chosen programming language.

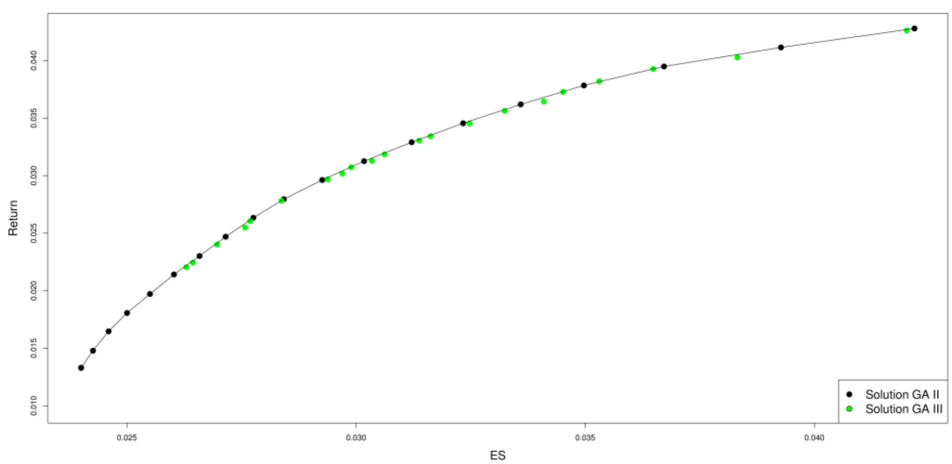


Figure 4.23: Comparison of GA II and GA III

Figure 4.23 compares the results of GA II and GA III. It can be seen that the results of GA III are slightly dominated by the ones of GA II. This results from the fact that GA III as a multi-objective algorithm tries to optimize a set of non-dominated solutions, whereas GA II as a single-objective algorithm tries to optimize a single solution and therefore has a search direction.

To emphasize the differences between VaR and ES, even more, the GA II was run with VaR as a risk measure. The obtained results are given below.

It can be seen in Figure 4.24 that the solution portfolios are indeed spread along the efficient frontier and have a better return risk ratio than the single assets, but compared to GA II with ES risk measure (Figure 4.18) the resulting "efficient" frontier seems to be flatter and even doubling the iterations did not result in a significantly better frontier. It is also recognized that the solution portfolios with low desired rates of return are not well distributed and seem to be limited by a barrier. The most significant differences appear if the resulting VaR portfolios are plotted against the ES portfolios. To do so we computed ES of the VaR portfolios. The results are illustrated in Figure 4.25.

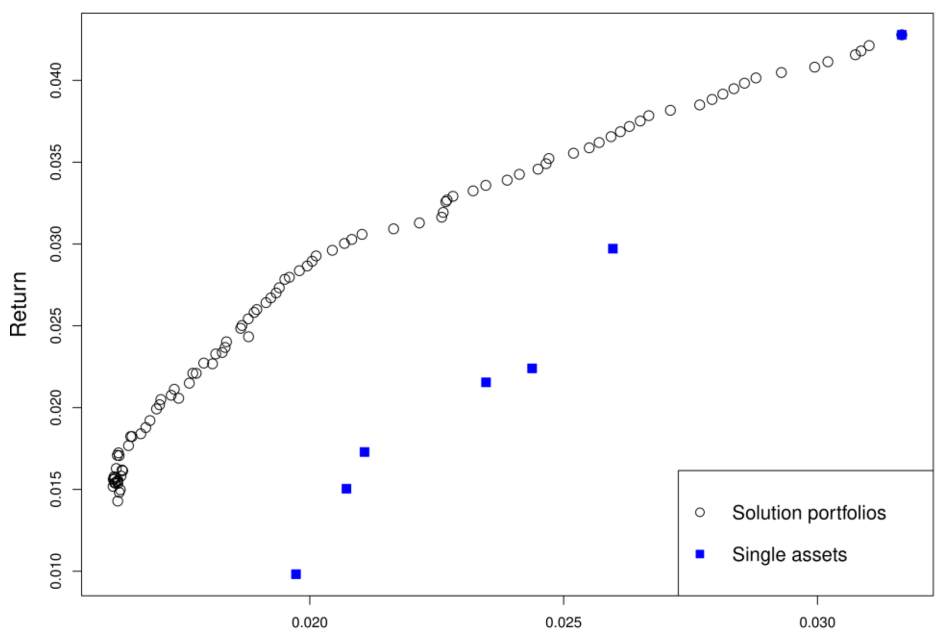


Figure 4.24: GA II with VaR instead of ES

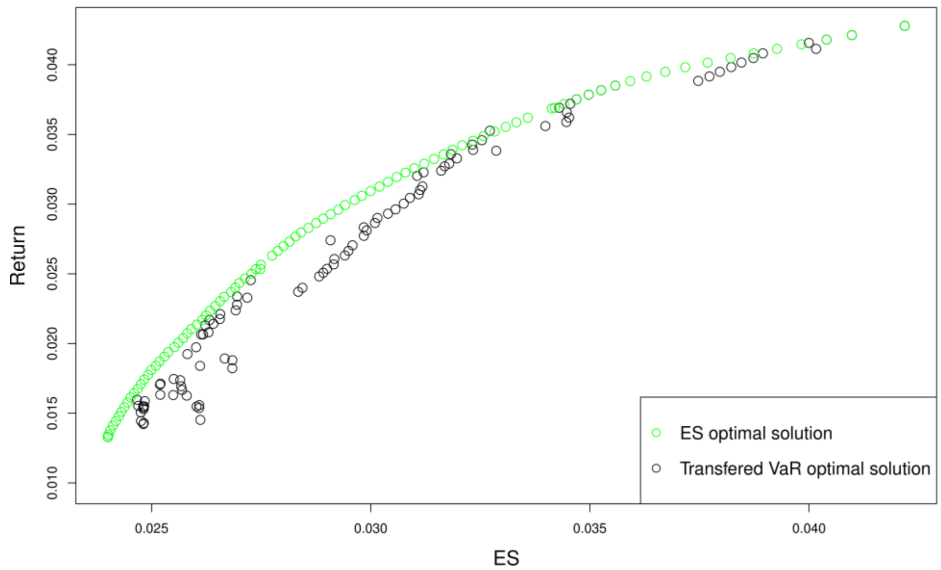


Figure 4.25: Transformed VaR optimal solutions vs ES optimal solutions

The converted VaR portfolios do not result in well distributed ES portfolios. The lack of solutions between an ES value of ~3.5% and ~ 3.7% is noticeable, but even more substantial is the distribution of the low return portfolios on the left side of the chart. Several portfolios have nearly the same expected rate of return, but the differences in ES are significant. This leads to the assumption that measuring risk with VaR for the given data set in combination with GA II leads to portfolios which underrate the tail risk. This is not the case if the resulted ES portfolios are converted to VaR portfolios. Figure 4.26 shows the results of plotting the VaR portfolios against the converted ES portfolios; therefore, we computed the VaR of the ES portfolios.

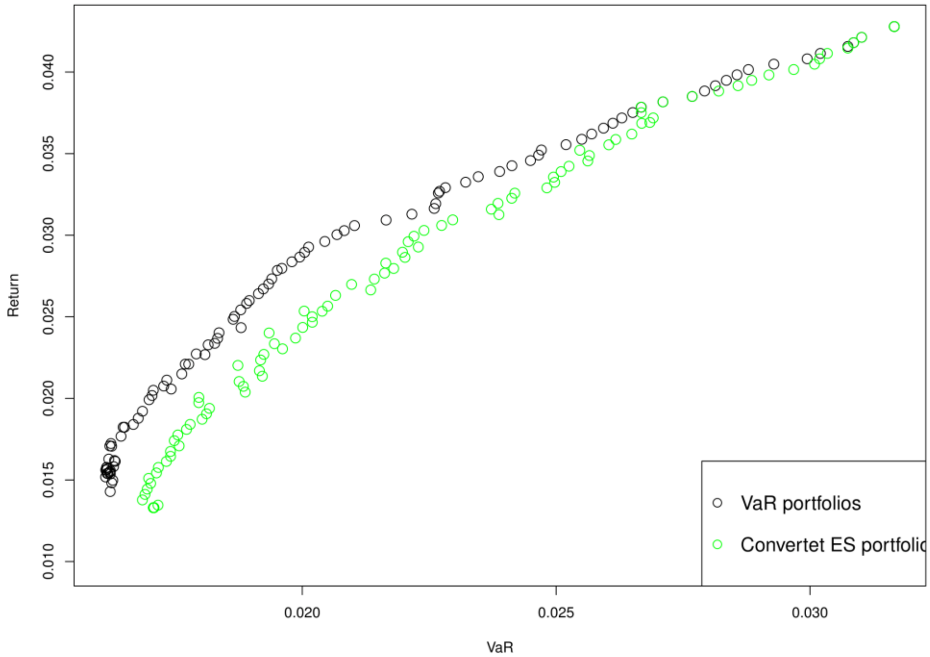


Figure 4.26: Transformed ES optimal solutions vs VaR optimal solutions

The resulting converted portfolios are way better distributed along the resulting frontier. Furthermore, both frontiers look very similar. The converted ES portfolios seem to converge towards the VaR portfolios and nearly coincide for high levels of expected return.

4.4 Conclusion and Further Research

In this paper, we used ES as a risk measure for the portfolio selection problem in combination with GA. We evaluated the best setting for the presented problem by combining different evolution schemes. When a crossover was set to single-point crossover, non-uniform mutation outperformed one-point and two-point mutation. Our results suggest that one-point mutation was outperformed heavily by the other two methods in terms of convergence. Conversely, when using linear crossover, one-point mutation outperformed the other two methods.

In general, the obtained results lead to the assumption that the use of linear crossover leads to better results in terms of convergence and final value. Therefore, this paper implements the combination of linear crossover and one-point mutation.

We simulated three different GAs; GA I transformed Markowitz's multi-objective model to a single-objective one by introducing λ as a risk-aversion parameter and solving the resulting problem for 101 equidistant levels of λ. GA II minimized risk for a given level of expected return. Again, we computed results for 101 different levels of return. To provide at least a certain level of expected return, we implemented a straightforward penalty term. To ensure solubility, the single asset solutions were added to the starting population. In addition, we included the best solution of the problem for the next highest level of return. Therefore, the algorithm started with the maximal return. Our results suggest that solutions of GA I are not as evenly distributed along the efficient frontier as the results of GA II, but lay on the same efficient frontier.

As for GA III, we implemented the SPEA2 as introduced by Zitzler et al. (2002). We performed two runs with different quantities of archive size, 21 and 250. The obtained solutions were well distributed along the resulting efficient frontier but seemed to lack on the low return side. On the other hand, for the test run with 250 archive individuals, solutions seemed to gather in the middle of the curve and at the high return side. Our results suggest that GA II slightly outperformed the results of GA III. In terms of computational time, GA I and GA II were much faster than GA III, whose second run took more than four days.

In the end, we ran GA II with VaR as the measure of risk. The resulting efficient frontier was flatter than the one obtained by using ES but also created well-distributed solutions. After that, ES for the solutions were computed and plotted against the optimal ES portfolios. We show that both frontiers differ from each other, especially at the low return side. The converted VaR solutions were not evenly distributed along the new frontier and even lacked for some ES values. Similarly, the optimal ES portfolios were transformed to VaR portfolios and plotted against the VaR optimal portfolios. This time, the resulting frontiers looked very similar; the converted ES solutions were well distributed and seemed to converge against the VaR optimal solutions. These results underline the differences between VaR and ES since they compute different solutions for the same data set.

We acknowledge several limitations in this research and recognize opportunities for future studies in this area. Firstly, the number of assets chosen for the algorithms is very small and random. Also, the tests of the different GA parameters were only performed for one single starting population and one lambda value. It is possible that diverse starting populations may lead to different results. Moreover, the population size was set to 50 and the number of generations was set to 100, whereas in Ranković et al. (2014) the settings were 200 and 500 respectively. However, we have insufficient evidence to state that an increase in the population size leads to significantly different results. We rejected the use of those settings since it would have led to unacceptable computation times. Also, the used penalty function in GA II is very simple in nature and should penalize solutions that are ‘far from acceptable’ more than ‘nearly acceptable’ solutions. Due to the simple structure of GA, it would be easy to add some real-world constraints, like cardinality constraints or short selling. Furthermore, a more in-depth investigation of possible operators for GA would be of interest. It would also be interesting to use bigger data sets, especially those which include market turmoil, and compare the results with VaR optimized solutions.

5 A Game Theory Application on Package Asset Auctions

The global financial crisis triggered by subprime mortgage loans posed many questions to researchers and regulators, as well as the management of big banks. One of the returning questions is how to manage and evaluate classes of assets that are illiquid or nonmarketable to retrieve liquidity to financial institutions or to remove unreflective or toxic assets from their balance sheets. Investors who put in effort to buy or sell assets that are not traded on centralized exchanges can face difficulties in finding a counterparty or an opportunity to make a trade. On the other side, the investor's inability to continuously trade represents an additional source of risk that cannot be hedged.

