Necessary and sufficient conditions for the existence of torsion-free covariant derivatives with prescribed curvature tensor

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Dissertation

Necessary and sufficient conditions for the existence of torsion-free covariant derivatives with prescribed curvature tensor

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Zusammenfassung

Diese Dissertation behandelt das Problem, torsionsfreie kovariante Ableitungen mit vorgeschriebener Krümmung zu konstruieren. Die Krümmung einer solchen kovarianten Ableitung erfüllt notwendigerweise die Bianchi-Identitäten. Das Hauptresultat der Arbeit besagt, dass diese Bedingungen bereits hinreichend sind: jede analytische Krümmungsabbildung, die die Bianchi-Identitäten erfüllt, kann lokal als Krümmung einer eindeutigen torsionsfreien kovarianten Ableitung realisiert werden.

Als Anwendungen dieses Resultats in der Holonomietheorie können wir die lokalen Existenzresultate für Riemannsche Metriken mit spezieller Holonomie von Calabi, Yau, Bryant etc. vereinheitlichen und wesentlich vereinfachen.

Abstract

This dissertation deals with the problem of constructing torsion-free covariant derivatives with prescribed curvature. The curvature of a such covariant derivative necessarily satisfies the Bianchi identities. The main result of the present work asserts that these identities are enough to achieve this: any analytic curvature map which satisfies the Bianchi identities, can locally be realized as the curvature of a unique torsion-free covariant derivative.

As applications of this result in Holonomy Theory, we can unify in a simple way all of the local existence results of Calabi, Yau, and Bryant, among others, for Riemannian metrics with special holonomy.

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«Aprendes de tu gente y de los extraños, y sobre todo, de los que tienen la paciencia de ver tu carita de circunstancia y de decir, sin juicios morales: "ve nada más este tonto, otra vez no entendió nada. Ven. Te explico" Y esos son los imprescindibles. Quienes poseen el cómo y te lo retacan a pesar de que a veces eres un poquito limitado, un tanto torpe, y un montón disperso. Todo el crédito para ellos. Tus maestros de vida.»

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... Por lo menos, confiesa que te he dado tema para una novela. ¿No, niño bueno?

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1

Introduction and historical context

The first account of the notion of holonomy can be traced back to the end of the 19th century, to the work of Heinrich Hertz, who in his 1894 *die Prinzipien der Mechanik in neuem Zusammenhange dargestellt* (“The principles of mechanics presented in a new form") coined the terms *holonomic* and *non-holonomic*, to distinguish between two kinds of velocity constraints in configuration spaces.

In a mathematical context, albeit related to Hertz’s notion, the concept of holonomy was introduced by Élie Cartan in his 1925 *La géométrie des espaces de Riemann* (“Introduction to Riemannian geometry"). The two reasons why Cartan became interested in holonomy were first because he claimed that by means of holonomy, his 1913 classification of compact subgroups of $\mathrm{SO}(n)$ could be significantly simplified, and secondly because he realized that the notion of holonomy could be helpful in his quest of classifying Riemannian symmetric spaces (see [1], [2], [3]).

For the better part of the next twenty years, further research in the field was relatively meager. It was not until the 1950s that the revival of the area took place. Starting in 1952, with A. Borel and A. Lichnerowicz showing that holonomy is always a Lie group (see [4]). Also in the same year, G. de Rham in [5] proved what is now called the *de Rham splitting Theorem*. Namely, if the holonomy of a Riemannian manifold is reducible then the metric must be a local product metric. The following year, W. Ambrose and I. Singer established a link between curvature and holonomy, see Proposition 3.7.5.

In actuality, the next big milestone came with M. Berger, who in his doctoral thesis [6], and based on the Theorem of Ambrose and Singer, established necessary conditions for a Lie algebra $\mathfrak{h} \subseteq \mathfrak{gl}(V)$ to be the Lie algebra of the holonomy group of a torsion-free covariant derivative, and used it to classify all of the possible irreducible, non-symmetric holonomy algebras of Riemannian metrics, i.e. the holonomy algebras of affine manifolds, which are entirely contained in $\mathfrak{so}(V)$, see Proposition 3.8.4. The imposed condition on the covariant derivative of it being torsion-free is what makes this classification problem non-trivial, since in 1956 J. Hano and H. Ozeki proved in just a few pages ([7]) that in dimension $n \geqslant 2$, *any* connected Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ can be realized as the holonomy of some affine covariant derivative (with non-vanishing torsion, in general). The classification of holonomy groups of torsion-free covariant derivatives is referred to in the modern literature as the *holonomy problem*.

It is worth mentioning that, in his original list, Berger also included the group Spin(9) as a possibility for it to occur as the holonomy group of a 16-dimensional manifold. However, sometime later, D. Alekseevsky ([8]), as well as R. Brown and A. Gray ([9]), showed that any Riemannian manifold having restricted holonomy contained in Spin(9) is necessarily locally symmetric, rendering it thus irrelevant for Berger's statement. Because the classification of the holonomy of Riemannian symmetric spaces was by the time a settled matter, together with the de Rham splitting Theorem, Berger's list was the final step in completely classifying the possible holonomy groups of a Riemannian simply-connected manifold.

The way in which Berger put his list together was by considering all of the connected, and irreducible subgroups of $\mathrm{SO}(V)$, where V denotes a finite-dimensional real vector space, which pass two concrete algebraic tests, nowadays known as *Berger's criteria*. The first of these tests determines all of the subgroups $H \subseteq \mathrm{SO}(V)$ having all of the previously mentioned algebraic properties which are *Berger subgroups*, that is, subgroups such that its Lie algebra is a *Berger algebra*, i.e. it satisfies

$$\mathfrak{h} = \langle R(x,y) |\ R \in K(\mathfrak{h});\ x, y \in V\rangle,$$

where the space $K(\mathfrak{h})$ denotes the space of *algebraic curvature tensors*, which is defined as

$$K(\mathfrak{h}) := \left\{ R \in \bigwedge^2 V^* \otimes \mathfrak{h} \,\middle|\, R(x,y)z + R(y,z)x + R(z,x)y = 0, \text{for } x, y, z \in V \right\}$$

and the notation $\langle S\rangle$ denotes the Lie algebra generated by the set S.

This test is necessary for the Lie algebra to occur as the holonomy algebra of a Riemannian manifold, in light of the Ambrose-Singer Theorem (see Proposition 3.7.5). The second test consists of analyzing whether a Berger algebra $\mathfrak{h}$ satisfies $K^1(\mathfrak{h}) = \{0\}$, where the space $K^1(\mathfrak{h})$ denotes the space of *algebraic curvature derivatives*, which is defined as

$$K^1(\mathfrak{h}) := \{\phi \in V^* \otimes K(\mathfrak{h}) |\ \phi(x)(y,z) + \phi(y)(z,x) + \phi(z)(x,y) = 0 \in \mathfrak{h}, \text{ for } x, y, z \in V\}.$$

This test is necessary for the manifold having holonomy algebra $\mathfrak{h}$ not to be locally symmetric (see definition 3.7.5).

As it turned out, Berger's list almost entirely coincides with that of the compact, connected Lie subgroups of $\mathrm{SO}(n)$ transitively acting on the sphere S^{n-1}, which was established a decade prior to Berger's work in [10]. The entries on the said list that elude Berger's are namely the groups $\mathrm{Sp}(m) \cdot \mathrm{U}(1)$ in $\mathrm{SO}(4m)$, and Spin(9) in dimension 16. The reason why the case Spin(9) is omitted from Berger's list was previously discussed. It can also be shown that a Riemannian manifold whose holonomy is contained in $\mathrm{Sp}(m) \cdot \mathrm{U}(1)$ is in reality already contained in $\mathrm{Sp}(m)$. By making use of the so-called *irreducible holonomy systems*, in [11] J. Simons proved an equivalent reformulation of Berger's Theorem in this context. In the same vein, albeit more than forty years after Simons' work was published, in [12] C. Olmos gave a completely geometric proof of the same result.

One apparent limitation of Berger's algebraic methods is that they were thought unable to show which of these candidates actually occur as holonomy groups. However, after almost 30 years of work, the occurrence of the entries in Berger's list as holonomy of torsion-free covariant derivatives was settled, when R. Bryant, by using techniques of Cartan-Kähler Theory, managed to show the occurrence of the exceptional groups. See [13, Theorem 3]

The first examples were only local ones, however as time went by, it was possible to construct geodesically complete and even compact examples for each of the entries in Berger's list. See for example [14], [15, Sections 2 and 3], and [16]. As it happens, the geometry of manifolds with special holonomy groups is of significant importance in many areas of differential geometry and theoretical physics, in particular in string theory, where Calabi-Yau 3-folds are necessary to define the notion of *mirror symmetry*. Dwelling on these physical notions is however beyond the scope of this work. For further reading on the topic see for example [17], and [16, Chapter 9].

In his doctoral thesis, Berger offered in addition to the list of possible holonomy groups of Riemannian manifolds a list of possible irreducible holonomy groups of simply-connected, non-symmetric pseudo-Riemannian manifolds (see [15, Tables 2 and 3]). In [18] he tackled the symmetric case. In contrast to the case of Riemannian holonomies, this list turned out to be incomplete and in fact, it was not until 1998 with the work of S. Merkulov and L. Schwachhöfer ([19]) when the complete classification of irreducible connected holonomies of torsion-free covariant derivatives was settled in a definite manner.

All this intensive work notwithstanding, as of this writing, a complete classification of the holonomy of pseudo-Riemannian manifolds is yet to be established. Technical difficulties like the lack of an analogous of the de Rham splitting Theorem in the pseudo-Riemannian setting make the classification of reducible holonomy groups significantly harder. At this point, it should be noted that the interest in explicitly tackling the reducible part of the holonomy problem has been gradually fading, since even if today such a list came to light, chances are that it would be far too extensive to be of any practical use.

The main purpose of this work is to provide necessary and sufficient conditions for the local existence of real analytic torsion-free covariant derivatives. Explicitly, our main result is the following

Theorem 1. *Let V a finite-dimensional $\mathbb{R}$-vector space and U an open neighborhood of 0 in V. Let $S\colon U \longrightarrow K(\mathrm{End}(V))$ be a real analytic map such that $dS : U \longrightarrow K^1(\mathrm{End}(V)) \subseteq V^* \otimes K(\mathrm{End}(V))$. Then there exists a unique torsion-free covariant derivative ∇ defined on a sufficiently small neighborhood of the origin $\mathcal{U} \subseteq U$ such that*

$$S(v) = P^{\nabla}_{1,0} R^{\nabla}_{\gamma_v(1)} \qquad \textit{for all} \quad v \in \mathcal{U}.$$

A particularly interesting application of this result is that it allows us to provide a criterion for the local existence of holonomy algebras of torsion-free covariant derivatives, which at the same time proves the fact that Berger's criteria truly are the only two obstructions for a Lie algebra to occur as holonomy algebra of torsion-free covariant derivatives. Indeed, we prove the following

Theorem 2. *Let V be a finite-dimensional real vector space. Let $S\colon U \longrightarrow K(\mathfrak{g})$ be a real analytic map defined in an open neighborhood U of the origin in V which satisfies that $dS\colon U \longrightarrow K^1(\mathfrak{g})$ and let ∇ be the covariant derivative given by Theorem 1. It holds that*

$$\mathfrak{hol}(\nabla) = \langle S(v)(x, y) |\ v \in U;\ x, y \in V\rangle.$$

An immediate consequence of Theorem 2 is the following criterion for the existence of holonomies of torsion-free covariant derivatives

Corollary. *In the situation of Theorem 2, let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. It holds that $\mathfrak{hol}(\nabla) \subseteq \mathfrak{h}$ if, and only if, the real analytic map S takes values in the space $K(\mathfrak{h})$.*

Also, if it happens that the Lie algebra $\mathfrak{h}$ admits elements of *full curvature*, that is, there exists $R \in K(\mathfrak{h})$ such that

$$\mathfrak{h} = \langle R(x, y) |\ x, y \in V\rangle,$$

we obtain the following existence result:

Corollary. *Let $\mathfrak{h}$ be a Berger algebra that admits elements of full curvature. Then it occurs as the holonomy algebra of a torsion-free covariant derivative.*

The standout feat of the previous result is that it produces conditions for the occurrence of a Lie algebra as holonomy of torsion-free covariant derivatives, which unify and simplify the previous work on the subject, since, as of this writing, there are no known examples of a Berger algebra which does not admit elements of full curvature, and thus, locally, every known example of a Berger algebra actually occurs as the holonomy algebra of a torsion-free covariant derivative.

The present work is structured as follows. Chapter 2 essentially provides the foundations for the proof of Theorem 1. In the first half, we present some standard results in the theory of ordinary differential equations in the real analytic setting. In the second half, we turn our attention to the necessary algebraic preliminaries from the theory of symmetric polynomials.

The third chapter is dedicated to collecting some relevant facts about the theory of principal fiber bundles, which provide us with the appropriate setting for the study of the curvature tensor, which will be helpful in the proof of the Main Result. It also provides the adequate setting to offer a simple proof of Theorem 2. It ends by giving a brief introduction to basic holonomy theory, a discussion about Berger's classification of Riemannian holonomies, and the geometry of manifolds with holonomy one of the entries in Berger's list.

The fourth chapter is the core chapter of the work. It starts by gathering some further required results from classical differential geometry, for then in the middle section of the chapter, namely section 4.3, provide a proof of the Main Result.

The final chapter is devoted to the study of some consequences of Theorem 1. In section 5.1 we prove Theorem 2 and its aforementioned corollaries, while in the last part we study some further ramifications of Theorem 1 in the case of some explicit Lie algebras.

2

Preliminaries

2.1 Ordinary differential equations

This section is dedicated to the study of real analytic solutions to systems of ordinary differential equations.

Despite the fact that the results presented are considered standard, we include them to be as self-contained as we possibly can.

The main goal of the section is to provide the necessary conditions for the initial value problem (IVP)

$$\begin{cases} u^{(m)}(t) = F(t, u(t), \dots, u^{(m-1)}(t)), \\ u^{(i)}(t_0) = x^i & i \in \{0, \dots, m-1\} \end{cases} \tag{2.1}$$

to have a unique solution given by a convergent power series which, in some suitable sense, depends on the initial conditions in a real analytic fashion.

To motivate the statement that momentarily will be proved, let us recall the standard result on the existence and uniqueness of continuously differentiable solutions for first-order systems of ordinary differential equations (see for example [20, Appendix D, Theorem D.6]):

Proposition 2.1.1. *Let* $J \subseteq \mathbb{R}$ *be an open interval,* $U \subseteq \mathbb{R}^m$ *an open subset, and let* $\mathbf{F} := (F^1, \dots, F^m)\colon J \times U \longrightarrow \mathbb{R}^m$ *be a smooth vector-valued function. Then for any* $t_0 \in J$, $\mathbf{x}_0 \in U$, *there exists an open interval* $t_0 \in J_0 \subseteq J$ *and an open subset* $\mathbf{x}_0 \in U_0 \subseteq U$, *such that for each* $s \in J_0$, $\mathbf{x} \in U_0$, *there is a* C^1 *map* $\mathbf{y}\colon J_0 \longrightarrow U$ *solving the IVP*

$$\begin{cases} \dot{\mathbf{y}}(t) = \mathbf{F}(t, \mathbf{y}(t)), \\ \mathbf{y}(s) = \mathbf{x}. \end{cases}$$

Furthermore, any two differentiable solutions to this IVP agree on their common domain and it smoothly depends on its initial conditions in the sense that the map $\Phi\colon J_0 \times J_0 \times U_0 \longrightarrow U$ *defined by* $\Phi(t, s, \mathbf{x}) := \mathbf{y}(t)$, *where* $\mathbf{y}\colon J_0 \longrightarrow U$ *is the unique solution to the previous IVP, is smooth.*

We notice the previous proposition can be reformulated in a much more succinct manner. Indeed, with the same assumptions of the previous proposition, finding a differentiable solution to the given IVP which smoothly depends on the initial data is equivalent to finding a smooth solution $\mathbf{u}\colon W \subseteq \mathbb{R}^{m+2} \longrightarrow \mathbb{R}^m$ to the IVP

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t}(t,s,\mathbf{x}) = \mathbf{F}(t,\mathbf{u}(t,s,\mathbf{x})), \\ \mathbf{u}(s,s,\mathbf{x}) = \mathbf{x}, \end{cases} \tag{2.2}$$

where $W \subseteq \mathbb{R}^{m+2}$ is an open neighborhood of $(t_0, t_0, \mathbf{x}_0)$.

The way we are going to tackle our original question is by proving a suitable analogue of the previous result. That is, by proving the existence and uniqueness of real analytic solutions of systems of first-order ordinary differential equations, which depend in a real analytic fashion on the initial conditions.

Even though all of this discussion can be reduced to a corollary of the well-known Cauchy-Kovalevskaya Theorem (cf. [21, Theorem 7.2.9]), we are nonetheless going to avoid taking this route and prove the real analytic version of the previous proposition as a standalone result.

In order to achieve this we are going to start by proving a series of simpler results, which later on can be put together and build the proof of our desired result.

Before we dive into the specifics, let us introduce a couple of necessary notions.

Definition 2.1.1. *Let $F, G \in C^\infty(U)$, with $0 \in U \subseteq \mathbb{R}^m$. We say that G* majorizes *F (alternatively, that G is a* majorant *of F), denoted $G \gg F$ or $F \ll G$, if*

$$D^\nu G(0) \geqslant |D^\nu F(0)| \qquad \textit{for all } \nu \in \mathbb{N}^m,$$

where for $\nu = (\nu_1, \dots, \nu_m)$, $D^\nu F$ denotes the $|\nu|$-order derivative of F

$$D^\nu F = \frac{\partial^{|\nu|} F}{\partial (x^1)^{\nu_1} \cdots \partial (x^m)^{\nu_m}}.$$

In the case of vector-valued functions, for $\mathbf{F} = (F^1, \dots, F^\ell)$, $\mathbf{G} = (G^1, \dots, G^\ell) \in C^\infty(U, \mathbb{R}^\ell)$, we say that $\mathbf{G}$ majorizes *$\mathbf{F}$, if $G^i \gg F^i$, for $i \in \{1, \dots, \ell\}$.*

Definition 2.1.2. *Let $U \subseteq \mathbb{R}^m$ be an open subset. A function $F\colon U \longrightarrow \mathbb{R}$ is said to be* real analytic *on U, written $f \in C^\omega(U)$, if for each $\mathbf{x}_0 \in U$ the function F may be represented by a convergent power series in some neighborhood of $\mathbf{x}_0$.*

A vector-valued function $\mathbf{F} = (F^1, \dots, F^\ell)\colon U \longrightarrow \mathbb{R}^\ell$ is called real analytic *if $F^i\colon U \longrightarrow \mathbb{R}$ is real analytic, for $i \in \{1, \dots, \ell\}$.*

The following result is a basic fact concerning real analytic functions.

Proposition 2.1.2 ([22, Theorem 1.1.17])**.** *Let $U \subseteq \mathbb{R}^m$ be an open subset and $f\colon U \longrightarrow \mathbb{R}$ be infinitely differentiable. The function f is real analytic if, and only if, for each $\mathbf{x}_0 \in U$ there exists an open neighborhood $\mathbf{x}_0 \in V \subseteq U$, and constants $C, r > 0$ such that*

$$\left| \frac{1}{\nu!} D^\nu f(\mathbf{x}) \right| \leqslant \frac{C}{r^{|\nu|}}$$

for all $\mathbf{x} \in V$ and $\nu \in \mathbb{N}^m$.

This boundedness result of the derivatives of real analytic functions can be easily generalized to vector-valued functions:

Proposition 2.1.3. *Let $U \subseteq \mathbb{R}^m$ be an open subset and $\mathbf{F}\colon U \longrightarrow \mathbb{R}^\ell$ be infinitely differentiable. The function $\mathbf{F}$ is real analytic if, and only if, for each $\mathbf{x}_0 \in U$ there exists an open neighborhood $\mathbf{x}_0 \in V \subseteq U$, and constants $C, r > 0$ such that*

$$\left\|\frac{1}{\nu!}D^\nu \mathbf{F}(\mathbf{x})\right\| \leqslant \frac{C}{r^{|\nu|}},$$

for all $\mathbf{x} \in V$ and $\nu \in \mathbb{N}^m$, and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^\ell$.

Proof. The result is in essence a consequence of the elementary estimate

$$\|\cdot\|_\infty \leqslant \|\cdot\| \leqslant \sqrt{\ell}\|\cdot\|_\infty$$

between the Euclidean and the ∞ norms on $\mathbb{R}^\ell$.

Indeed, assume $\mathbf{F} = (F^1, \dots, F^\ell)$ to be real analytic. The previous proposition implies that, for each $\mathbf{x}_0 \in U$, $i \in \{1, \dots, \ell\}$, there exist positive constants C_i, r_i and open neighborhoods $V_i \subseteq U$ such that

$$\left|\frac{1}{\nu!}D^\nu F^i(\mathbf{x})\right| \leqslant \frac{C_i}{r_i^{|\nu|}}$$

for all $\mathbf{x} \in V_i$, $\nu \in \mathbb{N}^m$.

Set $C' \coloneqq \max_i\{C_i\}$, $r \coloneqq \min_i\{r_i\}$, $V \coloneqq \bigcap_i V_i$.

It then holds for all $\mathbf{x} \in V$, $\nu \in \mathbb{N}^m$

$$\left\|\frac{1}{\nu!}D^\nu \mathbf{F}(\mathbf{x})\right\|_\infty = \max_i\left\{\left|\frac{1}{\nu!}D^\nu F^i(\mathbf{x})\right|\right\} \leqslant \frac{C'}{r^{|\nu|}}$$

and thus

$$\left\|\frac{1}{\nu!}D^\nu \mathbf{F}(\mathbf{x})\right\| \leqslant \sqrt{\ell}\left\|\frac{1}{\nu!}D^\nu \mathbf{F}(\mathbf{x})\right\|_\infty \leqslant \frac{C'\sqrt{\ell}}{r^{|\nu|}} \eqqcolon \frac{C}{r^{|\nu|}}.$$

If, on the other hand, the above estimate holds, we have that, for $i \in \{1, \dots, \ell\}$,

$$\left|\frac{1}{\nu!}D^\nu F^i(\mathbf{x})\right| \leqslant \left\|\frac{1}{\nu!}D^\nu \mathbf{F}(\mathbf{x})\right\|_\infty \leqslant \left\|\frac{1}{\nu!}D^\nu \mathbf{F}(\mathbf{x})\right\| \leqslant \frac{C}{r^{|\nu|}},$$

which in light of the previous proposition implies the real analyticity on U of each of the coordinate function F^i, and whence, that of $\mathbf{F}$. $\square$

We are now ready to begin with the buildup of the proof for the main result of the section:

Proposition 2.1.4. *Let $0 \in U \subseteq \mathbb{R}^m$ be an open set, $\mathbf{F} = (F^1, \dots, F^m)\colon U \longrightarrow \mathbb{R}^m$ be a real analytic vector-valued function. Then the unique solution $\mathbf{u} = (u^1, \dots, u^m)$ to the autonomous IVP*

$$\begin{cases} \frac{d\mathbf{u}}{dt}(t) = \mathbf{F}(\mathbf{u}(t)) \\ \mathbf{u}(0) = 0 \end{cases}$$

is real analytic near 0.

Proof. The smoothness near the origin of the solution $\mathbf{u}$ is a consequence of Proposition 2.1.1. The fact that, for $i \in \{1, \dots, m\}$,

$$\frac{du^i}{dt} = F^i(\mathbf{u}(t))$$

implies that, for $k \in \mathbb{N}_0$,

$$u^i_{k+1}(t) \coloneqq \frac{d^{k+1}u^i}{dt^{k+1}}(t) = \frac{d^k}{dt^k}(F^i \circ \mathbf{u})(t) \tag{2.3}$$

This recursion formula allows us to write each of the u^i_k in terms of the derivatives of the coordinate functions F^j in a particular way, which will be useful in showing the real analyticity of each of the u^i.

In order to achieve this we rely on the following generalization of the more or less well-known Faà di Bruno formula, for the higher-order derivatives of the composition of two one-variable functions:

Proposition 2.1.5 ([23, Theorem 2.1]). *Let* $f \in C^\infty(\mathbb{R}^\ell)$, $g^1, \dots, g^\ell \in C^\infty(\mathbb{R})$. *Then the derivatives of*
$h(t) \coloneqq f(g^1(t), \dots, g^\ell(t))$ *are given by*

$$h^{(k)}(t) = \sum_{\substack{\lambda \in \mathbb{N}_0^\ell \\ 1 \leqslant |\lambda| \leqslant k}} D^\lambda f(g^1(t), \dots, g^\ell(t)) \sum_{p(k,\lambda)} k! \prod_{j=1}^{k} \frac{1}{\mu_j!} \left(\frac{(g^1)^{(a_j)}(t)}{a_j!}, \dots, \frac{(g^\ell)^{(a_j)}(t)}{a_j!} \right)^{\mu_j},$$

where the set $p(k, \lambda)$ *is defined as*

$$\begin{aligned} p(k,\lambda) \coloneqq \Big\{ &(\mu_1, \dots, \mu_k; (a_1, \dots, a_k)) \in \left(\mathbb{N}_0^\ell\right)^k \times \mathbb{N}_0^k \,\Big|\, \text{for some } 1 \leqslant s \leqslant k, \\ &\mu_i = 0 \text{ and } a_i = 0 \text{ for } 1 \leqslant i \leqslant k-s;\ |\mu_i| > 0 \text{ for } k-s+1 \leqslant i \leqslant k; \\ &\text{and } 0 < a_{k-s+1} < \cdots < a_k \text{ are such that} \\ &\sum_{i=1}^k \mu_i = \lambda,\ \sum_{i=1}^k a_i |\mu_i| = k \Big\}. \end{aligned}$$

From the previous proposition we thus obtain, by setting $f = F^i$, $g^j(t) = u^j(t)$ for $j \in \{1, \dots, m\}$, together with (2.3), that for each $k \geqslant 1$, there exist a unique polynomial $P^i_k \in \mathbb{N}_0[X_1, \dots, X_\ell]$, with $\ell = m\binom{m+k-1}{m} = \#\big(\{\mu \in \mathbb{N}_0^m |\ |\mu| \leqslant k-1\} \times \{1, \dots, m\}\big)$ such that

$$u^i_k(t) = P^i_k\big(D^\mu F^j(\mathbf{u}(t))\big), \qquad \text{with } \mu \in \mathbb{N}_0^m,\ |\mu| \leqslant k-1,\ j = 1, \dots, m.$$

We also notice that these polynomials are, in a manner of speaking, universal, since they do not depend on the functions F^j in the sense that for any smooth function $\mathbf{G} = (G^1, \dots, G^m)$, $i \in \{1, \dots, m\}$, $P^i_k(D^\mu G^j(0)) \coloneqq v^i_k(0)$, where $\mathbf{v} = (v^1, \dots, v^m)$ is the solution to the IVP

$$\begin{cases} \frac{d\mathbf{v}}{dt}(t) = \mathbf{G}(\mathbf{v}(t)) \\ \mathbf{v}(0) = 0. \end{cases}$$

Now, since the map $\mathbf{F} \in C^\omega((-a,a)^m, \mathbb{R}^m)$, Proposition 2.1.3 guarantees the existence of an open neighborhood $0 \in V \subseteq (-a,a)^m$ and positive constants C, r such that

$$\|D^\mu \mathbf{F}(\mathbf{x})\| \leqslant \frac{C\mu!}{r^{|\mu|}},$$

for all $\mathbf{x} \in V$, $\mu \in \mathbb{N}_0^m$.

Let $H \coloneqq \left\{\mathbf{x} \in V \,\middle|\, \left|x^1 + \cdots + x^m\right| < r\right\}$ and set $\mathbf{G} = (G^1, \dots, G^m) =: (G_{C,r}, \dots, G_{C,r}) \in C^\omega(H, \mathbb{R}^m)$, with

$$\begin{aligned}
G_{C,r}(\mathbf{x}) =& \frac{Cr}{r - (x^1 + \cdots + x^m)} \\
=& C \sum_{\ell \geqslant 0} \left(\frac{x^1 + \cdots + x^m}{r}\right)^\ell \\
=& C \sum_{\ell \geqslant 0} \frac{1}{r^\ell} \sum_{\substack{\mu \in \mathbb{N}_0^m, \\ |\mu| = \ell}} \binom{\ell}{\mu} \mathbf{x}^\mu \\
=& \sum_{|\mu| \geqslant 0} \frac{1}{\mu!} \cdot \frac{C|\mu|!}{r^{|\mu|}} \mathbf{x}^\mu,
\end{aligned}$$

and thus, for $j \in \{1, \dots, m\}$,

$$\left|D^\mu F^j(0)\right| \leqslant \|D^\mu \mathbf{F}(0)\| \leqslant \frac{C\mu!}{r^{|\mu|}} \leqslant \frac{C|\mu|!}{r^{|\mu|}} = D^\mu G_{C,r}(0). \tag{$\star$}$$

Next we claim that, with $\mathbf{G} \colon H \longrightarrow \mathbb{R}^m$ given as before, the IVP

$$\begin{cases} \frac{d\mathbf{v}}{dt}(t) = \mathbf{G}(\mathbf{v}(t)) \\ \mathbf{v}(0) = 0 \end{cases}$$

has a unique solution that is real analytic near the origin.

Indeed, since for $\mathbf{v} = (v^1, \dots, v^m)$ it holds that

$$\dot{v}^i(t) = \dot{v}^j(t),$$

the fact that $\mathbf{v}(0) = 0$ implies that for all i, j, $v^i \equiv v^j =: v$. Thus, finding the solution to the above IVP boils down to solving the one variable IVP

$$\begin{cases} \frac{dv}{dt} = \frac{Cr}{r - mv} \\ v(0) = 0. \end{cases}$$

An elementary computation easily shows that the solution to this IVP is given by

$$v(t) = \frac{1}{m}\left(r - \sqrt{r^2 - 2Crmt}\right),$$

which is analytic on $\left(-\infty, \frac{r}{2Cm}\right)$, and hence the desired assertion is proved.

By making use of the fact that the previously found polynomials P^i_ℓ have positive coefficients together with the estimate ($\star$) we obtain, for $i \in \{1, \dots, m\}$, $k \in \mathbb{N}$

$$u^i_k(0) = P^i_k(D^\mu F^j(0)) \leqslant \left|u^i_k(0)\right| \leqslant P^i_k(\left|D^\mu F^j(0)\right|) \leqslant P^i_k(D^\mu G_{C,r}(0)) = \nu^{(k)}(0),$$

which implies the convergence of the power series

$$\sum_{k\geqslant 0} \frac{1}{k!}\left|u^i_k(0)\right|t^k,$$

what guarantees that in a sufficiently small neighborhood of the origin,

$$u^i(t) = \sum_{k\geqslant 0} \frac{1}{k!} u^i_k(0) t^k,$$

thus showing the real analyticity near 0 of the solution $\mathbf{u}$. □

We notice that an immediate consequence of the previous result is the existence of real analytic solutions of inhomogeneous systems of ordinary differential equations:

Corollary 2.1.1. *Let* $0 \in J \subset \mathbb{R}$ *be an open interval,* $0 \in U \subseteq \mathbb{R}^m$ *be an open set,* $\mathbf{F} = (F^1, \dots, F^m)\colon J \times U \longrightarrow \mathbb{R}^m$ *be a real analytic vector-valued function. Then the unique solution* $\mathbf{u} = (u^1, \dots, u^m)$ *to the IVP*

$$\begin{cases} \frac{d\mathbf{u}}{dt}(t) = \mathbf{F}(t, \mathbf{u}(t)) \\ \mathbf{u}(0) = 0 \end{cases}$$

is real analytic near 0*.*

Indeed, we prove this result by defining the auxiliary autonomous IVP

$$\begin{cases} \frac{d\mathbf{v}}{dt} = \mathbf{G}(\mathbf{v}(t)) \coloneqq (1, \mathbf{F}(\mathbf{v}(t))) \\ \mathbf{v}(0) = 0. \end{cases}$$

Clearly, the unique analytic solution to this auxiliary IVP gives us the unique analytic solution to the nonautonomous system.

Our next result is a slight generalization of the previous proposition, which in fact settles the matter regarding the particular system that is of interest to us.

Proposition 2.1.6. *Let* $\mathbf{u}_0\colon \mathbb{R}^m \longrightarrow \mathbb{R}^\ell$ *be real analytic near* 0*,* $\mathbf{F}\colon \mathbb{R}^{m+\ell} \longrightarrow \mathbb{R}^\ell$ *be real analytic near* $(0, \mathbf{u}_0(0))$*. Then the initial value problem*

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t)) \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \end{cases}$$

admits a unique real analytic solution near $0 \in \mathbb{R}^{m+1}$*.*

Proof. We proceed in a similar fashion as in the proof of the previous result. Firstly, by setting $\mathbf{v}(\mathbf{x}, t) \coloneqq \mathbf{u}(\mathbf{x}, t) - \mathbf{u}_0(\mathbf{x})$ we may assume, without loss of generality, that $\mathbf{u}_0 \equiv 0$. With this assumption, it easily follows that for a smooth solution $\mathbf{u} = (u^1, \dots, u^\ell)$ to the initial value problem

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t)) \\ \mathbf{u}(\mathbf{x}, 0) = 0, \end{cases}$$

any $\nu \in \mathbb{N}_0^m$, and $a \in \{1, \dots, \ell\}$,

$$D^\nu u^a(0, 0) = 0.$$

In a similar fashion to the proof of the previous proposition, by means of the generalized Faà di Bruno formula, we obtain for all $\nu \in \mathbb{N}_0^m$, $k \in \{1, \dots, \ell\}$ unique and universal polynomials with non-negative coefficients $P^a_{\nu,k}$ such that

$$D^\nu \frac{\partial^k}{\partial t^k} u^a(0,0) = P^a_{\nu,k}\left(D^\mu \frac{\partial^j}{\partial t^j} F^c(\mathbf{x}, \mathbf{u}(\mathbf{x}, t))\right)\bigg|_{(\mathbf{x},t,\mathbf{u})=0},$$

with $|\mu| + j \leqslant |\nu| + k - 1$, $c \in \{1, \dots, \ell\}$.

Next, let $C, r > 0$ such that on a suitable open neighborhood $0 \in V \subseteq \mathbb{R}^{m+\ell}$, for all $\alpha \in \mathbb{N}_0^{m+\ell}$,

$$\|D^\alpha \mathbf{F}(\mathbf{z})\| \leqslant \frac{C\alpha!}{r^{|\alpha|}},$$

for all $\mathbf{z} \in V$ and define the analytic map $\mathbf{G} = (G^1, \dots, G^\ell) =: (G_{C,r}, \dots, G_{C,r})\colon H \longrightarrow \mathbb{R}^\ell$, with $H \coloneqq \left\{(x^1, \dots, x^m, y^1, \dots, y^\ell) \in V \,\middle|\, \left|x^1 + \cdots + y^\ell\right| < r\right\}$, and

$$G_{C,r}(\mathbf{x}, \mathbf{y}) \coloneqq \frac{Cr}{r - (x^1 + \cdots + y^\ell)}.$$

We claim that the initial value problem

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t) = \mathbf{G}(\mathbf{x}, t) \\ \mathbf{v}(\mathbf{x}, 0) = 0 \end{cases}$$

has a real analytic solution near $0 \in \mathbb{R}^{m+1}$.

In order to achieve this, we notice that from the definition of $\mathbf{G}$ together with the fact that $\mathbf{v}(\mathbf{x}, 0) = 0$ we obtain, for any $a, b \in \{1, \dots, \ell\}$,

$$v^a(\mathbf{x}, t) = v^b(\mathbf{x}, t) =: v(\mathbf{x}, t),$$

where $\mathbf{v} = (v^1, \dots, v^\ell)$.

Thus, finding a solution to the above initial value problem boils down to finding a solution to the initial value problem

$$\begin{cases} \frac{\partial v}{\partial t}(\mathbf{x}, t) = \frac{Cr}{r - (x^1 + \cdots + x^m + \ell v(\mathbf{x}, t))} \\ v(\mathbf{x}, 0) = 0. \end{cases} \tag{2.4}$$

By setting $z = x^1 + \cdots + x^m$, finding a solution to the previous IVP is equivalent to finding a solution to the initial value problem

$$\begin{cases} \frac{\partial w}{\partial t}(z,t) = \frac{Cr}{r - z - \ell w(z,t)} \\ w(z,0) = 0. \end{cases}$$

We notice that this IVP is basically an analogue of the auxiliary initial value problem used in our previous proof. A direct computation shows that a solution for it is given by

$$w(z,t) = \frac{1}{\ell}\left(r - z - \sqrt{(r-z)^2 - 2Cr\ell t}\right).$$

Thus, a solution to (2.4) is then given by

$$v(\mathbf{x},t) = \frac{1}{\ell}\left(r - \sum_{j=1}^{m} x^j - \sqrt{\left(r - \sum_{j=1}^{m} x^j\right)^2 - 2Cr\ell t}\right),$$

which is analytic on $B_s(0) \subset \mathbb{R}^{m+1}$, for sufficiently small $s > 0$.

We finalize by noting that, in view of the straightforward estimate

$$|D^\alpha F^a(0)| \leqslant D^\alpha G_{C,r}(0),$$

for $\alpha \in \mathbb{N}_0^{m+\ell}$, $a \in \{1, \dots, \ell\}$, we obtain for every $\nu \in \mathbb{N}_0^m$, $k \in \mathbb{N}_0$,

$$\begin{aligned} \left| D^\nu \frac{\partial^k}{\partial t^k} u^a(0,0) \right| &= \left| P^a_{\nu,k}\left(D^\mu \frac{\partial^j}{\partial t^j} F^c(\mathbf{x}, \mathbf{u}(\mathbf{x},t)) \right)\Big|_{(\mathbf{x},t,\mathbf{u})=0} \right| \\ &\leqslant P^a_{\nu,k}\left(\left| D^\mu \frac{\partial^j}{\partial t^j} F^c(\mathbf{x}, \mathbf{u}(\mathbf{x},t)) \right| \right)\Big|_{(\mathbf{x},t,\mathbf{u})=0} \\ &\leqslant P^a_{\nu,k}\left(D^\mu \frac{\partial^j}{\partial t^j} G_{C,r}(\mathbf{x}, \mathbf{v}(\mathbf{x},t)) \right)\Big|_{(\mathbf{x},t,\mathbf{v})=0} \\ &= D^\nu \frac{\partial^k}{\partial t^k} v(0,0), \end{aligned}$$

which in turn implies the convergence in a sufficiently small neighborhood of $0 \in \mathbb{R}^{m+1}$ of the series

$$\sum_{\substack{\mu \in \mathbb{N}_0^{m+1}, \\ |\mu| \geqslant 0}} \frac{1}{\mu!} |D^\mu u^a(0,0)| (\mathbf{x}, t)^\mu$$

for $a \in \{1, \dots, \ell\}$, thus obtaining the real analyticity near the origin of the solution $\mathbf{u}$. The fact this analytic solution is indeed unique follows directly from the nature of the proof since we explicitly showed that there is only one possibility for the values of its derivatives at 0, which thus uniquely determines its Taylor series expansion at the origin. □

As it turns out, our question regarding real analytic solutions depending in a real analytic fashion on initial conditions can now easily be answered:

Corollary 2.1.2. *Let* $J \subseteq \mathbb{R}$ *be an open interval,* $U \subseteq \mathbb{R}^m$ *be an open subset,* $\mathbf{F}\colon J \times U \longrightarrow \mathbb{R}^m$ *a real analytic vector-valued function. Then for* $(t_0, t_0, \mathbf{x}_0) \in J \times J \times U$ *the initial value problem*

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t}(t, s, \mathbf{x}) = \mathbf{F}(t, \mathbf{u}(t, s, \mathbf{x})), \\ \mathbf{u}(t_0, t_0, \mathbf{x}) = \mathbf{x} \end{cases} \tag{2.5}$$

has a unique solution which is real analytic near $(t_0, t_0, \mathbf{x}_0) \in \mathbb{R}^{m+2}$.

Proof. By replacing $J \times J \times U$ by $J \times J \times U - \{(t_0, t_0, \mathbf{x}_0)\}$, we may assume, without loss of generality, that $(t_0, t_0, \mathbf{x}_0) = (0, 0, \mathbf{0}) = 0 \in \mathbb{R}^{m+2}$.

Now, proceeding as in our argument for the previous corollary, we obtain finding the solution to this IVP is equivalent to finding the solution to the initial value problem

$$\begin{cases} \frac{\partial \mathbf{w}}{\partial t}(t, \mathbf{y}) = \mathbf{G}(\mathbf{w}(t, \mathbf{y})) \\ \mathbf{w}(0, \mathbf{y}) = \mathbf{y}, \end{cases}$$

which, by our previous proposition has a unique real analytic solution near $0 \in \mathbb{R}^{m+2}$, hence proving our claim.

$\square$

As previously asserted at the beginning of the section, the previous result is equivalent to the real analytic analogue of Proposition 2.1.1. This allows us to provide the adequate setting for the original differential equation in which we were interested in the first place:

Proposition 2.1.7. *Let* $U \subseteq \mathbb{R}^{m+1}$ *be an open set,* $F\colon U \longrightarrow \mathbb{R}$ *be real analytic. Thus, for all* $(x^0, \dots, x^{m-1}, t_0) \in U$, *the initial value problem*

$$\begin{cases} u^{(m)}(t) = F(u(t), \dots, u^{(m-1)}(t), t) \\ u^{(i)}(t_0) = x^i & i \in \{0, \dots, m-1\} \end{cases}$$

has a unique real analytic solution near $(x^0, \dots, x^{m-1}, t_0)$, *which depends in a real analytic fashion on the initial conditions.*

Proof. By setting $u^i = u^{(i-1)}$, for $i \in \{1, \dots, m\}$ we obtain that a solution to this IVP is equivalent to a solution to the initial value problem

$$\begin{cases} \frac{d\mathbf{u}}{dt}(t) = \mathbf{G}(\mathbf{u}(t), t) := (u^2, \dots, u^m, F(\mathbf{u}(t), t)) \\ \mathbf{u}(t_0) = (x^0, \dots, x^{m-1}), \end{cases}$$

which in light of our previous result has a unique real analytic solution near $(x^0, \dots, x^{m-1}, t_0)$ depending in a real analytic fashion on the initial data. Proving thus our assertion. $\square$

In order to make some final remarks on the topics previously discussed, we formally introduce a concept of which we shall make intensive use in later steps in this work (and of which until this point we have only superficially used), namely, that of a formal power series.

Definition 2.1.3. *Let* $(A, +, \cdot)$ *be a commutative algebra over a field* $\mathbb{F}$. *A* formal power series *in the indeterminates* $x_1, \dots, x_m$ *with coefficients in* A *is an expression* $f = \sum_{\mu \in \mathbb{N}_0^m} a_\mu x_1^{\mu_1} \cdots x_m^{\mu_m}$, *in short* $\sum_\mu a_\mu \mathbf{x}^\mu$ *or* $\sum a_\mu \mathbf{x}^\mu$, *where* $a_\mu \in A$ *for every* μ. *The* a_μ*'s are called the* coefficients *of* $\sum a_\mu \mathbf{x}^\mu$. *The coefficient* $a_{(0,\dots,0)}$ *is denoted by* $f(0)$. *The set of all these formal power series will be denoted by* $A[[x_1, \dots, x_m]]$ *or* $A[[\mathbf{x}]]$.

We notice that the algebra structure of A naturally induces a commutative algebra structure on $A[[\mathbf{x}]]$. Indeed, by setting

$$\sum_\mu a_\mu \mathbf{x}^\mu + \sum_\mu b_\mu \mathbf{x}^\mu := \sum_\mu (a_\mu + b_\mu)\mathbf{x}^\mu,$$

$$\left(\sum_\mu a_\mu \mathbf{x}^\mu\right) \cdot \left(\sum_\mu b_\mu \mathbf{x}^\mu\right) := \sum_\mu \left(\sum_{\nu+\lambda=\mu} a_\nu \cdot b_\lambda\right)\mathbf{x}^\mu.$$

The proof of the following lemma is straightforward.

Lemma 2.1.1. *Let* $(A, +, \cdot, \leqslant)$ *be a commutative algebra over the field* $\mathbb{F}$ *which is additionally a partially ordered set. It holds that the commutative algebra* $A[[\mathbf{x}]]$ *is a partially ordered set as well, with the order relation given by*

$$f \leqslant g \text{ if and only if, for all } \mu \in \mathbb{N}^m, \ a_\mu \leqslant b_\mu,$$

where $f = \sum_\mu a_\mu \mathbf{x}^\mu$, $g = \sum_\mu b_\mu \mathbf{x}^\mu \in A[[\mathbf{x}]]$.

Remark. By using the proof of the previous proposition together with the previous lemma, we can deduce that, if $\mathbb{R}[[\mathbf{x}]] \ni F = \sum_\mu a_\mu \mathbf{x}^\mu \geqslant 0$ (in the sense of the previous lemma), and u is such that $u^{(m)} \leqslant F(t, u, u', \dots, u^{(m-1)})$, with $u(0) = 0, \dots, u^{(m-1)}(0) = 0$, then it holds that $u \leqslant v$, where $v^{(m)} = F(t, v, \dots, v^{(m-1)})$, with $v(0) = 0, \dots, v^{(m-1)}(0) = 0$. Indeed, the inequality $u^{(m)} \leqslant F(t, u, u', \dots, u^{(m-1)})$ with $u(0) = \dots = u^{(m-1)}(0) = 0$ is equivalent to the inequality

$$\begin{cases} \frac{d\mathbf{u}}{dt} \leqslant \mathbf{G}(t, \mathbf{u}) \\ \mathbf{u}(0) = 0, \end{cases}$$

with $\mathbf{G}$ and $\mathbf{u}$ defined as in the proof of Proposition 2.1.7.

The fact that $F \geqslant 0$ implies for $i = 1, \dots, m$, $k \in \mathbb{N}$,

$$\left(u^i\right)^{(k)}(0) \leqslant P_k^i(D^\mu F^j(0)) = \left(z^i\right)^{(k)}(0),$$

where $\mathbf{z} = (z^1, \dots, z^m)$ is a solution to the initial value problem

$$\begin{cases} \frac{d\mathbf{z}}{dt} = \mathbf{G}(t, \mathbf{z}) \\ \mathbf{z}(0) = 0 \end{cases}$$

This implies

$$w^i = \sum_{k \geqslant 0} \frac{\left(w^i\right)^{(k)}(0)}{k!} t^k \leqslant \sum_{k \geqslant 0} \frac{\left(z^i\right)^{(k)}(0)}{k!} t^k = z^i, \qquad i = 1, \dots, m.$$

Thus $\mathbf{w} \leqslant \mathbf{z}$, which implies in particular

$$u = w^1 \leqslant z^1 = v,$$

and from the definition of $\mathbf{G}$,

$$\begin{cases} v^{(m)} = F(t, v, \dots, v^{(m-1)}) \\ v^{(i)}(0) = 0 & i \in \{0, \dots, m-1\}. \end{cases}$$

An immediate consequence of the previous remark is the following

Proposition 2.1.8. *Let $U \subseteq \mathbb{R}^{m+1}$ be an open set containing 0, $F\colon U \longrightarrow \mathbb{R}$ be real analytic, whose Taylor series at 0 satisfies $F \geqslant 0$ in the sense of Lemma 2.1.1. If u is a function satisfying the differential inequality*

$$\begin{cases} u^{(m)}(t) \leqslant F(u(t), \dots, u^{(m-1)}(t), t) \\ u^{(i)}(0) = 0 & i \in \{0, \dots, m-1\} \end{cases}$$

then u is in fact a real analytic function near 0.

We finish up this section by making some further comments regarding the particular case in which $H = \mathbb{R}$. In particular, in the relationship between smooth functions and their formal Taylor series.

Let $f \in C^\infty(\mathbb{R}^m)$. We denote its formal Taylor series at the point $\mathbf{x} = 0$ by

$$T(f) := \sum_{\mu} \frac{1}{\mu!} D^\mu f(0) \mathbf{x}^\mu \in \mathbb{R}[[\mathbf{x}]].$$

A well-known fact is that the map

$$\begin{aligned} C^\infty(\mathbb{R}^m) &\longrightarrow \mathbb{R}[[\mathbf{x}]] \\ f &\longmapsto T(f) \end{aligned}$$

is in fact surjective. Explicitly, we have the following result (see [24, Theorem 1.5.4]):

Proposition 2.1.9 (Theorem of Borel)**.** *For each sequence $(c_\mu)_{\mu \in \mathbb{N}_0^m} \subseteq \mathbb{R}$ there exists a smooth function $f \in C^\infty(\mathbb{R}^m)$ such that*

$$\frac{1}{\mu!} D^\mu f(0) = c_\mu.$$

Of course, such an f is highly not uniquely given, not even in the case $m = 1$. This is due to the fact of the existence of smooth functions whose formal Taylor series at 0 is the 0-series, like the well-known case of the smooth function $e^{-\frac{1}{x^2}}$.

