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Optimal control of non-convex rate-independent systems via vanishing viscosity – The finite dimensional case *

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Dedicated to our teacher and good friend Fredi Tröltzsch on the occasion of his 70th birthday

Abstract

We investigate an optimal control problem governed by the evolution of a rate-independent system in finite dimensions. The rate-independent system is determined by a (possibly) non-convex energy, which contains the controllable, external load, and a dissipation potential, which is assumed to be positively homogenous of degree one. Under the several different concepts of solutions for these rate-independent systems, we bear on the so-called normalized parametrized BV solutions and prove the existence of a globally optimal solution of the optimal control problem constrained by this notion of solution. Our main result however concerns the approximation of optimal solutions by means of viscous regularization. The crucial issue in this context is that normalized parametrized BV solutions are in general non-unique and lack regularity, whereas the viscous solutions are unique and time-continuous. With the help of an additional regularity assumption on at least one optimal solution and a tailored penalization of the energy, one can nonetheless show that global minimizers of the viscous optimal control problems converge to an optimal solution of the original problem as the viscosity parameter tends to zero.

1 Introduction

The aim of this paper is to study an optimal control problem governed by the evolutionary system

$$0 \in \partial\mathcal{R}(\dot z(t)) + \partial_z\mathcal{I}(\ell(t), z(t)), \quad z(0) = z_0, \tag{RIS}$$

where the underlying space is $\mathbb{R}^n$. Since the dissipation potential $\mathcal{R}$ is assumed to be positive 1-homogeneous, the system behaves rate-independent, i.e., the equation is indeed independent of the rate with which the external load ℓ is applied, and any rescaling of the time leads to a likewise rescaled solution. The other constitutive component of (RIS) is the energy functional $\mathcal{I}$. The precise assumptions on $\mathcal{R}$ and $\mathcal{I}$ are specified in Section 2 below. As we will see, the energy is allowed to be non-convex and, for that reason, solutions are in general non-unique and discontinuous. Therefore, several distinct notions of solutions have been developed in the past two decades. The interested reader is referred to [Mie11, MR15] for a survey on the various concepts and a wide range of possible applications. Here, we focus on the so-called *normalized parametrized balanced viscosity (BV) solutions*, a concept, which has been introduced in [EM06] in a general form, but used earlier in e.g. [MSGM95, Bon96]. The precise

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definition of this notion of solution is given in Definition 3.6 below. It arises as limit of the following regularized viscous version of (RIS):

$$0 \in \partial\mathcal{R}(\dot z(t)) + \varepsilon \dot z + \partial_z \mathcal{I}(\ell(t), z(t)), \quad z(0) = z_0, \tag{RIS$_\varepsilon$}$$

where $\varepsilon > 0$ is the viscosity parameter. The passage to the limit $\varepsilon \searrow 0$ together with a reparametrization of the solution trajectory of (RIS$_\varepsilon$) allows to establish the existence of a normalized parametrized BV solution consisting of the triple $(S, \hat t, \hat z) \in [0,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^n)$, where the physical time $\hat t$ as well as the solution $\hat z$ are functions of an "artificial time" $s \in [0, S]$. In Section 3 below, the limit analysis is carried out in detail for loads in $\ell \in H^1(0,T;\mathbb{R}^n)$.

Based on the concept of normalized parametrized BV solutions, we introduce the following optimal control problem

$$\left.\begin{aligned}
\min \quad & J(S,z,\ell) := j(\hat z(S)) + \frac{\beta}{2}\,\|\ell\|^2_{H^1(0,T;\mathbb{R}^n)} \\
\text{s.t.} \quad & \ell \in H^1(0,T;\mathbb{R}^n), \quad (S,\hat t,\hat z) \in [0,\infty) \times W^{1,\infty}(0,S) \times W^{1,\infty}(0,S;\mathbb{R}^n), \\
& (S,\hat t,\hat z) \text{ is a normalized parametrized BV solution associated with } \ell, \\
& -\nabla_z \mathcal{I}(\ell(0), z_0) \in \partial\mathcal{R}(0), \quad -\nabla_z \mathcal{I}(\ell(T), \hat z(S)) \in \partial\mathcal{R}(0).
\end{aligned}\right\} \tag{1.1}$$

The assumptions on the data in (1.1) will be specified in Section 2 below. The motivation for this specific optimization problem is as follows: The end time objective could for instance be of the form $j(\hat z(S)) := \frac{1}{2}\|\hat z(S) - z_d\|^2$ in order to minimize the deviation of the state at end time to a given desired state z_d. Depending on the Tikhonov parameter $\beta > 0$, the second part of the objective measures the control cost and guarantees the needed regularity of the control variable ℓ. The additional constraints at initial and end time ensure the local stability of the initial and final state, which is physically motivated, see Remark 4.2 below.

Since the state equation (RIS) is in general not uniquely solvable as already mentioned above, there is no single-valued control-to-state mapping. For this reason, (1.1) is more an optimization problem in function space rather than an optimal control problem.

The main goal of this work is to approximate (optimal solutions of) (1.1) via viscous regularization, i.e., by replacing the normalized parametrized BV solution concept by the viscous equation (RIS$_\varepsilon$). In order to show that optimal solutions of the regularized problems converge to solutions of (1.1) (in a certain topology) for viscosity parameter tending to zero, the following steps have to be performed:

1. The existence of (weak) accumulation points of sequences of optimal solutions of the regularized problems have to be verified.

2. Weak limits have to be feasible for the original problem (1.1).

3. In order to show the optimality of the weak limit, one has to construct a *recovery sequence* for at least one optimal solution of the original problem.

The last item is also known as *reverse approximation* and might become a particularly challenging task in the context of optimization of rate-independent systems, see [MR09]. This also happens to be the case here: In contrast to (RIS), its viscous counterpart in (RIS$_\varepsilon$) admits a unique solution. It is therefore very unlikely that one can approximate *every* solution of (RIS) by means of vanishing viscosity and indeed, as the example in [MS20, Section 2.4] demonstrates, this is in fact not true. However, in the context of optimal control and optimization, respectively, we have an additional variable at hand in form of the control variable ℓ and, in order to construct a recovery sequence, we have to find a sequence of *tuples of state and control* that are feasible for the viscous system so that the associated objective function values converge to the optimal value of (1.1). This leads to much more flexibility in the construction of recovery sequences and is the essential ingredient for our reverse approximation argument. Nevertheless, even with the control as additional variable at hand we are only able to construct a recovery sequence under the fairly restrictive assumption that at least one optimal solution of (1.1) exists whose state is continuous in the physical time, which is in general not true for rate-independent systems with non-convex energy as explained above.

Let us put our work into perspective: Optimization and optimal control of rate-independent systems have been considered by various authors and we only refer to [Bro87, BK13, AO14, BK15, CHHM16, SWW17, AC18, GW18, Mün18] and the references therein. Albeit still nonsmooth, optimization problems of this type substantially simplify, if the underlying energy is uniformly convex. In this case, (RIS) admits a unique solution that is continuous in the physical time, which makes the construction of recovery sequences almost trivial. Nevertheless, the derivation of optimality conditions is still an intricate issue, see [Wac12, Wac15, Wac16]. While all contributions mentioned so far deal with uniformly convex energies, the literature becomes rather scarce, when it comes to energies that lack strict convexity. In [Rin09b, Ste12, ELS13, EL14, KT18] the existence of optimal solutions for problems with non-convex energies is addressed. The approximation of optimal control problems with non-convex energy by means of time-discretization is investigated in [MR09, Rin09a] for the concept of energetic solutions, which substantially differs from our notion of solution. For an overview over the various solution concepts for rate-independent systems, we refer to [MR15]. The discretization leads to incremental minimization problems and, in order to resolve the reverse approximation problem, the authors use ε-minimizers of these problems, i.e., solutions in each time step that are only optimal up to a constant $\varepsilon > 0$. In this way, the set of discrete solutions is sufficiently enlarged in order to contain a suitable recovery sequence. However, the optimal control problems discretized in this way are all but straight forward to solve, since, similar to (RIS) itself, there is still no single-valued control-to-state map (even worse, due to the ε-optimality, the set of discrete solutions is even enlarged). In contrast to this, the viscous optimal control problems, where (RIS) is replaced by (RIS_ε), provide a single-valued control-to-state operator. In combination with a potential further smoothing, the viscous optimal control problems are therefore amenable for standard adjoint-based optimization methods, see also Remark 6.5 at the very end of this paper.

The viscous regularization of an optimal control problem is also investigated in [MW20], where the rate-independent system of perfect plasticity is considered. Using the particular structure of this problem, the reverse approximation property is realized by means of an additional control variable, which is enforced to vanish as the viscosity parameter tends to zero. Here, we pursue a different approach and convexify the energy by adding a tailored quadratic penalization. It turns out that a *finite* penalization parameter suffice to obtain a uniformly convex energy, which is the essential for the vanishing viscosity analysis and the reverse approximation property, respectively.

Plan of the paper Before we investigate the state system and analyze the convergence behavior of viscous solutions for viscosity parameter tending to zero in Section 3, we introduce our notation and list the standing assumptions in Section 2. In Section 4, we then turn to the actual optimal control problem and establish the existence of global minimizer. Sections 5 and 6 are finally devoted to the viscous approximation of the optimal control problem. As already indicated above, the crucial result in this context is the reverse approximation property, which is established in Section 5. With this at hand, it is straightforward to obtain a convergence result for global minimizers of the viscous optimal control problem for viscosity parameter tending to zero, which is presented in Section 6. The paper ends with two appendices concerning auxiliary results connected to the chain rule for Sobolev functions and a uniqueness result for solutions to (RIS) in case of uniformly convex energies.

2 Notation and standing assumptions

Throughout this paper, $\|\cdot\|$ denotes the Euclidian norm and $\langle\cdot\,,\cdot\rangle$ the associated scalar product. Given a twice differentiable function $g : \mathbb{R}^n \to \mathbb{R}$, we denote its Hessian by $\nabla^2 g$ and, with a slight abuse of notation, the corresponding bilinear form is denoted by the same symbol. For a convex function $g : \mathbb{R}^n \to (-\infty, \infty]$, the convex subdifferential and the (Fenchel-)conjugate functional are denoted by ∂g and g^*, respectively. Moreover, given a point $v \in \mathbb{R}^n$ and a set $M \subset \mathbb{R}^n$, we define $\operatorname{dist}(v, M) := \inf_{m\in M} \|v - m\|$. Finally, $C > 0$ denotes a generic constant.

The following standing hypotheses on the data in (1.1) are tacitly assumed to hold throughout the whole paper without mentioning them every time.

Energy Throughout the paper, the energy functional $\mathcal{I} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$ is supposed to satisfy the following conditions:

$$\mathcal{I} \in C^1(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}), \quad \nabla_z \mathcal{I} \text{ locally Lipschitz on } \mathbb{R}^n \times \mathbb{R}^n, \tag{2.1}$$

$$\exists\, \lambda,\, \kappa,\, \mu > 0\ \forall\, \ell \in \mathbb{R}^n,\, z \in \mathbb{R}^n: \quad \|\nabla_\ell \mathcal{I}(\ell, z)\|^2 \le \lambda\big(\mathcal{I}(\ell, z) + \kappa\big) + \mu\|\ell\|^2. \tag{2.2}$$

Moreover, we assume that

$$\forall\, L > 0, R \in \mathbb{R} \ \text{ the sets } S_{L,R} := \{(\ell, z) \,:\, \|\ell\| \le L,\, \mathcal{I}(\ell, z) \le R\} \ \text{ are compact.} \tag{2.3}$$

Dissipation Concerning the dissipation potential, we suppose that $\mathcal{R} : \mathbb{R}^n \to [0, \infty]$ is convex, lower semicontinuous, positively homogeneous of degree one and satisfies

$$\exists\, c_{\mathcal{R}} > 0\ \forall\, z \in \mathbb{R}^n: \quad \mathcal{R}(z) \ge c_{\mathcal{R}}\, \|z\|\,. \tag{2.4}$$

Example 2.1. *Let us give an example fulfilling the above assumptions:*

$$\mathcal{I}(z) := \tfrac{1}{2}\langle Az, z\rangle + f(z) - \langle \ell, z\rangle, \quad \mathcal{R}(z) := \|z\|,$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite and $f : \mathbb{R}^n \to \mathbb{R}$ is bounded from below and continuously differentiable with locally Lipschitz derivative fulfilling the following growth condition:

$$\lim_{\|x\|\to\infty} \frac{\alpha}{2}\, \|z\|^2 + f(z) \to \infty \quad \textit{with} \quad \alpha < \lambda_{\min}(A),$$

where $\lambda_{\min}(A) > 0$ denotes the minimal eigenvalue of A.

Initial state and end time For the initial state $z_0 \in \mathbb{R}^n$ of the rate-independent evolution, we assume that there is a load vector $\ell_0 \in \mathbb{R}^n$ such that $-\nabla_z \mathcal{I}(\ell_0, z_0) \in \partial\mathcal{R}(0)$, i.e., the initial state is locally stable. Moreover, the end time $T > 0$ is fixed throughout the paper.

Data in the optimization problem The Tikhonov parameter β in the objective is a fixed positive real number. Moreover, the end time objective $j : \mathbb{R}^n \to \mathbb{R}$ is supposed to be continuous and bounded from below.

3 Analysis of the state system

Before we are in the position to investigate the optimal control problem in (1.1), we first need to study the rate-independent state system and its viscous regularization in detail. For the rest of this section, the control variable in terms of the applied loads is therefore a fixed function $\ell \in H^1(0,T;\mathbb{R}^n)$. We point out that the vanishing viscosity analysis for rate-independent systems with loads in H^1 has already been carried out in [KT18], even in the infinite dimensional case, where the underlying space is a Banach space an not just $\mathbb{R}^n$. However, for convenience of the reader and for the sake of later reference, we present the arguments in the following two subsections. Moreover, in the finite dimensional setting, some assumptions can be relaxed, e.g. the boundedness of the dissipation potential, cf. [KT18, Eq. (1.1)].

3.1 Existence and uniqueness for viscous regularized system

As already indicated in the introduction, the existence of a normalized parametrized BV solution can be shown by means of viscous regularization, which leads to following viscous problem

$$0 \in \partial\mathcal{R}(\dot{z}_\varepsilon(t)) + \varepsilon\, \dot{z}_\varepsilon(t) + \nabla_z \mathcal{I}(\ell(t), z_\varepsilon(t)), \quad z_\varepsilon(0) = z_0, \tag{RIS$_\varepsilon$}$$

with viscosity parameter $\varepsilon > 0$. The main advantage in (RIS$_\varepsilon$) compared to (RIS) is that the regularized version provides a unique solution. Indeed, we have the following existence and regularity result.

Proposition 3.1. *For every $\varepsilon > 0$, every $\ell \in H^1(0,T;\mathbb{R}^n)$ and all $z_0 \in \mathbb{R}^n$, there exists a unique function $z_\varepsilon \in H^2(0,T;\mathbb{R}^n)$ satisfying (RIS$_\varepsilon$).*

Moreover, the following estimates are valid with λ and c as in (2.2) and a further constant $C > 0$ that is independent of ε:

$$\mathcal{I}(\ell(t), z_\varepsilon(t)) \le e^{\lambda t}\big(\mathcal{I}(\ell(0), z_0) + \kappa\,\lambda\, T + \|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}\big), \tag{3.1a}$$

$$\|z_\varepsilon\|_{L^\infty(0,T;\mathbb{R}^n)} \le C\big(1 + \mathcal{I}(\ell(0), z_0) + \|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}\big), \tag{3.1b}$$

$$\int_0^T \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(r))\mathrm{d}r + \int_0^T \mathcal{R}^*_\varepsilon(-\nabla_z\mathcal{I}(\ell(r), z_\varepsilon(r)))\mathrm{d}r \le C\big(1 + \mathcal{I}(\ell(0), z_0) + \|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}\big). \tag{3.1c}$$

Proof. The arguments are rather standard, but, for convenience of the reader, we shortly sketch the proof. For $\varepsilon > 0$, let $\mathcal{R}^*_\varepsilon$ be the conjugate functional in the sense of convex analysis of

$$\mathcal{R}_\varepsilon(z) := \mathcal{R}(z) + \frac{\varepsilon}{2}\|z\|^2. \tag{3.2}$$

From the 1-homogeneity of $\mathcal{R}$ it follows that $\partial\mathcal{R}^*_\varepsilon$ is single-valued and globally Lipschitz continuous, and (RIS$_\varepsilon$) can be rewritten as

$$\dot{z}_\varepsilon(t) = \partial\mathcal{R}^*_\varepsilon(-\nabla_z\mathcal{I}(\ell(t), z_\varepsilon(t))), \quad z_\varepsilon(0) = z_0. \tag{3.3}$$

For given $M \in \mathbb{N}$ and $\delta > 0$, denote the orthogonal projection on the ball $\overline{B_M(0)}$ with radius M by Π_M and fix a regularized version thereof, denoted by Π^δ_M, with the following properties:

$$\Pi^\delta_M \in C^1(\mathbb{R}^n;\mathbb{R}^n), \quad \Pi^\delta_M(v) = \Pi_M(v) \quad \forall\, v \in \overline{B_M(0)}, \quad \|D\Pi^\delta_M\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le C, \tag{3.4}$$

where $D\Pi^\delta_M$ denotes the Jacobian of Π^δ_M. Given Π^δ_M, we consider the following modified ODE

$$\dot{\zeta}(t) = \partial\mathcal{R}^*_\varepsilon\Big(D\Pi^\delta_M\big(\zeta(t)\big)^\top\big(-\nabla_z\mathcal{I}[\ell(t), \Pi^\delta_M(\zeta(t))]\big)\Big), \quad \zeta(0) = z_0. \tag{3.5}$$

Due to the local Lipschitz continuity of $\nabla_z\mathcal{I}$, $\ell \in H^1(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$, and the boundedness assumption in (3.4), the function $\zeta \mapsto \partial\mathcal{R}^*_\varepsilon(D\Pi^\delta_M(\zeta)^\top\Pi_M(-\nabla_z\mathcal{I}[\ell(t), \Pi_M(\zeta)]))$ is globally Lipschitz continuous w.r.t. ζ, as locally Lipschitz continuous functions are globally Lipschitz on convex and compact sets. Therefore, the Picard-Lindelöf theorem implies the existence and uniqueness of a global solution $\zeta \in C^1([0,T];\mathbb{R}^n)$.