Different asset auction models are not new to researchers and regulators. On the contrary, treasury auctions trade a significant number of illiquid assets: The European Central Bank conducts weekly repo auctions, while the U.S. Federal Reserve does so every day, as a strongly regulated measure meant to provide liquidity to the financial system. One of the ongoing debates interrogates the transparency of information that central banks provide to private banks, especially in a crisis (Vives 2011). In this paper we will introduce a sustainable and fair action model that can be performed on relationship private seller – multiple private bidders that can guarantee competitive prices that are close to expected bidding outcome, as well as bidders' wanted assets, lowering their exposure in the process of bidding.

However, the introduction of the uniform price auction model, especially to the market of illiquid securities, might present a big challenge. In this paper, we start from the assumption that most of the bidders will be interested not in one part of the offered auction, but more in packages that will fulfil their bidding goals. The second assumption is that some parts of an offered asset are more attractive to some bidders than to others, but information about the attractiveness of one part of the asset to a bidder is not revealed to other bidders.

From the very start, a major problem that potential bidders might face is the exposure problem, which occurs when bidders want to acquire complementary parts of an offered asset and budget constraints which go hand in hand with the previous one. To clarify this statement, we can give an example: one bidder who aims to purchase the package of assets which consists of parts A, B, and C, in which case owning only two parts of the desired package is unaffordable or unprofitable, can spend the majority of budget for parts A and B, and then face a constraint in which the rest of the budget is not sufficient to win part C.

Package bidding, especially in cases where offered assets can be divided into a large number of parts, presented a big challenge in the past, mostly due to a large number of combinations involved. Additional factors include the fact that at any point in time, there could be no overlapping packages, or that every part of the offered asset can be included in only one package. Nowadays, the growing capabilities of technology, communications systems, and algorithms, offer more flexible possibilities for auctioneers and bidders.

There are numerous examples of successful and less successful ascending auctions with package bidding outside of the market of illiquid securities. The German 4G spectrum auction was a prime example of the large simultaneous ascending multi-band auction in which bidders presented both budget constraints and attraction towards complementing blocks of the spectrum, which ended after 224 rounds of competitive bidding (Cramton and Ockenfels 2014). Authors don't adequately clarify the fact that the outcome of the auction could have been better: "When the auction ended, the industry as a whole was more than €4 billion poorer and all bidders worse off compared to what probably might have been possible early in the auction."

The London Transportation authority was auctioneer of routes to private bus operators. The auction was conducted in the sealed-bid auction that allows bids on all combinations of routes within a particular auction (Cantillon and Pesendorfer, 2001). This paper states that 46% of winning bids involve package combinations.

An even better illustration of the practical application of package bidding auctions is the example of U.S. Advanced Wireless Service auction dated in the late summer of 2006, where the FCC auctioned 90 MHz of nationwide spectrum divided into 1122 parts. The auction attracted 168 bidders, where winning bids grossed 13.9 billion dollars (Bulow, Levin and Milgrom 2009).

In the world of illiquid assets, reverse auctions play a big part. The U.S. Treasury's autumn 2008 Troubled Asset Recovery Program (TARP) spent up to $700 billion buying "toxic assets" with a face value well in excess of $1 trillion. TARP purchased 25,000 similar securities with a similar price offered by 300 sellers. In this case, the weight of the sellers on the market played a significant role since the largest 10 held two-thirds of the total volume (Vives 2011).

5.1 Literature Review

Different auction models are well documented across the literature, investigating various applications, models, and bidding strategies. This study draws conclusions from three strands of previous research. The first strand draws conclusions from works on spectrum auctions. It is no coincidence that our literature review starts with practical examples of spectrum auctions, since we find that those types of auctions are the closest to how future illiquid assets auctions might look like, especially when it comes to package bidding, bidding strategies, attractiveness of certain packages, and arising practical problems, like exposure, budget constraints, and sellers payoff.

The literature documents that in large spectrum auctions, like ones conducted by FCC, information sufficient to forecast final price levels is often available early in the auction. Furthermore, the bidder's budgets, as opposed to their license values, are the ones that determine average prices in spectrum auction (Bulow, Levin and Milgrom 2009). Their experience points out that bidding activity in spectrum auctions often starts on the larger licenses and then moves on to smaller licenses (for which, there are strategical reasons) which can also be expected in illiquid assets auctions. Likewise, bidding strategies may be questioned for various reasons. Pagnozzi (2010) states that speculators often affect incentives to reduce demand of high-value bidders and develops a model to show that high-value bidders may strictly prefer to let speculators win some items and then buy these in the resale market, especially in the case when bidders' valuations are sufficiently different. Negotiations often involve costly delay, uncertainty, and inefficiency as a result of private information or some other factor (Myerson and Satterthwaite 1983, Cramton 1984). This is the reason why we specify our model with explicit no post-auction negotiations later on.

Practical examples show that reducing demand creates incentives for others to also reduce demand or even exit if one player can cause the auction to end (Cramton and Ockenfels 2014). Those authors argue that bidders that preferred different equilibria may have contributed to the conflict in the auction by creating a 'war of attrition' or 'chicken game', which may pose risks to the other bidders. In an illiquid assets auction game, we will assume there will be no 'war of attrition' or 'chicken game' (for the previously stated reasons) since we assume that holding one asset cannot improve bidders' market share like it is in spectrum auctions.

One of the major auction properties which should be considered even by the biggest bidders is exposure. We define exposure as the sum of all its bids in a given round, including its standing high bids from prior the round, and all of its new bids in the current round, whether or not the bidder is currently winning (Bulow, Levin and Milgrom 2009). As exposure rises closer to the budget constraint of the bidder, she will narrow the focus of bidding packages and less informational noise will be present. Some countries have adopted the combinatorial clock auction that allows bidders to bid on packages and thereby eliminate the exposure problem (Ausubel et al. 2006, Cramton 2013).

According to Bulow, Levin and Milgrom (2009), total exposure can be calculated as:

$$E = \pi_{all} \frac{D}{S}$$

where π_{all} is total price, D is demand and S is supply.

This work states that forecasting that total revenue would be equal to total exposure from that point forward would lead to errors that are mostly less than 10%, which is good enough to navigate crucial designations of every bidder in the game. A new entrant in any scenario can determine an exit strategy prior to final levels of a budget constraint if she ascertains that prices would become too high. Exposure tends to climb much faster than revenue, providing a useful forecast of final revenues early in the auction. Examples show that exposure reached 90% of final revenue at the time when revenue was under half of its final value.

The second strand of literature examines common knowledge factors in various auction game theory designs. Several recent works assume that the utility functions are common knowledge, but they are dependent on another parameter encoding the information of the bidder's type. Harsanyi (1967) and later Milgrom and Weber (1982) investigated models of auctions as games of incomplete information and adopted a Bayesian–Nash equilibrium as the solution concept. Lots of the following empirical works followed this methodology. Guerin and Tadjouddine (2006) present two special cases where common knowledge is achievable: auctions with identical players where the two highest bidders determine the price, and Groves mechanisms with a restriction on the pricing rule.

The third strand of literature documents the various auction designs. One of the most crucial works can be traced to the paper by William Vickery (1961) whose mechanism can be described as follows. Each bidder should report to the seller her entire demand schedule for all possible quantities. The seller will use that information to select the allocation that maximizes the total auction payoff. It then requires each buyer to pay an amount equal to the lowest total bid the buyer could have made to win her part of the final allocation, given the other bids. This way, the seller can be sure that each bidder will make a bid corresponding to her demand regardless of other bidders' offers. This auction model will be later generalized by Clarke (1971) and Groves (1973) who showed that this dominant strategy property can have a much wider range of applications lead to recognition of now widely used "Vickrey-Clarke-Groves (VCG) mechanism."

Despite the advantages of VCG auction model, it is not without fault. Researchers criticize that in some cases, revenues its yields can be very low or close to zero (Bulow, Levin and Milgrom 2009), in the case where one bidder is interested only in package with some bidding price x, while other bidders are interested in different single parts of that package with the same bidding price x. In that case, VCG model allocates a single piece of package to all latter bidders at price zero. They also point out its vulnerability to shill bidding and collusion, even by losing bidders (Sakurai, Yokoo and Matsubara, 1999 and Yokoo, Sakurai and Matsubara, 2000). In both works, the seller cannot determine the identities of bidders, the seller must be wary of whether bidders can profit by submitting additional bids under multiple identities.

One of the other weaknesses of Vickery auctions is that its dominant strategy property relies on unlimited bidder budgets, which are rarely the case (Che and Gale, 1998). Suppose that bidder A is willing to spend $2x$ on two parts of the package, of which every separate part can have a worth of $x+a$, $a<x$. Bidder B can offer $x-b$, $b<x$, for the first one and bidder C can offer $x+c$, where $c>b$ and $c<a$. If C continues to bid at some point, player A will hit the budget constraint even when he has the dominant strategy in both of the separate packages. This can be easily solved by the precise valuation of every part of the bided package, which leads us to the final Vickery model flaw.

In a combinatorial environment that is more complex, the determination of every package particle and its combination may present a big challenge for bidders. Lack of computational power may put some bidders in an inferior position. In the case of the package which consists of *A*, *B* and *C* parts in the fair game every bidder needs to know how valuable combinations *A*, *B*, *C*, *AB*, *BC*, *AC,* and *ABC* are to her. Combinations increase dramatically with the greater number of package particles. Rothkopf, Teisberg and Kahn (1990) show that reporting bidder valuations can hurt the fairness of the game, which can be solved by computerized auctions and end-to-end encryption.

All Vickery auction weaknesses lead researchers to modify and improve the auction model, moving to pay-as-bid auctions, which also show imperfections. One of the solutions is to introduce a custom bidding model, based on Vickery auction, for every specific market or auctioning case.

5.2 The Auction Model

Let there be a finite number I of different assets and finite number J of bidders in one auction. In this model we assume that bidders will form packages without any restrictions, meaning that bidders are free to make exclusive bids on as many assets which are formed into packages in every bidding round as they desire. This can cover a bigger spectrum of assets, while at the same time does not increase the bidder’s exposure. We also assume that the bidder has no mutually exclusive bids, meaning that there are no automatic options for the bidder who wants only asset A or only asset B, to reject asset B post to winning asset A if the bid was made.

The auction is a second price auction in the sense that the highest bid wins but pays only the second-highest price increased by the minimum increase amount. Hence, bidders can submit their truthful final bid price without being punished. This accelerates the auction time and protects the bidders' identities. All bids are firm offers. A bidder neither change nor withdraw an offer once it has been made, nor can they change the structure of the package in question.

Let us denote the bid on asset i from bidder j at moment $t \in (0, T)$ as $b_j^t(t)$ and the winning bid with same parameters as $B_j^t(t)$. All current and prior successful bids up to the time a bidder had her last successful bid are shown to her without the identity of any third bidder. Nevertheless, this is only the case if an asset were to be fully sold. If it is only sold in parts, the bidder can only see his own bid. If a bidder is currently not successful, she does not see the current winning bid. The bidders receive this information as a protocol.

After each round, the seller identifies a set of conditionally winning bids. In our case, that is the set of bids that maximizes the total profits of the seller. Every asset can have only one winning price and can be sold only once later. The auction proceeds in a sequence of bidding rounds until there is round in which there are new winning bids or no new entrants. The auction then ends, and the previous winning bids become part of final winnings.

The auction is open for a minimum time of 1 hour for all bidders and assets. Bidders who did not submit any bids exceeding the minimum start prices in the first 15 minutes are automatically excluded from the auction. If there is no activity for a period of 10 minutes on one particular asset that has a valid bid that is higher than the seller's reserve price R_i, the auction on that specific asset is closed. The auction time for such a particular asset extends for an additional 10 minutes if the higher bid is submitted. If one bidder has no successful bids on any open auctions for a time period of 30 minutes, she is excluded from further auctions.

In the end, the seller receives the information of finally successful bidders, including their names. In addition, the seller can see the highest unsuccessful bids exceeding the reserve price in case the winning bids are fewer, but without those bidders' names. The reserve prices are only known to the seller. The seller can potentially use this data anonymously for after-auction sales.

This auction model doesn't take *proxy agents* bidding into consideration, which means that every bidder needs to take care of their constraints, dependencies, and package desires[16]. In order to avoid collusion among bidders, the auction takes place at a supervised place without communication devices. The supervisor can also check whether any problems with the software occurs, which may justify extending the auction time for this or all current bidders.

5.2.1 Bidder's strategy

As we previously defined, let $b_j^t(i)$ be the bid of bidder j for asset i at round t. Also, let MI_i denote the minimum increase amount, which is a function of the notional and leading bid price, but in general, an immaterial price increase. This is set by the seller and remains the same for all items $i = 1 \dots I$. Let $B_j^t(i)$ denote winning bid of bidder j for asset i at round t.

$$B_j^t(i) = \begin{cases} \max_t B^t(i) - MI_i, \ if he\ won\ the\ asset \\ 0, \ otherwise. \end{cases} \quad (5.1)$$

For $t = 1, \dots, T$ a successful bid for the asset *i* is defined as

$$B^t(i) = \begin{cases} \max_j b_j^{t-1}(i), \ if\ b_j^{t-1}(i) \geq R_i \\ \emptyset, \ otherwise. \end{cases} \quad (5.2)$$

The threshold at time $t = 0$ is given by the auction's start price S_i. Thus for $t = 0$ it is:

$$B_j^0(i) = S_i \quad (3.3)$$

where S_i is auction start price and $S_i \leq R_i\ for\ all\ i = 1, \dots, I$.