It is also worth pointing out that thanks to the Theorem of Borel, and by working component-wise, we obtain a surjection

$$C^\infty(\mathbb{R}^m, \mathbb{R}^\ell) \longrightarrow \mathbb{R}^\ell[[\mathbf{x}]].$$

2.2 The vector spaces $\mathrm{Sym}^\bullet W$ and $\bigwedge^\bullet W$

Throughout this section, W will denote an inner-product finite-dimensional real vector space, whose inner product we denote by $\langle\cdot,\cdot\rangle$. The inner product on W induces an inner product on the tensor powers of W. Namely, by linearly extending the map

$$\begin{aligned}\langle\cdot,\cdot\rangle : \bigotimes^k W \times \bigotimes^k W &\longrightarrow \mathbb{R}\\ (v_1\otimes\cdots\otimes v_k, w_1\otimes\cdots\otimes w_k) &\longmapsto \prod_{i=1}^k \langle v_i, w_i\rangle .\end{aligned}$$

Next, we define the natural maps

$$\begin{aligned}\pi_{\mathrm{Sym}} : \bigotimes^k W &\longrightarrow \mathrm{Sym}^k W\\ w_1\otimes\cdots\otimes w_k &\longmapsto \frac{1}{k!}\sum_{\sigma\in S_k} w_{\sigma(1)}\otimes\cdots\otimes w_{\sigma(k)}\\ \pi_\wedge : \bigotimes^k W &\longrightarrow \bigwedge^k W\\ w_1\otimes\cdots\otimes w_k &\longmapsto \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma) w_{\sigma(1)}\otimes\cdots\otimes w_{\sigma(k)},\end{aligned}$$

We note that the maps π_{Sym} and $\pi_\wedge$ are self-adjoint. Indeed,

$$\begin{aligned}\left\langle \pi_{\mathrm{Sym}}(v_1\otimes\cdots\otimes v_k), w_1\otimes\cdots\otimes w_k\right\rangle &= \left\langle \frac{1}{k!}\sum_\sigma v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}, w_1\otimes\cdots\otimes w_k\right\rangle\\ &= \frac{1}{k!}\sum_\sigma\prod_i \left\langle v_{\sigma(i)}, w_i\right\rangle\\ &= \frac{1}{k!}\mathrm{perm}(\langle v_i, w_j\rangle)_{i,j}\\ &= \frac{1}{k!}\sum_\sigma\prod_i \left\langle v_i, w_{\sigma(i)}\right\rangle\\ &= \left\langle v_1\otimes\cdots\otimes v_k, \pi_{\mathrm{Sym}}(w_1\otimes\cdots\otimes w_k)\right\rangle,\end{aligned}$$

where $\mathrm{perm}\colon M_{k\times k}(\mathbb{R}) \longrightarrow \mathbb{R}$ denotes the *permanent* of the matrix, which is defined as

$$\mathrm{perm}(a_{ij})_{1\leqslant i,j\leqslant k} \coloneqq \sum_{\sigma\in S_k}\prod_{i=1}^k a_{i\sigma(i)}.$$

Similarly to the determinant, the permanent is invariant under transposition, that is, $\mathrm{perm}(A) = \mathrm{perm}(A^T)$, which translates into the equation

$$\sum_\sigma\prod_i a_{i\sigma(i)} = \sum_\sigma\prod_i a_{\sigma(i)i},$$

which is what justifies the second-to-last equality previously written.

That the map π_{Sym} is in fact self-adjoint follows from the fact that every element of $\bigotimes^k W$ is a finite sum of indecomposable tensors. The proof for $\pi_\wedge$ is completely analogous, due to the invariance of the determinant under transposition.

We also note that both of these maps are idempotent, that is, $\pi_{\mathrm{Sym}}^2 = \pi_{\mathrm{Sym}}$, $\pi_\wedge^2 = \pi_\wedge$:

$$\begin{aligned}
\pi_{\mathrm{Sym}}^2(w_1 \otimes \cdots \otimes w_k) &= \pi_{\mathrm{Sym}}\left(\frac{1}{k!}\sum_\sigma w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma_k}\right) \\
&= \frac{1}{(k!)^2}\sum_{\sigma,\tau} w_{\tau\sigma(1)} \otimes \cdots \otimes w_{\tau\sigma(k)} \\
&= \frac{1}{(k!)^2}\sum_{\sigma,\tau} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)} \\
&= \frac{1}{k!}\sum_{\sigma} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)} \\
&= \pi_{\mathrm{Sym}}(w_1 \otimes \cdots \otimes w_k).
\end{aligned}$$

For the idempotence of $\pi_\wedge$ we need to perform a similar calculation, noting that $\pi_\wedge(w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}) = \mathrm{sgn}(\sigma)\pi_\wedge(w_1 \otimes \cdots \otimes w_k)$.

The result now follows, as before, from the linearity of π_{Sym} and $\pi_\wedge$.

We also define $\langle\cdot,\cdot\rangle_{\mathrm{Sym}^k} : \mathrm{Sym}^k W \times \mathrm{Sym}^k W \longrightarrow \mathbb{R}$ as $\langle\cdot,\cdot\rangle_{\mathrm{Sym}^k} = \langle\cdot,\cdot\rangle|_{\mathrm{Sym}^k W \times \mathrm{Sym}^k W}$ and the analogous $\langle\cdot,\cdot\rangle_{\bigwedge^k}$. From the previous discussion we can obtain an explicit formula for these inner products:

$$\begin{aligned}
\langle v_1 \cdot \ldots \cdot v_k, w_1 \cdot \ldots \cdot w_k\rangle_{\mathrm{Sym}^k} &= \left\langle \pi_{\mathrm{Sym}}(v_1 \otimes \cdots \otimes v_k), \pi_{\mathrm{Sym}}(w_1 \otimes \cdots \otimes w_k)\right\rangle \\
&= \left\langle v_1 \otimes \cdots \otimes v_k, \pi_{\mathrm{Sym}}(w_1 \otimes \cdots \otimes w_k)\right\rangle \\
&= \frac{1}{k!}\operatorname{perm}(\langle v_i, w_j\rangle)_{i,j}; \\
\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k\rangle_{\bigwedge^k} &= \langle \pi_\wedge(v_1 \otimes \cdots \otimes v_k), \pi_\wedge(w_1 \otimes \cdots \otimes w_k)\rangle \\
&= \langle v_1 \otimes \cdots \otimes v_k, \pi_\wedge(w_1 \otimes \cdots \otimes w_k)\rangle \\
&= \frac{1}{k!}\det\big(\langle v_i, w_j\rangle\big)_{i,j}
\end{aligned}$$

Remark. It is worth mentioning that in most of the literature, the inner product on the spaces $\bigwedge^k W$, $\mathrm{Sym}^k W$ is defined without the factor $\frac{1}{k!}$. The reason why we are diverting from the usual convention is that by introducing this correction factor, we obtain appropriate estimates for the norms of specific operators, which are going to be used in later stages of the work.

From the self-adjointness and idempotence of π_{Sym} and $\pi_\wedge$ together with the Cauchy-Schwarz inequality it follows that

$$\left\|\pi_{\mathrm{Sym}}(x)\right\|^2 = \left\langle \pi_{\mathrm{Sym}}(x), \pi_{\mathrm{Sym}}(x)\right\rangle = \left\langle x, \pi_{\mathrm{Sym}}(x)\right\rangle \leqslant \|x\|\left\|\pi_{\mathrm{Sym}}(x)\right\|,$$

$$\|\pi_\wedge(x)\|^2 = \langle \pi_\wedge(x), \pi_\wedge(x)\rangle = \langle x, \pi_\wedge(x)\rangle \leqslant \|x\|\|\pi_\wedge(x)\|,$$

where, as usual, $\|\cdot\| \coloneqq \sqrt{\langle\cdot,\cdot\rangle}$. We also define $\|\cdot\|_k = \sqrt{\langle\cdot,\cdot\rangle_{\mathrm{Sym}^k}}$. By an abuse of notation, we are going to denote the norm induced on $\bigwedge^k W$ also by the symbol $\|\cdot\|_k$.

This means,

$$\left\|\pi_{\mathrm{Sym}}(x)\right\|_k,\ \|\pi_\wedge(x)\|_k \leqslant \|x\| \qquad \text{for all } x \in \bigotimes^k W.$$

From the definition of the norm on $\bigotimes^k W$, it is clear that, for $k, \ell \in \mathbb{N}$ and $x \in \bigotimes^k W$, $y \in \bigotimes^\ell W$,

$$\|x \otimes y\| = \|x\|\|y\|.$$

The natural way to relate polynomials (resp. forms) of different degrees is, of course, via the product, for which we use the projections π_{Sym} (resp. $\pi_\wedge$). Indeed, for $p \in \mathrm{Sym}^k W$, $q \in \mathrm{Sym}^\ell W$, the polynomial $p \cdot q \in \mathrm{Sym}^{k+\ell} W$ is defined by $\pi_{\mathrm{Sym}}(p \otimes q)$ (resp. for $\omega \in \bigwedge^k W$, $\eta \in \bigwedge^\ell W$, $\omega \wedge \eta \in \bigwedge^{k+\ell} W$ is defined by $\pi_\wedge(\omega \otimes \eta)$).

Thus we get

$$\|p \cdot q\|_{k+\ell} = \left\|\pi_{\mathrm{Sym}}(p \otimes q)\right\| \leqslant \|p \otimes q\| = \|p\|\|q\| = \|p\|_k\|q\|_\ell.$$

(resp. $\|\omega \wedge \eta\|_{k+\ell} \leqslant \|\omega\|_k\|\eta\|_\ell$).

Let $v \in W$. Next, we define the maps:

$$\begin{aligned}
v\cdot\colon\ \mathrm{Sym}^k W &\longrightarrow \mathrm{Sym}^{k+1} W\\
w_1 \cdot \ldots \cdot w_k &\longmapsto v \cdot w_1 \cdot \ldots \cdot w_k;\\
(v\cdot)^*\colon\ \mathrm{Sym}^{k+1} W &\longrightarrow \mathrm{Sym}^k W\\
w_1 \cdot \ldots \cdot w_{k+1} &\longmapsto \frac{1}{k+1}\sum_{i=1}^{k+1} \langle w_i, v\rangle\, w_1 \cdot \ldots \cdot \widehat{w}_i \cdot \ldots \cdot w_{k+1};\\
v\wedge\colon\ \bigwedge^k W &\longrightarrow \bigwedge^{k+1} W\\
w_1 \wedge \cdots \wedge w_k &\longmapsto v \wedge w_1 \wedge \cdots \wedge w_k;\\
(v\wedge)^*\colon\ \bigwedge^{k+1} W &\longrightarrow \bigwedge^k W\\
w_1 \wedge \cdots \wedge w_{k+1} &\longmapsto \frac{1}{k+1}\sum_{i=1}^{k+1} (-1)^{i-1}\langle w_i, v\rangle\, w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge w_{k+1}.
\end{aligned}$$

Next, we claim that the maps $(v\cdot)^*$, $(v\wedge)^*$ are the adjoint of the respective multiplication maps. That is, for every $p \in \mathrm{Sym}^{k+1} W$, $q \in \mathrm{Sym}^k W$, $\alpha \in \bigwedge^{k+1} W$, $\beta \in \bigwedge^k W$ the equations

$$\begin{aligned}
\langle (v\cdot)^* p, q\rangle_{\mathrm{Sym}^k} &= \langle p, v \cdot q\rangle_{\mathrm{Sym}^{k+1}},\\
\langle (v\wedge)^* \alpha, \beta\rangle_{\bigwedge^k} &= \langle \alpha, v \wedge \beta\rangle_{\bigwedge^{k+1}},
\end{aligned}$$

hold.

As usual by now, we deduce only the first equation, since the second one can be deduced in a totally similar fashion. Let $v_1 \cdots v_{k+1} \in \mathrm{Sym}^{k+1} W$, $w_1 \cdots w_k \in \mathrm{Sym}^k W$.

Then

$$\begin{aligned}
\langle (v\cdot)^*(v_1\cdots v_{k+1}), w_1\cdots w_k\rangle_{\mathrm{Sym}^k} &= \frac{1}{k+1}\sum_{i=1}^{k+1}\langle \langle v, v_i\rangle\, v_1\cdots \hat{v}_i\cdots v_{k+1}, w_1\cdots w_k\rangle_{\mathrm{Sym}^k}\\
&= \frac{1}{(k+1)!}\sum_{i=1}^{k+1}\langle v_i, v\rangle \operatorname{perm}\begin{pmatrix} \langle v_1, w_1\rangle & \cdots & \langle v_1, w_k\rangle\\ \vdots & \ddots & \vdots\\ \widehat{\langle v_i, w_1\rangle} & \cdots & \widehat{\langle v_i, w_k\rangle}\\ \vdots & \ddots & \vdots\\ \langle v_{k+1}, w_1\rangle & \cdots & \langle v_{k+1}, w_k\rangle\end{pmatrix}\\
&= \frac{1}{(k+1)!}\operatorname{perm}\begin{pmatrix} \langle v_1, v\rangle & \langle v_1, w_1\rangle & \cdots & \langle v_1, w_k\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle v_{k+1}, v\rangle & \langle v_{k+1}, w_1\rangle & \cdots & \langle v_{k+1}, w_k\rangle\end{pmatrix}\\
&= \langle v_1\cdots v_{k+1}, v\cdot w_1\cdots w_k\rangle_{\mathrm{Sym}^{k+1}}.
\end{aligned}$$

The result follows now from the linearity of these maps.

Now, the operator norm of the map $(v\cdot)^*$ is defined by the formula

$$\|(v\cdot)^*\| = \sup_{\|p\|_{k+1}=1}\|(v\cdot)^*p\|_k = \sup_{\|p\|_{k+1}=1, \|q\|_k=1}\left|\langle (v\cdot)^*p, q\rangle_{\mathrm{Sym}^k}\right|.$$

The above-described properties of this map allow us to obtain an estimate for its norm. Indeed, let $p\in\mathrm{Sym}^{k+1}W$, $q\in\mathrm{Sym}^k W$ both having length 1. It follows that

$$\begin{aligned}
\left|\langle (v\cdot)^*p, q\rangle_{\mathrm{Sym}^k}\right| &= \left|\langle p, v\cdot q\rangle_{\mathrm{Sym}^{k+1}}\right|\\
&\leqslant \|p\|_{k+1}\|v\cdot q\|_{k+1}\\
&= \|v\cdot q\|_{k+1}\\
&= \left\|\pi_{\mathrm{Sym}}(v\otimes q)\right\|\\
&\leqslant \|v\otimes q\|\\
&= \|v\|\|q\|_k\\
&= \|v\|.
\end{aligned}$$

This implies the estimate

$$\|(v\cdot)^*\|\leqslant\|v\|.$$

In a similar fashion, we obtain

$$\|v\wedge\|\leqslant\|v\|.$$

We also note that $\partial_v = (k+1)(v\cdot)^*$, where $\partial_v\colon \mathrm{Sym}^{k+1}W\longrightarrow\mathrm{Sym}^k W$ denotes the directional derivative in the direction of v.

Thus, for $v \in W$, $p \in \operatorname{Sym}^{k+1} W$ we get

$$\begin{aligned}\|\partial_v p\|_k &= (k+1)\|(v\cdot)^* p\|_k \\ &\leqslant (k+1)\|(v\cdot)^*\|\|p\|_{k+1} \\ &\leqslant (k+1)\|v\|\|p\|_{k+1}.\end{aligned}$$

All of what we have done so far could be done without the necessity of referring to a basis of these vector spaces. However, in order to obtain somewhat more refined estimates, we are going to choose some natural ones. In fact, let $\{e_1, \dots, e_m\}$ be an orthonormal basis for W. It is well-known that

$$\begin{aligned}&\{e^I \coloneqq e_1^{i_1} \cdot \dots \cdot e_m^{i_m} \mid I = (i_1, \dots, i_m) \in \mathbb{N}^m,\ |I| = k\}; \\ &\text{resp.,} \\ &\{e_J \coloneqq e_{j_1} \wedge \dots \wedge e_{j_\ell} \mid 1 \leqslant j_1 < \dots < j_\ell \leqslant m\}\end{aligned}$$

are bases for $\operatorname{Sym}^k W$, resp. $\bigwedge^\ell W$. In fact, with respect to the inner product just defined, it is easy to see that these bases are in fact orthogonal, with

$$\begin{aligned}\langle e^I, e^I\rangle &= \frac{I!}{k!}; \\ \langle e_J, e_J\rangle &= \frac{1}{\ell!}.\end{aligned}$$

We define now the map $\partial_a \colon \operatorname{Sym}^{k+1} W \otimes \bigwedge^\ell W \longrightarrow \operatorname{Sym}^k W \otimes \bigwedge^{\ell+1} W$, defined by

$$\partial_a c_I^J e^I \otimes e_J \coloneqq c_I^J i_a e^{I-E_a} \otimes e_a \wedge e_J,$$

where $E_a \coloneqq (0, \dots, 0, 1, 0, \dots, 0)$.

On $\operatorname{Sym}^{k+1} W \otimes \bigwedge^\ell W$ we define the natural inner product induced from the one we defined on each of the factors. Under this inner product, we obtain that

$$\begin{aligned}\langle c_I^J e^I \otimes e_J, c_\mu^\nu e^\mu \otimes e_\nu\rangle &= c_I^J c_\mu^\nu \langle e^I, e^\mu\rangle\langle e_J, e_\nu\rangle \\ &= \sum_{I,J} \left(c_I^J\right)^2 \frac{I!}{(k+1)!\ell!}.\end{aligned}$$

With this we get

$$\begin{aligned}\left\|\partial_a c_I^J e^I \otimes e_J\right\|^2 &\coloneqq \langle \partial_a c_I^J e^I \otimes e_J, \partial_a c_\mu^\nu e^\mu \otimes e_\nu\rangle = c_I^J c_\mu^\nu i_a \mu_a \langle e^{I-E_a}, e^{\mu-E_a}\rangle\langle e_a \wedge e_J, e_a \wedge e_\nu\rangle \\ &= \sum_{I,J}\left(c_I^J\right)^2 i_a^2 \frac{(I-E_a)!}{(\ell+1)!k!} \\ &= \sum_{I,J}\left(c_I^J\right)^2 i_a \frac{I!}{(\ell+1)!(k+1)!} \\ &\leqslant (k+1)^2 \sum_{I,J}\left(c_I^J\right)^2 \frac{I!}{\ell!(k+1)!} \\ &= (k+1)^2 \langle c_I^J e^I \otimes e_J, c_\mu^\nu e^\mu \otimes e_\nu\rangle \\ &\eqqcolon (k+1)^2 \left\|c_I^J e^I \otimes e_J\right\|^2.\end{aligned}$$

Now, for $v = v^a e_a \in W$, we define $\partial_v \coloneqq v^a \partial_a$. With the previous estimate, together with the Cauchy-Schwarz inequality on W, we obtain that, for $p \in \mathrm{Sym}^{k+1} W \otimes \bigwedge^\ell W$,

$$\begin{aligned}\|\partial_v p\| &= \|v^a \partial_a p\| \\ &\leqslant |v^a| \|\partial_a p\| \\ &\leqslant \|v\| \sum_{a=1}^m (k+1)\|p\| \\ &= m(k+1)\|v\|\|p\|.\end{aligned}$$

That is, for $v \in W$, the map $\partial_v \colon \mathrm{Sym}^{k+1} W \otimes \bigwedge^\ell W \longrightarrow \mathrm{Sym}^k W \otimes \bigwedge^{\ell+1} W$ satisfies that, for all p,

$$\|\partial_v p\| \leqslant m(k+1)\|v\|\|p\|. \tag{2.6}$$

We close this section with the introduction of a certain vector space which will be of great importance in the present work, as it will explicitly be described in later stages.

Define the vector space

$$K(W) \coloneqq \ker\left\{A \colon \bigwedge^2 W \otimes W \longrightarrow \bigwedge^3 W\right\},$$

where the map A denotes the natural anti-symmetrization map.

Let us now consider the map

$$\begin{aligned} d_{k+1} \colon \mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W &\longrightarrow \mathrm{Sym}^k W \otimes K(W), \\ \sum_{i,j} p_{ij} \otimes e_i e_j &\longmapsto \frac{1}{2}\sum_{i,j,\ell}(\partial_i p_{j\ell} - \partial_j p_{i\ell}) \otimes e_i \wedge e_j \otimes e_\ell.\end{aligned}$$

That the codomain of this map is rightly defined easily follows from the fact that

$$\sum_{\mathrm{cyc}\,(i,j,\ell)} \partial_i p_{j\ell} - \partial_j p_{i\ell} = \sum_{\mathrm{cyc}\,(i,j,\ell)} \partial_i (p_{j\ell} - p_{\ell j}) \overset{p_{j\ell}=p_{\ell j}}{=} 0.$$

Define the subspace $K^{(k)}(W) \coloneqq \operatorname{im} d_{k+1}$.

As usual, we will in general denote all of the operators d_k by the same symbol d.

Proposition 2.2.1. *With the above notation,*

$$\begin{aligned}K^{(k)}(W) = \ker\left\{\mathbb{1}_{\mathrm{Sym}^k} \otimes A\colon \mathrm{Sym}^k W \otimes \bigwedge^2 W \otimes W \longrightarrow \mathrm{Sym}^k W \otimes \bigwedge^3 W\right\}\\ \cap \ker\left\{\partial \coloneqq \partial \otimes \mathbb{1}_W\colon \mathrm{Sym}^k W \otimes \bigwedge^2 W \otimes W \longrightarrow \mathrm{Sym}^{k-1} W \otimes \bigwedge^3 W \otimes W\right\}.\end{aligned}$$

Proof. The inclusion $\subseteq$ immediately follows from the definition of $K^{(k)}(W) = \operatorname{im} d$. Indeed, for $\sum_{i,j} p_{ij} \otimes e_i e_j \in \mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W$, it holds that

$$\begin{aligned}\frac{1}{2}\sum_{i,j,\ell}\left(\partial_i p_{j\ell} - \partial_j p_{i\ell}\right) \otimes e_i \wedge e_j \otimes e_\ell \longmapsto &\frac{1}{2}\sum_{i,j,\ell}(\partial_i p_{j\ell} - \partial_j p_{i\ell}) \otimes e_i \wedge e_j \wedge e_\ell\\ =&\frac{1}{6}\sum_{i,j,\ell}\sum_{\mathrm{cyc}\,(i,j,\ell)}\left(\partial_i p_{j\ell} - \partial_j p_{i\ell}\right) \otimes e_i \wedge e_j \wedge e_\ell\\ =&0\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\sum_{i,j,\ell}\left(\partial_i p_{j\ell} - \partial_j p_{i\ell}\right) \otimes e_i \wedge e_j \otimes e_\ell \longmapsto &\frac{1}{2}\sum_{a,i,j,\ell}\partial_a(\partial_i p_{j\ell} - \partial_j p_{i\ell}) e_a \wedge e_i \wedge e_j \otimes e_\ell\\ =&\frac{1}{6}\sum_{a,i,j,\ell}\sum_{\mathrm{cyc}\,(a,i,j)}\partial_a\left(\partial_i p_{j\ell} - \partial_j p_{i\ell}\right) e_a \wedge e_i \wedge e_j \otimes e_\ell\\ =&0.\end{aligned}$$

In order to prove the reverse inclusion, we make use of the exactness of the *Koszul complex*[1], which is the vector space complex defined as

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Sym}^{k+2} W \arrow[r, "\partial"] & \mathrm{Sym}^{k+1} W \otimes W \arrow[r, "\partial"] & \mathrm{Sym}^k W \otimes \bigwedge^2 W \arrow[dll, "\partial"] \\ & \mathrm{Sym}^{k-1} W \otimes \bigwedge^3 W \arrow[r, "\partial"] & \mathrm{Sym}^{k-2} W \otimes \bigwedge^4 W \arrow[r, "\partial"] & \cdots . \end{tikzcd}$$

From it, we obtain the exactness of the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Sym}^{k+2} W \otimes W \arrow[r, "\partial"] & \mathrm{Sym}^{k+1} W \otimes W \otimes W \arrow[r, "\partial"] & \mathrm{Sym}^k W \otimes \bigwedge^2 W \otimes W \arrow[dll, "\partial"] \\ & \mathrm{Sym}^{k-1} W \otimes \bigwedge^3 W \otimes W \arrow[r, "\partial"] & \mathrm{Sym}^{k-2} W \otimes \bigwedge^4 W \otimes W \arrow[r, "\partial"] & \cdots . \end{tikzcd}$$

Let $R^{(k)} = \sum_{i,j,\ell} R^{(k)}_{ij,\ell} e_i \wedge e_j \otimes e_\ell$ be an element of the set on the right-hand side. Because of the exactness of the second sequence, we get that there exists $\Gamma^{(k+1)} = \sum_{i,j} \Gamma^{(k+1)}_{i,j} e_i \otimes e_j \in$

[1]For a proof of the exactness of the Koszul complex, see [25, Section 3]

$\mathrm{Sym}^{k+1} W \otimes W \otimes W$ such that

$$R^{(k)}_{ij,\ell} = \frac{1}{2}\left(\partial_i \Gamma^{(k+1)}_{j,\ell} - \partial_j \Gamma^{(k+1)}_{i,\ell}\right).$$

The fact that $R^{(k)} \in \ker\left\{\mathrm{Sym}^k W \otimes \bigwedge^2 W \otimes W \longrightarrow \mathrm{Sym}^k W \otimes \bigwedge^3 W\right\}$ implies that

$$\sum_{\mathrm{cyc}\,(i,j,\ell)} R^{(k)}_{ij,\ell} = 0.$$

Thus

$$\begin{aligned} 0 =& \frac{1}{2} \sum_{\mathrm{cyc}\,(i,j,\ell)} \partial_i \Gamma^{(k+1)}_{j,\ell} - \partial_j \Gamma^{(k+1)}_{i,\ell} \\ =& \frac{1}{2} \sum_{\mathrm{cyc}\,(i,j,\ell)} \partial_i \left(\Gamma^{(k+1)}_{j,\ell} - \Gamma^{(k+1)}_{l,j}\right) \\ =:& \sum_{\mathrm{cyc}\,(i,j,\ell)} \partial_i \Gamma^{(k+1)}_{-\,j,\ell}, \end{aligned}$$

that is, $\Gamma^{(k+1)}_{-} \in \ker\left\{\mathrm{Sym}^{k+1} W \otimes \bigwedge^2 W \longrightarrow \mathrm{Sym}^k W \otimes \bigwedge^3 W\right\}$.

The exactness of the first exact sequence implies that there exists $\phi^{(k+2)} \in \mathrm{Sym}^{k+2} W \otimes W$ such that

$$\Gamma^{(k+1)}_{-\,i,j} = \frac{1}{2}\left(\partial_i \phi^{(k+2)}_j - \partial_j \phi^{(k+2)}_i\right).$$

Define next $\Gamma^{(k+1)}_{+} \in \mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W$ by

$$\Gamma^{(k+1)}_{+\,i,j} \coloneqq \frac{1}{2}\left(\Gamma^{(k+1)}_{i,j} + \Gamma^{(k+1)}_{j,i}\right).$$

Thus

$$\Gamma^{(k+1)}_{i,j} = \Gamma^{(k+1)}_{+\,i,j} + \Gamma^{(k+1)}_{-\,i,j}.$$

Putting all of this together, we get then in sum

$$\begin{aligned} R^{(k)}_{ij,\ell} &= \frac{1}{4}\left[\partial_i \left(2\Gamma^{(k+1)}_{+\,j,\ell} + \partial_j \phi^{(k+2)}_\ell - \partial_\ell \phi^{(k+2)}_j\right)\right. \\ &\qquad \left. -\partial_j \left(2\Gamma^{(k+1)}_{+\,i,\ell} + \partial_i \phi^{(k+2)}_\ell - \partial_\ell \phi^{(k+2)}_i\right)\right] \\ &= \frac{1}{4}\left[\partial_i \left(2\Gamma^{(k+1)}_{+\,j,\ell} - \partial_j \phi^{(k+2)}_\ell - \partial_\ell \phi^{(k+2)}_j\right)\right. \\ &\qquad \left. -\partial_j \left(2\Gamma^{(k+1)}_{+\,i,\ell} - \partial_i \phi^{(k+2)}_\ell - \partial_\ell \phi^{(k+2)}_i\right)\right] \\ &=: \frac{1}{2}\left(\partial_i \Gamma^{(k+1)}_{j\ell} - \partial_j \Gamma^{(k+1)}_{i\ell}\right). \end{aligned}$$

It is clear that $\Gamma^{(k+1)} \coloneqq \sum_{i,j} \Gamma^{(k+1)}_{ij} \otimes e_i e_j \in \mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W$.
The equation above implies then that

$$R^{(k)} = d\Gamma^{(k+1)}$$

$\square$

Thinking a little bit more on the subject, we realize we can explicitly determine the $\Gamma^{(k+1)}$ at the end of the proof of the previous proposition. To achieve this, let $R^{(k)} \in K^{(k)}(W)$. Define $\widetilde{\Gamma} \in \mathrm{Sym}^{k+1} W \otimes W \otimes W$ given by

$$\widetilde{\Gamma}_{j,\ell} := \frac{1}{k+2} \sum_a e_a R^{(k)}_{aj,\ell}.$$

Thus we have

$$\begin{aligned}
\partial_i \widetilde{\Gamma}_{j,\ell} - \partial_j \widetilde{\Gamma}_{i,\ell} &= \frac{1}{k+2} \sum_a \left(\partial_i e_a R^{(k)}_{aj,\ell} - \partial_j e_a R^{(k)}_{ai,\ell} \right) \\
&= \frac{1}{k+2} \sum_a \left(2R^{(k)}_{ij,\ell} + e_a (\partial_i R^{(k)}_{aj,\ell} - \partial_j R^{(k)}_{ai,\ell}) \right) \\
&= R^{(k)}_{ij,\ell}.
\end{aligned}$$

We notice that $\widetilde{\Gamma}_{j,\ell} = \widetilde{\Gamma}^{+}_{j,\ell} + \widetilde{\Gamma}^{-}_{j,\ell}$, where

$$\begin{aligned}
\widetilde{\Gamma}^{+}_{j,\ell} &:= \frac{1}{2(k+2)} \sum_a e_a \left(R^{(k)}_{aj,\ell} + R^{(k)}_{a\ell,j} \right), \\
\widetilde{\Gamma}^{-}_{j,\ell} &:= \frac{1}{2(k+2)} \sum_a e_a \left(R^{(k)}_{aj,\ell} - R^{(k)}_{a\ell,j} \right).
\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}
\sum_{\mathrm{cyc}\,(i,j,\ell)} \partial_i \widetilde{\Gamma}^{-}_{j,\ell} =& \frac{1}{2(k+2)} \sum_{\mathrm{cyc}\,(i,j,\ell)} \sum_a \partial_i e_a R^{(k)}_{lj,a} \\
=& \frac{1}{2(k+2)} \sum_{\mathrm{cyc}\,(i,j,\ell)} \sum_a \left(R^{(k)}_{lj,i} + e_a \partial_i R^{(k)}_{lj,a} \right) \\
=& 0.
\end{aligned}$$

This means,

$$\widetilde{\Gamma}^{-} \in \ker \left\{ \mathrm{Sym}^{k+1} W \otimes \bigwedge^2 W \longrightarrow \mathrm{Sym}^{k} W \otimes \bigwedge^3 W \right\},$$

which implies that there exists $\phi \in \mathrm{Sym}^{k+2} W \otimes W$ such that

$$\frac{1}{2} (\partial_j \phi_\ell - \partial_\ell \phi_j) = \widetilde{\Gamma}^{-}_{j,\ell}.$$

An easy calculation shows that

$$\phi_j := -\frac{1}{(k+2)(k+3)} \sum_{a,b} e_a e_b R^{(k)}_{aj,b},$$

satisfies the desired equation.

Putting all of this together, we get

$$\begin{aligned}
R^{(k)}_{ij,\ell} &= \partial_i \widetilde{\Gamma}_{j,\ell} - \partial_j \widetilde{\Gamma}_{i,\ell} \\
&= \partial_i (\widetilde{\Gamma}^+_{j,\ell} + \widetilde{\Gamma}^-_{j,\ell}) - \partial_j (\widetilde{\Gamma}^+_{i,\ell} + \widetilde{\Gamma}^-_{i,\ell}) \\
&= \frac{1}{2}\Big(\partial_i (2\widetilde{\Gamma}^+_{j,\ell} + \partial_j \phi_\ell - \partial_\ell \phi_j) - \partial_j (2\widetilde{\Gamma}^+_{i,\ell} + \partial_i \phi_\ell - \partial_\ell \phi_i)\Big) \\
&= \frac{1}{2}\Big(\partial_i (2\widetilde{\Gamma}^+_{j,\ell} - \partial_j \phi_\ell - \partial_\ell \phi_j) - \partial_j (2\widetilde{\Gamma}^+_{i,\ell} - \partial_i \phi_\ell - \partial_\ell \phi_i)\Big) \\
&=: \frac{1}{2}\big(\partial_i \Gamma^R_{j\ell} - \partial_j \Gamma^R_{i\ell}\big)
\end{aligned}$$

Explicitly,

$$\Gamma^R_{j\ell} = \frac{1}{(k+2)(k+3)}\left[(k+4)\sum_a e_a\Big(R^{(k)}_{aj,\ell} + R^{(k)}_{a\ell,j}\Big) + \sum_{a,b} e_a e_b \Big(\partial_j R^{(k)}_{a\ell,b} + \partial_\ell R^{(k)}_{aj,b}\Big)\right] \tag{2.7}$$

The reason why we bothered in explicitly constructing these polynomials is that they are used in an elementary proof of the following

Proposition 2.2.2. *With the above notation, it holds that*

$$K^{(k)}(W) \cong \ker\left\{\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W \longrightarrow \mathrm{Sym}^{k+3} W\right\} =: \left(\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W\right)_0.$$

Proof. We define the map

$$\begin{aligned}
\Phi_k \colon K^{(k)}(W) &\longrightarrow \left(\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W\right)_0 \\
R^{(k)} &\longmapsto \Phi_k(R^{(k)}) := \sum_{j,\ell} \Gamma^R_{j\ell} \otimes e_j e_\ell.
\end{aligned}$$

The fact that $\operatorname{im}\Phi_k \subseteq \left(\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W\right)_0$ easily follows from the definition.

We claim that, for all $k \geqslant 0$,

$$d \circ \Phi_k = \mathbb{1}_{K^{(k)}(W)}, \qquad \Phi_k \circ d = \mathbb{1}_{\left(\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W\right)_0},$$

where we denote $d|_{\left(\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W\right)_0}$ simply by d.

The proof is simply a direct computation.

Let $R^{(k)} \in K^{(k)}(W)$. Then

$$
\begin{aligned}
d\Phi_k(R^{(k)}) &= \frac{1}{2}\sum_{i,j,\ell}\left(\partial_i \Gamma^R_{j\ell} - \partial_j \Gamma^R_{i\ell}\right) \otimes e_i \wedge e_j \otimes e_\ell \\
&= \frac{1}{2(k+2)(k+3)} \sum_{i,j,\ell} \Bigg[\partial_i \Bigg((k+4)\sum_a e_a (R^{(k)}_{aj,\ell} + R^{(k)}_{a\ell,j}) + \sum_{a,b} e_a e_b \left(\partial_j R^{(k)}_{a\ell,b} + \partial_\ell R^{(k)}_{aj,b}\right)\Bigg) \\
&\qquad - \partial_j \Bigg((k+4)\sum_a e_a (R^{(k)}_{ai,\ell} + R^{(k)}_{a\ell,i}) + \sum_{a,b} e_a e_b \left(\partial_i R^{(k)}_{a\ell,b} + \partial_\ell R^{(k)}_{ai,b}\right)\Bigg)\Bigg] \otimes e_i \wedge e_j \otimes e_k \\
&\overset{\substack{\sum_{\mathrm{cyc}\,(i,j,\ell)} R^{(k)}_{ij,\ell}=0;\\ \sum_{\mathrm{cyc}\,(a,i,j)} \partial_a R^{(k)}_{ij,\ell}=0}}{=} \frac{1}{2(k+2)} \sum_{i,j,\ell} \Bigg[(k+4) R^{(k)}_{ij,\ell} + \\
&\qquad \sum_a e_a \left(\partial_i R^{(k)}_{a\ell,j} + \partial_j R^{(k)}_{\ell a,i} + \partial_\ell R^{(k)}_{ij,a}\right)\Bigg] \otimes e_i \wedge e_j \otimes e_\ell \\
&= \frac{1}{2(k+2)} \sum_{i,j,\ell} \Bigg[(k+4) R^{(k)}_{ij,\ell} + \sum_a e_a \partial_a R^{(k)}_{ij,\ell} \Bigg] \otimes e_i \wedge e_j \otimes e_\ell \\
&= \sum_{i,j,\ell} R^{(k)}_{ij,\ell} \otimes e_i \wedge e_j \otimes e_\ell \\
&= R^{(k)}
\end{aligned}
$$

This shows the injectivity of the map Φ_k.

The identity $\Phi_k \circ d = \mathbb{1}_{(\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W)_0}$ easily follows from the fact that the equation $\sum_{a,b} e_a e_b \Gamma_{ab} = 0$ implies

$$
\sum_{a,b} e_a e_b \partial_{ij} \Gamma_{ab} = -2\Bigg(\Gamma_{ij} + \sum_a e_a (\partial_i \Gamma_{aj} + \partial_j \Gamma_{ai})\Bigg).
$$

□

Due to the fact that the map $\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W \longrightarrow \mathrm{Sym}^{k+3} W$ is surjective, the previous proposition allows us to compute the dimension of the vector space $K^{(k)}(W)$. Indeed, from the fact that

$$
\dim \left\{ \mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W \longrightarrow \mathrm{Sym}^{k+3} W \right\} = \binom{m+k}{k+1}\binom{m+1}{2} - \binom{m+k+2}{k+3}
$$

we obtain

$$
\dim\left(K^{(k)}(W)\right) = \frac{1}{2}\binom{m+k}{k+2}\frac{k+1}{k+3}(k + (m+1)(k+4)). \tag{2.8}
$$

Because the map Φ_k takes values in a finite-dimensional vector space, we infer that the map is bounded. We are interested in finding a bound for its norm.

We endow the vector spaces $\mathrm{Sym}^k W \otimes \bigwedge^2 W \otimes W$, $\mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W$ with the inner product induced by those on each of the factors, which at the same time induces a norm.

With this, we are able to define the operator norm on the space $\mathrm{Hom}(K^{(k)}(W), \mathrm{Sym}^{k+1} W \otimes \mathrm{Sym}^2 W)$ via the usual formula from Functional Analysis.

Namely, set

$$\|\psi\|_{\mathrm{op}} \coloneqq \sup_{\|R^{(k)}\|=1} \left\|\psi(R^{(k)})\right\|.$$

A general fact about the norm on tensor products of finite-dimensional inner product vector spaces, which can be easily verified, and of which we are going to make use in the following paragraphs is the following

Lemma 2.2.1. *Let* X, Y *be finite-dimensional inner product vector spaces, with orthogonal bases* $\{x_1, \dots, x_n\}, \{y_1, \dots, y_m\}$*. We denote the natural norm on* $X \otimes Y$ *coming from the inner product on* $X \otimes Y$*, which at the same time comes from the ones on each of the factors, by* $\|\cdot\|_\otimes$*. Then it holds,*

i) $\|\sum_{i=1}^m v_i \otimes y_i\|_\otimes = \|\sum_{i=1}^m \|v_i\|_X y_i\|_Y$

ii) $\|\sum_{i=1}^n x_i \otimes w_i\|_\otimes = \|\sum_{i=1}^n \|w_i\|_Y x_i\|_X$.

Using this lemma, we obtain that, for $R^{(k)} \in K^{(k)}(W)$,

$$\left\|R^{(k)}\right\| = \sqrt{\sum_{i,j,\ell} \frac{1}{2}\left\|R^{(k)}_{ij,\ell}\right\|^2}.$$

In particular, if $\|R^{(k)}\| = 1$, we conclude that, for all i, j, l, $\left\|R^{(k)}_{ij,\ell}\right\| \leqslant \sqrt{2}$.
Once more, the previous lemma implies that

$$\left\|\Phi_k(R^{(k)})\right\| \leqslant \sqrt{\sum_{i,j} \left\|\Gamma^R_{ij}\right\|^2}.$$

With all of what we have done so far, we obtain that

$$\begin{aligned}
\|\Gamma^R_{ij}\| \leqslant& \frac{1}{(k+2)(k+3)}\left[(k+4)\sum_a \left\|e_a(R^{(k)}_{ai,j} + R^{(k)}_{aj,i})\right\| + \sum_{a,b}\left\|e_a e_b(\partial_i R^{(k)}_{aj,b} + \partial_j R^{(k)}_{ai,b})\right\|\right] \\
\overset{\|e_a\|=1}{\leqslant}& \frac{1}{(k+2)(k+3)}\left[(k+4)\sum_a \left(\left\|R^{(k)}_{ai,j}\right\| + \left\|R^{(k)}_{aj,i}\right\|\right) + \sum_{a,b}\left(\left\|\partial_i R^{(k)}_{aj,b}\right\| + \left\|\partial_j R^{(k)}_{ai,b}\right\|\right)\right] \\
\overset{\substack{(2.6),\\ \left\|R^{(k)}_{ij,\ell}\right\|\leqslant\sqrt{2}}}{\leqslant}& \frac{1}{(k+2)(k+3)}\left[2\sqrt{2}m(k+4) + \sum_{a,b} 2\sqrt{2}mk\right] \\
=& \frac{2\sqrt{2}}{(k+2)(k+3)}\left[m(k+4) + m^3 k\right] \\
\leqslant& \frac{2\sqrt{2}m(m^2+1)(k+4)}{(k+2)(k+3)} \\
\leqslant& \frac{2\sqrt{2}m(m^2+1)}{k+1}.
\end{aligned}$$

Thus

$$\begin{aligned}\left\|\Phi_k(R^{(k)})\right\| \leqslant & \sqrt{\sum_{i,j}\left\|\Gamma^R_{ij}\right\|^2} \\ \leqslant & \sqrt{\sum_{i,j}\frac{8m^2(m^2+1)^2}{(k+1)^2}} \\ = & \frac{2\sqrt{2}m^2(m^2+1)}{k+1}\end{aligned}$$

In sum we get for all $k \in \mathbb{N}_0$,

$$\|\Phi_k\|_{op} \leqslant \frac{2\sqrt{2}m^2(m^2+1)}{k+1}. \tag{2.9}$$

3

Theory of connections

This chapter is a compendium of more or less well-known facts about the theory of connections on fiber bundles and some of their most relevant consequences in the context of the present work. Its contents are mainly built upon the references [26, Chapters 1-5], [27, Chapters I-IV], and [28, Chapter 6].

3.1 Lie groups, Lie subgroups, and relevant examples

For a good part of this work, Lie groups are going to play a prominent role. That is a compelling enough reason to dedicate a few lines to them. The main goal of the section is to provide some context for a particular list of examples of Lie groups that are going to appear in the later part of the work. Before this, we quickly go over the basic definitions. In addition to the main references provided at the beginning of the chapter, the contents of this section were built upon the references [29, Chapter 1], and [30, Chapter 1].

Definition 3.1.1. *A* Lie group G *is a group and a manifold so that the multiplication and inversion maps are smooth.*

Some of the most common examples of a Lie group are, of course, the group of all invertible matrices over a field $\mathbb{F}$, $\mathrm{GL}(n, \mathbb{F})$, as well as the special linear group $\mathrm{SL}(n, \mathbb{F})$.

Definition 3.1.2. *An* (immersed) submanifold N *of a manifold* M *is the image of a manifold* N' *under an injective immersion* $\varphi\colon N' \longrightarrow M$ *together with the manifold structure on* N *making* $\varphi\colon N' \longrightarrow N$ *a diffeomorphism.*

An imbedded (or regular) submanifold *is a submanifold* N *whose topology agrees with the relative topology.*

Definition 3.1.3. *A* Lie subgroup H *of a Lie group* G *is the image in* G *of a Lie group* H' *under an injective homomorphism* $\varphi\colon H' \longrightarrow G$ *together with the Lie group structure on* H *making* $\varphi\colon H' \longrightarrow H$ *a diffeomorphism.*

We now introduce a couple of examples of Lie groups.

The *(real) orthogonal group* $\mathrm{O}(n, \mathbb{R})$, or just $\mathrm{O}(n)$ is defined as the set

$$\mathrm{O}(n) := \left\{ M \in M_{n\times n}(\mathbb{R}) \,\middle|\, M^\top M = \mathbb{1}_n \right\}.$$

The *complex orthogonal group* $\mathrm{O}(n, \mathbb{C})$ is defined in an analogous fashion.

The *special (real) orthogonal group* $\mathrm{SO}(n, \mathbb{R}) = \mathrm{SO}(n)$ is defined as $\mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$. Similarly for $\mathrm{SO}(n, \mathbb{C})$.

The *unitary group* $\mathrm{U}(n)$ is the group defined as

$$\mathrm{U}(n) := \{M \in M_{n\times n}(\mathbb{C}) |\ M^* M = \mathbb{1}_n\}.$$

Notice that we can naturally consider the unitary group $\mathrm{U}(n)$ as a subgroup of $\mathrm{SO}(2n)$. We identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ via the map

$$\begin{aligned} \iota_{\mathbb{R}} \colon \mathbb{C}^n &\longrightarrow \mathbb{R}^{2n} \\ v + iw &\longmapsto \begin{pmatrix} v \\ w \end{pmatrix}. \end{aligned}$$

Under this identification, multiplication with i in $\mathbb{C}^n$ is thus multiplication with the matrix $J^0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R})$. By means of this identification we obtain the embedding of $\mathrm{GL}(n, \mathbb{C})$ in $\mathrm{GL}(2n, \mathbb{R})$

$$\begin{aligned} \iota_{\mathbb{R}} \colon \ \mathrm{GL}(n, \mathbb{C}) &\longrightarrow \mathrm{GL}(2n, \mathbb{C}) \\ A + iB &\longmapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \end{aligned}$$

for any given real matrices A, B. We notice that the image of this embedding is thus the set of all the matrices in $\mathrm{GL}(2n, \mathbb{R})$ which commute with J^0. In particular we obtain

$$\mathrm{U}(n) \mathrel{\widehat{=}} \iota_{\mathbb{R}}(\mathrm{U}(n)) = \left\{ M \in \mathrm{SO}(2n) |\ MJ^0 = J^0 M \right\}, \tag{3.1}$$

which is the way in which we consider $\mathrm{U}(n)$ a subgroup of $\mathrm{SO}(2n)$.

Since it easily follows that $\mathrm{U}(1) = \mathrm{SO}(2)$, one usually assumes that $n \geqslant 2$. The standard basis of $\mathbb{R}^{2n}$ takes the form

$$(e_1, \dots, e_{2n}) = (e_1, \dots, e_n, J^0 e_1, \dots, J^0 e_n).$$

The *special unitary group* $\mathrm{SU}(n)$ is defined as $\mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$.

By means of the previous embedding, we can consider $\mathrm{SU}(n)$ as a subgroup of $\mathrm{SO}(2n)$ as well.

The *symplectic group* $\mathrm{Sp}(2n, \mathbb{F})$ is the group defined as

$$\mathrm{Sp}(2n, \mathbb{F}) := \left\{ M \in M_{2n\times 2n}(\mathbb{F}) \,\middle|\, M^\top J M = J,\ J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right\}.$$

The *compact symplectic group* $\mathrm{Sp}(n)$ is defined as

$$\mathrm{Sp}(n) \coloneqq \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n).$$

In fact, there is an isomorphism

$$\mathrm{Sp}(n) \mathrel{\widehat{=}} \{M \in \mathrm{GL}(n, \mathbb{H}) |\ M^*M = \mathbb{1}_n\}$$

which allows us to consider $\mathrm{Sp}(n)$ as a subgroup of $\mathrm{SO}(4n)$ (see for example [30]). We identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ by means of the map

$$\begin{aligned} \iota_{\mathbb{C}} \colon \mathbb{H}^n &\longrightarrow \mathbb{C}^{2n} \\ z + wj &\longmapsto \begin{pmatrix} z \\ \overline{w} \end{pmatrix}, \end{aligned}$$

which induces the embedding of the group $\mathrm{GL}(n, \mathbb{H})$ in $\mathrm{GL}(2n, \mathbb{C})$

$$\begin{aligned} \iota_{\mathbb{C}} \colon\ \mathrm{GL}(n, \mathbb{H}) &\longrightarrow \mathrm{GL}(2n, \mathbb{C}) \\ Z + Wj &\longmapsto \begin{pmatrix} Z & -W \\ \overline{W} & \overline{Z} \end{pmatrix}, \end{aligned}$$

for arbitrary complex matrices Z, W. As before, let $J^0_{\mathbb{C}} = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{C})$ and thus obtain

$$\mathrm{GL}(n, \mathbb{H}) \mathrel{\widehat{=}} \iota_{\mathbb{C}}(\mathrm{GL}(n, \mathbb{H})) = \left\{M \in \mathrm{GL}(2n, \mathbb{C}) |\ \overline{M}J^0_{\mathbb{C}} = J^0_{\mathbb{C}}M\right\},$$

which in particular implies

$$\mathrm{Sp}(n) \mathrel{\widehat{=}} \iota_{\mathbb{C}}(\mathrm{Sp}(n)) = \left\{M \in \mathrm{SU}(2n) |\ \overline{M}J^0_{\mathbb{C}} = J^0_{\mathbb{C}}M\right\}. \tag{3.2}$$

With this we can therefore consider $\mathrm{Sp}(n)$ as a subgroup of $\mathrm{SU}(2n)$. Normally one considers $n \geqslant 2$, since $\mathrm{Sp}(1) = \mathrm{SU}(2)$.