We proceed with showing the estimates (3.1a)-(3.1c) for ζ provided that M is chosen sufficiently large. To this end, we use the Fenchel-Young equality as well as the chain rule for Sobolev functions, cf. e.g. [Zie12, Thm. 2.1.11], to rewrite (3.5) equivalently as

$$\begin{aligned}\mathcal{R}_\varepsilon(\dot\zeta(t)) &+ \mathcal{R}_\varepsilon^*\Big(D\Pi_M^\delta(\zeta(t))^\top\big(-\nabla_z\mathcal{I}[\ell(t),\Pi_M^\delta(\zeta(t))]\big)\Big)\\ &= \big\langle -\nabla_z\mathcal{I}[\ell(t),\Pi_M^\delta(\zeta(t))], D\Pi_M^\delta(\zeta(t))\dot\zeta(t)\big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}[\ell(t),\Pi_M^\delta(\zeta(t))] - \nabla_\ell\mathcal{I}[\ell(t),\Pi_M^\delta(\zeta(t))]\dot\ell(t)\end{aligned} \tag{3.6}$$

which holds for almost all $t\in[0,T]$. Integration of this equality implies the following energy-dissipation identity, that holds for all $t\in[0,T]$:

$$\begin{aligned}\mathcal{I}[\ell(t),\Pi_M^\delta(\zeta(t))] + \int_0^t \mathcal{R}_\varepsilon(\dot\zeta(r)) + \mathcal{R}_\varepsilon^*\big(D\Pi_M^\delta(\zeta(t))^\top(-\nabla_z\mathcal{I}[\ell(r),\Pi_M^\delta(\zeta(r))])\big)\mathrm{d}r&\\ = \mathcal{I}(\ell(0),z_0) + \int_0^t \nabla_\ell\mathcal{I}[\ell(r),\Pi_M^\delta(\zeta(r))]\dot\ell(r)\mathrm{d}r&,\end{aligned} \tag{3.7}$$

where we assumed that $M \geq \|z_0\|$. With Youngs inequality and assumption (2.2) the right hand side can be estimated as

$$\int_0^t \nabla_\ell\mathcal{I}[\ell(r),\Pi_M^\delta(\zeta(r))]\dot\ell(r)\mathrm{d}r \leq \|\ell\|^2_{H^1(0,t;\mathbb{R}^n)} + \int_0^t \lambda\big(\mathcal{I}[\ell(r),\Pi_M^\delta(\zeta(r))]+\kappa\big) + \mu\|\ell(r)\|^2\mathrm{d}r. \tag{3.8}$$

The non-negativity of $\mathcal{R}_\varepsilon$ as well as $\mathcal{R}_\varepsilon^*$ along with (3.7), (3.8), and Gronwall's lemma imply (3.1a) for $\Pi_M^\delta(\zeta)$ instead of z_ε, i.e.,

$$\mathcal{I}[\ell(t),\Pi_M^\delta(\zeta(t))] \leq e^{\lambda t}\big(\mathcal{I}(\ell(0),z_0) + \kappa\,\lambda\,T + (1+\mu)\|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}\big), \tag{3.9}$$

Together with (3.7) and (3.8), this leads to an estimate analogous to (3.1c):

$$\begin{aligned}\int_0^T \mathcal{R}_\varepsilon(\dot\zeta(r))\mathrm{d}r + \int_0^T \mathcal{R}_\varepsilon^*\big(D\Pi_M^\delta(\zeta(t))^\top(-\nabla_z\mathcal{I}[\ell(r),\Pi_M^\delta(\zeta(r))])\big)\mathrm{d}r&\\ \leq C(1+\mathcal{I}(\ell(0),z_0) + \|\ell\|^2_{H^1(0,T;\mathbb{R}^n)})&.\end{aligned} \tag{3.10}$$

Thanks to assumption (2.4), this in turn implies

$$\|\zeta(t)\| \leq \int_0^T \|\dot\zeta(r)\|\mathrm{d}r \leq \frac{1}{c_{\mathcal{R}}}\int_0^T \mathcal{R}_\varepsilon(\dot\zeta(r))\mathrm{d}r \leq C(1+\mathcal{I}(\ell(0),z_0)+\|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}). \tag{3.11}$$

So, if we choose $M \geq C(1+\mathcal{I}(\ell(0),z_0)+\|\ell\|^2_{H^1(0,T;\mathbb{R}^n)})$, then the truncation in (3.5) becomes superfluous, since $\Pi_M^\delta = \Pi_M =$ identity in $\overline{B_M(0)}$ by assumption. Therefore, (3.3) is equivalent to (3.5) and consequently, ζ also solves (3.3) in this case such that (3.3) indeed admits a unique solution as claimed. In addition, the estimates in (3.9)–(3.11) readily carry over to z_ε instead of ζ and $\Pi_M^\delta(\zeta)$, resp., if M is sufficiently large.

The claimed regularity of z_ε finally follows from (3.3) and the Lipschitz continuity of the right-hand side. □

Note that, due to the regularity of z_ε the ODE in (3.3) even holds for all $t\in[0,T]$. Beyond this, we have the following additional properties of solutions to (RIS$_\varepsilon$) that turn out to be useful for the derivation of the reverse approximation property, see Section 5 below.

Lemma 3.2. *Let $z_0\in\mathbb{R}^n$ satisfy $-\nabla_z\mathcal{I}(\ell(0),z_0)\in\partial\mathcal{R}(0)$. Assume further $\mathcal{I}\in C^2(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})$. If $z_\varepsilon\in H^2(0,T;\mathbb{R}^n)$ is a solution of (RIS$_\varepsilon$), then*

$$\varepsilon\langle\dot z_\varepsilon(t),\ddot z_\varepsilon(t)\rangle + \nabla^2_{zz}\mathcal{I}(\ell(t),z_\varepsilon(t))[\dot z_\varepsilon(t),\dot z_\varepsilon(t)] + \nabla^2_{\ell z}\mathcal{I}(\ell(t),z_\varepsilon(t))[\dot\ell(t),\dot z_\varepsilon(t)] = 0 \tag{3.12}$$

holds for almost all $t\in[0,T]$. Moreover, it holds $\dot z_\varepsilon(0)=0$.

Proof. The proof of (3.12) is similar to the proof of Lemma 5.6, see also [Mie11, Lem. 4.16].

For the second assertion, we again employ (3.3). Since $\dot z_\varepsilon \in H^1(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$ and $\partial\mathcal{R}_\varepsilon^*$ is Lipschitz-continuous, this equation holds for every $t \in [0,T]$ and hence,

$$\dot z_\varepsilon(0) = \partial\mathcal{R}_\varepsilon^*(-\nabla_z\mathcal{I}(\ell(0), z_0)). \tag{3.13}$$

Now, exploiting the inf-convolution formula we obtain

$$\mathcal{R}_\varepsilon^*(w) = \frac{1}{2\varepsilon}\operatorname{dist}(w, \partial\mathcal{R}(0))^2. \tag{3.14}$$

The assumption $-\nabla_z\mathcal{I}(\ell(0), z_0) \in \partial\mathcal{R}(0)$ thus yields

$$\mathcal{R}_\varepsilon^*(-\nabla_z\mathcal{I}(0,z_0)) + \langle 0, \eta + \nabla_z\mathcal{I}(0,z_0)\rangle = 0 \le \mathcal{R}_\varepsilon^*(\eta) \quad \forall \eta \in \mathbb{R}^n, \tag{3.15}$$

so that $0 \in \partial\mathcal{R}_\varepsilon^*(-\nabla_z\mathcal{I}(0,z_0))$ by the definition of the convex subdifferential. Since $\partial\mathcal{R}_\varepsilon^*$ is a singleton as seen above, (3.13) gives $\dot z_\varepsilon(0) = 0$. □

3.2 Vanishing viscosity and normalized parametrized BV solutions

We now turn to the convergence analysis for $\varepsilon \searrow 0$ in (RIS$_\varepsilon$), where, in view of Section 4, we allow for a further variation of the loads. This vanishing viscosity limit leads to the precise definition of normalized parametrized BV solutions, see Definition 3.6 below. The key tool for the limit analysis is a tailored reparametrization by a specific "energy arc-length". To this end, we follow closely the arguments from [MRS12]: For $v, w \in \mathbb{R}^n$ let

$$\mathfrak{p}(v,w) := \mathcal{R}(v) + \|v\| \operatorname{dist}\big(w, \partial\mathcal{R}(0)\big), \tag{3.16}$$

be the so-called *vanishing-viscosity contact potential*. We do not want to elaborate on the properties of $\mathfrak{p}$ here and refer to [MRS12] for details. However, let us mention that, for any $\varepsilon > 0$, it holds $\mathfrak{p}(v,w) \le \mathcal{R}_\varepsilon(v) + \mathcal{R}_\varepsilon^*(w)$. Indeed, by the Young inequality, we have

$$\mathfrak{p}(v,w) = \mathcal{R}(v) + \|v\| \operatorname{dist}\big(w, \partial\mathcal{R}(0)\big) \le \mathcal{R}(v) + \frac{\varepsilon}{2}\|v\|^2 + \frac{1}{2\varepsilon}\operatorname{dist}\big(w, \partial\mathcal{R}(0)\big)^2 = \mathcal{R}_\varepsilon(v) + \mathcal{R}_\varepsilon^*(w). \tag{3.17}$$

To proceed, we define

$$s_\varepsilon(t) := t + \int_0^t \mathfrak{p}(\dot z_\varepsilon(r), -\nabla_z\mathcal{I}(\ell_\varepsilon(r), z_\varepsilon(r)))\mathrm{d}r \quad \text{and} \quad S_\varepsilon := s_\varepsilon(T). \tag{3.18}$$

Since s_ε is a strictly increasing function in t it provides an inverse function $s_\varepsilon^{-1} : [0, S_\varepsilon] \to [0,T]$ by which we define the final reparameterization

$$\hat t_\varepsilon(s) := s_\varepsilon^{-1}(s) \quad \text{and} \quad \hat z_\varepsilon(s) := z_\varepsilon(\hat t_\varepsilon(s)). \tag{3.19}$$

It is easy to verify that $(\hat t_\varepsilon, \hat z_\varepsilon) : [0, S_\varepsilon] \to [0,T] \times \mathbb{R}^n$ are Lipschitz continuous and satisfy

$$\hat t_\varepsilon'(s) + \mathfrak{p}(\hat z_\varepsilon'(s), -\nabla_z\mathcal{I}(\ell_\varepsilon(\hat t_\varepsilon(s)), \hat z_\varepsilon(s))) = 1 \tag{3.20}$$

for almost all $s \in [0, S_\varepsilon]$.

Lemma 3.3. *Let $(\ell_n)_{n\in\mathbb{N}} \subset H^1(0,T;\mathbb{R}^n)$ and $(\hat t_n) : n \in \mathbb{N} \subset W^{1,\infty}(0,S)$, $S > 0$, be sequences with $\ell_n \rightharpoonup \ell$ weakly in $H^1(0,T;\mathbb{R}^n)$ and $\hat t_n \overset{*}{\rightharpoonup} \hat t$ in $W^{1,\infty}(0,S)$ satisfying $\hat t_n(0) = 0$, $\hat t_n(S) = T$, and $\hat t_n'(s) \ge 0$ f.a.a. $s \in (0,S)$ and all $n \in \mathbb{N}$. Then*

$$\ell_n \circ \hat t_n \rightharpoonup \ell \circ \hat t \quad \textit{weakly in } H^1(0,S;\mathbb{R}^n). \tag{3.21}$$

Proof. First observe that, by Lemma A.1, $\hat{\ell}_n = \ell_n \circ \hat{t}_n$ is an element of $H^1(0,S;\mathbb{R}^n)$ and $\|\hat{\ell}_n'\|_{L^2(0,S;\mathbb{R}^n)} \leq \|\hat{t}_n'\|_{L^\infty(0,S)}\|\dot{\ell}_n\|_{L^2(0,T;\mathbb{R}^n)}$, which implies that $(\hat{\ell}_n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,S;\mathbb{R}^n)$. Consequently, there exists a subsequence, denoted by the same symbol for simplicity, converging weakly to some $\bar{\ell}$ in $H^1(0,S;\mathbb{R}^n)$. On the other hand, due to the compact embeddings, it holds $\ell_n \to \ell$ uniformly in $C([0,T];\mathbb{R}^n)$ and $\hat{t}_n \to \hat{t}$ uniformly in $C([0,S])$. Hence, $\ell_n \circ \hat{t}_n$ converges uniformly to $\ell \circ \hat{t}$ in $C([0,S];\mathbb{R}^n)$ and also strongly in $L^2(0,S;\mathbb{R}^n)$. Now, since weak and strong limit coincide, we obtain $\bar{\ell} = \ell \circ \hat{t}$, which finishes the proof. □

As an immediate consequence of the above lemma and the compact embedding $H^1(0,S;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$ in combination with the continuity of $\nabla_\ell \mathcal{I}$, we obtain the following

Corollary 3.4. *Let $(\hat{z}_n)_{n\in\mathbb{N}} \subset C([0,S];\mathbb{R}^n)$ with $\hat{z}_n \to \hat{z}$ in $C([0,S];\mathbb{R}^n)$ and let $(\ell_n)_{n\in\mathbb{N}} \subset H^1(0,T;\mathbb{R}^n)$ and $(\hat{t}_n)_{n\in\mathbb{N}} \subset W^{1,\infty}(0,S)$ be sequences that converge as in Lemma 3.3. Then for all $s \in [0,S]$ it holds*

$$\lim_{n\to\infty} \int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}_n(r), \hat{z}_n(r))\hat{\ell}_n'(r)\mathrm{d}r = \int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}(r), \hat{z}(r))\hat{\ell}'(r)\mathrm{d}r, \tag{3.22}$$

where $\hat{\ell}_n := \ell_n \circ \hat{t}_n$ and $\hat{\ell} := \ell \circ \hat{t}$.

After these preparatory steps, we are now in the position to formulate the following Theorem for the vanishing viscosity limit ($\varepsilon \searrow 0$) under an additional weak convergence of the loads ℓ_ε in $H^1(0,T;\mathbb{R}^n)$.