[16] Ausubel and Milgrom, 2002 present auction design where bidding is required to be trough proxy agents. Bidders can instruct proxy agents to apply sincere strategy in which instructions are truthful or semi-sincere shading the proxy instructions which in our case leads to model complications.

A bidder's bid is equal to last successful bid B^{t-1} plus the minimum increase amount MI_i which implies:

$$b_j^t(i) = B^{t-1}(i) + MI_i \quad (5.4)$$

When we introduce the budget of bidder j as K_j we conclude that it must hold that that sum of all successful bids of bidder j does not exceed bidder's budget:

$$\sum_i B_j^t(i) \leq K_j \text{ for all } t = 1, \dots, T \quad (5.5)$$

Let us assume that every bidder has different valuation $v_j(i)$ of asset i in straightforward bidding which complies with bidder's views and $v_j(i) \geq 0 \; for \; all \; i = 1, \dots, I.$

The bidder's profit at the time t is defined as the difference between a bidder's valuation and the second last successful bid, given that he won the item:

$$\pi_j^t(i) = \begin{cases} v_j(i) - B_j^{t-1}(i), \; if \; asset \; i \; is \; won \\ 0, \; otherwise. \end{cases} \quad (5.6)$$

5.2.2 Seller's revenue

Given that the previously explained values comply, and that the seller accepts the maximum total value of all the offers at the time t, the seller's revenue at any round t can be defined as:

$$\pi^t = \sum_i B^t(i) \quad (5.7)$$

5.3 Asymmetric signaling game with two bidders

Let us assume there are two bidders 1 and 2 who are bidding on two types of assets A and B where seller of the goods is exogenous entity. Let us denote with K_1 and K_2 respectively bidders budgets in this game and U_{1A} utility of the asset A for the bidder 1 and similarly U_{1B}, U_{2A}, U_{2B} utility for the asset B as well as utilities for both A and B for the second bidder.

The following is the list of possible assumptions:

Q1: Bidder 1 values asset A more than bidder 2: $U_{1A} > U_{2A}$

Q2: Bidder 1 values asset B more than bidder 2: $U_{1B} > U_{2B}$

Q3: Bidder 2 values asset A more than bidder 1: $U_{1A} < U_{2A}$

Q4: Bidder 2 values asset B more than bidder 1: $U_{1B} < U_{2B}$

Q5: Bidder 1 has bigger budget than the bidder 2: K_1>K_2

Q6: Bidder 2 has bigger budget than the bidder 2: $K_1 < K_2$

Creating following list of possible scenarios:

S1: Q1, Q2, Q5

S2: Q1, Q2, Q6
S3: Q1, Q4, Q5
S4: Q1, Q4, Q6
S5: Q2, Q3, Q5
S6: Q2, Q3, Q6
S7: Q3, Q4, Q5
S8: Q3, Q4, Q6

Budgets of the bidders are the common knowledge.

It is trivial that there are certain games that quickly come to an end:

(S1) If $U_{1A} > U_{2A}$ and $U_{1B} > U_{2B}$ and $U_{1A} + U_{1B} < K_1$, then bidder 1 purchases both assets after the first round.

(S8) If $U_{2A} > U_{1A}$ and $U_{2B} > U_{1B}$ and $U_{2A} + U_{2B} < K_2$, then bidder 2 purchases both assets after the first round.

(S3) If $U_{1A} > U_{2A}$ and $U_{2B} > U_{1B}$ while $U_{1A} < K_1$ and $U_{2B} < K_2$ holds, then bidder 1 purchases asset A and bidder 2 purchases asset B and vice versa in case S6.

Let us assume a more complex state which doesn't reach equilibrium quickly. Let utility of the first asset A be higher for the bidder 1 $U_{1A} > U_{2A}$ and as well as utility for the asset B: $U_{1A} > U_{2A}$. Similarly

$$U_{1A} + U_{1B} > K_1 \text{ and } U_{2A} + U_{2B} < K_1 \text{ holds following S2.}$$

We will simplify this case by assuming that budgets of both bidders are common knowledge and relax that assumption later on.

We could observe behavior of bidder 2. She would obviously follow the price of asset A until it reaches U_{2A}. If she didn't win asset A on the price $U_{2A} + MI_A$ in that moment she realizes that $U_{1A} > U_{2A}$.

Since bidding is conducted through steps we can denote:

$$U_{1A} = n_{1A} MI_A \text{ and } U_{2A} = n_{2A} MI_A$$

which means that considering minimum increase amount for the asset A, bidder 1 can make a maximum of n_{1A} while bidder 2 can make n_{2A} where

$$n_{1A} = n_{2A} + l_A$$

5.3.1 Illustrative Examples: Two Asset Games

Consider a simple example of the game with two bidders, 1 and 2, and two assets, A and B. Let $U_{1A} = \$5$ be bidder 1's valuation for the asset A, $U_{2A} = \$4$ bidder 2's valuation for the same asset, as well as $U_{1B} = \$4$ and $U_{2B} = \$3$ respective valuations for the second asset. Let us also assume that the budget of bidder 1 is $K_1 = \$8$ while the budget of the second bidder is higher. The minimum increase amount for every asset is \$1. These assumptions cover S2.

To be more precise in analyzing this game we should point out that, while we denote $U_{2A} = \$4$,we keep in mind that the utility of asset A for player 1 can be, for example \$4.2, but since it will only make sense for computer-based bidding algorithms to follow bidding and stop on minimum increase amount stages, bidder 1 should recognize $U_{1A} = \$4 + \alpha_{1A}$ as $U_{1A} = \$4$ where $\alpha_{1A} \epsilon [0,1)$. This is important since bidder 1 winning asset A for the rate of \$4 wouldn't necessarily mean no profit, but profit of α_{1A}.

We will analyze the bidding game as such, during every step of the game we will analyze bidder's gains or losses in a two-dimensional system. Let us contemplate the following table:

Table 5.1: Outcome and equilibrium, two asset game with two bidders, S2

Bidder 1 \ Bidder 2	\$4	\$5
\$4	[0,0,0] / [0,1, α_{1B}]	[1,0,$-1 + \alpha_{2A}$] / [0,1, α_{1A}]
\$5	[0,0,0] / [1,0, α_{1A}]	[0,0,0] / [0,0,0]

If both players decide on \$4 as the highest price for asset A, on level \$4, the game will be equal, and as the price of that asset rises to the level of \$5, both players will opt out. In that case, since they are bidding simultaneously on asset B, bidder 1 has \$4 left in the budget, and he wins asset B (because his $U_{1B} = \$4$) for that exact amount – since price rose to that level and bidder 2 will automatically opted out after $U_{2B} = \$3$.

Resolution vectors are given in table 5.1 for both players. On a level where both of the players offer a maximum of \$4 for asset A, the resolution vector for bidder 1 is [0,1, α_{1A}], where the first dimension of the vector represents the case where the bidder obtained asset A (1 he did, 0 he didn't), the second dimension represents whether the bidder obtained asset B, and the third dimension denotes profit or loss in that case. In the case where both bidders offer \$4 as the highest price for asset A, bidder 2 will not obtain any assets, while bidder 1 will obtain only asset B and gain profit α_{1B}.

Similarly, when bidder 1 is ready to offer \$4 as the highest price for the asset A, while 2 is offering \$5 as the maximum, bidder 2 will obviously win asset A at the price of \$5, losing 1-α_{2A}, while bidder 1 will win asset B for the price of \$4 (since that is what is left in the budget, keeping in mind that $U_{2B} = \$3$, and that bidder 2 has no interest continuing the game after the price of \$3 for asset B). Correspondingly, outcomes for other combinations are given in table 5.1. In this simple example, outcome is a mixed strategy equilibrium where bidder 1 can choose both \$4 and \$5 as the maximum amount for asset A, while bidder 2 should always opt for \$4 for the same asset.

Consider now an example of the game with two bidders, 1 and 2 and two assets, A and B. Let $U_{1A} = \$6$ be bidder 1 valuation for the asset A, $U_{2A} = \$4$ bidder 2 valuation for the same asset, as well as $U_{1B} = \$5$ and $U_{2B} = \$3$ respective valuations for the second asset. Let us also assume that the budget of bidder 1 is $K_1 = \$9$ and $K_2 > K_1$ while the budget of the second bidder is higher. Minimum increase amount for every asset is \$1.

Table 5.2: Outcome and equilibrium, two asset game with two bidders, S2:

Bidder 1 \ Bidder 2	\$4	\$5	\$6
\$4	$[0,0,0]$ / $[0,1,\alpha_{1A}]$	$[1,0,-1+\alpha_{2A}]$ / $[0,1,\alpha_{1A}]$	$[1,0,-1+\alpha_{2A}]$ / $[0,1,1+\alpha_{1A}]$
\$5	$[0,0,0]$ / $[1,1,2+\alpha_{1A}+\alpha_{1B}]$	$[0,0,0]$ / $[0,1,1+\alpha_{1A}]$	$[1,0,-2+\alpha_{2A}]$ / $[1,0,1+\alpha_{1A}]$
\$6	$[0,0,0]$ / $[1,1,2+\alpha_{1A}+\alpha_{1B}]$	$[0,0,0]$ / $[1,1,\alpha_{1A}+\alpha_{1B}]$	$[0,0,0]$ / $[0,1,1+\alpha_{1A}]$

Outcomes are presented in table 5.2. In this case, even the budget of the bidder 2 is higher, higher valuation for both of the assets from bidder 1 puts bidder 2 in a subordinate position. Bidder 2, in this case, can obtain one asset in only two of nine cases, with expected losses.

In this case, a Nash equilibrium is reached, since bidder 1's strategy is to offer \$5 as the highest price for the asset A (to maximize the profit), while bidder 2's strategy is to offer \$4 for the same asset.

Let us widen the gap between bidder 1 and bidder 2 values for asset A. Let $U_{1A} = \$7$ be bidder 1 valuation for the asset A, $U_{2A} = \$5$ bidder 2 valuation for the same asset, as well as $U_{1B} = \$5$ and $U_{2B} = \$3$ respective valuations for the second asset. Let us also assume that the budget of bidder 1 is $K_1 = \$8$ and $K_2 > K_1$.

Outcomes are presented in table 5.3. In this case, like the second example, even when the budget of bidder 2 is higher, the higher valuation for both of the assets from bidder 1 puts bidder 2 in a subordinate position. It is clear, that bidder 1 will not opt for the maximum price of \$7 for the asset A, while she would opt less for price of \$4 than for \$5 or \$6. Bidder 2 would not opt for prices \$6 or \$7, which creates mixed strategy equilibrium. If we assume that bidder 2 chooses \$4 as the maximum price for asset A with probability p, we can calculate expected profits for bidder 1 and 2 as:

$$\mathrm{E}(\mathrm{P}_A) = \mathrm{p}(1 + \alpha_{1A}) + (1 - p)(1 + \alpha_{1B})$$

$$\mathrm{E}(\mathrm{P}_B) = \frac{1}{2}(1 - p)\,\alpha_{2B}$$

Table 5.3: Outcome and equilibrium, two asset game with two bidders, S2

Bidder 1 \ Bidder 2	\$4	\$5	\$6	\$7
\$4	[0,0,0] [0,1, α_{1B}]	[1,0,$-1+\alpha_{2A}$] [0,1,1 + α_{1A}]	[1,0,$-1+\alpha_{2A}$] [0,1,1+ α_{1A}]	[1,0,$-1+\alpha_{2A}$] [0,1,1+ α_{1A}]
\$5	[0,0,0] [1,0, 1 + α_{1A}]	[0,0,0] [0,1,1 + α_{1B}]	[1,0,-2+α_{2A}] [1,0,1 + α_{1A}]	[1,0,-2+α_{2A}] [1,0,1 + α_{1A}]
\$6	[0,0,0] [1,0, 1+ α_{1A}]	[0,1,α_{2B}] [1,0, 1 + α_{1A}]	[0,0,0] [0,1,1 + α_{1B}]	[1,0,$-3+\alpha_{2A}$] [0,1,1 + α_{1B}]
\$7	[0,0,0] [1,0, 1+ α_{1A}]	[0,1,α_{2B}] [1,0, 1 + α_{1A}]	[0,1,1 + α_{2B}] [1,0, α_{1A}]	[0,0,0] [0,1, α_{1B}]

Proposition 5.1. In a scenario where $U_{1A} > U_{2A}$, $U_{1B} > U_{2B}$ and $U_{1A} + U_{1B} > K_1$, $U_{2A} + U_{2B} < K_1$ bidder 1 will select $U_{1A} - MI_A$ as highest bidding price for asset A while bidder 2 will be in a subordinate position and can choose U_{2A} or lower as the highest bidding price for asset A.

