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Equivalence relations of quadratic forms in characteristic 2 and quasilinear *p*-forms

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Dissertation

Equivalence relations of quadratic forms in characteristic 2 and quasilinear p-forms

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Abstract

This thesis deals with quadratic forms and quasilinear p-forms in positive characteristic. In this setting, there are three well-known equivalence relations – similarity, birational equivalence, and stable birational equivalence. Inspired by an algebraic characterization of motivic equivalence of quadratic forms over fields of characteristic other than two, this thesis defines a new equivalence relation – the Vishik equivalence.

The thesis is divided into chapters based on the kind of forms treated: quasilinear p-forms (over fields of characteristic p), totally singular quadratic forms, nonsingular quadratic forms, and singular quadratic forms (all of them over fields of characteristic 2). The main goal is to compare the four above-mentioned equivalences for each of those kinds of forms. We also derive some consequences of two forms being equivalent for each of the four equivalences separately. In particular, we give a new characterization of the stable birational equivalence for quadratic forms. Moreover, we provide some new results regarding the isotropy of quasilinear p-forms over field extensions.

Keywords

Quadratic forms, quasilinear p-forms, finite characteristics, equivalence relations, quadrics.

Contents

Introduction

All students of mathematics meet quadratic forms right in their first year in a course on linear algebra. They will learn that any quadratic form can be expressed by a symmetric matrix, and, working over a field, this matrix can be diagonalized. But there is a small catch, an assumption: The field must not be of characteristic two. It is in the nature of a mathematician to ask what happens over the "forbidden" fields.

As we will see in Definition 1.2, there is no problem in defining quadratic forms over fields of characteristic two. The only difference to the usual definition (next to the unimportance of the signs, of course) is that when defining the polar form in part (2) of the definition, we have to drop the factor $\frac{1}{2}$. Unfortunately, this destroys the usual bijection between quadratic and symmetric bilinear forms; there is no such bijection in characteristic two, quadratic and bilinear forms build two mostly separate theories. Moreover, the impossibility to divide by two also causes that not all quadratic forms can be diagonalized. The best that can be achieved in general (if we view a quadratic form as a polynomial in the variables X_i, Y_i, Z_j) is a polynomial of the form

$$\sum_{i=1}^{r}(a_iX_i^2 + X_iY_i + b_iY_i^2) + \sum_{j=1}^{s} c_jZ_j^2 \tag{❃}$$

with $r, s \in \mathbb{N}\cup\{0\}$ and coefficients a_i, b_i, c_j from the base field F. Depending on the (non)zeroness of r and s, we distinguish between *nonsingular*, *singular* and *totally singular* quadratic forms (see Subsection 1.1.2). Nonsingular and totally singular quadratic forms have very different properties, and hence they are usually treated separately; singular forms are a mixture of both of these kinds, and so they are the most difficult to handle.

Even though the diagonalization, and quadratic forms in characteristic two in general, were already studied by Arf in 1941, the study of quadratic forms over fields of characteristic two is struggling, in a sense, to keep up with the study of the usual case. There is commonly a certain analogy between the two theories. In a certain sense, the case of characteristic two can be even viewed as the more general one; if you have a proof for an arbitrary quadratic form in characteristic two, you can often adjust it to the case of characteristic other than two. The other way around, starting with the usual case and translating the theorems and their proofs into characteristic two, is usually more complex and sometimes quite tricky. There are generally two problems: First, one has to deal with various kinds of quadratic forms, each of them having very different properties. Second, the theory of quadratic forms in characteristic two is less developed than in the usual case, so some of the necessary tools may not be available in characteristic two (yet).

Next to the quadratic forms in characteristic two, we will also devote a significant part of the thesis to the so-called *quasilinear p-forms*. These are a generalization of the totally singular quadratic forms – the ones with $r = 0$ in the representation (❃): We pick a field F of characteristic $p > 0$ and consider diagonal homogeneous polynomials of degree p over F, i.e., polynomials of the form

$$\sum_{j=1}^{s} c_j Z_j^p$$

with $c_j \in F$. These forms have been studied even less than quadratic forms in characteristic two so far.

In this thesis, we are particularly focused on equivalences of forms (that is, quadratic forms and quasilinear p-forms) and relations between them. We will properly define the equivalences in Section 1.2. For now, we only say that in the case of characteristic other than two, there are four equivalences we are interested in: *similarity* (sim), *birational equivalence* (bir), *stable birational equivalence* (stb) and *motivic equivalence* (mot). It is an easy observation that the similarity is the strongest equivalence between these four – two similar forms are also birationally, stably birationally and motivically equivalent. We depict these trivial relations in Figure 1.

$$\begin{tikzcd}
& \boxed{\text{bir}} & \\
\boxed{\text{sim}} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \boxed{\text{stb}} \\
& \boxed{\text{mot}} &
\end{tikzcd}$$

Figure 1: Trivial relations between the equivalences in characteristic $\neq 2$

Regarding the remaining relations between these four equivalences in characteristic other than two, a few more implications are known to hold, there are counterexamples for other ones, and some of them remain unsolved; see Figure 2.

We would like to point out that the motivic equivalence has a known criterion, called *Vishik's criterion* (Vishik), which is also depicted in Figure 2. This fact is important for us: Unlike the similarity, the birational and the stable birational equivalence, the motivic equivalence cannot be directly translated into characteristic two (at least not with the currently known tools). However, Vishik's criterion, being purely algebraic, has a natural transcription into the characteristic two case; we will call it *Vishik equivalence.*

The main goal of this thesis is to compare the similarity, the birational equivalence, the stable birational equivalence, and the Vishik equivalence in the case of quadratic forms over fields of characteristic two and in the case of quasilinear p-forms over fields of characteristic p. To our knowledge, there are few results known, and most of them consider the birational and the stable birational equivalence of quasilinear p-forms. The Vishik equivalence is a new concept (although arising naturally from the case of characteristic not two, where it is a characterization for the motivic equivalence); the

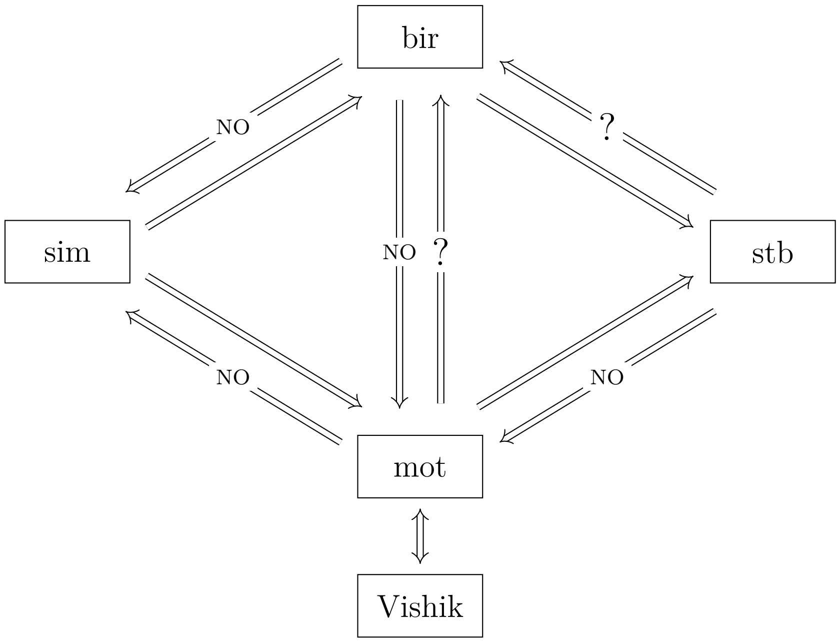

Figure 2: Equivalence relations in characteristic $\neq 2$

main question, which will accompany us throughout the whole thesis, is the following:

Question Q. *Are Vishik equivalent forms necessarily similar?*

Of course, we can conclude from Figure 2 that this question has a negative answer in characteristic other than two; however, we will see that in characteristic two, the answer is not clear.

Since the characteristic two case of quadratic forms is far less known than the usual one, we start the thesis with a quite comprehensive introduction to the topic in Section 1.1 of Chapter 1. The second part of Chapter 1, namely Section 1.2, is devoted to the definitions of the four equivalences and some of their properties. We would like to mention that from Subsection 1.1.3 further on, we include the quasilinear p-forms; in this sense, the content of this chapter is fairly unique in treating all kinds of quadratic forms and quasilinear p-forms uniformly.

One more chapter which treats all the forms uniformly can be found at the end of the thesis. In Appendix A, we look at the behavior of quadratic forms and quasilinear p-forms over fields with a discrete valuation. This chapter contains results that are easy to prove but are difficult (or even impossible) to be found in the current literature.

Each of the remaining chapters of the thesis is devoted to one specific kind of forms.

In Chapter 2, we look at the quasilinear p-forms. We start with a deeper introduction to the theory. Then we look at their isotropy properties and deduce some results on splitting patterns. Regarding the birational and the stable birational equivalence, we mostly just summarize known results. Finally, we look at the Vishik equivalence: We prove that Question Q has a negative answer if $p > 3$. Moreover, we draw some other interesting consequences of the Vishik equivalence of two quasilinear p-forms.

Setting $p = 2$, i.e., considering only totally singular quadratic forms, Chapter 2 gives an answer to all the relations between the equivalences except for Question Q. Therefore, that is the topic of Chapter 3. Unfortunately, we find neither a counterexample nor a proof applicable to all totally singular quadratic forms. However, we do give a positive answer for some families of totally singular quadratic forms. In particular, this chapter demonstrates the difficulty of the question.

In Chapter 4, we move on to a completely different kind of quadratic forms, the nonsingular ones. This kind of quadratic forms behave most analogously to the quadratic forms in characteristic other than two; hence, we make an exception for this chapter and do not impose any general assumption on the characteristic. Many of the results of this chapter are indeed independent of the characteristic.

Finally, Chapter 5 arch over all kinds of quadratic forms (but not the quasilinear p-forms). Since we could give a conclusive answer to Question Q neither for totally singular nor for nonsingular quadratic forms, we cannot expect to get an answer in this general case. Instead of that, we focus on the stable birational equivalence and provide a characterization of this phenomenon.

At this point should appear an acknowledgement. But to be honest, my biggest thanks belong (quite paradoxically) to the global pandemic of Covid-19, which provided so much life insecurity that continuing my doctoral studies seemed easier than giving up.

1. Preliminaries

If not said otherwise, F is an arbitrary field of characteristic 2, resp. of characteristic p in the case of quasilinear p-forms. Let $k \geq 1$; throughout the thesis, we denote:

- $F^* = F \setminus \{0\}$,
- $F^k = \{a^k \mid a \in F\}$, and
- $F^{*k} = \{a^k \mid a \in F^*\}$.

Note that if $\operatorname{char} F = p$, then $(a+b)^p = a^p + b^p$ for any $a, b \in F$, and hence F^p is a subfield of F.

1.1 Basic notions

We start by presenting the basic definitions and notations which will be used throughout the thesis. The standard literature for bilinear and quadratic forms is [EKM08], but we would like to point out that our notation sometimes differs from the one in this book, and follows rather [HL04]. The main source for an introduction to quasilinear p-forms is [Hof04]. If not said otherwise, any statements in this section are taken from these sources.

1.1.1 Bilinear forms

Definition 1.1. Let V be a finite-dimensional vector space over F. A *bilinear form* $\mathfrak{b}$ on V is a map $\mathfrak{b} : V \times V \to F$ satisfying for all $v, v', w, w' \in V$ and $a \in F$

$$\begin{aligned} \mathfrak{b}(v+v', w) &= \mathfrak{b}(v, w) + \mathfrak{b}(v', w), \\ \mathfrak{b}(v, w+w') &= \mathfrak{b}(v, w) + \mathfrak{b}(v, w'), \\ \mathfrak{b}(av, w) &= a\mathfrak{b}(v, w) = \mathfrak{b}(v, aw). \end{aligned}$$

For a bilinear form $\mathfrak{b}$ on a vector space V, we define the *dimension* of $\mathfrak{b}$ by $\dim \mathfrak{b} = \dim V$. We say that $\mathfrak{b}$ is *symmetric* if $\mathfrak{b}(v, w) = \mathfrak{b}(w, v)$ for any $v, w \in V$. We define the *radical* of a symmetric bilinear form $\mathfrak{b}$ by

$$\operatorname{rad} \mathfrak{b} = \{v \in V \mid \mathfrak{b}(v, w) = 0 \ \forall\, w \in V\}.$$

We call a symmetric bilinear form $\mathfrak{b}$ *nondegenerate* if $\operatorname{rad} \mathfrak{b} = \{0\}$.

If there exists a basis $\{e_1, \dots, e_n\}$ of V such that the Gramm matrix $G_\mathfrak{b}$ of $\mathfrak{b}$ with respect to this basis (i.e., the matrix $(\mathfrak{b}(e_i, e_j))_{i,j=1}^n$) is diagonal, then we call the bilinear form $\mathfrak{b}$ *diagonalizable*. In that case, we write $\mathfrak{b} \cong \langle c_1, \dots, c_n \rangle_b$, where $\operatorname{diag}(c_1, \dots, c_n) = G_\mathfrak{b}$.

1.1.2 Quadratic forms

Definition 1.2. Let V be a finite-dimensional vector space over F. We define a *quadratic form* over F as a map $\varphi : V \to F$ such that

(1) $\varphi(av) = a^2\varphi(v)$ for any $a \in F$ and $v \in V$,

(2) $\mathfrak{b}_\varphi : V \times V \to F$, defined by

$$\mathfrak{b}_\varphi(v,w) = \varphi(v+w) + \varphi(v) + \varphi(w)$$

for any $v, w \in V$, is a bilinear form.

We define the dimension of φ as $\dim V$, denoted $\dim \varphi$.