Theorem 3.5 (Vanishing viscosity limit)**.** *Let $(z_\varepsilon)_{\varepsilon>0}$ be solutions to (RIS$_\varepsilon$) with $\ell_\varepsilon \in H^1(0,T;\mathbb{R}^n)$ and fixed $z_0 \in \mathbb{R}^n$. Let further ℓ_ε weakly converge to some ℓ in $H^1(0,T;\mathbb{R}^n)$. Then there exists $S > 0$, $(\hat{t}, \hat{z}) \in W^{1,\infty}(0,S;\mathbb{R}\times\mathbb{R}^n)$ and a vanishing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ (i.e. $\varepsilon_n \searrow 0$) such that*

$$S_{\varepsilon_n} \to S, \quad (\hat{t}_{\varepsilon_n}, \hat{z}_{\varepsilon_n}) \overset{*}{\rightharpoonup} (\hat{t}, \hat{z}) \quad \text{in } W^{1,\infty}(0,S;\mathbb{R}\times\mathbb{R}^n) \tag{3.23}$$

together with uniform convergence on $[0,S]$. Together with $\hat{\ell} := \ell \circ \hat{t}$, the limit functions satisfy the following system:

$$\hat{t}(S) = T, \quad \hat{z}(0) = z_0, \tag{3.24}$$

$$\textit{for a.a. } s \in (0,S): \quad \hat{t}'(s) \geq 0, \quad \hat{t}'(s) + \mathfrak{p}(\hat{z}'(s), -\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s))) = 1, \tag{3.25}$$

$$\hat{t}'(s)\operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) = 0, \tag{3.26}$$

$$\textit{for all } s \in [0,S]: \quad \mathcal{I}(\hat{\ell}(s), \hat{z}(s)) + \int_0^s \mathcal{R}(\hat{z}'(\sigma)) + \|\hat{z}'(\sigma)\| \operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0))\mathrm{d}\sigma \tag{3.27}$$
$$= \mathcal{I}(\hat{\ell}(0), z_0) + \int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}(\sigma), \hat{z}(\sigma))\hat{\ell}'(\sigma)\mathrm{d}\sigma.$$

Moreover, every cluster point in the above sense of the sequence $(S_\varepsilon, \hat{t}_\varepsilon, \hat{z}_\varepsilon)_{\varepsilon>0}$ satisfies (3.25)–(3.27).

Note that the Theorem in particular includes the case where $\ell_\varepsilon \equiv \ell$. The proof is essentially identical to the arguments presented in [MRS12, Section 5]. For convenience, we repeat here the main steps.

Proof. First of all, we show that the arc-length S_ε is uniformly bounded. Therefore, we observe that by the weak convergence of ℓ_ε in $H^1(0,T;\mathbb{R}^n)$ the sequence is also bounded. Due to the compact embedding $H^1(0,T;\mathbb{R}^n) \hookrightarrow^{c} C([0,T];\mathbb{R}^n)$ and the continuity of $\mathcal{I}$, the same also holds true for $\mathcal{I}(\ell_\varepsilon(0), z_0)$. Now, thanks to (3.17) and estimate (3.1c) it holds

$$S_\varepsilon = T + \int_0^T \mathfrak{p}(\dot{z}_\varepsilon(r), -\nabla_z \mathcal{I}(\ell_\varepsilon(r), z_\varepsilon(r)))\mathrm{d}r \leq C\big(1 + \mathcal{I}(\ell(0), z_0) + \|\ell_\varepsilon\|^2_{H^1(0,T;\mathbb{R}^n)}\big). \tag{3.28}$$

By the boundedness of $\mathcal{I}(\ell_\varepsilon(0), z_0)$ and $\|\ell_\varepsilon\|_{H^1(0,T;\mathbb{R}^n)}$ we thus have $S_\varepsilon \leq C_S$ for some constant $C_S > 0$ independent of ε. Therefore, the end time S_ε is uniformly bounded and we can extract a subsequence converging to some S. Let further $\tilde{S} = \sup_{\varepsilon>0} S_\varepsilon$. Since every parametrized solution $\hat{z}_\varepsilon$ is defined on its own time horizon $[0, S_\varepsilon]$, we extend them to $[0, \tilde{S}]$ by constant continuation. Note that we still have $\tilde{S} \leq C_S$. From the normalization condition in (3.20) in combination with the definition of $\mathfrak{p}$ and assumption (2.4), we find that $(\hat{t}_\varepsilon, \hat{z}_\varepsilon)$ is uniformly bounded in $W^{1,\infty}(0, \tilde{S}; \mathbb{R} \times \mathbb{R}^n)$ independent of ε, so that the convergences in (3.23) immediately follow.

Therewith, the initial and end time condition in (3.24) as well as the sign condition in (3.25) follow directly from $\hat{z}_\varepsilon(0) = z_\varepsilon(0) = z_0$, $\hat{t}_\varepsilon(S_\varepsilon) = T$, and $\hat{t}'_\varepsilon(s) \geq 0$ for all $s \in [0, S_\varepsilon]$. Again, by the boundedness of ℓ_ε in $H^1(0,T;\mathbb{R}^n)$ and $\mathcal{I}(\ell_\varepsilon(0), z_\varepsilon(0))$, we infer that the estimates in Proposition 3.1 hold uniform in ε. Now let $\hat{\ell}_\varepsilon = \ell_\varepsilon \circ \hat{t}_\varepsilon$. In order to prove (3.26), we may argue as follows: Having in mind that $\mathcal{R}^*_\varepsilon(w) = \frac{1}{2\varepsilon} \operatorname{dist}(w, \partial\mathcal{R}(0))^2$ and $\mathcal{R}_\varepsilon(z) \geq 0$, the uniform estimate (3.1c) implies, after transformation to the new variable s, that

$$0 \leq \int_0^{S_\varepsilon} \hat{t}'_\varepsilon(s) \operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon(s), \hat{z}_\varepsilon(s)), \partial\mathcal{R}(0))^2 \mathrm{d}s$$
$$= \int_0^T \operatorname{dist}(-\nabla_z \mathcal{I}(\ell_\varepsilon(r), z_\varepsilon(r)), \partial\mathcal{R}(0))^2 \mathrm{d}r = \int_0^T 2\varepsilon \mathcal{R}^*_\varepsilon(-\nabla_z \mathcal{I}(\ell_\varepsilon(r), z_\varepsilon(r)) \mathrm{d}r \leq \varepsilon C, \quad (3.29)$$

where C is independent of ε. Since $-\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon, \hat{z}_\varepsilon)$ converges uniformly, the continuity of $\operatorname{dist}(\cdot, \partial\mathcal{R}(0))$ implies that $\operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon(\cdot), \hat{z}_\varepsilon(\cdot)), \partial\mathcal{R}(0))^2 \to \operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}(\cdot), \hat{z}(\cdot)), \partial\mathcal{R}(0))^2$ in $L^1(0, \tilde{S})$. In combination with the weak-* convergence of $\hat{t}'$ in $L^\infty(0, \tilde{S})$, this gives

$$0 = \lim_{\varepsilon \to 0} \int_0^{S_\varepsilon} \hat{t}'_\varepsilon(s) \big(\operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon(s), \hat{z}_\varepsilon(s)), \partial\mathcal{R}(0) \big)^2 \mathrm{d}s = \int_0^S \hat{t}'(s) \big(\operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0))^2 \mathrm{d}s.$$

Since $\hat{t}'$ (and therewith also the whole integrand) is non-negative, we find (3.26).

We proceed with proving (3.27). As in the proof of Proposition 3.1, one can show that (RIS$_\varepsilon$) is equivalent to the following energy-dissipation identity:

$$\mathcal{I}(\ell(t), z_\varepsilon(t)) + \int_0^t \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(r)) + \mathcal{R}^*_\varepsilon\big(-\nabla_z \mathcal{I}(\ell_\varepsilon(r), z_\varepsilon(r)) \big) \mathrm{d}r = \mathcal{I}(\ell_\varepsilon(0), z_0) + \int_0^t \nabla_\ell \mathcal{I}(\ell_\varepsilon(r), z_\varepsilon(r)) \dot{\ell}_\varepsilon(r) \mathrm{d}r$$

(cf. (3.7)). Thanks to (3.17), we can estimate the integrand on the left hand side from below by $\mathfrak{p}$ and the positive homogeneity of $\mathfrak{p}$ w.r.t. its first variable allows to rewrite the arising inequality in terms of the new variables $\hat{z}_\varepsilon$ and $\hat{t}_\varepsilon$ as follows:

$$\mathcal{I}(\hat{\ell}_\varepsilon(s), \hat{z}_\varepsilon(s)) + \int_0^s \mathfrak{p}\big(\hat{z}'_\varepsilon(s), -\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon(r), \hat{z}_\varepsilon(r)) \big) \mathrm{d}r \leq \mathcal{I}(\ell_\varepsilon(0), z_0) + \int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}_\varepsilon(r), \hat{z}_\varepsilon(r)) \hat{\ell}'_\varepsilon(r) \mathrm{d}r, \quad (3.30)$$

which holds for all $s \in [0, \tilde{S}]$. By Corollary 3.4, the integral term on the right hand side converges to $\int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}(r), \hat{z}(r)) \hat{\ell}'(r) \mathrm{d}r$. Clearly, $\mathcal{I}(\hat{\ell}_\varepsilon(s), \hat{z}_\varepsilon(s))$ converges to $\mathcal{I}(\hat{\ell}(s), \hat{z}(s))$ for $\varepsilon \to 0$ by the uniform convergence of ℓ_ε, $\hat{t}_\varepsilon$ and $\hat{z}_\varepsilon$ and the continuity of $\mathcal{I}$. For the integral on the left hand side of (3.30) we use the lower-semicontinuity result for $\mathfrak{p}(\cdot, \cdot)$ from [MRS09, Lemma 3.1]. Altogether, we obtain for all $s \in [0, S]$:

$$\begin{aligned}
&\mathcal{I}(\hat{\ell}(s), \hat{z}(s)) + \int_0^s \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r), \hat{z}(r))) \mathrm{d}r \\
&\leq \liminf_{\varepsilon \to 0} \Big(\mathcal{I}(\hat{\ell}_\varepsilon(s), \hat{z}_\varepsilon(s)) + \int_0^s \mathfrak{p}\big(\hat{z}'_\varepsilon(s), -\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon(r), \hat{z}_\varepsilon(r)) \big) \mathrm{d}r \Big) \\
&\leq \limsup_{\varepsilon \to 0} \Big(\mathcal{I}(\hat{\ell}_\varepsilon(s), \hat{z}_\varepsilon(s)) + \int_0^s \mathfrak{p}\big(\hat{z}'_\varepsilon(s), -\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon(r), \hat{z}_\varepsilon(r)) \big) \mathrm{d}r \Big) \\
&\leq \limsup_{\varepsilon \to 0} \Big(\mathcal{I}(\ell(0), z_0) + \int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}_\varepsilon(r), \hat{z}_\varepsilon(r)) \hat{\ell}'_\varepsilon(r) \mathrm{d}r \Big) \\
&= \mathcal{I}(\ell(0), z_0) + \int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}(r), \hat{z}(r)) \hat{\ell}'(r) \mathrm{d}r.
\end{aligned} \quad (3.31)$$

Consequently, (3.30) carries over to the limit and, by Lemma A.2, the inequality is in fact an equality, which, in view of the definition of $\mathfrak{p}$, is just the desired energy-dissipation identity in (3.27).

It remains to verify the last equation in (3.26). To this end, observe that the energy-dissipation identity implies that (3.31) holds with equality. In combination with the uniform convergence of $\mathcal{I}(\hat{\ell}_\varepsilon(\cdot), \hat{z}_\varepsilon(\cdot))$ and the weak convergence of $\hat{t}'$, this in particular yields that, for all $s \in [0, S]$,

$$s = \int_0^s \hat{t}_\varepsilon'(r) + \mathfrak{p}(\hat{z}_\varepsilon'(r), -\nabla_z \mathcal{I}(\hat{\ell}_\varepsilon(r), \hat{z}_\varepsilon(r)))\mathrm{d}r \to \int_0^s \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\mathrm{d}r. \tag{3.32}$$

Now assume that there is a Lebesgue measurable set $E \subset (0, S)$ such that $\hat{t}'(\cdot) + \mathfrak{p}(\hat{z}'(\cdot), -\nabla_z \mathcal{I}(\hat{\ell}(\cdot), \hat{z}(\cdot))) > 1$ a.e. on E and $|E| > 0$. Then for every finite union $U \subset (0, S)$ of disjoint open intervals, which contains E, (3.32) implies

$$|U| = \int_U \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\mathrm{d}r \geq \int_E \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r), \hat{z}(r)))\mathrm{d}r > |E|, \tag{3.33}$$

which contradicts the regularity of the Lebesgue measure. Hence, $\hat{t}'(\cdot) + \mathfrak{p}(\hat{z}'(\cdot), -\nabla_z \mathcal{I}(\hat{\ell}(\cdot), \hat{z}(\cdot))) \in [0, 1]$ a.e. in $(0, S)$ and (3.32) finally yields the last assertion of (3.25). □

Observe that every limit curve in the sense of Theorem 3.5 satisfies $S = T + \int_0^S \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r), \hat{z}(r))\mathrm{d}r$. The result of the former theorem directly leads us to the following definition:

Definition 3.6 (Normalized parametrized BV solution)**.** *Let $z_0 \in \mathbb{R}^n$, $\ell \in H^1(0, T; \mathbb{R}^n)$. A triple $(S, \hat{t}, \hat{z})$ with $S > 0$ and $(\hat{t}, \hat{z}) \in W^{1,\infty}(0, S; \mathbb{R} \times \mathbb{R}^n)$ is a normalized parametrized balanced viscosity (BV) solution of the system* (RIS) *if* (3.24)–(3.27) *are satisfied.*

Remark 3.7. *The denotation* normalized parametrized BV solution *indicates that we are dealing with a particular version of so-called balanced viscosity solutions and indeed, normalization and parametrization, respectively, is only one way of representing a BV solution, cf. [MRS12] for details on this solution concept.*

Remark 3.8. *The above Theorem shows the existence of normalized parametrized BV solutions. It generalizes slightly the results from [MRS12] to the case with $\ell \in H^1(0, T)$ and when $\mathcal{R}$ is unbounded. Note that, however, the case of an unbounded dissipation highly benefits from the fact that the spaces are finite dimensional. Normalized parametrized BV solutions in the sense of Definition 3.6 coincide with nondegenerate, surjective, normalized parametrized BV solutions in the sense of [MRS12, Definition 5.2].*

We underline that there are several other solution concepts for (RIS), among them the notion of global energetic solutions. For a detailed overview, we refer to [MR15]. The most rigorous solution concept is certainly the following:

Definition 3.9 (Differential solution)**.** *Let $z_0 \in \mathbb{R}^n$, $\ell \in H^1(0, T; \mathbb{R}^n)$ be given. We call $\tilde{z} \in W^{1,1}(0, T; \mathbb{R}^n)$ a differential solution of the system* (RIS) *if*

$$0 \in \partial\mathcal{R}(\dot{\tilde{z}}(t)) + \nabla_z \mathcal{I}(\ell(t), \tilde{z}(t)), \quad \tilde{z}(0) = z_0$$

is satisfied for almost all $t \in [0, T]$.

It is noted however that the existence of a differential solution cannot be guaranteed in case of a non-convex energy such that alternative, less rigorous solutions concepts such as normalized parametrized BV solutions are inevitable. We will return to the concept of a differential solution in connection with the reverse approximation problem in Section 5 below.

Thanks to [MRS12, Corollary 5.6] there is an equivalent characterization of normalized parametrized BV solutions as a differential inclusion, see also [KT18, Proposition 3.4] for a detailed proof within a similar setting.

Lemma 3.10. *A triple* $(S, \hat{t}, \hat{z})$ *with* $(\hat{t}, \hat{z}) \in W^{1,\infty}(0, S; \mathbb{R} \times \mathbb{R}^n)$ *that satisfies* (3.25) *is a normalized parametrized BV solution in the sense of Definition 3.6 if and only if there exists a measurable function* $\lambda : (0, S) \to [0, \infty)$ *such that*

$$0 \in \partial\mathcal{R}(\hat{z}'(s)) + \lambda(s)\hat{z}'(s) + \nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \qquad \hat{t}'(s)\lambda(s) = 0. \tag{3.34}$$

for almost all $s \in [0, S]$.

Testing (3.34) with $\hat{z}'$, integration with respect to s and applying the chain rule to the terms involving the energy $\mathcal{I}$ shows, after a comparison with the energy-dissipation identity (3.27), that

$$\text{for almost all } s \in [0, S]: \qquad \lambda(s)\|\hat{z}'(s)\|^2 = \|\hat{z}'(s)\| \operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)).$$

Since, by (3.34), $-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)) \in \partial\mathcal{R}(\hat{z}'(s)) \subset \partial\mathcal{R}(0)$ f.a.a. $s \in (0, S)$ with $\hat{z}'(s) = 0$, we even obtain

$$\text{for almost all } s \in [0, S]: \qquad \lambda(s)\|\hat{z}'(s)\| = \operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)). \tag{3.35}$$

This observation gives rise to the following distinction of physical regimes that may occur during a rate-independent evolution:

- *Sticking*:
  If $\hat{t}'(s) > 0$ and $\hat{z}'(s) = 0$ a.e. in an interval $I \subset [0, S]$, then the physical time proceeds in I, but the external loading is too small to change the system state such that the dissipation forces the system state to remain constant.

- *Rate-independent slip*:
  If $\hat{t}(s) > 0$ and $\hat{z}'(s) > 0$ a.e. in $I \subset [0, S]$, then the state indeed changes but in such a matter that the dissipation is strong enough to compensate the external loading.