Proposition 5.2. In scenario where $U_{1A} > U_{2A}$, $U_{1B} > U_{2B}$ and $U_{1A} + U_{1B} > K_1$, $U_{2A} + U_{2B} < K_1$ sellers revenue from (5.7) is

$$\pi_{AB}^t = U_{2A} + U_{2B} + MI_A \text{ when } \alpha_{1A} > \alpha_{1B} \text{ or}$$

$$\pi_{AB}^t = U_{2B} + MI_B \text{ when } \alpha_{1A} > \alpha_{1B}$$

It is clear that in the first case, the seller will sell both assets, while in the second one, she will sell only asset B. We can settle that case S7 is equivalent to case S2.

Moving on to S5, which also doesn't reach equilibrium quickly. Let utility of the first asset A be higher for the bidder 2 $U_{1A} < U_{2B}$ but utility of the second asset B be higher for the bidder 1: $U_{1B} > U_{2B}$. Also

$U_{1A} + U_{1B} > K_1$ *and* $U_{2A} + U_{2B} < K_1$ *holds while* $K_1 > K_2$

Table 5.4: Outcome and equilibrium, two asset game with two bidders, S5

Bidder 2 / Bidder 1	\$4	\$5	\$6
\$4	[0,0,0] [0,1,3 + α_{1B}]	[1,0,1 + α_{2A}] [0,1, 3 + α_{1B}]	[1,0,1 + α_{2A}] [0,1,3+ α_{1B}]
\$5	[0,0,0] [1,1, 2 + $\alpha_{1A} + \alpha_{1B}$]	[0,0,0] [0,1,3 + α_{1B}]	[1,0,α_{2A}] [0,1, 3 + α_{1B}]
\$6	[0,0,0] [1,1, 2+ $\alpha_{1A} + \alpha_{1B}$]	[0,0,0] [1,1, $\alpha_{1A} + \alpha_{1B}$]	[0,0,0] [0,1,3 + α_{1A}]

Let $U_{1A} = \$4$ be bidder 1 valuation for the asset A, $U_{2A} = \$6$ bidder 2 valuation for the same asset, as well as $U_{1B} = \$7$ and $U_{2B} = \$3$ respective valuations for the second asset. Let us also assume that the budget of bidder 1 is $K_1 = \$10$ and $K_2 = \$9$. Minimum increase amount for both assets is $1.

Outcomes are presented in table 5.4. It is obvious that bidder 2 will opt for a maximum price of $6 for the asset A, since the payoff in most of the cases is better. While keeping that in mind, bidder 1 would choose either $4 or $5 with same probabilities, since the payoff is the same in both cases. The result is a mixed strategy equilibrium with two outcomes. With probability of $\frac{1}{2}$, asset A will be sold to bidder 2 at price of $5 while the asset B will be sold to bidder 1 at the price of $4. In the other case, asset A will be sold to bidder 2 at price of $6, while the asset B will be sold to bidder 1 at the price of $4.

Proposition 5.3. In scenario where $U_{1A} < U_{2A}$, $U_{1B} > U_{2B}$ and $U_{1A} + U_{1B} > K_1$, $U_{2A} + U_{2B} < K_1$ and $K_1 > K_2$ bidder 1 can selected U_{1A} or higher as highest bidding price for asset A while bidder 2 must chose U_{2A} as highest bidding price for asset A. Bidder 1 will win asset B and bidder 2 will win asset A.

In case where bidder 1 opts for higher then U_{1A} she pushes bidder 2 to pay more for asset A (which can be important in case where with more than 2 assets) and also boosts sellers' revenue, but it can only bid until K_1- U_{2B}- MI_B.

Proposition 5.4. In scenario where $U_{1A} < U_{2A}$, $U_{1B} > U_{2B}$ and $U_{1A} + U_{1B} > K_1$, $U_{2A} + U_{2B} < K_1$ and $K_1 > K_2$ sellers' revenue from (5.7) is

$$\pi_{AB}^{t} = U_{1A} + U_{2B} + MI_A + MI_B$$

As a final not trivial case we analyze S4 where utility of the first asset A be higher for the bidder 1 $U_{1A} > U_{2B}$ but utility of the second asset B be higher for the bidder 2: $U_{1B} < U_{2B}$. Also

$U_{1A} + U_{1B} > K_1$ and $U_{2A} + U_{2B} < K_1$ holds while $K_1 < K_2$.

Let $U_{1A} = \$7$ be bidder 1 valuation for the asset A, $U_{2A} = \$4$ bidder 2 valuation for the same asset, as well as $U_{1B} = \$4$ and $U_{2B} = \$5$ respective valuations for the second asset. Let us also assume that budget of bidder 1 is $K_1 = \$10$ and $K_2 = \$12$. Minimum increase amount for both assets is $1.

Table 5.5: Outcome and equilibrium, two asset game with two bidders, S4

Bidder 1 \ Bidder 2	$4	$5	$6	$7
$4	$[0,1,\alpha_{2B}]$ / $[0,0,0]$	$[1,1,-1+\alpha_{2A}+\alpha_{2B}]$ / $[0,0,0]$	$[1,1,-1+\alpha_{2A}+\alpha_{2B}]$ / $[0,0,0]$	$[1,1,-1+\alpha_{2A}+\alpha_{2B}]$ / $[0,0,0]$
$5	$[0,1,\alpha_{2B}]$ / $[1,0,2+\alpha_{1A}]$	$[0,1,\alpha_{2B}]$ / $[0,0,0]$	$[1,1,-2+\alpha_{2A}+\alpha_{2B}]$ / $[0,0,0]$	$[1,1,-2+\alpha_{2A}+\alpha_{2B}]$ / $[0,0,0]$
$6	$[0,1,\alpha_{2B}]$ / $[1,0,2+\alpha_{1A}]$	$[0,1,\alpha_{2B}]$ / $[1,0,1+\alpha_{1A}]$	$[0,1,\alpha_{2B}]$ / $[0,0,0]$	$[1,1,-3+\alpha_{2A}]$ / $[0,0,0]$
$7	$[0,1,\alpha_{2B}]$ / $[1,0,2+\alpha_{1A}]$	$[0,1,\alpha_{2B}]$ / $[1,0,1+\alpha_{1A}]$	$[0,1,\alpha_{2B}]$ / $[1,0,\alpha_{1A}]$	$[0,1,1+\alpha_{2B}]$ / $[0,0,0]$

Outcomes are presented in table 5.5. Best payoffs for bidder 2 are lying when she presents strategy maximum price of $4 for the asset A. In that case she will obtain asset B and bidder 1 will win asset A for the price of $5, only if she chooses any of the strategies maximum price of $5 or higher for the asset A. In special case where both players are choosing maximum price of $7 for the asset A bidder 2 can expect that she will win asset B for price of $3 and block out bidder 1 from obtaining any of the assets, but that presents greater risk since bidder 1 chooses price $7 only with the probability of $\frac{1}{3}$.

Proposition 5.5. In scenario where $U_{1A} > U_{2A}$, $U_{1B} < U_{2B}$ and $U_{1A} + U_{1B} > K_1$, $U_{2A} + U_{2B} < K_1$ and $K_1 < K_2$ bidder 1 can selected $U_{2A} + MI_A$ or higher as highest bidding price for asset A while bidder 2 must chose U_{2A} as highest bidding price for asset A. Bidder 1 will win asset A and bidder 2 will win asset B.

Proposition 5.6. In scenario where $U_{1A} > U_{2A}$, $U_{1B} < U_{2B}$ and $U_{1A} + U_{1B} > K_1$, $U_{2A} + U_{2B} < K_1$ and $K_1 < K_2$ sellers' revenue from (5.7) is

$$\pi_{AB}^{t} = U_{2A} + MI_A + K_1 - (U_{2A} + MI_A) + MI_B =$$

$$= K_1 + MI_B$$

5.3.2 A genuine illustrative example

In this example we set more realistic example that can present any illiquid asset auction with 2 bidders and 2 assets where minimum increase amounts make around 5% of start price what became rule in most of first stages in asset auctions. Let us denote

$$\widetilde{U_{ji}} = \left(\left\lfloor \frac{U_{ji}}{MI_i} \right\rfloor + 1\right) * MI_i$$

as ceiling utility of asset i for bidder j, since we shod expect that every utility U_{ji} is in between MI_i steps. It is clear that every bidder needs to define their maximum bidding prices according to $\widetilde{U_{ji}}$.

Consider a bidding game for two assets with starting prices $S_A = \$260\ M$ and $S_B = \$180\ M$ where $MI_A = 20\ M$ and $MI_B = 10\ M$ and bidders' utilities and budgets are as following:

$U_{1A} = \$358M$; $U_{1B} = \$223M$; $K_1 = \$490M$

$U_{2A} = \$311M$; $U_{2B} = \$207M$; $K_2 = \$600M$

Outcomes are presented in table 5.6. The idea behind this game is that bidder 2 has a relatively bigger budget than bidder 1. This allows bidder 2 to push bidder 1 out of the competition for one asset. It is clear that bidder 1 will never obtain asset B, even though her utility U_{1B} is higher than the valuation of bidder 2. This is due to the starting bid of \$280M for asset A her remaining budget is only \$190M. Since the starting bid for asset B is $S_B = \$180\ M$ the only bid bidder 1 can turn is her reserve budget of \$190M (remember $MI_B = 10\ M$ and $U_{1B} = \$223M$). This leads to bidder 2 outbidding bidder 1 in every case for asset B. The question remains what the optimal betting strategy for bidder 1 should be, given she can only obtain asset A.

It turns out that the best strategy bidder 1 could play is bidding up to $\widetilde{U_{1A}} = \$360M - MI_A = \$340M$. If bidder 2 decides to play the same strategy, i.e. bidding up to $\widetilde{U_{2A}} = \$300M$. Since bidder 1 push up to \$340M bidder 2 will cut out at \$320M. This leaves both bidders with the optimal outcomes [1,0,38] for bidder 1 and [0,1,27] for bidder 2. If bidder 2 decides to make use of her higher budget and thus tries to outbid and pushes for prices above \$300 (e.g. \$320M, \$340M and \$360M) bidder 1 will either be left with a lower profit of 18 or not obtain any asset and leaves the auction with profits of zero (in case bidder 2 bids \$340M upwards).

Bidder 1 will not push higher than \$340M, because the chance of losing is higher, since bidder 2 could push for \$340M as well and thus leaving bidder 1 with a loss of -2. Bidder 1 will not push for \$380M since the chance of losing is even higher, resulting in -2 and -22 respectively, if bidder 2 decides to push for \$340M or \$360M.

Table 5.6: A genuine illustrative example, outcomes for 2 bidders. Utilities and budgets are following: $U_{1A} = \$358M$; $U_{1B} = \$223M$; $K_1 = \$490M$, $U_{2A} = \$311M$; $U_{2B} = \$207M$; $K_2 = \$600M$

Bidder 2 \ Bidder 1	\$300M	\$320M	\$340M	\$360M	\$380M	∑ of profits for bidder 2
\$300M	[0,0,0] / [0,1,7]	[1,0,38] / [0,1,27]	[1,0,38] / [0,1,27]	[1,0,38] / [0,1,27]	[1,0,38] / [0,1,27]	115
\$320M	[0,0,0] / [1,1,−9 + 7]	[0,0,0] / [0,1,27]	[1,0,18] / [0,1,27]	[1,0,18] / [0,1,27]	[1,0,18] / [0,1,27]	106
\$340M	[0,0,0] / [1,1,−9 + 7]	[0,0,0] / [1,1,−29 + 27]	[0,0,0] / [0,1,27]	[1,0,-2] / [0,1,27]	[1,0,-2] / [0,1,27]	77
\$360M	[0,0,0] / [1,1,−9 + 7]	[0,0,0] / [1,1,−29 + 27]	[0,0,0] / [1,1,−49 + 27]	[0,0,0] / [0,1,27]	[1,0,-22] / [0,1,27]	28
∑ of profits for bidder 1	0	38	56	54	32	

5.4 Three Asset Games

Let us consider case with three assets in bidding game with two bidders. Every bidder values ABC over all other packages. First bidder is betting on parts A, B and C and prefers some packages over some other. She also is pleased with obtaining a double consecutive asset package (i.e. AB, BC) in the auction, but gap package AC has no value. Second bidders values gap package AC less then consecutive ones AB, BC (as in spectrum auctions). This also means the utility for double packages must be equal or marginally greater (i.e. one unit of minimum increase amount) than the sum of single package utility. This holds even if she does not care about gap-packages since she has a valuation of the single packages. This is case two with $U_{1AC} = 0$. In such a case the utility for double packages must be equal to the difference from the double package and the missing gap-package.