For a quadratic form φ, the bilinear form $\mathfrak{b}_\varphi$ defined above is called the *polar form* of φ. On the other hand, if a bilinear form $\mathfrak{b}$ is given, then we can define its *associated quadratic form* $\varphi_\mathfrak{b}$ by $\varphi_\mathfrak{b}(v) = \mathfrak{b}(v,v)$, $v \in V$. Note that

$$\varphi_{\mathfrak{b}_\varphi}(v) = \varphi(v+v) + \varphi(v) + \varphi(v) = 0$$

for any $v \in V$. Similarly, for any $v, w \in V$,

$$\begin{aligned}\mathfrak{b}_{\varphi_\mathfrak{b}}(v,w) &= \varphi_\mathfrak{b}(v+w) + \varphi_\mathfrak{b}(v) + \varphi_\mathfrak{b}(w)\\ &= \mathfrak{b}(v+w, v+w) + \mathfrak{b}(v,v) + \mathfrak{b}(w,w) = \mathfrak{b}(v,w) + \mathfrak{b}(w,v);\end{aligned}$$

hence, if $\mathfrak{b}$ is symmetric, then we get $\mathfrak{b}_{\varphi_\mathfrak{b}}(v,w) = 0$ for any $v, w \in V$. Therefore, there is no correspondence between quadratic and bilinear forms in characteristic two. We will focus on quadratic forms.

Definition 1.3. Let φ and ψ be two quadratic forms on the vector spaces V_φ and V_ψ. We say that φ and ψ are *isometric*, and write $\varphi \cong \psi$, if there exists an F-linear isomorphism $f : V_\varphi \to V_\psi$ such that $\varphi(v) = \psi(f(v))$ for any $v \in V_\varphi$.

Let φ be a quadratic form on V over F, and let W be an F-linear subspace of V. We denote by $\varphi|_W$ the quadratic form given by the restriction of the map φ to W; the polar form of $\varphi|_W$ is $\mathfrak{b}_{\varphi|_W} = (\mathfrak{b}_\varphi)|_W$. If U is another F-linear subspace of V such that $V = U \oplus W$ and $\mathfrak{b}_\varphi(u,w) = 0$ for any $u \in U$ and $w \in W$, then $\varphi \cong \varphi|_U \perp \varphi|_W$.

For a positive integer n, we define

$$n \times \varphi = \underbrace{\varphi \perp \ldots \perp \varphi}_{n\text{-times}}.$$

Let us write $[a,b]$ for the quadratic form which corresponds to the polynomial $aX^2 + XY + bY^2$ and $\langle c \rangle$ for the one corresponding to cZ^2. Then any quadratic form φ can be written as

$$\varphi \cong [a_1, b_1] \perp \ldots \perp [a_r, b_r] \perp \langle c_1, \ldots, c_s \rangle \tag{1.1}$$

with $a_i, b_i, c_j \in F$, where $\langle c_1, \ldots, c_s \rangle$ is a short way to write $\langle c_1 \rangle \perp \ldots \perp \langle c_s \rangle$. We call the pair (r,s) the *type* of the form φ; it is invariant under isometry. Note that $\dim \varphi = 2r + s$. We say that the form φ is

- the *zero form* if $r = s = 0$,
- *nonsingular* if $r > 0$ and $s = 0$,
- *semisingular* if $r > 0$ and $s > 0$,
- *totally singular* (or *quasilinear*) if $r = 0$ and $s > 0$,
- and *singular* if $s > 0$ (this term covers both semisingular and totally singular forms).

In (1.1), we have $\langle c_1, \dots, c_s\rangle \cong \varphi|_{\mathrm{rad}\,\mathfrak{b}_\varphi}$, and hence $\langle c_1, \dots, c_s\rangle$ is determined uniquely up to isometry; it is called the *quasilinear part* and denoted by $\mathrm{ql}(\varphi)$.

Note that if φ is totally singular, i.e., $\varphi \cong \langle c_1, \dots, c_s\rangle$, then it can be viewed as a quadratic form associated to the bilinear form $\mathfrak{b} \cong \langle c_1, \dots, c_s\rangle_b$; it follows that $\mathfrak{b}_\varphi \cong s \times \langle 0\rangle_b$, i.e., the polar form of a totally singular quadratic forms is the zero map.

There are some isometries of small dimensional forms, called *standard relations*, which we will use implicitly; for any $a, b, c, d \in F$ and $x \in F^*$, the following hold:

$$\begin{array}{ll} [a,b] \cong [b,a], & \langle a,b\rangle \cong \langle b,a\rangle, \\ [a,b] \cong [ax^2, bx^{-2}], & \langle a\rangle \cong \langle x^2 a\rangle, \\ [a,b] \cong [a, b+ax^2+x], & \langle a,b\rangle \cong \langle a, a+b\rangle, \\ [a,b] \perp [c,d] \cong [a+c,b] \perp [c,b+d], & [a,b] \perp \langle c\rangle \cong [a,b+c] \perp \langle c\rangle. \end{array} \tag{1.2}$$

In particular, note that for any $x \in F^*$, we have

$$\mathbb{H} \cong [0,x].$$

It can be proved that any isometry of quadratic forms can be obtained by successive applications of the standard relations (1.2).

Additionally to the standard relations, we will also often use scalar multiples of quadratic forms, and it will be useful to have some explicit expressions: For any $a, b, c \in F$ and $\lambda \in F^*$, we have

$$\lambda[a,b] \cong [\lambda a, \lambda^{-1} b] \quad \text{and} \quad \lambda\langle c\rangle \cong \langle \lambda c\rangle. \tag{1.3}$$

Together with (1.1), it follows that any quadratic form φ can be written as

$$\varphi \cong a_1[1,b_1] \perp \ldots \perp a_r[1,b_r] \perp \langle c_1, \dots, c_s\rangle \tag{1.4}$$

for some $a_i \in F^*$ and $b_i, c_j \in F$.

The question arises if there is some kind of cancellation possible when the same quadratic form appears as an orthogonal summand on the left- and right-hand side of an isometry relation. The answer is “not always”, as one can see from

$$[1,1] \perp \langle 1\rangle \cong [0,0] \perp \langle 1\rangle \quad \text{whereas} \quad [1,1] \not\cong [0,0] \text{ over } \mathbb{F}_2.$$

But a cancellation is possible under some additional assumptions:

Theorem 1.4 ([HL04, Prop. 2.5 and Lemma 2.6])**.** *Let φ, ψ, τ be quadratic forms and $j \geq 0$.*

(i) *If $\varphi \perp j \times \langle 0\rangle \cong \psi \perp j \times \langle 0\rangle$, then $\varphi \cong \psi$.*

(ii) *(Witt Cancellation Theorem) Let τ be nonsingular. If $\varphi \perp \tau \cong \psi \perp \tau$, then $\varphi \cong \psi$.*

We say that a quadratic form φ on V is *isotropic* if there exists a nonzero vector $v \in V$ such that $\varphi(v) = 0$. A subspace $U \subseteq V$ such that $\varphi(u) = 0$ for each $u \in U$ is called *totally isotropic*. If the quadratic form is not isotropic, then it is called *anisotropic*.

For example, the quadratic form $[0,0]$, called *hyperbolic plane* and denoted by $\mathbb{H}$, is isotropic; so is the one-dimensional quadratic form $\langle 0 \rangle$. By Witt Decomposition Theorem [HL04, Prop. 2.4], any quadratic form φ can be written as

$$\varphi \cong i \times \mathbb{H} \perp \varphi_r \perp \varphi_s \perp j \times \langle 0 \rangle \tag{1.5}$$

with φ_r nonsingular, φ_s totally singular and $\varphi_r \perp \varphi_s$ anisotropic. In this decomposition, the form $\varphi_r \perp \varphi_s$ is unique up to isometry, it is called the *anisotropic part* of φ and denoted by φ_{an}. The form φ_s is also unique up to isometry. The form φ_r is generally not unique. Moreover, the numbers i and j are determined uniquely; we define

- the *Witt index* of φ as $\mathfrak{i}_{\mathrm{W}}(\varphi) = i$,
- the *defect* (or *quasilinear index*) of φ as $\mathfrak{i}_{\mathrm{d}}(\varphi) = j$,
- and the *total isotropy index* of φ as $\mathfrak{i}_{\mathrm{t}}(\varphi) = i + j$.

We call the form φ *nondefective* if $\mathfrak{i}_{\mathrm{d}}(\varphi) = 0$. For totally singular forms, being nondefective is equivalent to being anisotropic, whereas all nonsingular forms are nondefective. We define the *nondefective part* of φ as $\varphi_{\mathrm{nd}} \cong i \times \mathbb{H} \perp \varphi_r \perp \varphi_s$. Quadratic forms isometric to $k \times \mathbb{H}$ for some $k \geq 1$ are called *hyperbolic*. Note that $\mathfrak{i}_{\mathrm{t}}(\varphi)$ is the dimension of a *maximal isotropic subspace* (i.e., a totally isotropic subspace of maximal dimension) of V.

Definition 1.5. We say that quadratic forms φ and ψ are *Witt equivalent*, denoted by $\varphi \overset{\text{Witt}}{\sim} \psi$, if $\varphi_{\mathrm{an}} \cong \psi_{\mathrm{an}}$.

In the following example, we look at the isotropy of binary quadratic forms.

Example 1.6. Let $a, b \in F$, and we consider the binary quadratic forms $\langle 1, a \rangle$ and $[1, b]$. Obviously, the form $\langle 1, a \rangle$ is isotropic if and only if $a \in F^2$; in such case, we have $\langle 1, a \rangle \cong \langle 1, 0 \rangle$. On the other hand, the form $[1, b]$ is isotropic if and only if there exists $x \in F$ such that $b = x^2 + x$; moreover, note that if $[1, b]$ is isotropic, then necessarily $[1, b] \cong \mathbb{H}$.

Inspired by the previous example, we define the *Artin–Schreier* map:

$$\wp : F \to F, \quad x \mapsto x^2 + x.$$

It follows that the quadratic form $[1, b]$ is isotropic if and only if $b \in \wp(F)$.

Let φ and σ be quadratic forms over F. We say that σ is *dominated by* φ, denoted by $\sigma \preccurlyeq \varphi$, if $\sigma \cong \varphi|_U$ for some vector space $U \subseteq V$. Moreover, we say that σ is a *subform* of φ, denoted by $\sigma \subseteq \varphi$, if there exists a quadratic form ψ (possibly the zero form) such that $\varphi \cong \sigma \perp \psi$. Note that any subform of φ is dominated by φ. If σ is nonsingular or both φ and σ are totally singular, then the notions of subform and dominance are equivalent, but not so in general: For example, $\langle 0 \rangle \preccurlyeq \mathbb{H}$, but $\langle 0 \rangle \perp \langle a \rangle \not\cong \mathbb{H}$ for any $a \in F$, because isometry preserves the type.

Proposition 1.7 ([HL04, Lemma 3.1]). *Let σ, φ be quadratic forms over F. Then the following are equivalent:*

(i) $\sigma \preccurlyeq \varphi$,

(ii) *there exist nonsingular forms σ_r and τ over F, nonnegative integers $s' \leq s \leq s''$ and $c_i, d_j \in F$ (with $1 \leq i \leq s''$ and $1 \leq j \leq s'$) such that*

$$\sigma \cong \sigma_r \perp \langle c_1, \dots, c_s \rangle, \text{ and}$$
$$\varphi \cong \sigma_r \perp \tau \perp [c_1, d_1] \perp \dots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1}, \dots, c_{s''} \rangle.$$

For a symmetric bilinear form $\mathfrak{b}$ on V and a quadratic form φ on W, we define the *tensor product* of $\mathfrak{b}$ and φ as the unique quadratic form $\mathfrak{b} \otimes \varphi$ that satisfies

$$(\mathfrak{b} \otimes \varphi)(v \otimes w) = \mathfrak{b}(v, v)\varphi(w)$$

for any $v \in V$ and $w \in W$, and with the polar form of $\mathfrak{b} \otimes \varphi$ defined as $\mathfrak{b} \otimes \mathfrak{b}_\varphi$. For example, if $a \in F$, then $\langle a \rangle_b \otimes \varphi \cong a\varphi$.

Consider now totally singular quadratic forms φ and ψ. We can find a bilinear form $\mathfrak{b}$ such that φ is the associated quadratic form to $\mathfrak{b}$: for $\varphi \cong \langle c_1, \dots, c_s \rangle$ set $\mathfrak{b} \cong \langle c_1, \dots, c_s \rangle_b$. We define the tensor product $\varphi \otimes \psi$ as $\mathfrak{b} \otimes \psi$, i.e.,

$$\langle c_1, \dots, c_s \rangle \otimes \psi \cong c_1\psi \perp \dots \perp c_s\psi.$$

Let φ, ψ be quadratic forms over F. We say that φ is *divisible* by ψ if there exists a bilinear form $\mathfrak{b}$ (or a totally singular quadratic form γ) over F such that $\varphi \cong \mathfrak{b} \otimes \psi$ (or $\varphi \cong \gamma \otimes \psi$).

1.1.3 Quasilinear p-forms

Definition 1.8. Let F be a field of characteristic $p > 0$ and V a finite-dimensional vector space over F. A *quasilinear p-form* is a map $\varphi : V \to F$ with the following properties:

(1) $\varphi(av) = a^p\varphi(v)$ for any $a \in F$ and $v \in V$,

(2) $\varphi(v + w) = \varphi(v) + \varphi(w)$ for any $v, w \in V$.

We define the dimension of φ as $\dim \varphi = \dim V$.

Note that if $p = 2$ and σ is a totally singular quadratic form, then it satisfies property (1). Moreover, recall that the polar form of σ is the zero map; hence, property (2) is fulfilled as well. So any totally singular quadratic form is a quasilinear 2-form.

On the other hand, since the polar form of a totally singular quadratic form is the zero map, we can, in this sense, also talk about a polar form of a quasilinear p-form. So, to avoid case distinction, we define the *polar form* $\mathfrak{b}_\varphi$ of a quasilinear p-form φ as $\mathfrak{b}_\varphi = \dim \varphi \times \langle 0 \rangle_b$. With this definition, it follows that the quasilinear 2-forms are exactly the totally singular quadratic forms.

Let F be a field of characteristic $p > 0$, V a vector space over F with a basis $\{e_1, \dots, e_n\}$, and φ a quasilinear p-form on V. Let $a_i = \varphi(e_i)$; then, for a vector $v = \sum_{i=1}^n x_i e_i \in V$, we have

$$\varphi(v) = \sum_{i=1}^n a_i x_i^p,$$

and hence we can associate φ with the "diagonal" homogeneous polynomial $\sum_{i=1}^{n} a_i X_i^p \in F[X_1, \ldots, X_n]$. Analogously as in the case of totally singular quadratic forms, we denote such a quasilinear p-form by $\langle a_1, \ldots, a_n \rangle$.