- *Jump sets*:
  If $\hat{t}(s) = 0$ and $\hat{z}'(s) > 0$ a.e. in I, then the physical time does not proceed, although the system state changes, and thus and we observe a discontinuous behavior of the system. Then we have to distinguish between two different cases:

  – *Viscous jump*:
    If in addition $\lambda(s) > 0$ a.e. in I, then, according to (3.34), an additional viscous term arises in the state equation. Moreover, due to (3.35), we have $\operatorname{dist}(-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)), \partial\mathcal{R}(0)) > 0$ such that the state is no longer *locally stable* in I. There is thus a viscous transition through the complement of the region of local stability, which is seen as a jump in physical time, cf. Example 3.11 below.

  – *Rate-independent jump*:
    Discontinuities may even occur in the area of local stability, where $-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)) \in \partial\mathcal{R}(0)$ and thus $\lambda(s) = 0$, as [Sie20, Example 2.3.5] demonstrates. If $I \subset [0, S]$ is an interval, where $\hat{t}'(s) = 0$ and $-\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s)) \in \partial\mathcal{R}(0)$ a.e. in I, then (3.25) implies $\mathcal{R}(\hat{z}'(s)) = 1$ a.e. in I such that, according to the energy identity in (3.27), the energy has to decay linearly along this part of the trajectory of $\hat{z}$, which is probably a rather pathological situation.

If the normalization condition in (3.25) is relaxed to an inequality, i.e., $\hat{t}'(s) + \mathfrak{p}(\hat{z}'(s), -\nabla_z \mathcal{I}(\hat{\ell}(s), \hat{z}(s))) \leq 1$ f.a.a. $t \in [0, T]$, such that $\hat{t}'$ and $\hat{z}'$ may vanish at the same time, there is an additional regime that might occur:

- *Removable arcs*:
  If $\hat{t}(s) = 0$ and $\hat{z}'(s) = 0$ a.e. in I, then I can simply be removed from the evolution without changing the system behavior.

The following example is taken from [Sie20, Example 2.4.8] and visualizes the prior comments.

Example 3.11. *We consider* $\mathcal{I} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *with*

$$\mathcal{I}(t, z) = \mathcal{E}(z) - \ell(t)\, z \;\; \textit{with} \;\; \mathcal{E}(z) = \begin{cases} \frac{1}{2}(z+4)^2, & z \leq -2, \\ 4 - \frac{1}{2}z^2, & |z| < 2, \\ \frac{1}{2}(z-4)^2, & z \geq 2, \end{cases}$$

as well as $\mathcal{R}(z) = |z|$*, cf. [MR15, Ex. 1.8.3]. We additionaly set* $z_0 = -2$ *and* $\ell(t) = t + 1$*. Then, direct calculations lead to the corresponding parametrized BV solution*

$$\hat{z}(s) = \begin{cases} -2, & s \in [0,2], \\ s-4, & s \in (2,10], \\ (s+2)/2, & s \in (10,16], \end{cases} \quad \textit{and} \quad \hat{t}(s) = \begin{cases} s, & s \in [0,2], \\ 2, & s \in (2,10], \\ (s-6)/2, & s \in (10,16], \end{cases}$$

and the multiplier

$$\lambda(s) = \begin{cases} 0, & s \in [0,2], \\ 2-s, & s \in (2,6], \\ s-10, & s \in (6,10], \\ 0, & s \in (10,16]. \end{cases}$$

Note that during the viscous jump, we obtain the additional viscous dissipation $\lambda(s)\|\hat{z}'(s)\|^2$.

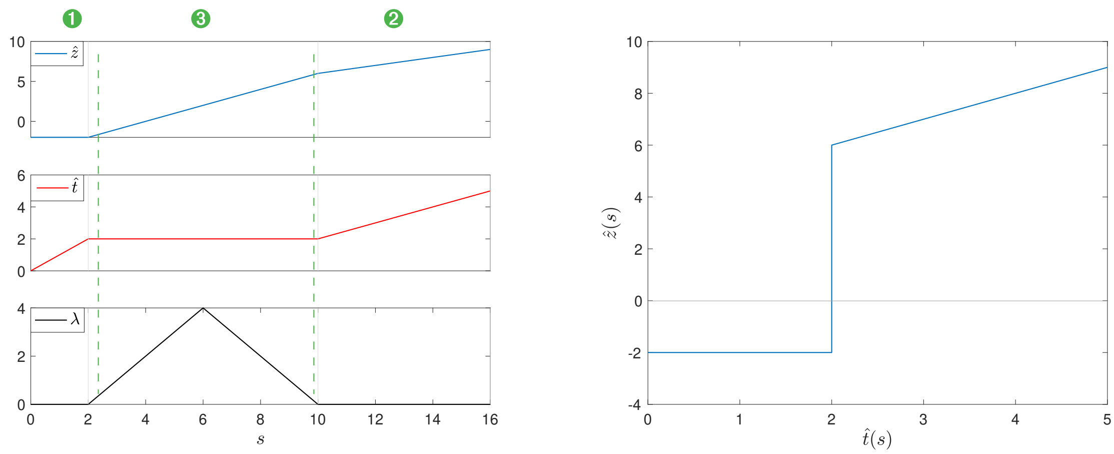

Figure 1: Left: Plot of the functions $\hat{z}$, $\hat{t}$ and λ (from top to bottom) depending on the artificial time s. The numbers indicate the different regimes *Sticking* ❶, *Rate-independent slip* ❷ and *Viscous jump* ❸. Right: Graph $\{(\hat{t}(s), \hat{z}(s)) \, : \, s \in [0, S]\} \subset [0, T] \times \mathbb{R}$ of the parametrized BV solution $(\hat{t}, \hat{z})$.

In preparation of the investigation of the optimization problem governed by (RIS), we provide the following boundedness results for normalized parametrized BV solutions:

Lemma 3.12. *For all $z_0 \in \mathbb{R}^n$, $\ell \in H^1(0,T;\mathbb{R}^n)$ and all normalized parametrized BV solutions $(S,\hat{t},\hat{z})$ associated with (z_0,ℓ) it holds with λ, κ, and μ from (2.2):*

$$\begin{aligned}\mathcal{I}(\hat{\ell}(s),\hat{z}(s)) + \int_0^s \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r),\hat{z}(r)))\mathrm{d}r \\ \leq \Big(\mathcal{I}(\ell(0),z_0) + \tfrac{\lambda\mu+1}{2}\|\ell\|^2_{H^1((0,T);\mathbb{R}^n)} + \tfrac{\lambda\kappa T}{2}\Big)\left(1+\tfrac{\lambda T}{2}\exp(\tfrac{\lambda T}{2})\right)\end{aligned} \tag{3.36}$$

for all $s \in [0,S]$.

Proof. The estimate is a consequence of the Gronwall inequality applied to the energy-dissipation identity (3.27). Indeed, the integral on the right hand side of (3.27) can be estimated as follows by using (A.1) from the appendix and assumption (2.2):

$$\begin{aligned}\int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}(r),\hat{z}(r))\hat{\ell}'(r)\mathrm{d}r &\leq \frac{1}{2}\int_0^s \|\nabla_\ell \mathcal{I}(\hat{\ell}(r),\hat{z}(r))\|^2 \hat{t}'(r)\mathrm{d}r + \frac{1}{2}\|\dot{\ell}\|^2_{L^2(0,T;\mathbb{R}^n)} \\ &\leq \int_0^s \frac{\lambda}{2}\big(\mathcal{I}(\hat{\ell}(r),\hat{z}(r)) + \kappa + \mu\|\hat{\ell}(r)\|^2\big)\hat{t}'(r)\mathrm{d}r + \frac{1}{2}\|\dot{\ell}\|^2_{L^2(0,T;\mathbb{R}^n)} \\ &\leq \int_0^s \frac{\lambda}{2}\big(\mathcal{I}(\hat{\ell}(r),\hat{z}(r)) + \kappa\big)\hat{t}'(r)\mathrm{d}r + \frac{\lambda\mu+1}{2}\|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}\end{aligned} \tag{3.37}$$

With $\alpha(s) := \mathcal{I}(\ell(0),z_0) + \frac{\lambda\mu+1}{2}\|\ell\|^2_{H^1((0,T);\mathbb{R}^n)} + \frac{\lambda\kappa}{2}\,\hat{t}(s)$, relation (3.27) leads to the following estimate:

$$\mathcal{I}(\hat{\ell}(s),\hat{z}(s)) \leq \alpha(s) + \int_0^s \frac{\lambda}{2}\,\hat{t}'(r)\,\mathcal{I}(\hat{\ell}(r),\hat{z}(r))\mathrm{d}r.$$

Hence, by the Gronwall inequality we find

$$\mathcal{I}(\hat{\ell}(s),\hat{z}(s)) \leq \alpha(s)\exp\Big(\int_0^s \frac{\lambda}{2}\,\hat{t}'(r)\mathrm{d}r\Big) \leq \alpha(S)\exp\big(\tfrac{\lambda T}{2}\big).$$

Inserting this into (3.37) and exploiting once more the energy equality from (3.27) we end up with (3.36). □

Corollary 3.13. *For every $L > 0$ there exists a constant $C_L > 0$ such that for all $z_0 \in \mathbb{R}^n$, $\ell \in H^1(0,T;\mathbb{R}^n)$ with $\|z_0\| + \|\ell\|_{H^1(0,T;\mathbb{R}^n)} \leq L$ and all normalized parametrized BV solutions $(S,\hat{t},\hat{z})$ associated with (z_0,ℓ), it holds*

$$S + \|\hat{z}\|_{L^\infty(0,S;\mathbb{R}^n)} + \int_0^S \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r),\hat{z}(r)))\mathrm{d}r \leq C_L. \tag{3.38}$$

Proof. By the normalization, i.e. $\hat{t}'(s) + \mathfrak{p}(\hat{z}'(s), -\nabla_z \mathcal{I}(\hat{\ell}(s),\hat{z}(s))) = 1$, the artificial end time fulfills

$$S = \int_0^S \hat{t}'(r) + \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r),\hat{z}(r)))\mathrm{d}r = T + \int_0^S \mathfrak{p}(\hat{z}'(r), -\nabla_z \mathcal{I}(\hat{\ell}(r),\hat{z}(r)))\mathrm{d}r.$$

Exploiting (3.36) and the compactness assumption in (2.3) gives (3.38). □

A natural question is, whether BV solutions are unique and whether or not every normalized parametrized BV solution can be approximated by the above viscosity scheme. In the non-convex case there exist examples that show that both questions have a negative answer. Such an example is for instance given in [MS20, Section 2.4]. However, if we also allow the function ℓ to vary with the parameter ε, then the question of approximability can be answered positively, at least partly. This essentially forms the basis for our reverse approximation result in Section 5. Prior to that, we will, however, take a look at an optimization problem whose constraint is given by the rate-independent evolution (RIS). The main result here is the existence of an optimal solution.

4 Existence of globally optimal solutions

The existence of globally optimal solutions to optimal control problems governed by rate-independent systems is already addressed in [KT18], even in a spatially distributed setting, where the underlying spaces are Banach spaces and not just $\mathbb{R}^n$. However, since one can employ much easier and more direct arguments in the finite dimensional case and the results are slightly sharper, we present the proof of existence in detail for convenience of the reader.

Our control variable is the applied load ℓ, while we assume the initial state z_0 to be fixed for the rest of the paper. For the precise formulation of our optimal control problem, let us introduce the following set:

$$\begin{aligned}\mathcal{L}(\ell) := \big\{(S,\hat{t},\hat{z}) \in\ & [T,\infty)\times W^{1,\infty}(0,S)\times W^{1,1}(0,S;\mathbb{R}^n) :\\ & (S,\hat{t},\hat{z}) \text{ is a } \textit{normalized parametrized BV solution} \text{ of (RIS) associated with } \ell\big\},\end{aligned} \tag{4.1}$$

cf. Definition 3.6. Then the optimal control problem under consideration reads as follows:

$$\left.\begin{aligned} \min\quad & J(S,\hat{z},\ell) := j(\hat{z}(S)) + \frac{\beta}{2}\,\|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}\\ \text{s.t.}\quad & \ell\in H^1(0,T;\mathbb{R}^n), \quad (S,\hat{t},\hat{z})\in\mathcal{L}(\ell),\\ & -\nabla_z\mathcal{I}(\ell(0),z_0)\in\partial\mathcal{R}(0), \quad -\nabla_z\mathcal{I}(\ell(T),\hat{z}(S))\in\partial\mathcal{R}(0). \end{aligned}\right\} \tag{OCP}$$

Remark 4.1. *The objective in* (OCP) *provides an end time observation of the state variable z, which is meaningful in many applications, where the goal is to reach a desired final state. Other types of objectives such as integral type functionals can easily be incorporated into the subsequent analysis. To keep the discussion concise we however focus on objectives of the form in* (OCP)*.*

Remark 4.2. *The additional end time constraint in* (OCP) *enforces the final state to be locally stable. This is to ensure that the optimal trajectory does not stop during a viscous jump, which would mean that the final state is not seen in the physical time, which would certainly make no sense from an application point of view.*

It is to be noted that, given a load $\ell\in H^1(0,T;\mathbb{R}^n)$ and an associated normalized parametrized BV solution $(S,\hat{t},\hat{z})\in\mathcal{L}(\ell)$, one can always find a point $s^*\in(0,S]$ such that $\hat{t}(s^*)=T$ and $-\nabla_z\mathcal{I}(\hat{\ell}(s^*),\hat{z}(s^*))\in\partial\mathcal{R}(0)$, i.e., if we restrict the trajectory to $[0,s^*]$, then the final time condition is fulfilled. This is seen as follows: We define s^* by

$$s^* := \arg\max\Big\{s\in[0,S] : \operatorname{dist}\big(-\nabla_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s),\partial\mathcal{R}(0)\big)=0\Big\}.$$

Note that the set $\{s\in[0,S] : \operatorname{dist}(-\nabla_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s),\partial\mathcal{R}(0))=0\}$ is non-empty, since otherwise $\hat{t}'=0$ a.e. in $(0,S)$ by the complementarity in (3.26)*, which contradicts $\hat{t}(S)=T$. Moreover, this set is compact by the continuity of ℓ, $\hat{t}$, and $\hat{z}$ due to the compactness of $H^1(0,T;\mathbb{R}^n)\hookrightarrow C([0,T];\mathbb{R}^n)$ and thus, s^* is well defined. If $s^*=S$, then the assertion is fulfilled because $\hat{t}(S)=T$. If $s^*<S$, then $\operatorname{dist}(-\nabla_z\mathcal{I}(\hat{\ell}(s),\hat{z}(s),\partial\mathcal{R}(0))>0$ for all $s\in(s^*,S]$ and again by the complementarity in* (3.26)*, we have $\hat{t}'=0$ a.e. in $(s^*,S]$. Hence $\hat{t}$ is constant in $(s^*,S]$ and the continuity of $\hat{t}$ along with $\hat{t}(S)=T$ gives $\hat{t}(s^*)=T$.*

Remark 4.3. *The initial time constraint in* (OCP) *allows us to employ the results of Lemma 3.2 that will be useful in Section 5 below. For the mere existence of optimal solutions, this is not needed, but, for reasons of clarity, we already included it here.*

Given a load $\ell\in H^1(0,T;\mathbb{R}^n)$ and an associated normalized parametrized BV solution $(S,\hat{t},\hat{z})\in\mathcal{L}(\ell)$, we will frequently use the notation $\hat{\ell}:=\ell\circ\hat{t}$ in the following.

Theorem 4.4 (Existence of optimal solutions)**.** *There exists a globally optimal solution*

$$(\hat{S}^*,\hat{t}^*,\hat{z}^*,\ell^*)\in[T,\infty)\times W^{1,\infty}(0,S^*)\times W^{1,\infty}(0,S^*;\mathbb{R}^n)\times H^1(0,T;\mathbb{R}^n)$$

to the optimization problem (OCP)*.*

Proof. The proof is very similar to the one of Theorem 3.5. Nevertheless, we shortly sketch the arguments for convenience of the reader. First of all, the feasible set of (OCP) is non-empty, since, by our standing assumption $-\nabla_z\mathcal{I}(\ell_0, z_0) \in \partial\mathcal{R}(0)$ so that the constant tuple (z_0, ℓ_0) together with $\hat{t} = \mathrm{id}$ (identity) is a normalized parametrized BV solution that, additionally, also fulfills the initial and end time condition in (OCP) and is therefore feasible.