Table 5.7: Utilities for specific packages in three asset game

A	**$6**	**$5**
B	$7	$6
C	$4	$3
AB	$14	$12
BC	$10	$9
AC	$0	$10
ABC	$19	$20
Budget	$15	$18

For an illustration, consider following utilities for both bidders for every possible package, as well as their budgets in three asset auction (table 5.7):

Table 5.8: Outcomes, three asset game

p_{1B} \ p_{2B}	\$5	\$6	\$7
\$5	[0,0,0,0] / $[0,0,1,\alpha_{1C}]$	$[1,1,0,\alpha_{2AB}]$ / $[0,0,1, \alpha_{1C}]$	$[1,1,0,\alpha_{2AB}]$ / $[0,0,1, \alpha_{1C}]$
\$6	[0,0,0,0] / $[1,1,0,2+\alpha_{1AB}]$	$[0,1,0, \alpha_{2A}]$ / $[0,0,1,\alpha_{1C}]$	$[1,1,0, \alpha_{2AB}]$ / $[0,0,1,\alpha_{1C}]$
\$7	[0,0,0,0] / $[1,1,0,2+\alpha_{1AB}]$	$[0,1,0,-1+\alpha_{2A}]$ / $[1,0,0,\alpha_{1B}]$	$[0,0,1,1+\alpha_{2C}]$ / $[0,1,0,\alpha_{1A}]$

Focus of building strategy for both bidders is on asset B. Let us study every combination for price of asset B, $5 \leq p_{1B}, p_{1B} \leq 7$. Every pair of strategies for the assets B creates equilibrium in game for A and C, resulting in outcomes presented in table 5.8 where vectors $[b_B, b_A, b_C, \pi]$, $b_B, b_A, b_C \in \{0,1\}$ represent did the bidder obtained assets A, B or C and which is her profit

$$\pi = U_x - b_A p_A + b_B p_B + b_C p_C$$

Subgame equilibriums are presented in tables 5.9 – 5.15.

In this case, bidder 1 is in subordinate position, since most of his subgame strategies are annulled due to the high exposure that gap-package brings with its zero utility. Bidder 1 can obtain only one asset and will set \$6 for the asset B if $\alpha_{1C} > \alpha_{1A}$ and \$7 in other case. Bidder 2 can obtain one or two other assets only if he sets his maximum bidding price for asset B at \$7. It is important to notice that bidder’s budgets again don’t play important role in outcome of the game. Bidder 2 is in subordinate position in all combinations where he opts for \$5 as his maximum bidding price for asset B. In this example it is clear that bidder 1 strategy [A, B, C] is [4, 6, 4] if $\alpha_{1C} > \alpha_{1A}$ and [6, 7, 4] if $\alpha_{1C} < \alpha_{1A}$. It is also important to emphasize that bidder 1 will opt for second strategy if she has benefit of pushing bidder 2 out of the obtaining asset C, making bidding for asset B lose-

lose game. Bidder 2 strategy [A, B, C] is [5, 7, 3] and she will obtain assets A and B or asset C, depending on if $\alpha_{1C} > \alpha_{1A}$ stands and bidder 1's strategy.

Proposition 5.7. In three asset auction game with two bidders where $U_{1x} > U_{2x}$ and x is A, B, C, AB, BC, ABC and $U_{1AC} = 0$ and $U_{2AC} > 0$ and $K_1 < K_2$ bidder 1 will select

$$[U_{1A} - \big((K_2 - K_1) - (U_{1B} - U_{2B})\big), U_{2B}, U_{1C}\ ] \text{ if } \alpha_{1C} > \alpha_{1A}$$

$$\text{and chose } [U_{1A}, U_{1B}, U_{1C}\ ] \text{ if } \alpha_{1C} < \alpha_{1A}$$

as her optimal strategy, and bidder B will select $[U_{2A}, U_{2B} + MI_B, U_{2C}\ ]$ as her optimal strategy. Bidder 1 will obtain C and bidder 2 will obtain A and B if $\alpha_{1C} > \alpha_{1A}$ or bidder 1 will obtain A and bidder 2 will obtain C, if $\alpha_{1C} < \alpha_{1A}$.

Proposition 5.8. In three asset auction game with two bidders where U_{1x}

$> U_{2x}$ and x is A, B, C, AB, BC, ABC and $U_{1AC} = 0$ and $U_{2AC} > 0$ and $K_1 < K_2$ sellers' revenue from (5.7) is

$$\pi_{AB}^{t} = U_{2A} + U_{2B} + U_{1C} + MI_B \text{ if } \alpha_{1C} > \alpha_{1A} \text{ or}$$

$$\pi_{AB}^{t} = K_1 - U_{1A} - U_{1B} + MI_C \text{ if } \alpha_{1C} < \alpha_{1A}$$

Table 5.9: Subgame, three asset game, for $p_{1B} = 5$ *and* $p_{2B} = 5$

	Bidder 2 / Bidder 1	\$4	\$5	\$6
$p_{1B} = 5$ $p_{2B} = 5$ $K_{1l} = 10$ $K_{2l} = 13$	\$4	[0,0,0] [0,1,α_{1C}]	[1,0,α_{2A}] [0,1, α_{1C}]	[1,0,α_{2A}] [0,1, α_{1C}]
	\$5	[0,0,0] [1,1,−10]	[0,0,0] [0,1,α_{1C}]	[1,0,$-1 + \alpha_{2A}$] [0,1, α_{1C}]
	\$6	[0,0,0] [1,1, -10]	[0,0,0] [1,1, -10]	[0,0,0] [0,1, α_{1C}]

Table 5.10: Subgame, three asset game, for $p_{1B} = 5$ and $p_{2B} = 6$, player 2 wins asset B

	Bidder 2 / Bidder 1	\$4	\$5	\$6
$p_{1B} = 5$ $p_{2B} = 6$, wins B $K_{1l} = 10$ $K_{2l} = 12$	\$4	[0,0,0] / $[0,1,\alpha_{1C}]$	$[1,0,1 + \alpha_{2A}]$ / $[0,1, \alpha_{1C}]$	$[1,0,1 + \alpha_{2AB}]$ / $[0,1, \alpha_{1c}]$
	\$5	[0,0,0] / $[1,1,-9]$	[0,0,0] / $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2AB}]$ / $[0,1, \alpha_{1C}]$
	\$6	[0,0,0] / [1,1, -9]	[0,0,0] / [1,1, -10]	$[1,0,\alpha_{2AB}]$ / $[0,1, \alpha_{1C}]$

Table 5.11: Subgame, three asset game, for $p_{1B} = 6$ and $p_{2B} = 5$, player 1 wins asset B

	Bidder 2 / Bidder 1	\$4	\$5	\$6
$p_{1B} = 6$, wins B $p_{2B} = 5$ $K_{1l} = 10$ $K_{2l} = 12$	\$4	[0,0,0] / $[0,1,\alpha_{1BC}]$	$[1,0,\alpha_{2A}]$ / $[0,1,\alpha_{1BC}]$	$[1,0,1 + \alpha_{2AB}]$ / $[0,1, \alpha_{1c}]$
	\$5	[0,0,0] / $[1,1,4 + \alpha_{1ABC}]$	[0,0,0] / $[0,1,\alpha_{1BC}]$	$[1,0,-1 + \alpha_{2A}]$ / $[0,1, \alpha_{1C}]$
	\$6	[0,0,0] / $[1,1, 4 + \alpha_{1ABC}]$	[0,0,0] / $[1,0, 2+\alpha_{1AB}]$	[0,0,0] / [0,0,0]

Table 5.12: Subgame, three asset game, for $p_{1B} = 6$ and $p_{2B} = 6$

	Bidder 2 / Bidder 1	\$4	\$5	\$6
$p_{1B} = 6$ $p_{2B} = 6$ $K_{1l} = 9$ $K_{2l} = 12$	\$4	[0,0,0] $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2A}]$ $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2A}]$ $[0,1, \alpha_{1c}]$
	\$5	[0,0,0] $[1,1,-9]$	[0,0,0] $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2A}]$ $[0,1, \alpha_{1C}]$
	\$6	[0,0,0] $[1,1,-9]$	[0,0,0] $[1,0, \alpha_{1A}]]$	[0,0,0] [0,0,0]

Table 5.13: Subgame, three asset game, for $p_{1B} = 6$ and $p_{2B} = 7$, player 2 wins asset B

	Bidder 2 / Bidder 1	\$4	\$5	\$6
$p_{1B} = 6$ $p_{2B} = 7$ wins B $K_{1l} = 9$ $K_{2l} = 11$	\$4	[0,0,-1] $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2AB}]$ $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2AB}]$ $[0,1, \alpha_{1c}]$
	\$5	[0,0,-1] $[1,1,-9]$	[0,0,-1] $[0,1,\alpha_{1C}]$	$[1,0,-1+\alpha_{2AB}]$ $[0,1, \alpha_{1C}]$
	\$6	$[0,0,-1]$ $[1,1,-9]$	[0,0,-1] $[1,0, \alpha_{1A}]$	[0,0,-1] [0,0,0]

Table 5.14: Subgame, three asset game, for $p_{1B} = 7$ and $p_{2B} = 6$, player 1 wins asset B

	Bidder 2 / Bidder 1	\$4	\$5	\$6
$p_{1B} = 7$ $p_{2B} = 6$, wins B $K_{1l} = 8$ $K_{2l} = 12$	\$4	$[0,0,0]$ $[0,1,-1+\alpha_{1BC}]$	$[1,0,\alpha_{2A}]$ $[0,1,-1+\alpha_{1C}]$	$[1,0,-1+\alpha_{2AB}]$ $[0,1,\alpha_{1B}]$
	\$5	$[0,0,0]$ $[1,1,1+\alpha_{1AB}]$	$[0,0,0]$ $[0,0,\alpha_{1B}]$	$[1,0,-1+\alpha_{2B}]$ $[0,0,\alpha_{1B}]$
	\$6	$[0,0,0]$ $[1,0,1+\alpha_{1AB}]$	$[0,1,\alpha_{2C}]$ $[1,0,\alpha_{1AB}]$	$[0,1,\alpha_{2C}]$ $[0,0,-1+\alpha_{1B}]$

Table 5.15: Subgame, three asset game, for $p_{1B} = 7$ and $p_{2B} = 7$

	Bidder 2 / Bidder 1	\$4	\$5	\$6
$p_{1B} = 7$ $p_{2B} = 7$, $K_{1l} = 8$ $K_{2l} = 11$	\$4	$[0,0,0]$ $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2A}]$ $[0,1,\alpha_{1C}]$	$[1,0,\alpha_{2A}]$ $[0,1,\alpha_{1C}]$
	\$5	$[0,0,0]$ $[1,0,1+\alpha_{1A}]$	$[0,0,0]$ $[0,0,\alpha_{1B}]$	$[1,0,-1+\alpha_{2A}]$ $[0,0,0]$
	\$6	$[0,0,0]$ $[1,0,1+\alpha_{1AB}]$	$[0,1,1+\alpha_{2C}]$ $[1,0,\alpha_{1A}]$	$[0,1,1+\alpha_{2C}]$ $[0,0,0]$

5.5 Empirical evidence

5.5.1 FCC RSA Auction with 3 assets and 7 bidders

FCC auctions can provide a good illustration how even smaller auctions with lower number of bidders and a lower number of assets offered can turn into longer game dynamics with subgames and outcomes that can be explained by propositions 5.1 to 5.8. In May of 2002 FCC offered for sale 25 MHz band in rural areas divided into 3 licenses based on location. Since rural location varied in population as well as in operational costs for different bidders, those 3 licenses had significantly diverse utilities for different bidders. The auction attracted several bidders of which 7 qualified for auction: CCPR Services, Inc, MilkyWay Wireless, LLC, Paris Tower, Inc, Robert Price, Southwestern Bell Wireless, Trent Cellular, LLC and WWC Holding Co., Inc.

Trent Cellular, LLC and MilkyWay Wireless were passive participants in this particular auction, since they didn't post any biddings during all rounds, which we can, for simplifying purposes observe as three asset game with five bidders denoting them as presented in table 5.16.

Table 5.16: List of bidders, FCC RSA Auction with 3 assets and 7 bidders

Active bidders	
CCPR Services, Inc.	A
WWC Holding Co., Inc.	B
Southwestern Bell Wireless	C
Paris Tower, Inc	D
Robert Price	E

Assets	
Arkansas 9 – Polk A	*m*
North Dakota 3 – Barnes A	*n*
Puerto Rico 5 – Ceiba A	*l*

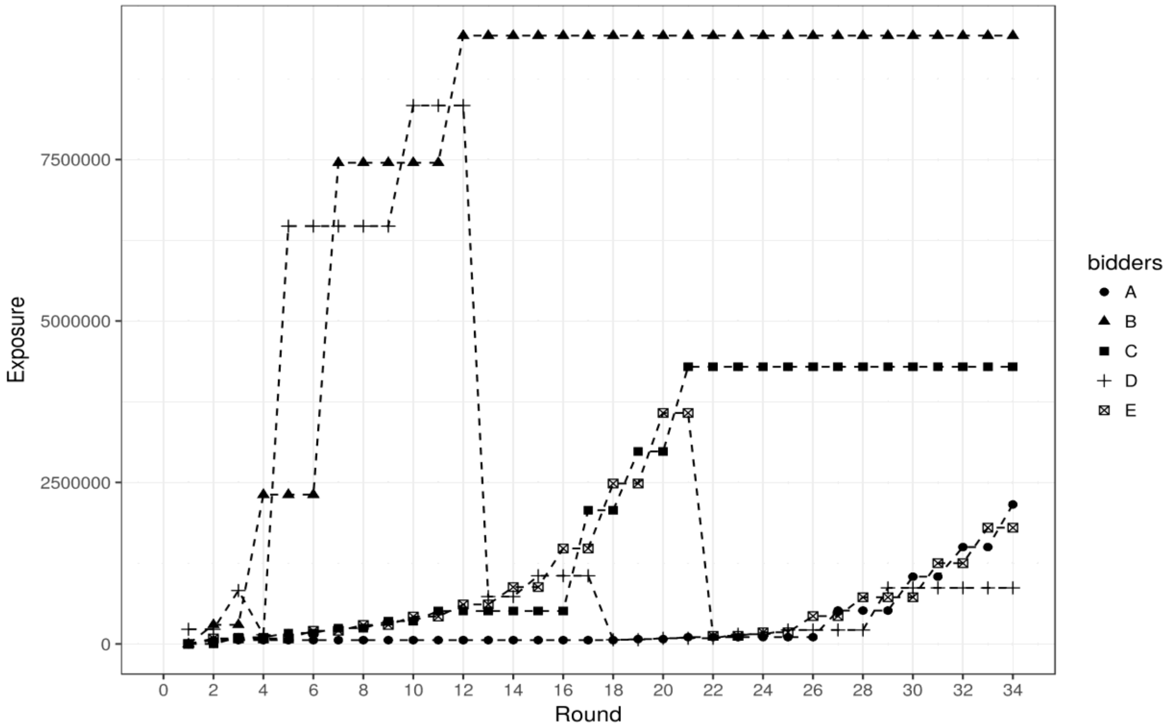


Figure 5.1: Bidders exposure, FCC RSA Auction with 3 assets and 7 bidders. Bidder D's exposure is highest just before round 12, securing her asset won, but on the highest price per capita.