Now identify $\mathbb{C}^{2n}$ with $\mathbb{R}^{4n}$ via

$$\begin{aligned} \iota'_{\mathbb{R}} \colon \mathbb{C}^{2n} &\longrightarrow \mathbb{R}^{4n} \\ \begin{pmatrix} z \\ w \end{pmatrix} &\longmapsto \begin{pmatrix} \iota_{\mathbb{R}}(z) \\ \iota_{\mathbb{R}}(w) \end{pmatrix}. \end{aligned}$$

Thus, for real matrices A, B, C, D and $M = A + Bi + (C + Dj)j \in \mathrm{GL}(n, \mathbb{H})$ we obtain

$$\iota'_{\mathbb{R}}\iota_{\mathbb{C}}(M) = \left(\begin{array}{cc|cc} A & -B & -C & D \\ B & A & -D & -C \\ \hline C & D & A & B \\ -D & C & -B & A \end{array}\right).$$

We also obtain that

$$\iota'_{\mathbb{R}}\iota_{\mathbb{C}}(a+bi+cj+dk) = \begin{pmatrix} a \\ b \\ c \\ -d \end{pmatrix} \in \mathbb{R}^{4n}.$$

With respect to the identification $\iota'_{\mathbb{R}}\iota_{\mathbb{C}}$, the right multiplication by an arbitrary unitary quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathrm{Sp}(1)$ on $\mathbb{H}^n$ corresponds to the orthogonal matrix $R_q \in \mathrm{SO}(4n)$

$$R_q = \left(\begin{array}{cc|cc} x_0 & -x_1 & -x_2 & x_3 \\ x_1 & x_0 & x_3 & x_2 \\ \hline x_2 & -x_3 & x_0 & -x_1 \\ -x_3 & -x_2 & x_1 & x_0 \end{array}\right),$$

where each entry $\pm x_\ell$ is in fact the matrix $\pm x_\ell \mathbb{1}_n$.

With this notation, we introduce the matrices $J_1^0 \coloneqq R_i$, $J_2^0 \coloneqq R_j$ and $J_3^0 \coloneqq R_{-k}$. It easily follows that $J_1^0 J_2^0 = J_3^0$, $(J_\ell^0)^2 = -\mathbb{1}_{4n}$ and

$$\mathrm{Sp}(n) \,\widehat{=}\, \iota'_{\mathbb{R}}\iota_{\mathbb{C}}(\mathrm{Sp}(n)) = \left\{ M \in \mathrm{SO}(4n) \mid MJ_\ell^0 = J_\ell^0 M,\ \ell = 1,2,3 \right\}. \tag{3.3}$$

The standard basis of $\mathbb{R}^{4n}$ takes thus the form

$$(e_1, \ldots, e_{4n}) = (e_1, \ldots, e_n, J_1^0 e_1, \ldots, J_3^0 e_n).$$

The multiplication $\mathbb{H}^n \times \mathrm{Sp}(1) \longrightarrow \mathbb{H}^n$ previously discussed actually defines a right action, which can be used to define a further subgroup of $\mathrm{SO}(4n)$. Firstly, we notice that the Lie algebra of the group

$$\mathrm{Sp}(1) \,\widehat{=}\, \{R_q \mid q \in \mathrm{Sp}(1)\} \subseteq \mathrm{SO}(4n)$$

is given by

$$\begin{aligned} \mathfrak{sp}(1) =& \{R_a \mid a \in \mathfrak{sp}(1) = \operatorname{Im}\mathbb{H}\} \\ =& E^0 \coloneqq \operatorname{span}_{\mathbb{R}} \left\{ J_1^0, J_2^0, J_3^0 \right\} \subseteq \mathfrak{so}(4n). \end{aligned}$$

Thus, define the subgroup $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subseteq \mathrm{SO}(4n)$ as

$$\begin{aligned} \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) =& \{M \cdot R_q \mid M \in \mathrm{Sp}(n) \subseteq \mathrm{SO}(4n),\ q \in \mathrm{Sp}(1)\} \\ =& \left\{ L \in \mathrm{SO}(4n) \mid \mathrm{Ad}(L)E^0 = LE^0L^{-1} = E^0 \right\}. \end{aligned}$$

Once again, due to the fact that $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) = \mathrm{SO}(4)$, one normally considers the case $n \geqslant 2$.

Another important example is the so-called *exceptional group* G_2, which is the 14-dimensional, simply-connected, compact Lie subgroup of $\mathrm{SO}(7)$ defined as

$$G_2 \coloneqq \{A \in \mathrm{SO}(7) \mid A^*\omega_0 = \omega_0\},$$

where $\omega_0 \in \bigwedge^3(\mathbb{R}^7)^*$ is the 3-form defined as

$$\omega_0 := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{3.4}$$

The last family of examples we introduce are the so-called *spin groups*. The group $\mathrm{Spin}(n)$ is defined as the double cover of the group $\mathrm{SO}(n)$ such that, for $n \geqslant 3$, there exists a short exact sequence of Lie groups

$$\{1\} \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow \{1\}.$$

For $n \geqslant 3$, $\mathrm{Spin}(n)$ is simply-connected. The first of the members of this family that is not already included as a member of some another family of classical Lie groups is the group $\mathrm{Spin}(7)$, since one has the isomorphisms (see [31, Chapter 1, Theorem 8.1])

$$\begin{aligned}
\mathrm{Spin}(2) &\cong S^1 \\
\mathrm{Spin}(3) &\cong \mathrm{SU}(2) \\
\mathrm{Spin}(4) &\cong \mathrm{SU}(2) \times \mathrm{SU}(2) \\
\mathrm{Spin}(5) &\cong \mathrm{Sp}(2) \\
\mathrm{Spin}(6) &\cong \mathrm{SU}(4)
\end{aligned}$$

The group $\mathrm{Spin}(7)$ can be realized as a compact subgroup of $\mathrm{SO}(8)$. To do so, we define the 4-form $\sigma_0 \in \bigwedge^4(\mathbb{R}^8)^*$

$$\begin{aligned}
\sigma_0 :=& e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1368} - e^{1458} - e^{1467} \\
& - e^{2358} - e^{2367} - e^{2457} + e^{2468} + e^{3456} + e^{3478} + e^{5678}.
\end{aligned} \tag{3.5}$$

Thus, one can show that

$$\mathrm{Spin}(7) = \{A \in \mathrm{SO}(8) |\; A^*\sigma_0 = \sigma_0\}.$$

3.2 Principal and associated fiber bundles

A central part of this work is devoted to the study in a systematic way of the curvature tensor on an affine manifold. Before beginning this, we are going to establish some standard background on the theory of (smooth) fiber bundles. For the purposes of this work, it suffices to focus on a special class of fiber bundles, namely, on principal fiber bundles. Most of the results that are about to be discussed on the following paragraphs are well-known, and can be in principle found within the principal references given at the beginning of the chapter, the only reason to include them is that of self-containment.

First we recall the general definition of a general fiber bundle together with some of their elementary properties.

Definition 3.2.1. *Let* $E, M,$ *and* F *be smooth manifolds and let* $\pi\colon E \longrightarrow M$ *be a smooth map. The quadruple* $(E, \pi, M; F)$ *is called a (locally trivial)* smooth fiber bundle *if for each point* $x \in M$ *there is an open neighborhood* U *of* x *and a smooth diffeomorphism*

$$\phi_U \colon \pi^{-1}(U) \longrightarrow U \times F$$

such that $\mathrm{pr}_1 \circ \phi_U = \pi$.

From the definition, we obtain that the map π is in fact a submersion and E_x a submanifold of E, which is diffeomorphic to F. Indeed, given a pair (U, ϕ_U) as before, the map

$$\phi_{Ux} \coloneqq \mathrm{pr}_2 \circ \phi_U|_{E_x} \colon E_x \longrightarrow F$$

is a diffeomorphism.

The manifold E is called the *total space*, M is called the *base space*, π is called the *bundle projection* and F is called the *typical fiber*. For each $x \in M$, the set $E_x \coloneqq \pi^{-1}(x)$ is called the *fiber* over x.

The pair (U, ϕ_U) in the definition above is called a *local trivialization* over U or a *bundle chart*. A family $\{(U_i, \phi_i)\}_{i\in I}$ of bundle charts such that $\{U_i\}_{i\in I}$ is a cover of M is said to be a *bundle atlas*. Given two bundle charts $(U_i, \phi_i), (U_k, \phi_k)$, whose domains overlap, we define the *transition maps*

$$\phi_i \circ \phi_k^{-1} \colon (U_i \cap U_k) \times F \longrightarrow (U_i \cap U_k) \times F.$$

These transition maps define the maps

$$\begin{aligned} \phi_{ik} \colon U_i \cap U_k &\longrightarrow \mathrm{Diff}(F) \\ x &\longmapsto \phi_{ix} \circ \phi_{kx}^{-1}, \end{aligned}$$

which clearly satisfy the "cocycle conditions":

$$\begin{aligned} \phi_{ii}(x) &= \mathbb{1}_F && \text{for } x \in U_i, \\ \phi_{ik}(x) \circ \phi_{kj}(x) &= \phi_{ij}(x) && \text{for } x \in U_i \cap U_j \cap U_k. \end{aligned}$$

The maps $\{\phi_{ik}\}_{i,k\in I}$ are called the *cocycles of the the bundle*.

The transition maps are, in a way, the building blocks of a fiber bundle as the next proposition asserts.

Proposition 3.2.1. *Let* M, F *be smooth manifolds,* E *a set, and* $\pi \colon E \longrightarrow M$ *a surjective map. Let* $\{(U_i, \phi_i)\}_{i\in I}$ *be a system of local trivializations (that is,* $\{U_i\}_{i\in I}$ *is an open cover of* M *and, for every* i, $\phi_i \colon \pi^{-1}(U_i) \longrightarrow U_i \times F$ *is a bijective map with* $\mathrm{pr}_1 \circ \phi_i = \pi|_{E_{U_i}}$*) such that all of the transition maps* $\phi_i \circ \phi_k^{-1}$ *are smooth. Then there is a unique smooth structure on* E *such that* $(E, \pi, M; F)$ *is a smooth locally trivial fiber bundle with bundle atlas* $\{(U_i, \phi_i)\}_{i\in I}$.

Given a smooth fiber bundle $\pi \colon E \longrightarrow M$, we define the space of *smooth sections* of the fiber bundle as the set

$$\Gamma(E) \coloneqq \{s \in C^\infty(M, E) |\ \pi \circ s = \mathbb{1}_M\}.$$

We now introduce a very important class of fiber bundles. Namely, the principal fiber bundles.

Definition 3.2.2. *Let* $\pi\colon P \longrightarrow M$ *be a smooth fiber bundle with typical fiber a Lie group* G*. The bundle* $(P, \pi, M; G)$ *is called a* principal G-bundle *if there is smooth free right action of* G *on* P *such that*

1. *The action is fiber-preserving:* $\pi(ug) = \pi(u)$ *for all* $u \in P$ *and* $g \in G$.

2. *For each* $x \in M$*, there exists a local trivialization* $\phi_U\colon \pi^{-1}(U) \longrightarrow U \times G$ *with* $x \in U$ *such that*

$$\phi_U(ug) = \phi_U(u)g$$

for all $u \in \pi^{-1}(U)$ *and* $g \in G$*, where the Lie group* G *acts on* $U \times G$ *as* $(x, a)g \coloneqq (x, ag)$.

From the definition, it is straightforward to see that the action of the group G is *transitive on fibers*: that is, given u_1, u_2 in the same fiber, there exists a $g \in G$ such that $u_1 = u_2 g$. Which is equivalent to saying that the fibers of $P \longrightarrow M$ are exactly the orbits of the group action.

As in the case of general fiber bundles, we can reconstruct the structure of a principal bundle from local information.

Proposition 3.2.2. *Let* G *be a Lie group and* $\pi\colon P \longrightarrow M$ *a smooth map. The quadruple* $(P, \pi, M; G)$ *is a principal bundle if, and only if*

i) There is a right action of G *on* P*, which is fiber-preserving and transitive on fibers.*

ii) There is an open cover $\{U_i\}_i$ *of* M *and local sections* $s_i\colon U_i \longrightarrow P$ *for every* i.

One of the most important examples of a principal bundle is the so called frame bundle.

For this, let $E \longrightarrow M$ be a rank k vector bundle with fiber $V = \mathbb{F}^k$. For every $x \in M$, let $GL(V, E_x)$ denote the set of linear isomorphisms from V to E_x. If we choose a fixed basis $(e_1, \dots, e_k)$ for V, then each frame $(u_1, \dots, u_k)$ over x gives rise to an element $u \in GL(V, E_x)$ defined by

$$u(v) = v^i u_i,$$

where $v = v^i e_i$. We identify u with $(u_1, \dots, u_k)$ and refer to it as a frame. With this identification, define σ_α, the *local frame field coming from the local trivialization* $\phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times V$ as

$$\sigma_\alpha(x) = \phi_{\alpha x}^{-1} \qquad \text{for } x \in U_\alpha.$$

Define now

$$F(E) \coloneqq \bigsqcup_{x \in M} GL(V, E_x)$$

Let $\pi\colon F(E) \longrightarrow M$ be the projection map defined by $\pi(u) = x$, for $u \in GL(V, E_x)$. Notice that the group $GL(V)$ acts on the set $F(E)$: $F(E) \times GL(V) \longrightarrow F(E)$ is given by $(u, g) \longmapsto ug \coloneqq$

$u \circ g$. It easily follows that the orbit of a frame at x is exactly the set $\pi^{-1}(x) = GL(V, E_x)$ and that the action is free. For each local trivialization (U, ϕ) for the vector bundle E, let σ_ϕ be the associated frame field. Define $f_\phi \colon U \times GL(V) \longrightarrow \pi^{-1}(U)$ by $f_\phi(x, g) = \sigma_\phi(x)g$. The map f_ϕ is a bijection. Let now be $\tilde{\phi} := f_\phi^{-1}$. We have $\tilde{\phi} = (\pi, \tilde{\Phi})$, where $\tilde{\Phi}$ is uniquely determined by $\tilde{\phi}$. Thus, a system of local trivializations $\{(U_\alpha, \phi_\alpha)\}$ for the vector bundle E induces a system of local trivializations $\{(U_\alpha, \tilde{\phi}_a)\}$ for $F(E) \longrightarrow M$. Which, by Proposition 3.2.1, induces a smooth structure on $F(E)$, making it into a smooth manifold.

Definition 3.2.3. *The* $GL(V)$*-principal bundle constructed above is called the* linear frame bundle *of* E. *The frame bundle for the tangent bundle of a manifold* M *is usually denoted by* $F(M)$ *rather than by* $F(TM)$.

A special mention to the case $E = TM$ is due, since it provides us with several concrete examples of geometric importance.

By the above discussion, we notice that

$$F(M)_x = \{v_x := (v_1, \dots, v_n) |\ v_x \text{ is a basis of } T_xM\} = GL(\mathbb{R}^n, T_xM)$$

In this case we can explicitly describe the action of $GL(n, \mathbb{R})$ on $F(M)$. Namely,

$$(v_1, \dots, v_n) \cdot A = \left(\sum_i A_{i1} v_i, \dots, \sum_i A_{in} v_n \right) = v_x \circ A \in GL(\mathbb{R}^n, T_xM),$$

where $A = (A_{ij})_{i,j} \in GL(n, \mathbb{R})$.

In the same vein we can show that, depending on the structure of our base manifold, we can define certain useful subbundles:

a) Let $(M, \mathcal{O}_M)$ is an oriented manifold, it makes sense to consider the set of all positive oriented bases of its tangent spaces:

$$F(M)_x^+ := \{v_x \in F(M)_x |\ v_x \text{ is a positive oriented basis of } T_xM\}.$$

With this we obtain the $GL(n, \mathbb{R})^+$-principal bundle of positive oriented frames, which we denote by $(F(M)^+, \pi, M; GL(n, \mathbb{R})^+)$.

b) Let $(M^{p,q}, g)$ be a pseudo-Riemannian manifold of signature (p, q). In this case we consider the orthonormal bases of the tangent spaces of M:

$$O(M, g)_x := \left\{ v_x = (v_1, \dots, v_n) \in F(M)_x \;\middle|\; \big(g_x(v_i, v_j)\big)_{i,j} = \begin{pmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{pmatrix} \right\}$$

and so we obtain the $O(p, q)$-principal bundle $(O(M, g), \pi, M; O(p, q))$ of all orthonormal frames.

c) Let (M, ω) be a symplectic manifold of dimension $2n$. We consider the sets

$$Sp(M, \omega)_x := \left\{ v_x = (v_1, \dots, v_{2n}) \;\middle|\; \big(\omega_x(v_i, v_j)\big)_{i,j} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right\}$$

and so we obtain the $Sp(2n, \mathbb{R})$-principal bundle $(Sp(M, \omega), \pi, M; Sp(2n, \mathbb{R}))$ of symplectic frames.

Definition 3.2.4. *Two* G- *principal bundles* $(P, \pi, M; G)$, $(\widetilde{P}, \widetilde{\pi}, M; G)$ *over the same base are called* isomorphic *if there exists a* G*-equivariant diffeomorphism* $\Psi \colon P \longrightarrow \widetilde{P}$ *such that* $\widetilde{\pi} \circ \Psi = \pi$.

A G-principal bundle $P \longrightarrow M$ is called *trivial* if it is isomorphic to the trivial G-principal bundle $(M \times G, pr_1, M; G)$.

The existence of the action of the Lie group G has the following strong consequence.

Proposition 3.2.3. *A* G*-principal bundle* $P \longrightarrow M$ *is trivial if, and only if* $\Gamma(P) \neq \emptyset$.

A nice application of principal bundles is that we can "replace" its fibers in order to construct new fiber bundles.

Let $(P, \pi, M; G)$ be a principal bundle, and let F be a smooth manifold on which there exists a left action of the Lie group G. Thus, on the cartesian product $P \times F$ we have a free right action of G, namely:

$$(p, v) \cdot g \coloneqq (pg, g^{-1}v).$$

We define

$$E \coloneqq (P \times F)/G \eqqcolon P \times_G F,$$

and the projection

$$\begin{aligned} \widehat{\pi} \colon \quad E &\longrightarrow M \\ [p, v] &\longmapsto \pi(p). \end{aligned}$$

This gives rise to a new fiber bundle.

Proposition 3.2.4 ([26, Satz 2.7])**.** *The tuple* $(E, \widehat{\pi}, M; F)$ *is a smooth fiber bundle.*

Definition 3.2.5. *The bundle* $E \longrightarrow M$ *in the previous proposition is called the* associated fiber bundle *to the* G*-principal bundle* $P \longrightarrow M$.

Notice that, on the associated fiber bundle E we have a particular kind of fiber diffeomorphisms:

Definition 3.2.6. *Let* $p \in P_x$*, for* $x \in M$*. The map*

$$\begin{aligned} [p] \colon F &\longrightarrow P_x \times_G F = E_x \\ v &\longmapsto [p, v] \end{aligned}$$

is called the fiber diffeomorphism defined by p.

From the definition, it clearly follows that

$$[pg] = [p] \circ l_g, \qquad \text{for } p \in P,\ g \in G.$$

The inverse of the previous map is given by

$$\begin{aligned}[p]^{-1}\colon P_x \times_G F &\longrightarrow F\\ [q,\nu] &\longmapsto g_q \nu,\end{aligned}$$

where $g_q \in G$ is the unique element of the group such that $q = pg_q$.

Now we see that there is a useful interpretation of the sections of the associated fiber bundle of a principal bundle.

Let $C^\infty(P,F)^G$ denote the set of the smooth G-equivariant maps from P to F:

$$C^\infty(P,F)^G := \{\bar{s} \in C^\infty(P,F) | \ \bar{s}(pg) = g^{-1}\bar{s}(p) \text{ for all } p \in P,\ g \in G\}.$$

The following proposition establishes the one-to-one correspondence between G-equivariant maps and sections of the associated bundle.

Proposition 3.2.5 ([26, Satz 2.9])**.** *Let $E = P \times_G F$ be the associated fiber bundle to the G-principal bundle $P \longrightarrow M$. Then there is a one-to-one correspondence between the sets*

$$\Gamma(E) \overset{1:1}{\longleftrightarrow} C^\infty(P,F)^G.$$

As a useful example of associated fiber bundles, let us consider the frame bundle of a smooth manifold M. It is not difficult to show the vector bundle isomorphism

$$\begin{aligned}F(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n &\longrightarrow TM\\ [(\nu_1,\dots,\nu_n), c^i e_i] &\longmapsto c^i \nu_i,\end{aligned}$$

which in turn implies the vector bundle isomorphism

$$F(M) \times_{GL(n,\mathbb{R})} T^{(r,s)}\mathbb{R}^n \longrightarrow T^{(r,s)}M,$$

by tensorially extending the standard action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ to an action on $T^{(r,s)}\mathbb{R}^n := \bigotimes^r \mathbb{R}^n \otimes \bigotimes^s (\mathbb{R}^n)^*$.

Some additional remarks on vector bundles as associated bundles can be made by making use of so-called *bundle metrics*:

Definition 3.2.7. *A* bundle metric *on the real (resp. complex) vector bundle $E \longrightarrow M$ is a section $\langle\cdot,\cdot\rangle \in \Gamma(E^* \otimes \overline{E}^*)$, which assigns each $x \in M$ to a non-degenerate symmetric bilinear form (resp. to a non-degenerate Hermitian form)*

$$\langle\cdot,\cdot\rangle_{E_x} := \langle\cdot,\cdot\rangle(x)\colon E_x \times E_x \longrightarrow \mathbb{F}.$$

A partition of the unity argument readily shows that on every vector bundle there exists a positive-definite bundle metric.

The existence of bundle metrics on vector bundles implies the following

Proposition 3.2.6. *i) Every real rank k vector bundle is associated to an $O(k)$-principal bundle.*

ii) Every complex rank k vector bundle is associated to a $U(k)$-principal bundle.

Proof. Let $E \longrightarrow M$ be a rank k vector bundle. Let $\langle \cdot, \cdot \rangle$ be a fixed positive-definite bundle metric on E and define the set

$$P_x := \left\{ s_x = (s_1, \dots, s_k) \middle| \, s_x \text{ is a basis of } E_x \text{ with } \langle s_i, s_j \rangle = \delta_{ij} \right\}.$$

We thus get that

$$\begin{aligned} \pi \colon P := \bigsqcup_{x \in M} P_x &\longrightarrow M \\ s_x &\longmapsto x \end{aligned}$$

is an $O(k)$-principal bundle over M (resp. a $U(k)$-principal bundle over M) and we obtain that

$$P \times_{O(k)} \mathbb{R}^k \cong E \quad \text{resp.} \quad P \times_{U(k)} \mathbb{C}^k \cong E,$$

where the vector bundle isomorphism is given by

$$[(s_1, \dots, s_k), x^i e_i] \longmapsto x^i s_i.$$

□

The next result shows us how to obtain an explicit bundle metric on associated vector bundles.

Proposition 3.2.7. *Let $(P, \pi, M; G)$ be a principal bundle, $\rho \colon G \longrightarrow GL(V)$ a representation of G and $\langle \cdot, \cdot \rangle_V$ a G-invariant symmetric ($\mathbb{F} = \mathbb{R}$), resp. Hermitian ($\mathbb{F} = \mathbb{C}$) scalar product on V. Then, on $E = P \times_{(G,\rho)} V$, the maps*

$$\langle e, \widehat{e} \rangle_{E_x} := \langle v, \widehat{v} \rangle_V \qquad \text{for } e, \widehat{e} \in E_x,$$

where $e = [p, v], \widehat{e} = [p, \widehat{v}]$, for a $p \in P_x$ define a bundle metric on E. The scalar products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_{E_x}$ have the same signature.

3.3 Reduction of principal fiber bundles

The main idea behind the concept of associated fiber bundles is to use the particular features that are in general only enjoyed by principal bundles and by exploiting them, we found ways to "change the fiber" of our original bundle.

A more or less natural question would be whether it would be possible to do the same with the structure group of our bundle rather than with the fiber. It turns out that in a way, such a manoeuvre can be achieved. This is precisely what we briefly introduce in the next couple of paragraphs.

Definition 3.3.1. *Let* $(P, \pi_P, M; G)$ *be a* G*-principal bundle and* $\lambda\colon H \longrightarrow G$ *a Lie group homomorphism. A* λ-reduction *of* P *is a pair* (Q, f) *consisting of an* H*-principal bundle* $(Q, \pi_Q, M; H)$ *and a smooth map* $f\colon Q \longrightarrow P$ *which satisfies:*

i) $\pi_P \circ f = \pi_Q$, *and*

ii) $f(q \cdot h) = f(q) \cdot \lambda(h)$ *for every* $q \in Q$, $h \in H$.

In other words, the pair (Q, f) is a λ-reduction of the principal G-bundle P if the diagram

$$\begin{tikzcd} Q \times H \arrow[r, "\cdot"] \arrow[d, "f\times\lambda"] & Q \arrow[d, "f"] \arrow[dr, "\pi_Q"] & \\ P \times G \arrow[r, "\cdot"] & P \arrow[r, "\pi_P"] & M \end{tikzcd}$$

commutes.

In the particular case in which $H \subseteq G$ is a Lie subgroup and λ the inclusion map, a λ-reduction (Q, f) is simply called an H-*reduction* of P. For a Lie subgroup $H \subseteq GL(n, \mathbb{R})$, an H-reduction of the frame bundle $F(M)$ is called an H-*structure* on M.

Definition 3.3.2. *Two* λ*-reductions* (Q, f), $(\widetilde{Q}, \widetilde{f})$ *of the principal bundle* P *are called* isomorphic *if there is a principal bundle isomorphism* $\phi\colon Q \longrightarrow \widetilde{Q}$ *such that* $\widetilde{f} \circ \phi = f$. *We denote the set of all the isomorphism classes of* λ*-reductions of* P *by* $\mathrm{Red}_\lambda(P)$.

The following result provides a criterion for identifying subsets of a principal bundle, which can be given the structure of reductions (see [26, Satz 2.14]).

Proposition 3.3.1. *Let* $H \subseteq G$ *be a Lie subgroup,* $(P, \pi, M; G)$ *a* G*-principal bundle and* $Q \subseteq P$ *a subset with the following properties:*

i) $R_h(Q) = Q$ *for all* $h \in H$.

ii) For $q, \widetilde{q} \in Q_x := Q \cap P_x$ *such that* $q = R_g(\widetilde{q}) = \widetilde{q}g$, *then we have that* $g \in H$.

iii) For each $x \in M$ *there exists an open neighborhood* $U_x \subseteq M$ *and a smooth section* $s\colon U_x \longrightarrow P$ *with* $s(U_x) \subseteq Q$.

Then it follows that Q *is indeed a smooth submanifold of* P, $(Q, \pi|_Q, M; H)$ *an* H*-principal bundle and* (Q, ι) *an* H*-reduction of* P.

Another useful criterion for reducibility relies on the existence of sections of certain associated fiber bundles. First we provide the necessary set-up.

Let $(P, \pi_P, M; G)$ be a G-principal bundle, and $H \subseteq G$ a closed subgroup. We consider the induced left action of the Lie group G on the homogeneous space G/H.

That is,

$$\begin{aligned} G \times G/H &\longrightarrow G/H \\ (g, [a]) &\longmapsto [ga] \end{aligned}$$

Let E denote the associated fiber bundle with respect to this action, i.e.

$$\begin{aligned} E \coloneqq P \times_G G/H \simeq & P/H \\ [p, gH] \longmapsto & (pg)H \end{aligned}$$

Proposition 3.3.2. *For the closed subgroup $H \subseteq G$, there exists an H-reduction of the G-principal bundle $(P, \pi_P, M; G)$ if, and only if the associated fiber bundle $(E, \pi_E, M; G/H)$ has a global section.*

Proof. Let $s \in \Gamma(E)$. According to Proposition 3.2.5, there exists exactly one G-equivariant smooth map $\overline{s} \in C^\infty(P, G/H)^G$ to which s corresponds. Let $Q \subseteq P$ be the subset defined by

$$Q \coloneqq \{p \in P | \, \overline{s}(p) = eH\}.$$

We claim that the tuple $(Q, \pi_Q \coloneqq \pi_P|_Q, M; H)$ is an H-principal bundle and that (Q, ι) is an H-reduction of P.

Because of the G-equivariance of the map $\overline{s}$, it follows that for all $p \in P$, $h \in H$,

$$\overline{s}(ph) = h^{-1}\overline{s}(p) = h^{-1}eH = eH,$$

which implies that the subgroup H acts on the right on the set Q. Let $q, \widetilde{q} \in Q \cap P_x$. Then because the action of G on P_x is simply transitive, we have that there exists exactly one $g \in G$ such that $q = \widetilde{q}g$. Since by definition

$$\overline{s}(q) = eH = \overline{s}(\widetilde{q}g) = g^{-1}\overline{s}(\widetilde{q}) = g^{-1}eH = g^{-1}H$$

we conclude that $g \in H$. From which we conclude that the action of H on Q is fiber-preserving and simply transitive on the fibers. Let $\{(U_i, s_i)\}_{i \in I}$ be a cover of P by local sections that correspond to bundle charts of the G-principal bundle, and $\sigma_i \colon W_i \subseteq G/H \longrightarrow G$ local sections of the homogeneous bundle $p \colon G \longrightarrow G/H$ such that $\overline{s} \circ s_i(U_i) \subseteq W_i$. It then follows that $g_i \coloneqq \sigma_i \circ \overline{s} \circ s_i \colon U_i \longrightarrow G$ is a smooth map. Define next the smooth local section $\widetilde{s}_i \colon U_i \longrightarrow P$ by

$$\widetilde{s}_i(x) \coloneqq s_i(x)g_i(x).$$

The invariance of the map $\overline{s}$ implies

$$\overline{s}(\widetilde{s}_i(x)) = g_i(x)^{-1}\overline{s}(s_i(x)) = g_i(x)^{-1}(g_i(x)H) = eH.$$

Thus the smooth maps $\widetilde{s}_i \colon U_i \longrightarrow Q$ are in fact local sections. The claim follows now from the previous proposition.

Suppose on the other hand, that $(Q, \pi_Q, M; H)$ is an H-principal bundle and that (Q, f) is an H-reduction of P. It follows that the map $f \colon Q \longrightarrow P$ is an embedding. Since the subgroup H acts on the left on the group G, the tuple $(Q \times_H G, \widehat{\pi}, M; G)$ is a G-principal bundle, and in fact it is easy to verify that the map

$$\begin{aligned} Q \times_H G \longrightarrow & P \\ [q, g] \longmapsto & f(q)g \end{aligned}$$

is a well-defined isomorphism of G-principal bundles. We define the smooth map

$$\begin{aligned} \overline{s}\colon P \simeq Q \times_H G &\longrightarrow G/H \\ [q, g] &\longmapsto g^{-1}H. \end{aligned}$$

We notice that for every $g, a \in G$, $q \in Q$,

$$\overline{s}([q, g]a) = \overline{s}([q, ga]) = a^{-1}g^{-1}H = a^{-1}\overline{s}([q, g]),$$

i.e. $\overline{s} \in C^\infty(P, G/H)^G$, which uniquely determines a smooth section of the fiber bundle $(E, \pi_E, M; G/H)$. □

As an application of the previous result, we prove that every G-principal bundle with a non-compact structure group G can indeed be reduced to a compact group.

In order to achieve this, we recall the following profound result concerning the existence of maximal compact subgroups of connected Lie groups (cf. [32, Satz III.7.3, Satz III.7.21])

Proposition 3.3.3 (Fundamental Theorem about the existence of maximal compact subgroups)**.**

i) Every connected Lie-Group G *contains a maximal compact subgroup* K*. Any other compact subgroup* $\widehat{K} \subseteq G$ *is conjugated in* K*, i.e. there exists* $g \in G$ *such that* $g\widehat{K}g^{-1} \subseteq K$*.*

ii) Let $K \subseteq G$ *be a maximal compact subgroup of a connected Lie group* G*. Then there exists a submanifold* $N \subseteq G$*, which is diffeomorphic to some* $\mathbb{R}^r$*, such that the map*

$$\begin{aligned} N \times K &\longrightarrow G \\ (n, k) &\longmapsto nk \end{aligned}$$

is a diffeomorphism.

This result allows us to prove the claimed result from the previous paragraph:

Proposition 3.3.4. *Let* G *be a connected, non-compact Lie group and* $(P, \pi, M; G)$ *be a* G*-principal bundle. This principal bundle is reducible to any maximal compact subgroup* $K \subseteq G$*.*

Proof. Let $K \subseteq G$ be a maximal compact subgroup. As a consequence of *ii)* in the previous proposition, we have that the homogeneous space G/K is diffeomorphic to some $\mathbb{R}^r$. Thus the associated bundle $E = P \times_G G/K$ is a fiber bundle, whose typical fiber is diffeomorphic to some $\mathbb{R}^r$. We claim that $\Gamma(E) \neq \emptyset$, which implies the stated assertion, according to Proposition 3.3.2.

That $\Gamma(E) \neq \emptyset$ is in fact a consequence of a more general fact on locally trivial fiber bundles. Namely, we have the following result (cf. [27, Chapter I, Theorem 5.7]):

Lemma 3.3.1. *Let* $(B, \pi, M; F)$ *be a smooth locally trivial fiber bundle, whose typical fiber is diffeomorphic to some* $\mathbb{R}^\ell$*, and let* $A \subseteq M$ *be a closed subset. Then it is possible to extend any smooth section* $s\colon A \longrightarrow B$ *to a smooth global section.*

A trivial consequence of this lemma is that any smooth fiber bundle with typical fiber diffeomorphic to some $\mathbb{R}^\ell$ has a global section (this is the special case in which the closed set in the above lemma is just the empty set). Therefore, we conclude that, in the setting of our proposition, $\Gamma(E) \neq \emptyset$ and whence the claim made follows. $\square$

3.4 Connections on principal and associated fiber bundles

We now introduce the necessary tools that will lead us to the study of notions of differential calculus in the language of principal bundles.

For that we first recall that a smooth rank k *distribution* on a smooth manifold N is a smooth rank k vector subbundle $E \longrightarrow N$ of the tangent bundle.

Let us now consider a principal bundle $(P, \pi, M; G)$. We notice that on P there is a canonical smooth distribution coming from the fibers of the bundle. Indeed, since $\pi\colon P \longrightarrow M$ is a submersion, each fiber P_x is a topological submanifold of P. For $u \in P_x$ we define

$$T\nu_u P := T_u(P_x) \subseteq T_u P.$$

The space $T\nu_u P$ is called the *vertical space of* P at the point u.

By means of the Lie algebra of the group G and the exponential map, we obtain the following useful characterisations of the vertical subspaces:

Proposition 3.4.1. *With the above notation, it holds that:*

i) $T\nu_u P = \ker d_u\pi$.

ii) The map

$$X \in \mathfrak{g} \longmapsto \widetilde{X}(u) := \left.\frac{d}{dt}\right|_0 u \cdot \exp(tX) \in T\nu_u P$$

is a linear isomorphism. That is,

$$T\nu_u P = \left\{ \widetilde{X}(u) \,\middle|\, X \in \mathfrak{g} \right\}.$$

For $X \in \mathfrak{g}$, the vector field $\widetilde{X} \in \Gamma(TP)$ defined here is called the *fundamental vector field* generated by X.

Notice that the second assertion shows that $T\nu P := \bigsqcup_{u\in P} T\nu_u P$ is a smooth distribution on P, while the first one shows that the distribution is right invariant, that is, it holds that $d_u R_g(T\nu_u P) = T\nu_{ug} P$, where $R_g\colon P \longrightarrow P$ is simply defined by $R_g(u) = ug$.

Indeed, since the action on P is fiber preserving ($\pi \circ R_g = \pi$), we get on one hand that

$$(d_{ug}\pi)(d_u R_g)\widetilde{X}(u) = d_u\pi(\widetilde{X}(u)) = 0,$$

while, on the other hand we get that

$$\widetilde{X}(ug) = (d_u R_g)(d_{ug} R_{g^{-1}})\widetilde{X}(ug) \in d_u R_g(T\nu_u P).$$

The distribution $\mathrm{T}v\mathrm{P} \subseteq \mathrm{TP}$ is called the *vertical tangent bundle* of P. A complementary vector space to $\mathrm{T}v_u\mathrm{P} \subseteq \mathrm{T}_u\mathrm{P}$ is called a *horizontal tangent space* in P at the point $u \in \mathrm{P}$.

Definition 3.4.1. *A* connection *on the principal bundle* $(\mathrm{P}, \pi, \mathrm{M}; \mathrm{G})$ *is a smooth distribution* $\mathrm{ThP} := \bigsqcup_{u \in \mathrm{P}} \mathrm{Th}_u\mathrm{P} \subseteq \mathrm{TP}$ *of right invariant horizontal tangent spaces. That is, for every* $g \in \mathrm{G}$ *and* $u \in \mathrm{P}$

$$\begin{aligned} \mathrm{T}_u\mathrm{P} &= \mathrm{T}v_u\mathrm{P} \oplus \mathrm{Th}_u\mathrm{P}, \\ d_u R_g(\mathrm{Th}_u\mathrm{P}) &= \mathrm{Th}_{ug}\mathrm{P}. \end{aligned}$$

The distribution $\mathrm{ThP} \subseteq \mathrm{TP}$ is called the *horizontal tangent bundle*. From the definition, it follows that the projections $\mathrm{pr}_v : \mathrm{TP} \longrightarrow \mathrm{T}v\mathrm{P}$ and $\mathrm{pr}_h : \mathrm{TP} \longrightarrow \mathrm{ThP}$ are smooth, and because of the fact that $\mathrm{T}v_u\mathrm{P} = \ker d_u\pi$, it follows that

$$d_u\pi|_{\mathrm{Th}_u\mathrm{P}} : \mathrm{Th}_u\mathrm{P} \longrightarrow \mathrm{T}_{\pi(u)}\mathrm{M}$$

is a linear isomorphism.

It is well-known that connections on principal bundles are determined by their associated connection forms.

Definition 3.4.2. *Let* $(\mathrm{P}, \pi, \mathrm{M}; \mathrm{G})$ *be a principal bundle. A* connection form *on the bundle* $\mathrm{P} \longrightarrow \mathrm{M}$ *is a 1-form* $\omega \in \Omega^1(\mathrm{P}, \mathfrak{g})$ *which satisfies:*

1. $R_g^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ *for every* $g \in \mathrm{G}$,
2. $\omega(\widetilde{X}) = X$ *for every* $X \in \mathfrak{g}$.

We denote by $\mathcal{C}(\mathrm{P})$ the set of all the connection forms on P.

Proposition 3.4.2 ([27, Chapter II, Proposition 1.1])**.** *The connections and connection forms on the principal bundle* $(\mathrm{P}, \pi, \mathrm{M}; \mathrm{G})$ *are in one-to-one correspondence:*

i) Let ThP *be a connection on* P. *Then* $\omega \in \Omega^1(\mathrm{P}, \mathfrak{g})$ *given by*

$$\omega_u(\widetilde{X}(u) \oplus Y_h) := X \qquad \text{for all } u \in \mathrm{P},\ X \in \mathfrak{g},\ Y_h \in \mathrm{Th}_u\mathrm{P},$$

is a connection form on P.

ii) If $\omega \in \Omega^1(\mathrm{P}, \mathfrak{g})$ *is a connection form on* P, *then*

$$\mathrm{ThP} = \bigsqcup_{u \in \mathrm{P}} \mathrm{Th}_u\mathrm{P} := \bigsqcup_{u \in \mathrm{P}} \ker \omega_u$$

defines a connection on P.

In order to state a further characterization of connections on principal bundles we introduce some additional notation.

Let $\omega \in \Omega^1(\mathrm{P}, \mathfrak{g})$ be a connection form on the principal bundle P and $s : \mathrm{U} \subseteq \mathrm{M} \longrightarrow \mathrm{P}$ a local section. We define the *local connection form* defined by s as the 1-form $\omega^s \in \Omega^1(\mathrm{U}, \mathfrak{g})$ given by

$$\omega^s := s^*\omega = \omega \circ ds.$$

Let now $s_i \colon U_i \longrightarrow P$ and $s_j \colon U_j \longrightarrow P$ be two local sections with $U_i \cap U_j \neq \emptyset$. Then, because the action of the group G on P is smooth and transitive on fibers, there exists a smooth transition function $g_{ij} \colon U_i \cap U_j \longrightarrow G$ such that

$$s_i(x) = s_j(x) g_{ij}(x) \qquad \text{for } x \in U_i \cap U_j.$$

Let also $\mu_G \in \Omega^1(G, \mathfrak{g})$ denote the Maurer-Cartan form of the Lie group G, that is,

$$(\mu_G)(Y_g) \coloneqq (dL_{g^{-1}})(Y_g),$$

for all $Y_g \in T_g G$, where $L_g \colon G \longrightarrow G$ denotes the left multiplication map by g. Let $\mu_{ij} \in \Omega^1(U_i \cap U_j, \mathfrak{g})$ be the 1-form

$$\mu_{ij} = g_{ij}^* \mu_G,$$

that is, for $X \in T_x(U_i \cap U_j)$,

$$\mu_{ij}(X) = dL_{g_{ij}(x)^{-1}}(dg_{ij}(X)).$$

With all of this we can now state a characterization of connections on principal bundles via local connection forms:

Proposition 3.4.3 ([27, Chapter II, Proposition 1.4])**.** *Let $(P, \pi, M; G)$ be a principal fiber bundle. With the above notation, it holds that:*

i) for a connection form $\omega \in \Omega^1(P, \mathfrak{g})$ and local sections $s_i \in \Gamma_{U_i}(P)$, $s_j \in \Gamma_{U_j}(P)$ with $U_i \cap U_j \neq \emptyset$,

$$\omega^{s_i} = \mathrm{Ad}(g_{ij}^{-1}) \circ \omega^{s_j} + \mu_{ij},$$

ii) if $\{U_i\}_i$ is an open cover of M with local sections $s_i \in \Gamma_{U_i}(P)$, and $\left\{\omega^i \in \Omega^1(U_i, \mathfrak{g})\right\}_i$ is a given family of $\mathfrak{g}$-valued 1-forms such that, for $U_i \cap U_j \neq \emptyset$,

$$\omega_i = \mathrm{Ad}(g_{ij}^{-1}) \circ \omega_j + \mu_{ij},$$

then there exists a connection form ω on P such that $\omega^{s_i} = \omega_i$ for all i.

As a useful application of the previous result, we establish in an explicit manner the well-known fact that there exists a one-to-one correspondence between the set of connections on the frame bundle of a smooth manifold and the set of covariant derivatives on its tangent bundle:

Let M be an n-dimensional smooth manifold and $F(M) \longrightarrow M$ its frame bundle. Let $\omega \in \Omega^1(F(M), \mathfrak{gl}(n, \mathbb{R}))$ be a connection form. We denote by B^j_k the $n \times n$-matrix whose (j, k) entry is 1 and the rest are 0. Then we can express the connection form ω in the form

$$\omega = \omega^k_j B^j_k,$$

for unique 1-forms $\omega^k_j \in \Omega^1(F(M))$. Let $s \coloneqq (s_1, \ldots, s_n) \colon U \longrightarrow F(M)$ be a local section of the frame bundle. We define the covariant derivative ∇^ω corresponding to ω by linearly extending the expressions

$$\nabla^{\omega}_X s_j := \sum_{i=1}^{n} s^*\omega^k_j(X)s_k, \qquad j = 1, \dots, n$$

and defining the product rule

$$\nabla^{\omega}_X f s_j := X(f)s_j + f\nabla^{\omega}_X s_j \qquad f \in C^\infty(U).$$

The transformation formulas described in Proposition 3.4.3 show that ∇^ω is well-defined, that is, it does not depend on the choice of local section we initially choose.

On the other hand, let ∇ be a covariant derivative on TM and $s\colon U \longrightarrow F(M)$ a local section of the frame bundle. Then we have that there exist $\omega^k_j \in \Omega^1(U)$ such that

$$\nabla s_j = \omega^k_j \otimes s_k.$$

We define now the local 1-form $\omega^\nabla_s \in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R}))$ by

$$\omega^\nabla_s := (\omega^k_j B^j_k)^T = \sum_{k,j} \omega^k_j B^k_j.$$

The family $\{(\omega^\nabla_s, s)|\ s \text{ is a local section on } F(M)\}$ satisfies the transformation rules given in Proposition 3.4.3, which implies that it uniquely defines a connection form ω^∇ on the frame bundle $F(M)$. It should also be noted that, essentially the same argument shows in fact the existence of a one-to-one correspondence between the set covariant derivatives on a vector bundle E and its frame bundle $F(E)$. See for example [33, Section 9.19].

Due to the fact that on the trivial principal bundle $(M \times G, pr_1, M; G)$ there exists a canonical connection, namely the one associated to the Maurer-Cartan form in the sense of the previous proposition, a partition of unity argument shows the following

Proposition 3.4.4. *On every principal bundle there exists a connection.*

As we saw before, one can identify sections of the associated bundle to a G-principal bundle with the G-invariant maps on the total space. An analogue for k-forms with values in the associated vector bundle is possible.

First we define:

Definition 3.4.3. *Let $(P, \pi, M; G)$ be a principal bundle, V a vector space, and $\rho\colon G \longrightarrow GL(V)$ a representation. A V-valued k-form $\omega \in \Omega^k(P, V)$ is called*

1. horizontal *if $\omega_p(X_1, \dots, X_k) = 0$ in case one of the $X_i \in T_pP$ is a vertical vector.*

2. of type ρ, *if $R^*_a\omega = \rho(a^{-1}) \circ \omega$ for all $a \in G$.*

We denote the set of the horizontal k-forms of type ρ by

$$\Omega^k_{hor}(P, V)^{(G,\rho)}.$$

From the definition of a connection form, we notice that $\mathcal{C}(P)$ is an affine space modelled over the vector space $\Omega^1_{hor}(P, \mathfrak{g})^{(G,Ad)}$.

Proposition 3.4.5. *With the same notation as before, let* $E := P \times_{(G,\rho)} V$ *be the associated vector bundle to the* G*-principal bundle* $P \longrightarrow M$*. Then*

$$\Omega^k(M,E) \cong \Omega^k_{hor}(P,V)^{(G,\rho)}$$

as vector spaces.

The proof of this result is somewhat standard. One defines the linear map $\Psi_k\colon \Omega^k_{hor}(P,V)^{(G,\rho)} \longrightarrow \Omega^k(M,E)$ by $\Psi_k(\overline{\omega}) = \omega$, where

$$\omega_x(t_1,\dots,t_k) := [p, \overline{\omega}_p(X_1,\dots,X_k)],$$

for $\pi(p) = x$, and $t_i \in T_xM$, $X_i \in T_pP$ with $d\pi_p(X_i) = t_i$ and shows that this is the desired linear isomorphism.

As a corollary of the previous proposition we get that $\mathcal{C}(P)$ is an affine space modelled over the vector space $\Omega^1(M, \mathrm{Ad}(P))$, where

$$\mathrm{Ad}(P) := P \times_G \mathfrak{g}$$

is the so-called *adjoint bundle*.

3.5 Parallel transport and covariant derivatives

Connections on principal bundles enable us to introduce the notion of parallel transport, which will allow us to relate the fibers of a principal bundle with each other.

As usual, we fix $(P, \pi, M; G)$ a principal bundle with connection ThP associated to the connection form $\omega \in \Omega^1(P, \mathfrak{g})$.

Definition 3.5.1. *Let* X *be a vector field on* M*. A vector field* $\overline{X}$ *on* P *is called a* horizontal lift *of* X *if*

i) $\overline{X}_u \in Th_uP$ *and*

ii) $d_u\pi(\overline{X}_u) = X_{\pi(u)}$

for every $u \in P$*.*

The next proposition encloses the most important properties of horizontal lifts.

Proposition 3.5.1. *i) For every vector field* X *on* M*, there exists a unique horizontal lift* $\overline{X}$ *on* P*. This horizontal lift is right-invariant.*

ii) Given a right-invariant horizontal vector field Z *on* P*. Then there exists exactly one vector field* X *on* M *such that* $\overline{X} = Z$*.*

iii) Let $X, Y \in \Gamma(TM)$*,* $f \in C^\infty(M)$*. Then,*

$$\begin{aligned} \overline{X} + \overline{Y} &= \overline{X+Y}, \\ \overline{fX} &= (f \circ \pi)\overline{X}, \\ \overline{[X,Y]} &= \mathrm{pr}_h[\overline{X}, \overline{Y}]. \end{aligned}$$

iv) If Z is a horizontal and $\widetilde{B}$ a fundamental vector field on P, then $[\widetilde{B}, Z]$ is a horizontal vector field on P as well. If X a vector field on M, then $[\widetilde{B}, \overline{X}] = 0$.