Thus there exists an infimal sequence. Let $(\ell_n, S_n, \hat{t}_n, \hat{z}_n)$ be such a sequence, i.e.,

$$J(S_n, \hat{z}_n, \ell_n) \to J^* := \inf\{J(S, \hat{z}, \ell) : \ell \in H^1(0, S; \mathbb{R}^n), (S, \hat{t}, \hat{z}) \in \mathcal{L}(\ell)\}$$

with $\mathcal{L}(\ell)$ from (4.1). Due to the Tikhonov-term in the objective and the boundedness assumption on j, the sequence $\{\ell_n\}$ is bounded in $H^1(0, T; \mathbb{R}^n)$ so that there exists a weakly converging subsequence, which we denote by the same symbol for simplicity, i.e.,

$$\ell_n \rightharpoonup \ell^* \quad \text{in } H^1(0, T; \mathbb{R}^n). \tag{4.2}$$

Therefore, Corollary 3.13 implies that $\{S_n\} \subset [T, \infty)$ is also bounded, and consequently, there is a converging subsequence, too, again unrelabeled for simplicity, i.e.,

$$S_n \to S^* \in [T, \infty). \tag{4.3}$$

Moreover, the conditions in (3.25) imply the boundedness of $\{\|\hat{t}_n\|_{W^{1,\infty}(0,S_n)}\}$ and, thanks to the assumption on $\mathcal{R}$ in (2.4), the last equation in (3.25) together with the definition of $\mathfrak{p}$ yields that $\{\|\hat{z}_n'\|_{L^\infty(0,S_n;\mathbb{R}^n)}\}$ is bounded, too. Corollary 3.13 and the weak convergence of $\{\ell_n\}$ imply the same for $\{\|\hat{z}_n\|_{L^\infty(0,S_n;\mathbb{R}^n)}\}$. Therefore, possibly after an extension of $\hat{t}_n$ and $\hat{z}_n$ to the time interval $[0, S^*]$, for instance by constant continuation, we may select weakly convergent subsequences, w.l.o.g. again denoted by the same symbols, i.e.,

$$\hat{t}_n \overset{*}{\rightharpoonup} \hat{t}^* \quad \text{in } W^{1,\infty}(0, S^*) \quad \text{and} \quad \hat{z}_n \overset{*}{\rightharpoonup} \hat{z}^* \quad \text{in } W^{1,\infty}(0, S^*; \mathbb{R}^n). \tag{4.4}$$

By the weak-$*$ closedness of the set of non-negative functions in $W^{1,\infty}(0, S^*)$ and compactness of the embedding $W^{1,\infty}(0, S^*) \hookrightarrow C([0, S^*])$, we see that the weak limit $\hat{t}^*$ fulfills $(\hat{t}^*)'(s) \geq 0$ a.e. in $(0, S^*)$ and $\hat{t}^*(S^*) = T$, i.e., the final time condition in (3.24) and the sign condition in (3.25). Moreover, the uniform convergence of $\hat{z}_n$ yields $\hat{z}^*(0) = z_0$, i.e., the initial condition in (3.24). In order to pass to the limit in the energy identity (3.27), let $s \in [0, S^*)$ be fixed, but arbitrary. Then, thanks to (4.3), there is an index $N_s \in \mathbb{N}$ (depending on s) so that $s < S_n$ for all $n \geq N_s$. Consequently, (3.27) holds for $n \geq N_s$, i.e.

$$\begin{aligned}
\mathcal{I}(\ell_n(\hat{t}_n(s)), \hat{z}_n(s)) + \int_0^s \Big(\mathcal{R}(\hat{z}_n'(\sigma)) + \|\hat{z}_n'(\sigma)\| \operatorname{dist}\big(-\nabla_z\mathcal{I}(\ell_n(\hat{t}_n(\sigma)), \hat{z}_n(\sigma)), \partial\mathcal{R}(0)\big)\Big)\mathrm{d}\sigma \\
= \mathcal{I}(\ell_n(\hat{t}_n(0)), z_0) + \int_0^s \nabla_\ell \mathcal{I}(\hat{\ell}_n(\sigma), \hat{z}_n(\sigma))\hat{\ell}_n'(\sigma)\mathrm{d}\sigma.
\end{aligned}$$

In view of the uniform convergence of $\{\hat{t}_n\}$, $\{\hat{z}_n\}$, and $\{\ell_n\}$ by the compact embedding $H^1(0, S^*) \hookrightarrow C([0, S^*])$, the passage to the limit in this energy identity follows exactly by the same arguments as in the proof of Theorem 3.5. The convergence of the integral involving the load follows again from Corollary 3.4 and the lower-semicontinuity result for $\mathfrak{p}(\cdot, \cdot)$ from [MRS09, Lemma 3.1] again yield an inequality analogous to (3.31). This first gives an energy inequality, which, thanks to Lemma A.2, indeed holds with equality, which is the desired energy identity in the limit. Since $s \in [0, S^*)$ was arbitrary, the energy identity holds for every $s \in [0, S^*)$ and, as $\hat{t}^*$, ℓ^*, and $\hat{z}^*$ are continuous, also for $s = S^*$.

The normalization condition in (3.25) is derived as in the proof of Theorem 3.5, too. Completely analogously to (3.32), one deduces

$$s = \int_0^s (\hat{t}^*)'(r) + \mathfrak{p}\big((\hat{z}^*)'(r), -\nabla_z\mathcal{I}(\ell^*(\hat{t}^*(r)), \hat{z}^*(r))\big)\mathrm{d}r$$

and an argument by contradiction analogous to (3.33) gives the last equation in (3.26).

From the uniform convergence of $\{\hat{t}_n\}$, $\{\hat{z}_n\}$, and $\{\ell_n\}$, the weak convergence of $\{\hat{t}_n'\}$, and the Lipschitz continuity of dist, we obtain

$$0 = \int_0^{S_n} \hat{t}_n'(r) \operatorname{dist}\big(-\nabla_z \mathcal{I}(\ell_n(\hat{t}_n(r)), \hat{z}_n(r)), \partial\mathcal{R}(0)\big) \mathrm{d}r \\ \to \int_0^{S^*} (\hat{t}^*)'(r) \operatorname{dist}\big(-\nabla_z \mathcal{I}(\ell^*(\hat{t}^*(r)), \hat{z}^*(r)), \partial\mathcal{R}(0)\big) \mathrm{d}r.$$

Since the integrand is non-negative almost everywhere, we obtain (3.26) such that $(S^*, \hat{t}^*, \hat{z}^*)$ is indeed a normalized parametrized BV solution associated with ℓ^*, i.e., $(S^*, \hat{t}^*, \hat{z}^*) \in \mathcal{L}(\ell^*)$.

Finally, the initial and end time condition follows from (4.3), the uniform convergence of $\{\ell_n\}$ and $\{\hat{z}_n\}$, the continuity of $\nabla_z \mathcal{I}$, and the closedness of $\partial\mathcal{R}(0)$. □

5 Reverse approximation

For the rest of this paper we assume that the energy is linear in ℓ, i.e.

$$\mathcal{I}(\ell, z) = \mathcal{E}(z) - \langle \ell, z \rangle \tag{5.1}$$

with a function $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ satisfying the following

Assumption 5.1 (Structural assumptions on the energy functional)**.**

(a) The nonlinear part $\mathcal{E}$ of the energy fulfills $\mathcal{E} \in C^2(\mathbb{R}^n; \mathbb{R})$. Moreover, $\mathcal{E}$ is such that (2.2) and (2.3) hold true and its Hessian is Lipschitz continuous on bounded sets, i.e.,

$$\|\nabla^2 \mathcal{E}(z_1) - \nabla^2 \mathcal{E}(z_2)\| \leq L(r)\|z_1 - z_2\| \quad \forall\, z_1, z_2 \in B(0, r) \tag{5.2}$$

with a constant $L(r) \geq 0$ depending only on the radius $r > 0$.

(b) Moreover, we assume that there is a constant $K \geq 0$ such that $\|\nabla^2 \mathcal{E}(z)\| \leq K$ for all $z \in \mathbb{R}^n$. This boundedness assumption can be avoided, see Corollary 5.10 below, but in order to ease the following arguments, let us suppose it holds for the time being.

Note that the energy functional in Example 2.1 is exactly of this form provided that the nonlinearity f is twice continuously differentiable with a Hessian that fulfills the above Lipschitz and boundedness condition.

As already announced, this section is devoted to the approximation of normalized parametrized BV solutions by means of viscous regularization. Since the set of normalized parametrized BV solutions associated with a given load is in general no singleton, while the viscous equation always admits a unique solution as seen in Section 3.1, there is no hope that every normalized parametrized BV solution can be approximated via viscous regularization. However, in the context of optimal control, we can also vary the loads, which gives us additional flexibility, as already explained in the introduction. Unfortunately, this does still not suffice to construct an approximating sequence of viscous solutions. We additionally need the following

Assumption 5.2 (Critical continuity assumption)**.** *Let $\ell \in H^1(0, T; \mathbb{R}^n)$ and $z_0 \in \mathbb{R}^n$ with $-\nabla_z \mathcal{I}(\ell(0), z_0) \in \partial\mathcal{R}(0)$ be given. We assume that, among all normalized parametrized BV solutions associated with (z_0, ℓ), there is at least one, denoted by $(S, \hat{t}, \hat{z})$, that fulfills*

$$\exists\, \delta \in (0, 1] : \quad \hat{t}'(s) \geq \delta \quad \textit{f.a.a. } s \in [0, S]. \tag{5.3}$$

This is cleary a very strong assumption on the regularity of the solution $(S, \hat{t}, \hat{z})$. It implies that the physical time always proceeds and no (viscous or rate-independent) jump occurs. In principle, Assumption 5.2 is equivalent to the existence of a differential solution, as defined in Definition 3.9. This is seen as follows: Thanks to (5.3), $\hat{t}$ is invertible and the inverse function $\sigma = \hat{t}^{-1}$ is Lipschitz continuous with a Lipschitz constant that is bounded by δ^{-1}. Hence, the function $\tilde{z} := \hat{z} \circ \sigma$ belongs to $W^{1,\infty}(0,T;\mathbb{R}^n)$ and, since (5.3) yields that the associated multiplier λ vanishes almost everywhere (cf. the second equation in (3.34)), we deduce from the first equation in (3.34) that, indeed,

$$0 \in \partial\mathcal{R}(\dot{\tilde{z}}(t)) + \nabla_z \mathcal{I}(\ell(t), \tilde{z}(t)), \quad \tilde{z}(0) = z_0 \tag{5.4}$$

is satisfied for almost all $t \in [0,T]$.

Remark 5.3. *As already emphasized in the context of Definition 3.9, the existence of a differential solution can in general not be guaranteed in case of a non-convex energy. Assumption 5.2 is therefore rather restrictive. In context of our optimal control problem, we need to assume that there is at least one optimal solution, which fulfills Assumption 5.2, see Theorem 6.3 below. We underline that we do not require every optimal solution to be of this form, which would really be very restrictive.*

Remark 5.4. *It is to be noted that (5.4) along with the embedding of $H^1(0,T;\mathbb{R}^n)$ in $C([0,T];\mathbb{R}^n)$ implies that $-\nabla_z \mathcal{I}(\ell(t), \tilde{z}(t)) \in \partial\mathcal{R}(0)$ for every $t \in [0,T]$. Thus the initial time constraint in (OCP) is a mandatory prerequisite on the initial value of ℓ for the existence of a differential solution.*

The construction of the approximation sequence is not only based on viscous regularization, but also includes a *quadratic penalty term*, which we add to the energy. The penalized energy reads $\mathcal{I}(\ell, z) + \frac{\eta}{2}\|z - \tilde{z}(t)\|^2$ with a penalization parameter $\eta > 0$ that is potentially large, but *finite*. This leads to the following regularized system

$$0 \in \partial\mathcal{R}(\dot{z}_\varepsilon(t)) + \varepsilon\, \dot{z}_\varepsilon(t) + \nabla_z \mathcal{I}(\ell(t), z_\varepsilon(t)) + \eta(z_\varepsilon(t) - \tilde{z}(t)), \quad z_\varepsilon(0) = z_0 \tag{5.5}$$

where $\varepsilon > 0$ is arbitrary and $\eta > 0$ is chosen such that the new energy

$$\mathcal{I}_\eta(\ell, z) := \mathcal{I}(\ell, z) + \frac{\eta}{2}\|z\|^2 \tag{5.6}$$

is uniformly convex with constant $\alpha > 0$, i.e.,

$$\nabla^2_{zz}\mathcal{I}_\eta(\ell, z)[s,s] \geq \alpha\|s\|^2 \quad \forall\, (\ell, z, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n.$$

Note that this is always possible, since $\nabla^2_{zz}\mathcal{I}_\eta(\ell, z) = \nabla^2\mathcal{E}(z) + \eta$ and $\nabla^2\mathcal{E}$ is bounded by Assumption 5.1, see also Corollary 5.10 below. With $\mathcal{I}_\eta$ at hand, we can rewrite the above inclusion equivalently as

$$0 \in \partial\mathcal{R}(\dot{z}_\varepsilon(t)) + \varepsilon\, \dot{z}_\varepsilon(t) + \nabla_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\varepsilon(t)), \quad z_\varepsilon(0) = z_0. \tag{RIS$_{\varepsilon,\eta}$}$$

From Proposition 3.1, we know that there exists a solution $z_\varepsilon \in H^2(0,T;\mathbb{R}^n)$ of (RIS$_{\varepsilon,\eta}$). From Lemma 3.2, we furthermore have

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{z}_\varepsilon\|^2 + \nabla^2_{zz}\mathcal{I}_\eta(\ell + \eta\tilde{z}, z_\varepsilon)[\dot{z}_\varepsilon, \dot{z}_\varepsilon] - \langle \dot{\ell} + \eta\dot{\tilde{z}}, \dot{z}_\varepsilon\rangle = 0 \quad \text{a.e. in } [0,T]. \tag{5.7}$$

If we now define

$$\mathcal{E}_\eta(z) := \mathcal{E}(z) + \frac{\eta}{2}\|z\|^2$$

and exploit once more the explicit form of the energy $\mathcal{I}$, then we can rewrite (5.7) as

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{z}_\varepsilon\|^2 + \nabla^2\mathcal{E}_\eta(z_\varepsilon)[\dot{z}_\varepsilon, \dot{z}_\varepsilon] - \langle \dot{\ell} + \eta\dot{\tilde{z}}, \dot{z}_\varepsilon\rangle = 0 \quad \text{a.e. in } [0,T], \tag{5.8}$$

which is the starting point for the following auxiliary result. Note that $\nabla^2\mathcal{E}_\eta(z) = \nabla^2_{zz}\mathcal{I}_\eta(\ell, z)$, such that $\mathcal{E}_\eta$ is also uniformly convex.

Lemma 5.5. *Under the Assumption 5.2 the solutions z_ε of* ($\mathrm{RIS}_{\varepsilon,\eta}$) *fulfill*

$$z_\varepsilon \rightharpoonup \tilde{z} \quad \text{in } H^1(0,T;\mathbb{R}^n) \quad \text{as } \varepsilon \searrow 0, \tag{5.9}$$

where $\tilde{z} = \hat{z} \circ \hat{t}^{-1}$ is the solution satisfying (5.4).

Proof. We split the proof into three steps.

(i) Boundedness of z_ε:

We first use (5.8) to show that z_ε is bounded in $H^1(0,T;\mathbb{R}^n)$. For this we integrate (5.8) from 0 to T and exploit $\|\dot{z}_\varepsilon(0)\|_{\mathbb{V}} = 0$ from Lemma 3.2 (which is applicable due to $-\nabla_z \mathcal{I}(\ell(0), z_0) \in \partial\mathcal{R}(0)$ by Assumption 5.2) and the uniform convexity of $\nabla^2 \mathcal{E}_\eta$ with constant α to obtain

$$\begin{aligned} 0 &= \frac{\varepsilon}{2}\|\dot{z}_\varepsilon(T)\|^2 - \frac{\varepsilon}{2}\|\dot{z}_\varepsilon(0)\|^2 + \int_0^T \nabla^2 \mathcal{E}_\eta(z_\varepsilon(t))[\dot{z}_\varepsilon(t), \dot{z}_\varepsilon(t)] - \langle \dot{\ell}(t) + \eta\dot{\tilde{z}}(t), \dot{z}_\varepsilon(t)\rangle \mathrm{d}t \\ &\geq \alpha \int_0^T \|\dot{z}_\varepsilon(t)\|^2 - \langle \dot{\ell}(t) + \eta\dot{\tilde{z}}(t), \dot{z}_\varepsilon(t)\rangle \mathrm{d}t. \end{aligned}$$

Hence

$$\|\dot{z}_\varepsilon\|_{L^2(0,T;\mathbb{R}^n)} \leq \frac{1}{\alpha}\left(\|\dot{\ell}\|_{L^2(0,T;\mathbb{R}^n)} + \eta\|\dot{\tilde{z}}\|_{L^2(0,T;\mathbb{R}^n)}\right), \tag{5.10}$$

which, together with the initial conditions, gives the claimed boundedness in $H^1(0,T;\mathbb{R}^n)$. There is thus a weakly convergent subsequence and w.l.o.g. we assume that the whole sequence convergence weakly to simplify the notation, i.e., $z_\varepsilon \rightharpoonup z^*$ in $H^1(0,T;\mathbb{R}^n)$ as $\varepsilon \searrow 0$. (At the end of the proof, we will see that the weak limit is unique so that indeed the whole sequence converges.)