Auction lasted for 35 rounds during 5 days in which bidders had enough time to change and adopt new bidding strategies resulting in revenue of $15,871,000. Three bidders won three licenses.

Figure 5.1 illustrates bidder's exposure during every round of auction, summing all winning bids in a specific round. As described in Bulow, et al., 2009 early bidding tends to focus on the largest licenses with the biggest utility. The RSA 2002 auction was no exception: initial bidding centered on the Channel A of North Dakota 3 - Barnes license. However, an interesting subgame occurred: Bidder D, who, at the end of the auction will not obtain any asset, already in round 4 opted for strong biding offer on the asset *n*, increasing price beyond $6M. At round 11 bidder D decides to drop out from subgame for asset *n*, reviling his budget. At round 12 bidder B offers the price for asset *n* which will stay final until the end of auction. The explanation for this might be "chicken game" strategy in which bidder D has an agreement with some other bidders to attack position of the player with biggest budget, maxing out her budget on the asset with highest utility, keeping her away from obtaining all assets. In Figure 5.1 we can comprehend that, if bidder D didn't enter the for the asset *n* strong in round 5, bidder B would obtain asset *n* for a much lower price of $2.312.000 and had enough budget to participate in subgames for assets *m* and *l*.

It is clear that bidder D tried to compete in subgames for every three assets, either without a clear strategy or with third party agreement. Bidder E competed in two subgames. Both bidder D and bidder E didn't obtain any assets while reviling their budgets along the way. Bidder A was interestingly passive until very far in the game. Her first offer for asset *l* was in round 27, soon pushing out bidder D and winning that asset over bidder E for a total of $2,160,000. The passive position allowed her to obtain information about bidder D and bidder E's budgets without increasing the price of asset *l* – obviously, the only one that had attractive utility for her.

Although outcome is not package bidding, it could easily turn out so. Bidder B had enough budget to obtain all three assets if bidding for asset *n* turned out differently, while bidders D and E participated in bidding for three and two assets respectively. By our proposition 5.1. better strategy for bidder A is to start bidding for asset *l* in round 18 since budget of bidder E would limit her more in subgame for asset *l*. In round 18, bidder E would be pressured to bid on package *ml*. Outcome is given in table 5.17.

Table 5.17: Outcome, FCC RSA Auction with 3 assets and 7 bidders. Bidder CCPR Services, Inc. secured lowest price per capita, but won the asset the latest in the game.

Active bidders	**Assets won**	**Population**	**Gross price**	**Price per capita**	**Round**
CCPR Services, Inc.	Puerto Rico 5 – Ceiba A	39,765	$2,160,000	$54.32	34
WWC Holding Co., Inc.	North Dakota 3 – Barnes A	94,616	$9,419,000	$99.55	12
Southwestern Bell Wireless	Arkansas 9 – Polk A	62,480	$4,292,000	$68.69	21
Paris Tower, Inc	-		-		
Robert Price	-		-		

5.5.2 FCC VHF Public Coast Auction with 46 assets and 8 bidders

Auctions with significantly more assets than bidders present fertile ground for package auctions and development of more complex bidding strategies. In March of 1998 FCC offered for sale 156-162 MHz VHF Public Coast Service divided into 46 licenses based on location. The auction attracted several bidders of which 8 qualified for auction. All 8 bidders were active in the auction (table 5.18).

Table 5.18: List of bidders, FCC VHF Public Coast Auction

Bidder		**Bidder**	
WJG MariTEL Corporation	A	American Pacific Inc.	F
Warren C. Havens	H	RadioLink Corporation	E
DATA RADIO MANAGEMENT	D	SCOTT COMMUNICATIONS	C
SMR Systems, Inc	G	SHARRON L ROTH	B

Auction lasted for 44 rounds during 8 days in which bidders had enough time to change and adopt new bidding strategies resulting in revenue of $7,812,000. 4 bidders won 26 licenses (3 small business bidders won 17 licenses which were all Inland VPCs 1 other bidder won 9 Maritime Border and Maritime Non-Border VPCs). Bidding for 16 licenses was unsuccessful and they were held by FCC.

Figure 5.2 illustrates bidder's exposure during every round of auction, displaying logarithm of the sum of all winning bids in a specific round. As described in Bulow, et al., 2009 early bidding tends to focus on the largest licenses with the biggest utility. In this case bidders F and A (as suggested in our proposition 5.7.) opened strong on asset 4, which not only has biggest population, but geographically also has biggest utility for every bidder since it has border with 16 consecutive assets Figure 5.3).

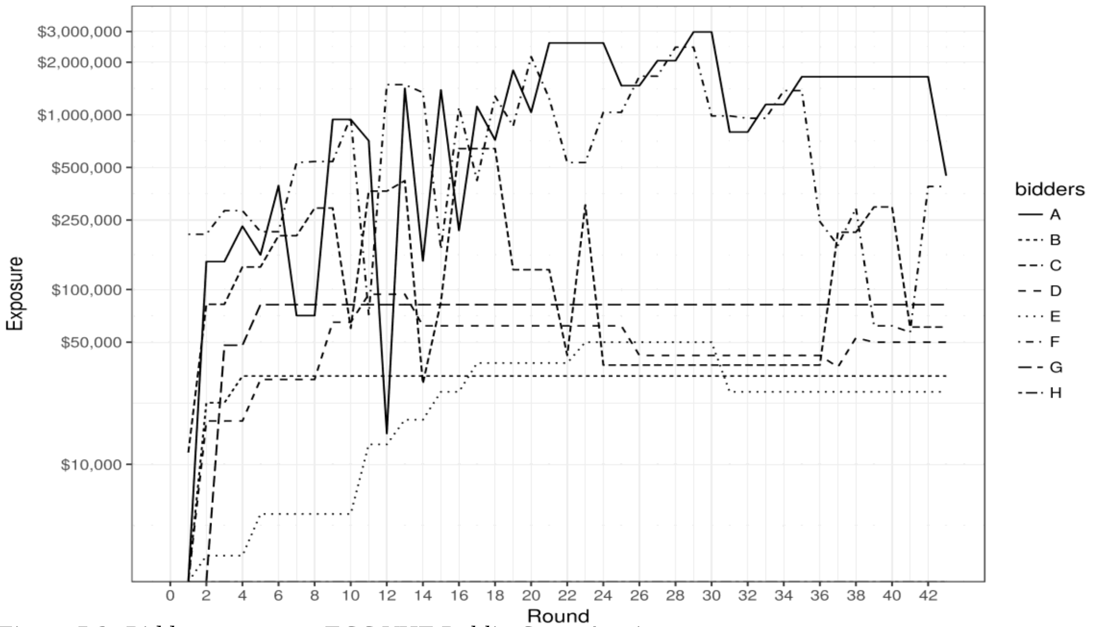


Figure 5.2: Bidders exposure, FCC VHF Public Coast Auction

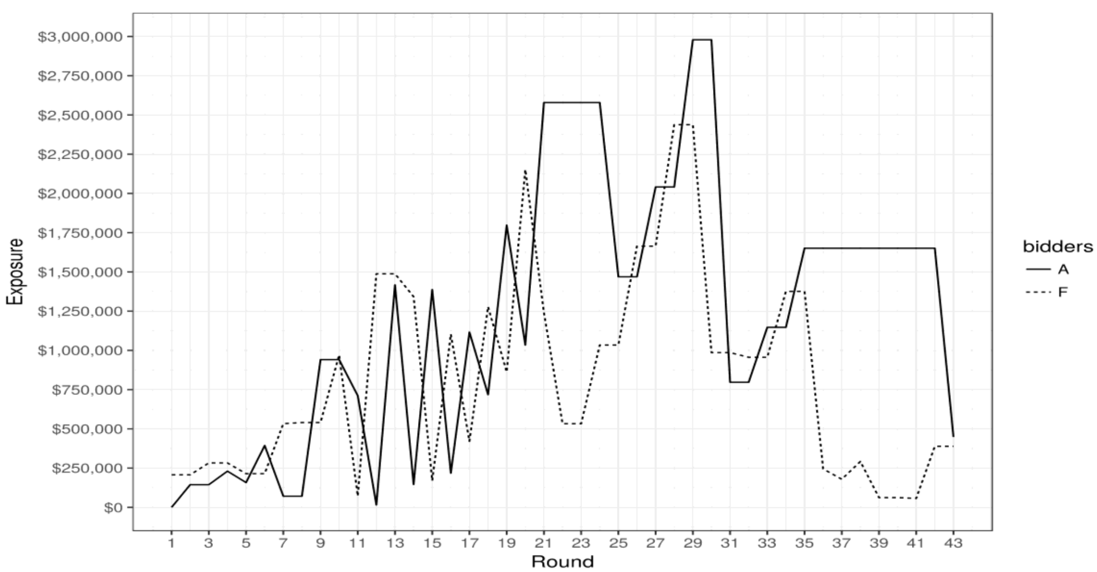


Figure 5.3: Bidder A and F exposure by rounds, FCC VHF Public Coast Auction

44 rounds of bidding over 46 assets by many bidders resulted in dozens of sub-games, of which we will focus on crucial ones with biggest exposure and utilities. Figure 5.3 proves that bidders F and A have similar budgets, but bidding rounds show that bidder A used strategy suggested by proposition 5.7. to win costal assets 1-7 (Figure 5.4). Simple round analysis proves that asset 4 was valued high by bidder A and that was focus of her strategy. Assets 6 and 7 had strategic value lover than their utility (like asset A in proposition 5.7.), while bidder A participated in sub-game bidding over assets 24, 25 and 36 with bidder F, but setting it on the same strategy as her utility. By proposition 5.7 it was expected that bidder A will lose 24, 25 and 36 to bidder F, which happened early for the asset 24 (in round 13) and late in the game for assets 25 and 36 (39^{th} and 41^{st} round respectively) straining bidders F budget, and at the same time completing the mission to win all the costal assets.

Assets 1, 2, 3 and 5 were won by bidder A at round 21, exciding the utility of bidder F, nonetheless bidder F stayed in the game for asset 4, which still has high utility since it is consecutive package to assets 24 and 25 (Figure 5.4). Bidder A will win asset 4 in round 29, after which she will win rest of the costal package – assets 6 and 7 in rounds 35 and 41.

Outcomes are presented in table 5.19. WJG MariTEL Corporation can be considered as winner of this auction, since they acquired assets with biggest population, resulting in lowest price per capita. Their direct competitor Data Radio Management, although reviling similar budget ended with two times higher price per capita. Warren C. Havens had a luck of their competitors not being too much focused in some of the assets that were their first focus, ending overall second in price per capita, and in population covered by assets they won.