The notions of being *isometric*, *(an)isotropic* or *nondefective*, the notion of *defect* and of being a *subform* are completely analogous to the ones for totally singular quadratic forms. In particular, as in the case of totally singular quadratic forms, it holds for any quasilinear p-form φ that $\varphi \cong \varphi_{\mathrm{an}} \perp \mathfrak{i}_{\mathrm{d}}(\varphi) \times \langle 0 \rangle$ and $\varphi_{\mathrm{an}} \cong \varphi_{\mathrm{nd}}$. The Witt cancellation works as in the case of totally singular quadratic forms, namely only $\langle 0 \rangle$'s can be canceled.

The *standard relations* for quasilinear p-forms are also analogous to the ones for totally singular quadratic forms; namely, for $a, b, \lambda \in F$ and $x \in F^*$, we have

$$\langle a \rangle \cong \langle a x^p \rangle, \quad \langle a, b \rangle \cong \langle a, a+b \rangle, \quad \lambda \langle a \rangle \cong \langle \lambda a \rangle.$$

In particular, note that $-1 = (-1)^p$, and hence $\langle -a \rangle \cong \langle a \rangle$ for any $a \in F$.

The *tensor product* of two quasilinear p-forms $\varphi \cong \langle a_1, \ldots, a_n \rangle$ and ψ over F is defined as

$$\varphi \otimes \psi \cong a_1 \psi \perp \ldots \perp a_n \psi.$$

In the following, we write *p-form* for short instead of quasilinear p-form and use *forms* as a common term for both quadratic forms and p-forms.

1.1.4 Field extensions

Let $\mathfrak{b}$ be a bilinear form on the vector space V over F. If E/F is a field extension, then we write $\mathfrak{b}_E$ for the bilinear form on the vector space $V_E = E \otimes_F V$ that satisfies $\mathfrak{b}_E(e \otimes v, f \otimes w) = ef\mathfrak{b}(v, w)$ for any $e, f \in E$ and $v, w \in V$. Furthermore, if φ is a form on the vector space V over F with the polar form $\mathfrak{b}_\varphi$, then we write φ_E for the form on the vector space V_E that satisfies $\varphi_E(a \otimes v) = a^p \varphi(v)$ for any $a \in E$ and $v \in V$, and with the polar form $\mathfrak{b}_{\varphi_E} = (\mathfrak{b}_\varphi)_E$.

We start with a few basic observations from linear algebra which have some important effects on quadratic forms over field extensions.

Lemma 1.9. *Let V be a vector space over F, and let E/F be a field extension. Let $v_1, \ldots, v_n \in V$ be F-linearly independent vectors. Then the vectors $1 \otimes v_1, \ldots, 1 \otimes v_n$ are linearly independent over E. In particular, $\dim_E V_E = \dim_F V$.*

Proof. Let $\{\alpha_j \mid j \in J\}$ for a suitable index set J be a basis of E as an F-vector space. Assume that

$$0 = \sum_{i=1}^{n} \gamma_i (1 \otimes v_i)$$

for some $\gamma_i \in E$, and for each i write

$$\gamma_i = \sum_{j \in J} x_{ij}\alpha_j$$

with $x_{ij} \in F$, such that $x_{ij} = 0$ for almost all $j \in J$. Then we have

$$0 = \sum_{i=1}^{n} \gamma_i(1 \otimes v_i) = \sum_{i=1}^{n}\sum_{j \in J} x_{ij}\alpha_j \otimes v_i = \sum_{j \in J} \alpha_j \otimes \underbrace{\left(\sum_{i=1}^{n} x_{ij}v_i\right)}_{\in V}.$$

Since $\{\alpha_j \mid j \in J\}$ is linearly independent over F, we get $\sum_{i=1}^{n} x_{ij}v_i = 0$ for each $j \in J$. But now we are in V, where $v_1, \dots, v_n$ are linearly independent; therefore, $x_{ij} = 0$ for each i and j. It follows that $\gamma_i = 0$ for each i, and hence the vectors $1 \otimes v_1, \dots, 1 \otimes v_n$ are linearly independent over E.

It is easy to see that if we have $\operatorname{span}_F\{v_1, \dots, v_n\} = V$, it follows that $\operatorname{span}_E\{1 \otimes v_1, \dots, 1 \otimes v_n\} = V_E$. Together with the first part of the lemma, we get $\dim_F V = \dim_E V_E$. □

Lemma 1.10. *Let $\mathfrak{b}$ be a symmetric bilinear form on an F-vector space and E/F a field extension. Then $E \otimes \operatorname{rad}\mathfrak{b} \simeq \operatorname{rad}\mathfrak{b}_E$.*

Proof. First of all, note that

$$E \otimes \operatorname{rad}\mathfrak{b} = \left\{ \sum_i e_i \otimes v_i \in V_E \;\middle|\; \mathfrak{b}(v_i, u) = 0 \;\forall\, i,\ \forall\, u \in V \right\},$$

$$\operatorname{rad}\mathfrak{b}_E = \left\{ \sum_i e_i \otimes v_i \in V_E \;\middle|\; \sum_{i,j} e_i f_j \mathfrak{b}(v_i, w_j) = 0 \;\forall \sum_j f_j \otimes w_j \in V_E \right\}.$$

Thus, the inclusion $E \otimes \operatorname{rad}\mathfrak{b} \subseteq \operatorname{rad}\mathfrak{b}_E$ is trivial. Let $v \in \operatorname{rad}\mathfrak{b}_E$ with $v = \sum_{i=1}^{n} e_i \otimes v_i$ for some $n > 0$, $e_i \in E$ and $v_i \in V$. Let $\{\alpha_j \mid j \in J\}$ for a suitable index set J be an F-basis of E, and for each i, write

$$e_i = \sum_{j \in J} x_{ij}\alpha_j$$

with $x_{ij} \in F$ and $x_{ij} = 0$ for almost all $j \in J$. Then

$$v = \sum_{i=1}^{n} e_i \otimes v_i = \sum_{i=1}^{n}\sum_{j \in J} x_{ij}\alpha_j \otimes v_i = \sum_{j \in J} \alpha_j \otimes \left(\sum_{i=1}^{n} x_{ij}v_i\right). \tag{1.6}$$

Thus, for each $u \in V$, we have

$$0 = \mathfrak{b}_E\left(\sum_{i=1}^{n} e_i \otimes v_i, 1 \otimes u\right) = \sum_{j \in J} \alpha_j \mathfrak{b}\left(\sum_{i=1}^{n} x_{ij}v_i, u\right).$$

From the linear independence of $\{\alpha_j \mid j \in J\}$, it follows $\mathfrak{b}\left(\sum_i x_{ij}v_i, u\right) = 0$ for each $j \in J$. Therefore, $\sum_i x_{ij}v_i \in \operatorname{rad}\mathfrak{b}$. Hence, using (1.6) again, we obtain $v \in E \otimes \operatorname{rad}\mathfrak{b}$. □

Note that the previous lemma implies that the bilinear form $\mathfrak{b}_E$ is nondegenerate if and only if $\mathfrak{b}$ is nondegenerate.

Corollary 1.11. *The type of a quadratic form is invariant under field extensions.*

Proof. Let φ be a quadratic form over F and let E/F be a field extension. The dimension of $\mathrm{ql}(\varphi_E)$ equals the dimension of the E-vector space $\mathrm{rad}\,\mathfrak{b}_{\varphi_E}$, which is by Lemma 1.10 isometric to the vector space $E \otimes \mathrm{rad}\,\mathfrak{b}_\varphi$. Thus,

$$\dim \mathrm{ql}(\varphi_E) = \dim_E E \otimes \mathrm{rad}\,\mathfrak{b}_\varphi = \dim_F \mathrm{rad}\,\mathfrak{b}_\varphi = \dim \mathrm{ql}(\varphi),$$

where the second equality follows from Lemma 1.9. By the same lemma, we have $\dim \varphi = \dim \varphi_E$. The claim follows. $\square$

Corollary 1.12. *Let φ be a form over F and E/F a field extension. Then $\mathfrak{i}_\mathrm{W}(\varphi_E) \geq \mathfrak{i}_\mathrm{W}(\varphi)$, $\mathfrak{i}_\mathrm{d}(\varphi_E) \geq \mathfrak{i}_\mathrm{d}(\varphi)$ and $\mathfrak{i}_\mathrm{t}(\varphi_E) \geq \mathfrak{i}_\mathrm{t}(\varphi)$.*

Proof. We prove the statement only for the Witt index, the other cases are analogous. Let V be the underlying F-vector space of φ and $U \subseteq V$ be such that $V \simeq U \perp \mathrm{rad}\,\mathfrak{b}_\varphi$. Let $W \subseteq U$ be a maximal isotropic subspace of U, and denote $W_E = E \otimes W$. Let $\sum_{i=1}^n e_i \otimes w_i \in W_E$ with $w_i \in W$ for each i. Then $\varphi(w_i) = 0$, and also

$$\mathfrak{b}_\varphi(w_i, w_j) = \varphi(w_i + w_j) + \varphi(w_i) + \varphi(w_j) = 0$$

for all i, j. Thus, we have

$$\varphi\left(\sum_{i=1}^n e_i \otimes w_i\right) = \sum_{i=1}^n e_i^2 \varphi(w_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^n e_i e_j \mathfrak{b}_\varphi(w_i, w_j) = 0,$$

and hence W_E is a totally isotropic subspace of φ_E, which means that $\mathfrak{i}_\mathrm{t}(\varphi_E) \geq \dim_E W_E$. Moreover, we have

$$W_E \cap \mathrm{rad}\,\mathfrak{b}_{\varphi_E} \simeq E \otimes (W \cap \mathrm{rad}\,\mathfrak{b}_\varphi) = E \otimes \{0\} \simeq \{0\};$$

therefore, the form $(\varphi_E)|_{W_E}$ is nonsingular, and so $\mathfrak{i}_\mathrm{W}(\varphi_E) \geq \dim_E W_E$. Since we know from Lemma 1.9 that $\dim_E W_E = \dim_F W$, we get the desired inequality $\mathfrak{i}_\mathrm{W}(\varphi_E) \geq \mathfrak{i}_\mathrm{W}(\varphi)$. $\square$

At this point, let us recall some basic algebra about field extensions in finite characteristic.

We call a field extension E/F *purely transcendental* if there exists a subset $S \subseteq E$ of algebraically independent elements such that $E = F(S)$. The typical example is the extension $E = F(X_1, \ldots, X_n)$ over F with X_i variables; in that case, we call n the *transcendence degree* and denote $\mathrm{trdeg}_F E = n$.

Purely transcendental extensions do not change the isotropy of forms.

Lemma 1.13 ([EKM08, Lemma 7.15] and [Hof04, Prop. 5.3])**.** *Let φ be a form over F and E/F a purely transcendental extension. Then it holds that $\mathfrak{i}_\mathrm{W}(\varphi_E) = \mathfrak{i}_\mathrm{W}(\varphi)$ and $\mathfrak{i}_\mathrm{d}(\varphi_E) = \mathfrak{i}_\mathrm{d}(\varphi)$. In particular, φ is isotropic over E if and only if φ is isotropic over F.*

An algebraic extension E/F is called *separable* if the minimal polynomial over F of any element of E is separable; it is called *inseparable* otherwise.

Lemma 1.14 ([Hof04, Prop.5.3] and [EKM08, Lemma 22.13])**.** *Let E/F be a separable extension.*

(i) *If* $\operatorname{char} F = p$ *and φ is a p-form over F, then $\mathfrak{i}_\mathrm{d}(\varphi_E) = \mathfrak{i}_\mathrm{d}(\varphi)$. In particular, if φ is anisotropic over F, then it remains anisotropic over E.*

(ii) *Let* $\operatorname{char} F = 2$ *and ψ be a quadratic form over F. Then we have $\mathfrak{i}_\mathrm{d}(\psi_E) = \mathfrak{i}_\mathrm{d}(\psi)$. In particular, if* $\mathrm{ql}(\psi)$ *is anisotropic over F, then it remains anisotropic over E.*

However, a singular quadratic form can become isotropic over a separable field extension, as we will see in Subsection 1.1.5.

We call an extension L/F *purely inseparable* if for each $\alpha \in L$ there exists $n(\alpha) \geq 0$ such that $\alpha^{p^{n(\alpha)}} \in F$; the minimal $n(\alpha)$ with this property is called the *exponent of α over F*. If there exists the maximum of exponents of all elements of L, then it is called the *exponent of L over F*. The degree of any finite purely inseparable extension is a power of p.

Lemma 1.15 ([Pic39, Th. 4 on pg. 220])**.** *Let L/F be a finite purely inseparable field extension. Then there exist $a_1, \dots, a_r \in F$ and $n_1, \dots, n_r \geq 0$ such that*

$$L \simeq F\left(\sqrt[p^{n_1}]{a_1}, \dots, \sqrt[p^{n_r}]{a_r}\right).$$

In that case, $[L : F] \leq p^{n_1 + \dots + n_r}$.

Any algebraic field extension L/F can be realized as a separable extension K/F followed by a purely inseparable extension L/K. Here we have

$$K = \{\alpha \in L \mid \alpha \text{ separable over } F\},$$

and K is called a *separable closure of F in L* ([Bos20, Th. 4 on pg. 161]). If $L \simeq \overline{L}$, i.e., L is algebraically closed, then we call K simply a *separable closure of F*.