Proof. *i)* Let $\mathrm{pr}_h \colon TP \longrightarrow ThP$ denote the projection to the horizontal bundle. As previously discussed, the map $d_u\pi \colon Th_uP \longrightarrow T_{\pi(u)}M$ is a linear isomorphism, which implies that the unique choice of a horizontal lift is given by the formula

$$\overline{X}_u \coloneqq (d\pi|_{ThP})^{-1}\big(X_{\pi(u)}\big).$$

We claim that $\overline{X}$ actually satisfies the defining properties of a smooth horizontal lift. First of all we verify the smoothness of $\overline{X}$. Let $\phi \colon P_U \simeq U \times G$ be a local trivialization about the point $\pi(u)$. Let Y be the smooth vector field $Y \coloneqq d\phi^{-1}(X \oplus 0) \in \Gamma_U(P) = \Gamma(P_U)$. It follows that $d\pi(Y) = X$, which in turn means $\overline{X} = \mathrm{pr}_h Y$. The smoothness of $\overline{X}$ follows now from that of pr_h and Y. The right invariance of the connection Th implies that $dR_g(\overline{X}_u) \in Th_{ug}P$, from which it follows that $d\pi(dR_g(\overline{X}_u)) = d\pi(\overline{X}_u) = X_{\pi(u)}$. The uniqueness of the horizontal lift implies $dR_g(\overline{X}_u) = \overline{X}_{ug}$. That is, $\overline{X}$ is right-invariant.

ii) For a horizontal right-invariant vector field Z on P we define a vector field on M by the formula

$$X_x \coloneqq d_u\pi(Z_u) \qquad \text{for a } u \in P_x.$$

The right-invariance of the vector field Z implies that the definition of the vector field X does not depend on the choice of $u \in P_x$, and it satisfies $\overline{X} = Z$.

iii) The first two equations are a direct and simple computation. The third one is a consequence of the naturality of the Lie bracket, that is, the fact that $\overline{X}, \overline{Y}$ are π-related to X, Y respectively implies that $[\overline{X}, \overline{Y}]$ is π-related to $[X, Y]$ as well, and the uniqueness of horizontal lifts:

$$d\pi \circ \mathrm{pr}_h[\overline{X}, \overline{Y}] \overset{TvP = \ker d\pi}{=} d\pi \circ [\overline{X}, \overline{Y}] = [X, Y] \circ \pi = d\pi \circ \overline{[X, Y]}.$$

iv) Let Z a horizontal vector field on P, $B \in \mathfrak{g}$ and $\widetilde{B}$ the fundamental vector field on P generated by B. The nature of the action of the Lie group G on the smooth manifold P implies that, in this case, the flow of $\widetilde{B}$ is given by the family of diffeomorphisms (cf. [20, Cahpter 20, Proposition 20.8])

$$\begin{aligned} \Phi_t^{\widetilde{B}} \colon P &\longrightarrow P \\ u &\longmapsto u \exp(tB) = R_{\exp(tB)}(u). \end{aligned}$$

Thus,

$$\begin{aligned}[\widetilde{B}, Z]_u =& (\mathcal{L}_{\widetilde{B}} Z)_u \\ =& \frac{d}{dt}\bigg|_0 \left(\Phi_t^{\widetilde{B}}\right)^* Z_u \\ =& \frac{d}{dt}\bigg|_0 d\Phi_{-t}^{\widetilde{B}}\left(Z_{\Phi_t^{\widetilde{B}}(u)}\right) \\ =& \frac{d}{dt}\bigg|_0 dR_{\exp(-tB)}(Z_{u\exp(tB)}).\end{aligned}$$

Since the vector field Z is horizontal and the connection Th is right-invariant, we obtain that the curve defined in the last equation is a curve in $Th_u P$. This implies in sum that $[\widetilde{B}, Z]$ is a horizontal vector field. In the particular case in which $Z = \overline{X}$, for a vector field X on M, is the vector field Z also right-invariant, and thus the curve $t \longmapsto dR_{\exp(-tB)}(Z_{u\exp(tB)})$ is the constant curve Z_u. Which implies $[\widetilde{B}, \overline{X}] = 0$. □

We can also consider horizontal lifts of piecewise smooth curves in the base M of the bundle P.

Definition 3.5.2. *A curve* $\overline{\gamma}\colon I \longrightarrow P$ *is called a* horizontal lift *of the curve* $\gamma\colon I \longrightarrow M$ *if*

i) $\pi(\overline{\gamma}(t)) = \gamma(t)$ *for every* $t \in I$ *and*

ii) The tangent vectors $\dot{\overline{\gamma}}(t)$ *are horizontal for all* $t \in I$, *that is,* $\dot{\overline{\gamma}}(t) \in Th_{\gamma(t)}P$.

As the next result shows, for a fixed initial point on the the fiber over a point on the curve γ, there exists exactly one horizontal lift.

Proposition 3.5.2 ([27, Chapter II, Proposition 3.1])**.** *Let* $\gamma\colon I \longrightarrow M$ *be a piecewise smooth curve in* M, $t_0 \in I$ *and* $u \in P_{\gamma(t_0)}$. *Then there exists exactly one horizontal lift* $\overline{\gamma}_u$ *of* γ *with* $\overline{\gamma}_u(t_0) = u$.

It is precisely this uniqueness result the one that allows us to compare the fibers of a principal bundle:

Definition 3.5.3. *Let* $\gamma\colon [a, b] \longrightarrow M$ *be a piecewise smooth curve in* M. *The map*

$$\begin{aligned}P_\gamma^\omega\colon P_{\gamma(a)} &\longrightarrow P_{\gamma(b)} \\ u &\longmapsto \overline{\gamma}_u(b)\end{aligned}$$

is called the parallel transport *along* γ *with respect to the connection* ω.

From the properties of the horizontal lift $\overline{\gamma}_u$ of γ, it follows that the parallel transport is independent of the parametrization of the curve γ.

Let now $\gamma\colon [a, b] \longrightarrow M$, $\mu\colon [c, d] \longrightarrow M$ be two piecewise smooth curves in M with $\gamma(b) = \mu(c)$. We define the *concatenation* of the curves γ, μ as the piecewise smooth curve $\mu * \gamma\colon [0, 1] \longrightarrow M$, with

$$\mu * \gamma(t) := \begin{cases} \gamma(a + 2t(b - a)), & \text{for } t \in [0, \frac{1}{2}] \\ \mu(c + (2t - 1)(d - c)), & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

We also define the *inverse* of the curve γ as the curve $\gamma^-\colon [0,1] \longrightarrow M$, which runs through the same path as the curve γ but in reversed order. We define γ^- by the formula

$$\gamma^-(t) := \gamma(b - t(b-a)).$$

We now summarize some of the basic properties of the parallel transport, which are consequences of the uniqueness of horizontal lifts and the right invariance of the horizontal space.

Proposition 3.5.3. *i) Let γ, μ be two piecewise smooth curves as before. Then*

$$P^{\omega}_{\mu*\gamma} = P^{\omega}_{\mu} \circ P^{\omega}_{\gamma}.$$

ii) The parallel transport is a diffeomorphism. For γ as before,

$$(P^{\omega}_{\gamma})^{-1} = P^{\omega}_{\gamma^-}.$$

iii) The parallel transport is G-equivariant, that is,

$$P^{\omega}_{\gamma} \circ R_g = R_g \circ P^{\omega}_{\gamma} \qquad \text{for every } g \in G.$$

We notice that the G-equivariance of the parallel transport allows us to define a parallel transport on the associated fiber bundle $E = P \times_G F$. Indeed, let $\gamma\colon [a,b] \longrightarrow M$ be a piecewise smooth curve in M. The map

$$\begin{aligned} P^{E,\omega}_{\gamma}\colon E_{\gamma(a)} &\longrightarrow E_{\gamma(b)} \\ [p,v] &\longmapsto [P^{\omega}_{\gamma}(p), v] \end{aligned}$$

is well-defined because of the fact that the parallel transport is G-equivariant. The map $P^{E,\omega}_{\gamma}$ is called the *parallel transport on E induced by the connection ω*. We have the following description of the parallel transport in terms of the special fiber diffeomorphisms of the associated bundle:

Lemma 3.5.1. *Let $\overline{\gamma}\colon [a,b] \longrightarrow P$ be a horizontal lift of the piecewise smooth curve γ. Then*

$$P^{E,\omega}_{\gamma} = [\overline{\gamma}(b)] \circ [\overline{\gamma}(a)]^{-1}.$$

Proof. Let $[q,v] \in E_{\gamma(a)} = P_{\gamma(a)} \times_G F$. Let $g_q \in G$ be the unique group element such that $q = \overline{\gamma}(a) g_q$.

Thus

$$
\begin{aligned}
P^{E,\omega}_{\gamma}([q,v]) &= [P^{\omega}_{\gamma}(q), v] \\
&= [P^{\omega}_{\gamma}(\overline{\gamma}(a) g_q), v] \\
&\overset{\text{Prop.3.5.3}}{=} [P^{\omega}_{\gamma}(\overline{\gamma}(a)) g_q, v] \\
&= [P^{\omega}_{\gamma}(\overline{\gamma}(a)), g_q v] \\
&= [\overline{\gamma}_{\overline{\gamma}(a)}(b), g_q v] \\
&\overset{\text{Prop.3.5.2}}{=} [\overline{\gamma}(b), g_q v] \\
&= [\overline{\gamma}(b)][\overline{\gamma}(a)]^{-1}[q,v].
\end{aligned}
$$

□

In the particular case in which E is a vector bundle, we obtain that the parallel transport $P^{E,\omega}_{\gamma}$ is a linear isomorphism.

Now we establish how from connections on principal bundles we can obtain covariant derivatives on associated vector bundles.

For that, recall that a *covariant derivative* on the vector bundle $E \longrightarrow M$ is a linear map

$$\nabla \colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

that satisfies the Leibniz rule, that is,

$$\nabla(fe) = df \otimes e + f\nabla e \qquad \text{for } f \in C^{\infty}(M),\ e \in \Gamma(E).$$

Definition 3.5.4. *Let $(P, \pi, M; G)$ be a principal bundle with connection form ω, and V a vector space. The linear map $D_\omega \colon \Omega^k(P, V) \longrightarrow \Omega^{k+1}(P, V)$ defined by*

$$(D_\omega \eta)_p(t_0, \dots, t_k) \coloneqq d\eta_p(\mathrm{pr}_h t_0, \dots, \mathrm{pr}_h t_k), \qquad \textit{for } t_i \in T_pP, \tag{3.6}$$

where $d \colon \Omega^k(P, V) \longrightarrow \Omega^{k+1}(P, V)$ denotes the usual differential on k-forms is called the absolute differential *on P defined by ω.*

The importance of this modified differential is that it enjoys a feature not shared by the usual one, namely, that it maps horizontal forms of type ρ into differential forms of the same kind, which because of the isomorphism $\Omega^k(M, E = P \times_{(G,\rho)} V) \cong \Omega^k_{hor}(P, V)^{(G,\rho)}$, is a minimum requirement for obtaining covariant derivatives on the associated vector bundle:

Proposition 3.5.4. *With the above notation, it holds that*

$$D_\omega \colon \Omega^k_{hor}(P, V)^{(G,\rho)} \longrightarrow \Omega^{k+1}_{hor}(P, V)^{(G,\rho)}.$$

For $\eta \in \Omega^k_{hor}(P, V)^{(G,\rho)}$ it holds that

$$D_\omega \eta = d\eta + \rho_*(\omega) \wedge \eta, \tag{3.7}$$

where the second summand is defined as

$$(\rho_*(\omega) \wedge \eta)(t_0, \dots, t_k) \coloneqq \sum_{i=0}^{k} (-1)^i \rho_*(\omega(t_i))\big(\eta(t_0, \dots, \widehat{t}_i, \dots, t_k)\big).$$

With help of the previous proposition, together with the linear isomorphism

$$\Psi_\bullet \colon \Omega^\bullet_{hor}(P, V)^{(G,\rho)} \longrightarrow \Omega^\bullet(M, E),$$

we define the linear operator

$$d_\omega \colon \Omega^k(M, E) \longrightarrow \Omega^{k+1}(M, E)$$

as the unique map such that the diagram

$$\begin{tikzcd} \Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)} \arrow[r, "D_\omega"] \arrow[d, "\Psi_k"'] & \Omega^{k+1}_{\mathrm{hor}}(P, V)^{(G,\rho)} \arrow[d, "\Psi_{k+1}"] \\ \Omega^k(M, E) \arrow[r, "d_\omega"'] & \Omega^{k+1}(M, E) \end{tikzcd}$$

commutes.

The differential induced by the connection form ω satisfies the usual product rule for the wedge product:

Proposition 3.5.5. *Let* $d_\omega \colon \Omega^\bullet(M, E) \longrightarrow \Omega^{\bullet+1}(M, E)$ *be the differential induced by the connection form* ω*. Then, for* $\sigma \in \Omega^k(M), \eta \in \Omega^l(M, E)$ *with* $k, l \geqslant 0$*, it holds that*

$$d_\omega(\sigma \wedge \eta) = d\sigma \wedge \eta + (-1)^k \sigma \wedge d_\omega \eta. \tag{3.8}$$

We notice that in the special case $k = 0$ we obtain a linear operator

$$d_\omega \colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E),$$

which, by Proposition 3.5.5, satisfies the product rule

$$d_\omega(fe) = df \otimes e + f d_\omega e, \qquad \text{for } f \in C^\infty(M),\ e \in \Gamma(E),$$

which implies that this linear map is in fact a covariant derivative on the associated vector bundle E.

Definition 3.5.5. *We call the map*

$$\nabla^\omega \coloneqq d_\omega|_{\Omega^0(M,E)} \colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

the covariant derivative *on* E *induced by the connection form* ω*.*

Next, we provide an explicit formula for the covariant derivative ∇^ω.

Proposition 3.5.6. *Let* $P \longrightarrow M$ *be a* G*-principal bundle,* $\rho\colon G \longrightarrow GL(V)$ *a* G*-representation, and* $E = P \times_{(G,\rho)} V$ *the associated vector bundle. Let* ω *be a connection form on* P *and* ∇^ω *the induced covariant derivative on* E*. Then, for* $e \in \Gamma(E), X \in \Gamma(TM)$ *the following local formula holds:*

$$(\nabla^\omega_X e)(x) = [s(x), dv_x(X_x) + \rho_*(\omega^s(X_x))v(x)] \in E_x, \tag{3.9}$$

where $s \in \Gamma_U(P)$, $x \in U \subseteq M$, $v \in C^\infty(U, V)$ *is a smooth map with* $e|_U = [s, v]$*, and* $\omega^s = s^*\omega$ *is the local connection form.*

Proof. Let $s\colon U \longrightarrow P_U$ be a local section. For any section $e \in \Gamma(E)$, define the smooth map $v\colon U \longrightarrow V$ as the map $v := \overline{e} \circ s$, where $\overline{e} \in C^\infty(P, V)^G$ is the unique G-equivariant map to which the section e corresponds. That the map v satisfies the identity $e|_U = [s, v]$ trivially follows.

We thus obtain

$$\begin{aligned}(\nabla^\omega_X e)(x) =&(d_\omega e)(X_x)\\ =&[s(x), (D_\omega \overline{e})(d_x s(X_x))]\\ =&[s(x), d\overline{e}(d_x s(X_x)) + \rho_*(\omega(d_x s(X_x)))\overline{e}(s(x))]\\ =&[s(x), d_x v(X_x) + \rho_*(\omega^s(X_x))v(x)].\end{aligned}$$

□

Corollary 3.5.1. *Let* $\overline{X}$ *be the horizontal lift of the vector field* X*. For* $e \in \Gamma(E)$ *we get that*

$$(\nabla^\omega_X e)(x) = [s(x), d\overline{e}(\overline{X}_{s(x)})],$$

for any local section $s \in \Gamma_U(P)$*, with* $x \in U \subseteq M$*, and* $\overline{e} \in C^\infty(P, V)^{(G,\rho)}$ *the unique* G*-equivariant* V*-valued map corresponding to the section* e*.*

Proof. First we note that X and $\overline{X}$ are s-related:

$$\begin{aligned}\overline{X} \circ s &= (d\pi|^{-1}_{ThP} \circ X \circ \pi) \circ s\\ &= d\pi|^{-1}_{ThP} \circ d(\pi|_{ThP} \circ s) \circ X\\ &= ds \circ X.\end{aligned}$$

The result follows now from Proposition 3.5.6, and by noting that the second summand in equation (3.9) vanishes by the observation that X is s-related to $\overline{X}$. □

Back in Proposition 3.2.7 we obtained an explicit bundle metric on the associated vector bundle E coming from a G-invariant scalar product on the vector space V. A nice property of the induced covariant derivative ∇^ω is that it is *metric* with respect to this special bundle metric:

Proposition 3.5.7. *Let* $g_E := \langle \cdot, \cdot \rangle_E$ *be a bundle metric on the associated vector bundle* $E = P \times_G V$ *as in Proposition 3.2.7. It holds that* g_E *is a constant tensor with respect to the induced covariant derivative* ∇^ω*. That is,*

$$\nabla^\omega g_E \equiv 0.$$

As a final remark on the covariant derivative ∇^ω, we also have a way to recover it from the parallel transport, in an analogous fashion as in the case of any covariant derivative on a vector bundle.

In order to achieve this, let γ be a piecewise smooth curve in M with initial value $\gamma(0) = x$. We define now the map

$$P^{E,\omega}_{t,0} \colon E_{\gamma(t)} \longrightarrow E_{\gamma(0)}$$

to be the parallel transport along the inverse curve $\gamma^- \colon [0,1] \longrightarrow M$.

Proposition 3.5.8. *Let $e \in \Gamma(E)$. Then for any piecewise smooth curve γ we have that*

$$\frac{d}{dt} P^{E,\omega}_{t,0} e(\gamma(t)) = P^{E,\omega}_{t,0} \nabla^\omega_{\dot\gamma(t)} e(\gamma(t)).$$

Proof. Let $\overline{\gamma}$ be any horizontal lift of γ. Lemma 3.5.1 implies that

$$P^{E,\omega}_{t,0} = [\overline{\gamma}(0)] \circ [\overline{\gamma}(t)]^{-1}.$$

Let also $\overline{e} \in C^\infty(P,V)^{(G,\rho)}$ be the invariant V-valued function corresponding to the section e.

By making use of the previous corollary, we thus obtain

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 P^{E,\omega}_{t,0}(e(\gamma(t))) &= \left.\frac{d}{dt}\right|_0 [\overline{\gamma}(0)]\big([\overline{\gamma}(t)]^{-1} e(\gamma(t))\big) \\ &= \left[\overline{\gamma}(0), \left.\frac{d}{dt}\right|_0 \overline{e}(\overline{\gamma}(t))\right] \\ &= [\overline{\gamma}(0), d\overline{e}(\dot{\overline{\gamma}}(0))] \\ &= (\nabla^\omega_{\dot\gamma(0)} e)(\gamma(0)). \end{aligned}$$

We notice that this special case is the key step in proving our claim. Define the piecewise smooth curve $\gamma_t(u) := \gamma(t+u)$. From our previous discussion, we thus obtain

$$\begin{aligned} \nabla^\omega_{\dot\gamma(t)} e(\gamma(t)) = \nabla^\omega_{\dot\gamma_t(0)} e(\gamma_t(0)) &= \left.\frac{d}{du}\right|_0 P^{E,\omega}_{u,0} e(\gamma_t(u)) \\ &= \left.\frac{d}{du}\right|_0 P^{E,\omega}_{t+u,t} e(\gamma(t+u)) \\ &= \left.\frac{d}{du}\right|_0 P^{E,\omega}_{0,t} \circ P^{E,\omega}_{t+u,0} e(\gamma(t+u)) \\ &= P^{E,\omega}_{0,t} \frac{d}{dt} P^{E,\omega}_{t,0} e(\gamma(t)). \end{aligned}$$

□

3.6 Introduction to curvature

In this section, we make some remarks about the curvature on principal fiber bundles.

As usual, $(P, \pi, M; G)$ is a principal bundle with a fixed connection $\mathrm{Th}P$ with respect to the associated connection form ω. $\rho\colon G \longrightarrow GL(V)$ is a representation of the Lie group G, and $E = P \times_{(G,\rho)} V$ is the associated vector bundle over M.

Definition 3.6.1. *The* curvature form *of the connection form* ω *is the* $\mathfrak{g}$*-valued 2-form*

$$F^{\omega} \coloneqq D_{\omega}\omega \in \Omega^2(P, \mathfrak{g}) \tag{3.10}$$

By definition of the absolute differential D_{ω}, we obtain that F^{ω} is a horizontal form, and because of the fact that the connection form ω is of type Ad, F^{ω} is of type Ad as well. That is, it holds that

$$F^{\omega} \in \Omega^2_{hor}(P, \mathfrak{g})^{(G,\mathrm{Ad})}.$$

Let $s \in \Gamma_U(P)$. The $\mathfrak{g}$-valued 2-form on $U \subseteq M$

$$F^s \coloneqq s^*F^{\omega} \in \Omega^2(U, \mathfrak{g})$$

is called the *local curvature form* with respect to s.

If $\tau \in \Gamma_U(P)$ is another local section on P, and $\tau = s \cdot g$, for a smooth map $g\colon U \longrightarrow G$, then we obtain the following transformation rule:

$$F^{\tau} = \mathrm{Ad}(g^{-1}) \circ F^s.$$

Before we state further properties of the curvature form, we recall how the commutator of differential forms taking values on a Lie algebra is defined. Let N be a smooth manifold and $\mathfrak{g}$ a Lie algebra. We fix a basis $(a_1, \dots, a_r)$ of $\mathfrak{g}$. Then, every $\eta \in \Omega^k(N, \mathfrak{g})$, $\tau \in \Omega^l(N, \mathfrak{g})$ are represented in terms of this basis as

$$\eta = \eta^i a_i \qquad \text{and} \qquad \tau = \tau^i a_i,$$

where $\eta^i \in \Omega^k(N)$, $\tau^i \in \Omega^l(N)$.

We then define the commutator of η and τ by the formula

$$[\eta, \tau] \coloneqq (\eta^i \wedge \tau^j)[a_i, a_j]_{\mathfrak{g}} \in \Omega^{k+l}(N, \mathfrak{g}). \tag{3.11}$$

Notice that this definition is independent of the choice of basis of the Lie algebra. Indeed, let $(b_1, \dots, b_r)$ be another basis of $\mathfrak{g}$ with change of basis matrix $A = (\alpha^i_j)_{i,j}$. Then the bilinearity of the Lie bracket implies

$$(\eta^i \wedge \tau^j)[a_i, a_j]_{\mathfrak{g}} = (\eta^i \alpha^k_i \wedge \tau^j \alpha^{\ell}_j)[b_k, b_{\ell}]_{\mathfrak{g}}.$$

In a local coordinate system $(U, (x^i))$ of N, a $\mathfrak{g}$-valued differential form $\eta \in \Omega^{\bullet}(N, \mathfrak{g})$ has the representation

$$\eta = \eta_I \, dx^I$$

for unique smooth functions $\eta_I\colon U \longrightarrow \mathfrak{g}$.

In this case, the formula (3.11) is equivalent to

$$[\eta,\tau] = [\eta_I,\tau_J]_{\mathfrak{g}}\, dx^I \wedge dx^J .$$

Some basic properties of the commutator $[\cdot,\cdot]\colon \Omega^k(N,\mathfrak{g}) \times \Omega^l(N,\mathfrak{g}) \longrightarrow \Omega^{k+l}(N,\mathfrak{g})$ are:

i) $[\eta,\tau] = (-1)^{kl+1}[\tau,\eta]$.

ii) $d[\eta,\tau] = [d\eta,\tau] + (-1)^k[\eta, d\tau]$.

iii) If η is a 1-form, it holds that $[\eta,\eta](X,Y) = 2[\eta(X),\eta(Y)]_{\mathfrak{g}}$.

Next, we prove some of the basic properties of the curvature form.

Proposition 3.6.1. *Let $F^\omega \in \Omega^2(P,\mathfrak{g})$ be the curvature form of the connection form ω. Then the following identities hold:*

i) The structure equation*:* $F^\omega = d\omega + \frac{1}{2}[\omega,\omega]$.

ii) The (differential) Bianchi identity*:* $D_\omega F^\omega = 0$.

iii) For a horizontal k-form of type ρ, $\eta \in \Omega^k_{hor}(P,V)^{(G,\rho)}$,

$$D_\omega D_\omega \eta = \rho_*(F^\omega) \wedge \eta.$$

Proof. To prove the structure equation, it suffices to check it on horizontal and vertical vectors $X,Y \in T_pP = Tv_pP \oplus Th_pP$. If both X,Y are horizontal, we have that $\omega(X) = \omega(Y) = 0$ and, by definition, $F^\omega(X,Y) = D_\omega\omega(X,Y) = d\omega(X,Y)$, as desired. If X is horizontal and Y vertical, then we have that $F^\omega(X,Y) = 0$, since F^ω is a horizontal form. Now, Propositions 3.4.1 and 3.5.1 allow us to write $X = \overline{V}_p, Y = \widetilde{T}(p)$, for a $V \in \Gamma(TM)$ and a $T \in \mathfrak{g}$. Once more, Proposition 3.5.1 implies that $[\overline{V},\widetilde{T}] = 0$. Because $\omega(\overline{V}) = 0$ and $\omega(\widetilde{T}) = T$ are constant, we get that $d\omega(X,Y) = 0$, which verifies the structure equation in this particular case. Lastly, suppose both X,Y are vertical vectors and write $X = \widetilde{T}(p), Y = \widetilde{S}(p)$, for $T,S \in \mathfrak{g}$. Then, on one hand we get $F^\omega(X,Y) = 0$, and on the other hand

$$\begin{aligned}
d\omega &= X(\omega(\widetilde{S})) - Y(\omega(\widetilde{T})) - \omega([\widetilde{T},\widetilde{S}])\\
&= -\omega(\widetilde{[T,S]})\\
&= -[T,S]_{\mathfrak{g}}\\
&= -[\omega(\widetilde{T}(p)),\omega(\widetilde{S}(p))]_{\mathfrak{g}}\\
&= -[\omega(X),\omega(Y)]_{\mathfrak{g}}\\
&= -\frac{1}{2}[\omega,\omega](X,Y).
\end{aligned}$$

Now, to prove the Bianchi identity we differentiate the structure equation

$$dF^\omega = dd\omega + \frac{1}{2}\, d[\omega,\omega] = [d\omega,\omega],$$

from which we obtain that

$$D_\omega F^\omega = dF^\omega \circ \mathrm{pr}_h = [d\omega \circ \mathrm{pr}_h, \omega \circ \mathrm{pr}_h] = 0,$$

since $T hP = \ker \omega$.

Finally, we show item *iii)*. Since both $\eta, D_\omega \eta$ are horizontal of type ρ we make use of Proposition 3.5.4 and thus obtain

$$\begin{aligned} D_\omega(D_\omega \eta) &= d(d\eta + \rho_*(\omega) \wedge \eta) + \rho_*(\omega) \wedge (d\eta + \rho_*(\omega) \wedge \eta) \\ &= d(\rho_*(\omega)) \wedge \eta + \rho_*(\omega) \wedge \rho_*(\omega) \wedge \eta \\ &= \rho_*(d\omega) \wedge \eta + \rho_*(\omega) \wedge \rho_*(\omega) \wedge \eta, \end{aligned}$$

and since

$$\begin{aligned} (\rho_*(\omega) \wedge \rho_*(\omega))(X, Y) &= \rho_*(\omega(X)) \circ \rho_*(\omega(Y)) - \rho_*(\omega(Y)) \circ \rho_*(\omega(X)) \\ &= [\rho_*(\omega(X)), \rho_*(\omega(Y))]_{\mathrm{End}(V)} \\ &= \rho_*([\omega(X), \omega(Y)]_{\mathfrak{g}}) \\ &= \frac{1}{2}\rho_*([\omega, \omega](X, Y)), \end{aligned}$$

it follows that

$$D_\omega D_\omega \eta = \rho_*\left(d\omega + \frac{1}{2}[\omega, \omega]\right) \wedge \eta = \rho_*(F^\omega) \wedge \eta.$$

□

Previously we verified that the curvature form is a horizontal $\mathfrak{g}$-valued 2-form of type Ad, thus it can be identified with a 2-form on M with values in the adjoint bundle $\mathrm{Ad}(P)$, which we will denote by the same symbol F^ω. Now we translate the properties of the curvature form just proved to this 2-form. First of all, we consider the following bundle morphism induced by the representation $\rho\colon G \longrightarrow GL(V)$:

$$\rho_*\colon \mathrm{Ad}(P) \longrightarrow \mathrm{End}(E).$$

Let $\varphi \in \mathrm{Ad}(P)_x$ and $e \in E_x$. For fixed $p \in P_x$ it holds that $\varphi = [p, X]$ and $e = [p, v]$ for a $X \in \mathfrak{g}$ and $v \in V$. Then we define

$$\rho_*(\varphi)e \coloneqq [p, \rho_*(X)v].$$

Because the action on P is free and transitive on fibers, we see that the definition of the bundle morphism ρ_* does not depend on the choice of the fiber element $p \in P_x$.

Whit this morphism, we define the wedge product of differential forms with values in $\mathrm{Ad}(P)$ with differential forms with values in E:

$$\begin{aligned} \wedge\colon \Omega^k(M, \mathrm{Ad}(P)) \times \Omega^l(M, E) &\longrightarrow \Omega^{k+l}(M, E) \\ (\eta, \tau) \qquad\qquad &\longmapsto \quad \eta \wedge \tau, \end{aligned}$$

with

$$\begin{aligned} &(\eta \wedge \tau)_x(X_1, \dots, X_{k+l}) \\ &\quad := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) \rho_*\big(\eta_x(X_{\sigma(1)}, \dots, X_{\sigma(k)})\big) \tau_x\big(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}\big). \end{aligned}$$

This together with the identities in Proposition 3.6.1 allow us to prove analogous ones for the curvature form $F^\omega \in \Omega^2(M, \mathrm{Ad}(P))$:

Proposition 3.6.2. *Let* $F^\omega \in \Omega^2(M, \mathrm{Ad}(P))$ *be the curvature form of the connection form* ω. *Then* F^ω *satisfies the (second) Bianchi identity*

$$d_\omega F^\omega = 0.$$

For the differential $d_\omega\colon \Omega^k(M, E) \longrightarrow \Omega^{k+1}(M, E)$ *it holds that*

$$d_\omega d_\omega \eta = F^\omega \wedge \eta.$$

From the previous proposition, we conclude that the curvature form of a connection form measures how much $d_\omega \circ d_\omega$ fails to vanish.

Some additional properties of the curvature form, which immediately follow from the definition of a curvature form and the fact that $ThP = \ker \omega$, are given in the next

Proposition 3.6.3. *Let* X, Y *be horizontal vector fields on* P *and* $F^\omega \in \Omega^2(P, \mathfrak{g})$ *be the curvature form associated to the connection form* ω. *Then*

i) $F^\omega(X, Y) = -\omega([X, Y])$.

ii) $\mathrm{pr}_v([X, Y]) = -\widetilde{F^\omega(X, Y)}$.

The last property described in the previous proposition let us establish some deeper results on connections on principal bundles. But before we get into further specifics, we recall some definitions on geometric distributions and Frobenius Theorem. A distribution $\mathcal{D} \subseteq TN$ on the smooth manifold N is called *involutive* if, for every vector fields X, Y on N that take their values in $\mathcal{D}$, the Lie bracket $[X, Y]$ also takes all of its values in $\mathcal{D}$. An *integral manifold* of $\mathcal{D}$ is a submanifold $Q \subseteq N$ such that $T_q Q = D_q$ for all $q \in Q$. A distribution $\mathcal{D}$ is called *integrable* if for every $x \in N$ there is a maximal connected integral manifold of $\mathcal{D}$ containing x. Frobenius Theorem claims that a distribution $\mathcal{D}$ is integrable if, and only if, it is involutive.

Proposition 3.6.4.

i) The vertical tangent bundle $TvP \subseteq TP$ *is involutive.*

ii) The horizontal tangent bundle $\mathrm{ThP} \subseteq \mathrm{TP}$ *is involutive if, and only if,* $F^{\omega} \equiv 0$.

Proof. The first claim follows from the fact that $[\widetilde{T}, \widetilde{S}] = \widetilde{[T, S]}$ for any two fundamental vector fields $\widetilde{T}, \widetilde{S}$. This implies then that the Lie bracket of vertical vector fields is again a vertical vector field. To prove the second assertion, for X, Y horizontal vector fields it holds, from the previous proposition, that $\mathrm{pr}_v([X, Y]) = -\widetilde{F^{\omega}(X, Y)}$, which implies that $[X, Y]$ is horizontal if, and only if, $F^{\omega}(X, Y) = 0$. The involutivity of the horizontal tangent bundle is then equivalent to the vanishing of the curvature form F^{ω}. $\square$

What we just proved shows that the vanishing of the curvature form is equivalent to saying that through every point of P there is a maximal submanifold $Q \subseteq P$ that is transversal to the fibers of the bundle whose tangent bundle is $TQ = \mathrm{ThP}|_Q$. As an example, let us consider the trivial principal bundle $P_0 = M \times G$ over M together with the canonical flat connection ThP_0 corresponding to the connection form

$$
\begin{aligned}
(\omega_0)_{(x,g)} \colon T_{(x,g)}(M \times G) \cong T_x M \times T_g G &\longrightarrow \mathfrak{g} \\
(X, Y) &\longmapsto \mu_G(Y) = dL_{g^{-1}} Y.
\end{aligned}
$$

The maximal integral manifold of ThP_0 through the point (x, g) is the submanifold $M \times \{g\} \subseteq M \times G$, which implies that ThP_0 is involutive and thus, by Proposition 3.6.4, that $F^{\omega_0} \equiv 0$.

Definition 3.6.2. *A connection, resp. the corresponding connection form* ω *on the* G*-principal bundle* $P \longrightarrow M$ *is called* flat *if* $F^{\omega} \equiv 0$.

For Principal bundles over simply-connected base spaces, we have the following global result.

Proposition 3.6.5 ([26, Satz 3.20])**.** *Let* $(P, \pi, M; G)$ *be a principal bundle with* M *simply connected and connection form* ω. *The connection form* ω *is flat if, and only if,*

$$
(P, \omega) \cong (P_0, \omega_0)
$$

as principal fiber bundles.

We now collect some relevant results about the behavior of connection and curvature forms under reductions.

Proposition 3.6.6. *Let* $(P, \pi_P, M; G)$ *be a principal* G*-bundle,* $\lambda \colon H \longrightarrow G$ *be a Lie group homomorphism, and* $\big((Q, \pi_Q, M; H), f\big)$ *be a* λ*-reduction of* P. *Let also* ω *be a connection form on* Q. *There exists exactly one connection form* $\widetilde{\omega}$ *on* P *such that*

$$
d_q f(\mathrm{Th}^{\omega}_q Q) = \mathrm{Th}^{\widetilde{\omega}}_{f(q)} P.
$$

For the connection and curvature forms we have

$$
\begin{aligned}
f^* \widetilde{\omega} &= \lambda_* \circ \omega, \\
f^* F^{\widetilde{\omega}} &= \lambda_* \circ F^{\omega}.
\end{aligned}
$$

Proof. Let $x \in M$ and $p \in P_x$. For $q \in Q_x$ we have that $f(q) \in P_x$, since $\pi_Q = \pi_P \circ f$. Let $g \in G$ be the unique element such that $p = f(q)g$. Define

$$\mathrm{Th}_p P \coloneqq d_q(R_g \circ f)(\mathrm{Th}^\omega_q Q) \subseteq T_p P.$$

We claim that the distribution

$$\mathrm{Th}\colon p \in P \longmapsto \mathrm{Th}_p P \subseteq T_p P$$

defines a connection on P. To achieve this we need to verify that the subspaces $\mathrm{Th}_p P$ are well-defined in the sense that they do not depend on the choice of the point $q \in Q$. Let $(q', g') \in Q \times G$ such that $p = f(q')g'$. The fact that $q, q' \in Q_x$ implies that there exists a unique $h \in H$ such that $q' = qh$. Then we obtain $p = f(qh)g' = f(q)\lambda(h)g' = f(q)g$. Since the action of G on the fiber P_x is simply transitive we deduce that $\lambda(h)g' = g$, which implies

$$\begin{aligned} d_{q'}(R_{g'} \circ f)\big(\mathrm{Th}^\omega_{q'} Q\big) &= d_{q'}(R_{g'} \circ f)\big(\mathrm{Th}^\omega_{qh} Q\big) \\ &= d_{q'}(R_{g'} \circ f)\big(d_q R_h\big(\mathrm{Th}^\omega_q Q\big)\big) \\ &\overset{f \circ R_h = R_{\lambda(h)} \circ f}{=} d_q(R_{g'} \circ R_{\lambda(h)} \circ f)\big(\mathrm{Th}^\omega_q Q\big) \\ &= d_q(R_g \circ f)\big(\mathrm{Th}^\omega_q Q\big) \\ &= \mathrm{Th}_p P. \end{aligned}$$

We also have that the assignment $p \longmapsto \mathrm{Th}_p P$ is right-invariant, since for all $a \in G$,

$$d_p R_a(\mathrm{Th}_p P) = d_p R_a d_q(R_g \circ f)\big(\mathrm{Th}^\omega_q Q\big) = d_q(R_{ga} \circ f)\big(\mathrm{Th}^\omega_q Q\big) = \mathrm{Th}_{pa} P.$$

That the subspace $\mathrm{Th}_p P \subseteq T_p P$ is complementary to the vertical space $\mathrm{T}\nu_p P$ follows from the fact that $(d\pi_P \circ df)|_{\mathrm{Th}^\omega_q Q} = d\pi_Q|_{\mathrm{Th}^\omega_q Q}$ is an isomorphism between $\mathrm{Th}^\omega_q Q$ and $T_x M$ and $d_q f\colon \mathrm{Th}^\omega_q Q \longrightarrow \mathrm{Th}_{f(q)} P$ is a surjective map. The smoothness of the distribution $\mathrm{Th}P$ follows from the smoothness of both, the distribution $\mathrm{Th}^\omega Q$ and the map f. This proves our claim. The uniqueness of the connection $\mathrm{Th}P$ necessarily follows from the invariance conditions it needs to satisfy, and which our previously defined one already fulfills.

Let $\widetilde{\omega} \in \mathcal{C}(P)$ be the associated connection form to the connection $\mathrm{Th}P$ we have just defined. For a horizontal vector $X \in \mathrm{Th}^\omega_q Q$ we have that

$$\lambda_*(\omega(X)) = 0 \quad \text{and} \quad (f^*\widetilde{\omega})(X) = \widetilde{\omega}(df(X)) = 0.$$

For a vertical vector $\widetilde{Y}_q \in \mathrm{T}\nu_q Q$, with $Y \in \mathfrak{h}$ it holds that

$$\begin{aligned} \lambda_*\Big(\omega(\widetilde{Y}_q)\Big) &= \lambda_* Y; \\ (f^*\widetilde{\omega})(\widetilde{Y}_q) &= \widetilde{\omega}\Big(df(\widetilde{Y}_q)\Big) = \widetilde{\omega}\Big(\widetilde{\lambda_* Y}_{f(q)}\Big) = \lambda_* Y. \end{aligned}$$

Thus, the identity $\lambda_* \circ \omega = f^*\widetilde{\omega}$ holds on all of $T_q Q$.

For the last identity we make use of the structure equation which is satisfied by the curvature forms.

We obtain

$$\begin{aligned}
f^*F^{\widetilde{\omega}} &= f^*d\widetilde{\omega} + \frac{1}{2}f^*[\widetilde{\omega}, \widetilde{\omega}] \\
&= d(f^*\widetilde{\omega}) + \frac{1}{2}[f^*\widetilde{\omega}, \widetilde{\omega}] \\
&= d(\lambda_* \circ \omega) + \frac{1}{2}[\lambda_* \circ \omega, \lambda_* \circ \omega] \\
&= \lambda_* \circ d\omega + \frac{1}{2}\lambda_*[\omega, \omega] \\
&= \lambda_* \circ F^{\omega}.
\end{aligned}$$

□

Definition 3.6.3. *The connection form* $\widetilde{\omega} \in \mathcal{C}(P)$ *from the previous proposition is called the* λ-extension *of* $\omega \in \mathcal{C}(Q)$. *If* $\widetilde{\omega} \in \mathcal{C}(P)$ *is given and there exists a connection form* $\omega \in \mathcal{C}(Q)$, *which satisfies the relations above, the connection form* ω *is called the* λ-reduction *of* $\widetilde{\omega}$. *If* $H \subseteq G$ *is a Lie subgroup and* $Q \subseteq P$ *an* H*-reduction of* P, *we call the connection form* ω *the reduction of* $\widetilde{\omega}$ *on* Q, *and we say that* $\widetilde{\omega}$ *is reducible to* Q, *whenever such a connection form* $\omega \in \mathcal{C}(Q)$ *exists.*

Next we provide some equivalent formulations of reducibility of connection forms for the special case in which the Lie group H is a Lie subgroup of the group G.

Proposition 3.6.7. *Let* $H \subseteq G$ *be a Lie subgroup,* $Q \subseteq P$ *an* H*-reduction of the* G*-principal bundle* P *and* $\widetilde{\omega}$ *a connection form on* P. *The following statements are equivalent:*

i) $\widetilde{\omega}$ *is reducible to* Q.

ii) $\widetilde{\omega}|_{TQ}$ *takes values in the Lie subalgebra* $\mathfrak{h}$.

iii) $Th^{\widetilde{\omega}}_q P \subseteq T_q Q$ *for all* $q \in Q$.

Proof. For the submanifolds $Q \subseteq P$, $H \subseteq G$ we make the usual identifications of $d_q\iota T_q Q$ with $T_q Q$ and $\iota_*\mathfrak{h}$ with $\mathfrak{h}$, where ι denote the respective inclusion maps. Setting f and λ in the previous proposition to these inclusion maps, the condition $d_q f(Th^{\omega}_q Q) = Th_{f(q)}^{\widetilde{\omega}} P$ is equivalent in this case to $Th^{\omega}_q Q \subseteq Th^{\widetilde{\omega}}_q P$ for all $q \in Q$, whereas the condition $f^*\widetilde{\omega} = \lambda_* \circ \omega$ is equivalent in this setting to $\widetilde{\omega}|_{TQ} = \omega$.

If $\widetilde{\omega}$ is reducible to Q, Proposition 3.6.6 implies the existence of a connection form ω on Q such that $\widetilde{\omega}|_{TQ} = \omega$, which clearly implies *ii)*. If on the other hand, we have that the connection form $\widetilde{\omega}$ on P satisfies that $\widetilde{\omega}|_{TQ} \subseteq \mathfrak{h}$, the $\mathfrak{h}$-valued 1-form $\omega := \widetilde{\omega}|_{TQ}$ defines a connection form on Q, which by our previous discussion implies that ω satisfies the compatibility condition $f^*\widetilde{\omega} = \lambda_* \circ \omega$ in Proposition 3.6.6, which precisely defines the reducibility of the connection form $\widetilde{\omega}$ to Q. With this we obtain the equivalence between *i)* and *ii)*.

Now, if $\widetilde{\omega}$ is reducible to Q, then there exists a connection form ω on Q such that $Th^{\omega}_q Q \subseteq Th^{\widetilde{\omega}}_q P$ for all $q \in Q$, according to our discussion at the beginning of the proof. It even holds that $Th^{\omega}_q Q = Th^{\widetilde{\omega}}_q P$ for all $q \in Q$, since for all $X \in Th^{\widetilde{\omega}}_q P$ there exists exactly one horizontal lift $Y \in Th^{\omega}_q Q$ with respect to the connection form ω of $d_q\pi(X)$, which because of the condition

$\mathrm{Th}^{\omega}_q Q \subseteq \mathrm{Th}^{\widetilde{\omega}}_q P$ is horizontal with respect to the connection form $\widetilde{\omega}$ as well. The uniqueness of horizontal lifts implies therefore that $Y = X$, which in turn implies $\mathrm{Th}^{\widetilde{\omega}}_q P \subseteq \mathrm{Th}^{\omega}_q Q \subseteq T_q Q$.

Suppose on the other hand that $\mathrm{Th}^{\widetilde{\omega}}_q P \subseteq \mathrm{Th}^{\omega}_q Q$. Thus the assignment $Q \ni q \longmapsto \mathrm{Th}_q Q \coloneqq \mathrm{Th}^{\widetilde{\omega}}_q P$ defines a connection on Q which, by the discussion at the beginning of our proof, implies the reducibility of $\widetilde{\omega}$ to Q, having thus shown the equivalence between items *i)* and *iii)*. $\square$

In analogy to the case of covariant derivatives, on every vector bundle we can define a curvature endomorphism.

Definition 3.6.4. *Let* $\nabla\colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ *be a covariant derivative on the vector bundle* $E \longrightarrow M$*. The* $\mathrm{End}(E)$*-valued 2-form*

$$R^{\nabla} \in \Gamma\Big(\bigwedge^2 T^*M \otimes \mathrm{End}(E)\Big)$$

defined by

$$R^{\nabla}(X,Y) \coloneqq \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for vector fields X, Y *on* M *is called the* curvature (endomorphism) *of the covariant derivative* ∇*.*

Before we established how a connection form ω on the principal bundle $(P, \pi, M; G)$ induces a covariant derivative ∇^{ω} on the associated vector bundle $E = P \times_{(G,\rho)} V \longrightarrow M$. The next result shows the analogous link between the curvature form F^{ω} and the curvature $R^{\nabla^{\omega}}$.

Proposition 3.6.8 ([27, Chapter III, Theorem 5.1])**.** *Let* $p \in P_x$ *be a point in the fiber of* P *over* x *and* $[p]\colon V \longrightarrow E_x$ *the corresponding fiber isomorphism. Then*

$$R^{\nabla^{\omega}}_x(X,Y) = [p] \circ \rho_*(F^{\omega}_p(\overline{X}, \overline{Y})) \circ [p]^{-1},$$

where $X, Y \in T_x M$ *and* $\overline{X}, \overline{Y} \in T_p P$ *are their respective horizontal lifts.*

For an $\mathrm{End}(E)$-valued k-form $H \in \Omega^k(M, \mathrm{End}(E))$ and an E-valued l-form $\eta \in \Omega^l(M, E)$ we define the wedge product as

$$\begin{aligned}(H \wedge \eta)&(X_1, \ldots, X_{k+l}) \\ &\coloneqq \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) H(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\big(\eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})\big).\end{aligned}$$

With this wedge product, together with Proposition 3.6.2, we obtain that

$$d_\omega d_\omega \eta = R^{\nabla^{\omega}} \wedge \eta, \qquad \text{for } \eta \in \Omega^k(M, E).$$

At last, we observe how connections, curvatures, and parallel transports behave under automorphisms of principal bundles.

Definition 3.6.5. *A* gauge transformation *on the principal bundle* $(P, \pi, M; G)$ *is a fiber-preserving,* G*-equivariant diffeomorphism* $f\colon P \longrightarrow P$*, i.e.,* *i)* $\pi \circ f = \pi$,

ii) $f(pg) = f(p)g$ *for every* $p \in P$ *and* $g \in G$.

The group of gauge transformation on the bundle $P \longrightarrow M$ *is denoted by* $\mathcal{G}(P) \subseteq \mathrm{Diff}(P)$.

We notice that there is a one-to-one correspondence between $\mathcal{G}(P)$ and $C^\infty(P, G)^G$, the set of the G-equivariant smooth maps on P with values in G, that is

$$C^\infty(P, G)^G := \left\{\sigma \in C^\infty(P, G) | \ \sigma(pg) = g^{-1}\sigma(p)g, \text{ for } p \in P, \ g \in G\right\}.$$

The correspondence is given by $f \in \mathcal{G}(P) \longmapsto \sigma_f \in C^\infty(P, G)^G$, with

$$f(p) = p\sigma_f(p), \qquad \text{for } p \in P.$$

Proposition 3.6.9 ([26, Satz 3.22]). *Let* ω *be a connection form on the principal bundle* $P \longrightarrow M$, *and* $f \in \mathcal{G}(P)$ *a gauge transformation. Then* $f^*\omega$ *is also a connection form on* P *and the following holds:*

1. $f^*\omega = \mathrm{Ad}(\sigma_f^{-1}) \circ \omega + \sigma_f^* \mu_G$.

2. $f \circ P_\gamma^{f^*\omega} = P_\gamma^\omega \circ f$.

3. $D_{f^*\omega} = f^* \circ D_\omega \circ f^{*-1}$.

4. $F^{f^*\omega} = f^* F^\omega = \mathrm{Ad}(\sigma_f^{-1}) \circ F^\omega$.

3.7 Introduction to holonomy theory

In this section, we provide a quick review of holonomy theory and state some standard results.

Throughout this section $(P, \pi, M; G)$ denotes a G-principal bundle over a connected manifold M, and ω denotes a fixed connection form on P. Let $\gamma \colon [0, 1] \longrightarrow M$ be a piecewise-smooth path in M. As usual $P_\gamma^\omega \colon P_{\gamma(0)} \longrightarrow P_{\gamma(1)}$ the parallel transport along the path γ with respect to the connection $\ker \omega$. For fixed $x \in M$ we defined the loop space at x

$$\begin{aligned} \Omega(x) &:= \{\gamma | \ \gamma \text{ is a path in M with } \gamma(0) = \gamma(1) = x\}, \\ \Omega_0(x) &:= \{\gamma \in \Omega(x) | \ \gamma \text{ is null-homotopic}\}. \end{aligned}$$

Let $\gamma \in \Omega(x)$ and $u \in P_x$. The fact that the action of the Lie group G is simply transitive on P_x implies the existence of a unique $\mathrm{hol}_u(\gamma) \in G$ such that

$$P_\gamma^\omega(u) = u \, \mathrm{hol}_u(\gamma).$$

The element $\mathrm{hol}_u(\gamma) \in G$ is called the *holonomy* of γ based at u.