Table 5.19: Outcome, FCC VHF Public Coast Auction

Winning bidders	Assets won	Population	Gross Price	Price per capita	Rounds
WJG MariTEL Corporation	1-7	235,863,260	$6,804,000.00	$0.03	21,21,21,29,21,35,43
Warren C. Havens	27 – 35, 37, 39, 40, 41, 42	9,226,584	$390,650.00	$0.04	17 - 41
DATA RADIO MANAGEMENT	24, 25, 36	4,221,966	$247,650.00	$0.06	23, 39, 41
SMR Systems, Inc.	17	213,430	$16,900.00	$0.08	31

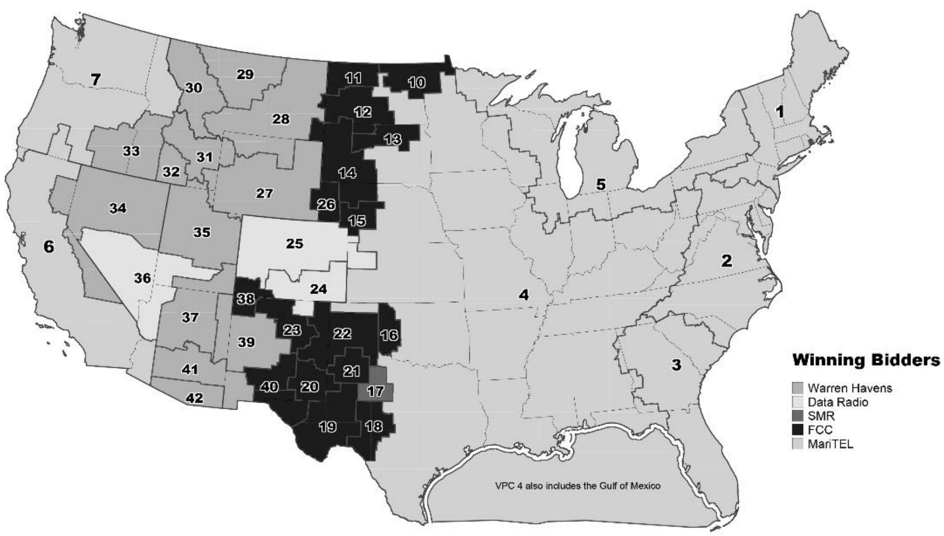


Figure 5.4: Map of the assets won by bidders, FCC VHF Public Coast Auction

5.6 Conclusion

We have investigated Vickery based auctions of illiquid assets with a focus on package bidding through game theory optimal strategies. Theoretical considerations suggest that every case that can be presented as a two asset or three asset game, as well as longer games that can be presented through two and three asset subgames, has a strong equilibrium if bidders' budgets and utilities for every asset are common knowledge. Guerrin and Tadjouddine (2006) presented mechanisms of achieving common knowledge of asset utility, while we proved in empirical examples that some key bidders may reveal their budgets early in the game.

Furthermore, we have shown that outcomes strongly rely more on bidders' utilities for certain assets than their budgets, proving that bidders with higher utilities, but lower budgets, may have dominant strategies. Depending on the relationship between certain utilities and budgets, every version of the game shows that every player can create an optimal strategy, which does not require bidding until her utility for every asset, lowering exposure, and preserving her from obtaining unwanted packages. Additionally, we have shown that certain differences between asset utilities and lower closest step of minimum increase amounts play a big role in creating optimal strategy and game outcome for every bidder. Similarly, for every version of the game, the seller can determine her expected auction revenue, as well as possible unsold assets.

The empirical evidence shows the existence and importance of a strategically most important asset, but only in the case where utility on consecutive package is higher than the sum of utilities of separate assets. It deepens findings of Bulow et al. (2009) evidencing that early bidding does not only tend to focus on the largest assets, but also on assets with the best strategical position. The empirical evidence also shows the presence of bidders that opted for a "chicken game" – being involved in multiple subgames, without purchasing any of the assets, and thus strategically straining the budgets of particular bidders.

6 Conclusion

The first part of the thesis discusses the signaling game in entrepreneurship funding between entrepreneur, angel and venture capitalist. Our study proposes a two-stage financing model that considers output elasticities of all three players through the Cobb-Douglas utility function, as well as a synergy coefficient between players. Our findings suggest that a higher complementary coefficient between players in both stages leads to the higher level of effort from all three players, taking game dynamics away from the moral hazard problem and causing higher exit stage payoffs. The previous track record of the angel and VC, and output elasticity of the entrepreneur, in combination with the share of the company, offered to angel and VC, impact the three-player game dynamic causing some players to reduce their efforts after certain funding rounds.

The next section empirically expands on the theoretical proposal with the focus on the venture capital syndication. Our results show that VC syndication increases the average amount of funding offered to entrepreneurs as well as that syndicated ventures have a higher number of funding rounds, resulting in a higher number of possible entry-points provided by those start-ups. At the same time, investors capitalize on less exposure to a specific company's risk, enabling them to fund additional projects and increase their expected returns. In this chapter, we also investigated the influence of academic level of founder and employees on the chance of syndication. Our empirical results suggest that academic titles of higher grade, like PhDs. have an influence on the opportunity of syndication, implying that knowledge of entrepreneurs, portrayed by degrees, serves as a signaling factor for venture capitalists and possibly also angels.

Chapter 4 deals with portfolio optimization under Expected Shortfall as a risk measure, using genetic algorithms. We implemented the combination of linear crossover and one-point mutation. Our results suggest that the proposed combination slightly outperforms SPEA2 as introduced by Zitzler et al. (2002). In the end, we ran a proposed algorithm with Value-at-Risk as the measure of risk. The resulting efficient frontier was flatter than the one obtained by using Expected Shortfall but also created well-distributed solutions. After that, Expected Shortfall for the solutions were computed and plotted against the optimal Expected Shortfall portfolios. We show that both frontiers differ from each other, especially at the low return side. The converted Value-at-Risk solutions were not evenly distributed along the new frontier and were even missing for some ES values.

Chapter 5 explores a Vickery based package asset auction model for illiquid assets using a game theory approach. We emphasize two difficulties facing bidders: exposure problems, which occur when bidders wish to acquire complementary licenses, and budget constraints, which we argue are ubiquitous. The theoretical part of chapter 2 has shown that outcomes strongly rely more on bidders' utilities for certain assets than their budgets, proving that bidders with higher utilities, but lower budgets, may have dominant strategies. Empirical evidence proves the existence and importance of a strategically most important asset, but only in the case where utilities of consecutive package is higher than the sum of utilities of separate assets. It deepens findings of Bulow, et al. (2009) evidencing that not only early bidding tends to focus on the largest assets but also on assets with the best strategical position. Empirical evidence also showed the presence of bidders that opted for a "chicken game" – being involved in multiple subgames, without purchasing any of the assets, thus strategically straining budgets of particular bidders.

Bibliography

Abreu, D., Milgrom, P., & Pearce, D. (1991). Information and Timing in Repeated Partnerships. *59*(6), 1713-1733.

Acerbi, C., & Tasche, D. (2002). Expected shortfall: a natural coherent alternative to value at risk. *Economic notes, 31*, 379-388.

Acerbi, C., & Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance, 26*, 1487-1503.

Acerbi, C., Nordio, C., & Sirtori, C. (2001). Expected shortfall as a tool for financial risk management. *arXiv preprint cond-mat/0102304*.

Amit, R., & Antweiler, W. (2002). Venture-Capital Syndication: Improved Venture Selection vs. the Value-Added Hypothesis. *Journal of Economics & Management Strategy, 11 (3)*, 422–451.

Anagnostopoulos, K. P., & Mamanis, G. (2011). Multiobjective evolutionary algorithms for complex portfolio optimization problems. *Computational Management Science, 8*, 259-279.

Armantier, O., Holt, C. A., & Plott, C. R. (2013). A Procurement Auction for Toxic Assets with Asymmetric Information. *American Economic Journal: Microeconomics, 5*(4), 142-162.

Arnone, S., Loraschi, A., & Tettamanzi, A. (1993). A genetic approach to portfolio selection. *Neural Network World, 3*, 597-604.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1997). Thinking Coherently. *Mathematical finance, 10*, 68-71.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent measures of risk. *Mathematical finance, 9*, 203-228.

Arumugam, M. S., & Rao, M. V. (2005). Novel hybrid approaches for real coded genetic algorithm to compute the optimal control of a single stage hybrid manufacturing systems. *Int. J. Computat. Intelligence, 1*, 189-206.

Ausubel, L. M., & Milgrom, P. (2006). The lovely but lonely Vickrey auction. *Combinatorial auctions, 17*, 22-26.

Bonatti, A., & Hörner, J. (2011). Collaborating. *American Economic Review, 101*(2), 632-663.

Bowden, & Roger, J. (1994). Bargaining, size, and return in venture capital funds. *Journal of Business Venturing, 9*(4), 307-330.

Bradley, M., Desai, A., & Kim, E. (1988). Synergistic Gains from Corporate Acquisitions an Division between the Stockholders of Target and Acquiring Firms. *Journal of Financial Economics,* 3-40.

Branke, J., Scheckenbach, B., Stein, M., Deb, K., & Schmeck, H. (2009). Portfolio optimization with an envelope-based multi-objective evolutionary algorithm. *European Journal of Operational Research, 199*, 684-693.

Bulow, J., Levin, J., & Milgrom, P. (2009). Winning play in spectrum auctions. *National Bureau of Economic Research*.

Burke, E. K., Kendall, G., & others. (2005). *Search methodologies.* Springer.

Bygrave, D., William, A., & Timmons, J. (1991). *Venture and risk capital: Practice and performance, promises and policy.* Boston: Harvard Business School Press.

Cable, D., & Shane, S. (1997). A Prisoner's Dilemma Approach to Entrepreneur-Venture Capitalist Relationship. *Academy of Management Review, 22*(1).

Caccioli, F., Farmer, J. D., Foti, N., & Rockmore, D. (2015, 2). Overlapping portfolios, contagion, and financial stability. *Journal of Economic Dynamics and Control, 51*, 50-63. doi:10.1016/j.jedc.2014.09.041

Casamatta, C., & Haritchabalet, C. (2007). Experience, screening and syndication in venture capital investments. *Journal of Financial Intermediation 16 (3)*, 368–398.

Chan, T. C., Mahmoudzadeh, H., & Purdie, T. G. (2014, 11). A robust-CVaR optimization approach with application to breast cancer therapy. *European Journal of Operational Research, 238*, 876-885. doi:10.1016/j.ejor.2014.04.038

Chang, T.-J., Meade, N., Beasley, J. E., & Sharaiha, Y. M. (2000). Heuristics for cardinality constrained portfolio optimisation. *Computers & Operations Research*, *27*, 1271-1302.

Chang, T.-J., Yang, S.-C., & Chang, K.-J. (2009). Portfolio optimization problems in different risk measures using genetic algorithm. *Expert Systems with Applications, 36*, 10529-10537.

Che, Y.-K., & Gale, I. (1998). Standard auctions with financially constrained bidders. *The Review of Economic Studies, 65*(1), 1-21.

Christensen, & Clayton, M. (1997). *The Innovator's Dilemma: When New Technologies Cause Great Firms to Fail.* Boston: Harvard Business School Press.

Ciliberti, S., Kondor, I., & Mézard, M. (2007, 8). On the feasibility of portfolio optimization under expected shortfall. *Quantitative Finance, 7*, 389-396. doi:10.1080/14697680701422089

Clarke, E. H. (1971). Multipart pricing of public goods. *Public choice, 11*(1), 17-33.

Conti, A., Thursby, J., & Thursby, M. (2013). Patents as Signals for Startup Financing. *Symposium on Patents, Entrepreneurship and Innovation, 61*(3), 469-470.

Corne, D. W., Knowles, J. D., & Oates, M. J. (2000). The Pareto Envelope-Based Selection Algorithm for Multiobjective Optimization. In *Parallel Problem Solving from Nature PPSN VI* (pp. 839-848). Springer Berlin Heidelberg. doi:10.1007/3-540-45356-3_82

Cramton, P. (2013). Spectrum auction design. *Review of Industrial Organization, 42*(2), 161-190.

Cramton, P. C. (1984). Bargaining with incomplete information: An infinite-horizon model with two-sided uncertainty. *The Review of Economic Studies, 51*(4), 579-593.

Cramton, P., & Ockenfels, A. (2014). The German 4G Spectrum Auction: Design and Behavior.

Dallagnol, V. A., Berg, J., & Mous, L. (2009). Portfolio management using value at risk: A comparison between genetic algorithms and particle swarm optimization. *International Journal of Intelligent Systems, 24*, 766-792.

Danielsson, J., Embrechts, P., Goodhart, C., Keating, C., Muennich, F., Renault, O., . . . others. (2001). An academic response to Basel II. *Special Paper-LSE Financial Markets Group*.

Darwin, C. R. (1859). *On the Orngin of Species.* London: John Murray.

De Jong, K. A. (1975). *An Analysis of the Behavior of a Class of Genetic Adaptive Systems.* Ann Arbor, MI, USA: University of Michigan.

Deb, K., & Deb, D. (2014). Analysing mutation schemes for real-parameter genetic algorithms. *International Journal of Artificial Intelligence and Soft Computing, 4*, 1-28.

Dutta, S., & Folta, T. B. (2016). A comparison of the effect of angels and venture capitalists on innovation and value creation. *Journal of Business Venturing 31 (1),* 39–54.

Elitzur, R., & Gavious, A. (2003). Contracting, signaling, and moral hazard: a model of entrepreneurs, ‘angels,’ and venture capitalists. *Journal of Business Venturing, 18*(6), 709-725.

Eshelman, L. J., Caruana, R. A., & Schaffer, J. D. (1989). Biases in the crossover landscape. *Proceedings of the third international conference on Genetic algorithms*, (pp. 10-19).

Estrada, J. (2015). Mean-semivariance optimization: A heuristic approach. *Journal of Applied Finance*, 57-72.

Fairchild, R. (2009). An entrepreneur's choice of venture capitalist or angel-financing: A behavioural game-theoretic approach. *Journal of Business Venturing,* 1-16.