Lemma 1.16. *Let* $\operatorname{char} F = 2$*. Let φ be a quadratic form over F and ψ a nonsingular quadratic form over F such that $\varphi \cong \psi \perp \mathrm{ql}(\varphi)$. Moreover, let K be a separable closure of F. Then ψ is hyperbolic over K; in particular, $\mathfrak{i}_\mathrm{W}(\varphi_K) = \frac{1}{2}\dim\psi$.*

Proof. Let

$$\psi \cong \mathop{\perp}_{i=1}^{n} a_i[1, b_i]$$

for some $n \geq 0$ and $a_i, b_i \in F$. Then, for each i, we have $\wp^{-1}(b_i) \in K$, and hence $[1, b_i]_K \cong \mathbb{H}_K$. It follows that ψ_K is hyperbolic. □

1.1.5 Function fields

Let $f \in F[X_0, \dots, X_n]$ be a homogeneous irreducible polynomial; if $n \geq 1$, then the polynomial $f(1, X_1, \dots, X_n) \in F[X_1, \dots, X_n]$ is also irreducible. So, we define the *function field* $F(f)$ as

$$F(f) = \operatorname{Quot}\left(F[X_1, \dots, X_n]/f(1, X_1, \dots, X_n)\right).$$

If $g \in F[X_1, \dots, X_n]$ is irreducible, but not homogeneous, then we define its function field $F(g)$ as

$$F(g) = \mathrm{Quot}\left(F[X_1, \dots, X_n]/g(X_1, \dots, X_n)\right).$$

Note that this is consistent with the definition for homogeneous polynomials: Indeed, we could consider the homogenization $\widetilde{g}$ of g, i.e., the homogeneous polynomial $\widetilde{g}(X_0, \dots, X_n) = X_0^{\deg g} g\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$, and then construct the field $F(\widetilde{g})$ as above. But we have $\widetilde{g}(1, X_1, \dots, X_n) = g(X_1, \dots, X_n)$, and hence $F(\widetilde{g}) = F(g)$.

For a quadratic form φ of dimension $n+1$, the polynomial $\varphi(X_0, \dots, X_n)$ is reducible if and only if the nondefective part φ_{nd} of φ is either of the type $(0,1)$, or of the type $(1,0)$ and $\varphi_{\mathrm{nd}} \cong \mathbb{H}$, see [Ahm97]. If φ is an $(n+1)$-dimensional p-form, then the polynomial $\varphi(X_0, \dots, X_n)$ is reducible if and only if $\dim \varphi_{\mathrm{an}} = 1$, see [Hof04, Lemma 7.1].

We say that the form φ is *irreducible* if the polynomial $\varphi(X_0, \dots, X_n)$ is irreducible; in that case, we have $n \geq 1$ and we can define the function field $F(\varphi)$ as for irreducible homogeneous polynomials; we will see in Subsection 1.1.10 that this definition agrees with the function field of the projective scheme given by φ. If φ is reducible, then we define $F(\varphi) = F(X_1, \dots, X_j)$, where $j = \mathfrak{i}_{\mathrm{d}}(\varphi)$; in particular, $F(\mathbb{H}) = F$ and $F(\langle a \rangle) = F$ for any $a \in F^*$. This definition is consistent with the following lemma:

Lemma 1.17. *Let φ be a form over F. Then $F(\varphi) \simeq F(\varphi_{\mathrm{nd}})(Y_1, \dots, Y_j)$, where $j = \mathfrak{i}_{\mathrm{d}}(\varphi)$. In particular, the extension $F(\varphi)/F(\varphi_{\mathrm{nd}})$ is purely transcendental.*

Proof. Let $\dim \varphi_{\mathrm{nd}} = n$ and write $\boldsymbol{X} = (X_1, \dots, X_n)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_j)$. Then $\varphi(\boldsymbol{X}, \boldsymbol{Y}) = \varphi_{\mathrm{nd}}(\boldsymbol{X})$ as polynomials; hence, $F(\varphi) \simeq F(\varphi_{\mathrm{nd}})(\boldsymbol{Y})$. □

A slightly different situation occurs for nondefective forms; recall that only nonsingular and semisingular quadratic forms can be nondefective and isotropic at once.

Lemma 1.18 ([EKM08, Prop. 22.9])**.** *Let φ be a nondefective quadratic form. Then the field extension $F(\varphi)/F$ is purely transcendental if and only if φ is isotropic.*

Note that it follows from the definition of a function field that $\varphi_{F(\varphi)}$ is isotropic for any form φ over F with $\dim \varphi_{\mathrm{nd}} > 1$.

The following lemma follows directly from the definitions.

Lemma 1.19. *Let φ be a form (or, more generally, an irreducible polynomial) over F, and let $c \in F^*$. Then $F(c\varphi) = F(\varphi)$.*

Let us look at function fields of anisotropic binary forms; with the lemma above, we can assume that $\varphi \cong \langle 1, a \rangle$ or $\varphi \cong [1, a]$ for some $a \in F$.

First, let $\varphi \cong \langle 1, a \rangle$ be a p-form. In order for φ to be anisotropic, we must have $a \notin F^p$. Then

$$F(\langle 1, a \rangle) = \mathrm{Quot}\left(F[X]/(1 + aX^p)\right) = F[X]/(1 + aX^p) \simeq F(\sqrt[p]{a}).$$

Let $\varphi \cong [1, a]$, so in particular char $F = 2$ for now. Recall that this form is anisotropic if and only if $a \notin \wp(F)$. For such a, we have

$$F([1,a]) = \mathrm{Quot}\left(F[X]/(X^2 + X + a)\right) = F[X]/(X^2+X+a) \simeq F(\wp^{-1}(a)),$$

where under $\wp^{-1}(a)$, we understand an element α from an algebraic closure $\overline{F}$ of F such that $\wp(\alpha) = a$.

More generally, one can prove the following lemma:

Lemma 1.20. *Let φ be an irreducible form. Then $F(\varphi)$ can be realized as a purely transcendental extension K/F of transcendence degree* $\dim \varphi - 2$*, followed by an algebraic extension L/K of degree p.*

(i) *If φ is a semisingular or a nonsingular quadratic form, then L/K is separable; in particular, if we have $\varphi \cong a_0[1, b_0] \perp \ldots \perp a_r[1, b_r]$ with $a_0, \ldots, a_r \neq 0$, then*

$$F(\varphi) \cong F(X_1, Y_1, \ldots, X_r, Y_r)\left(\wp^{-1}(\alpha)\right)$$

where

$$\alpha = a_0^{-1}\left(a_1(X_1^2 + X_1Y_1 + b_1Y_1^2) + \cdots + a_r(X_r^2 + X_rY_r + b_rY_r^2)\right) + b_0.$$

(ii) *If φ is a p-form, then L/K is purely inseparable; in particular, if $\varphi \cong \langle c_0, \ldots, c_s\rangle$ with $c_0 \neq 0$, then*

$$F(\varphi) \cong F(X_2, \ldots, X_s)\left(\sqrt[p]{c_0^{-1}(c_1 + c_2X_2^p + \cdots + c_sX_s^p)}\right).$$

Note that if ψ is a quadratic form over F which is not totally singular, then by Lemma 1.20, the extension $F(\psi)/F$ is separable. By Lemma 1.14, any totally singular form anisotropic over F remains anisotropic over $F(\psi)$. Thus, we get the following lemma.

Lemma 1.21 ([Lag02, Cor. 3.3])**.** *Let φ, ψ be anisotropic quadratic forms over F. If φ is totally singular and ψ is not totally singular, then $\varphi_{F(\psi)}$ is anisotropic.*

Finally, the following lemma will be useful when dealing with more than one form.

Lemma 1.22 ([EKM08, Ex. 22.3], [Scu16b, Sec. 7.2])**.** *Let φ, ψ be forms over F. Then $F(\varphi)(\psi_{F(\varphi)}) \simeq F(\psi)(\varphi_{F(\psi)})$.*

1.1.6 Sets and groups generated by the represented elements

For a form φ on V over F, we denote

$$D_F(\varphi) = \{\varphi(v) \mid v \in V\},$$

i.e., $D_F(\varphi)$ is the set of all elements of F represented by φ (including zero). Furthermore, we set $D_F^*(\varphi) = D_F(\varphi) \setminus \{0\}$. If E/F is a field extension, then we write $D_E(\varphi)$ (resp. $D_E^*(\varphi)$) for short instead of $D_E(\varphi_E)$ (resp. $D_E^*(\varphi_E)$). Note that $D_F(\varphi) = D_F(\varphi_{\mathrm{nd}})$ and $D_F^*(\varphi) = D_F^*(\varphi_{\mathrm{nd}})$.

Let $\varphi \cong \langle a_1, \dots, a_n \rangle$ be a p-form over F. Recall that F^p is a field; then the set

$$D_F(\varphi) = \{a_1 x_1^p + \dots + a_n x_n^p \mid x_i \in F, 1 \le i \le n\}$$

is a vector space over F^p generated by $\{a_1, \dots, a_n\}$. It follows that

$$\langle a, b \rangle \cong \langle a, a+b \rangle$$

for any $a, b \in F$; this is one of the standard relations we have seen. In particular, as $-1 = (-1)^p$, we have

$$\langle a, a \rangle \cong \langle a, -a \rangle \cong \langle a, 0 \rangle,$$

and hence the form $\langle a, a \rangle$ is isotropic for any $a \in F$. It is actually easy to see that φ is anisotropic if and only if $\dim \varphi = \dim_{F^p} D_F(\varphi)$. More precisely, we have the following lemma:

Lemma 1.23 ([Hof04, Prop. 2.6]). *Let φ be a p-form over F, and let $\{c_1, \dots, c_k\}$ be any F^p-basis of the vector space $D_F(\varphi)$. Then we have $\varphi_{\mathrm{an}} \cong \langle c_1, \dots, c_k \rangle$.*

For $k \ge 1$ and forms $\varphi_1, \dots, \varphi_k$ over F, we define

$$D_F(\varphi_1) \cdots D_F(\varphi_k) = \left\{ \prod_{i=1}^k a_i \;\middle|\; a_i \in D_F(\varphi_i), 1 \le i \le k \right\}.$$

If $\varphi_1 = \dots = \varphi_k = \varphi$, then we write $D_F(\varphi)^k$ for short; analogously for $D_F^*(\varphi_1) \cdots D_F^*(\varphi_k)$ and $D_F^*(\varphi)^k$. In particular,

$$D_F^*(\varphi)^k = \left\{ \prod_{i=1}^k a_i \;\middle|\; a_i \in D_F^*(\varphi), 1 \le i \le k \right\}.$$

Finally, by $\langle D_F^*(\varphi)^k \rangle$, we denote the multiplicative subgroup of F^* generated by $D_F^*(\varphi)^k$, i.e.,

$$\langle D_F^*(\varphi)^k \rangle = \left\{ \prod_{i=1}^n b_i \;\middle|\; n \ge 0, b_1, \dots, b_n \in D_F^*(\varphi)^k \right\}.$$

1.1.7 Pfister forms and Pfister neighbors

Let $n \ge 0$ and $a_1, \dots, a_n \in F^*$, $b \in F$. We define:

- the *n-fold bilinear Pfister form* for $n = 0$ as $\langle 1 \rangle_b$, and for $n > 0$ by

$$\langle\langle a_1, \dots, a_n \rangle\rangle_b = \langle 1, a_1 \rangle_b \otimes \dots \otimes \langle 1, a_n \rangle_b;$$

- the $(n+1)$*-fold quadratic Pfister form* by

$$\langle\langle a_1, \dots, a_n; b]] = \langle\langle a_1, \dots, a_n \rangle\rangle_b \otimes [1, b];$$

- the *n-fold quasi-Pfister form* for $n = 0$ as $\langle 1\rangle$, and for $n > 0$ by

$$\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, a_1, a_1^2, \dots, a_1^{p-1}\rangle \otimes \cdots \otimes \langle 1, a_n, a_n^2, \dots, a_n^{p-1}\rangle.$$

We will use *Pfister form* as a common name for all three kinds. Note that, in all cases, the dimension of an n-fold Pfister form is equal to p^n.

A form φ over F is called a *(quadratic/quasi-) Pfister neighbor* if there exists a (quadratic/quasi-) Pfister form π over F such that $c\varphi \preccurlyeq \pi$ for some $c \in F^*$ and $\dim\varphi > \frac{1}{p}\dim\pi$. If such a form π is anisotropic or nonsingular, then it is determined uniquely up to isometry (see [EKM08, Rem. 23.11] and [Hof04, Prop. 4.14]).

Lemma 1.24. *Let V be a vector space over F and π be a Pfister form on V. If E/F is an extension such that π_E is isotropic, then there exists a totally isotropic subspace $U \subseteq V_E$ such that $\dim U \geq \frac{p-1}{p}\dim V$.*

Proof. Bilinear Pfister forms: [EKM08, Cor. 6.3]; quadratic Pfister forms: [EKM08, Cor. 9.10]; quasi-Pfister forms: [Hof04, Cor. 4.10]. □

Corollary 1.25. *A quadratic Pfister form is either anisotropic or hyperbolic.*

Lemma 1.26. *Let π be a Pfister form over F and φ a Pfister neighbor of π. Then:*
(i) *$\varphi_{F(\pi)}$ is isotropic;*
(ii) *for any field extension E/F, φ_E is isotropic if and only if π_E is isotropic.*

Proof. Since $\pi_{F(\pi)}$ is isotropic, (i) is a consequence of (ii). To prove (ii), assume that $\dim\pi = p^n$, let V be the underlying vector space of π, W the one of φ, and let $c \in F^*$ be such that $c\varphi \preccurlyeq \pi$. Obviously, if φ_E is isotropic, then π_E is also isotropic. Thus, assume that π_E is isotropic. By Lemma 1.24, there exists a subspace $U \subseteq V_E$ such that $\dim U \geq (p-1)p^{n-1}$ and $\pi_E(u) = 0$ for any $u \in U$. Since φ is a Pfister neighbor of π, we have $\dim W > p^{n-1}$. For dimension reasons, $U \cap W_E \neq \{0\}$. Therefore, there exists a nonzero vector $w \in U \cap W_E$, and so $\varphi_E(w) = 0$. It follows that φ_E is isotropic. □

1.1.8 Excellent forms

Let E/F be a field extension and φ be a form over E. We say that φ is *defined over F* if there exists a form ψ over F such that $\psi_E \cong \varphi$.

Let ψ be a form over F; we say that ψ is *excellent* if, for any field extension E/F, the form $(\psi_E)_{\mathrm{an}}$ is defined over F.