It is not difficult to show the following properties of the holonomy of paths (cf. [26, Lemma 4.1]):

Proposition 3.7.1. *Let* $\gamma, \delta \in \Omega(x)$, $u \in P_x$, $a \in G$ *and* $\mu\colon [0,1] \longrightarrow M$ *a path in* M *starting at the point* x. *The holonomy of paths satisfies:*

i) $\mathrm{hol}_u(\gamma * \delta) = \mathrm{hol}_u(\gamma)\,\mathrm{hol}_u(\delta)$.

ii) $\mathrm{hol}_{ua}(\gamma) = a^{-1}\,\mathrm{hol}_u(\gamma) a$.

iii) $\mathrm{hol}_{P^{\omega}_{\mu}(u)}(\mu * \gamma * \mu^{-}) = \mathrm{hol}_u(\gamma)$.

Definition 3.7.1. *Let* $u \in P_x$. *The group*

$$\mathrm{Hol}_u(\omega) := \{\mathrm{hol}_u(\gamma) |\ \gamma \in \Omega(x)\} \subseteq G$$

is called the holonomy group *of* ω *based at* u. *The group*

$$\mathrm{Hol}^0_u(\omega) := \{\mathrm{hol}_u(\gamma) |\ \gamma \in \Omega_0(x)\} \subseteq G$$

is called the reduced holonomy group *of* ω *based at* u.

Notice that the holonomy groups $\mathrm{Hol}_u(\omega)$, $\mathrm{Hol}^0_u(\omega)$ are indeed subgroups of G thanks to *i)* in the previous proposition. The fact that for $\gamma \in \Omega(x)$, $\eta \in \Omega_0(x)$, the path $\gamma^{-} * \eta * \gamma$ is nullhomotopic implies that $\mathrm{hol}_u(\gamma^{-} * \eta * \gamma) = \mathrm{hol}_u(\gamma)^{-1}\,\mathrm{hol}_u(\eta)\,\mathrm{hol}_u(\gamma) \in \mathrm{Hol}^0_u(\omega)$. This implies that $\mathrm{Hol}^0_u(\omega)$ is normal in $\mathrm{Hol}_u(\omega)$. *ii)* in the previous proposition implies that any two holonomy groups based at two points in the same fiber are conjugated. That is, for any $u \in P_x$, $a \in G$,

$$\mathrm{Hol}_{ua}(\omega) = a^{-1}\,\mathrm{Hol}_u(\omega) a.$$

For $x, y \in M$, $u \in P_x$ and μ a path between x and y, it holds according to *iii)* in the previous proposition that

$$\mathrm{Hol}_u(\omega) = \mathrm{Hol}_{P^{\omega}_{\mu}(u)}(\omega).$$

It can be shown in fact, that the holonomy groups are in fact Lie subgroups of G. In order to show this we make use of the following result (see [26, Satz 1.23]):

Proposition 3.7.2. *Let* G *be a Lie group and* $H \subseteq G$ *a subgroup such that for all* $h \in H$ *there is a piecewise smooth curve* $\gamma\colon I \longrightarrow G$ *with* $\gamma(I) \subseteq H$, *which connects* h *with the identity element in* G. *It follows that* H *is in fact a Lie subgroup.*

Proposition 3.7.3. *The holonomy group* $\mathrm{Hol}_u(\omega)$ *is a Lie subgroup of* G. *The reduced holonomy group* $\mathrm{Hol}^0_u(\omega)$ *is the connected component of the identity element of* $\mathrm{Hol}_u(\omega)$. *In particular, if* M *is simply connected, the holonomy group* $\mathrm{Hol}_u(\omega)$ *is a connected Lie subgroup.*

Proof. First, we show that the reduced holonomy group is a connected Lie subgroup of G. Let $g \in \mathrm{Hol}^0_u(\omega)$, $x = \pi(u)$ and $\gamma \in \Omega_0(x)$ such that $g = \mathrm{hol}_u(\gamma)$. Let $H\colon [0,1] \times [0,1] \longrightarrow M$ be a homotopy between the constant loop at x and the loop γ, with $H_s = H(\cdot, s) \in \Omega_0(x)$. The fact that γ is piecewise smooth implies that we can choose the homotopy H to be piecewise smooth as well. Let $\overline{H}_s$ denote the horizontal lift of H_s starting at $u \in P_x$. The fact that the curves $\overline{H}_s$ are given as the solution of an ordinary differential equation, which depends piecewise smoothly on the parameter s, implies that the curve $s \longmapsto \overline{H}_s(1) \in P$ is also piecewise smooth.

Let g_s denote the unique group element such that $P^{\omega}_{H_s}(u) = \overline{H}_s(1) = ug_s$. Thus the map $s \in [0,1] \longmapsto g_s \in G$ is a piecewise smooth curve in G, whose image lies in $\mathrm{Hol}^0_u(\omega)$ and which connects the identity element e with g ($g_0 = e$, $g_1 = g$). Since the element $g \in \mathrm{Hol}^0_u(\omega)$ was arbitrarily chosen, the claim follows from the previous proposition.

Let us consider the map

$$\begin{aligned} \rho\colon \pi_1(M,x) &\longrightarrow \mathrm{Hol}_u(\omega)/\,\mathrm{Hol}^0_u(\omega) \\ [\gamma] &\longmapsto \mathrm{hol}_u(\gamma)\,\mathrm{Hol}^0_u(\omega). \end{aligned}$$

Because of *i)* in Proposition 3.7.1, it is easy to deduce that ρ is well defined.

The map ρ is a group homomorphism. Indeed, for $\gamma, \delta \in \Omega(x)$,

$$\rho([\gamma]\cdot[\delta]) = \rho([\gamma * \delta]) = [\mathrm{hol}_u(\gamma * \delta)] = [\mathrm{hol}_u(\gamma)\,\mathrm{hol}_u(\delta)] = \rho([\gamma])\rho([\delta]).$$

Since the map ρ is clearly surjective and $\pi_1(M,x)$ is at most countable, we conclude that the factor space $\mathrm{Hol}_u(\omega)/\,\mathrm{Hol}^0_u(\omega)$ is at most countable. In particular, the group $\mathrm{Hol}_u(\omega)$ is the union of at most countably many orbits $g_n\,\mathrm{Hol}^0_u(\omega)$, with $g_n \in \mathrm{Hol}_u(\omega)$. Since the smooth structure on $\mathrm{Hol}^0_u(\omega)$ can be transferred to each of the orbits, and there are at most countably many of them, this provides the desired smooth structure on $\mathrm{Hol}_u(\omega)$. The multiplication and inversion are smooth with respect to this differential structure, which makes $\mathrm{Hol}_u(\omega)$ a Lie subgroup of G. That the reduced holonomy group is the connected component of the identity follows now from the definition. In the case in which M is simply connected, the fact that $\pi_1(M,x) = \{e\}$ implies that $\mathrm{Hol}_u(\omega) = \mathrm{Hol}^0_u(\omega)$, whence $\mathrm{Hol}_u(\omega)$ is a connected Lie subgroup. □

A classical result in holonomy theory is the following one, which proves that every connection on a G-principal bundle is reducible to its holonomy group (cf. [27, Chapter 2, Theorem 7.1]):

Proposition 3.7.4 (Reduction Theorem). *Let $(P,\pi,M;G)$ be a principal G-bundle over a connected manifold M with connection form ω. For $u \in P$ we define the set*

$$P^{\omega}(u) := \{p \in P \mid \textit{there exists an } \omega\textit{-horizontal path from } u \textit{ to } p\}.$$

The following holds:

i) $(P^{\omega}(u), \pi|_{P^{\omega}(u)}, M; \mathrm{Hol}_u(\omega))$ *is a principal bundle.*

ii) The G-principal bundle P together with its connection form ω are reducible to the $\mathrm{Hol}_u(\omega)$-principal bundle $P^{\omega}(u)$.

Definition 3.7.2. *The principal bundle $(P^{\omega}(u),\pi,M;\mathrm{Hol}_u(\omega))$ is called the* holonomy bundle *of ω based at u.*

Definition 3.7.3. *Let $(P,\pi,M;G)$ be a principal bundle on a connected manifold M. We say that a connection form $\omega \in \mathcal{C}(P)$ is* irreducible *if (P,ω) can not be reduced to a proper Lie subgroup.*

A consequence of the Reduction Theorem is the following

Corollary 3.7.1. *Let* (P, ω) *be a* G*-principal bundle and* $(Q, \widehat{\omega})$ *be a reduction of* P *to a Lie subgroup of* G*. The following holds:*

i) $P^{\omega}(u) \subseteq Q$ *for all* $u \in Q$*.*

ii) $\widehat{\omega}|_{TP^{\omega}(u)} = \omega|_{TP^{\omega}(u)}$*, this means that the connection* $\widehat{\omega}$ *reduces to the holonomy bundle of the connection* ω*.*

In this sense, one could think of the holonomy bundle as the smallest possible reduction of a principal bundle.

This observation, together with the Reduction Theorem implies that a connection form $\omega \in \mathcal{C}(P)$ is irreducible if, and only if, $P = P^{\omega}(u)$ and $G = \mathrm{Hol}_u(\omega)$ for all $u \in P$.

One fundamental result in holonomy theory is the so-called *Holonomy Theorem of Ambrose-Singer*, which provides evidence of how big the holonomy group can be.

Proposition 3.7.5 (Ambrose-Singer Holonomy Theorem)**.** *Let* $(P, \pi, M; G)$ *be a principal bundle on a connected manifold* M *and* ω *a connection form with associated curvature form* $F^{\omega} = D_{\omega}\omega$ *and let* $\mathfrak{hol}_u(\omega)$ *denote the Lie algebra of the Lie subgroup* $\mathrm{Hol}_u(\omega)$ *(which of course coincides with the Lie algebra of the connected Lie subgroup* $\mathrm{Hol}^0_u(\omega)$*). It holds that*

$$\mathfrak{hol}_u(\omega) = \left\langle F^{\omega}_p(X, Y) \middle| \, p \in P^{\omega}(u),\, X, Y \in Th^{\omega}_p P \right\rangle \subseteq \mathfrak{g}.$$

If G *is a connected Lie group and* M *simply connected, the connection form* ω *is irreducible if, and only if*

$$\mathfrak{g} = \left\langle F^{\omega}_p(X, Y) \middle| \, p \in P^{\omega}(u),\, X, Y \in Th^{\omega}_p P \right\rangle.$$

Proof. Without loss of generality we can assume that $G = \mathrm{Hol}_u(\omega)$ and $P = P^{\omega}(u)$, otherwise, we can reduce the principal bundle (P, ω) to the holonomy bundle $P^{\omega}(u)$, according to the Reduction Theorem 3.7.4.

Set

$$\mathfrak{b} := \left\langle F^{\omega}_p(X, Y) \middle| \, p \in P,\, X, Y \in Th^{\omega}_p P \right\rangle.$$

Our first claim is that $\mathfrak{b} \trianglelefteq \mathfrak{g}$. Let $F^{\omega}_p(X, Y) \in \mathfrak{b}$ and $W \in \mathfrak{g}$. We define the curve

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathfrak{b} \\ t &\longmapsto (R^*_{\exp(tW)} F^{\omega})_p(X, Y). \end{aligned}$$

From the invariance property of the curvature form we obtain

$$\begin{aligned} \mathfrak{b} \ni \left.\frac{d}{dt}\right|_0 \left(F^{\omega}_{p\exp(tW)}(d_p R_{\exp(tW)} X, d_p R_{\exp(tW)} Y)\right) = &\left.\frac{d}{dt}\right|_0 \left(\mathrm{Ad}(\exp(-tW))(F^{\omega}_p(X, Y))\right) \\ = &-\mathrm{ad}(W)(F^{\omega}_p(X, Y)) \\ = &[F^{\omega}_p(X, Y), W], \end{aligned}$$

which implies that $\mathfrak{b} \trianglelefteq \mathfrak{g}$.

Next, we consider the smooth distribution

$$E \colon p \in P \longmapsto E_p := Th^{\omega}_p P \oplus \left\{ \widetilde{W}(p) | \, W \in \mathfrak{b} \right\} \subseteq T_p P.$$

Our second claim is that this distribution is involutive. Let X be a horizontal vector field on P and $\widetilde{W}$ the fundamental vector field generated by $W \in \mathfrak{b}$. Because of *iv)* in Proposition 3.5.1 we conclude that $[X, \widetilde{W}]$ is also a horizontal vector field, which implies that for all $p \in P$, $[X, \widetilde{W}]_p \in E_p$. Now, for any two fundamental vector fields $\widetilde{V}, \widetilde{W}$ with $V, W \in \mathfrak{b}$, it holds that $[\widetilde{V}, \widetilde{W}] = \widetilde{[V, W]}$ which implies, because of the fact that $\mathfrak{b}$ is an ideal of $\mathfrak{g}$, that $[V, W] \in \mathfrak{b}$, and thus $\widetilde{[V, W]}_p \in E_p$, for every $p \in P$. Finally, if X, Y are any two horizontal vector fields, we have previously shown that

$$\mathrm{pr}_v[X, Y] = -\widetilde{F^\omega(X, Y)},$$

where by definition $F^\omega(X, Y) \in \mathfrak{b}$. This implies then that $[X, Y]_p \in E_p$, for every $p \in P$. This concludes the proof of our second claim. Frobenius' Theorem implies thus the existence of a maximal integral manifold $Q \subseteq P$ of E through $u \in P$. A point in P lies in Q if, and only if, there exists a path $\gamma \colon [0, 1] \longrightarrow P$ between u and q such that $\dot\gamma(t) \in E_{\gamma(t)}$ for every $t \in [0, 1]$. Since $ThP \subseteq TQ$, it follows from the definition of $P^\omega(u)$, that $P = P^\omega(u) \subseteq Q$, which implies that $P = Q$, and thus $E = TP$. This clearly implies that $\mathfrak{b} = \mathfrak{g}$. □

In the case of a vector bundle $E \longrightarrow M$ with a given covariant derivative ∇, it is possible to define the (reduced) holonomy group in a similar fashion as we previously did in the case of principal fiber bundles.

Definition 3.7.4. *Let $E \longrightarrow M$ be a smooth vector bundle with a given covariant derivative ∇. For $x \in M$ we define the* holonomy group *of ∇ based at x as the subgroup*

$$\mathrm{Hol}_x(\nabla) := \left\{P^\nabla_\gamma \mid \gamma \in \Omega(x)\right\} \subseteq GL(E_x).$$

The group

$$\mathrm{Hol}^0_x(\nabla) := \left\{P^\nabla_\gamma \mid \gamma \in \Omega_0(x)\right\} \subseteq GL(E_x)$$

is called the reduced holonomy group *of ∇ based at x.*

We notice that, in the case in which the manifold M is connected, any two holonomy groups are conjugate. Indeed let $x, y \in M$ and $\mu \colon [0, 1] \longrightarrow M$ any path between them. For $\gamma \in \Omega(y)$ we obtain the loop $\mu^- * \gamma * \mu \in \Omega(x)$. Thus

$$\mathrm{Hol}_x(\nabla) \ni P_{\mu^- * \gamma * \mu} = P_\mu^{-1} \circ P_\gamma \circ P_\mu,$$

which implies

$$\mathrm{Hol}_x(\nabla) = P_\mu^{-1} \circ \mathrm{Hol}_y(\nabla) \circ P_\mu.$$

As it is well known, there exists a tight relation between the holonomy groups of connections on principal fiber bundles and the holonomy groups of the associated covariant derivatives on associated vector bundles:

Proposition 3.7.6. *Let P be a G-principal bundle on the manifold M, $\rho \colon G \longrightarrow GL(V)$ a representation of the Lie group G and $E = P \times_G V$ the associated vector bundle. Further, let ω be a connection form on P and let ∇^ω denote the associated covariant derivative on E. For $u \in P_x$, $[u] \colon V \longrightarrow E_x$ denotes the fiber diffeomorphism defined by u. It holds that*

$$\mathrm{Hol}_x(\nabla^\omega) = [u] \circ \rho(\mathrm{Hol}_u(\omega)) \circ [u]^{-1}.$$

In the particular case in which the map ρ *is injective, we obtain that the holonomy groups* $\mathrm{Hol}_u(\omega)$ *and* $\mathrm{Hol}_x(\nabla^\omega)$ *are isomorphic.*

Let R^{∇^ω} *denote the curvature endomorphism of the covariant derivative* ∇^ω*, and* $P_\gamma := P_\gamma^{\nabla^\omega}$ *the parallel transport map in* E *with respect to* ∇^ω*. It holds for the Lie algebra of the holonomy group* $\mathrm{Hol}_x(\nabla^\omega)$ *that*

$$\mathfrak{hol}_x(\nabla^\omega) = \left\langle P_\gamma^{-1} \circ R_y^{\nabla^\omega}(v,w) \circ P_\gamma \middle| \,{}_{\gamma \text{ path between } x \text{ and } y}^{v,w \in T_yM,} \right\rangle.$$

Proof. Let $\delta \in \Omega(x)$ and $e = [u,z] \in E_x$. It holds that

$$\begin{aligned} P_\delta^{\nabla^\omega}(e) =& [P_\delta^\omega(u), z] = [u\,\mathrm{hol}_u(\delta), z] = [u, \rho(\mathrm{hol}_u(\delta))z] \\ =& \big([u] \circ \rho(\mathrm{hol}_u(\delta)) \circ [u]^{-1}\big)(e), \end{aligned}$$

from which we immediately obtain the first claim.

Let $\overline{\gamma}$ be an ω-horizontal curve from u to $p \in P^\omega(u)_y$ and $\gamma = \pi \circ \overline{\gamma}$. It then holds that $P_\gamma^{\nabla^\omega} \circ [u] = [p]$. Let $v, w \in T_yM$ and $\overline{v}, \overline{w} \in T_pP$ their ω-horizontal lifts. From Proposition 3.6.8 we obtain that

$$\begin{aligned} \rho_*(F_p^\omega(\overline{v},\overline{w})) =& [p]^{-1} \circ R_y^{\nabla^\omega}(v,w) \circ [p] \\ =& [u]^{-1} \circ \Big(P_\gamma^{-1} \circ R_y^{\nabla^\omega}(v,w) \circ P_\gamma\Big) \circ [u] \end{aligned}$$

This identity together with the first claim in the proposition and the Theorem of Ambrose-Singer implies that

$$\begin{aligned} \mathfrak{hol}_x(\nabla^\omega) =& [u] \circ \rho_*(\mathfrak{hol}_u(\omega)) \circ [u]^{-1} \\ =& [u] \circ \big\langle \rho_*(F_p^\omega(X,Y)) \big| \; p \in P^\omega(u),\, X, Y \in T_pP \big\rangle \circ [u]^{-1} \\ =& \left\langle P_\gamma^{-1} \circ R_y^{\nabla^\omega}(v,w) \circ P_\gamma \middle| \,{}_{\gamma \text{ path between } x \text{ and } y}^{v,w \in T_yM,} \right\rangle. \end{aligned}$$

□

We finish up this section by discussing how holonomy groups can be used to find parallel sections of vector bundles.

As a reminder, given a real or complex vector bundle (E, π, M) over a connected manifold with a covariant derivative ∇. The set of parallel sections is defined as

$$\mathrm{Par}(E, \nabla) := \{\sigma \in \Gamma(E) |\; \nabla\sigma = 0\}.$$

As usual, let P denote a G-principal bundle over the connected manifold M with a connection form ω. Let $\rho \colon G \longrightarrow GL(V)$ be a Lie group representation and $E := P \times_G V$ the associated vector bundle with induced covariant derivative ∇^E.

Proposition 3.7.7 (The holonomy principle)**.** *There exists a one-to-one correspondence between the set of parallel sections in* E *and the set of holonomy invariant vectors in* V*:*

$$\mathrm{Par}(E, \nabla^E) \quad \overset{1:1}{\longleftrightarrow} \quad V_{\mathrm{Hol}_u(\omega)} := \{v \in V |\; \rho(\mathrm{Hol}_u(\omega))v = v\}.$$

Furthermore, if M *is simply connected, it holds that*

$$V_{\mathrm{Hol}_u(\omega)} = \{v \in V |\ \rho_*(\mathfrak{hol}_u(\omega))v = 0\}.$$

Proof. Let $x = \pi(u)$, $v \in V_{\mathrm{Hol}_u(\omega)}$. Define the section σ_v in E by

$$\sigma_v : y \in M \longmapsto \sigma_v(y) = P^{E,\omega}_{\gamma}([u,v]) = [P^{\omega}_{\gamma}(u), v] \in E_y,$$

where γ denotes a path between x and y. For this definition to make sense we need it to be independent of the chosen path, which is in turn the case. Let μ be another path between x and y. Then $\mu^- * \gamma$ is a loop based at x and

$$P^{\omega}_{\mu^- * \gamma}(u) = P^{\omega}_{\mu^-}(P^{\omega}_{\gamma}(u)) = uh,$$

for some $h \in \mathrm{Hol}_u(\omega)$. The right invariance of the parallel transport thus implies $P^{\omega}_{\gamma}(u) = P^{\omega}_{\mu}(uh) = P^{\omega}_{\mu}(u)h$ and thus

$$[P^{\omega}_{\gamma}(u), v] = [P^{\omega}_{\mu}(u)h, v] = [P^{\omega}_{\mu}(u), \rho(h)v] = [P^{\omega}_{\mu}(u), v].$$

That $\sigma_v \in \Gamma(E)$ follows from the smooth dependence on the initial conditions of the system of ordinary differential equations that define the parallel transport (cf. Proposition 4.2.4). The fact that σ_v is a parallel section automatically follows from its mere definition:

$$\begin{aligned}\left(\nabla^E_X \sigma_v\right)(y) &= \frac{d}{dt}\bigg|_0 P^{E,\omega}_{t,0}(\sigma_v(\gamma(t))) \\ &= \frac{d}{dt}\bigg|_0 P^{E,\omega}_{t,0} P^{E\,\omega}_{0,t}[u,v] \\ &=0,\end{aligned}$$

for any path γ with $\gamma(0) = y$.

Conversely, let σ be a parallel section in E. The Reduction Theorem implies that (P, ω) reduces to the holonomy bundle $P^{\omega}(u)$. Thus, we obtain the vector bundle isomorphism

$$E = P \times_G V \cong P^{\omega}(u) \times_{\mathrm{Hol}_u(\omega)} V.$$

This implies the existence of unique invariant maps $\overline{\sigma} \in C^{\infty}(P, V)^G$, $\overline{\tau} \in C^{\infty}(P^{\omega}(u), V)^{\mathrm{Hol}_u(\omega)}$, with $\overline{\sigma}|_{P^{\omega}(u)} = \overline{\tau}$ such that for every $p \in P$,

$$\sigma(\pi(p)) = [p, \overline{\sigma}(p)] = [p, \overline{\tau}(p)].$$

Now, by the corollary of Proposition 3.5.6 we have the formula

$$(\nabla^E_X \sigma)(\pi(p)) = [p, \overline{X}(\overline{\sigma})(p)]$$

which implies that $\nabla \sigma = 0$ if, and only if, the map $\overline{\sigma}$ is constant along horizontal curves in P. That is to say, there exists some $v \in V$ such that $\overline{\tau} = \overline{\sigma}|_{P^{\omega}(u)} \equiv v$.

Thus, for every $q \in P^{\omega}(u)$, $h \in \mathrm{Hol}_u(\omega)$,

$$v = \overline{\tau}(qh) = \rho(h^{-1})\overline{\tau}(q) = \rho(h^{-1})v,$$

that is, $v \in V_{\mathrm{Hol}_u(\omega)}$. From the way the v is defined, it is evident that $\sigma_v = \sigma$.

Finally, for $v \in V_{\mathrm{Hol}_u(\omega)}$, we obtain for all $X \in \mathfrak{hol}_u(\omega)$ that $\rho(\exp(tX))v = v$, which immediately implies $d_e\rho(X)(v) = \rho_*(X)v = 0$. If, on the other hand $\rho_*(X)v = 0$ for every $X \in \mathfrak{hol}_u(\omega)$, we obtain

$$\rho(\exp(X))v = \exp(\rho_*(X))v = e^{\rho_*(X)}v = v.$$

In the case in which the manifold is simply connected, we already showed that the holonomy group $\mathrm{Hol}_u(\omega)$ is connected, which implies

$$\mathrm{Hol}_u(\omega) = \langle \exp(X) | \, X \in \mathfrak{hol}_u(\omega) \rangle \, .$$

Thus, from the considerations in the preceding paragraph, we obtain

$$V_{\mathrm{Hol}_u(\omega)} = \{v \in V | \, \rho_*(\mathfrak{hol}_u(\omega))v = 0\}.$$

$\square$

In the particular case, ∇ is the Levi-Civita covariant derivative of a (pseudo-)Riemannian manifold, the holonomy principle can be reformulated as follows:

Proposition 3.7.8. *Let (M, g) be a (pseudo-)Riemannian manifold with Levi-Civita covariant derivative ∇, $\mathcal{T}$ any tensor bundle over M, whose tensorial extension of the Levi-Civita covariant derivative on M we denote by ∇ as well, and let $x \in M$.*

i) Let $T \in \Gamma(\mathcal{T})$ be a parallel tensor field. It holds that $\mathrm{Hol}_x(g) \cdot T(x) = T(x)$, where $\mathrm{Hol}_x(g) := \mathrm{Hol}_x(\nabla)$.

ii) Let $T_x \in \mathcal{T}_x$ be a tensor that is invariant under the action of $\mathrm{Hol}_x(g)$. Then there exists a unique tensor field $T \in \Gamma(\mathcal{T})$ such that $T(x) = T_x$. In fact, $T(y) := P^{\nabla}_{\gamma}(T_x)$, where $y \in M$ and γ is a curve connecting x and y.

We finalize this section by making some further remarks on the holonomy of torsion-free covariant derivatives.

Let M be a smooth connected n-manifold. As a consequence of Proposition 3.7.6, all of the important results which were obtained for the holonomy groups of connection forms on principal bundles have an analogous version in the setting of smooth vector bundles. In particular, in the case in which $E = TM$.

Up next we collect some of the properties of holonomy groups of affine covariant derivatives, some of which we already encountered in the context of holonomy groups of principal fiber bundles:

i) The restricted holonomy group $\mathrm{Hol}^0_x(\nabla)$ is a connected Lie subgroup of $GL(T_xM)$, which is the connected component at the identity and a normal subgroup of $\mathrm{Hol}_x(\nabla)$.

This result holds in fact for any covariant derivative on any finite-rank vector bundle over M (cf. Proposition 3.7.3).

ii) If $\pi\colon \widetilde{M} \longrightarrow M$ is the universal cover and $\widetilde{\nabla}$ is the lift of the covariant derivative ∇ to $\widetilde{M}$, then $\mathrm{Hol}_{\tilde{x}}(\widetilde{\nabla}) = \mathrm{Hol}^0_{\tilde{x}}(\widetilde{\nabla}) \cong \mathrm{Hol}^0_{\pi(\tilde{x})}(\nabla)$. This implies we can assume, without loss of generality, that the holonomy group is connected.

iii) Since the manifold M is connected, we previously showed that the holonomy groups based at any two points in the manifold are conjugated. If we fix a frame $\iota\colon T_xM \longrightarrow V$, the conjugacy class of $\iota(\mathrm{Hol}_x(\nabla)) \subseteq \mathrm{GL}(V)$ depends neither on the point $x \in M$ nor on the linear isomorphism ι. Taking advantage of this fact, we refer to the conjugacy class of $\mathrm{Hol}(\nabla) \coloneqq \iota(\mathrm{Hol}_x(\nabla)) \subseteq \mathrm{GL}(V)$ (resp. $\mathrm{Hol}^0(\nabla) \coloneqq \iota(\mathrm{Hol}^0_x(\nabla)) \subseteq \mathrm{GL}(V)$) as the holonomy group (resp. restricted holonomy group) of ∇. Similarly, we refer to the Lie algebra $\mathfrak{hol}(\nabla) \subseteq \mathrm{End}(V)$ of $\mathrm{Hol}(\nabla) \subseteq \mathrm{GL}(V)$ as the holonomy algebra of ∇. The standard action of $\mathrm{Hol}(\nabla)$ on V is referred to as the *holonomy representation*. The holonomy group is called *reducible, resp. irreducible* if its holonomy representation is reducible, resp. irreducible.

We notice as well that for any torsion-free affine covariant derivative on M, using the fact that the tangent bundle is naturally associated to the frame bundle of M, as previously discussed, and that the parallel transport maps are linear isomorphisms, we can reformulate Proposition 3.7.6 and obtain

$$\mathfrak{hol}_x(\nabla) = \langle (P_\gamma \cdot R)(v, w) |\ v, w \in T_xM,\ \gamma \text{ is a path with } \gamma(1) = x \rangle\,,$$

where $(P_\gamma \cdot R)(v,w) \coloneqq P_\gamma \circ R_{\gamma(0)}(P_\gamma^{-1}v, P_\gamma^{-1}w) \circ P_\gamma^{-1}$.

From the definition, it is easy to verify that for any such path, $P_\gamma \cdot R \in K(\mathfrak{hol}_x(\nabla))$, where $K(\mathfrak{hol}_x(\nabla)) \subseteq K(T_x^*M) \otimes T_xM$ denotes the set of the so-called *algebraic curvature tensors*, which we formally introduce in the next chapter. Now, thanks to this reformulated version of the Ambrose-Singer Theorem we obtain that

$$\underline{\mathfrak{hol}_x(\nabla)} = \mathfrak{hol}_x(\nabla),$$

where for any Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g} = \mathrm{End}(V)$,

$$\underline{\mathfrak{h}} \coloneqq \langle R(v,w) |\ R \in K(\mathfrak{h}),\ v, w \in V \rangle \trianglelefteq \mathfrak{h}.$$

Definition 3.7.5. *A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called a* Berger algebra *if $\underline{\mathfrak{h}} = \mathfrak{h}$. A Berger algebra is called* symmetric *if its space of formal curvature derivatives is trivial, that is, $K^1(\mathfrak{h}) = \{0\}$ and* non-symmetric *otherwise.*

*A Lie subgroup $H \subseteq G = \mathrm{GL}(V)$ is called a (*symmetric*, resp.* non-symmetric*)* Berger group *if its Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a (symmetric, resp. non-symmetric) Berger algebra.*

The motivation for the name symmetric comes from the fact that, as one can readily show, if ∇ is a torsion-free covariant derivative on M, the map $X \longmapsto \nabla_X R$ is an element of

$K^1(\mathfrak{hol}_x(\nabla))$. Thus, if $K^1(\mathfrak{hol}_x(\nabla)) = \{0\}$ we obtain in particular that $\nabla R \equiv 0$, which implies that the covariant derivative is locally symmetric (see [34, Theorem 3.5]).

As a consequence of the previous discussion, we obtain the following classical result in holonomy theory. These are the criteria used by Berger in his classification work [6].

Proposition 3.7.9 (Berger's criteria). *Let $H \subseteq G$ be a Lie subgroup that occurs as the holonomy group of a torsion-free affine covariant derivative on some manifold M. Then H must be a Berger group. If the covariant derivative is not locally symmetric, then H must be a non-symmetric Berger group.*

3.8 Riemannian holonomy groups

In this section, we go into some detail about the classification of Riemannian holonomy groups.

Let (M, g) be a Riemannian n-manifold with Levi-Civita covariant derivative ∇. An immediate consequence of Proposition 3.2.6 and corollary 3.7.1 is that the holonomy group $\mathrm{Hol}(g) := \mathrm{Hol}(\nabla)$ is a subgroup of $O(n)$, and consequently, $\mathrm{Hol}^0(\nabla) \subseteq \mathrm{SO}(n)$.

In fact, one can show the following

Proposition 3.8.1 ([16, Proposition 3.1.5]). *Let (M, ∇) be an n-dimensional affine manifold, ∇ torsion-free. Then ∇ is the Levi-Civita covariant derivative of a Riemannian metric on M if, and only if, $\mathrm{Hol}(\nabla)$ is conjugate in $\mathrm{GL}(n, \mathbb{R})$ to a subgroup of $O(n)$.*

Definition 3.8.1. *A Riemannian manifold (M, g) is said to be* (locally) reducible *if every point has an open neighborhood isometric to a Riemannian product $(M_1 \times M_2, g_1 \times g_2)$, and* irreducible *if it is not locally reducible.*

It is not difficult to show that the following result for the holonomy of a Riemannian product holds:

Proposition 3.8.2 ([16, Proposition 3.2.1]). *Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds. Then the product metric $g_1 \times g_2$ has holonomy $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$.*

In fact, under the right assumptions we obtain the following converse of the previous proposition, which is referred to as the *de Rham splitting Theorem*:

Proposition 3.8.3 ([16, Proposition 3.2.7]). *Let (M, g) be a complete, simply-connected Riemannian manifold. Then there exist complete, simply-connected Riemannian manifolds (M_j, g_j), for $j = 1, \ldots, k$, such that the holonomy representation of $\mathrm{Hol}(g_j)$ is irreducible, (M, g) is isometric to $(M_1 \times \cdots \times M_k, g_1 \times \cdots \times g_k)$, and $\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \cdots \times \mathrm{Hol}(g_k)$.*

Up next we precisely introduce the so-called *Berger's List*, which was the first serious attempt at classifying Riemannian holonomies, and in fact, the list was, up to a single entry, complete (see, for example, [34, Theorem 3.6]).

Proposition 3.8.4 (Berger). *Let M be an n-dimensional simply-connected Riemannian manifold, which is irreducible and nonsymmetric. Then exactly one of the following seven cases holds.*

i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$, *ii)* $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(n)$,

iii) $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(n)$,

iv) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(n)$,

v) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ *in* $\mathrm{SO}(n)$,

vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*

vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

The path Berger followed to prove his classification result was by applying the two criteria enclosed in Proposition 3.7.9 to the closed, connected Lie subgroups of $\mathrm{SO}(n)$, whose classification follows from the classification of Lie groups. Being the groups in the list the only ones that pass such tests.

The geometric features a manifold needs to have for its holonomy group to be one of the groups appearing in Berger's list are very well understood and we briefly summarize them in the following paragraphs. It is worth noting that all of these results are essentially consequences of the Reduction Theorem and the holonomy principle 3.7.8.

***i)*:** $\mathrm{SO}(n)$ is the holonomy group of a generic Riemannian manifold.

***ii)*:** *The case* $\mathrm{U}(m)$ (The *Kähler case*).

Definition 3.8.2. *A vector bundle homomorphism* $J \colon TM \longrightarrow TM$ *such that* $J^2 = -\mathbb{1}_{TM}$ *is called an* almost complex structure *on* M.

An almost Hermitian manifold *is a triple* (M, g, J) *consisting of a Riemannian manifold* (M, g) *together with an orthogonal almost complex structure, that is, an almost complex structure on* M *such that*

$$g(JX, JY) = g(X, Y)$$

for all vector fields X, Y.

An Hermitian manifold *is an almost Hermitian manifold, whose associated almost complex structure is integrable.*

Definition 3.8.3. *A* Kähler manifold *is an almost Hermitian manifold* (M, g, J), *whose almost complex structure is parallel with respect to the Levi-Civita covariant derivative of* (M, g), *i.e.* $\nabla J \equiv 0$.

An immediate consequence of the Newlander-Nirenberg Theorem implies that every Kähler manifold is a Hermitian manifold, since for any two vector fields X, Y,

$$\begin{aligned} -N_J(X, Y) =& [JX, JY] - J([X, JY] + [JX, Y]) - [X, Y] \\ \overset{T^{\nabla} \equiv 0}{=} & (\nabla_X J)(JY) - (\nabla_Y J)(JX) + (\nabla_{JX} J)(Y) - (\nabla_{JY} J)(X) \\ =& 0. \end{aligned}$$

On an Hermitian manifold (M, g, J) we define its *Kähler form* as the 2-form ω, with

$$\omega(X, Y) := g(JX, Y).$$

We have the following characterization of Kähler manifolds:

Proposition 3.8.5. *An almost Hermitian manifold (M,g,J) is Kähler if, and only if,* J *is integrable and its Kähler form is closed.*

Proof. Let ∇ denote the Levi-Civita covariant derivative of g. The fact that J is integrable implies that the identities

$$\begin{aligned} d\omega(X,Y,Z) =& g((\nabla_X J)Y,Z) + g((\nabla_Y J)Z,X) + g((\nabla_Z J)X,Y), \\ 2g((\nabla_X J)Y,Z) =& d\omega(X,Y,Z) - d\omega(X,JY,JZ) \end{aligned}$$

hold for any vector fields X,Y,Z (see [35, Proposition 4.16]). The claim follows now immediately. ☐

Proposition 3.8.6 ([26, Satz 5.22])**.** *A Riemannian manifold* (M^{2m}, g) *is Kähler if, and only if, its holonomy group is contained in* $U(m)$.

The proof of this proposition is indeed an easy consequence of the holonomy principle, as well as of the observation that $U(m) \subseteq SO(2m)$ made in (3.1). For a Kähler manifold (M,g,J), the fact that J is parallel implies, for every $x, y \in M$ and any curve between them, $P_\gamma \circ J_x = J_y \circ P_\gamma$. This readily implies $\mathrm{Hol}_x(\nabla) \subseteq U(T_xM, g_x, J_x)$, while on the other hand, if we assume $\mathrm{Hol}_x(\nabla) \subseteq U(T_xM, g_x, J_x)$, item *ii)* in the holonomy principle 3.7.8 allows to construct a parallel orthogonal complex on (M,g).

***iii)*: *The case* $SU(m)$ (The *special Kähler case*).

The fact that $SU(m) \subseteq U(m)$ immediately implies that a manifold having holonomy contained in $SU(m)$ is automatically Kähler. The characterization of manifolds with holonomy contained in $SU(m)$ is given by the following

Proposition 3.8.7 ([26, Satz 5.23])**.** *A Kähler manifold* (M^{2m}, g, J) *is Ricci-flat if, and only if, its holonomy group is contained in* $SU(m)$.

Definition 3.8.4. *A* Calabi-Yau manifold *is a compact Kähler manifold of dimension* $2m \geqslant 4$ *whose holonomy group is exactly the group* $SU(m)$.

***iv)*: *The case* $Sp(m)$ (The *hyper-Kähler case*).

We begin with the following definition.

Definition 3.8.5. *An* almost quaternionic structure *on a manifold* M *is a triple* $\underline{J} := (J_1, J_2, J_3)$ *of anti-commuting almost complex structures on* M *with* $J_1J_2 = J_3$. *If, in addition, each of the almost complex structures is orthogonal with respect to a Riemannian metric* g*, we call* $\underline{J}$ *an* orthogonal almost quaternionic structure *on* (M,g).

An almost hyper-Kähler manifold *is a Riemannian manifold* (M,g) *together with an orthogonal almost quaternionic structure* $\underline{J}$.

Definition 3.8.6. *An almost quaternionic structure* (J_1, J_2, J_3) *on a Riemannian manifold* (M,g) *is called* parallel *if* $\nabla J_\ell \equiv 0$*, for* $\ell = 1,2,3$. *An* hyper-Kähler manifold *is a Riemannian manifold of dimension* $4m \geqslant 8$ *with a parallel orthogonal almost-quaternionic structure.*

Proposition 3.8.8 ([26, Satz 5.24])**.** *A $4m$-dimensional Riemannian manifold is hyper-Kähler if, and only if, its holonomy group is contained in* $\mathrm{Sp}(m)$.

From the fact that $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m)$, as described in (3.2), we obtain that a hyper-Kähler manifold is necessarily Ricci-flat and Kähler.

***v)*:** *The case* $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ (The *quaternionic-Kähler case*).

We recall the definition of a quaternionic-Kähler manifold.

Definition 3.8.7. *A Riemannian manifold* (M, g) *of dimension* $4m \geqslant 8$ *is called an* almost quaternionic-Kähler manifold, *if there exists a 3-dimensional subbundle* $E \subseteq \mathfrak{so}(TM, g)$, *which locally is generated by almost quaternionic structures in the following sense: For every* $x \in M$ *there is a neighborhood* $x \in U$ *and an orthogonal almost quaternionic structure* $\underline{J}_U = (J_1, J_2, J_3)$ *on* $(U, g|_U)$ *with* $E_U = \mathrm{span}_{\mathbb{R}}\{J_1, J_2, J_3\}$. *If in addition, the bundle* E *is parallel in the sense that* $\nabla\Gamma(E) \subseteq \Gamma(E)$, *the triple* (M, g, E) *is called a* quaternionic Kähler manifold.

The holonomy principle thus implies

Proposition 3.8.9 ([26, Satz 3.25])**.** *A Riemannian manifold of dimension* $4m \geqslant 8$ *is a quaternionic-Kähler manifold if, and only if, its holonomy group is contained in* $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$.

Quaternionic-Kähler manifolds are Einstein spaces, but not Ricci-flat. See for example [36].

***vi)*:** *The case* G_2.

Let (M^7, g) denote an orientable 7-dimensional Riemannian manifold. For $x \in M$, we define the set

$$\mathcal{F}^3_x M := \left\{ \varphi \in \bigwedge\nolimits^3 T^*_x M \;\middle|\; \begin{array}{l} \text{there exists an orientation-preserving isometry} \\ L\colon (\mathbb{R}^7, \langle\cdot,\cdot\rangle_{\mathbb{R}^7}) \longrightarrow (T_x M, g_x) \text{ such that } L^*\varphi = \omega_0 \end{array} \right\},$$

where $\omega_0 \in \bigwedge^3(\mathbb{R}^7)^*$ is the 3-form given in equation (3.4).

Definition 3.8.8. *A 3-form* $\omega \in \Omega^3(M)$ *on* (M^7, g) *is called* admissible *if* $\omega_x \in \mathcal{F}^3_x M$ *for all* $x \in M$.

Definition 3.8.9. *A 7-dimensional oriented Riemannian manifold* (M^7, g) *is called a* G_2-manifold *if there exists an admissible parallel 3-form* $\omega \in \Omega^3(M)$.

Proposition 3.8.10 ([26, Satz 5.26])**.** *A 7-dimensional oriented Riemannian manifold is a* G_2-*manifold if, and only if, its holonomy group is contained in* G_2.

It can also be proved that Riemannian metrics with holonomy G_2 are Ricci-flat. See for example [34, Section 3.5].

***vii)*:** *The case* $\mathrm{Spin}(7)$.

Let (M^8, g) be an 8-dimensional orientable Riemannian manifold. In a similar fashion as in the previous case, we define for $x \in M$ the set

$$\mathcal{F}^4_x M := \left\{ \varphi \in \bigwedge\nolimits^4 T^*_x M \;\middle|\; \begin{array}{l} \text{there exists an orientation-preserving isometry} \\ L\colon (\mathbb{R}^8, \langle\cdot,\cdot\rangle_{\mathbb{R}^8}) \longrightarrow (T_x M, g_x) \text{ such that } L^*\varphi = \sigma_0 \end{array} \right\},$$

where $\sigma_0 \in \bigwedge^4(R^8)^*$ is the 4-form defined in equation (3.5).

Definition 3.8.10. *A 4-form* $\sigma \in \Omega^4(M)$ *on* (M^8, g) *is called* admissible *if* $\sigma_x \in \mathcal{F}^4_x M$ *for all* $x \in M$.

Definition 3.8.11. *An 8-dimensional oriented Riemannian manifold* (M^8, g) *is called a* Spin(7)-manifold *if there exists an admissible parallel 4-form* $\sigma \in \Omega^4(M)$.

Proposition 3.8.11 ([26, Satz 5.27])**.** *An 8-dimensional oriented Riemannian manifold is a* Spin(7)-*manifold if, and only if, its holonomy group is contained in* Spin(7)*.*

Similarly as in the G_2 case, metrics with holonomy Spin(7) are necessarily Ricci-flat. See for example [34, Section 3.5].

4

The curvature tensor of a torsion-free affine manifold

The main goal of the chapter is to state and prove the core results of the present work, as well as their applications to the problem of the classification of holonomy groups.

Said main results are indeed local ones, so in order to simplify the matter by a reasonable amount we are going to resort to a fundamental tool in Differential Geometry, namely, that of normal coordinates.

The first part of the chapter is devoted to the introduction of the necessary notions to properly establish the adequate framework.

4.1 The exponential map on affine manifolds

Let (M, ∇) be an affine manifold. We define the *torsion* of the covariant derivative as the tensor $T^{\nabla} \in \Gamma(\bigwedge^2 T^*M \otimes TM) \subseteq \mathcal{T}^{(1,2)}(M)$ defined by

$$T^{\nabla}(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

The covariant derivative ∇ is said to be *symmetric* or *torsion-free* if its torsion T^{∇} identically vanishes.

Let $\gamma\colon I \longrightarrow M$ be a smooth curve. The set of vector fields along the curve γ is defined as

$$\Gamma_{\gamma}(TM) := \{X \in C^{\infty}(I, TM) |\ \pi_{TM} \circ X = \gamma\}.$$

The covariant derivative ∇ can help us make sense of the notion of directional derivatives of vector fields along curves. The following standard result provides the adequate setting for it (see [37, Lemma 4.9]):

Lemma 4.1.1. *Let $\gamma\colon I \longrightarrow M$ be a curve. The covariant derivative ∇ determines a unique operator*

$$\frac{D}{dt}\colon \Gamma_{\gamma}(TM) \longrightarrow \Gamma_{\gamma}(TM)$$

which satisfies the following properties:

i) $\frac{D}{dt}$ *is* $\mathbb{R}$*-linear.*

ii) For all $X \in \Gamma_\gamma(TM)$, $f \in C^\infty(I)$ *it holds that*

$$\frac{D}{dt}(fX) = \dot{f}X + f\frac{D}{dt}X.$$

iii) If X *is* extendible, *that is, if there exists a vector field* $\widetilde{X}$ *on a neighborhood of the image of* γ*, such that for all* $t \in I$, $X(t) = \widetilde{X}_{\gamma(t)}$*. Then for any extension* $\widetilde{X}$ *of X it holds that*

$$\frac{D}{dt}X(t) = \nabla_{\dot{\gamma}(t)}\widetilde{X}.$$

Let $(U, \varphi = (x^1, \ldots, x^n))$ be a chart of the manifold and $X = X^i\partial_i$ be the local representation of $X \in \Gamma_\gamma(TM)$ with respect to this chart. Since the coordinate vectors ∂_i are extendible, item *iii)* in the previous proposition implies that the local representation of the vector field $\frac{D}{dt}X$ in these coordinates is given by the formula

$$\frac{D}{dt}X(t) = \dot{X}^i(t)\partial_i|_{\gamma(t)} + X^i(t)\nabla_{\dot{\gamma}(t)}\partial_i|_{\gamma(t)} \tag{4.1}$$

for all t such that $\gamma(t) \in U$.

A vector field $X \in \Gamma_\gamma(TM)$ along γ is called *parallel along* γ if $\frac{D}{dt}X \equiv 0$. The curve γ is called a *geodesic* if

$$\frac{D}{dt}\dot{\gamma} \equiv 0.$$

Standard results from the theory of ordinary differential equations show that for every $x \in M$ and every $v \in T_xM$ there exists exactly one maximal geodesic $\gamma_v\colon I_v \longrightarrow M$ such that $\gamma_v(0) = x$, $\dot{\gamma}_v(0) = v$. The uniqueness of such geodesics implies that, for any $c, t \in \mathbb{R}$,

$$\gamma_{cv}(t) = \gamma_v(ct),$$

whenever either side is defined.

This rescaling property immediately proves that the set

$$D_x \coloneqq \{v \in T_xM \mid 1 \in I_v\} \subseteq T_xM$$

is star-shaped with respect to 0. It can also be shown that D_x is open ([37, Prop. 5.7]).

Definition 4.1.1. *With the above notation we define the* exponential map *of* (M, ∇) *at the point* $x \in M$ *as the map*

$$\begin{aligned} \exp_x\colon D_x &\longrightarrow M \\ v &\longmapsto \gamma_v(1). \end{aligned}$$

The rescaling property of the geodesic γ_v implies that

$$\gamma_v(t) = \exp_x(tv)$$

whenever either side is defined.

From the results obtained in the previous paragraphs, we easily conclude that for all $s, t \in I_v$ such that $s + t \in I_v$,

$$\gamma_v(t+s) = \gamma_{\dot{\gamma}_v(t)}(s),$$

or equivalently, in terms of the exponential map,

$$\exp_x((t+s)v) = \exp_{\gamma_v(t)}(s\dot{\gamma}_x(t)) = \exp_{\gamma_v(s)}(t\dot{\gamma}_x(s))$$

as long as any of the expressions on the right is defined.

Now, we also have that, as a consequence of the inverse function theorem, the map $\exp_x : D_x \longrightarrow M$ is a local diffeomorphism at the point $0 \in T_xM$. Indeed, this follows from the fact that $d_0 \exp_x$ is an invertible map. To see this, we note that we have the natural identification $T_0T_xM \cong T_xM$. Under this identification we have that $d_0 \exp_x$ is simply the identity map on T_xM:

$$d_0 \exp_x v = d_0 \exp_x \left.\frac{d}{dt}\right|_0 tv = \left.\frac{d}{dt}\right|_0 \exp_x(tv) = \left.\frac{d}{dt}\right|_0 \gamma_v(t) = v.$$

For $\mathcal{V} \subseteq D_x$ a star-shaped neighborhood with respect to 0 such that $\exp_x : \mathcal{V} \longrightarrow \exp_x(\mathcal{V})$ is a diffeomorphism, we call $\mathcal{U} := \exp_x(\mathcal{V})$ a *normal neighborhood* of $x \in M$ and the coordinate system defined by $\exp_x^{-1}$ and a frame of T_xM are called *normal coordinates* on $\mathcal{U}$. The geodesics $\gamma_v : [0,1] \longrightarrow M$ starting at $x \in \mathcal{U}$, with $v \in \mathcal{V}$ are called *radial geodesics*. The motivation for the term radial geodesic is provided in Proposition 4.1.1.