(ii) The weak limit is a differential solution:

In order to show that z^* is a differential solution of (5.4), we reformulate ($\mathrm{RIS}_{\varepsilon,\eta}$) as

$$\mathcal{R}(\dot{z}_\varepsilon(t)) - \mathcal{R}(v) + \varepsilon\langle \dot{z}_\varepsilon(t), \dot{z}_\varepsilon(t) - v\rangle + \langle \nabla_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\varepsilon(t)), \dot{z}_\varepsilon(t) - v\rangle \leq 0 \quad \forall\, v \in \mathbb{R}^n,$$

which holds for almost all $t \in [0,T]$. Now let $\varphi \in C_c^\infty(0,T)$ with $\varphi \geq 0$ be arbitrary. Then the above inequality implies for all $v \in \mathbb{R}^n$

$$\int_0^T \Big(\mathcal{R}(\dot{z}_\varepsilon(t)) - \mathcal{R}(v) + \varepsilon\langle \dot{z}_\varepsilon(t), \dot{z}_\varepsilon(t) - v\rangle + \langle \nabla_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\varepsilon(t)), \dot{z}_\varepsilon(t) - v\rangle\Big)\varphi(t)\,\mathrm{d}t \leq 0. \tag{5.11}$$

Since $\nabla_z \mathcal{I}_\eta$ as well as ℓ and $\tilde{z}$ are continuous by assumption and z_ε converges uniformly to z^* because of the compactness of $H^1(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$, we have that $\nabla_z \mathcal{I}_\eta(\ell + \eta\tilde{z}, z_\varepsilon) \to \nabla_z \mathcal{I}_\eta(\ell + \eta\tilde{z}, z^*)$ uniformly in $[0,T]$. Together with the weak lower semicontinuity of the mapping $z \mapsto \int_0^T \mathcal{R}(z)\varphi \mathrm{d}t$, the boundedness of z_ε in $H^1(0,T;\mathbb{R}^n)$, and the weak convergence of $\dot{z}_\varepsilon$, this allows us to pass to the limit $\varepsilon \searrow 0$ in (5.11) to obtain

$$\begin{aligned} 0 &\geq \liminf_{\varepsilon \searrow 0} \int_0^T \Big(\mathcal{R}(\dot{z}_\varepsilon(t)) - \mathcal{R}(v) + \varepsilon\langle \dot{z}_\varepsilon(t), \dot{z}_\varepsilon(t) - v\rangle + \langle \nabla_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z_\varepsilon(t)), \dot{z}_\varepsilon(t) - v\rangle\Big)\varphi(t)\,\mathrm{d}t \\ &\geq \int_0^T \Big(\mathcal{R}(\dot{z}^*(t)) - \mathcal{R}(v) + \langle \nabla_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z^*(t)), \dot{z}^*(t) - v\rangle\Big)\varphi(t)\,\mathrm{d}t. \end{aligned}$$

Since this holds for all $\varphi \in C_c^\infty(0,T)$ with $\varphi \geq 0$, the fundamental theorem of calculus of variations implies

$$0 \in \partial\mathcal{R}(\dot{z}^*(t)) + D_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z^*(t)) \quad \text{f.a.a. } t \in [0,T] \tag{5.12}$$

such that z^* is indeed a differential solution.

(ii) $z^* = \tilde{z}$:

By construction of $\mathcal{I}_\eta$, there holds $\nabla_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), z) = \nabla_z \mathcal{I}(\ell(t), z) + \eta(z - \tilde{z}(t))$ for all $t \in [0,T]$ and all

$z \in \mathbb{R}^n$. Thus, $\nabla_z \mathcal{I}_\eta(\ell(t) + \eta\tilde{z}(t), \tilde{z}(t)) = \nabla_z \mathcal{I}(\ell(t), \tilde{z}(t))$ and therefore, (5.4) implies that $\tilde{z}$ is a differential solution of (5.12), too. However, according to [MR07, MT04], differential solutions are *unique* in case of uniformly convex energies, cf. also Appendix B. Note that, at this point, the Lipschitz continuity of $\nabla^2\mathcal{E}$ on bounded sets is needed, which is ensured by Assumption 5.1. Thus, we obtain $z^* = \tilde{z}$ and a well known argument by contradiction gives the weak convergence of the whole sequence. □

In view of Section 6 the weak convergence is not enough to ensure approximibability of the optimal control by viscous regularization. We in fact need strong convergence of z_ε for which we need the following Lemma:

Lemma 5.6. *The solution $\tilde{z} \in W^{1,\infty}(0,T;\mathbb{R}^n)$ of* (5.4) *(whose existence is guaranteed by Assumption* 5.2*) satisfies*

$$\nabla^2\mathcal{E}(\tilde{z}(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)] - \langle \dot{\ell}(t), \dot{\tilde{z}}(t)\rangle = 0 \quad \text{f.a.a. } t \in [0,T]. \tag{5.13}$$

Proof. Due to $\partial\mathcal{R}(v) \subset \partial\mathcal{R}(0)$ for all $v \in \mathbb{R}^n$, the embedding $W^{1,\infty}(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$, the continuity of $D_z\mathcal{I}$ and the closedness of $\partial\mathcal{R}(0)$, we deduce from (5.4) that

$$0 \in \partial\mathcal{R}(0) + \nabla_z\mathcal{I}(\ell(t), \tilde{z}(t)) \quad \forall\, t \in [0,T].$$

Exploiting the 1-homogeneity of $\mathcal{R}$ we can rewrite this inclusion and (5.4) by

$$\mathcal{R}(\dot{\tilde{z}}(t)) = \langle -\nabla_z\mathcal{I}(\ell(t), \tilde{z}(t)), \dot{\tilde{z}}(t)\rangle \quad \text{f.a.a. } t \in [0,T] \tag{5.14}$$

$$\forall\, v \in \mathbb{R}^n: \qquad \mathcal{R}(v) \geq \langle -\nabla_z\mathcal{I}(\ell(\tau), \tilde{z}(\tau)), v\rangle \qquad \forall\, \tau \in [0,T]. \tag{5.15}$$

Now, let $t \in (0,T)$ be a Lebesgue point of $\dot{\tilde{z}}$ for which (5.14) holds. Testing (5.15) with $\dot{\tilde{z}}(t)$, inserting $\tau = t \pm h$ as well as the explicit form of $\mathcal{I}$ and substracting (5.14) thereof, we obtain

$$0 \leq \langle \nabla\mathcal{E}(\tilde{z}(t \pm h)) - \nabla\mathcal{E}(\tilde{z}(t)), \dot{\tilde{z}}(t)\rangle - \langle \ell(t \pm h) - \ell(t), \dot{\tilde{z}}(t)\rangle.$$

We then divide by $h > 0$, pass to the limit $h \searrow 0$ and obtain, using Lebesgue's differentiation theorem,

$$\nabla^2\mathcal{E}(\tilde{z}(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)] - \langle \dot{\ell}(t), \dot{\tilde{z}}(t)\rangle = 0.$$

Since this holds for almost all $t \in (0,T)$, we end up with (5.13). □

Lemma 5.7. *Let the Assumption* 5.2 *hold and let $\tilde{z}$ denote the corresponding solution satisfying* (5.4)*. Then the solutions z_ε of* (RIS$_{\varepsilon,\eta}$) *fulfill*

$$z_\varepsilon \to \tilde{z} \quad \text{in } H^1(0,T;\mathbb{R}^n). \tag{5.16}$$

Proof. We start with the equality in Lemma 5.6 and rewrite it using the new energy $\mathcal{E}_\eta$:

$$0 = \nabla^2\mathcal{E}(\tilde{z}(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)] - \langle \dot{\ell}(t), \dot{\tilde{z}}(t)\rangle = \nabla^2\mathcal{E}_\eta(\tilde{z}(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)] - \langle \dot{\ell}(t) + \eta\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)\rangle \tag{5.17}$$

for almost all $t \in [0,T]$. Subtracting (5.17) from (5.8) and integrating yields

$$\begin{aligned}
0 &= \frac{\varepsilon}{2}\|\dot{z}_\varepsilon(T)\|^2 - \frac{\varepsilon}{2}\|\dot{z}_\varepsilon(0)\|^2 \\
&\quad + \int_0^T \nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{z}_\varepsilon(t), \dot{z}_\varepsilon(t)] - \nabla^2\mathcal{E}_\eta(\tilde{z}(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)]\,\mathrm{d}t - \int_0^T \langle \dot{\ell} + \eta\dot{\tilde{z}}, \dot{z}_\varepsilon(t) - \dot{\tilde{z}}(t)\rangle\,\mathrm{d}t \\
&= \frac{\varepsilon}{2}\|\dot{z}_\varepsilon(T)\|^2 + \int_0^T \nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{z}_\varepsilon(t) - \dot{\tilde{z}}(t), \dot{z}_\varepsilon(t) - \dot{\tilde{z}}(t)]\,\mathrm{d}t - \int_0^T \langle \dot{\ell} + \eta\dot{\tilde{z}}, \dot{z}_\varepsilon(t) - \dot{\tilde{z}}(t)\rangle\,\mathrm{d}t \\
&\quad + \int_0^T 2\nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{\tilde{z}}(t), \dot{z}_\varepsilon(t)] - \nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)] - \nabla^2\mathcal{E}_\eta(\tilde{z}(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)]\,\mathrm{d}t \\
&\geq \alpha \int_0^T \|\dot{z}_\varepsilon(t) - \dot{\tilde{z}}(t)\|^2\mathrm{d}t - \int_0^T \langle \dot{\ell} + \eta\dot{\tilde{z}}, \dot{z}_\varepsilon(t) - \dot{\tilde{z}}(t)\rangle\,\mathrm{d}t \\
&\quad + \int_0^T 2\nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{\tilde{z}}(t), \dot{z}_\varepsilon(t)] - \nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)] - \nabla^2\mathcal{E}_\eta(\tilde{z}(t))[\dot{\tilde{z}}(t), \dot{\tilde{z}}(t)]\,\mathrm{d}t,
\end{aligned}$$

where we exploited the uniform convexity of $\mathcal{E}_\eta$, as well as $\|\dot z_\varepsilon(0)\| = 0$ from Lemma 3.2. The goal now is to show that the last two integral-terms on the right-hand side converge to zero as $\varepsilon \searrow 0$, which yields that in fact z_ε strongly converges to $\tilde z$ in $H^1(0,T;\mathbb{R}^n)$. The first integral involving $\dot\ell$ clearly converges to zero due to the weak convergence of z_ε by Lemma 5.6. Furthermore, thanks to the compactness of $H^1(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$, z_ε converges uniformly to z^* and therefore the continuity of $\nabla^2\mathcal{E}_\eta$ yields that

$$\nabla^2\mathcal{E}_\eta(z_\varepsilon(t))\dot{\tilde z}(t) \to \nabla^2\mathcal{E}_\eta(\tilde z(t))\dot{\tilde z}(t) \quad \text{pointwise a.e. in } [0,T] \text{ as } \varepsilon \searrow 0.$$

Moreover, the boundedness assumption on $\nabla^2\mathcal{E}$ implies $\|\nabla^2\mathcal{E}_\eta(z_\varepsilon(t))\dot{\tilde z}(t)\| \le (K+\eta)\|\dot{\tilde z}(t)\|$ f.a.a. $t \in [0,T]$. Since $\|\dot{\tilde z}(t)\| \in L^2(0,T)$, Lebesgue's dominated convergence theorem implies

$$\nabla^2\mathcal{E}_\eta(z_\varepsilon)\dot{\tilde z} \to \nabla^2\mathcal{E}_\eta(\tilde z)\dot{\tilde z} \quad \text{in } L^2(0,T;\mathbb{R}^n).$$

Combining this with the weak convergence of $\dot z_\varepsilon$ in $L^2(0,T;\mathbb{R}^n)$ we finally obtain

$$\int_0^T 2\nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{\tilde z}(t), \dot z_\varepsilon(t)] - \nabla^2\mathcal{E}_\eta(z_\varepsilon(t))[\dot{\tilde z}(t), \dot{\tilde z}(t)] - \nabla^2\mathcal{E}_\eta(\tilde z(t))[\dot{\tilde z}(t), \dot{\tilde z}(t)]\,\mathrm{d}t \to 0,$$

which proves the claim. □

Lemma 5.8. *Let the Assumption 5.2 hold and let $\tilde z = \hat z \circ \hat t^{-1}$ denote the corresponding solution satisfying (5.4). Then there is a constant $\tilde C > 0$, depending on ℓ and $\tilde z$, but not on ε, such that*

$$\operatorname{dist}\big(-\nabla\mathcal{E}(z_\varepsilon(T)) - \ell(T) - \eta(z_\varepsilon(T) - \tilde z(T)), \partial\mathcal{R}(0)\big) \le \tilde C\,\varepsilon^{1/4}. \tag{5.18}$$

Proof. We first derive an estimate of z_ε in $W^{1,\infty}(0,T;\mathbb{R}^n)$. For this purpose, let us return to (5.8), which together with Lemma 3.2 gives the following estimate for almost every $t \in [0,T]$:

$$\begin{aligned} \frac{\varepsilon}{2}\|\dot z_\varepsilon(t)\|^2 &= -\int_0^t \nabla^2\mathcal{E}_\eta(z_\varepsilon(r))[\dot z_\varepsilon(r), \dot z_\varepsilon(r)] - \langle \dot\ell(r) + \eta\dot{\tilde z}(r), \dot z_\varepsilon(r)\rangle \mathrm{d}r \\ &\le \|\dot\ell + \eta\dot{\tilde z}\|_{L^2(0,T;\mathbb{R}^n)}\|\dot z_\varepsilon\|_{L^2(0,T;\mathbb{R}^n)} \le \alpha^{-1}\big(\|\dot\ell\|_{L^2(0,T;\mathbb{R}^n)} + \eta\|\dot{\tilde z}\|_{L^2(0,T;\mathbb{R}^n)}\big)^2, \end{aligned} \tag{5.19}$$

where we used the convexity of $\mathcal{E}_\eta$ and the bound for $\|\dot z_\varepsilon\|_{L^2(0,T;\mathbb{R}^n)}$ by (5.10). Now, recall the regularized equation (5.5), which, thanks to (5.1), can be reformulated as

$$\dot z_\varepsilon(t) = \partial\mathcal{R}_\varepsilon^*\big(-\nabla\mathcal{E}(z_\varepsilon(t)) - \ell(t) - \eta(z_\varepsilon(t) - \tilde z(t))\big), \quad z_\varepsilon(0) = z_0. \tag{5.20}$$

In view of the Fenchel-Young-equality, this in turn is equivalent to

$$\mathcal{R}_\varepsilon(\dot z_\varepsilon(t)) + \mathcal{R}_\varepsilon^*\big(-\nabla\mathcal{E}(z_\varepsilon(t)) - \ell(t) - \eta(z_\varepsilon(t) - \tilde z(t))\big) = \langle -\nabla\mathcal{E}(z_\varepsilon(t)) - \ell(t) - \eta(z_\varepsilon(t) - \tilde z(t)), \dot z_\varepsilon(t)\rangle$$

f.a.a. $t \in [0,T]$. Using (3.14), this leads to

$$\begin{aligned} &\frac{1}{2\varepsilon}\operatorname{dist}\big(-\nabla\mathcal{E}(z_\varepsilon(T)) - \ell(T) - \eta(z_\varepsilon(T) - \tilde z(T)), \partial\mathcal{R}(0)\big)^2 \\ &\quad = \mathcal{R}_\varepsilon^*\big(-\nabla\mathcal{E}(z_\varepsilon(T)) - \ell(T) - \eta(z_\varepsilon(T) - \tilde z(T))\big) \\ &\quad \le \|\nabla\mathcal{E}(z_\varepsilon) - \ell - \eta(z_\varepsilon - \tilde z)\|_{L^\infty(0,T;\mathbb{R}^n)}\|\dot z_\varepsilon\|_{L^\infty(0,T;\mathbb{R}^n)} \\ &\quad \le C\,\frac{1}{\sqrt{\varepsilon}}, \end{aligned} \tag{5.21}$$

where we used the positivity of $\mathcal{R}_\varepsilon$ and the estimate from (5.19). Note that $\|z_\varepsilon\|_{L^\infty(0,T;\mathbb{R}^n)}$ can be estimated by (5.10) due to $H^1(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$. Therefore, the constant in (5.21) and thus also the one in (5.18) only depends on ℓ and $\tilde z$, but not on ε. □

We collect the findings of this section in the following

Theorem 5.9 (Reverse approximation property)**.** *Suppose that the energy functional satisfies Assumption 5.1. Let ℓ and a normalized parametrized BV solution $(S, \hat{t}, \hat{z})$ be given such that Assumption 5.2 is fulfilled and define $\tilde{z} := \hat{z} \circ \hat{t}^{-1}$. Then there exists a sequence $\{(\ell_\varepsilon, z_\varepsilon)\}_{\varepsilon>0} \subset H^1(0,T;\mathbb{R}^n) \times H^2(0,T;\mathbb{R}^n)$ that fulfills*

$$0 \in \partial\mathcal{R}(\dot{z}_\varepsilon(t)) + \varepsilon\, \dot{z}_\varepsilon(t) + \nabla_z \mathcal{I}(\ell_\varepsilon(t), z_\varepsilon(t)), \quad z_\varepsilon(0) = z_0, \tag{5.22}$$

$$z_\varepsilon \to \tilde{z} \quad \textit{in } H^1(0,T;\mathbb{R}^n), \quad \ell_\varepsilon \to \ell \quad \textit{in } H^1(0,T;\mathbb{R}^n) \quad \textit{as } \varepsilon \searrow 0, \tag{5.23}$$

and

$$\ell_\varepsilon(0) = \ell(0), \quad \operatorname{dist}\big(-\nabla\mathcal{I}(\ell_\varepsilon(T), z_\varepsilon(T)), \partial\mathcal{R}(0)\big) \leq \tilde{C}\,\varepsilon^{1/4} \tag{5.24}$$

with a constant $\tilde{C} > 0$ not depending on ε.