Lemma 1.27. *All p-forms are excellent.*

Proof. Let $\varphi \cong \langle a_1, \dots, a_n\rangle$ be a p-form over F, and consider a field extension E/F. Then $D_E(\varphi)$ is a vector space over E^p generated by the set $\{a_1, \dots, a_n\}$, and hence we can pick $i_1, \dots, i_k \in \{1, \dots, n\}$ such that $\{a_{i_1}, \dots, a_{i_k}\}$ is a basis of $D_E(\varphi)$. By Lemma 1.23, $(\varphi_E)_{\mathrm{an}} \cong \langle a_{i_1}, \dots, a_{i_k}\rangle_E$, and hence it is defined over F. □

Lemma 1.28. *Quadratic Pfister forms are excellent.*

Proof. Let π be a quadratic Pfister form over F and E/F a field extension. By Corollary 1.25, π_E is either anisotropic or hyperbolic; in both cases, $(\pi_E)_{\mathrm{an}}$ is defined over F. $\square$

1.1.9 Splitting patterns

For a form φ over F, we define the *standard* (sometimes also called *Knebusch*) *splitting tower* of φ as follows: We put $F_0 = F$ and $\varphi_0 \cong \varphi_{\mathrm{an}}$, and then inductively for $n \geq 1$, we set $F_n = F_{n-1}(\varphi_{n-1})$ and $\varphi_n \cong ((\varphi_{n-1})_{F_n})_{\mathrm{an}}$. The *standard height* $h(\varphi)$ of φ is the smallest integer h such that $\dim \varphi_h \leq 1$. For $1 \leq k \leq h(\varphi)$, we call the form φ_k the *k-th kernel form*. Moreover, we set $\mathfrak{i}_k(\varphi) = \mathfrak{i}_{\mathrm{t}}(\varphi_{F_k})$, and define the *standard splitting pattern* of φ as

$$\mathrm{sSP}(\varphi) = (\dim(\varphi_{F_0})_{\mathrm{an}}, \dim(\varphi_{F_1})_{\mathrm{an}}, \dots, \dim(\varphi_{F_{h(\varphi)}})_{\mathrm{an}}).$$

Proposition 1.29 ([HL04, Prop. 4.6])**.** *Let φ be an anisotropic quadratic form over F of type (r_0, s_0), and let (r_i, s_i) be the type of the i-th kernel form φ_i in the standard splitting tower $F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_h$ of φ with $h = h(\varphi)$. Let E/F be any field extension and (r, s) be the type of $(\varphi_E)_{\mathrm{an}}$. Then there exists $i \in \{0, \dots, h\}$ such that $r = r_i$.*

In other words, the standard splitting tower produces all possible Witt indices of the given form. But not so for the possible defects: For example, let $\operatorname{char} F = 2$ and $\varphi \cong \langle\langle a, b\rangle\rangle \perp \langle c\rangle$ be an anisotropic totally singular quadratic form over F. Note that it is a quasi-Pfister neighbor of the quasi-Pfister form $\langle\langle a, b, c\rangle\rangle$, and hence $\mathrm{sSP}(\varphi) = (5, 4, 2, 1)$ by [HL04, Th. 8.11]. On the other hand, we have $(\varphi_{F(\sqrt{a})})_{\mathrm{an}} \cong \langle\langle b\rangle\rangle \perp \langle c\rangle$, i.e., $\dim(\varphi_{F(\sqrt{a})})_{\mathrm{an}} = 3$.

As shown above, the standard splitting pattern does not carry all the information about the isotropy of a given form. Hence, we define the *full splitting pattern* of a form φ over F as

$$\mathrm{fSP}(\varphi) = \{\dim(\varphi_E)_{\mathrm{an}} \mid E/F \text{ a field extension}\}.$$

Note that by Proposition 1.29, the standard and full splitting pattern coincide for nonsingular quadratic forms.

1.1.10 Some algebraic geometry

Two out of the four equivalence relations we are going to define have their origin in algebraic geometry. Hence, in this subsection, we define a scheme given by a form and rational maps. But we avoid most of the details since we will translate the geometric conditions into algebraic ones and work almost exclusively with them. A much more detailed background on algebraic geometry can be found, e.g., in [Vak17] or [Har77].

Let φ be a form over F (or, more generally, a homogeneous polynomial) of dimension $n + 1$; let us view φ as a homogeneous polynomial $\varphi \in F[X_0, \dots, X_n]$. Set

$$S(\varphi) = F[X_0, \dots, X_n]/(\varphi);$$

we will apply the Proj construction on $S(\varphi)$: Since $S(\varphi)$ is a graded ring (with respect to the total degree), we can write $S(\varphi) = \bigoplus_{k\geq 0} S(\varphi)_k$; denote $S(\varphi)_+ = \bigoplus_{k>0} S(\varphi)_k$.

First, we define $\operatorname{Proj} S(\varphi)$ as a set by

$$\operatorname{Proj} S(\varphi) = \{P \mid P \subseteq S(\varphi) \text{ a homogeneous prime ideal s.t. } S(\varphi)_+ \not\subseteq P\}.$$

Second, we can define on $\operatorname{Proj} S(\varphi)$ a topology, called the *Zariski topology*: The closed sets are of the form

$$V(I) = \{P \in \operatorname{Proj} S(\varphi) \mid I \subseteq P\} \tag{1.7}$$

where $I \subseteq S(\varphi)$ is a homogeneous ideal.

Finally, to make $\operatorname{Proj} S(\varphi)$ into a scheme, we construct a *structure sheaf* on $\operatorname{Proj} S(\varphi)$: For an open subset $U \subseteq \operatorname{Proj} S(\varphi)$, we define the ring

$$\mathcal{O}_X(U) = \Big\{ f : U \to \bigcup_{P\in U} S(\varphi)_{(P)} \;\Big|\; \forall P \in U \text{ cond. (1) and (2) hold} \Big\}^1 \tag{1.8}$$

where

(1) $f(P) \in S(\varphi)_{(P)}$,

(2) there exists an open subset $V \subseteq U$ containing P, and $s, t \in S(\varphi)$ homogeneous of the same degree, such that for each $Q \in V$, we have $t \notin Q$ and $f(Q) = \frac{s}{t}$.

We have obtained the scheme

$$X_\varphi = \operatorname{Proj} S(\varphi);$$

we call X_φ a *quadric* if φ is a quadratic form, and a *quasilinear p-hypersurface* if φ is a p-form.

The following lemma explains why some quadratic forms are called singular, resp. totally singular.

Lemma 1.30 ([EKM08, Prop. 22.1])**.** *Let φ be a nonzero quadratic form of dimension at least 2. Then the quadric X_φ is smooth if and only if* $\dim \operatorname{rad} \mathfrak{b}_\varphi \leq 1$*, i.e., if and only if* $\dim \operatorname{ql}(\varphi) \leq 1$.

Example 1.31. (i) Suppose that $\varphi \cong (n+1) \times \langle 0 \rangle$ for some $n \geq 1$. Then, with the notation from the construction above, $S(\varphi) = F[X_0, \dots, X_n]$ and $X_\varphi \simeq \mathbb{P}^n$, the n-dimensional projective space.

(ii) Let $\varphi \cong \psi \perp n \times \langle 0 \rangle$ for some irreducible form ψ over F and $n \geq 0$. Then $S(\psi) = S(\varphi)[Y_1, \dots, Y_n]$, and $X_\varphi \simeq X_\psi \times \mathbb{P}^n$ (the fibre product of the schemes X_ψ and $\mathbb{P}^n$).

The scheme X_φ is called *irreducible* if it is irreducible as a topological space: If $X_\varphi = Y_1 \cup Y_2$ for some closed subsets $Y_1, Y_2 \subseteq X_\varphi$, then $X_\varphi = Y_i$ for some $i \in \{1, 2\}$.

If the scheme X_φ is irreducible, then it is equidimensional of dimension $\dim X_\varphi = \dim \varphi - 2$.

Note that for any irreducible form φ, the ring $\mathcal{O}_X(U)$ defined in (1.8) is reduced (contains no nonzero nilpotent elements) for each nonempty open subset $U \subseteq X_\varphi$; hence, the scheme X_φ is *reduced*. Moreover, a scheme is called *integral* if it is irreducible and reduced.

[1]Here, $S(\varphi)_{(P)} = \big\{ \frac{s}{t} \mid s, t \in S(\varphi) \text{ homogeneous of the same degree, } t \notin P \big\}$.

Lemma 1.32 ([EKM08, Sec. 22] and [Scu13, Sec. 4]). *The form φ is irreducible if and only if X_φ is an integral scheme.*

Proof of the "if" part. Suppose that φ is reducible, and write $\varphi = fg$ for some polynomials $f, g \in F[X_0, \ldots, X_n]$; necessarily, f and g must be homogeneous. Consider the closed sets $V(f) = V((f))$, $V(g) = V((g))$ defined as in (1.7). Obviously, it holds that $V(f) \cup V(g) \subseteq \operatorname{Proj} S(\varphi)$. On the other hand, for any $P \in \operatorname{Proj} S(\varphi)$, we have $(\varphi) \subseteq P$, i.e., $fg = \varphi \in P$. Since P is a prime ideal, we must have $f \in P$ or $g \in P$; thus, $P \in V(f) \cup V(g)$. Therefore, $X_\varphi = V(f) \cup V(g)$. It follows that for reducible φ, the scheme X_φ is also reducible. $\square$

Let φ, ψ be forms over the same field F with φ irreducible. A *rational map* between the schemes X_φ and X_ψ, denoted by $X_\varphi \dashrightarrow X_\psi$, is a morphism (of schemes) on a dense open set, with the following equivalence relation: If U, V are dense open subsets of X_φ and $f : U \to X_\psi$, $g : V \to X_\psi$ morphisms, then $(f : U \to X_\psi) \sim (g : V \to X_\psi)$ if there exists a dense open set $W \subseteq U \cap V$ such that $f|_W = g|_W$.

The rational map $X_\varphi \dashrightarrow X_\psi$ is *dominant* if for some representative $U \to X_\psi$, the image is dense in X_ψ.

Next to the rational map, we could define a *rational function* on X_φ; then the set of all rational functions on X_φ forms a ring, and if X_φ is integral, then this ring is actually a field, called the *function field* of X_φ and denoted by $F(X_\varphi)$. Since this field happens to be isomorphic to the function field $F(\varphi)$ defined in Subsection 1.1.5, we omit the definition of a rational function and set simply $F(X_\varphi) = F(\varphi)$.

If φ is an irreducible form over F, it can be shown that X_φ is a *projective F-variety*, and in particular, that X_φ is of *finite type* ([Vak17, Ex. 5.3.E(b)]). For those "nice" schemes, there is a correspondence between rational maps and homomorphisms of their function fields.

Lemma 1.33 ([Vak17, Ex. 6.5.B and Prop. 6.5.7]). *Let φ, ψ be irreducible forms over F. There exists a dominant rational map $X_\psi \dashrightarrow X_\varphi$ if and only if there exists an F-homomorphism $F(\varphi) \hookrightarrow F(\psi)$.*

As it is well known, we can translate the geometric condition on the existence of a rational map into an algebraic one. The translation will be crucial for this thesis.

Lemma 1.34 ([EKM08, Sec. 22.A]). *Let φ, ψ be irreducible forms over F. The form $\varphi_{F(\psi)}$ is isotropic if and only if there exists a rational map $X_\psi \dashrightarrow X_\varphi$.*

1.2 Equivalence relations

One of the goals of this thesis is to compare different equivalence relations of forms. We devote this section to their definitions and some basic properties.

1.2.1 Similarity

We have seen in Lemma 1.19 that for a form φ over F and $c \in F^*$, the function fields of φ and $c\varphi$ coincide. It is also clear that φ is isotropic if and only if $c\varphi$ is isotropic. These are two examples of situations, where we are interested in forms not up to isometry but only up to scalar multiples. This leads to the following definition.

Definition 1.35. Let φ, ψ be forms over F. We say that they are *similar* and write $\varphi \overset{\text{sim}}{\sim} \psi$ if there exists $c \in F^*$ such that $\varphi \cong c\psi$.

We start with an obvious lemma.

Lemma 1.36. *If $\varphi \overset{\text{sim}}{\sim} \psi$, then φ and ψ are of the same type; in particular,* $\dim \varphi = \dim \psi$.

Let $c \in F^*$. Looking back at (1.3), we get $c\langle 0\rangle \cong \langle 0\rangle$ and $c\mathbb{H} \cong \mathbb{H}$. With this information at hand, we can prove:

Lemma 1.37. *Let φ, ψ be quadratic forms of the same type or p-forms of the same dimension over F. Then $\varphi \overset{\text{sim}}{\sim} \psi$ if and only if $\varphi_{\text{an}} \overset{\text{sim}}{\sim} \psi_{\text{an}}$.*

Proof. If $\varphi \overset{\text{sim}}{\sim} \psi$, then also $\varphi_{\text{an}} \overset{\text{sim}}{\sim} \psi_{\text{an}}$ by Witt Decomposition Theorem.

We illustrate the proof of the opposite implication in the case of semisingular quadratic forms. Since $\varphi_{\text{an}} \overset{\text{sim}}{\sim} \psi_{\text{an}}$, the quadratic forms φ_{an} and ψ_{an} must be of the same type by Lemma 1.36. As φ and ψ are also of the same type, we get by Witt Decomposition Theorem that $\mathfrak{i}_{\text{W}}(\varphi) = \mathfrak{i}_{\text{W}}(\psi)$ and $\mathfrak{i}_{\text{d}}(\varphi) = \mathfrak{i}_{\text{d}}(\psi)$. Let $c \in F^*$ is such that $\varphi_{\text{an}} \cong c\psi_{\text{an}}$. Since $c\mathbb{H} \cong \mathbb{H}$ and $c\langle 0\rangle \cong \langle 0\rangle$, it follows that $\varphi \cong c\psi$. □

The forms $\mathbb{H}$ and $\langle 0\rangle$ are not the only ones that can "absorb" some scalars; for example, let $a \in F^*$, then

$$a\langle 1, a\rangle \cong \langle a, a^2\rangle \cong \langle a, 1\rangle \cong \langle 1, a\rangle.$$

This example leads us to the following definition.

Definition 1.38. Let φ be a form over F. We call $x \in F^*$ a *similarity factor* if $x\varphi \cong \varphi$. The set of all similarity factors of φ over F is denoted by $G_F^*(\varphi)$. Furthermore, we set $G_F(\varphi) = G_F^*(\varphi) \cup \{0\}$.