Some of the most elementary properties of normal coordinate systems are enclosed in the following

Proposition 4.1.1. *Let $(\mathcal{U}, \varphi = (x^i))$ be any normal coordinate chart in the n-dimensional affine manifold (M, ∇) determined by the local section $s = (\partial_1, \ldots, \partial_n) \in \Gamma_{\mathcal{U}}(F(M))$ at $x \in M$. Let also $\left\{ \Gamma^k_{ij} \,\middle|\, 1 \leqslant i,j,k \leqslant n \right\} \subseteq C^\infty(\mathcal{U})$ denote the set of* Christoffel symbols *of ∇ with respect to the local frame s, that is, for all $i, j \in \{1, \ldots, n\}$, Γ^k_{ij} are the smooth functions on $\mathcal{U}$ such that*

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k.$$

i) For any $X = X^i \partial_i|_x \in T_xM$, the geodesic γ_X starting at x with initial velocity vector X is represented in normal coordinates by the radial line segment

$$\gamma_X(t) \mathrel{\widehat{=}} (tX^1, \ldots, tX^n)$$

as long as γ_X stays within $\mathcal{U}$.

ii) The coordinates of x are $(0, \ldots, 0)$.

iii) For all i, j, k *it holds that*

$$\Gamma^k_{ij}(x) + \Gamma^k_{ji}(x) = 0.$$

Proof. The proof of items *i)* and *ii)* follows from the definition of a normal coordinate system: by definition

$$\varphi = s_x^{-1} \circ (\exp_x)^{-1} \colon \mathcal{U} \longrightarrow \mathbb{R}^n.$$

Thus for all t such that $\gamma_X(t) \in \mathcal{U}$,

$$\varphi(\gamma_X(t)) = s_x^{-1}((\exp_x)^{-1}(\exp_x(tX))) = s_x^{-1}(tX) = (tX^1, \dots, tX^n),$$

which proves *i)*.

ii) follows immediately from *i)*, since $x = \gamma_X(0)$.

For the last assertion, let $(a^1, \dots, a^n) \in \mathbb{R}^n$ be arbitrary. The curve γ such that $\varphi(\gamma(t)) = (ta^1, \dots, ta^n) =: (\gamma^1(t), \dots, \gamma^n(t))$ is then a geodesic starting at x.

The geodesic equations

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = 0$$

transform themselves in this case into

$$\Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = \Gamma^k_{ij}(\gamma(t))a^i a^j = 0,$$

and in particular for $t = 0$,

$$\Gamma^k_{ij}(x)a^i a^j = \frac{1}{2}(\Gamma^k_{ij}(x) + \Gamma^k_{ji}(x))a^i a^j = 0.$$

Because the above equation holds for any $(a^1, \dots, a^n)$, we conclude in sum that

$$\Gamma^k_{ij}(x) + \Gamma^k_{ji}(x) = 0.$$

□

A direct computation shows that the components of the torsion tensor with respect to a coordinate system $(U, (x^1, \dots, x^n))$ are given by

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

The previous proposition implies then, that in the case in which the covariant derivative is torsion-free and the given coordinate system is normal and centered at x, it holds that

$$\Gamma^k_{ij}(x) = 0.$$

Before we proceed, we make some further comments concerning the Christoffel symbols of the covariant derivative ∇. Given the coordinate system $(U, (x^i))$ on the manifold M determined by a local section $s \in \Gamma_U(F(M))$, an elementary computation shows that the components of the curvature tensor are given in terms of the Christoffel symbols of its covariant derivative. Explicitly, we have the formula

$$R^{\ell}_{ijk}(x) = \partial_i \Gamma^{\ell}_{jk}(x) - \partial_j \Gamma^{\ell}_{ik}(x) + \Gamma^{\ell}_{i\alpha}(x)\Gamma^{\alpha}_{jk}(x) - \Gamma^{\ell}_{j\alpha}(x)\Gamma^{\ell}_{ik}(x), \tag{4.2}$$

for all $x \in U$, or equivalently,

$$R_{ij}(x) = \partial_i \Gamma_j(x) - \partial_j \Gamma_i(x) + \left[\Gamma_i(x), \Gamma_j(x)\right] \in \mathfrak{gl}(n, \mathbb{R}), \tag{4.3}$$

where for all i, j we have defined the smooth maps $\Gamma_i, R_{ij} \colon U \longrightarrow \mathfrak{gl}(n, \mathbb{R})$ by the formulas

$$\begin{aligned} \Gamma_i(x) &:= (\Gamma^k_{ij}(x))_{k,j}, \\ R_{ij}(x) &:= (R^{\ell}_{ijk}(x))_{\ell,k}, \end{aligned}$$

and $[\cdot, \cdot]$ denotes the usual matrix commutator.

In order to further simplify the expressions given above, we introduce the differential forms

$$\begin{aligned} \Gamma &\in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R})), \\ F &\in \Omega^2(U, \mathfrak{gl}(n, \mathbb{R})) \end{aligned}$$

defined as

$$\Gamma := \Gamma_i \, dx^i \,, \tag{4.4}$$

$$F := \frac{1}{2} R_{ij} \, dx^i \wedge dx^j \,. \tag{4.5}$$

From here it is trivial to obtain the relation

$$F = d\Gamma + \frac{1}{2}[\Gamma, \Gamma], \tag{4.6}$$

where $[\cdot, \cdot]$ denotes the commutator of $\mathfrak{gl}(n, \mathbb{R})$-valued differential forms defined in equation (3.11).

From the discussion in the previous chapter, in which we obtained the one-to-one correspondence between connections on the frame bundle and covariant derivatives on the tangent bundle of the smooth manifold M, we notice that

$$\Gamma = s^* \omega^{\nabla},$$

where $\omega^{\nabla} \in \Omega^1(F(M), \mathfrak{gl}(n, \mathbb{R}))$ denotes the connection form associated to ∇.

This observation allows us to establish the relation between the 2-form F and the curvature form associated to the covariant derivative ∇:

$$F = d\Gamma + \frac{1}{2}[\Gamma, \Gamma] = ds^* \omega^{\nabla} + \frac{1}{2}[s^* \omega^{\nabla}, s^* \omega^{\nabla}] = s^* \left(d\omega^{\nabla} + \frac{1}{2}[\omega^{\nabla}, \omega^{\nabla}] \right) = s^* F^{\omega^{\nabla}},$$

where we have made use of the structure equation that the curvature form $F^{\omega^{\nabla}} \in \Omega^2(F(M), \mathfrak{gl}(n, \mathbb{R}))$ satisfies, according to Proposition 3.6.1.

Among the many special features a normal coordinate system enjoys, one particularly nice is the fact that in the case the covariant derivative ∇ is torsion-free, it allows us to consider covariant differentiation as partial differentiation at the point about the normal coordinate system is centered. To be explicit, we recall first the standard fact that a covariant derivative on TM can naturally be extended to the space of tensor fields $\mathcal{T}^{(r,s)}(M)$ (see [37, Lemma 4.6]). Indeed, let $K^{i_1\cdots i_r}_{j_1\cdots j_s}$ denote the components of the tensor field $K \in \mathcal{T}^{(r,s)}(M)$ with respect to the coordinates $(x^1, \dots, x^n)$. That is,

$$K = K^{i_1\cdots i_r}_{j_1\cdots j_s} dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes \partial_{i_1} \otimes \cdots \otimes \partial_{i_r}.$$

The covariant derivative of the tensor field K is then defined as the tensor field $\nabla K \in \mathcal{T}^{(r,s+1)}(M)$ with components $K^{i_1\cdots i_r}_{j_1\cdots j_s;k} \coloneqq \nabla_k K^{i_1\cdots i_r}_{j_1\cdots j_s}$, where

$$\nabla_k K^{i_1\cdots i_r}_{j_1\cdots j_s} = \partial_k K^{i_1\cdots i_r}_{j_1\cdots j_s} + \sum_{\mu=1}^{r} \Gamma^{i_\mu}_{k\alpha} K^{i_1\cdots i_{\mu-1}\alpha i_{\mu+1}\cdots i_r}_{j_1\cdots j_s} - \sum_{\nu=1}^{s} \Gamma^{\alpha}_{kj_\nu} K^{i_1\cdots i_r}_{j_1\cdots j_{\nu-1}\alpha j_{\nu+1}\cdots j_s}. \tag{4.7}$$

Thus, Proposition 4.1.1 implies that if $x \in M$ is the center of a normal coordinate chart and ∇ is torsion-free, then

$$\nabla_k K^{i_1\cdots i_r}_{j_1\cdots j_s}(x) = \partial_k K^{i_1\cdots i_r}_{j_1\cdots j_s}(x).$$

We finalize this section by stating some basic facts about a special class of vector fields along smooth curves. The so-called *Jacobi fields*.

Definition 4.1.2. *Let γ be a geodesic of (M, ∇). A vector field $X \in \Gamma_\gamma(TM)$ is called a* Jacobi field *along γ, if it satisfies the differential equation*

$$\frac{D}{dt}\frac{D}{dt}X + R(X, \dot\gamma)\dot\gamma = 0,$$

where $R \in \Gamma(\bigwedge^2 T^*M \otimes \mathrm{End}(TM))$ is the curvature tensor of the covariant derivative ∇.

For a smooth curve $\gamma\colon I \longrightarrow M$ a smooth map $H\colon I \times (-\varepsilon, \varepsilon) \longrightarrow M$ with $H(\cdot, 0) = \gamma$ is called a *variation* of γ. For fixed $t \in I$ we can consider the smooth curve $H_t(s) \coloneqq H(t, s)$ and derive with respect to s. We denote such a derivative as

$$\frac{\partial}{\partial s}H(t,s) \coloneqq H_t'(s) \in T_{H(t,s)}M.$$

The vector field $Y \in \Gamma_\gamma(TM)$ given by

$$Y(t) \coloneqq \left.\frac{\partial}{\partial s}\right|_0 H(t,s)$$

is called the *variation vector field* of H. In the case in which all of the curves $H(\cdot, s)$ for $s \in (-\varepsilon, \varepsilon)$ are geodesics, we call H a *geodesic variation*.

In the particular case in which the covariant derivative ∇ is torsion-free, a simple computation in local coordinates shows that, for any variation $H\colon I \times (-\varepsilon, \varepsilon) \longrightarrow M$ of a smooth curve, it holds that

$$\frac{D}{dt}\frac{\partial}{\partial s}H = \frac{D}{ds}\frac{\partial}{\partial t}H. \tag{4.8}$$

In this case we get the following

Proposition 4.1.2. *Let* M *be a smooth manifold endowed with a torsion-free covariant derivative,* $H\colon I \times (-\varepsilon, \varepsilon) \longrightarrow M$ *be a geodesic variation of the geodesic* γ. *Then it holds that the variation vector field* $Y = \frac{\partial}{\partial s} H(\cdot, 0)$ *is a Jacobi field along* γ.

An interesting property of Jacobi fields is that they characterize the differential of the exponential map, as the next result shows, whose proof can be found in [38, Chapter 8, Lemma 5 and Proposition 6]

Proposition 4.1.3. *Let* $\gamma\colon [0,1] \longrightarrow M$ *be a geodesic in* M *with* $\gamma(0) = x$ *and* $\dot{\gamma}(0) = v$. *Then*

i) For any given vectors $u, w \in T_x M$ *there exists exactly one Jacobi field* J_w *along* γ *with* $J_w(0) = u$ *and* $\frac{D}{dt} J_w(0) = w$.

ii) For the Jacobi field with initial conditions $J_w(0) = 0$ *and* $\frac{D}{dt} J_w(0) = w$ *it holds that*

$$J_w(t) = t d_{tv} \exp_x(w) \in T_{\gamma(t)} M.$$

An immediate consequence of the previous proposition is that for the geodesic γ starting at x with initial velocity v holds

$$\dot{\gamma}(t) = d_{tv} \exp_x(v).$$

This follows from the fact that the vector field along γ defined by

$$J(t) := t\dot{\gamma}(t)$$

is a Jacobi field with $J(0) = 0$, $\frac{D}{dt} J(0) = v$.

For $v, w \in T_x M$, we notice that for a sufficiently small $\varepsilon > 0$, the map $H\colon I \times (-\varepsilon, \varepsilon) \longrightarrow M$ given by $H(t, s) := \gamma_{v+sw}(t)$ defines a geodesic variation of the geodesic γ_v, and thus, as a consequence of Proposition 4.1.2, we conclude that $J_w(t) := \frac{\partial}{\partial s}\big|_0 \gamma_{v+sw}(t)$ is a Jacobi field along γ_v.

Furthermore, in light of (4.8) it follows that

$$J_w(0) = 0, \qquad \frac{D}{dt} J_w(0) = w,$$

and thus, item *ii)* in the previous proposition implies

$$\frac{\partial}{\partial s}\bigg|_0 \exp_x(t(v + sw)) = J_w(t) = t d_{tv} \exp_x(w), \tag{4.9}$$

from which we easily obtain the identity

$$\frac{\partial}{\partial s} \exp_x(t(v + sw)) = t d_{t(v+sw)} \exp_x(w),$$

whenever either side is defined.

4.2 The curvature tensor revisited

Despite the convenient and systematic way in which the topic of covariant derivatives and the notion of parallel translation of associated vector bundles was already treated, in the case of the tangent bundle of a n-dimensional manifold such an approach is not the most customary way to proceed. Indeed, we have the following

Definition 4.2.1. *Let* (M, ∇) *be an affine manifold, and* $\gamma\colon [a, b] \longrightarrow M$ *a smooth curve. The* parallel transport *along the curve* γ *is defined as the map*

$$\begin{aligned} P_\gamma\colon T_{\gamma(a)}M &\longrightarrow T_{\gamma(b)}M \\ v &\longmapsto X_v(b), \end{aligned}$$

where $X_v \in \Gamma_\gamma(TM)$ *is the unique vector field along the curve* γ *such that*

$$X_v(a) = v \qquad \textit{and} \qquad \frac{D}{dt}X_v \equiv 0.$$

The uniqueness of the vector field in the previous definition is assured thanks to standard theory of systems of ordinary differential equations (see [37, Theorem 4.11]).

The way to conciliate this seemingly unrelated definition to the one when we look at the tangent bundle as the associated vector bundle $F(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$ is by noticing that the diagram

$$\begin{tikzcd} F(M)_{\gamma(a)} \times_{GL(n,\mathbb{R})} \mathbb{R}^n \arrow[r, "P_\gamma^{E,\omega^\nabla}"] \arrow[d, "\cong"'] & F(M)_{\gamma(b)} \times_{GL(n,\mathbb{R})} \mathbb{R}^n \arrow[d, "\cong"] \\ T_{\gamma(a)}M \arrow[r, "P_\gamma"'] & T_{\gamma(b)}M \end{tikzcd}$$

commutes.

By an abuse of notation we write from now on $P_\gamma^{E,\omega^\nabla} = P_\gamma^\nabla = P_\gamma$, and similarly $\nabla^{\omega^\nabla} = \nabla$.

The one-to-one correspondence between covariant derivatives and connection forms on the frame bundle allows us to establish the following analogue of Proposition 3.5.8:

Proposition 4.2.1. *Let* $e \in \Gamma(TM)$*. Then for any piecewise smooth curve* γ *we have that*

$$\frac{d}{dt}P_{t,0}e(\gamma(t)) = P_{t,0}\nabla_{\dot\gamma(t)}e(\gamma(t)). \tag{4.10}$$

Proof. Set $\gamma(0) = x$ and choose a basis $\{e_1, \dots, e_n\}$ of T_xM. Define $e_i(t) := P_{\gamma|_{[0,t]}}e_i$. Since $P_{\gamma|_{[0,t]}}$ is a linear isomorphism between T_xM and $T_{\gamma(t)}M$, we guarantee the existence of smooth coefficients a^i such that $e(\gamma(t)) = a^i(t)e_i(t) = P_{\gamma|_{[0,t]}}(a^i(t)e_i)$, which implies that

$$P_{t,0}e(\gamma(t)) = a^i(t)e_i,$$

and thus

$$\frac{d}{dt}P_{t,0}e(\gamma(t)) = \dot a^i(t)e_i = P_{t,0}(\dot a^i(t)e_i(t)).$$

On the other hand, it holds that

$$\begin{aligned}\nabla_{\dot\gamma(t)} e(\gamma(t)) &= \frac{D}{dt}\big(a^i(t)e_i(t)\big)\\ &= \dot a^i(t)e_i(t) + a^i(t)\frac{D}{dt}e_i(t)\\ &\overset{e_i(t)=P_{\gamma|_{[0,t]}}e_i}{=} \dot a^i(t)e_i(t).\end{aligned}$$

By collecting both calculations we immediately obtain the claim. □

In the special case in which the smooth curve in the previous proposition is a geodesic, we can easily generalize this result.

Proposition 4.2.2. *Let* $e \in \Gamma(TM)$ *and* $\gamma\colon I \longrightarrow M$ *a geodesic. Then, for every* $k \in \mathbb{N}$ *it holds that*

$$\frac{d^k}{dt^k} P_{t,0} e(\gamma(t)) = P_{t,0}\nabla^k_{\dot\gamma(t),\dots,\dot\gamma(t)} e(\gamma(t)), \tag{4.11}$$

where $\nabla^k\colon \Gamma(TM) \longrightarrow \Gamma(\bigotimes^k T^*M \otimes TM)$ *denotes the* k*-fold iterated covariant derivative of the vector field* e.[1]

Proof. We prove our result by induction over the order of the derivative. The base case is already taken care of, being it precisely equation (4.10).

Suppose now that, for a $k \in \mathbb{N}$, it holds that

$$\frac{d^k}{dt^k} P_{t,0} e(\gamma(t)) = P_{t,0}\nabla^k_{\dot\gamma(t),\dots,\dot\gamma(t)} e(\gamma(t)).$$

Then

$$\begin{aligned}\frac{d^{k+1}}{dt^{k+1}} P_{t,0} e(\gamma(t)) &= \frac{d}{dt}\frac{d^k}{dt^k} P_{t,0} e(\gamma(t))\\ &= \frac{d}{dt} P_{t,0}\nabla^k_{\dot\gamma(t),\dots,\dot\gamma(t)} e(\gamma(t))\\ &\overset{(4.10)}{=} P_{t,0}\nabla_{\dot\gamma(t)}\nabla^k_{\dot\gamma(t),\dots,\dot\gamma(t)} e(\gamma(t))\\ &= P_{t,0}\nabla^{k+1}_{\dot\gamma(t),\dots,\dot\gamma(t)} e(\gamma(t)),\end{aligned}$$

where the last equation follows from the fact that γ is a geodesic, which means in particular that $\nabla_{\dot\gamma}\dot\gamma = 0$, thus getting rid of said summands in the definition of ∇^{k+1}. □

It is worth mentioning that, in the case of sections along curves of smooth vector bundles, it is no longer required for the curve γ to be a geodesic to get an analogue of the previous proposition, whose proof is an almost verbatim reproduction of the one previously given:

[1] For a covariant derivative ∇ on the vector bundle $E \longrightarrow M$ we inductively define the k-fold iterated covariant derivative $\nabla^k\colon \Gamma(E) \longrightarrow \Gamma(\bigotimes^k T^*M \otimes E)$ by $\nabla^0 s := s$, and

$$\nabla^{k+1}_{X_1,\dots,X_{k+1}} s := \nabla_{X_1}\nabla^k_{X_2,\dots,X_{k+1}} s - \sum_{i=2}^{k+1} \nabla^k_{X_2,\dots,\nabla_{X_1}X_i,\dots,X_{k+1}} s.$$

Proposition 4.2.3 ([39, Lemma A.1.1]). *Let $E \longrightarrow M$ be a smooth vector bundle with a covariant derivative ∇^E. Let also $\gamma\colon I \longrightarrow M$ be a smooth curve on M and $s \in \Gamma_\gamma(E)$ a smooth section of E along the curve γ. For $t \in I$ and $k \in \mathbb{N}_0$ it holds that*

$$\frac{d^k}{dt^k} P^E_{t,0} s(t) = P^E_{t,0}\left(\frac{D}{dt}\cdots\frac{D}{dt}s(t)\right) =: P^E_{t,0}\left(\frac{D^k}{dt^k}s(t)\right), \tag{4.12}$$

where, as usual, $P^E_{\gamma|_{[0,t]}}\colon E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$ denotes the parallel transport map with respect to the covariant derivative ∇^E, and $P^E_{t,0} = (P^E_{\gamma|_{[0,t]}})^{-1}$.

As an application of the previous result, and by considering $E = TM$, we obtain the formal Taylor series at $0 \in T_xM$ of the map $v \longmapsto P_{1,0} d_v \exp_x(w) \in T_xM$. Firstly, we notice that, for a fixed $v \in T_xM$, the map

$$s\colon t \longmapsto d_{tv}\exp_x(w)$$

defines a smooth section along the curve γ_v, since by Proposition 4.1.3, we have that $J_w(t) = ts(t)$. From this relation, we obtain that

$$\frac{D}{dt}J_w(t) = s(t) + t\frac{D}{dt}s(t),$$

and thus, an induction proof readily implies that, for all $\ell \in \mathbb{N}_0$,

$$\frac{D^\ell}{dt^\ell}s(0) = \frac{1}{\ell+1}\frac{D^{\ell+1}}{dt^{\ell+1}}J_w(0).$$

We thus get from equation (4.12) that the formal Taylor series expansion at $t = 0$ of the map $t \longmapsto P_{t,0}s(t)$ is given by

$$\begin{aligned} P_{t,0}s(t) &\approx \sum_{\ell\geqslant 0}\frac{t^\ell}{\ell!}\frac{D^\ell}{dt^\ell}s(0) \\ &= \sum_{\ell\geqslant 0}\frac{t^\ell}{\ell!}\frac{1}{\ell+1}\frac{D^{\ell+1}}{dt^{\ell+1}}J_w(0). \end{aligned}$$

On the other hand, since J_w satisfies the differential equation

$$\frac{D}{dt}\frac{D}{dt}J_w(t) = R_{\gamma_v(t)}(\dot\gamma_v(t), J_w(t))\dot\gamma_v(t),$$

yet another induction yields, for all $\ell \in \mathbb{N}_0$,

$$\frac{D^\ell}{dt^\ell}\frac{D}{dt}\frac{D}{dt}J_w(t) = \sum_{k=0}^{\ell}\binom{\ell}{k}\left(\nabla^{\ell-k}_{\dot\gamma_v(t),\dots,\dot\gamma_v(t)}R\right)_{\gamma_v(t)}\left(\dot\gamma_v(t), \frac{D^k}{dt^k}J_w(t)\right)\dot\gamma_v(t).$$

This recursion formula together with fact that $J_w(0) = 0$, and $\frac{D}{dt}J_w(0) = w$ implies the existence of unique symmetric polynomials $Q^{(\ell)}_w \in \mathrm{Sym}^\ell\, T^*_xM \otimes T_xM$, which depend on the curvature tensor of ∇ and its iterated covariant derivatives at the point x such that

$$P_{t,0}d_{tv}\exp_x(w) \approx \sum_{\ell\geqslant 0}\frac{t^\ell}{\ell!}\frac{1}{\ell+1}\frac{D^{\ell+1}}{dt^{\ell+1}}J_w(0) =: \sum_{\ell\geqslant 0}\frac{t^\ell}{\ell!}Q^{(\ell)}_w(v,\dots,v). \tag{4.13}$$

From the first terms of the above expansion we notice that the way the polynomials $Q_w^{(\bullet)}$ depend on the various covariant derivatives of the curvature tensor is far from obvious:

$$\begin{aligned}
P_{t,0}d_{tv}\exp_x(w) \approx\, & w + \frac{t^2}{2}\left(\frac{1}{3}R_x(v,w)v\right) + \frac{t^3}{3!}\left(\frac{1}{2}(\nabla_v R)_x(v,w)v\right) \\
& + \frac{t^4}{4!}\left(\frac{1}{5}(3(\nabla^2_{v,v}R)_x(v,w)v + R_x(v,R_x(v,w)v)v)\right) \\
& + \frac{t^5}{5!}\left(\frac{1}{6}(4(\nabla^3_{v,v,v}R)_x(v,w)v + 4(\nabla_v R)_x(v,R_x(v,w)v)v + 2R_x(v,(\nabla_v R)_x(v,w)v)v)\right) \\
& + \cdots
\end{aligned}$$

From this, we obtain that, near $0 \in T_xM$,

$$P_{1,0}d_v \exp_x(w) \approx w + \frac{1}{6}R_x(v,w)v + \frac{1}{12}(\nabla_v R)_x(v,w)v + \frac{1}{40}(\nabla^2_{v,v}R)_x(v,w)v + \cdots$$

It should however be noted, despite the relative simplicity of the computations leading to the determination of the iterated covariant derivatives of the Jacobi field J_w at $t = 0$, the right-hand side formal expansion will in general not converge in any reasonable sense (cf. [40, Lemma 3.1], [39, Theorem A.2.9] and the discussion thereafter).

As a further application of the preceding results, Proposition 4.2.2 implies that the formal Taylor series around $t = 0$ of the map

$$\begin{aligned}
[0,1] &\longrightarrow T_xM \\
t &\longmapsto P_{t,0}e(\gamma(t))
\end{aligned}$$

is the formal power series

$$P_{t,0}e(\gamma(t)) \approx \sum_{k\geqslant 0} \frac{t^k}{k!}\nabla^k_{v,\dots,v}e_x \in T_xM[[t]]$$

where $\gamma(0) = x, \dot\gamma(0) = v$.

This result can easily be generalized to any tensor bundle $\mathcal{T}$ over M. For simplicity, let us assume $\mathcal{T} = T^{(r,s)}M$, for some $r, s \in \mathbb{N}_0$. The covariant derivative we consider in this case is the unique extension of the covariant derivative ∇ on TM to $T^{(r,s)}M$. This implies that the parallel transport along any smooth curve γ on this vector bundle is thus given by the natural tensorial extension of the parallel transport on TM we just discussed, which we will also denote by P_γ. The compatibility earlier discussed in the case $(1,0)$, which can naturally be established for arbitrary (r,s), together with the vector bundle isomorphism

$$T^{(r,s)}M \cong F(M) \times_{GL(n,\mathbb{R})} T^{(r,s)}\mathbb{R}^n,$$

implies that, for any geodesic γ and for any tensor field $\alpha \in \mathcal{T}^{(r,s)}M := \Gamma(T^{(r,s)}M)$,

$$\frac{d^k}{dt^k}P_{t,0}\alpha_{\gamma(t)} = P_{t,0}\nabla^k_{\dot\gamma(t),\dots,\dot\gamma(t)}\alpha_{\gamma(t)}.$$

Thus, the formal Taylor series of the map

$$\begin{aligned}[0,1] &\longrightarrow T^{(r,s)}_{\gamma(0)}M\\ t &\longmapsto P_{t,0}\alpha_{\gamma(t)}\end{aligned}$$

around $t = 0$ is given by

$$P_{t,0}\alpha_{\gamma(t)} \approx \sum_{k\geqslant 0} \frac{t^k}{k!}\nabla^k_{\dot\gamma(0),\dots,\dot\gamma(0)}\alpha_{\gamma(0)} \in T^{(r,s)}_{\gamma(0)}M[[t]]. \tag{4.14}$$

In the closing remarks of the preceding section, we pointed out, that for the geodesic γ starting at x with $\dot\gamma(0) = v$ it holds

$$\dot\gamma(t) = d_{tv}\exp_x(v).$$

The uniqueness of the parallel transport thus implies

$$P^{\nabla}_{\gamma|_{[0,t]}}(v) = d_{tv}\exp_x(v).$$

Even though such a convenient expression for the parallel transport is not possible in general, we can at least explicitly compute the formal Taylor series around $t = 0$ of the map $t \longmapsto P^{\nabla}_{\gamma|_{[0,t]}}(w)$, for any $w \in T_{\gamma(0)}M$.

For $x \in M$, let $\mathcal{U} = \exp_x(\mathcal{V})$ be a normal coordinate system centered at x and $\{e_1, \dots, e_n\}$ an orthonormal basis of T_xM. Let $\gamma_v\colon [0,1] \longrightarrow M$ be the radial geodesic $\gamma_v(t) = \exp_x(tv)$, for $v \in \mathcal{V}$.

We define the map $\sigma\colon \mathcal{U} \longrightarrow F(M)$ by

$$\sigma(\exp_x(v)) := \big(\sigma_1(\exp_x(v)), \dots, \sigma_n(\exp_x(v))\big) := \big(P^{\nabla}_{\gamma_v}e_1, \dots, P^{\nabla}_{\gamma_v}e_n\big) \in F(M)_{\exp_x(v)}.$$

For $v \in \mathcal{V},\ t \in [0,1]$ we define

$$E_i(t) := d_{tv}\exp_x(e_i) \in T_{\gamma_v(t)}M.$$

Proposition 4.2.4. *With the above notation, σ is a smooth local section of the frame bundle $F(M)$ on the normal neighborhood $\mathcal{U}$ centered at x, which will be called an* exponential framing at x.

Proof. This result boils down to proving the fact that, in this setting, the map

$$\begin{aligned}\sigma_w\colon \mathcal{U} &\longrightarrow TM\\ \exp_x(v) &\longmapsto P_{\gamma_v}w\end{aligned}$$

is smooth, for all $w = w^i e_i \in T_xM$.

In order to achieve this we notice that, by definition of the parallel transport map, $\tau_w \in \Gamma_{\gamma_v}(TM)$ defined as $\tau_w(t) = P_{\gamma_{tv}}w$ is the parallel section along the geodesic γ_v starting at w. That is,

$$\frac{D}{dt}\tau_w(t) = 0, \qquad \tau_w(0) = w.$$

With respect to the frame $(E_1(t),\dots,E_n(t)) \in F(M)_{\gamma_v(t)}$ the above initial value problem can be written as

$$\begin{cases} \frac{d\tau_w^k}{dt} = -\Gamma_{ij}^k(\gamma_v(t))\frac{d\gamma^i}{dt}\tau_w^j(t), \quad k=1,\dots,n \\ \tau_w^k(0) = w^k, \end{cases}$$

where $\tau_w(t) = \tau_w^i(t)E_i(t) \mathrel{\widehat{=}} \tau_w^i(t)e_i$.

Since $\mathcal{U}$ is a normal neighborhood, we know that as long as $\operatorname{im}(\gamma_v) \subseteq \mathcal{U}$, $\gamma_v(t) \mathrel{\widehat{=}} tv$. Therefore, solving the previous initial value problem is equivalent to solving

$$\begin{cases} \frac{d\tau_w^k}{dt} = -\Gamma_{ij}^k(tv)v^i\tau_w^j(t), \quad k=1,\dots,n \\ \tau_w^k(0) = w^k, \end{cases}$$

where $v = v^i e_i$.

By setting $v\colon I \longrightarrow T_xM$ as $v(t) \equiv v$, we obtain that this initial value problem is equivalent to

$$\begin{cases} \frac{dv^\ell}{dt} = 0 & \ell = 1,\dots,n, \\ \frac{d\tau_w^k}{dt} = -\Gamma_{ij}^k(tv(t))v^i(t)\tau_w^j(t), & k=1,\dots,n \\ v^\ell(0) = v^\ell, \\ \tau_w^k(0) = w^k. \end{cases}$$

By making the identification $T_xM \mathrel{\widehat{=}} \mathbb{R}^n$ and defining the smooth map

$$\begin{aligned} F\colon I \times \mathcal{V} \times \mathcal{V} &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (t,v,\tau) &\longmapsto (0,\dots,0,-\Gamma_{ij}^1(tv)v^i\tau^j,\dots,-\Gamma_{ij}^n(tv)v^i\tau^j) \end{aligned}$$

we can succinctly write the above system as

$$\begin{cases} \frac{dz}{dt} = F(t,z(t)) = F(t,v(t),\tau_w(t)) \\ z(0) = (v,w). \end{cases}$$

Because of Proposition 2.1.1, we obtain that there exists an interval $0 \in I_0 \subseteq I$, and an open subset $0 \in \mathcal{V}_0 \subseteq \mathcal{V}$ such that the map

$$\begin{aligned} \mathbf{u}\colon I_0 \times I_0 \times \mathcal{V}_0 \times \mathcal{V}_0 &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (t,s,v,w) &\longmapsto z(t) = (v(t),\tau(t)) \end{aligned}$$

with

$$\begin{cases} \frac{dz}{dt} = F(t,z(t)), \\ z(s) = (v,w) \end{cases}$$

is smooth. The smoothness of the map σ_w follows directly, since

$$\sigma_w \mathrel{\widehat{=}} \mathrm{pr}_2 \circ \mathbf{u} \circ f_w,$$

where $\mathrm{pr}_2 \colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the projection onto the second factor, and $f_w \colon \mathcal{U} \longrightarrow I_0 \times I_0 \times \mathcal{V}_0 \times \mathcal{V}_0$ is the smooth map defined by $f_w(q) = (1, 0, \exp_x^{-1}(q), w)$.

Therefore, each of the coordinate maps $\sigma_i = \sigma_{e_i} \colon \mathcal{U} \longrightarrow TM$ is smooth, whence $\sigma \in \Gamma_{\mathcal{U}}(F(M))$.

□

Let us introduce now the formal power series

$$P_v \coloneqq \sum_{k \geqslant 0} \frac{(-1)^k t^k}{k!} \nabla^k_{v,\dots,v} \eqqcolon \sum_{k \geqslant 0} \frac{t^k}{k!} P^{(k)}(v, \dots, v) \in \operatorname{End}(T_x M)[[t]]$$

We notice that this series is in fact invertible in the sense that

$$P_v \cdot P_{-v} = \mathbb{1}_{T_x M} \in \operatorname{End}(T_x M)[[t]],$$

where the product in $\operatorname{End}(T_x M)[[t]]$ is given by the composition of coefficients.

Any given exponential framing $\sigma \colon \mathcal{U} \longrightarrow F(M)$ at the point $x \in M$ allows us to construct the following smooth map.

Let $\mathcal{T}$ be any tensor bundle over M. Thus, there is a vector bundle isomorphism $\mathcal{T} \cong E \coloneqq F(M) \times_G W$, for some finite-dimensional vector space W on which the group $G = GL(n, \mathbb{R})$ acts on the left. For any smooth section $R \in \Gamma(E) \simeq C^\infty(F(M), W)^G$ we define the smooth map $R^\sigma \coloneqq \overline{R} \circ \sigma \colon \mathcal{U} \longrightarrow W$, where $\overline{R} \in C^\infty(F(M), W)^G$ is the G-equivariant smooth map corresponding to the section R.

Now, the exponential framing, being a local section of a principal bundle, induces a local trivialization

$$\begin{aligned} \psi_\sigma \colon \mathcal{U} \times G &\longrightarrow \pi^{-1}(\mathcal{U}) \\ (y, g) &\longmapsto \sigma(y) \cdot g. \end{aligned}$$

Define

$$F(M) \supset \pi^{-1}(\mathcal{U}) = \psi_\sigma(\mathcal{U} \times G) \eqqcolon F_G = \bigsqcup_{y \in \mathcal{U}} (F_G)_y \coloneqq \bigsqcup_{y \in \mathcal{U}} \{\sigma(y) \cdot g \mid g \in G\}.$$

Since by definition the principal bundle is locally trivial, $(F_G, \pi|_{\pi^{-1}(\mathcal{U})}, \mathcal{U}; G)$ is a G-principal bundle isomorphic to the trivial G-bundle $(\mathcal{U} \times G, \mathrm{pr}_1, \mathcal{U}; G)$. In general, for any Lie subgroup $H \subseteq G$, Proposition 3.3.1 implies that $(F_H, \pi|_{\mathrm{im}\,(\psi_\sigma)}, \mathcal{U}; H)$ is an H-principal bundle and the pair (F_H, ι) an H-reduction of F_G.

Before we let the bundle F_G for some time, we make a further observation about it. Let $\omega^\nabla \in \Omega^1(F(M), \mathfrak{g})$ be the connection form induced by the covariant derivative ∇. This implies that $\sigma^* \omega^\nabla = \Gamma$, where $\Gamma \in \Omega^1(\mathcal{U}, \mathfrak{g})$ is the local connection form associated to the local section σ, the exponential framing at x, which, as we quickly can corroborate, satisfies the identity

$$\nabla_X \sigma_j = \Gamma(X) \sigma_j = \Gamma_j^k(X) \sigma_k \quad \text{for all } X \in \Gamma(TM).$$

The fact that $d_{\sigma(y)}\pi\colon \mathrm{Th}_{\sigma(y)}F(M) \longrightarrow T_yM$ is an isomorphism, implies that, for all $y \in \mathcal{U}$,

$$\mathrm{Th}_{\sigma(y)}F(M) \subseteq T_{\sigma(y)}F_G,$$

which at the same time implies that, due to the nature of the action of G on $F(M)$, for every $g \in G$,

$$\mathrm{Th}_{\sigma(y)g}F(M) \subseteq T_{\sigma(y)g}F_G,$$

Thus, Proposition 3.6.7 implies that ω^{∇} reduces to F_G.

In the case in which ∇ is a torsion-free covariant derivative on the tangent bundle, the curvature tensor $R \in \mathcal{T}^{(1,3)}M$ and its covariant derivative $\nabla R \in \mathcal{T}^{(1,4)}M$ satisfy the *Bianchi identities*: for all $X, Y, Z \in \Gamma(TM)$,

i) *First (or algebraic) Bianchi identity:*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \in \Gamma(TM),$$

ii) *Second (or differential) Bianchi identity*

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0 \in \Gamma(\mathrm{End}(TM)).$$

In the setting of a (pseudo-)Riemannian manifold (M^n, g) with Levi-Civita covariant derivative ∇^g we define the *Riemann curvature* as the tensor field $\mathrm{Rm} \in \Gamma\Big(\mathrm{Sym}^2\left(\bigwedge^2 T^*M\right)\Big) \subseteq \mathcal{T}^{(0,4)}M$ defined by

$$\mathrm{Rm}(X,Y,U,V) \coloneqq g(R^{\nabla^g}(X,Y)U,V).$$

That Rm is indeed a section of said subbundle is a consequence of the first Bianchi identity that the tensor R^{∇^g} satisfies.

Another important related tensor is the so called *Ricci curvature*, which is defined as $\mathrm{Ric} \in \Gamma(\mathrm{Sym}^2 T^*M) \subseteq \mathcal{T}^{(0,2)}M$ given by the formula

$$\mathrm{Ric}(Y,Z) \coloneqq \mathrm{tr}\Big(X \longmapsto R^{\nabla^g}(X,Y)Z\Big).$$

That the components of the Ricci tensor are in fact symmetric in their indices follows from the fact that

$$R^{\ell}_{ijk} = g^{\ell a}R_{ijka}$$

and thus

$$\mathrm{Ric}_{ba} \eqqcolon R_{ba} = R^{i}_{iba} = g^{ij}R_{ibaj} = g^{ij}R_{ajib} = g^{ij}R_{jabi} = R_{ab}.$$

The metric g is said to be *Ricci-flat* if $\mathrm{Ric} \equiv 0$.

The *scalar curvature* g is defined as

$$s \coloneqq \mathrm{tr}_g \mathrm{Ric} = g^{ab}R_{ab},$$

Finally, the metric g is said to be *Einstein* if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} = \lambda g,$$

or equivalently,

$$s = \lambda n.$$

The core part of this work is connected to the study of the following vector spaces:

Definition 4.2.2. *Let* $\mathfrak{h} \subseteq \mathfrak{g} = \mathrm{End}(V)$ *be a Lie subalgebra. The* space of formal curvature maps *is defined as*

$$K(\mathfrak{h}) := \ker\left\{\bigwedge^2 V^* \otimes \mathfrak{h} \longrightarrow \bigwedge^3 V^* \otimes V\right\},$$

where The map $\bigwedge^2 V^* \otimes \mathfrak{h} \longrightarrow \bigwedge^3 V^* \otimes V$ *is the composition of the natural maps*

$$\bigwedge^2 V^* \otimes \mathfrak{h} \longrightarrow \bigwedge^2 V^* \otimes V^* \otimes V \longrightarrow \bigwedge^3 V^* \otimes V.$$

The space of formal curvature derivatives *is defined as*

$$K^1(\mathfrak{h}) := \ker\left\{V^* \otimes K(\mathfrak{h}) \longrightarrow \bigwedge^3 V^* \otimes \mathfrak{h}\right\},$$

where the map $V^* \otimes K(\mathfrak{h}) \longrightarrow \bigwedge^3 V^* \otimes \mathfrak{h}$ *denotes the composition of the natural maps*

$$V^* \otimes K(\mathfrak{h}) \longrightarrow V^* \otimes \bigwedge^2 V^* \otimes \mathfrak{h} \longrightarrow \bigwedge^3 V^* \otimes \mathfrak{h}.$$

In analogy to the spaces $K^{(m)}$ *introduced in the second chapter, we define in general for* $m \in \mathbb{N}_0$ *the* space of formal m-th order curvature derivatives *as*

$$\begin{aligned} K^{(m)}(\mathfrak{h}) :=& \ker\left\{\mathrm{Sym}^m V^* \otimes \bigwedge^2 V^* \otimes \mathfrak{h} \longrightarrow \mathrm{Sym}^m V^* \otimes \bigwedge^3 V^* \otimes V\right\} \\ &\cap \ker\left\{\mathrm{Sym}^m V^* \otimes \bigwedge^2 V^* \otimes \mathfrak{h} \longrightarrow \mathrm{Sym}^{m-1} V^* \otimes \bigwedge^3 V^* \otimes \mathfrak{h}\right\}. \end{aligned}$$

For $m \in \mathbb{N}$, it is not difficult to see that

$$\begin{aligned} K^{(m)}(\mathfrak{h}) =& (\mathrm{Sym}^m V^* \otimes K(\mathfrak{h})) \cap \left(\mathrm{Sym}^{m-1} V^* \otimes K^1(\mathfrak{h})\right) \\ \subseteq& \,\mathrm{Sym}^{m-1} V^* \otimes V^* \otimes K(\mathfrak{h}). \end{aligned}$$

We notice that $K^{(0)}(\mathfrak{h}) = K(\mathfrak{h})$, $K^{(1)}(\mathfrak{h}) = K^1(\mathfrak{h})$, and so, for $m \geqslant 1$, by setting $K^{(-1)}(\mathfrak{h}) := \mathfrak{h}$, we obtain the exact sequence

$$0 \longrightarrow K^{(m)}(\mathfrak{h}) \longrightarrow V^* \otimes K^{(m-1)}(\mathfrak{h}) \longrightarrow \bigwedge^2 V^* \otimes K^{(m-2)}(\mathfrak{h}). \tag{4.15}$$

With this definition, we introduce the tensor bundles

$$K(M) := \bigsqcup_{x\in M} K(\mathrm{End}(T_xM)) \subseteq \bigwedge^2 T^*M \otimes \mathrm{End}(TM),$$
$$K^{(m)}(M) := \bigsqcup_{x\in M} K^{(m)}(\mathrm{End}(T_xM)) \subseteq \mathrm{Sym}^{m-1} T^*M \otimes T^*M \otimes K(M),$$

all of which are associated to the frame bundle $F(M)$.

In this setting, the Bianchi identities translate thus to

i) $R \in \Gamma(K(M))$,

ii) $\nabla R \in \Gamma(K^1(M))$.

Proposition 4.2.5. *With the above notation, it holds that*

$$\nabla^{(m)} R \in \Gamma(K^{(m)}(M)).$$

Proof. Because of the definition of the spaces $K^{(m)}$, to prove the claim it suffices to show

i) $\nabla^m_{A_1,\dots,A_m} R_x \in K(\mathrm{End}(T_xM))$,

ii) $\nabla^{m-1}_{A_1,\dots,A_{m-1}} \nabla R_x \in K^1(\mathrm{End}(T_xM))$

for any tangent vectors $A_i \in T_xM$.

We argue by induction on m. The claims for $m = 0, 1$ are already taken care of, since they are simply the Bianchi identities.

Suppose the claim holds for an $m \in \mathbb{N}_0$. Then for any tangent vectors $X, Y, Z \in T_xM$

$$(\nabla^{m+1}_{A_1,\dots,A_{m+1}} R_x)(X,Y)Z = \nabla_{A_1}\Big((\nabla^m_{A_2,\dots,A_{m+1}} R_x)(X,Y)Z\Big) - \sum_{i=2}^{m+1} (\nabla^m_{A_2,\dots,\nabla_{A_1}A_i,\dots,A_{m+1}} R_x)(X,Y)Z.$$

The induction hypothesis implies then

$$\sum_{\mathrm{cyc}\,(X,Y,Z)} (\nabla^{m+1}_{A_1,\dots,A_{k+1}} R_x)(X,Y)Z = 0$$

which concludes the proof for item *i)*.

The proof of item *ii)* follows in a completely analogous fashion.

The proposition follows immediately from the definition of the symmetrized covariant derivative. □

For a fixed $p \in M$ with normal neighborhood $\mathcal{U} = \exp_p(\mathcal{V})$, we set $V := T_pM$ with orthonormal basis $\{e_1, \dots, e_n\}$ and dual basis $\{e^1, \dots, e^n\}$. Considering the curvature tensor $R \in \Gamma(K(M))$ and the exponential framing at p, we get the smooth map

$$\begin{aligned} S\colon \mathcal{V} &\longrightarrow K(\mathrm{End}(V)) \\ v &\longmapsto R^\sigma(\exp_p(v)). \end{aligned}$$

Now, since it is clear from the definition of the vector bundle isomorphism

$$\Psi\colon F(M) \times_G K(\mathrm{End}(V)) \longrightarrow K(M)$$

that

$$R_{\exp_p(v)} = \Psi_{\exp_p(v)}\Big(\Big[(\sigma_1(\exp_p(v)), \dots, \sigma_n(\exp_p(v))), P_{1,0}R_{\exp_p(v)}\Big]\Big),$$

we obtain, by recalling that $\mathcal{V} \subseteq T_pM$ is star-shaped with respect to 0,

$$S(tv) = P_{t,0}R_{\exp_p(tv)} \approx \sum_{m\geqslant 0} \frac{t^m}{m!}\nabla^m_{v,\dots,v}R_p =: \sum_{m\geqslant 0}\frac{t^m}{m!}S^{(m)}(v,\dots,v) \in K(\mathrm{End}(V))[[t]],$$

where, for $m \in \mathbb{N}_0$ we defined the polynomials $S^{(m)} \in K^{(m)}(\mathrm{End}(V))$ by the formula

$$S^{(m)} \coloneqq \nabla^{(m)}R_p.$$

This implies in particular that the differential $dS\colon \mathcal{V} \longrightarrow V^* \otimes K(\mathrm{End}(V))$ takes values in the subspace $K^1(\mathrm{End}(V))$.

Notice as well that the natural tensorial extension of the parallel transport map on TM implies that the parallel transport map on the vector bundle $K(M)$ along any piecewise smooth curve $\gamma\colon [0,1] \longrightarrow M$ starting at p is given by the formula

$$P_{\gamma|_{[0,t]}}(R_p) =: P_{0,t}(R_p) = P_{0,t}\cdot R_p \in K(M)_{\gamma(t)} = K(\mathrm{End}(T_{\gamma(t)}M)),$$

where we define for all $X, Y \in \Gamma(TM)$

$$P_{0,t}\cdot R_p(X_{\gamma(t)}, Y_{\gamma(t)}) \coloneqq P_{0,t} \circ R_p(P_{t,0}X_{\gamma(t)}, P_{t,0}Y_{\gamma(t)}) \circ P_{t,0}.$$

Now, since we are considering a system of normal coordinates centered at p, Proposition 4.1.1 implies that we can soundly restrict ourselves to the special case $M = V$, $p = 0 \in V$, whose radial geodesics are given by $\gamma_v(t) = tv$. In this case we obtain $\exp_0\colon T_0V \cong V \longrightarrow V$ is in fact the identity map, which implies that

$$\begin{aligned} P^{\gamma_v}_{0,t} \approx& P_v(t) = \sum_{\ell\geqslant 0}\frac{(-1)^\ell t^\ell}{\ell!}\nabla^\ell_{v\dots,v} \in \mathrm{End}(V)[[t]],\\ P_{t,0} \approx& P_{-v}(t) \in \mathrm{End}(V)[[t]]. \end{aligned}$$

In this setting we define the map $R\colon V \longrightarrow K(\mathrm{End}(V)) \subseteq \bigwedge^2 V^* \otimes \mathrm{End}(V)$ as $R(v) \coloneqq R_{\gamma_v(1)} = R_{ij}(v)e^i \wedge e^j \coloneqq R^\ell_{ijk}(v)e^i \wedge e^j \otimes e^k \otimes e_\ell$, which gives rise to the map

$$t \longmapsto R_{\gamma_v(t)} =: R(tv),$$

for all $v \in V$.

We also introduce the left action of $\mathrm{GL}(V)$ on the vector space $K(\mathrm{End}(V))$ defined as

$$\begin{aligned}\cdot\colon\ \mathrm{GL}(V)\times K(\mathrm{End}(V)) &\longrightarrow K(\mathrm{End}(V))\\ (T,R) &\longmapsto T\cdot R\colon (x,y)\longmapsto T\circ R(T^{-1}(x),T^{-1}(y))\circ T^{-1}\in \mathrm{End}(V),\end{aligned}$$

which naturally induces the Lie group representation of the group $\mathrm{GL}(V)$ on the vector space $K(\mathrm{End}(V))$

$$\begin{aligned}\rho\colon\ \mathrm{GL}(V) &\longrightarrow \mathrm{GL}(K(\mathrm{End}(V)))\\ T &\longmapsto \rho(T) \coloneqq T\cdot\,.\end{aligned}$$

By means of all of these maps, we define the application $S\colon V\longrightarrow K(\mathrm{End}(V))$ defined by

$$S(v)\coloneqq P(-v)\cdot R(v),$$

which gives rise to the map

$$t\longmapsto S(tv)$$

for all $v\in V$.