Proof. The assertion follows from Proposition 5.7 and Lemma 5.8 by setting $\ell_\varepsilon := \ell + \eta(z_\varepsilon - \tilde{z})$. □

The prior theorem guarantees the existence of a sequence of loads $\{\ell_\varepsilon\}_{\varepsilon>0}$ such that $\tilde{z}$ can be approximated by solutions of the viscous regularized problem. In fact, this so-called "reverse approximation property" is an essential ingredient in Theorem 6.2 below. However, before we step on to the next section, let us investigate on the assumption of the boundedness of the Hessian of $\mathcal{E}$. For this purpose, we introduce the following variant of the penalized energy functional

$$\mathcal{J}_\eta(t,\ell,z) := \mathcal{E}(z) + \frac{\eta}{2}\,\|z - \tilde{z}(t)\|^2 - \langle \ell, z\rangle$$

and rewrite (5.5) by means of the Fenchel-Young equality as

$$\mathcal{R}_\varepsilon(\dot{z}_\varepsilon(t)) + \mathcal{R}_\varepsilon^*\big(-\nabla_z \mathcal{J}_c(t,\ell(t),z_\varepsilon(t))\big) = \langle -\nabla_z \mathcal{J}_c(t,\ell(t),z_\varepsilon(t)), \dot{z}_\varepsilon(t)\rangle.$$

Now we can argue exactly as in the proof of Proposition 3.1, when we derived (3.11) from (3.6) to obtain

$$\|z_\varepsilon\|_{L^\infty(0,T;\mathbb{R}^n)} \leq C\big(1 + \mathcal{J}_c(0,\ell(0),z_0) + \|\ell\|^2_{H^1(0,T;\mathbb{R}^n)}\big). \tag{5.25}$$

Note in this context that $\mathcal{J}_c$ satisfies (2.2) with the same constants as $\mathcal{I}$ does such that the constant C in the above estimate is the same as in (3.11) and thus independent of η. For the energy at the initial time, $\tilde{z}(0) = z_0$ implies that $\mathcal{J}_c(0,\ell(0),z_0) = \mathcal{I}(\ell(0),z_0)$ and therefore (5.25) yields that

$$\|z_\varepsilon\|_{L^\infty(0,T;\mathbb{R}^n)} \leq R \tag{5.26}$$

with a bound $R > 0$ that is independent of ε and η. Revisiting now the arguments of this section, we see that the boundedness of $\nabla^2\mathcal{E}$ is only needed in the points $z_\varepsilon(t)$. In view of (5.26), we can therefore drop the boundedness assumption on $\nabla^2\mathcal{E}$ and replace the constant K from Assumption 5.1 by

$$K := \max_{v \in \overline{B(0,R)}} \|\nabla^2\mathcal{E}(v)\|.$$

This yields the following

Corollary 5.10. *The assertions of Theorem 5.9 also hold without the boundedness assumption on $\nabla^2\mathcal{E}$ from Assumption 5.1.*

6 Approximation of optimal control problems via viscous regularization

With the results of Section 5, we are now in the position to prove our final result, the approximation of optimal solutions to (OCP) by minimizers of "viscous" optimal control problems. The latter read as follows:

$$\left.\begin{aligned} \min \quad & J_\varepsilon(z_\varepsilon, \ell) := j(z_\varepsilon(T)) + \frac{\beta}{2}\,\|\ell\|^2_{H^1(0,T;\mathbb{R}^n)} \\ \text{s.t.} \quad & \ell \in H^1(0,T;\mathbb{R}^n), \quad z_\varepsilon \in H^2(0,T;\mathbb{R}^n), \\ & 0 \in \partial\mathcal{R}(\dot z_\varepsilon(t)) + \varepsilon \dot z_\varepsilon(t) + \partial_z \mathcal{I}(\ell(t), z_\varepsilon(t)), \quad z(0) = z_0, \\ & -\nabla_z \mathcal{I}(\ell(0), z_0) \in \partial\mathcal{R}(0), \quad \operatorname{dist}\big(-\nabla_z\mathcal{I}(\ell(T), z_\varepsilon(T)), \partial\mathcal{R}(0)\big) \le \varepsilon^{1/8}. \end{aligned}\right\} \tag{vOPC$_\varepsilon$}$$

While the objective is the same as in (OCP), we replaced the rate-independent system by its viscous regularization. Moreover, the end time constraint is relaxed in order to guarantee the feasibility of the recovery sequence from Theorem 5.9, see the proof of Proposition 6.2 below.

On the basis of the equivalent dual formulation in (3.3), it is easy to see that the solution map $\ell \mapsto z_\varepsilon$ of the viscous equation is continuous from $L^\infty(0,T;\mathbb{R}^n)$ to $W^{1,\infty}(0,T;\mathbb{R}^n)$. Together with the compactness of $H^1(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$, the standard direct method of the calculus of variations immediately gives the following

Lemma 6.1. *For every $\varepsilon > 0$, there exists an optimal solution $(\ell^*_\varepsilon, z^*_\varepsilon) \in H^1(0,T;\mathbb{R}^n) \times H^2(0,T;\mathbb{R}^n)$ of* (vOPC$_\varepsilon$).

In view of Section 5, we moreover have the following:

Proposition 6.2. *Let Assumption 5.1 (a) be fulfilled. Then there holds the following "Mosco-type" convergence of the viscous optimal control problem:*

(i) Weak lower semicontinuity:
Let $\{\ell^*_\varepsilon\}_{\varepsilon>0}$ be a sequence of global minimizers for (vOPC$_\varepsilon$) *with corresponding states $\{z^*_\varepsilon\}_{\varepsilon>0}$. Denote the reparametrized solution according to* (3.18) *and* (3.19) *by $(S^*_\varepsilon, \hat t^*_\varepsilon, \hat z^*_\varepsilon) \in [0,\infty) \times W^{1,\infty}(0,S^*_\varepsilon) \times W^{1,\infty}(0,S^*_\varepsilon;\mathbb{R}^n)$. Suppose moreover that $\ell^*_\varepsilon \rightharpoonup \ell^*$ in $H^1(0,T;\mathbb{R}^n)$.*

Then, there is a subsequence (denoted w.l.o.g. by the same symbol) such that

$$S^*_\varepsilon \to S^*, \quad \hat t^*_\varepsilon \overset{*}{\rightharpoonup} \hat t^*, \quad \hat z^*_\varepsilon \overset{*}{\rightharpoonup} \hat z^* \quad \text{in } W^{1,\infty}(0,S^*;\mathbb{R}\times\mathbb{R}^n), \tag{6.1}$$

where $(S^*, \hat t^*, \hat z^*)$ is a normalized parametrized BV solution associated with ℓ^*, i.e. $(S^*, \hat t^*, \hat z^*) \in \mathcal{L}(\ell^*)$. Furthermore, the initial and end time constraint in (OCP) *is fulfilled, i.e., $(S^*, \hat t^*, \hat z^*, \ell^*)$ is feasible for* (OCP)*, and there holds*

$$\liminf_{\varepsilon\searrow 0} J_\varepsilon(z^*_\varepsilon, \ell^*_\varepsilon) \ge J(S^*, \hat z^*, \ell^*). \tag{6.2}$$

(ii) Reverse approximation property:
Assume that $(\overline{S}, \overline{t}, \overline{z}, \overline{\ell})$ is global minimizer of (OCP) *that fulfills Assumption 5.2, i.e., there is a constant $\delta > 0$ such that $\overline{t}'(s) \ge \delta$ f.a.a. $t \in [0, \overline{S}]$. Then there exists a sequence $\{\overline{\ell}_\varepsilon\}_{\varepsilon>0} \subset H^1(0,T;\mathbb{R}^n)$ with associated viscous solution $\overline{z}_\varepsilon \in H^2(0,T;\mathbb{R}^n)$ such that $(\overline{z}_\varepsilon, \overline{\ell}_\varepsilon)$ is feasible for* (vOPC$_\varepsilon$) *for all $\varepsilon > 0$ sufficiently small and*

$$\lim_{\varepsilon\searrow 0} J_\varepsilon(\overline{z}_\varepsilon, \overline{\ell}_\varepsilon) = J(\overline{S}, \overline{z}, \overline{\ell}). \tag{6.3}$$

Proof. Let a sequence $\{\ell^*_\varepsilon\}_{\varepsilon>0}$ as in (i) be given. From Theorem 3.5 we directly obtain a subsequence such that (6.1) holds true with a limit in $\mathcal{L}(\ell^*)$. By compact embeddings, ℓ^*_ε and $\hat z^*_\varepsilon$ converge uniformly to ℓ^* and $\hat z^*$, respectively. Therefore, the initial and final time constraints in (OCP) are fulfilled, too, and thus the weak limit is feasible for (OCP). The uniform convergence of $\hat{z}_\varepsilon^*$ furthermore gives $j(z_\varepsilon^*(T)) = j(\hat{z}_\varepsilon^*(S_\varepsilon^*)) \to j(\hat{z}^*(S^*))$. Together with weak lower-semicontinuity of the squared norm, this implies (6.2) as claimed.

The second assertion is an immediate consequence of Theorem 5.9 and Corollary 5.10, respectively. Because of (5.24), the initial and finial time constraint in (vOPC$_\varepsilon$) are fulfilled for all sufficiently small $\varepsilon > 0$. Moreover, the strong convergence in (5.23) along with $H^1(0,T;\mathbb{R}^n) \hookrightarrow C([0,T];\mathbb{R}^n)$ implies (6.3). □

With all this at hand, we can now prove our main result:

Theorem 6.3 (Main result – approximation of global minimizers via vanishing viscosity)**.** *Let Assumption 5.1 (a) be fulfilled and denote by $\{\ell_\varepsilon^*\}_{\varepsilon>0}$ a sequence of global minimizers for* (vOPC$_\varepsilon$) *with corresponding states $\{z_\varepsilon^*\}_{\varepsilon>0}$. Assume moreover that a global minimizer $(\overline{S}, \overline{t}, \overline{z}, \overline{\ell})$ of* (OCP) *exists, which fulfills Assumption 5.2, i.e., there is a $\delta > 0$ such that $\overline{t}'(s) \geq \delta$ f.a.a. $t \in [0, \overline{S}]$.*

Then there is a weak accumulation point $(S^*, \hat{t}^*, \hat{z}^*, \hat{\ell}^*)$ in the sense that

$$\ell_\varepsilon^* \rightharpoonup \ell^* \quad in\ H^1(0,T;\mathbb{R}^n), \quad S_\varepsilon^* \to S^*, \quad \hat{t}_\varepsilon^* \overset{*}{\rightharpoonup} \hat{t}^* \quad in\ W^{1,\infty}(0,S), \quad \hat{z}_\varepsilon^* \overset{*}{\rightharpoonup} \hat{z}^* \quad in\ W^{1,\infty}(0,S;\mathbb{R}^n), \tag{6.4}$$

where $(S_\varepsilon^*, \hat{t}_\varepsilon^*, \hat{z}_\varepsilon^*)$ again denote the reparametrized solution according to (3.18) *and* (3.19)*, and every such accumulation point is a global minimizer of* (OCP)*. Moreover, with respect to the control variable, every weak accumulation point is automatically a strong one, i.e., $\ell_\varepsilon^* \to \ell^*$ in $H^1(0,T;\mathbb{R}^n)$.*

Proof. According to our standing assumptions, there exists ℓ_0 such that $-D_z\mathcal{I}(\ell_0, z_0) \in \partial\mathcal{R}(0)$. Thus, the constant tuple (z_0, ℓ_0) forms a solution of the viscous equation and is feasible for (vOPC$_\varepsilon$) for every $\varepsilon > 0$. Thus $J_\varepsilon(z_\varepsilon^*, \ell_\varepsilon^*)$ is bounded independent of ε. Therefore, since j is bounded from below, we have that $\|\ell_\varepsilon^*\|_{H^1(0,T;\mathbb{R}^n)}$ is also bounded independent of ε and we can extract a subsequence converging weakly to some ℓ^* in $H^1(0,T;\mathbb{R}^n)$. Now, by Proposition 6.2(i), there exists a further subsequence (denoted by the same symbol) such that (6.4) and (6.2) hold true and the limit $(S^*, \hat{t}^*, \hat{z}^*, \ell^*)$ is feasible for (OCP).

On the other hand, the reverse approximation property along with the optimality of $(z_\varepsilon^*, \ell_\varepsilon^*)$ for the viscous optimal control problem implies

$$J(S^*, \hat{z}^*, \ell^*) \leq \liminf_{\varepsilon \searrow 0} J_\varepsilon(z_\varepsilon^*, \ell_\varepsilon^*) \leq \limsup_{\varepsilon \searrow 0} J_\varepsilon(z_\varepsilon^*, \ell_\varepsilon^*) \leq \lim_{\varepsilon \searrow 0} J_\varepsilon(\overline{z}_\varepsilon, \overline{\ell}_\varepsilon) = J(\overline{S}, \overline{z}, \overline{\ell}) = \min\,(\text{OCP}) \tag{6.5}$$

so that $(S^*, \hat{t}^*, \hat{z}^*, \ell^*)$ is indeed optimal for (OCP) as claimed.

Moreover, (6.5) implies the convergence of $\{J_\varepsilon(z_\varepsilon^*, \ell_\varepsilon^*)\}$ and, due to the uniform convergence of $\{z_\varepsilon^*\}$, we obtain

$$\|\ell_\varepsilon^*\|^2_{H^1(0,T;\mathbb{R}^n)} = \frac{2}{\beta}\big(J_\varepsilon(z_\varepsilon^*, \ell_\varepsilon^*) - j(z_\varepsilon^*(T))\big) \to \frac{2}{\beta}\big(J(S^*, \hat{z}^*, \ell^*) - j(\hat{z}^*(S^*))\big) = \|\ell^*\|^2_{H^1(0,T;\mathbb{R}^n)}.$$

Since weak and norm convergence yield strong convergence, this completes the proof. □

Remark 6.4. *As already mentioned in Remark 5.3, Assumption 5.2 is rather restrictive, but need not be satisfied by every global minimizer of* (OCP)*, but just by at least one. Nevertheless, if we restrict to control variables in $H^1(0,T;\mathbb{R}^n)$, then an assumption of this type seems to be indispensable, at least if the recovery sequence in the reverse approximation argument is constructed in the way we did it. This is seen as follows: the approximating control is given by the derivative of the penalized energy, i.e., $\overline{\ell}_\varepsilon = \overline{\ell} + \eta(z_\varepsilon - \tilde{z})$, cf. the proof of Theorem 5.9. Now, $\tilde{z}$ is a function in the physical time, where also ℓ "lives", i.e., we have to transform $\overline{z}$ back to the physical time t. If, however, this transformed function is no differential solution and provides (countably many) jumps in the physical time, then it will no longer be an element of $H^1(0,T;\mathbb{R}^n)$ (as required for the control variable), but only in $BV(0,T;\mathbb{R}^n)$. We therefore need to consider control functions in $BV(0,T;\mathbb{R}^n)$. However, the mere existence of normalized parametrized BV solutions for external loadings in BV has been addressed only recently in the literature, see e.g. [KZ19], and therefore, the weakening of Assumption 5.2 is subject to future research.*

Remark 6.5. *The regularized optimal control problem in* (vOPC$_\varepsilon$) *offers ample possibilities for numerical algorithms. By dualization, the state equation in* (vOPC$_\varepsilon$) *is equivalent to*

$$\dot{z}_\varepsilon(t) = \varepsilon\Big[-\nabla_z\mathcal{I}(\ell(t), z_\varepsilon(t)) - \Pi_{\partial\mathcal{R}(0)}\big(-\nabla_z\mathcal{I}(\ell(t), z_\varepsilon(t))\big)\Big], \quad z_\varepsilon(0) = z_0,$$

where $\Pi_{\partial\mathcal{R}(0)}$ *denotes the orthogonal projection on* $\partial\mathcal{R}(0)$*, cf.* (3.3) *and* (3.14)*. Due to the projection, this is still a non-smooth equation, but, by means of a further smoothing of the projection, one obtains a smooth control-to-state mapping. The Gâteaux-derivative of the objective (reduced to the control variable only) can then be computed with the standard adjoint approach. This gives rise to gradient-based optimization algorithms for the solution of* (vOPC$_\varepsilon$).