Lemma 1.39. *Let φ be a form over F. Then $G_F^*(\varphi)$ is a multiplicative group.*

Proof. Obviously $1 \in G_F^*(\varphi)$. Let $x, y \in G_F^*(\varphi)$; then

$$(xy)\varphi \cong x(y\varphi) \cong x\varphi \cong \varphi,$$

and hence $xy \in G_F^*(\varphi)$. Furthermore,

$$\varphi \cong x^{-1}x\varphi \cong x^{-1}\varphi;$$

thus, $x^{-1} \in G_F^*(\varphi)$. □

The proof of the following lemma is trivial.

Lemma 1.40. *Let φ, ψ be forms over F. If $\varphi \overset{\text{sim}}{\sim} \psi$, then $G^*_F(\varphi) = G^*_F(\psi)$.*

A form φ over F is called *round* if $D^*_F(\varphi) = G^*_F(\varphi)$.

Lemma 1.41 ([EKM08, Cor. 9.9] and [Hof04, Prop. 4.6])**.**
(i) *Quadratic Pfister forms are round.*
(ii) *A p-form φ is round if and only if φ_{an} is isometric to a quasi-Pfister form.*

Lemma 1.42. *Let π be a (quadratic/quasi-) Pfister form over F.*
(i) *If $c \in F^*$ is such that $1 \in D^*_F(c\pi)$, then $c \in G^*_F(\pi)$.*
(ii) *Let φ be a quadratic form over F such that $\varphi \overset{\text{sim}}{\sim} \pi$. If $1 \in D^*_F(\varphi)$, then $\varphi \cong \pi$.*

Proof. (i) If $1 \in D_F(c\pi)$, then $c^{-1} \in D^*_F(\pi) = G^*_F(\pi)$ (where the equality follows from Lemma 1.41). Since $G^*_F(\pi)$ is a group by Lemma 1.39, it follows that $c \in G^*_F(\pi)$.

(ii) Let $x \in F^*$ be such that $x\varphi \cong \pi$. Then $x \in D^*_F(x\varphi) = D^*_F(\pi)$, and hence $x \in G^*_F(\pi)$ by Lemma 1.41. Then $\varphi \cong x\pi \cong \pi$. □

1.2.2 Birational equivalence

The birational equivalence is the first one with geometric flavor.

Definition 1.43. Let φ, ψ be irreducible forms.
(i) A rational map $f : X_\varphi \dashrightarrow X_\psi$ is said to be *birational* if it is dominant, and there exists a dominant rational map $g : X_\psi \dashrightarrow X_\varphi$ such that $g \circ f$ is in the same equivalence class as the identity on X_φ and $f \circ g$ is in the same equivalence class as the identity on X_ψ.
(ii) The schemes X_φ and X_ψ are *birational*, denoted by $X_\varphi \overset{\text{bir}}{\sim} X_\psi$, if there exists a birational map $X_\varphi \dashrightarrow X_\psi$.
(iii) The forms φ, ψ are called *birationally equivalent*, denoted by $\varphi \overset{\text{bir}}{\sim} \psi$, if X_φ and X_ψ are birational.

Let us translate the definition into more algebraic terms.

Proposition 1.44. *Let φ, ψ be irreducible forms over F. Then $\varphi \overset{\text{bir}}{\sim} \psi$ if and only if $F(\varphi) \simeq_F F(\psi)$.*

Idea of the proof. Essentially Lemma 1.33 but a concrete dependence of the map $F(\varphi) \hookrightarrow F(\psi)$ on the rational map $X_\psi \dashrightarrow X_\varphi$ is needed. □

Lemma 1.45. *Let φ, ψ be irreducible forms over F. If $\varphi \overset{\text{bir}}{\sim} \psi$, then $\dim \varphi = \dim \psi$.*

Proof. If $\varphi \overset{\text{bir}}{\sim} \psi$, then we have $F(\varphi) \simeq_F F(\psi)$ by Proposition 1.44; in particular, we have $\operatorname{trdeg}_F F(\varphi) = \operatorname{trdeg}_F F(\psi)$. Invoking Lemma 1.20, we get $\dim \varphi = \dim \psi$. □

To avoid case distinctions, we extend the definition of the birational equivalence to reducible forms.

Definition 1.46. Let φ, ψ be forms over the same field F (possibly reducible). We say that they are *birationally equivalent*, denoted by $\varphi \overset{\text{bir}}{\sim} \psi$, if $\dim \varphi = \dim \psi$ and $F(\varphi) \simeq_F F(\psi)$.

By Proposition 1.44 and Lemma 1.45, the Definitions 1.43 and 1.46 are equivalent for irreducible forms. For reducible forms, we have set $F(\mathbb{H}) = F$ and $F(\langle a \rangle) = F$ but the requirement on the dimension guarantees that $\mathbb{H} \overset{\text{bir}}{\nsim} \langle a \rangle$ for any $a \in F$.

Having defined two equivalence relations, we can start with their comparison.

Lemma 1.47. *If $\varphi \overset{\text{sim}}{\sim} \psi$, then $\varphi \overset{\text{bir}}{\sim} \psi$.*

Proof. Let $c \in F^*$ be such that $\varphi \cong c\psi$. Then $\dim \varphi = \dim \psi$, and by Lemma 1.19, $F(\varphi) = F(\psi)$. Hence, $\varphi \overset{\text{bir}}{\sim} \psi$. $\square$

The other implication does not hold in general, as we will see later for p-forms in Remark 2.73.

1.2.3 Stable birational equivalence

Also the stable birational equivalence has its origin in algebraic geometry.

Definition 1.48. Let φ, ψ be irreducible forms over F.

(i) The schemes X_φ and X_ψ are called *stably birational* if there exist $m, n \geq 0$ such that $X_\varphi \times \mathbb{P}^n \overset{\text{bir}}{\sim} X_\psi \times \mathbb{P}^m$. We denote this fact by $X_\varphi \overset{\text{stb}}{\sim} X_\psi$.

(ii) We say that the forms φ and ψ are *stably birationally equivalent*, denoted by $\varphi \overset{\text{stb}}{\sim} \psi$, if the schemes X_φ and X_ψ are stably birational.

As a direct consequence of the definitions, we get:

Lemma 1.49. *Let φ, ψ be irreducible forms. If $\varphi \overset{\text{bir}}{\sim} \psi$, then $\varphi \overset{\text{stb}}{\sim} \psi$.*

The opposite direction, that is, the question of whether stable birationally equivalent forms are birationally equivalent, is, in general, a difficult problem; in the case of quadratic forms and under the additional assumption that the forms are of the same dimension, this is called the *Quadratic Zariski problem.*

The Lemma 1.49 together with Lemma 1.47 gives the following:

Lemma 1.50. *Let φ, ψ be irreducible forms. If $\varphi \overset{\text{sim}}{\sim} \psi$, then $\varphi \overset{\text{stb}}{\sim} \psi$.*

Similarly as for the birational equivalence, we translate the geometric definition into algebraic terms.

Proposition 1.51. *Let φ, ψ be irreducible forms over F. Then $\varphi \overset{\text{stb}}{\sim} \psi$ if and only if $F(\varphi)(T_1, \dots, T_n) \simeq_F F(\psi)(T_1, \dots, T_m)$ for some $m, n \geq 0$.*

Proof. We have $\varphi \overset{\text{stb}}{\sim} \psi$ if and only if $X_\varphi \times \mathbb{P}^n \overset{\text{bir}}{\sim} X_\psi \times \mathbb{P}^m$ for some $m, n \geq 0$ if and only if $F(X_\varphi \times \mathbb{P}^n) \simeq_F F(X_\psi \times \mathbb{P}^m)$ for some $m, n \geq 0$. Since it holds $F(X_\varphi \times \mathbb{P}^n) \simeq_F F(\varphi)(T_1, \dots, T_n)$ and $F(X_\psi \times \mathbb{P}^m) \simeq_F F(\psi)(T_1, \dots, T_m)$, the claim follows. $\square$

Let $\varphi \cong \varphi_{\text{nd}} \perp n \times \langle 0 \rangle$ be an irreducible form over F; then, by Lemma 1.17, $F(\varphi) \simeq_F F(\varphi_{\text{nd}})(T_1, \dots, T_n)$. Invoking Proposition 1.51, we get:

Lemma 1.52 ([Scu16b, Rem. 4.2(ii)] for p-forms)**.** *Let φ be an irreducible form over F. Then $\varphi \overset{\text{stb}}{\sim} \varphi_{\text{nd}}$.*

The previous lemma allows us to produce an easy example showing that stably birationally equivalent forms are not necessarily birationally equivalent: Consider an irreducible isotropic p-form φ; then $\varphi \overset{\text{stb}}{\sim} \varphi_{\text{an}}$ but $\varphi \overset{\text{bir}}{\not\sim} \varphi_{\text{an}}$ by Lemma 1.45, because $\dim \varphi_{\text{an}} < \dim \varphi$.

Lemma 1.53. *Let φ, ψ be irreducible forms. If $\varphi \overset{\text{stb}}{\sim} \psi$, then $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic.*

Proof. We have $F(\varphi)(T_1, \dots, T_n) \simeq_F F(\psi)(T_1, \dots, T_m)$ for some $m, n \geq 0$ by Proposition 1.51. Since φ is isotropic over $F(\varphi)$, it is also isotropic over $F(\varphi)(T_1, \dots, T_n)$, and hence over $F(\psi)(T_1, \dots, T_m)$. Since the field extension $F(\psi)(T_1, \dots, T_m)/F(\psi)$ is purely transcendental, it follows from Lemma 1.13 that φ must be isotropic over $F(\psi)$. The proof of the isotropy of $\psi_{F(\varphi)}$ is analogous. $\square$

Proposition 1.54. *Let φ, ψ be nondefective irreducible quadratic forms over F. Then the following are equivalent:*

(i) *$\varphi \overset{\text{stb}}{\sim} \psi$;*
(ii) *there exist $m, n \geq 0$ such that $F(\varphi)(T_1, \dots, T_n) \simeq_F F(\psi)(T_1, \dots, T_m)$;*
(iii) *$\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic;*
(iv) *there exist dominant rational maps $X_\varphi \dashrightarrow X_\psi$ and $X_\psi \dashrightarrow X_\varphi$.*

Proof. (i) $\Leftrightarrow$ (ii) is covered by Proposition 1.51. (iii) $\Leftrightarrow$ (iv) is an application of Lemma 1.34. (i) $\Rightarrow$ (iii) follows from Lemma 1.53.

Finally, we show (iii) $\Rightarrow$ (i). If φ were totally singular and ψ not, then $\varphi_{F(\psi)}$ would be anisotropic by Lemma 1.21, a contradiction. Vice versa, if ψ were totally singular and φ not, then $\psi_{F(\varphi)}$ would be anisotropic, a contradiction again. Therefore, we have two possibilities: Either both φ and ψ are totally singular, or both φ and ψ are not totally singular. In the former case, we are done by [Tot08, Theorem 6.5]. Therefore, consider the latter case. Then, again by Lemma 1.21, $\text{ql}(\varphi)$ is anisotropic over $F(\psi)$, and hence we must have $\mathfrak{i}_{\text{W}}(\varphi_{F(\psi)}) > 0$; analogously, $\mathfrak{i}_{\text{W}}(\psi_{F(\varphi)}) > 0$. Denote by $K = F(\varphi)(\psi_{F(\varphi)})$; by Lemma 1.22, we have $K \simeq F(\psi)(\varphi_{F(\psi)})$. It follows that both $K/F(\varphi)$ and $K/F(\psi)$ are purely transcendental extensions, i.e.,

$$F(\varphi)(T_1, \dots, T_n) \simeq_F K \simeq_F F(\psi)(T_1, \dots, T_m)$$

for some $m, n \geq 0$. Thus, (ii) holds, and (i) follows by the previous part of the proof. $\square$

We provide a counterexample to implication (iii) $\Rightarrow$ (i) in the case of defective quadratic forms.

Example 1.55. Let τ be an anisotropic totally singular quadratic form over F of dimension 2^n for some $n \geq 1$, and set $\varphi \cong \tau \perp \langle 0 \rangle$ and $\psi \cong \tau \perp \langle T \rangle$ over the rational function field $E = F(T)$. As φ is isotropic over F, it is also isotropic over $E(\psi)$. Moreover, since ψ is obviously isotropic over $E(\tau)$ and $E(\varphi)$ is a purely transcendental extension of $E(\tau)$, it follows that ψ is isotropic over $E(\varphi)$.

Suppose that $\varphi \overset{\text{stb}}{\sim} \psi$; as we have $\varphi \overset{\text{stb}}{\sim} \tau$ by Lemma 1.52, we must have $\psi \overset{\text{stb}}{\sim} \tau$. Hence, $\tau_{E(\psi)}$ is isotropic by Lemma 1.53; but that contradicts the Separation theorem (see Theorem 4.6). Therefore, φ is not stably birational to ψ.

Remark 1.56. The implication (iii) $\Rightarrow$ (i) of Proposition 1.54 also holds for nondefective p-forms in the case of $p = 3$; see [Scu13, Theorem 7.7]. However, it is not known whether this implication holds for $p > 3$.

1.2.4 Vishik equivalence

At this point, one could expect the *motivic equivalence*. But as it is not available in our case, we take its algebraic characterization from the case of characteristic other than two, called *Vishik's criterion* (see [Hof15]), and adjust it to quadratic forms in arbitrary characteristic, resp. to p-forms in characteristic p. Here we include characteristic zero because we will use it in Chapter 4.

Definition 1.57. Let φ, ψ be forms over a field F of any characteristic (including zero). We say that they are *Vishik equivalent* and write $\varphi \overset{v}{\sim} \psi$ if $\dim \varphi = \dim \psi$ and the following holds:

$$\mathfrak{i}_{\mathrm{W}}(\varphi_E) = \mathfrak{i}_{\mathrm{W}}(\psi_E) \quad \& \quad \mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E) \quad \forall\ E/F. \tag{v}$$

At the first glance, it may seem that the equality of the dimensions follows from the conditions (v), but it is not necessarily so: Consider the forms $\varphi \cong \mathbb{H} \perp \langle 1, 0 \rangle$ and $\psi \cong \mathbb{H} \perp \langle 0 \rangle$. Then ($v$) is fulfilled but obviously $\dim \varphi \neq \dim \psi$.