Assuming the covariant derivative is torsion-free, we also define the map $\Gamma\colon V\longrightarrow \mathrm{Sym}^2 V^*\otimes V\subseteq V^*\otimes \mathrm{End}(V)$ as $\Gamma(v)\coloneqq \Gamma_i(v)e^i\coloneqq \Gamma_{ij}^k(v)e^ie^j\otimes e_k$, which in a similar fashion gives rise to the map

$$t\longmapsto \Gamma(tv),$$

for all $v\in V$.

Now, since the maps R, P, and Γ come from smooth sections of vector bundles, we deduce that both of these applications are in fact smooth maps, which in turn implies the smoothness of the map S, whose formal Taylor series around 0 we write as

$$\begin{aligned}P(v)&\approx\sum_{m\geqslant 0}\frac{1}{m!}P^{(m)}(v,\dots,v),\\ R(v)&\approx\sum_{m\geqslant 0}\frac{1}{m!}R^{(m)}(v,\dots,v),\\ S(v)&\approx\sum_{m\geqslant 0}\frac{1}{m!}S^{(m)}(v,\dots,v),\\ \Gamma(v)&\approx\sum_{m\geqslant 0}\frac{1}{m!}\Gamma^{(m)}(v,\dots,v),\end{aligned}$$

where for each $m\in\mathbb{N}$, $\Gamma^{(m)}\in \mathrm{Sym}^m V^*\otimes \mathrm{Sym}^2 V^*\otimes V$, $R^{(m)}, S^{(m)}\in \mathrm{Sym}^m V^*\otimes K(\mathrm{End}(V))$, $P^{(m)}\in \mathrm{Sym}^m V^*\otimes \mathrm{End}(V)$ are the unique symmetric polynomials given by the m-th order partial derivatives of the maps P, R, S and Γ.

Explicitly, it is easy to obtain the formulas

$$P^{(m)} = \sum_{|\mu|=m} \binom{m}{\mu} D^{\mu}P(0)(e^1)^{\mu_1}\cdots(e^n)^{\mu_n},$$
$$R^{(m)} = \sum_{|\mu|=m} \binom{m}{\mu} D^{\mu}R(0)(e^1)^{\mu_1}\cdots(e^n)^{\mu_n},$$
$$S^{(m)} = \sum_{|\mu|=m} \binom{m}{\mu} D^{\mu}S(0)(e^1)^{\mu_1}\cdots(e^n)^{\mu_n},$$
$$\Gamma^{(m)} = \sum_{|\mu|=m} \binom{m}{\mu} D^{\mu}\Gamma(0)(e^1)^{\mu_1}\cdots(e^n)^{\mu_n},$$

where the right-hand side on each of these equations is to be understood as component-wise differentiation. Explicitly,

$$\begin{aligned} D^{\mu}P(0) &:= D^{\mu}P^{i}_{j}(0)e^j \otimes e_i, \\ D^{\mu}R(0) &:= D^{\mu}R^{\alpha}_{ij\beta}(0)e^i \wedge e^j \otimes e^{\beta} \otimes e_{\alpha}, \\ D^{\mu}S(0) &:= D^{\mu}S^{\alpha}_{ij\beta}(0)e^i \wedge e^j \otimes e^{\beta} \otimes e_{\alpha}, \\ D^{\mu}\Gamma(0) &:= D^{\mu}\Gamma^{k}_{ij}(0)e^i e^j \otimes e_k. \end{aligned}$$

In a similar fashion, the map $\Gamma\colon V \longrightarrow \operatorname{Sym}^2 V^* \otimes V$ gives rise to the smooth maps $d\Gamma, [\Gamma,\Gamma]\colon V \longrightarrow K(\operatorname{End}(V)) \subseteq \bigwedge^2 V^* \otimes \operatorname{End}(V)$ defined by the formulas

$$\begin{aligned} d\Gamma(v) &:= \frac{1}{2}(\partial_i \Gamma^{\alpha}_{j\beta}(v) - \partial_j \Gamma^{\alpha}_{i\beta}(v))e^i \wedge e^j \otimes e^{\beta} \otimes e_{\alpha}, \\ [\Gamma,\Gamma](v) &:= \big(\Gamma^{\alpha}_{i\lambda}(v)\Gamma^{\lambda}_{j\beta}(v) - \Gamma^{\alpha}_{j\lambda}(v)\Gamma^{\lambda}_{i\beta}(v)\big)e^i \wedge e^j \otimes e^{\beta} \otimes e_{\alpha} = [\Gamma_i(v), \Gamma_j(v)]e^i \wedge e^j, \end{aligned}$$

whose formal Taylor series around 0 are given by the formulas

$$\begin{aligned} d\Gamma(v) &\approx \sum_{m\geqslant 0} \frac{1}{m!} d_{m+1}\Gamma^{(m+1)}(v,\dots,v), \\ [\Gamma,\Gamma](v) &\approx \sum_{m\geqslant 0} \frac{1}{m!} \sum_{a+b=m} \binom{m}{a} \Big[\Gamma^{(a)}_i(v,\dots,v), \Gamma^{(b)}_j(v,\dots,v)\Big] e^i \wedge e^j \\ &=: \sum_{m\geqslant 0} \frac{1}{m!} \sum_{a+b=m} \binom{m}{a} \Big[\Gamma^{(a)}, \Gamma^{(b)}\Big](v,\dots,v), \end{aligned}$$

where $d_{\bullet}\colon \operatorname{Sym}^{\bullet} V^* \otimes \operatorname{Sym}^2 V^* \otimes V \longrightarrow \operatorname{Sym}^{\bullet-1} V^* \otimes K(\operatorname{End}(V))$ is the natural map previously defined, whereas the map

$$[\cdot,\cdot]\colon \big(\operatorname{Sym}^a V^* \otimes \operatorname{Sym}^2 V^* \otimes V\big) \times \big(\operatorname{Sym}^b V^* \otimes \operatorname{Sym}^2 V^* \otimes V\big) \longrightarrow \operatorname{Sym}^{a+b} V^* \otimes K(\operatorname{End}(V))$$

is defined by the formula given in the following polarization result for symmetric polynomials:

Proposition 4.2.6 ([22, Lemma F.2.15]). *Let $\mathbb{F}$ be a field and let U, W be $\mathbb{F}$-vector spaces. Then for* $f \in \operatorname{Pol}^k(U, W)$, *with*

$$\operatorname{Pol}^k(U, W) := \left\{ f\colon U \longrightarrow W \,|\, \text{there exists } \phi \in \bigotimes^k U^* \otimes W \text{ such that } f(u) = \phi(u, \dots, u) \right\},$$

there exists a unique $\phi \in \operatorname{Sym}^k U^* \otimes W$ *such that*

$$f(u) = \phi(u, \dots, u).$$

Moreover, for $u_1, \dots, u_k \in U$ *we have*

$$\phi(u_1, \dots, u_k) = \frac{1}{k!} \sum_{\ell=1}^{k} \sum_{\{j_1, \dots, j_\ell\} \subseteq \{1, \dots, k\}} (-1)^{k-\ell} \phi\big(u_{j_1} + \dots + u_{j_\ell}, \dots, u_{j_1} + \dots + u_{j_\ell}\big). \quad (4.16)$$

By a slight abuse of notation we define

$$\begin{aligned}
P &:= \sum_{m \geqslant 0} \frac{1}{m!} P^{(m)} \in \prod_{m \geqslant 0} (\operatorname{Sym}^m V^* \otimes \operatorname{End}(V)), \\
R &:= \sum_{m \geqslant 0} \frac{1}{m!} R^{(m)} \in \prod_{m \geqslant 0} (\operatorname{Sym}^m V^* \otimes K(\operatorname{End}(V))), \\
S &:= \sum_{m \geqslant 0} \frac{1}{m!} S^{(m)} \in \prod_{m \geqslant 0} (\operatorname{Sym}^m V^* \otimes K(\operatorname{End}(V))), \\
\Gamma &:= \sum_{m \geqslant 0} \frac{1}{m!} \Gamma^{(m)} \in \prod_{m \geqslant 0} \big(\operatorname{Sym}^m V^* \otimes \operatorname{Sym}^2 V^* \otimes V\big), \\
d\Gamma &:= \sum_{m \geqslant 0} \frac{1}{m!} d_{m+1} \Gamma^{(m+1)} = \sum_{m \geqslant 0} \frac{1}{m!} d\Gamma^{(m+1)} \in \prod_{m \geqslant 0} (\operatorname{Sym}^m V^* \otimes K(\operatorname{End}(V))), \\
[\Gamma, \Gamma] &:= \sum_{m \geqslant 0} \frac{1}{m!} \sum_{a+b=m} \binom{m}{a} [\Gamma^{(a)}, \Gamma^{(b)}] \in \prod_{m \geqslant 0} (\operatorname{Sym}^m V^* \otimes K(\operatorname{End}(V))),
\end{aligned}$$

By Construction, it actually holds that $\Gamma^{(m)} \in (\operatorname{Sym}^m V^* \otimes \operatorname{Sym}^2 V^*)_0 \otimes V$, as well as $R^{(m)}, S^{(m)}, d\Gamma^{(m+1)}, \sum_{a+b=m} \binom{m}{a} [\Gamma^{(a)}, \Gamma^{(b)}] \in \prod_{m \geqslant 0} K^{(m)}(\operatorname{End}(V))$.

Making use of the structure equation[2]

$$R = 2d\Gamma + [\Gamma, \Gamma]$$

and equating coefficients on the formal power series, we get the relations

$$R^{(m)} = 2d\Gamma^{(m+1)} + \sum_{a+b=m} \binom{m}{a} \Big[\Gamma^{(a)}, \Gamma^{(b)}\Big] \in K^{(m)}(\operatorname{End}(V)) \quad (4.17)$$

for every $m \geqslant 0$.

[2]Here it should be noted that the usual convention is to consider $\frac{1}{2}R$ as the curvature form, exactly as is equation (4.5). The deviation made here is just to make computations slightly less cumbersome.

On the other hand, the equation

$$S(tv) = P(-tv) \cdot R(tv) = \rho(P(-tv))(R(tv))$$

becomes now

$$\begin{aligned}\sum_{m\geqslant 0} \frac{t^m}{m!} S^{(m)}(v,\dots,v) =& \rho\left(\sum_{m\geqslant 0} \frac{(-1)^m t^m}{m!} P^{(m)}(v,\dots,v)\right)\left(\sum_{m\geqslant 0} \frac{t^m}{m!} R^{(m)}(v,\dots,v)\right)\\ =& \left(\sum_{m\geqslant 0} \frac{(-1)^m t^m}{m!} \rho_*(P^{(m)}(v,\dots,v))\right)\left(\sum_{m\geqslant 0} \frac{t^m}{m!} R^{(m)}(v,\dots,v)\right)\\ =& \sum_{m\geqslant 0} \frac{t^m}{m!} \sum_{a+b=m} \binom{m}{a} (-1)^a \rho_*\Big(P^{(a)}(v,\dots,v)\Big)\Big(R^{(b)}(v,\dots,v)\Big),\end{aligned}$$

where as usual, $\rho_* \coloneqq d_{\mathbb{1}_V}\rho$.

With this we obtain that, for every $m \in \mathbb{N}_0$, and every $v \in V$,

$$S^{(m)}(v,\dots,v) = \sum_{a+b=m} \binom{m}{a} (-1)^a \rho_*\Big(P^{(a)}(v,\dots,v)\Big)\Big(R^{(b)}(v,\dots,v)\Big),$$

which is enough to fully determine the polynomials $S^{(m)}$, according to Proposition 4.2.6.

And so we get:

$$\begin{aligned}S^{(m)} =& \sum_{a+b=m} \binom{m}{a} (-1)^a P^{(a)} \cdot R^{(b)} = R^{(m)} + \sum_{\substack{a+b=m\\ b<m}} \binom{m}{a} (-1)^a P^{(a)} \cdot R^{(b)}\\ =& 2d\Gamma^{(m+1)} + \sum_{a+b=m} \binom{m}{a} \Big[\Gamma^{(a)}, \Gamma^{(b)}\Big] + \sum_{\substack{a+b=m\\ b<m}} \binom{m}{a} (-1)^a P^{(a)} \cdot R^{(b)},\end{aligned}$$

which then implies

$$d\Gamma^{(m+1)} = \frac{1}{2}\left(S^{(m)} - \sum_{a+b=m} \binom{m}{a} \Big[\Gamma^{(a)}, \Gamma^{(b)}\Big] - \sum_{\substack{a+b=m\\ b<m}} \binom{m}{a} (-1)^a P^{(a)} \cdot R^{(b)}\right).$$

Now, in Proposition 2.2.2 we established the linear isomorphism

$$\Phi_m \colon K^{(m)}(W) \longrightarrow \ker\left\{\operatorname{Sym}^{m+1} W \otimes \operatorname{Sym}^2 W \longrightarrow \operatorname{Sym}^{m+3} W\right\} \eqqcolon (\operatorname{Sym}^{m+1} W \otimes \operatorname{Sym}^2 W)_0$$

for any finite-dimensional vector space W. We even got an explicit bound for the norm of this linear isomorphism in terms of the dimension of W. Indeed, from equation (2.9) we obtain the estimate

$$\|\Phi_m\|_{\mathrm{op}} \leqslant \frac{2\sqrt{2}(\dim W)^2\big((\dim W)^2 + 1\big)}{m+1}.$$

Replacing W for V^* and denoting the map $\Phi_m \otimes \mathbb{1}_V$ again by Φ_m we thus obtain the following linear isomorphisms, which are inverse of each other,

$$\begin{aligned}\Phi_m \colon K^{(m)}(\mathrm{End}(V)) &\longrightarrow \left(\mathrm{Sym}^{m+1} V^* \otimes \mathrm{Sym}^2 V^*\right)_0 \otimes V\\ d \colon \left(\mathrm{Sym}^{m+1} V^* \otimes \mathrm{Sym}^2 V^*\right)_0 \otimes V &\longrightarrow K^{(m)}(\mathrm{End}(V)).\end{aligned}$$

Notice as well that the map Φ_m is again bounded by the same bound as before since it is easy to see that

$$\|\Phi_m \otimes \mathbb{1}_V\|_{op} = \|\Phi_m\|_{op}\|\mathbb{1}_V\|_{op} = \|\Phi_m\|_{op} \leqslant \frac{2\sqrt{2}n^2(n^2+1)}{m+1}.$$

With this we thus get

$$\Gamma^{(m+1)} = \frac{1}{2}\Phi_m\left(S^{(m)} - \sum_{a+b=m}\binom{m}{a}\left[\Gamma^{(a)},\Gamma^{(b)}\right] - \sum_{\substack{a+b=m\\ b<m}}\binom{m}{a}(-1)^a P^{(a)}\cdot R^{(b)}\right) \tag{4.18}$$

In this setting, we can also adapt the covariant derivative along the smooth curve γ_v. Specifically, let $\tau \in \Gamma_{\gamma_v}(TM)$ and write $T_{\gamma_v(t)}M \ni \tau(t) = \tau^i(t)\partial_i|_{\gamma_v(t)} \mathrel{\widehat{=}} \tau^i(t)e_i \in V$. Equation (4.1) becomes then

$$\begin{aligned}T_{\gamma_v(t)}M \ni \frac{D}{dt}\tau(t) =& \frac{d\tau^i}{dt}\partial_i|_{\gamma_v(t)} + \tau^i(t)\nabla_{\dot\gamma_v(t)}\partial_i|_{\gamma_v(t)}\\ =& \frac{d\tau^\ell}{dt}\partial_\ell|_{\gamma_v(t)} + \Gamma^\ell_{ij}(\gamma_v(t))\dot\gamma^i_v(t)\tau^j(t)\partial_\ell|_{\gamma_v(t)}\\ \mathrel{\widehat{=}}& \frac{d\tau^\ell}{dt}e_\ell + \Gamma^\ell_{ij}(tv)v^i\tau^j(t)e_\ell\\ =& \frac{d\tau}{dt} + \Gamma(tv)(v)\tau(t) \in V.\end{aligned}$$

What we do now is to make a similar analysis to the formal power series of the smooth section along the curve τ and establish its relation to the formal power series of the map Γ. In order to achieve this, we begin with an elementary result from the theory of ordinary differential equations:

Proposition 4.2.7. *Let* $g \colon J \longrightarrow \mathbb{R}$ *be a smooth map with* $J \subseteq \mathbb{R}$ *an interval containing* 0 *and* $g(0) = 0$*. Then the unique smooth solution to the initial value problem*

$$\begin{cases}\dot x(t) = g(t)x(t),\\ x(0) = x_0\end{cases}$$

is the smooth function $x \colon J \longrightarrow \mathbb{R}$ *which satisfies for all* $m \in \mathbb{N}_0$

$$x^{(m)}(0) = x_0\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\frac{1}{k!}\sum_{\substack{I\in\mathbb{N}_0^k,\\ |I|=m}}\binom{m}{I}g^{(i_1-1)}(0)\cdots g^{(i_k-1)}(0). \tag{4.19}$$

Proof. The result follows from the fact that the unique solution to the given initial value problem is given by

$$x(t) = x_0 e^{\int_0^t g(s)ds}.$$

To see this, we write the formal power series of the smooth map g, x around $t = 0$ as

$$g(t) \approx \sum_{m \geqslant 0} \frac{t^m}{m!} g^{(m)}(0) =: \sum_{m \geqslant 0} \frac{t^m}{m!} g_m$$
$$x(t) \approx \sum_{m \geqslant 0} \frac{t^m}{m!} x^{(m)}(0) =: \sum_{m \geqslant 0} \frac{t^m}{m!} x_m.$$

The formal power series of the smooth function $h(t) := \int_0^t g(s)\, ds$ at $t = 0$ is then given by

$$h(t) \approx \sum_{m \geqslant 0} \frac{t^m}{m!} g_{m+1} =: \sum_{m \geqslant 0} \frac{t^m}{m!} h_m,$$

with $h_0 = 0$, $h_m = g_{m-1}$ for all $m \geqslant 1$.

With this we get

$$\begin{aligned} e^{h(t)} &= \sum_{k \geqslant 0} \frac{1}{k!} (h(t))^k \\ &\approx \sum_{k \geqslant 0} \frac{1}{k!} \left(\sum_{m \geqslant 0} \frac{t^m}{m!} h_m \right)^k \\ &= \sum_{k \geqslant 0} \frac{1}{k!} \sum_{m \geqslant 0} \frac{t^m}{m!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I| = m}} \binom{m}{I} h_{i_1} \cdots h_{i_k} \\ &= \sum_{m \geqslant 0} \frac{t^m}{m!} \sum_{k \geqslant 0} \frac{1}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I| = m}} \binom{m}{I} h_{i_1} \cdots h_{i_k}. \end{aligned}$$

The formula $x(t) = x_0 e^{h(t)}$ implies then

$$x_m = x_0 \sum_{k \geqslant 0} \frac{1}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I| = m}} \binom{m}{I} h_{i_1} \cdots h_{i_k}.$$

Now, since $g(0) = g_0 = h_1 = 0 = h_0$, we obtain that the summands on the right-hand side of the above equation do not vanish only when $i_k \geqslant 2$ for all k. Equivalently, $m = |I| \geqslant 2k$. This immediately implies (4.19). □

This result can in fact be generalized to the setting of matrix differential equations:

Proposition 4.2.8. *Let* $A \colon J \longrightarrow \mathfrak{gl}(n, \mathbb{R})$ *be a smooth map, with* $J \subseteq \mathbb{R}$ *an interval containing* 0 *such that* $A(0) = 0$*. Then the unique solution to the initial value problem*

$$\begin{cases} \dot{x}(t) = A(t)x(t) \\ x(0) = x_0 \end{cases}$$

is given by the smooth map $x\colon J \longrightarrow \mathbb{R}^n$ *which satisfies for all* $m \in \mathbb{N}_0$

$$x^{(m)}(0) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=m}} \binom{m}{I} A^{(i_1-1)}(0) \cdots A^{(i_k-1)}(0) x_0. \tag{4.20}$$

Proof. Even though in this case we can not really argue that the solution of the initial value problem is given by the analogous formula $x(t) = e^{\int_0^t A(s)ds} x_0$, since for the generic smooth map $A\colon J \longrightarrow \mathfrak{gl}(n, \mathbb{R})$, the matrices $A(t), \int_0^t A(s)ds$ do not commute, we can still prove this claim via mathematical induction. The base case $m = 0$ is trivially satisfied.

By setting $A^{(i)}(0) =: A_i, x^{(i)}(0) =: x_i$ we obtain the formal Taylor series around 0

$$A(t) \approx \sum_{m \geqslant 0} \frac{t^m}{m!} A_m,$$
$$x(t) \approx \sum_{m \geqslant 0} \frac{t^m}{m!} x_m.$$

Since $x\colon J \longrightarrow \mathbb{R}^n$ is the unique solution to the given initial value problem, we obtain, at the level of formal power series,

$$\sum_{m \geqslant 0} \frac{t^m}{m!} x_{m+1} = \sum_{m \geqslant 0} \frac{t^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} A_{m-\ell} x_\ell,$$

from which we obtain the recursion formula

$$x_{m+1} = \sum_{\ell=0}^{m} \binom{m}{\ell} A_{m-\ell} x_\ell, \qquad \text{for all } m \in \mathbb{N}_0$$

Suppose now, that (4.20) holds for all $\ell \leqslant m$.

From the established recursion formula we thus obtain

$$\begin{aligned}
x_{m+1} &= \sum_{\ell=0}^{m} \binom{m}{\ell} A_{m-\ell} x_\ell \\
&= \sum_{\ell=0}^{m} \binom{m}{\ell} A_{m-\ell} \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{1}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=\ell}} \binom{\ell}{I} A_{i_1-1} \cdots A_{i_k-1} x_0 \\
&= \sum_{\ell=0}^{m} \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{1}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=\ell}} \binom{m}{\ell} \binom{\ell}{I} A_{m-\ell} A_{i_1-1} \cdots A_{i_k-1} x_0 \\
&= \sum_{\ell=0}^{m} \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{m-\ell+1}{(m+1)k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=\ell}} \binom{m+1}{m-\ell+1, i_1, \ldots, i_k} A_{m-\ell} A_{i_1-1} \cdots A_{i_k-1} x_0
\end{aligned}$$

The fact that $A(0) = A_0 = 0$ readily implies

$$
\begin{aligned}
x_{m+1} &= \sum_{\ell=0}^{m} \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{m-\ell+1}{(m+1)k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=\ell}} \binom{m+1}{m-\ell+1, i_1, \dots, i_k} A_{m-\ell} A_{i_1-1} \cdots A_{i_k-1} x_0 \\
&= \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{1}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=m+1}} \binom{m+1}{i_1, \dots, i_k} A_{i_1-1} \cdots A_{i_k-1} x_0.
\end{aligned}
$$

□

Notice that even though the solution of the IVP in the previous proposition cannot in general be given by a closed formula analogous to the one in the case of Proposition 4.2.7, the type of algebraic manipulations used in both of the proofs are almost identical. Yet, in the appropriate setting, it is possible to find a partial conciliation of both of these results.

Let W be a finite-dimensional real vector space with a given inner product. We define the map

$$
\begin{aligned}
N\colon W[[t]] &\longrightarrow \mathbb{R}[[t]] \\
\sum_{m \geqslant 0} \frac{t^m}{m!} f_m &\longmapsto \sum_{m \geqslant 0} \frac{t^m}{m!} \|f_m\|,
\end{aligned}
$$

where the norm on the right-hand side is the one induced by the inner product on W. It is clear that the map N depends on the vector space W. However, most of the time the corresponding W will be clear from the context, which motivates us to omit this dependence from the notation. We also note, that apart from the fact that N does not take values on $\mathbb{R}$, it has all of the properties which define a norm.

In the classical theory of bounded operators on Hilbert spaces, we have that the operator norm is in fact a sub-multiplicative norm. In the case of the map N does not make much sense to talk about sub-multiplicativity, since the composition of formal power series of one variable with coefficients on a real vector space is not well-defined. However, something that from a distance resembles the sub-multiplicative property of the map N is given by the following result.

Proposition 4.2.9. *Let W be a finite-dimensional real vector space with a given inner product and let $f \in W[[t]]$, $g \in \mathbb{R}[[t]]$ with $g(0) = 0$. Then it holds that*

$$
N_W(f(g(t))) \leqslant N_W(f)(N_{\mathbb{R}}(g(t))) \in \mathbb{R}[[t]].
$$

Proof. Let us write

$$
\begin{aligned}
f &= \sum_{m \geqslant 0} \frac{t^m}{m!} f_m \in W[[t]], \\
g &= \sum_{m \geqslant 0} \frac{t^m}{m!} g_m \in \mathbb{R}[[t]].
\end{aligned}
$$

The fact that $g(0) = g_0 = 0$ implies that the substitution

$$f(g(t)) = \sum_{m \geqslant 0} \frac{f_m}{m!}(g(t))^m \in W[[t]]$$

is well-defined.

Indeed,

$$\begin{aligned}
f(g(t)) &= \sum_{m \geqslant 0} \frac{f_m}{m!}(g(t))^m \\
&= \sum_{m \geqslant 0} \frac{f_m}{m!} \left(\sum_{\ell \geqslant 0} \frac{t^\ell}{\ell!} g_\ell \right)^m \\
&= \sum_{m \geqslant 0} \frac{f_m}{m!} \sum_{\ell \geqslant 0} \frac{t^\ell}{\ell!} \sum_{\substack{I \in \mathbb{N}_0^m, \\ |I| = \ell}} \binom{\ell}{I} g_{i_1} \cdots g_{i_m} \\
&\overset{g_0 = 0}{=} \sum_{\ell \geqslant 0} \frac{t^\ell}{\ell!} \sum_{m=0}^{\ell} \frac{f_m}{m!} \sum_{\substack{I \in \mathbb{N}_0^m, \\ |I| = \ell}} \binom{\ell}{I} g_{i_1} \cdots g_{i_m}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
N_W(f(g(t))) &= \sum_{\ell \geqslant 0} \frac{t^\ell}{\ell!} \left\| \sum_{m=0}^{\ell} \frac{f_m}{m!} \sum_{\substack{I \in \mathbb{N}_0^m, \\ |I| = \ell}} \binom{\ell}{I} g_{i_1} \cdots g_{i_m} \right\| \\
&\leqslant \sum_{\ell \geqslant 0} \frac{t^\ell}{\ell!} \sum_{m=0}^{\ell} \frac{\|f_m\|}{m!} \sum_{\substack{I \in \mathbb{N}_0^m \\ |I| = \ell}} \binom{\ell}{I} |g_{i_1}| \cdots |g_{i_m}| \\
&= \sum_{m \geqslant 0} \frac{\|f_m\|}{m!} \left(\sum_{\ell \geqslant 0} \frac{t^\ell}{\ell!} |g_\ell| \right)^m \\
&= N_W(f)(N_{\mathbb{R}}(g(t))).
\end{aligned}$$

□

We notice as well that, by identifying a smooth function $f \in C^\infty(J, W)$, where $0 \in J \subseteq \mathbb{R}$, with its formal power series expansion around $t = 0$ we can think of $C^\infty(J, W)$ as a subset of the domain of the map N.

In this context, we have the following result, which establishes the desired relation between both of the previous propositions.

Proposition 4.2.10. *Suppose we are in the situation described in Proposition 4.2.8. Let $\mathbb{R}^n$ be endowed with the standard inner product, which induces a natural norm $\|\cdot\|_{op}$ on $\mathfrak{gl}(n, \mathbb{R})$. Then it holds that*

$$N(x) \leqslant \|x_0\| e^{\int_0^t N(A)} \in \mathbb{R}[[t]], \tag{4.21}$$

where the integral of a formal power series $\sum_{m\geqslant 0}\frac{a_m}{m!}t^m \in \mathbb{R}[[t]]$ *is defined as*

$$\int_0^t \sum_{m\geqslant 0}\frac{a_m}{m!}z^m\,dz := \sum_{m\geqslant 0}\frac{a_m}{(m+1)!}t^{m+1}.$$

Proof. This result is essentially a consequence of the previous result. Namely, in the proof of Proposition 4.2.8 we obtained the recursion formula for all $m \in \mathbb{N}_0$

$$x^{(m+1)}(0) = \sum_{a+b=m}\binom{m}{a}A^{(a)}(0)x^{(b)}(0),$$

from which we obtain

$$\left\|x^{(m+1)}(0)\right\| \leqslant \sum_{a+b=m}\binom{m}{a}\left\|A^{(a)}(0)\right\|_{op}\left\|x^{(b)}(0)\right\|.$$

The partial ordering on $\mathbb{R}[[t]]$ defined in Lemma 2.1.1 implies then

$$\sum_{m\geqslant 0}\frac{t^m}{m!}\left\|x^{(m+1)}(0)\right\| \leqslant \sum_{m\geqslant 0}\frac{t^m}{m!}\sum_{a+b=m}\binom{m}{a}\left\|A^{(a)}(0)\right\|_{op}\left\|x^{(b)}(0)\right\|.$$

That is,

$$N(x)' \leqslant N(A)N(x) \in \mathbb{R}[t]],$$

which, due to the fact that all of these formal power series have non-negative coefficients, implies the desired estimate, according to Proposition 2.1.8. □

In the current context, the importance of the previous proposition is that it allows us to establish the concrete relationship between the parallel translation map and the Christoffel symbols of the underlying covariant derivative.

Specifically, let $v, w \in V$ and set $\tau_w : I \longrightarrow V$ by the formula $\tau_w(t) := P^{\gamma_v}_{0,t}(w) = P(tv)w$.

As previously discussed, the expansion as formal power series of the map τ_w is given by the formula

$$\tau_w(t) = \sum_{m\geqslant 0}\frac{t^m}{m!}P^{(m)}(v,\ldots,v)(w).$$

The fact the τ_w is parallel along the radial geodesic γ_v, together with the fact that $\tau_w(0) = w$, gives rise to the initial value problem

$$\begin{cases}\frac{d\tau_w}{dt} = -\Gamma(tv)(v)\tau_w(t)\\ \tau_w(0) = w\end{cases}$$

The previous proposition implies then

$$P^{(m)}(v,\ldots,v)(w) = \sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\frac{(-1)^k}{k!}\sum_{\substack{I\in\mathbb{N}_0^k,\\ |I|=m}}\binom{m}{I}\Gamma^{(i_1-1)}(v,\ldots,v)(v)\cdots\Gamma^{(i_k-1)}(v,\ldots,v)(v)(w).$$

Since this formula holds for any $w \in V$ we thus obtain the identity

$$P^{(m)}(v, \ldots, v) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=m}} \binom{m}{I} \Gamma^{(i_1-1)}(v, \ldots, v)(v) \cdots \Gamma^{(i_k-1)}(v, \ldots, v)(v)$$

as elements of the vector space $\mathrm{End}(V)$.

Now, we define the map

$$\begin{aligned} *\colon \mathrm{Sym}^{a_1} V^* \otimes \mathrm{Sym}^2 V^* \otimes V \times \cdots \times \mathrm{Sym}^{a_k} V^* \otimes \mathrm{Sym}^2 V^* \otimes V &\longrightarrow \mathrm{Sym}^{a_1+\cdots+a_k+k} V^* \otimes \mathrm{End}(V) \\ (Q_1, \ldots, Q_k) &\longmapsto Q_1 * \cdots * Q_k \end{aligned}$$

for all $a_1, \ldots, a_k, k \in \mathbb{N}_0$ given by the formula

$$Q_1 * \cdots * Q_k(v, \ldots, v) \coloneqq Q_1(v, \ldots, v)(v) \circ \cdots \circ Q_k(v, \ldots, v)(v) \in \mathrm{End}(V)$$

for all $v \in V$. The polarization formula (4.16) guarantees this is enough to completely define the map $*$.

Thus, since the vector $v \in V$ was arbitrarily chosen, we get the identity

$$P^{(m)} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k}{k!} \sum_{\substack{I \in \mathbb{N}_0^k, \\ |I|=m}} \binom{m}{I} \Gamma^{(i_1-1)} * \cdots * \Gamma^{(i_k-1)} \in \mathrm{Sym}^m \otimes \mathrm{End}(V) \tag{4.22}$$

One of the consequences of the considerations made during the last couple of paragraphs is thus that given an analytic covariant derivative on V with curvature tensor R, then the map $S\colon V \longrightarrow K(\mathrm{End}(V))$ defined as $S(v) \coloneqq P^{\nabla}_{1,0} \cdot R_{\gamma_v(1)}$, where $P^{\nabla}_{1,0} = (P^{-1}_{\gamma_v})$, is real analytic and satisfies that $dS\colon V \longrightarrow K^1(\mathrm{End}(V))$. That these constructions can so to speak be reversed are the contents of the Main Result of this work.

4.3 Sufficient conditions for the existence of torsion-free covariant derivatives

The last part of the previous section dealt with some algebraic generalities that led to the special map S, intimately related to the first and second Bianchi identities, which turned out to be real analytic, provided the covariant derivative was real analytic as well.

The core result of the present work guarantees that the existence of such a particular analytic map is the only obstruction to the existence of real analytic torsion-free covariant derivatives. This section is devoted to proving this result.

Theorem 1. *Let V a finite-dimensional $\mathbb{R}$-vector space and U an open neighborhood of 0 in V. Let $S\colon U \longrightarrow K(\mathrm{End}(V))$ be a real analytic map such that $dS : U \longrightarrow K^1(\mathrm{End}(V)) \subseteq V^* \otimes K(\mathrm{End}(V))$. Then there exists a unique torsion-free covariant derivative ∇ defined on a sufficiently small neighborhood of the origin $\mathcal{U} \subseteq U$ such that*

$$S(v) = P^{\nabla}_{1,0} R^{\nabla}_{\gamma_v(1)} \qquad \textit{for all} \quad v \in \mathcal{U}.$$

Before we discuss the proof of the theorem we first need the following auxiliary result, which at the same time serves as a justification for our particular interest in formal power series of one variable.

Lemma 4.3.1. *Let V, W be finite-dimensional inner-product vector spaces. The map $F\colon V \longrightarrow W$ is real-analytic near $0 \in V$ if and only if the map $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$f(t) \approx \sum_{m \geqslant 0} \frac{1}{m!} \left\| F^{(m)} \right\|_{\mathrm{op}} t^m \in \mathbb{R}[[t]]$$

is real-analytic in an open neighborhood of 0, where the polynomial $F^{(m)} \in \operatorname{Sym}^m V^* \otimes W$ is the unique symmetric polynomial such that

$$F(x) \approx \sum_{m \geqslant 0} \frac{1}{m!} F^{(m)}(x, \dots, x),$$

while $\|\cdot\|_{\mathrm{op}}$ denotes the norm on $\operatorname{Hom}(\operatorname{Sym}^m V, W) \cong \operatorname{Sym}^m V^* \otimes W$ given by the formula

$$\begin{aligned} \|Q\|_{\mathrm{op}} &:= \max_{V \ni v \neq 0} \frac{\|Q(v^m)\|_W}{\|v^m\|_{\mathrm{Sym}}} = \max_{V \ni v \neq 0} \frac{\|Q(v^m)\|_W}{\|v\|^m} \\ &= \inf\{c \geqslant 0 \mid \|Q(v^m)\|_W \leqslant c\|v\|^m \text{ for all } v \in V\} \end{aligned}$$

for all $m \in \mathbb{N}_0$.

Proof. First of all, we need to verify that $\|\cdot\|_{\mathrm{op}}\colon \operatorname{Sym}^m V^* \otimes W \longrightarrow \mathbb{R}$ does indeed define a norm. In reality, the only property that somewhat may not be evident is the positive definiteness, since both the absolute homogeneity and the triangle inequality readily follow from the analogous properties the norm on W satisfies.

Suppose then that $\|Q\|_{\mathrm{op}} = 0$. This implies that for all $v \in V$, $Q(v^m) = 0$. Now, thanks to Proposition 4.2.6 we know that every monomial $v_1 \cdots v_m \in \operatorname{Sym}^m V$ can be expressed as a finite sum of monomials of the form v^m. Indeed, thanks to said result, we have the explicit formula

$$v_1 \cdots v_m = \frac{1}{m!} \sum_{\ell=1}^{m} \sum_{\{j_1, \dots, j_\ell\} \subseteq \{1, \dots, m\}} (-1)^{m-\ell} \big(v_{j_1} + \dots + v_{j_\ell}\big)^m.$$

The fact that every polynomial $\xi \in \operatorname{Sym}^m V$ can be written as a finite sum of monomials of this form implies that $Q(\xi) = 0$, which in turn implies $Q \equiv 0 \in \operatorname{Sym}^m V^* \otimes W$.

Now to the actual proof of the lemma.

One of the implications is relatively light to see. Namely, let $f\colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ be real-analytic for some $\varepsilon > 0$. Since it holds that

$$\left\| F^{(m)}(x, \dots, x) \right\| = \left\| F^{(m)}(x^m) \right\| \leqslant \left\| F^{(m)} \right\|_{\mathrm{op}} \|x^m\|_{\mathrm{Sym}} = \left\| F^{(m)} \right\|_{\mathrm{op}} \|x\|^m,$$

we then obtain, for $\|x\| < \varepsilon$, that

$$\begin{aligned}\sum_{m\geqslant 0}\left\|\frac{1}{m!}F^{(m)}(x,\dots,x)\right\| &\leqslant \sum_{m\geqslant 0}\frac{1}{m!}\left\|F^{(m)}\right\|_{op}\|x\|^m \\ &<\infty,\end{aligned}$$

which implies the absolute convergence of the Taylor series

$$\sum_{m\geqslant 0}\frac{1}{m!}F^{(m)}(x,\dots,x)$$

for all $x \in B_\varepsilon(0) \subseteq V$, that is, $F|_{B_\varepsilon(0)}$ is real-analytic, i.e.

$$F(x) = \sum_{m\geqslant 0}\frac{1}{m!}F^{(m)}(x,\dots,x) \qquad \text{for all } x \in B_\varepsilon(0).$$

Suppose, on the other hand, that the map $F\colon V \longrightarrow W$ is real-analytic near the origin. The Taylor Theorem implies that

$$F(x) = \sum_{m\geqslant 0}\frac{1}{m!}F^{(m)}(x,\dots,x) = \sum_{m\geqslant 0}\frac{1}{m!}\sum_{|\mu|=m}\binom{m}{\mu}D^\mu F(0)x^\mu.$$

Thus we obtain the explicit formula for the m-th Taylor polynomial of the map F

$$F^{(m)} = \sum_{|\mu|=m}\binom{m}{\mu}D^\mu F(0)(e^1)^{\mu_1}\cdots(e^n)^{\mu_n} \in \operatorname{Sym}^m V^* \otimes W, \tag{4.23}$$

where in a similar fashion to the way we argued before, $D^\mu F(0)$ denotes component-wise differentiation, that is

$$D^\mu F(0) = D^\mu F^i(0) f_i$$

with $F(x) = F^i(x)f_i$ for some orthonormal basis $\{f_1,\dots,f_{\dim W}\}$ of W and analytic component functions $F^i\colon V \longrightarrow \mathbb{R}$.

Now, since F is real analytic near the origin, Proposition 2.1.3 guarantees the existence of a neighborhood $0 \in U \subseteq V$, and positive constants C, r such that

$$\left\|\frac{1}{\mu!}D^\mu F(x)\right\| \leqslant \frac{C}{r^{|\mu|}}$$

for all $x \in U$, and $\mu \in \mathbb{N}^n$, where n is the dimension of V.

Let us recall that, with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathrm{Sym}}$ previously defined, the elements of the set

$$\left\{\sqrt{\frac{m!}{\mu!}}e^\mu = \sqrt{\frac{m!}{\mu!}}(e_1)^{\mu_1}\cdots(e_n)^{\mu_n}\;\middle|\; \mu \in \mathbb{N}_0^n,\ |\mu| = m\right\}$$

form an orthonormal basis of the vector space $\operatorname{Sym}^m V$.

Thus, for every $\xi \in \operatorname{Sym}^m V$

$$\begin{aligned}
F^{(m)}(\xi) &= \sum_{|\mu|=m} \binom{m}{\mu} \langle \xi, e^\mu \rangle F^{(m)}(e^\mu) \\
&\overset{(4.23)}{=} \sum_{|\mu|=m} \langle \xi, e^\mu \rangle D^\mu F(0).
\end{aligned}$$

Therefore, by denoting the norms on $\operatorname{Sym}^m V$ and W with the same symbol to avoid cumbersome notation, we obtain that

$$\begin{aligned}
\frac{1}{m!}\left\| F^{(m)}(\xi) \right\| =& \left\| \sum_{|\mu|=m} \frac{1}{m!} \langle \xi, e^\mu \rangle D^\mu F(0) \right\| \\
\leqslant& \sum_{|\mu|=m} |\langle \xi, e^\mu \rangle| \left\| \frac{1}{\mu!} D^\mu F(0) \right\| \\
\leqslant& \sum_{|\mu|=m} \|\xi\| \|e^\mu\| \frac{C}{r^m} \\
\leqslant& \sum_{|\mu|=m} \frac{C}{r^m} \|\xi\| \\
=& \frac{C}{r^m} \|\xi\| \binom{m+n-1}{m} \\
=& \frac{C}{r^m} \|\xi\| \frac{(m+1)\cdots(m+n-1)}{(n-1)!} \\
=& \frac{C}{r^m} \|\xi\| \left(\frac{m+1}{1}\right)\left(\frac{m+2}{2}\right)\cdots\left(\frac{m+n-1}{n-1}\right) \\
\leqslant& \frac{C}{r^m}(m+1)^{n-1}\|\xi\|.
\end{aligned}$$

Therefore, since the above estimate holds in particular for the monomials $x^m \in \operatorname{Sym}^m V$, we obtain that, for all $m \in \mathbb{N}_0$,

$$\frac{1}{m!}\left\| F^{(m)} \right\|_{\mathrm{op}} \leqslant \frac{C}{r^m}(m+1)^{n-1}.$$

Hence

$$\begin{aligned}
\sum_{m \geqslant 0} \frac{1}{m!}\left\| F^{(m)} \right\|_{\mathrm{op}} t^m \leqslant& C \sum_{m \geqslant 0} (m+1)^{n-1} \left(\frac{t}{r}\right)^m \\
=& C\left(t\frac{d}{dt} + \mathbb{1}\right)^{n-1} \frac{r}{r-t},
\end{aligned}$$

for all $t \in (-r, r)$, which by the elementary comparison criterion of power series in one variable implies the real analyticity of the map $f\colon (-r, r) \longrightarrow \mathbb{R}$. □

Proof of Theorem 1. The way we prove the theorem is by formally defining the symmetric polynomials that uniquely determine the power series expansion of the Christoffel symbols of the desired covariant derivative and showing that this power series actually converges in a sufficiently small neighborhood of the origin.

Firstly, we write the formal power series of the map S as

$$S = \sum_{m\geqslant 0} \frac{1}{m!} S^{(m)} \in \prod_{m\geqslant 0} (\mathrm{Sym}^m V^* \otimes K(\mathrm{End}(V))).$$

The fact that $dS \colon U \longrightarrow K^1(\mathrm{End}(V))$ implies that in reality $S^{(m)} \in K^{(m)}(\mathrm{End}(V))$, for all $m \geqslant 0$.

We set $\Gamma^{(0)} = 0$ and define $\Gamma^{(m+1)}$ by the formula (4.18), for all $m \geqslant 0$, where the polynomials $R^{(m)} \in K^{(m)}(\mathrm{End}(V))$, $P^{(m)} \in \mathrm{Sym}^m V^* \otimes \mathrm{End}(V)$ are determined by the the polynomials $\Gamma^{(\bullet)}$ in the sense we previously established. That is, we define

$$R^{(m)} := 2d\Gamma^{(m+1)} + \sum_{a+b=m} \binom{m}{a} \left[\Gamma^{(a)}, \Gamma^{(b)}\right],$$

$$P^{(m)} := \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k}{k!} \sum_{\substack{I\in\mathbb{N}_0^k,\\ |I|=m}} \binom{m}{I} \Gamma^{(i_1-1)} * \cdots * \Gamma^{(i_k-1)}.$$

By means of these definitions, we set the formal power series Γ, $d\Gamma$, R, P by the previously established formulas.

Notice as well that from the definition of P and R, it is straightforward to obtain the identity

$$R(tv) = P(tv) \cdot S(tv) = \rho(P(tv))(S(tv)), \tag{4.24}$$

for all t, v, and $\rho\colon \mathrm{GL}(V) \longrightarrow \mathrm{GL}(K(\mathrm{End}(V)))$ the Lie group representation of the group $\mathrm{GL}(V)$ on the space of formal curvature maps previously defined.

Additionally, we define the formal power series s, g, p, $r \in \mathbb{R}[[t]]$ by the formulas

$$s = s(t) = \sum_{m\geqslant 0} \frac{t^m}{m!} \left\| S^{(m)} \right\|_{op},$$

$$g = g(t) = \sum_{m\geqslant 0} \frac{t^m}{m!} \left\| \Gamma^{(m)} \right\|_{op},$$

$$p = p(t) = \sum_{m\geqslant 0} \frac{t^m}{m!} \left\| P^{(m)} \right\|_{op},$$

$$r = r(t) = \sum_{m\geqslant 0} \frac{t^m}{m!} \left\| R^{(m)} \right\|_{op},$$

where $\|\cdot\|_{op}$ denotes the operator norm on the spaces $\mathrm{Sym}^\bullet V^* \otimes W$, with $W \in \left\{K(\mathrm{End}(V)), \mathrm{Sym}^2 V^* \otimes V, \mathrm{End}(V)\right\}$. The previous lemma implies that s is in fact a real analytic function, whose expansion as a power series converges in a sufficiently small neighborhood of the origin.

The core of the proof is to prove that g is in fact a convergent power series near the origin.

Firstly, since, by definition $\Gamma^{(m+1)} \in \left(\mathrm{Sym}^{m+1} V^* \otimes \mathrm{Sym}^2 V^*\right)_0 \otimes V$, for all $m \in \mathbb{N}_0$, we have the identity

$$\Gamma^{(m+1)} = \Phi_m \, d\Gamma^{(m+1)} .$$

Thus, making use of the estimate for the norm of the linear isomorphism Φ_m, we obtain

$$\begin{aligned}
\left\|\Gamma^{(m+1)}\right\|_{\mathrm{op}} \leqslant & \|\Phi_m\|_{\mathrm{op}} \left\|d\Gamma^{(m+1)}\right\|_{\mathrm{op}} \\
\leqslant & \frac{2\sqrt{2}n^2(n^2+1)}{m+1}\left\|d\Gamma^{(m+1)}\right\|_{\mathrm{op}} \\
\leqslant & 2\sqrt{2}n^2(n^2+1)\left\|d\Gamma^{(m+1)}\right\|_{\mathrm{op}}.
\end{aligned}$$

From the definition of the polynomials $P^{(\bullet)}$, it is straightforward to see that

$$\left\|P^{(m)}\right\|_{\mathrm{op}} \leqslant \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{k!} \sum_{\substack{r_1+\cdots+r_k=m \\ r_i \in \mathbb{N}_0}} \binom{m}{r_1,\ldots,r_k} \left\|\Gamma^{(r_1-1)}\right\|_{\mathrm{op}} \cdots \left\|\Gamma^{(r_k-1)}\right\|_{\mathrm{op}}.$$

Now, by using the partial ordering defined in Lemma 2.1.1, Proposition 4.2.7 implies

$$p(t) \leqslant \left\|P^{(0)}\right\|_{\mathrm{op}} e^{\int_0^t g(z)dz} = e^{\int_0^t g(z)dz} \in \mathbb{R}[[t]], \tag{4.25}$$

since $p(0) = \|P^{(0)}\|_{\mathrm{op}}$ and $P^{(0)} = \mathbb{1}_V$.

It is not difficult to see that there exists a constant $C(n) > 0$, which only depends on $n = \dim V$ such that, for all $a, b \in \mathbb{N}_0$,

$$\left\|\left[\Gamma^{(a)}, \Gamma^{(b)}\right]\right\|_{\mathrm{op}} \leqslant C(n)\left\|\Gamma^{(a)}\right\|_{\mathrm{op}}\left\|\Gamma^{(b)}\right\|_{\mathrm{op}}. \tag{4.26}$$

In the same vein, we also require to find a meaningful estimate for the operator norm of the polynomials $R^{(\bullet)} \in K^{(\bullet)}(\mathrm{End}(V)) \subseteq \mathrm{Sym}^{\bullet} V^* \otimes K(\mathrm{End}(V))$. This is not a difficult task, but the details are convoluted enough, that it is best to enclose them in a stand-alone lemma:

Lemma 4.3.2. *Let $v \in V$ and define the formal power series*

$$\begin{aligned}
S_v &:= \sum_{m \geqslant 0} \frac{t^m}{m!} S^{(m)}(v, \ldots, v) = \sum_{m \geqslant 0} \frac{t^m}{m!} S^{(m)}(v^m) \in K(\mathrm{End}(V))[[t]], \\
P_v &:= \sum_{m \geqslant 0} \frac{t^m}{m!} P^{(m)}(v, \ldots, v) = \sum_{m \geqslant 0} \frac{t^m}{m!} P^{(m)}(v^m) \in \mathrm{End}(V)[[t]], \\
R_v &:= \sum_{m \geqslant 0} \frac{t^m}{m!} R^{(m)}(v, \ldots, v) = \sum_{m \geqslant 0} \frac{t^m}{m!} R^{(m)}(v^m) \in K(\mathrm{End}(V))[[t]],
\end{aligned}$$

where the polynomials $S^{(m)}$ are the ones given at the beginning of the proof of Theorem 1, whereas the polynomials $P^{(m)}$, $R^{(m)}$ are the ones defined shortly thereafter. It holds that

$$N_{K(\mathrm{End}(V))}(R_v) \leqslant \sqrt{2}n\binom{n}{2}\left(N_{\mathrm{End}(V)}(P_v)\right)^4 N_{K(\mathrm{End}(V)}(S_v) \in \mathbb{R}[[t]].$$

Proof of the lemma. Since by definition $\mathsf{K}(\mathrm{End}(V)) \subseteq \bigwedge^2 V^* \otimes \mathrm{End}(V)$, we can assume, without loss of generality that $\dim V \geqslant 2$. Let $x, y, z \in V$ unitary vectors such that $\langle x, y \rangle = 0$.