Fairchild, R. (2011). An entrepreneur's choice of venture capitalist or angel-financing: A behavioral game-theoretic approach. *Journal of Business Venturing 26 (3),* 359–374.

Fulghieri, P., & Sevilir, M. (2009). Size and Scope of Human Capital Intensive Firms.

Gerdes, I., Klawonn, F., & Kruse, R. (2004). *Evolutionäre Algorithmen-Genetische Algorithmen-Strategien und Optimierungsverfahren-Beispielanwendungen.* Vieweg.

Grossmann, S. (1981). The informational role of warranties and private disclosure about product quality. *Journal of Law and Economics, 24,* 462-483.

Groves, T. (1973). Incentives in teams. *Econometrica: Journal of the Econometric Society*, 617-631.

Hahn, G., Kim, K., & Kwon, J. (n.d.). Two-Stage Startup Financing with Signaling under Ambiguity. Retrieved from http://kea.ne.kr/conference-2017/download/S6-1-2_Joon%20Yeop%20Kwon.pdf

Higashide, H., & Birley, S. (2002). The consequences of conflict between the venture capitalist and the entrepreneurial team in the United Kingdom from the perspective of the venture capitalist. *Journal of Business Venturing, 17*(1), 59-81.

Holland, J. H. (1975). *Adaptation in natural and artificial systems: an introductory analysis with applications to biology, control, and artificial intelligence.* U Michigan Press.

Hornby, A. P., & Lewis, J. W. (1974). *Oxford Advanced Learners Dictionary of Current English* (7 ed.). Oxford University Press.

Hustedde, R. J., & Pulver, G. C. (1992). Factors affecting equity capital acquisition: The demand side. *Journal of Business Venturing, 7*(5), 363-374.

Kandori, M., & Obbara, I. (2006). Efficiency in Repeated Games Revisited: The Role of Private Strategies. *Econometrica, 74*(2), 499-519.

Kenney, M., & Hsu, D. H. (2005). Organizing venture capital: the rise and demise of American Research & Development Corporation, 1946–1973. *Industrial and Corporate Change, 14*(4).

Kim, J., & Wagman, L. (2016). Early-stage entrepreneurial financing: A signaling perspective. *Journal of Banking & Finance 67*, 12–22.

Kim, J.-H., & Wagman, L. (2014). Portfolio size and information disclosure: An analysis of startup accelerators. *Journal of Corporate Finance, 29*, 520-534.

Kim, J.-H., & Wagman, L. (2016). Early-stage entrepreneurial financing: A signaling perspective. *Journal of Banking & Finance, 67*, 12-22.

Kingdon, J., & Feldman, K. (1995). Genetic algorithms and applications to finance. *Applied Mathematical Finance, 2*, 89-116.

Kolm, P. N., Tütüncü, R., & Fabozzi, F. J. (2014). 60 Years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research, 234*, 356-371.

Koza, J. R. (1990). *Genetic programming: A paradigm for genetically breeding populations of computer programs to solve problems.* Stanford University, Department of Computer Science.

Krokhmal, P., Palmquist, J., & Uryasev, S. (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of risk, 4*, 43-68.

Krokhmal, P., Zabarankin, M., & Uryasev, S. (2011). Modeling and optimization of risk. *Surveys in Operations Research and Management Science, 16*, 49-66.

Labini, P. S. (1995). Why the interpretation of the Cobb-Douglas production function must be radically changed. *Structural Change and Economic Dynamics, 6*(4), 485-504.

Lehmann, E. (2006). Does Venture Capital Syndication Spur Employment Growth and Shareholder Value? Evidence from German IPO Data. *Small Bus Econ 26 (5)*, 455–464.

Lim, A. E., Shanthikumar, J. G., & Vahn, G.-Y. (2011). Conditional value-at-risk in portfolio optimization: Coherent but fragile. *Operations Research Letters, 39*, 163-171.

Lin, C.-C., & Liu, Y.-T. (2008, 2). Genetic algorithms for portfolio selection problems with minimum transaction lots. *European Journal of Operational Research, 185*, 393-404. doi:10.1016/j.ejor.2006.12.024

Loraschi, A., & Tettamanzi, A. (1995). An evolutionary algorithm for portfolio selection in a downside risk framework. *Working Papers in Financial Economics, 6*, 8-12.

Loraschi, A., Tomassini, M., Tettamanzi, A., & Verda, P. (1995). Distributed genetic algorithms with an application to portfolio selection problems. *Artificial neural nets and genetic algorithms*, (pp. 384-387).

Markides, C. (2006). Disruptive Innovation: In Need of Better Theory. *Product Innovation Management, 23*(1), 19-25.

Markowitz, H. (1952). Portfolio selection. *The journal of finance, 7*, 77-91.

Markowitz, H. M. (1959). *Portfolio Selection: Efficient Diversification of Investments.* Yale University Press. Retrieved from http://www.jstor.org/stable/j.ctt1bh4c8h

Markowitz, H. M. (1991). *Portfolio Selection: Efficient Diversification of Investments* (2 ed.). Mass.: Blackwell.

Markowitz, H., Todd, P., Xu, G., & Yamane, Y. (1993). Computation of mean-semivariance efficient sets by the critical line algorithm. *Annals of operations research, 45*, 307-317.

McNeil, A. J., Frey, R., & Embrechts, P. (2015). *Quantitative risk management: Concepts, techniques and tools* (2 ed.). Princeton university press.

Metaxiotis, K., & Liagkouras, K. (2012, 10). Multiobjective Evolutionary Algorithms for Portfolio Management: A comprehensive literature review. *Expert Systems with Applications, 39*, 11685-11698. doi:10.1016/j.eswa.2012.04.053

Michalewicz, Z. (1996). *Genetic Algorithms + Data Structures = Evolution Programs (3rd Ed.).* London, UK, UK: Springer-Verlag.

Milgrom, P. (1981). Good News and Bad News: Representation Theorems and Applications. *Bell Journal of Economics, 12*, 380-391.

Mitchell, M. (1998). *An introduction to genetic algorithms.* MIT press.

Mühlenbein, H. (1992). How Genetic Algorithms Really Work: Mutation and Hillclimbing. *PPSN, 92,* pp. 15-25.

Myerson, R. B., & Satterthwaite, M. A. (1983). Efficient mechanisms for bilateral trading. *Journal of Economic Theory, 29*(2), 265-281.

Nissen, V. (1997). *Einführung in evolutionäre Algorithmen: Optimierung nach dem Vorbild der Evolution.* Vieweg+Teubner Verlag.

Pagnozzi, M. (2010). Are speculators unwelcome in multi-object auctions? *American Economic Journal: Microeconomics, 2*(2), 97-131.

Pendharkar, P. C., Rodger, J. A., & Subramanian, G. H. (2008). An empirical study of the Cobb–Douglas production function properties of software development effort. *Information and Software Technology, 50*(12), 1181-1188.

Pereira, R. (2000). *Genetic Algorithm Optimisation for Finance and Investment.* Tech. rep., School of Economics, La Trobe University.

Pesendorfer, M., & Cantillon, E. (2003). Combination bidding in multi-unit auctions. Harvard Business School Working Draft .

Pflug, G. C. (2000). Some remarks on the value-at-risk and the conditional value-at-risk. In *Probabilistic constrained optimization* (pp. 272-281). Springer.

Quaranta, A. G., & Zaffaroni, A. (2008). Robust optimization of conditional value at risk and portfolio selection. *Journal of Banking & Finance, 32*, 2046-2056.

Radner, R., Myerson, R., & Maskin, E. (1986). An Example of a Repeated Partnership Game with Discounting and with Uniformly Inefficient Equilibria. *The Review of Economic Studies, 53*(1), 59-69.

Ranković, V., Drenovak, M., Stojanović, B., Kalinić, Z., & Arsovski, Z. (2014). The mean-Value at Risk static portfolio optimization using genetic algorithm. *Computer Science and Information Systems, 11*, 89-109.

Rechenberg, E. (1973). Optimierung technischer Systeme nach Prinzipien der biologischen Evolution. *Frommann-Holzboog Verlag, Stuttgart*.

Rifki, O., & Ono, H. (2012). A survey of computational approaches to portfolio optimization by genetic algorithms. *18th International Conference Computing in Economics and Finance.*

Rockafellar, R. T., & Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of risk, 2*, 21-42.

Rosen, J. B. (1965). Existence and Uniqueness of Equilibrium Points for Concave N-Person Games. *Econometrica, 33*(3), 520-534.

Roth, A. E. (2002). The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics. *Econometrica*.

Rothkopf, M. H., Teisberg, T. J., & Kahn, E. P. (1990). Why are Vickrey auctions rare? *Journal of Political Economy*, 94-109.

Sakurai, Y., Yokoo, M., & Matsubara, S. (1999). A Limitation of the Generalized Vickrey Auction in Electronic Commerce: Robustness against False-name Bids.

Samuelson, P. (1958). The fundamental approximation theorem of portfolio analysis in terms of means variances and higher moments. *Review of Economic Studies, 25*, 65-86.

Sarin, S. C., Sherali, H. D., & Liao, L. (2013, 10). Minimizing conditional-value-at-risk for stochastic scheduling problems. *Journal of Scheduling, 17,* 5-15. doi:10.1007/s10951-013-0349-6

Schwefel, H.-P. (1975). *Evolutionsstrategie und numerische Optimierung.* Technische Universität Berlin.

Silverman, B. W. (1986). *Density estimation for statistics and data analysis* (Vol. 26). CRC press.

Sivanandam, S. N., & Deepa, S. N. (2007). *Introduction to genetic algorithms.* Springer Science & Business Media.

Skolpadungket, P., Dahal, K., & Harnpornchai, N. (2007). Portfolio optimization using multi-objective genetic algorithms., (pp. 516-523).

Spence, A. M. (1973). Time and Communication in Economic and Social Interaction. *The Quarterly Journal of Economics, 87*(4), 651-660.

Srinivas, M., & Patnaik, L. M. (1994). Adaptive probabilities of crossover and mutation in genetic algorithms. *IEEE Transactions on Systems, Man, and Cybernetics, 24*, 656-667.

Szegö, G. (2002). Measures of risk. *Journal of Banking & Finance, 26*, 1253-1272.

Takeda, A. (2009, 6). Generalization performance of -support vector classifier based on conditional value-at-risk minimization. *Neurocomputing, 72*, 2351-2358. doi:10.1016/j.neucom.2008.11.022

Takeda, A., & Kanamori, T. (2009, 10). A robust approach based on conditional value-at-risk measure to statistical learning problems. *European Journal of Operational Research, 198*, 287-296. doi:10.1016/j.ejor.2008.07.027

Takeda, A., Fujiwara, S., & Kanamori, T. (2014, 11). Extended Robust Support Vector Machine Based on Financial Risk Minimization. *Neural Computation, 26*, 2541-2569. doi:10.1162/neco_a_00647

Team, R. C. (2015). *R: A Language and Environment for Statistical Computing.* Vienna. Retrieved from http://www.R-project.org/

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance, 16*(1), 8-37.

Wang, Y., Dang, C., & Wang, S. (2015, 9). Robust Novelty Detection via Worst Case CVaR Minimization. *IEEE Transactions on Neural Networks and Learning Systems, 26*, 2098-2110. doi:10.1109/tnnls.2014.2378270

Wong, A., Bhatia, M., & Freeman, Z. (2009). Angel finance: the other venture capital. *Entrepreneurial Finance* , 7-8.

Wright, A. H., & others. (1991). Genetic algorithms for real parameter optimization. *Foundations of genetic algorithms, 1*, 205-218.

Yamai, Y., & Yoshiba, T. (2005). Value-at-risk versus expected shortfall: A practical perspective. *Journal of Banking & Finance, 29*, 997-1015.

Yokoo, M., Sakurai, Y., & Matsubara, S. (2000). The effect of false-name declarations in mechanism design: Towards collective decision making on the Internet. *Distributed Computing Systems*.

Zacharakis, A. L., & Meyer, G. D. (2000). The potential of actuarial decision models: Can they improve the venture capital investment decision? *Journal of Business Venturing, 15*(4), 323-346.

Zacharakis, A. L., & Shepherd, D. A. (2001). The nature of information and overconfidence on venture capitalists' decision making. *Journal of Business Venturing, 16*(4), 311-332.

Zhu, S. (2016). Research on the Portfolio Optimization Model under Quantitative Constraint Based on Genetic Algorithm. *Journal of Mathematical Finance, 6*, 465-470.

Zitzler, E., & Thiele, L. (1999). Multiobjective evolutionary algorithms: a comparative case study and the strength Pareto approach. *IEEE transactions on Evolutionary Computation, 3*, 257-271.

Zitzler, E., Laumanns, M., & Thiele, L. (2002). SPEA2: Improving the Strength Pareto Evolutionary Algorithm for Multiobjective Optimization. In K. C. Giannakoglou, & others (Ed.), *Evolutionary Methods for Design, Optimisation and Control with Application to Industrial Problems (EUROGEN 2001)* (pp. 95-100). International Center for Numerical Methods in Engineering (CIMNE).