Since we are working with different kinds of forms, and the important information is carried by another part of the definition for each of them, we provide a table with an overview, see Table 1.1. We also include a translation from isotropy indices to the dimension (or type when necessary) of the anisotropic part.

Remark 1.58. Instead of (v), we could have used for the definition of the Vishik equivalence the condition

$$\mathfrak{i}_{\mathrm{t}}(\varphi_E) = \mathfrak{i}_{\mathrm{t}}(\psi_E) \qquad \forall\ E/F, \tag{v'}$$

i.e., the equality of the dimensions of maximal totally isotropy subspaces over any field. But we would have to strengthen the assumption on the

char F	Form	Condition on the isotropy $\forall\, E/F$	Condition on the dimension $\forall\, E/F$
$p>0$	Quasilinear p-forms	$\mathfrak{i}_{\mathrm{d}}(\varphi_E)=\mathfrak{i}_{\mathrm{d}}(\psi_E)$	$\dim(\varphi_E)_{\mathrm{an}}=\dim(\psi_E)_{\mathrm{an}}$
2	Totally singular quadratic forms	$\mathfrak{i}_{\mathrm{d}}(\varphi_E)=\mathfrak{i}_{\mathrm{d}}(\psi_E)$	$\dim(\varphi_E)_{\mathrm{an}}=\dim(\psi_E)_{\mathrm{an}}$
2	Semisingular quadratic forms	$\mathfrak{i}_{\mathrm{W}}(\varphi_E)=\mathfrak{i}_{\mathrm{W}}(\psi_E)$ & $\mathfrak{i}_{\mathrm{d}}(\varphi_E)=\mathfrak{i}_{\mathrm{d}}(\psi_E)$	$(\varphi_E)_{\mathrm{an}}$ and $(\psi_E)_{\mathrm{an}}$ are of the same type
2	Nonsingular quadratic forms	$\mathfrak{i}_{\mathrm{W}}(\varphi_E)=\mathfrak{i}_{\mathrm{W}}(\psi_E)$	$\dim(\varphi_E)_{\mathrm{an}}=\dim(\psi_E)_{\mathrm{an}}$
$\neq 2$	Quadratic forms	$\mathfrak{i}_{\mathrm{W}}(\varphi_E)=\mathfrak{i}_{\mathrm{W}}(\psi_E)$	$\dim(\varphi_E)_{\mathrm{an}}=\dim(\psi_E)_{\mathrm{an}}$

Table 1.1: Conditions on $\varphi \overset{v}{\sim} \psi$, assuming $\dim\varphi=\dim\psi$.

equality of dimensions to the equality of types: For $\varphi\cong\mathbb{H}$ and $\psi\cong\langle 1,0\rangle$, we have $\dim\varphi=\dim\psi$ and $\mathfrak{i}_{\mathrm{t}}(\varphi_E)=\mathfrak{i}_{\mathrm{t}}(\psi_E)$ for any field E, but $\varphi\overset{v}{\not\sim}\psi$.

Of course, if we consider only nonsingular quadratic forms or only quasilinear p-forms (including totally singular quadratic forms), then there is no difference between (v) and (v'). So, let us focus on semisingular quadratic forms. We know from Lemma 1.14 that the defect does not change over separable field extension while such extension can change the Witt index. It means that we can compute the defect (and hence also the Witt index) from the knowledge of the total isotropy index: Let φ be a semisingular form over F of type (r,s). Let E/F be a field extension, and denote E_s the separable closure of E. Then $(\varphi_{E_s})_{\mathrm{an}}$ is totally singular by Lemma 1.16, and we know that $\mathfrak{i}_{\mathrm{d}}(\varphi_{E_s})=\mathfrak{i}_{\mathrm{d}}(\varphi_E)$. Thus, we get

$$\mathfrak{i}_{\mathrm{t}}(\varphi_{E_s})=\mathfrak{i}_{\mathrm{W}}(\varphi_{E_s})+\mathfrak{i}_{\mathrm{d}}(\varphi_{E_s})=r+\mathfrak{i}_{\mathrm{d}}(\varphi_E),$$

and then of course $\mathfrak{i}_{\mathrm{W}}(\varphi_E)=\mathfrak{i}_{\mathrm{t}}(\varphi_E)-\mathfrak{i}_{\mathrm{d}}(\varphi_E)$. Therefore, under the assumption that φ and ψ are of the same type, the conditions (v) and (v') are equivalent even for semisingular quadratic forms.

Lemma 1.59. *Let φ, ψ be forms over F such that $\varphi\overset{v}{\sim}\psi$, and E/F a field extension. Then $(\varphi_E)_{\mathrm{an}}\overset{v}{\sim}(\psi_E)_{\mathrm{an}}$ and $(\varphi_E)_{\mathrm{nd}}\overset{v}{\sim}(\psi_E)_{\mathrm{nd}}$. In particular, $\varphi_{\mathrm{an}}\overset{v}{\sim}\psi_{\mathrm{an}}$ and $\varphi_{\mathrm{nd}}\overset{v}{\sim}\psi_{\mathrm{nd}}$.*

Proof. Denote $\varphi'=(\varphi_E)_{\mathrm{nd}}$ and $\psi'=(\psi_E)_{\mathrm{nd}}$. Let K/E be another field extension. It holds that $\mathfrak{i}_{\mathrm{d}}(\varphi_K)=\mathfrak{i}_{\mathrm{d}}(\psi_K)$ and $\mathfrak{i}_{\mathrm{d}}(\varphi_E)=\mathfrak{i}_{\mathrm{d}}(\psi_E)$. Thus,

$$\mathfrak{i}_{\mathrm{d}}(\varphi'_K)=\mathfrak{i}_{\mathrm{d}}(\varphi_K)-\mathfrak{i}_{\mathrm{d}}(\varphi_E)=\mathfrak{i}_{\mathrm{d}}(\psi_K)-\mathfrak{i}_{\mathrm{d}}(\psi_E)=\mathfrak{i}_{\mathrm{d}}(\psi'_K).$$

Moreover,

$$\mathfrak{i}_{\mathrm{W}}(\varphi'_K) = \mathfrak{i}_{\mathrm{W}}(\varphi_K) = \mathfrak{i}_{\mathrm{W}}(\psi_K) = \mathfrak{i}_{\mathrm{W}}(\psi'_K).$$

Thus, $\varphi' \overset{v}{\sim} \psi'$. The proof of $(\varphi_E)_{\mathrm{an}} \overset{v}{\sim} (\psi_E)_{\mathrm{an}}$ is analogous. □

Since similar forms must be of the same dimension and have the same Witt indices and defects over any field, we get:

Lemma 1.60. *If $\varphi \overset{\mathrm{sim}}{\sim} \psi$, then $\varphi \overset{v}{\sim} \psi$.*

Among others, the Vishik equivalence implies that a form is isotropic over the function field of any Vishik equivalent form. Recall that this condition is closely connected to the stable birational equivalence.

Lemma 1.61. *Let φ, ψ be forms over F of dimension at least two. If $\varphi \overset{v}{\sim} \psi$, then $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic.*

Proof. Since $\varphi_{F(\varphi)}$ is isotropic and we know that $\mathfrak{i}_{\mathrm{t}}(\psi_{F(\varphi)}) = \mathfrak{i}_{\mathrm{t}}(\varphi_{F(\varphi)})$, we get that $\psi_{F(\varphi)}$ is isotropic. Symmetrically, $\varphi_{F(\psi)}$ is isotropic. □

Now we can apply the previous lemma on quadratic forms to see that the Vishik equivalence is stronger than the stable birational equivalence:

Lemma 1.62. *Let φ, ψ be quadratic forms over F. If $\varphi \overset{v}{\sim} \psi$, then $\varphi \overset{\mathrm{stb}}{\sim} \psi$.*

Proof. If one of the forms is reducible, then the other one must be reducible, too, and the equality of the Witt indices and the defects implies that $\varphi \overset{\mathrm{sim}}{\sim} \psi$. Therefore, assume that φ and ψ are irreducible. By Lemma 1.59, we have $\varphi_{\mathrm{nd}} \overset{v}{\sim} \psi_{\mathrm{nd}}$. By Lemma 1.61, we get that $(\psi_{\mathrm{nd}})_{F(\varphi_{\mathrm{nd}})}$ and $(\varphi_{\mathrm{nd}})_{F(\psi_{\mathrm{nd}})}$ are isotropic. Therefore, $\varphi_{\mathrm{nd}} \overset{\mathrm{stb}}{\sim} \psi_{\mathrm{nd}}$ by Proposition 1.54. Invoking Lemma 1.52, we get $\varphi \overset{\mathrm{stb}}{\sim} \psi$. □

Remark 1.63 (Vishik equivalence on bilinear forms)**.** We could define the Vishik equivalence on bilinear forms analogously as in the case of quadratic forms. As $D_F(\mathfrak{b}) = D_F(\varphi_{\mathfrak{b}})$ for any bilinear form $\mathfrak{b}$ defined over F, so asking about the isotropy of $\mathfrak{b}$ is the same as asking about the isotropy of its associated quadratic form $\varphi_{\mathfrak{b}}$.

So, let $F = \mathbb{F}_2(a, b)$ with a, b algebraically independent over $\mathbb{F}_2$, and consider the bilinear forms $\mathfrak{b}_1 \cong \langle a, b\rangle_b$ and $\mathfrak{b}_2 \cong \langle a, a+b\rangle_b$. Note that $\varphi_{\mathfrak{b}_1} \cong \langle a, b\rangle \cong \langle a, a+b\rangle \cong \varphi_{\mathfrak{b}_2}$, and so, for any field extension E/F, $(\varphi_{\mathfrak{b}_1})_E$ is isotropic if and only if $(\varphi_{\mathfrak{b}_2})_E$ is isotropic. Therefore, $(\mathfrak{b}_1)_E$ is isotropic if and only if $(\mathfrak{b}_2)_E$ is isotropic. Since the only possible values of the Witt index of a two-dimensional bilinear form are zero and one, it follows that $\mathfrak{b}_1 \overset{v}{\sim} \mathfrak{b}_2$. But by comparing the determinants, we get that $\mathfrak{b}_1 \overset{\mathrm{sim}}{\not\sim} \mathfrak{b}_2$.

1.2.5 Summary

So far, we have proved the following relations between the equivalences:

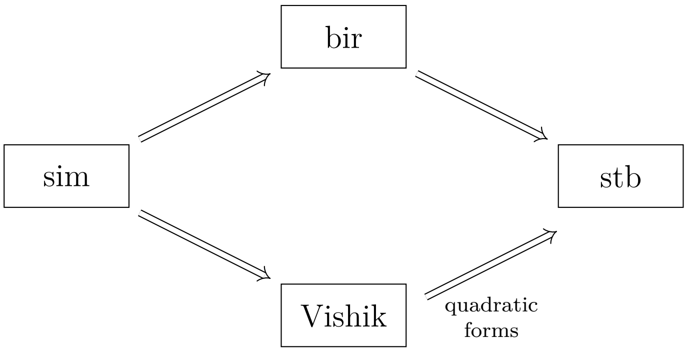

Figure 1.1: Trivial relations between the equivalences

2. Quasilinear p-forms

All the fields in this chapter are of characteristic $p > 0$ (including the possibility $p = 2$) and all the forms are quasilinear p-forms which have been defined in Subsection 1.1.3.

We devote this chapter to the p-forms. Some of the statements presented here will not be used until the next chapter, which will treat the special case of $p = 2$; however, we prefer to provide them in their full strength. In Section 2.1, we recall many known properties of p-forms and add a few new ones. In particular, we define a new class of p-forms, the so-called minimal p-forms (see Subsection 2.1.3).

Section 2.2 diverts a bit from the main goal of the thesis, and examines isotropy properties of p-forms, mainly over purely inseparable field extensions. We would like to point out Theorem 2.28, in which we show that the defect of a p-form φ over a purely separable field extension L/F equals the defect of φ over the maximal purely inseparable subfield K of L which has exponent one over F. Later in this section, we provide the full splitting pattern of a few families of p-forms; the most interesting and ingenious one is a special type of quasi-Pfister neighbors; see Theorem 2.42.

Section 2.3 looks at the birational and the stable birational equivalence; it mostly just summarizes known results from [Scu13].

In Section 2.4, we examine the Vishik equivalence and, above all, its relation to the similarity. In particular, we provide examples of p-forms, which are Vishik equivalent but not similar; see Examples 2.65 and 2.66. However, we prove in Proposition 2.68 that Vishik equivalent p-forms have the same similarity factors, which will allow us in the next chapter to reduce the problem to forms that are not divisible by any nontrivial quasi-Pfister form. Moreover, in Example 2.72, we provide a pair of p-forms which are birationally equivalent but not Vishik equivalent; this, in its consequence, rejects three of the unknown implications from Figure 1.1.

Finally, in Section 2.4, we summarize what we know about the relations between the four studied equivalences, and by doing so, we complete the diagram from Figure 1.1.

2.1 Preliminaries

As we have already mentioned before, the peculiar property of a field F of characteristic p is that F^p is a field, too. Recall that for a p-form φ on V over F, the set $D_F(\varphi)$ is a vector space over F^p. Conversely, any finite-dimensional vector space over F^p inside F corresponds to a unique (up to isometry) anisotropic p-form over F (see [Hof04, Prop. 2.12]). Moreover, the set $W = \{v \in V \mid \varphi(v) = 0\}$ is also an F^p-vector space, because for any $v, w \in W$, we have $\varphi(v + w) = \varphi(v) + \varphi(w) = 0$; it follows that W is the unique maximal isotropic subspace of V.

As a consequence of Lemma 1.23, we get:

Lemma 2.1. *Let φ be a p-form over F. If $a_1, \dots, a_m \in D_F(\varphi)$, then $\langle a_1, \dots, a_m\rangle_{\mathrm{an}} \subseteq \varphi$.*

The following lemma is just a reinterpretation of Lemma 1.27.