From the definition of the action $\rho\colon \mathrm{GL}(V) \longrightarrow \mathrm{GL}(\mathsf{K}(\mathrm{End}(V))$ previously defined, we obtain

$$\begin{aligned}(P(tv) \cdot S(tv))(x,y)z =& P(tv)\big(S(tv)\big(P(tv)^{-1}x, P(tv)^{-1}y\big)P(tv)^{-1}z\big)\\ =& P(tv)(S(tv)(P(-tv)x, P(-tv)y)P(-tv)z).\end{aligned}$$

Clearly, the formal power series S_v, P_v, R_v correspond to formal power series expansion of the maps $t \longmapsto S(tv)$, $t \longmapsto P(tv)$, $t \longmapsto R(tv)$.

In terms of such correspondence, the functional equation

$$R(tv)(x,y)z = (P(tv) \cdot S(tv))(x,y)z$$

transforms itself into the equation

$$\begin{aligned}
&R(tv)(x,y)z = \sum_{m\geqslant 0} \frac{t^m}{m!} R^{(m)}(v^m)(x,y)z\\
=&P(tv)S(tv)\left(\sum_{m\geqslant 0} \frac{(-1)^m t^m}{m!} P^{(m)}(v^m)x, \sum_{m\geqslant 0} \frac{(-1)^m t^m}{m!} P^{(m)}(v^m)y\right) \sum_{m\geqslant 0} \frac{(-1)^m t^m}{m!} P^{(m)}(v^m)z\\
=&P(tv) \sum_{m\geqslant 0} \frac{(-1)^m t^m}{m!} \sum_{\substack{\mu\in\mathbb{N}_0^3,\\ |\mu|=m}} \binom{m}{\mu} S(tv)\Big(P^{(\mu_1)}(v^{\mu_1})x, P^{(\mu_2)}(v^{\mu_2})y\Big)P^{(\mu_3)}(v^{\mu_3})z\\
=&P(tv) \sum_{m\geqslant 0} \frac{t^m}{m!} \sum_{\substack{\mu\in\mathbb{N}_0^4,\\ |\mu|=m}} \binom{m}{\mu} (-1)^{\mu_2+\mu_3+\mu_4} S^{(\mu_1)}(v^{\mu_1})\Big(P^{(\mu_2)}(v^{\mu_2})x, P^{(\mu_3)}(v^{\mu_3})y\Big)P^{(\mu_4)}(v^{\mu_4})z\\
=&\sum_{m\geqslant 0} \frac{t^m}{m!} \sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}} \binom{m}{\mu} (-1)^{\mu_3+\mu_4+\mu_5} P^{(\mu_1)}(v^{\mu_1})\Big(S^{(\mu_2)}(v^{\mu_2})\Big(P^{(\mu_3)}(v^{\mu_3})x, P^{(\mu_4)}(v^{\mu_4})y\Big)P^{(\mu_5)}(v^{\mu_5})z\Big)
\end{aligned}$$

in the set $V[[t]]$, which therefore implies the identity

$$\begin{aligned}
&R^{(m)}(v^m)(x,y)z = R^{(m)}(v^m)((x\wedge y)\otimes z) =\\
&\sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}} \binom{m}{\mu} (-1)^{\mu_3+\mu_4+\mu_5} P^{(\mu_1)}(v^{\mu_1})\Big(S^{(\mu_2)}(v^{\mu_2})\Big(P^{(\mu_3)}(v^{\mu_3})x, P^{(\mu_4)}(v^{\mu_4})y\Big)P^{(\mu_5)}(v^{\mu_5})z\Big) \in V.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\left\|R^{(m)}(\nu^m)(x,y)z\right\| &\leqslant \sum_{\substack{\mu\in\mathbb{N}_0^5,\\|\mu|=m}}\binom{m}{\mu}\left\|P^{(\mu_1)}(\nu^{\mu_1})\left(S^{(\mu_2)}(\nu^{\mu_2})\left(P^{(\mu_3)}(\nu^{\mu_3})x,P^{(\mu_4)}(\nu^{\mu_4})y\right)P^{(\mu_5)}(\nu^{\mu_5})z\right)\right\|\\
&\overset{\text{op. Norm on End}(V)}{\leqslant} \sum_{\substack{\mu\in\mathbb{N}_0^5,\\|\mu|=m}}\binom{m}{\mu}\left\|P^{(\mu_1)}(\nu^{\mu_1})\right\|_{op}\left\|S^{(\mu_2)}(\nu^{\mu_2})\left(P^{(\mu_3)}(\nu^{\mu_3})x,P^{(\mu_4)}(\nu^{\mu_4})y\right)P^{(\mu_5)}(\nu^{\mu_5})z\right\|\\
&\overset{\text{op. Norm on K}(\text{End}(V))}{\leqslant} \sum_{\substack{\mu\in\mathbb{N}_0^5,\\|\mu|=m}}\binom{m}{\mu}\left\|P^{(\mu_1)}(\nu^{\mu_1})\right\|_{op}\left\|S^{(\mu_2)}(\nu^{\mu_2})\right\|_{op}\left\|P^{(\mu_3)}(\nu^{\mu_3})x\wedge P^{(\mu_4)}(\nu^{\mu_4})y\otimes P^{(\mu_5)}(\nu^{\mu_5})z\right\|\\
&\leqslant \sum_{\substack{\mu\in\mathbb{N}_0^5,\\|\mu|=m}}\binom{m}{\mu}\left\|P^{(\mu_1)}(\nu^{\mu_1})\right\|_{op}\left\|S^{(\mu_2)}(\nu^{\mu_2})\right\|_{op}\left\|P^{(\mu_3)}(\nu^{\mu_3})x\right\|\left\|P^{(\mu_4)}(\nu^{\mu_4})y\right\|\left\|P^{(\mu_5)}(\nu^{\mu_5})z\right\|\\
&\leqslant \sum_{\substack{\mu\in\mathbb{N}_0^5,\\|\mu|=m}}\binom{m}{\mu}\left\|S^{(\mu_1)}(\nu^{\mu_1})\right\|_{op}\prod_{i=2}^{5}\left\|P^{(\mu_i)}(\nu^{\mu_i})\right\|_{op}\|x\|\|y\|\|z\|\\
&\overset{\|x\|=\|y\|=\|z\|=1}{=} \sum_{\substack{\mu\in\mathbb{N}_0^5,\\|\mu|=m}}\binom{m}{\mu}\left\|S^{(\mu_1)}(\nu^{\mu_1})\right\|_{op}\prod_{i=2}^{5}\left\|P^{(\mu_i)}(\nu^{\mu_i})\right\|_{op}.
\end{aligned}$$

Now, we recall that with respect to the inner product $\langle\cdot,\cdot\rangle_{\bigwedge^2}$ earlier defined, the elements of the set

$$\left\{\sqrt{2}e_i\wedge e_j \mid 1\leqslant i<j\leqslant n\right\}$$

form an orthonormal basis of the vector space $\bigwedge^2 V$.

Thus for a given $\xi\in\bigwedge^2 V\otimes V$ we have the decomposition

$$\xi = 2\sum_{\substack{1\leqslant i<j\leqslant n,\\1\leqslant k\leqslant n}}\langle\xi, e_i\wedge e_j\otimes e_k\rangle e_i\wedge e_j\otimes e_k,$$

which in turn implies

$$R^{(m)}(\nu^m)\xi = 2\sum_{\substack{1\leqslant i<j\leqslant n,\\1\leqslant k\leqslant n}}\langle\xi, e_i\wedge e_j\otimes e_k\rangle R^{(m)}(\nu^m)(e_i,e_j)e_k.$$

By making use of the previous estimate, with the triple (e_i, e_j, e_k) taking the place of the triple (x, y, z) above, we obtain

$$\begin{aligned}
\left\|R^{(m)}(\nu^m)\xi\right\| &\leqslant 2 \sum_{\substack{1\leqslant i<j\leqslant n,\\ 1\leqslant k\leqslant n}} \|\xi\|\|e_i \wedge e_j \otimes e_k\| \left\|R^{(m)}(\nu^m)(e_i, e_j)e_k\right\| \\
&= \sqrt{2} \sum_{\substack{1\leqslant i<j\leqslant n,\\ 1\leqslant k\leqslant n}} \|\xi\| \left\|R^{(m)}(\nu^m)(e_i, e_j)e_k\right\| \\
&\leqslant \sqrt{2} \sum_{\substack{1\leqslant i<j\leqslant n,\\ 1\leqslant k\leqslant n}} \left(\sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}} \binom{m}{\mu} \left\|S^{(\mu_1)}(\nu^{\mu_1})\right\|_{op} \prod_{\ell=2}^{5} \left\|P^{(\mu_\ell)}(\nu^{\mu_\ell})\right\|_{op} \right) \|\xi\| \\
&= \sqrt{2} n \binom{n}{2} \left(\sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}} \binom{m}{\mu} \left\|S^{(\mu_1)}(\nu^{\mu_1})\right\|_{op} \prod_{\ell=2}^{5} \left\|P^{(\mu_\ell)}(\nu^{\mu_\ell})\right\|_{op} \right) \|\xi\|
\end{aligned}$$

We thus conclude that, for all $m \in \mathbb{N}_0$,

$$\left\|R^{(m)}(\nu^m)\right\|_{op} \leqslant \sqrt{2} n \binom{n}{2} \sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}} \binom{m}{\mu} \left\|S^{(\mu_1)}(\nu^{\mu_1})\right\|_{op} \prod_{i=2}^{5} \left\|P^{(\mu_i)}(\nu^{\mu_i})\right\|_{op}. \tag{4.27}$$

Therefore, at the level of formal power series, we obtain that

$$\sum_{m\geqslant 0} \frac{t^m}{m!} \left\|R^{(m)}(\nu^m)\right\|_{op} \leqslant \sqrt{2} n \binom{n}{2} \sum_{m\geqslant 0} \frac{t^m}{m!} \sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}} \binom{m}{\mu} \left\|S^{(\mu_1)}(\nu^{\mu_1})\right\|_{op} \prod_{i=2}^{5} \left\|P^{(\mu_i)}(\nu^{\mu_i})\right\|_{op},$$

which by definition is exactly the desired inequality of formal power series in $\mathbb{R}[[t]]$, that is,

$$N_{K(\mathrm{End}(V))}(R_\nu) \leqslant \sqrt{2} n \binom{n}{2} \left(N_{\mathrm{End}(V)}(P_\nu)\right)^4 N_{K(\mathrm{End}(V))}(S_\nu).$$

$\triangledown$

Notice that the previous lemma is precisely what we need to find the desired estimate for the norms of the polynomials $R^{(m)}$. Indeed, equation (4.27) together with the fact that $\|\nu^m\|_{\mathrm{Sym}} = \|\nu\|^m$ imply

$$\left\|R^{(m)}\right\|_{op} \leqslant \sqrt{2} n \binom{n}{2} \sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}} \binom{m}{\mu} \left\|S^{(\mu_1)}\right\|_{op} \prod_{i=2}^{5} \left\|P^{(\mu_i)}\right\|_{op}. \tag{4.28}$$

This series of estimates was in fact the missing part in our proof.

Putting all of this together we obtain in sum

$$
\begin{aligned}
\left\|\Gamma^{(m+1)}\right\|_{op} &\leqslant 2\sqrt{2}n^2(n^2+1)\|d\Gamma^{m+1}\|_{op} \\
&\overset{\text{Def. of } R^{(m)}}{\leqslant} \sqrt{2}n^2(n^2+1)\left(\left\|R^{(m)}\right\|_{op} + \sum_{a+b=m}\binom{m}{a}\left\|\left[\Gamma^{(a)},\Gamma^{(b)}\right]\right\|_{op}\right) \\
&\leqslant \sqrt{2}n^2(n^2+1)\left(\left\|R^{(m)}\right\|_{op} + C(n)\sum_{a+b=m}\binom{m}{a}\left\|\Gamma^{(a)}\right\|_{op}\left\|\Gamma^{(b)}\right\|_{op}\right) \\
&\leqslant \sqrt{2}n^2(n^2+1)\left(\sqrt{2}n\binom{n}{2}\sum_{\substack{\mu\in\mathbb{N}_0^5,\\ |\mu|=m}}\binom{m}{\mu}\left\|S^{(\mu_1)}\right\|_{op}\prod_{i=2}^{5}\left\|P^{(\mu_i)}\right\|_{op}\right. \\
&\qquad\left. + C(n)\sum_{a+b=m}\binom{m}{a}\left\|\Gamma^{(a)}\right\|_{op}\left\|\Gamma^{(b)}\right\|_{op}\right) \\
&\overset{\text{Partial ordering on } \mathbb{R}[[t]]}{=} \sqrt{2}n^2(n^2+1)\left[\frac{t^m}{m!}\right]\left(\sqrt{2}n\binom{n}{2}p(t)^4 s(t) + C(n)g(t)^2\right) \\
&\overset{(4.25)}{\leqslant} \sqrt{2}n^2(n^2+1)\left[\frac{t^m}{m!}\right]\left(\sqrt{2}n\binom{n}{2}e^{4\int_0^t g(z)dz}s(t) + C(n)g(t)^2\right),
\end{aligned}
$$

where $\left[\frac{t^m}{m!}\right]\sum_{\ell\geqslant 0}\frac{a_\ell}{\ell!}t^\ell := a_m$, for $\sum_{\ell\geqslant 0}\frac{a_\ell}{\ell!}t^\ell \in \mathbb{R}[[t]]$.

Thus, by definition of the partial ordering on $\mathbb{R}[[t]]$, we obtain

$$g'(t) \leqslant \sqrt{2}n^2(n^2+1)\left(\sqrt{2}n\binom{n}{2}e^{4\int_0^t g(z)dz}s(t) + C(n)g(t)^2\right).$$

We define $h(t) := \int_0^t g(z)dz$ and thus the above inequality becomes

$$h''(t) \leqslant \sqrt{2}n^2(n^2+1)\left(\sqrt{2}n\binom{n}{2}e^{4h(t)}s(t) + C(n)h'(t)^2\right).$$

Now, since the function $F_s\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$(x,y,z) \longmapsto \sqrt{2}n^2(n^2+1)\left(\sqrt{2}n\binom{n}{2}e^{4y}s(x) + C(n)z^2\right)$$

is a real analytic map for $(x,y,z) \in U \times \mathbb{R}^2 \subseteq \mathbb{R}^3$, with U an open neighborhood of the origin in which the Taylor series at $x = 0$ of the map s converges to it, Proposition 2.1.7 implies that the initial value problem

$$\begin{cases} u''(t) = F_s(t, u(t), u'(t)) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

has a unique real analytic solution defined in an open neighborhood of the origin, and thus, since the coefficients of the Taylor series at $(x,y,z) = (0,0,0)$ of the map F_s are non-negative, we conclude, by Proposition 2.1.8, that the map h, and consequently $h' = g$, is a real analytic map defined in a sufficiently small neighborhood of the origin.

Therefore, Lemma 4.3.1 implies that the map $\Gamma\colon V \longrightarrow \operatorname{Sym}^2 V^* \otimes V$, whose formal Taylor series expansion at $0 \in V$ is given by

$$\Gamma(v) \approx \sum_{m \geqslant 0} \frac{1}{m!} \Gamma^{(m)}(v, \dots, v)$$

is in fact a real analytic map in a sufficiently small neighborhood of $0 \in V$. That is, there exists an open neighborhood $0 \in \mathcal{U} \subseteq V$ such that

$$\Gamma(v) = \sum_{m \geqslant 0} \frac{1}{m!} \Gamma^{(m)}(v, \dots, v)$$

for all $v \in \mathcal{U}$.

The desired covariant derivative is thus the one defined by the real analytic Christoffel symbols we just found. That the desired relations between the parallel transport map and its associated curvature tensor follow is just a consequence of the fact that the polynomials $R^{(\bullet)} \in K^{(\bullet)}(\operatorname{End}(V))$, and $P^{(\bullet)} \in \operatorname{Sym}^\bullet V^* \otimes \operatorname{End}(V)$, by the way they are defined, in actuality denote the Taylor polynomials of the power series expansion of the curvature tensor R and the parallel transport map P_{γ_v}. The torsion-freeness of the covariant derivative follows from the fact that the Christoffel symbols generated by the function Γ are symmetric. $\square$

5
Applications

This chapter is devoted to the study of some consequences of Theorem 1.

5.1 The holonomy of torsion-free covariant derivatives

The goal of this section is to establish necessary and sufficient conditions for the existence of holonomy algebras of torsion-free covariant derivatives. The main result of the section is the following

Theorem 2. *Let* V *be a finite-dimensional real vector space. Let* $S\colon U \longrightarrow K(\mathfrak{g})$ *be a real analytic map defined in an open neighborhood* U *of the origin in* V *which satisfies that* $dS\colon U \longrightarrow K^1(\mathfrak{g})$ *and let* ∇ *be the covariant derivative given by Theorem 1. It holds that*

$$\mathfrak{hol}(\nabla) = \langle S(v)(x,y) |\, v \in U;\ x,y \in V\rangle.$$

Proof. Let us denote the right-hand side as $\mathfrak{h}$. The Theorem of Ambrose-Singer and Proposition 3.7.6 together with Theorem 1 immediately imply that $\mathfrak{h} \subseteq \mathfrak{hol}(\nabla)$.

For the reverse implication, we make use of the theory of Lie derivatives. Indeed, by the natural definition of Lie derivatives of vector-valued differential forms, we obtain that for σ, the exponential framing at 0 induced by the covariant derivative ∇ and $v \in \mathcal{W} := \mathrm{dom}(\sigma) \subseteq U$ (see [20, Chapter 12, proposition 12.36]),

$$\frac{d}{dt}(\Phi^E_{-t})_*\Gamma(\Phi^E_t(v)) = (\Phi^E_{-t})_*\Big((\mathcal{L}_E\Gamma)_{\Phi^E_t(v)}\Big),$$

where E denotes the Euler vector field on V, that is, the vector field whose flow is given by $\Phi^E\colon \mathbb{R} \times V \longrightarrow V$, with $\Phi^E(t,v) = e^t v$, and $\Gamma = \sigma^*\omega^\nabla$, with ω^∇ the connection form associated to the covariant derivative ∇.

Now, $\Phi_0^E = \mathbb{1}_V$, and according to the proof of Theorem 1, $\Gamma(0) = 0$. Thus we obtain by integrating the previous equation,

$$\Gamma(v) = \int_{-\infty}^{0} (\Phi_{-t}^E)_*(\mathcal{L}_E \Gamma)_{e^t v} dt. \tag{5.1}$$

Now, the fact that, by definition, im $(S) \subseteq K(\mathfrak{h})$ implies that the associated curvature form F^∇ actually takes values in the Lie subalgebra $\mathfrak{h}$, that is,
$F^\nabla \in \Omega^2(\sigma(\mathcal{W}), \mathfrak{h})$. From (5.1) we thus obtain $\omega^\nabla \in \Omega^1(\sigma(\mathcal{W}), \mathfrak{h})$. We extend this form to a 1-form on the previously defined H-subbundle

$$F_H = \bigsqcup_{u \in \sigma(\mathcal{W})} \{\sigma(u)h |\ h \in H\},$$

with H the connected Lie subgroup such that $\mathrm{Lie}(H) = \mathfrak{h}$. Namely by setting

$$\omega_{uh}^\nabla = \mathrm{Ad}(h^{-1}) \circ \omega_u^\nabla.$$

This implies that the connection form ω^∇, on TF_H, takes values in $\mathfrak{h}$, from which *ii)* in Proposition 3.6.7 implies that the connection ω^∇ is reducible to a connection form on the subbundle F_H, which we denote by ω_H. In this context, the corollary 3.7.1 reads:

i) $P^{\omega^\nabla}(u) \subseteq F_H$ for all $u \in F_H$.

ii) $\omega_H|_{TP^{\omega^\nabla}(u)} = \omega^\nabla|_{TP^{\omega^\nabla}(u)}$

which in turn implies that ω_H is reducible to the holonomy bundle of the connection form ω^∇, and therefore it follows that the holonomy algebra of the connection form ω^∇, and whence $\mathfrak{hol}(\nabla)$, is contained in $\mathfrak{h}$. □

An immediate consequence of the previous result is thus the following

Corollary 5.1.1. *In the situation of Theorem 2, let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. It holds that $\mathfrak{hol}(\nabla) \subseteq \mathfrak{h}$ if, and only if, the real analytic map S takes values in the space $K(\mathfrak{h})$.*

Definition 5.1.1. *Let $\mathfrak{h} \subseteq \mathfrak{g} = \mathrm{End}(V)$ be a Lie subalgebra. The algebraic curvature tensor $R \in K(\mathfrak{h})$ is said to have* $\mathfrak{h}$-full curvature *if*

$$\mathfrak{h} = \langle R(x, y) |\ x, y \in V \rangle.$$

In case a Berger algebra admits elements of full curvature, Theorem 2 guarantees:

Corollary 5.1.2. *Let $\mathfrak{h} \subseteq \mathrm{End}(V)$ be a Berger algebra that admits elements of full curvature. Then it occurs as the holonomy algebra of a torsion-free covariant derivative.*

Proof. Let $R \in K(\mathfrak{h})$ be an element of full curvature. Define the real analytic map
$S \colon V \longrightarrow K(\mathrm{End}(V))$ as the constant map $S \equiv R$. Since $dS \equiv 0 \in K^1(\mathrm{End}(V))$, Theorem 2

implies

$$\begin{aligned}\mathfrak{hol}(\nabla) =& \langle S(v)(x,y)|\ v,x,y \in V\rangle \\ =& \langle R(x,y)|\ x,y \in V\rangle \\ =& \mathfrak{h}.\end{aligned}$$

□

The existence of elements of full curvature is a rather mild condition. All of the known Berger algebras admit elements of full curvature, in particular, the Lie algebras appearing in Berger's list in Proposition 3.8.4 (see [15]). Thus, the previous corollary shows, in a unified way, that all of the elements of Berger's list actually occur as holonomy of torsion-free covariant derivatives. Not only that, it does so, without the need of using sophisticated and previously used methods, like those from Cartan-Kähler theory Bryant used to show the same local existence results we just obtained.

Another relevant consequence of these previous results is a proof of the fact that Berger's criteria really are the only obstructions for a Lie algebra to occur as the holonomy algebra of a torsion-free covariant derivative.

A further remarkable feat of the results in this section is the fact that even though they are completely simple to grasp and relatively simple to prove, they also offer a pleasant enough sense of generality inasmuch as they hold irrespective of the (non-)reducibility of the holonomy representation.

5.2 Explicit examples

For the final part of the work we explore some further consequences of Theorem 1, but now in the context of Berger's classification of Riemannian holonomies. In particular, we will establish some further facts about the geometry of manifolds with specific holonomy. In the spirit of Berger's list we will go through the classical groups in the list, namely, items *i)-iv)*.

Due to the fact that the map

$$\begin{aligned}\bigwedge^2 V \longrightarrow& \mathfrak{so}(V) \\ x \wedge y \longmapsto& x \wedge y : z \longmapsto \langle x,z\rangle y - \langle y,z\rangle x\end{aligned}$$

is a linear isomorphism and, by means of the scalar product, we have an isomorphism between V and V^*, we thus obtain, by writing $\mathfrak{so}(V) = \bigwedge^2 V^*$,

$$K(\mathfrak{so}(V)) = \ker\left\{\bigwedge^2 V^* \otimes \bigwedge^2 V^* \longrightarrow \bigwedge^3 V^* \otimes V^*\right\},$$

where the map $\bigwedge^2 V^* \otimes \bigwedge^2 V^* \longrightarrow \bigwedge^3 V^* \otimes V^*$ denotes the natural anti-symmetrization map

$$\begin{aligned}A\colon\ \bigwedge^2 V^* \otimes \bigwedge^2 V^* \longrightarrow& \bigwedge^3 V^* \otimes V^* \\ \eta \longmapsto& A(\eta)\colon x \wedge y \wedge z \otimes u \longmapsto \sum_{\mathrm{cyc}\,(x,y,z)} \eta(x \wedge y \otimes z \wedge u).\end{aligned}$$

In a similar fashion, we have

$$K^1(\mathfrak{so}(V)) = \ker\left\{B\colon V^* \otimes K(\mathfrak{so}(V)) \longrightarrow \bigwedge^3 V^* \otimes \bigwedge^2 V^*\right\},$$

where

$$\begin{aligned} B\colon V^* \otimes K(\mathfrak{so}(V)) &\longrightarrow \bigwedge^3 V^* \otimes \bigwedge^2 V^* \\ \zeta \longmapsto & B(\zeta)\colon x \wedge y \wedge z \otimes u \wedge w \longmapsto \sum_{\mathrm{cyc}\,(x,y,z)} \zeta(x \otimes y \wedge z \otimes u \wedge w). \end{aligned}$$

For the spaces $K^{(m)}(\mathfrak{so}(V))$ we obtain the following characterization:

Proposition 5.2.1. *For* $m \in \mathbb{N}_0$ *define the maps*

$$\begin{aligned} \psi_m\colon \operatorname{Sym}^{m+2} V^* \otimes \operatorname{Sym}^2 V^* &\longrightarrow \operatorname{Sym}^m V^* \otimes \operatorname{Sym}^2\left(\bigwedge^2 V^*\right) \\ \eta \longmapsto & \psi_m(\eta)\colon p \otimes x \wedge y \otimes z \wedge w \longmapsto \eta(pxz \otimes yw) - \eta(pxw \otimes yz) \\ & \qquad + \eta(pyw \otimes xz) - \eta(pyz \otimes xw) \\ \mu_m\colon \operatorname{Sym}^{m+2} V^* \otimes \operatorname{Sym}^2 V^* &\longrightarrow \operatorname{Sym}^{m+3} V^* \otimes V^* \\ \eta \longmapsto & \mu_m(\eta)\colon v_1 \cdots v_{m+3} \otimes x \longmapsto \sum_{i=1}^{m+3} \eta(v_1 \cdots \widehat{v_i} \cdots v_{m+3} \otimes v_i x). \end{aligned}$$

It then holds that

$$\begin{aligned} K^{(m)}(\mathfrak{so}(V)) &= \operatorname{im} \psi_m \\ &\cong \ker \mu_m. \end{aligned}$$

Furthermore, the dimension of this vector subspace is given by the formula

$$\dim\left(K^{(m)}(\mathfrak{so}(V))\right) = \frac{m+1}{m+3}\binom{n+m+1}{m+2}\binom{n}{2}.$$

Proof. Before we begin with our proof we show that, for all $m \in \mathbb{N}_0$,

$$K^{(m)}(\mathfrak{so}(V)) \subseteq \operatorname{Sym}^m V^* \otimes \operatorname{Sym}^2\left(\bigwedge^2 V^*\right).$$

Indeed, let us recall that the subspace $K^{(m)}(\mathfrak{so}(V))$ is defined as

$$\begin{aligned} K^{(m)}(\mathfrak{so}(V)) = & \ker\left\{A_m\colon \operatorname{Sym}^m V^* \otimes \bigwedge^2 V^* \otimes \bigwedge^2 V^* \longrightarrow \operatorname{Sym}^m V^* \otimes \bigwedge^3 V^* \otimes V^*\right\} \\ & \cap \ker\left\{B_m\colon \operatorname{Sym}^m V^* \otimes \bigwedge^2 V^* \otimes \bigwedge^2 V^* \longrightarrow \operatorname{Sym}^{m-1} V^* \otimes \bigwedge^3 V^* \otimes \bigwedge^2 V^*\right\}, \end{aligned}$$

where the maps A_m, B_m denote the obvious generalizations of the maps A, B in the definition of the spaces $K(\mathfrak{so}(V)), K^1(\mathfrak{so}(V))$. We say that $R \in \ker(A_m)$ satisfies the *first Bianchi identity* *(or 1BI)*, whereas for $R \in \ker(B_m)$ we say that it satisfies *the second Bianchi identity (or 2BI)*. For $R \in K^{(m)}(\mathfrak{so}(V))$, it holds in particular that R satisfies the first Bianchi identity, that is, $A_m \circ R \equiv 0$. Explicitly, for any $p \otimes x \wedge y \wedge z \otimes u \in \mathrm{Sym}^m V \otimes \bigwedge^3 V \otimes V$,

$$\sum_{\mathrm{cyc}\,(x,y,z)} R(p \otimes x \wedge y \otimes z \wedge u) = 0.$$

Thus

$$\begin{aligned} 0 = & \sum_{\mathrm{cyc}\,(x,y,z)} R(p \otimes x \wedge y \otimes z \wedge u) + \sum_{\mathrm{cyc}\,(z,x,u)} R(p \otimes z \wedge x \otimes u \wedge y) \\ & + \sum_{\mathrm{cyc}\,(u,y,x)} R(p \otimes u \wedge y \otimes x \wedge z) + \sum_{\mathrm{cyc}\,(z,u,y)} R(p \otimes z \wedge u \otimes y \wedge x) \\ = & 2R(p \otimes x \wedge y \otimes z \wedge u) - 2R(p \otimes z \wedge u \otimes x \wedge y), \end{aligned}$$

which immediately implies the additional symmetry

$$R(p \otimes x \wedge y \otimes z \wedge u) = R(p \otimes z \wedge u \otimes x \wedge y),$$

and in turn the inclusion $K^{(m)}(\mathfrak{so}(V)) \subseteq \mathrm{Sym}^m V^* \otimes \mathrm{Sym}^2(\bigwedge^2 V^*)$.

Now to the actual proof. We begin by showing the first of the characterizations. This proof is essentially a "coordinate-free" analogue of the proof of Proposition 2.2.1. We directly show both of the inclusions $\mathrm{im}\,(\psi_m) \subseteq K^{(m)}(\mathfrak{so}(V))$, $K^{(m)}(\mathfrak{so}(V)) \subseteq \mathrm{im}\,(\psi_m)$.

Let $\eta \in \mathrm{Sym}^{m+2} V^* \otimes \mathrm{Sym}^2 V^*$. Thus, for given $p \in \mathrm{Sym}^m V$, $x, y, z, u \in V$,

$$\begin{aligned} A_m \psi_m(\eta)(p \otimes x \wedge y \wedge z \otimes u) = & \sum_{\mathrm{cyc}\,(x,y,z)} \psi_m(\eta)(p \otimes x \wedge y \otimes z \wedge u) \\ = & \sum_{\mathrm{cyc}\,(x,y,z)} \big(\eta(pxz \otimes yu) - \eta(pxu \otimes yz) \\ & \qquad + \eta(pyu \otimes xz) - \eta(pyz \otimes xu)\big) \\ = & \sum_{\mathrm{cyc}\,(x,y,z)} \big(-\eta(pxy \otimes zu) + \eta(pyx \otimes zu) \\ & \qquad - \eta(pxu \otimes yz) + \eta(pxu \otimes zy)\big) \\ = & 0, \end{aligned}$$

Similarly, for $q \otimes x \wedge y \wedge z \otimes u \wedge w \in \operatorname{Sym}^{m-1} V \otimes \bigwedge^3 V \otimes \bigwedge^2 V$,

$$\begin{aligned} B_m \psi_m(\eta)(q \otimes x \wedge y \wedge z \otimes u \wedge w) &= \sum_{\text{cyc}\,(x,y,z)} \psi_m(\eta)(qx \otimes y \wedge z \otimes u \wedge w) \\ &= \sum_{\text{cyc}\,(x,y,z)} \big(\eta(qxyu \otimes zw) - \eta(qxyw \otimes zu) \\ &\qquad + \eta(qxzw \otimes yu) - \eta(qxzu \otimes yw)\big) \\ &= \sum_{\text{cyc}\,(x,y,z)} \big(\eta(qxyu \otimes zw) - \eta(qyxu \otimes zw) \\ &\qquad - \eta(qxyw \otimes zu) + \eta(qyxw \otimes zu)\big) \\ &= 0, \end{aligned}$$

which hence implies

$$\operatorname{im}(\psi_m) \subseteq K^{(m)}(\mathfrak{so}(V)).$$

For the second inclusion, we make use of the Koszul sequences we introduced in the proof of Proposition 2.2.1.

Let $R \in K^{(m)}(\mathfrak{so}(V))$. For $u, w \in V$, define the 2-form $\alpha_{u,w} \in \operatorname{Sym}^m V^* \otimes \bigwedge^2 V^*$ defined as

$$\alpha_{u,w}(p \otimes x \wedge y) := R(p \otimes x \wedge y \otimes u \wedge w).$$

For $q \otimes x \wedge y \wedge z \in \operatorname{Sym}^{m-1} V \otimes \bigwedge^3 V$ holds

$$\begin{aligned} \partial \alpha_{u,w}(q \otimes x \wedge y \wedge z) &= \sum_{\text{cyc}\,(x,y,z)} \alpha_{u,w}(qx \otimes y \wedge z) \\ &= \sum_{\text{cyc}\,(x,y,z)} R(qx \otimes y \wedge z \otimes u \wedge w) \\ &= B_m(R)(q \otimes x \wedge y \wedge z \otimes u \wedge w) \\ &= 0. \end{aligned}$$

That is, $\alpha_{u,w} \in \ker \left\{\partial \colon \operatorname{Sym}^m V^* \otimes \bigwedge^2 V^* \longrightarrow \operatorname{Sym}^{m-1} V^* \otimes \bigwedge^3 V^*\right\}$, which implies the existence of some $\beta_{u,w} \in \operatorname{Sym}^{m+1} V^* \otimes V^*$ such that

$$\partial \beta_{u,w} = \alpha_{u,w}.$$

Equivalently,

$$R(p \otimes x \wedge y \otimes u \wedge w) = \beta_{u,w}(px \otimes y) - \beta_{u,w}(py \otimes x).$$

Define now the 2-form $\gamma_w \in \operatorname{Sym}^{m+1} V^* \otimes \bigwedge^2 V^*$ by the formula

$$\gamma_w(r \otimes x \wedge y) := \beta_{x,w}(r \otimes y) - \beta_{y,w}(r \otimes x).$$

It holds that

$$\begin{aligned}
\partial\gamma_w(p\otimes x\wedge y\wedge z) &= \sum_{\mathrm{cyc}\,(x,y,z)} \gamma_w(px\otimes y\wedge z)\\
&= \sum_{\mathrm{cyc}\,(x,y,z)} \big(\beta_{y,w}(px\otimes z)-\beta_{z,w}(px\otimes y)\big)\\
&= \sum_{\mathrm{cyc}\,(x,y,z)} \big(\beta_{x,w}(pz\otimes y)-\beta_{x,w}(py\otimes z)\big)\\
&= \sum_{\mathrm{cyc}\,(x,y,z)} R(p\otimes z\wedge y\otimes x\wedge w)\\
&= -\sum_{\mathrm{cyc}\,(x,y,z)} R(p\otimes x\wedge y\otimes z\wedge w)\\
&= 0.
\end{aligned}$$

That is, $\gamma_w \in \ker\left\{\partial\colon \operatorname{Sym}^{m+1} V^* \otimes \bigwedge^2 V^* \longrightarrow \operatorname{Sym}^m V^* \otimes \bigwedge^3 V^*\right\}$, which implies the existence of some $\eta_w \in \operatorname{Sym}^{m+2} V^* \otimes V^*$ such that

$$\partial\eta_w = \gamma_w.$$

Equivalently,

$$\gamma_w(r\otimes x\wedge y) = \eta_w(rx\otimes y) - \eta_w(ry\otimes x).$$

Define further the forms $\eta \in \operatorname{Sym}^{m+2} V^* \otimes V^* \otimes V^*$, $\eta^+ \in \operatorname{Sym}^{m+2} V^* \otimes \operatorname{Sym}^2 V^*$, $\eta^- \in \operatorname{Sym}^{m+2} V^* \otimes \bigwedge^2 V^*$ by the formulas

$$\begin{aligned}
\eta(s\otimes x\otimes w) &:= \eta_w(s\otimes x),\\
\eta^+(s\otimes xw) &:= \eta(s\otimes x\otimes w) + \eta(s\otimes w\otimes x),\\
\eta^-(s\otimes x\wedge w) &:= \eta(s\otimes x\otimes w) - \eta(s\otimes w\otimes x),
\end{aligned}$$

from which we immediately obtain

$$\eta = \frac{1}{2}(\eta^+ + \eta^-).$$

Combining all of the above relations we obtain

$$\begin{aligned}
\psi_m(\eta^+)(p\otimes x\wedge y\otimes u\wedge w) &= \eta^+(pxu\otimes yw)-\eta^+(pxw\otimes yu)\\
&\quad +\eta^+(pyw\otimes xu)-\eta^+(pyu\otimes xw)\\
&\overset{\eta^-|_{\mathrm{Sym}^{m+2}V^*\otimes \mathrm{Sym}^2 V^*}\equiv 0}{=} 2(\eta(pxu\otimes yw)-\eta(pxw\otimes yu)\\
&\quad +\eta(pyw\otimes xu)-\eta(pyu\otimes xw))\\
&\overset{\partial\eta_w=\gamma_w}{=} 2\big(\beta_{x,w}(pu\otimes y)-\beta_{y,w}(pu\otimes x)\\
&\quad +\beta_{u,y}(px\otimes w)-\beta_{w,y}(px\otimes u)\\
&\quad +\beta_{y,u}(pw\otimes x)-\beta_{x,u}(pw\otimes y)\\
&\quad +\beta_{w,x}(py\otimes u)-\beta_{u,x}(py\otimes w)\big)\\
&\overset{\beta_{u,w}=-\beta_{w,u}}{=} 2\big(\beta_{x,w}(pu\otimes y)-\beta_{x,w}(py\otimes u)\\
&\quad +\beta_{y,w}(px\otimes u)-\beta_{y,w}(pu\otimes x)\\
&\quad +\beta_{y,u}(pw\otimes x)-\beta_{y,u}(px\otimes w)\\
&\quad +\beta_{x,u}(py\otimes w)-\beta_{x,u}(pw\otimes y)\big)\\
&\overset{\partial\beta_{u,w}=\alpha_{u,w}}{=} 4(R(p\otimes u\wedge y\otimes x\wedge w)+R(p\otimes y\wedge w\otimes x\wedge u))\\
&\overset{1\mathrm{BI}}{=} 4R(p\otimes x\wedge y\otimes u\otimes w),
\end{aligned}$$

which yields $K^{(m)}(\mathfrak{so}(V))\subseteq \mathrm{im}\,(\psi_m)$, and therefore

$$K^{(m)}(\mathfrak{so}(V))=\mathrm{im}\,(\psi_m).$$

In order to show the second characterization, define the map

$$\begin{aligned}
\phi_m\colon K^{(m)}(\mathfrak{so}(V)) &\longrightarrow \ker(\mu_m)\\
R \longmapsto \phi_m(R)\colon v_1\cdots v_{m+2}\otimes xy &\longmapsto \sum_{i,j} R(v_1\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots v_{m+2}\otimes v_i\wedge x\otimes v_j\wedge y)\\
&= 2\sum_{i<j} R(v_1\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots v_{m+2}\otimes v_i\wedge x\otimes v_j\wedge y).
\end{aligned}$$

We claim that ϕ_m is an isomorphism by explicitly finding its inverse. Indeed, a direct computation involving the Bianchi identities shows that there exists a constant $C(m)$, depending only on the degree m such that

$$\psi_m\circ\phi_m=C(m)\mathbb{1}_{K^{(m)}(\mathfrak{so}(V))}.$$

The fact that the map μ_m is clearly surjective immediately implies the claim regarding the dimension of $K^{(m)}(\mathfrak{so}(V))$:

$$\begin{aligned}\dim\left(K^{(m)}(\mathfrak{so}(V))\right) &= \dim(\ker(\mu_m))\\ &= \dim\left(\operatorname{Sym}^{m+2} V^* \otimes \operatorname{Sym}^2 V^*\right) - \dim\left(\operatorname{Sym}^{m+3} V^* \otimes V^*\right)\\ &= \binom{n+m+1}{m+2}\binom{n+1}{2} - n\binom{n+m+2}{m+3}\\ &= \frac{m+1}{m+3}\binom{n}{2}\binom{n+m+1}{m+2}.\end{aligned}$$

□

Notice that in light of this result, combined with Theorems 1 and 2, we have proved the following

Proposition 5.2.2. *In the situation of Theorem 1, assume the real analytic map* S *takes values in* $K(\mathfrak{so}(V))$*, and let* ∇ *be the torsion-free covariant derivative given by it. Then there exists a unique* $\eta_S\colon \mathcal{U} \longrightarrow \operatorname{Sym}^2 V^* \otimes \operatorname{Sym}^2 V^*$ *such that*

$$\psi_0 \circ \eta_S = \mathrm{Rm}^\nabla,$$

where $\mathrm{Rm}^\nabla\colon \mathcal{U} \longrightarrow \operatorname{Sym}^2(\bigwedge^2 V^*)$ *denotes the Riemann curvature of* ∇*.*

As a final example, let us now turn our attention to the Lie algebra $\mathfrak{u}(n)$.

Let $V = \mathbb{C}^{2n} = \operatorname{span}\{e_1, \ldots, e_{2n}\}$ and define the non-degenerate, skew-symmetric bilinear form

$$\begin{aligned} Q\colon V \times V &\longrightarrow \mathbb{C}\\ (x, y) &\longmapsto x^\top J y,\end{aligned}$$

where

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

We immediately notice that this bilinear form preserves the decomposition

$$\begin{aligned} V &= \operatorname{span}_{\mathbb{C}}\{e_1, \ldots, e_n\} \oplus \operatorname{span}_{\mathbb{C}}\{e_{n+1}, \ldots, e_{2n}\}\\ &=: \operatorname{span}_{\mathbb{C}}\{e_1, \ldots, e_n\} \oplus \operatorname{span}_{\mathbb{C}}\{\bar{e}_1, \ldots, \bar{e}_n\}\\ &=: V' \oplus V''\end{aligned}$$

The non-degeneracy of the bilinear form Q allows us to naturally identify V with V^*.

For $p, q \in \mathbb{N}_0$, we define the set of polynomials

$$\operatorname{Sym}^{(p,q)} V := \operatorname{Sym}^p V' \otimes \operatorname{Sym}^q V''.$$

Let us consider the group

$$\mathrm{GL}(n,\mathbb{C}) \mathrel{\hat{=}} \mathsf{G} := \left\{ \begin{pmatrix} X & 0 \\ 0 & (X^{-1})^{\top} \end{pmatrix} \middle| \, X \in \mathrm{GL}(n,\mathbb{C}) \right\} \subseteq \mathrm{Sp}(2n,\mathbb{C}).$$

Thus, by complexifying the exact sequence defining the space $K(\mathfrak{u}(n))$ together with the fact that $\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C})$, we obtain

$$K(\mathfrak{u}(n))_{\mathbb{C}} \cong K(\mathfrak{g}),$$

where $\mathfrak{g}$ denotes the Lie algebra of G, which is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^{\top} \end{pmatrix} \middle| \, X \in \mathfrak{gl}(n,\mathbb{C}) \right\}.$$

Proposition 5.2.3. *With this notation, we have the isomorphisms*

i) $K(\mathfrak{g}) \cong \mathrm{Sym}^{(2,2)} V^*$,

ii) $K^{(m)}(\mathfrak{g}) \cong \mathrm{Sym}^{(m+2,2)} V^* \oplus \mathrm{Sym}^{(2,m+2)} V^*$, *for* $m \in \mathbb{N}$.

Proof. Before proving the claims, we make a couple of relevant remarks about the nature of the elements of $K(\mathfrak{g})$. By definition of the Lie algebra $\mathfrak{g}$, we obtain that, for $x, y \in V$, $v \in V'$, $w \in V''$,

$$\begin{aligned} R(x,y)v &\in V' \\ R(x,y)w &\in V'' \end{aligned}$$

for every $R \in K(\mathfrak{g})$. In addition, because of the first Bianchi identity, we obtain maps

$$\begin{aligned} R\colon V' &\longrightarrow \mathrm{Sym}^2(V'')^* \\ v &\longmapsto R(v,\cdot)\cdot, \\ R\colon V'' &\longrightarrow \mathrm{Sym}^2(V')^* \\ w &\longmapsto R(w,\cdot)\cdot\,. \end{aligned}$$

Indeed, for $v \in V'$, $w_1, w_2 \in V''$ we have,

$$V' \ni R(w_1,w_2)v = R(v,w_2)w_1 + R(w_1,v)w_2 \in V'',$$

which implies

$$R(w_1,w_2)v = 0$$

and in turn

$$R(v,w_1)w_2 = R(v,w_2)w_1.$$

In fact, since $\mathfrak{g} \subseteq \mathfrak{sp}(2n, \mathbb{C})$ we obtain for all $v \in V'$, $w_i \in V''$,

$$\begin{aligned} 0 =& Q(R(w_1, w_2)w_3, v) + Q(w_3, R(w_1, w_2)v) \\ =& Q(R(w_1, w_2)w_3, v), \end{aligned}$$

and thus, due to the non-degeneracy of Q we obtain

$$R(w_1, w_2)w_3 = 0,$$

therefore

$$R(V'', V'') \equiv 0.$$

A similar computation shows that for $v_i \in V', w \in V''$,

$$R(w, v_1)v_2 = R(w, v_2)v_1,$$

whence

$$R(V', V') \equiv 0.$$

i) Define the map

$$\begin{aligned} \xi \colon K(\mathfrak{g}) &\longrightarrow \operatorname{Sym}^{(2,2)} V^* \\ R &\longmapsto \xi(R), \end{aligned}$$

where for $v_i \in V', w_i \in V''$,

$$\xi(R)(v_1, v_2, w_1, w_2) := Q(R(v_1, w_1)v_2, w_2).$$

That $\xi(R) \in \operatorname{Sym}^{(2,2)} V^*$ follows from the symmetry properties we showed earlier:

$$\begin{aligned} \xi(R)(v_2, v_1, w_1, w_2) =& Q(R(v_2, w_1)v_1, w_2) \\ =& Q(R(v_1, w_1)v_2 + R(v_2, v_1)w_1, w_2) \\ =& Q(R(v_1, w_1)v_2, w_2) \\ =& \xi(R)(v_1, v_2, w_1, w_2), \end{aligned}$$

with a similar computation for the symmetry in the last two components.

That ξ is an injective map immediately follows from our previous considerations. Indeed, for $R \in \ker(\xi)$, $v_i \in V'$, $w_i \in V''$ we have

$$Q(R(v_1, w_1)v_2, w_2) = 0 = -Q(v_2, R(v_1, w_1)w_2),$$

from which we obtain

$$\begin{aligned} R(v_1, w_1)v_2 =& 0 \\ R(v_1, w_1)w_2 =& 0, \end{aligned}$$

and so

$$R(V', V'') \equiv 0,$$

which, combined with the fact that $R(V', V') \equiv 0 \equiv R(V'', V'')$ implies $R \equiv 0$.

The fact that Q is non-degenerate together with all of the invariance properties an element of $K(\mathfrak{g})$ must fulfill (this essentially means reversing all of the computations previously made) implies that for any $\tau \in \mathrm{Sym}^{(2,2)} V^*$, there exists a unique $R^\tau \in K(\mathfrak{g})$ such that

$$\tau(v_1 v_2 \otimes w_1 w_2) = Q(R^\tau(v_1, w_1)v_2, w_2),$$

which readily implies the surjectivity of the map ξ. For item *ii)*, a direct computation shows that the following generalization of the map ξ from item *i)*,

$$\begin{aligned}
\xi_m \colon K^{(m)}(\mathfrak{g}) \subseteq \mathrm{Sym}^m V^* \otimes K(\mathfrak{g}) &\longrightarrow \mathrm{Sym}^{(m+2,2)} V^* \oplus \mathrm{Sym}^{(2,m+2)} V^* \\
R &\longrightarrow \xi_m(R) \colon v_1 \cdots v_{m+2} \otimes w_1 w_2 + x_1 x_2 \otimes y_1 \cdots y_{m+2} \\
&\longmapsto \frac{1}{2} \sum_{i,j} Q(R(v_1 \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots v_{m+2} \otimes v_i \wedge w_1)v_j, w_2) \\
&+ \frac{1}{2} \sum_{i,j} Q(R(y_1 \cdots \widehat{y}_i \cdots \widehat{y}_j \cdots y_{m+2} \otimes x_1 \wedge y_i)x_2, y_j)
\end{aligned}$$

is the desired isomorphism.

$\square$

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