A Appendix: Chain rules

Lemma A.1. *Let* $\ell \in H^1(0,T;\mathbb{R}^n)$ *and* $\hat{t} \in W^{1,\infty}(0,S)$ *with* $\hat{t}(0) = 0$, $\hat{t}(S) = T$, *and* $\hat{t}'(s) \geq 0$ *f.a.a.* $s \in (0,S)$ *be given. Then* $\hat{\ell} := \ell \circ \hat{t} \in H^1(0,S;\mathbb{R}^n)$ *and* $\|\hat{\ell}'\|^2_{L^2(0,S;\mathbb{R}^n)} \leq \|\hat{t}'\|_{L^\infty(0,S)}\|\dot{\ell}\|^2_{L^2(0,T;\mathbb{R}^n)}$.

Moreover, for every $\hat{z} \in C([0,S];\mathbb{R}^n)$*, there holds*

$$\int_0^s \partial_\ell\mathcal{I}(\hat{\ell}(r), \hat{z}(r))\hat{\ell}'(r)\mathrm{d}r \leq \frac{1}{2}\int_0^s \|\partial_\ell\mathcal{I}(\hat{\ell}(r), \hat{z}(r))\|^2\hat{t}'(r)\mathrm{d}r + \frac{1}{2}\|\dot{\ell}\|^2_{L^2(0,T;\mathbb{R}^n)}. \tag{A.1}$$

Proof. Let $\delta \in (0,1]$ be fixed but arbitrary. We extend ℓ and $\hat{t}$ to $\mathbb{R}$ by constant continuation and denote these extensions by the same symbols for simplicity. Define the function $\hat{t}_\delta(s) := \hat{t}(s) + \delta\, s$. Then $\hat{t}_\delta : \mathbb{R} \to \mathbb{R}$ is clearly strictly monotone increasing, and thus bi-Lipschitz. Thus [Zie12, Thm. 2.2.2] implies that $\ell \circ \hat{t}_\delta \in H^1(\hat{t}_\delta^{-1}(0, T+\delta S);\mathbb{R}^n) = H^1(0,S;\mathbb{R}^n)$ and its derivative is given by

$$(\ell \circ \hat{t}_\delta)'(s) = \dot{\ell}(\hat{t}_\delta(s))(\hat{t}'(s) + \delta) \quad \text{f.a.a. } s \in (0,S).$$

This leads to

$$\begin{aligned}
\|(\ell \circ \hat{t}_\delta)'\|^2_{L^2(0,S;\mathbb{R}^n)} &\leq \big(\|\hat{t}'\|_{L^\infty(0,S)} + \delta\big)\int_0^S \big(\dot{\ell}(\hat{t}_\delta(s))\big)^2\hat{t}_\delta'(s)\, ds \\
&= \big(\|\hat{t}'\|_{L^\infty(0,S)} + \delta\big)\int_0^{T+\delta S} \dot{\ell}(t)^2 dt \\
&= \big(\|\hat{t}'\|_{L^\infty(0,S)} + \delta\big)\|\dot{\ell}\|^2_{L^2(0,T;\mathbb{R}^n)} \\
&\leq \big(\|\hat{t}'\|_{L^\infty(0,S)} + 1\big)\|\dot{\ell}\|^2_{L^2(0,T;\mathbb{R}^n)} \leq C \neq C(\delta) < \infty.
\end{aligned} \tag{A.2}$$

where we exploited the constant continuation of ℓ. Now, if we consider a sequence $\delta \searrow 0$, then $\{\ell \circ \hat{t}_\delta\}$ is bounded in $H^1(0,S;\mathbb{R}^n)$ and consequently, there is a subsequence converging weakly in $H^1(0,S;\mathbb{R}^n)$. On the other hand, $\hat{t}_\delta$ converges uniformly to $\hat{t}$ on $[0,S]$ so that the continuity of ℓ (due to the compactness of $H^1(0,T+S;\mathbb{R}^n) \hookrightarrow C([0,T+S];\mathbb{R}^n)$) gives the pointwise convergence of $\ell\circ\hat{t}_\delta$ to $\ell\circ\hat{t}$ on $[0,S]$. Since the weak and the pointwise limit coincide, this gives the claimed regularity of $\hat{\ell}$. The estimate of its norm directly follows from (A.2) together with the weak convergence of $\ell\circ\hat{t}_\delta$ to $\ell\circ\hat{t}$ and the lower weak semicontinuity of $\|\cdot\|^2_{L^2(0,S;\mathbb{R}^n)}$.

The inequality in (A.1) is proven similarly: using the same notation as above, it holds

$$\begin{aligned}
&\int_0^s \partial_\ell\mathcal{I}\big(\ell(\hat{t}_\delta(r)), \hat{z}(r)\big)\dot{\ell}(\hat{t}_\delta(r))\hat{t}_\delta'(r)\mathrm{d}r \\
&\qquad \leq \frac{1}{2}\int_0^s \big\|\partial_\ell\mathcal{I}\big(\ell(\hat{t}_\delta(r)), \hat{z}(r)\big)\big\|^2\, \hat{t}_\delta'(r)\mathrm{d}r + \frac{1}{2}\int_0^s \|\dot{\ell}(\hat{t}_\delta(r))\|^2\hat{t}_\delta'(r)\mathrm{d}r
\end{aligned} \tag{A.3}$$

For the right hand side, we have as above

$$\int_0^S \|\dot{\ell}(\hat{t}_\delta(r))\|^2\hat{t}_\delta'(r)\mathrm{d}r = \int_0^{T+\delta S} \|\ell(t)\|^2\, \mathrm{d}t = \|\dot{\ell}\|^2_{L^2(0,T;\mathbb{R}^n)}$$

and, by $\hat{t}_\delta(s) \to \hat{t}(s)$ in $W^{1,\infty}(0,S)$ and the continuity of ℓ, $\hat{z}$, and $\partial_\ell \mathcal{I}$, see (2.1),

$$\frac{1}{2}\int_0^s \big\|\partial_\ell \mathcal{I}\big(\ell(\hat{t}_\delta(r)),\hat{z}(r)\big)\big\|^2 \hat{t}_\delta'(r)\mathrm{d}r \to \frac{1}{2}\int_0^s \big\|\partial_\ell \mathcal{I}\big(\ell(\hat{t}(r)),\hat{z}(r)\big)\big\|^2 \hat{t}'(r)\mathrm{d}r.$$

For the left hand side, the weak convergence $\ell\circ\hat{t}_\delta \rightharpoonup \ell\circ\hat{t}$ in $H^1(0,S;\mathbb{R}^n)$ implies

$$\begin{aligned}\int_0^s \partial_\ell \mathcal{I}\big(\ell(\hat{t}_\delta(r)),\hat{z}(r)\big)\dot{\ell}(\hat{t}_\delta(r))\hat{t}_\delta'(r)\mathrm{d}r &= \int_0^s \partial_\ell \mathcal{I}\big(\ell(\hat{t}_\delta(r)),\hat{z}(r)\big)(\ell\circ\hat{t}_\delta)'(r)\mathrm{d}r\\ &\to \int_0^s \partial_\ell \mathcal{I}\big(\ell(\hat{t}(r)),\hat{z}(r)\big)(\ell\circ\hat{t})'(r)\mathrm{d}r,\end{aligned}$$

which finally proves (A.1). □

Lemma A.2. *Let $(\hat{t},\hat{z})$ be a pair with $\hat{t}\in W^{1,\infty}(0,S)$ and $\hat{z}\in W^{1,1}(0,S;\mathbb{R}^n)$. Then, the energy equality (3.27) is equivalent to the following energy inequality:*

$$\begin{aligned}&\mathcal{I}(\ell(\hat{t}(s)),\hat{z}(s)) + \int_0^s \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\|\operatorname{dist}\{-D_z\mathcal{I}(\hat{t}(r),\hat{z}(r)),\partial\mathcal{R}(0)\}\mathrm{d}r\\ &\qquad\le \mathcal{I}(\ell(0),z_0) + \int_0^s \partial_\ell\mathcal{I}(\ell(\hat{t}(r)),\hat{z}(r))\ell'(\hat{t}(r))\hat{t}'(r)\mathrm{d}r \qquad \forall\, s\in[0,S].\end{aligned} \tag{A.4}$$

Proof. The proof of this Lemma is based on [KRZ13, Lem. 6.6]. Clearly, if (3.27) holds, then so does the above inequality (A.4). It therefore suffices to proof the opposite direction. Hence, let $(\hat{t},\hat{z})$ be given as in the assumptions, which fulfills the inequality (A.4). Applying the chain rule for Sobolev functions, cf. e.g. [Zie12, Thm. 2.1.11], gives for almost all $s\in[0,S]$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\hat{t}(s),\hat{z}(s)) = \langle D_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\hat{z}'(s)\rangle + \partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))\,\hat{t}'(s). \tag{A.5}$$

Now, let $\xi(s)\in\partial\mathcal{R}(0)$ such that $\operatorname{dist}\{-D_z\mathcal{I}(\hat{t}(r),\hat{z}(r)),\partial\mathcal{R}(0)\} = \|-D_z\mathcal{I}(\hat{t}(s),\hat{z}(s))-\xi(s)\|$ for almost all $s\in(0,S)$. Exploiting that $\mathcal{R}(v)\ge\langle\xi,v\rangle$ for all $\xi\in\partial\mathcal{R}(0)$, we can consequently estimate

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(\hat{t}(s),\hat{z}(s)) + \partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))\,\hat{t}'(s)\\ &\qquad= \langle -D_z\mathcal{I}(\hat{t}(s),\hat{z}(s))-\xi(s),\hat{z}'(s)\rangle + \langle\xi(s),\hat{z}'(s)\rangle\\ &\qquad\le \|-D_z\mathcal{I}(\hat{t}(s),\hat{z}(s))-\xi(s)\|\|\hat{z}'(s)\| + \mathcal{R}(\hat{z}'(s))\\ &\qquad= \|\hat{z}'(s)\|\operatorname{dist}\{-D_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}(0)\} + \mathcal{R}(\hat{z}'(s)).\end{aligned}$$

Integration with respect to time and inserting the energy inequality, we obtain

$$\begin{aligned}&\mathcal{I}(0,z_0) - \mathcal{I}(\hat{t}(s),\hat{z}(s)) + \int_0^s \partial_t\mathcal{I}(\hat{t}(r),\hat{z}(r))\hat{t}'(r)\mathrm{d}r\\ &\qquad\le \int_0^s \|\hat{z}'(r)\|\operatorname{dist}\{-D_z\mathcal{I}(\hat{t}(r),\hat{z}(r)),\partial\mathcal{R}(0)\} + \mathcal{R}(\hat{z}'(r))\mathrm{d}r\\ &\qquad\le \mathcal{I}(0,z_0) + \int_0^s \partial_t\mathcal{I}(\hat{t}(r),\hat{z}(r))\hat{t}'(r)\mathrm{d}r - \mathcal{I}(\hat{t}(s),\hat{z}(s))\end{aligned}$$

for all $s\in[0,S]$. Hence, (A.4) holds in fact with equality which is (3.27). □

B Appendix: Uniqueness of differential solutions for $\ell\in H^1(0,T;\mathbb{R}^n)$

For the sake of completeness, let us briefly comment on the uniqueness of differential solutions in case of uniformly convex energies for loads in $H^1(0,T;\mathbb{R}^n)$, since this case is not covered by the results from the literature (which in general require $\ell \in W^{1,\infty}(0,T;\mathbb{R}^n)$, cf. e.g. [MT04]). To this end, let $z_1, z_2 \in W^{1,1}(0,T,\mathbb{R}^n)$ be two differential solutions. We introduce the following distance measure

$$\gamma(t) := \langle \nabla_z \mathcal{I}(\ell(t), z_1(t)) - \nabla_z \mathcal{I}(\ell(t), z_2(t)), z_1(t) - z_2(t) \rangle.$$

Note that by the uniform convexity of $\mathcal{I}$, we have $\gamma(t) \geq \alpha \|z_1(t) - z_2(t)\|^2$ with some $\alpha > 0$. Clearly, by the structure of $\mathcal{I}$, i.e., $\mathcal{I}(\ell, z) = \mathcal{E}(z) - \langle \ell, z \rangle$, we have $\gamma(t) = \langle \nabla \mathcal{E}(z_1(t)) - \nabla \mathcal{E}(z_2(t)), z_1(t) - z_2(t) \rangle$. Now, using the symmetry of $\nabla^2 \mathcal{E}$, we calculate

$$\begin{aligned}\gamma'(t) = {}& \langle \nabla^2 \mathcal{E}(t, z_1(t))[z_1(t) - z_2(t)], z_1'(t) \rangle \\ &- \langle \nabla^2 \mathcal{E}(t, z_2(t))[z_1(t) - z_2(t)], z_2'(t) \rangle + \langle \nabla^2 \mathcal{E}(t, z_1(t)) - D_z \mathcal{I}(t, z_2(t)), z_1'(t) - z_2'(t) \rangle.\end{aligned}$$

Rearranging terms, we arrive at

$$\begin{aligned}\gamma'(t) = {}& \langle \nabla^2 \mathcal{E}(t, z_1(t))[z_1(t) - z_2(t)] + \nabla \mathcal{E}(t, z_2(t)) - \nabla \mathcal{E}(t, z_1(t)), z_1'(t) \rangle \\ &- \langle \nabla^2 \mathcal{E}(t, z_2(t))[z_1(t) - z_2(t)] + \nabla \mathcal{E}(t, z_1(t)) - \nabla \mathcal{E}(t, z_2(t)), z_2'(t) \rangle \\ &+ 2 \langle \nabla \mathcal{E}(t, z_1(t)) - \nabla \mathcal{E}(t, z_2(t)), z_1'(t) - z_2'(t) \rangle.\end{aligned}$$

Now, due to $z_1, z_2 \in W^{1,1}(0,T;\mathbb{R}^n)$ and the regularity on $\mathcal{E}$ (see (5.2)), we find that

$$\begin{aligned}\gamma'(t) \leq {}& C \|z_1(t) - z_2(t)\|^2 \|z_1'(t)\| + C \|z_1(t) - z_2(t)\|^2 \|z_2'(t)\| \\ &+ 2 \langle \nabla_z \mathcal{I}(\ell(t), z_1(t)) - \nabla_z \mathcal{I}(\ell(t), z_2(t)), z_1'(t) - z_2'(t) \rangle.\end{aligned} \tag{B.1}$$

Since z_1 and z_2 are differential solutions, it holds

$$0 \in \partial \mathcal{R}(\dot{z}_i(t)) + \nabla_z \mathcal{I}(\ell(t), z_i(t)) \quad i = 1, 2$$

and testing these equations with $z_1' - z_2'$ and adding them lead to $\langle \nabla_z \mathcal{I}(t, z_1(t)) - \nabla_z \mathcal{I}(t, z_2(t)), z_1'(t) - z_2'(t) \rangle \leq 0$. Hence, inserting this and the fact that $\|z_1(t) - z_2(t)\|^2 \leq \gamma(t)/\alpha$ into (B.1) and integrating over $[0, t]$ gives

$$\gamma(t) \leq \gamma(0) + C \int_0^t (\|z_1'(\tau)\| + \|z_2'(\tau)\|) \gamma(\tau) \mathrm{d}\tau.$$

Applying Gronwall's lemma we eventually end up with

$$\gamma(t) \leq \gamma(0) \exp \left\{ C \int_0^t (\|z_1'(\tau)\| + \|z_2'(\tau)\|) \mathrm{d}\tau \right\}.$$

Since $z_1(0) = z_2(0) = z_0$ we have $\gamma(0) = 0$ and thus $\gamma(t) = 0$ for all $t \in [0, T]$ which proves the uniqueness of differential solutions.

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