Lemma 2.2. *Let $\tau = \langle a_1, \dots, a_n\rangle$ be a p-form defined over F and E/F a field extension. Then there exist $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$ such that $(\tau_E)_{\mathrm{an}} \cong \langle a_{i_1}, \dots, a_{i_m}\rangle_E$. In this situation, $a_k \in \mathrm{span}_{E^p}\{a_{i_1}, \dots, a_{i_m}\}$ for every $1 \leq k \leq n$.*

2.1.1 Field theory

Since F^p is a field, we can compare extensions of F and of F^p. We will use the following lemma in this chapter repeatedly.

Lemma 2.3 ([Hof04, Lemma 7.5])**.** *Let $\operatorname{char} F = p$.*

(i) *Let E/F be a field extension and $\alpha \in E$. Then α is algebraic over F if and only if α^p is algebraic over F^p. Moreover, it holds that*

$$[F(\alpha) : F] = [F^p(\alpha^p) : F^p].$$

(ii) *If $a_1, \dots, a_r \in F$, then*

$$[F(\sqrt[p]{a_1}, \dots, \sqrt[p]{a_r}) : F] = [F^p(a_1, \dots, a_r) : F^p].$$

We say about a finite set $\{a_1, \dots, a_n\} \subseteq F$ that it is *p-independent over F* if $[F^p(a_1, \dots, a_n) : F^p] = p^n$; otherwise, we call the set (or the elements of the set) *p-dependent over F*. We say that a set $S \subseteq F$ is *p-independent over F* if any finite subset of S is p-independent over F.

Lemma 2.4 ([Pic50, pg. 27])**.** *Let $a_1, \dots, a_n \in F$. The following are equivalent:*

(i) *The set $\{a_1, \dots, a_n\}$ is p-independent over F,*
(ii) *for any $1 \leq i \leq n$, we have $a_i \notin F^p(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$,*
(iii) *the system $\left(a_1^{i_1} \cdots a_n^{i_n} \;\middle|\; (i_1, \dots, i_n) \in \{0, \dots, p-1\}^n\right)$ is F^p-linearly independent.*

Example 2.5. (i) It is clear directly form the definition that $\{a\} \subseteq F$ is p-independent if and only if $a \notin F^p$.

(ii) Let $a, b \in F \setminus F^p$. Then, by Lemma 2.4, a, b are p-dependent if and only if $a \in F^p(b)$ if and only if $b \in F^p(a)$.

(iii) Let $X_1, \ldots, X_n$ be variables (i.e., algebraically independent) over F. Then $\{X_1, \ldots, X_n\}$ is p-independent not only over F (that is obvious), but also over $F(X_1, \ldots, X_n)$: Assume that

$$\sum_{i_1=0}^{p-1} \cdots \sum_{i_n=0}^{p-1} f_{i_1,\ldots,i_n}^p X_1^{i_1} \cdots X_n^{i_n} = 0 \tag{2.1}$$

for some $f_{i_1,\ldots,i_n} \in F(X_1, \ldots, X_n)$. We can assume $f_{i_1,\ldots,i_n} \in F[X_1, \ldots, X_n]$. Then we can rewrite (2.1) as

$$\sum_{i_1=0}^{p-1} \underbrace{\left(\sum_{i_2=0}^{p-1} \cdots \sum_{i_n=0}^{p-1} f_{i_1,\ldots,i_n}^p X_2^{i_1} \cdots X_n^{i_n} \right)}_{g_{i_1}} X_1^{i_1} = 0;$$

note that $g_{i_1} \in F(X_1, \ldots, X_n)^p[X_2, \ldots, X_n]$. Considering the expression in (2.1) as a polynomial in X_1 over $F(X_1)^p(X_2, \ldots, X_n)$, we conclude that $g_{i_1} = 0$ for each $0 \leq i_1 \leq p-1$. Iterating this process, we get $f_{i_1,\ldots,i_n}^p = 0$ for each $0 \leq i_j \leq p-1$, $1 \leq j \leq n$. The claim follows by Lemma 2.4.

Corollary 2.6. *Let $a_1, \ldots, a_n \in F^*$. The set $\{a_1, \ldots, a_n\}$ is p-independent over F if and only if the quasi-Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ is anisotropic over F.*

Proof. First of all, note that

$$\langle\langle a_1, \ldots, a_n \rangle\rangle \cong \bigperp_{(i_1,\ldots,i_n) \in \{0,\ldots,p-1\}^n} \langle a_1^{i_1} \cdots a_n^{i_n} \rangle.$$

Now $\{a_1, \ldots, a_n\}$ is p-independent over F if and only if (by Lemma 2.4) the system $(a_1^{i_1} \cdots a_n^{i_n} \mid (i_1, \ldots, i_n) \in \{0, \ldots, p-1\}^n)$ is F^p-linearly independent if and only if $\dim_{F^p} D_F(\langle\langle a_1, \ldots, a_n \rangle\rangle) = p^n$ if and only if $\langle\langle a_1, \ldots, a_n \rangle\rangle$ is anisotropic over F. □

Let E be a field with $F^p \subseteq E \subseteq F$. Note that the extension E/F^p is purely inseparable of exponent one. We call a set $\mathcal{B} \subseteq E$ a *p-basis of E over F* if $\mathcal{B}$ is p-independent over F and $F^p(\mathcal{B}) = E$.

Lemma 2.7 ([GS06, Cor. A.8.9])**.** *Let E be a field with $F^p \subseteq E \subseteq F$. Then the following hold:*

(i) *There exists a p-basis of E over F.*
(ii) *Assume that E/F^p is finite and $\mathcal{B}$ is a p-basis of E over F. Then $|\mathcal{B}| = \log_p[E : F^p]$.*
(iii) *If $\{a_1, \ldots, a_n\} \subseteq E$ is p-independent over F, then there exists a set $\mathcal{A} \subseteq E$ which is p-independent over F and such that $\{a_1, \ldots, a_n\} \cup \mathcal{A}$ is a p-basis of E over F.*
(iv) *Let $\mathcal{A} \subseteq E$ be such that $F^p(\mathcal{A}) = E$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ which is a p-basis of E over F.*

Proof. Parts (i), (iii), and (iv) are easy exercises on Zorn's lemma: We seek a maximal element of $\mathcal{S}$, where:

(i) $\mathcal{S} = \{A \mid A \subseteq E,\ A\ p\text{-independent over } F\}$,

(iii) $\mathcal{S} = \{A \mid \{a_1, \dots, a_n\} \subseteq A \subseteq E,\ A\ p\text{-independent over } F\}$,

(iv) $\mathcal{S} = \{A \mid A \subseteq \mathcal{A},\ A\ p\text{-independent over } F\}$.

To prove (ii), note that for any $\{a_1, \dots, a_n\} \subseteq E$, we have $F^p \subseteq F^p(a_1, \dots, a_n) \subseteq E$. Therefore, any subset of E which is p-independent over F can have at most $\log_p([E : F^p])$ elements. The rest follows. $\square$

Let E be as above and $\mathcal{B} = \{b_i \mid i \in I\}$ be a p-basis of E over F. Then, by Lemma 2.4,

$$\widehat{\mathcal{B}} = \left\{ \prod_{i \in I} b_i^{\lambda(i)} \;\middle|\; \lambda : I \to \{0, \dots, p-1\}, \lambda(i) = 0 \text{ for almost all } i \in I \right\}$$

is an F^p-linear basis of E, i.e., any $a \in E$ can be expressed uniquely as

$$a = \sum_{\lambda} x_\lambda^p \prod_{i \in I} b_i^{\lambda(i)}$$

for some $x_\lambda \in F$, almost all of them zero. By abuse of notation, we say that *a can be expressed uniquely with respect to* $\mathcal{B}$.

2.1.2 Quasi-Pfister forms and norm forms

Let φ be a p-form over F. We define the *norm field* of φ over F as the field

$$N_F(\varphi) = F^p\left(\frac{a}{b} \;\middle|\; a, b \in D_F^*(\varphi)\right).$$

Note that $N_F(\varphi)$ is a finite field extension of F^p; we define the *norm degree* of φ over F as $\operatorname{ndeg}_F \varphi = [N_F(\varphi) : F^p]$.

It is obvious from the definition that $N_F(\varphi) = N_F(\varphi_{\mathrm{an}})$ and also that $N_F(\varphi) = N_F(c\varphi)$ for any $c \in F^*$. Moreover, if ψ is another p-form over F such that $\psi \cong \varphi$, then $N_F(\psi) = N_F(\varphi)$. Finally, if $\tau_{\mathrm{an}} \subseteq c\varphi_{\mathrm{an}}$ for some p-form τ over F and $c \in F^*$, then $N_F(\tau) \subseteq N_F(\varphi)$.

Note that the norm degree is always a p-power. Let us assume that $N_F(\varphi) = F^p(b_1, \dots, b_n)$ for some $b_i \in F$; then $\{b_1, \dots, b_n\}$ is p-independent over F if and only if $\operatorname{ndeg}_F \varphi = p^n$. By Lemma 2.7, such a p-basis of $N_F(\varphi)$ over F always exists.

Lemma 2.8 ([Hof04, Lemma 4.2 and Cor. 4.3])**.** *Let φ be a p-form over F.*
(i) *If $\varphi \cong \langle a_0, \dots, a_n \rangle$ for some $n \geq 1$ and $a_i \in F$, $0 \leq i \leq n$, with $a_0 \neq 0$, then $N_F(\varphi) = F^p\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right)$.*
(ii) *Suppose $N_F(\varphi) = F^p(b_1, \dots, b_m)$ for some $b_i \in F^*$, $1 \leq i \leq m$, and let E/F be a field extension. Then $N_E(\varphi) = E^p(b_1, \dots, b_m)$.*

Recall that, by Lemma 2.7, any p-generating set contains a p-basis. Therefore, part (i) of the previous lemma implies the following:

Corollary 2.9. *Let $\varphi \cong \langle a_0, \dots, a_n \rangle$ for some $n \geq 1$ and $a_0, \dots, a_n \in F$ with $a_0 \neq 0$. Moreover, suppose that $\operatorname{ndeg}_F \varphi = p^k$. Then there exists a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ such that $\left\{\frac{a_{i_1}}{a_0}, \dots, \frac{a_{i_k}}{a_0}\right\}$ is a p-basis of $N_F(\varphi)$ over F.*

In particular, if $1 \in D_F^*(\varphi)$, then there exist $b_1, \dots, b_n \in F$ such that $\varphi \cong \langle 1, b_1, \dots, b_n \rangle$ and $N_F(\varphi) = F^p(b_1, \dots, b_k)$.

The following proposition provides an alternative possibility for a determination of the norm field.

Proposition 2.10. *Let $\varphi = \langle 1, a_1, \dots, a_n \rangle$ be a p-form over F, and let $b_1, \dots, b_m \in F$. Then*

$$\left(\varphi_{F(\sqrt[p]{b_1}, \dots, \sqrt[p]{b_m})}\right)_{\mathrm{an}} \cong \langle 1 \rangle \quad \iff \quad N_F(\varphi) \subseteq F^p(b_1, \dots, b_m).$$

If, moreover, $\{b_1, \dots, b_m\} \subseteq \{a_1, \dots, a_n\}$, then

$$\left(\varphi_{F(\sqrt[p]{b_1}, \dots, \sqrt[p]{b_m})}\right)_{\mathrm{an}} \cong \langle 1 \rangle \quad \iff \quad N_F(\varphi) = F^p(b_1, \dots, b_m).$$

In particular, if m is minimal with this property, then $\operatorname{ndeg}_F \varphi = p^m$.

Proof. We set $E = F\left(\sqrt[p]{b_1}, \dots, \sqrt[p]{b_m}\right)$.

First, $(\varphi_E)_{\mathrm{an}} \cong \langle 1 \rangle$ is equivalent to $\operatorname{span}_{E^p}\{1\} = \operatorname{span}_{E^p}\{1, a_1, \dots, a_n\}$, which holds if and only if $a_i \in E^p$ for all $1 \leq i \leq n$. As $E^p = F^p(b_1, \dots, b_m)$, the latter condition is equivalent to $F^p(a_1, \dots, a_n) \subseteq F^p(b_1, \dots, b_m)$. But $F^p(a_1, \dots, a_n) = N_F(\varphi)$, so we are done.

If $\{b_1, \dots, b_m\} \subseteq \{a_1, \dots, a_n\}$, then $F^p(b_1, \dots, b_m) \subseteq N_F(\varphi)$, and the claim follows by the previous case. $\square$

Note that it follows from Lemma 2.8 that $D_F(\varphi) \subseteq cN_F(\varphi)$ for any $c \in D_F^*(\varphi)$; in particular, if $1 \in D_F(\varphi)$, then $D_F(\varphi) \subseteq N_F(\varphi)$. The forms, for which an equality holds, are precisely the quasi-Pfister forms:

Proposition 2.11 ([Hof04, Prop. 4.6])**.** *Let F be a field.*

(i) *Let $\varphi \cong \langle\langle a_1, \dots, a_n \rangle\rangle$ be a quasi-Pfister form over F. Then we have $D_F(\varphi) = F^p(a_1, \dots, a_n) = N_F(\varphi)$.*

(ii) *There is a natural bijection between anisotropic n-fold quasi-Pfister forms over F and purely inseparable field extensions E of F^p inside F with $[E : F^p] = p^n$. This bijection is given by*

$$\langle\langle a_1, \dots, a_n \rangle\rangle \leftrightarrow F^p(a_1, \dots, a_n).$$

Combining the previous proposition with Lemma 2.7, we get an interesting statement about changing "slots" in a quasi-Pfister form.

Corollary 2.12. *Let $\pi \cong \langle\langle a_1, \dots, a_n \rangle\rangle$ be an anisotropic quasi-Pfister form over F. If $\{b_1, \dots, b_m\}$ is a p-basis of $F^p(a_1, \dots, a_n)$, then we have $m = n$ and $\pi \cong \langle\langle b_1, \dots, b_m \rangle\rangle$. In particular, if $x \in D_F(\pi) \setminus F^p$, then there exist $b_2, \dots, b_n \in D_F^*(\pi)$ such that $\pi \cong \langle\langle x, b_2, \dots, b_n \rangle\rangle$.*

Roughly said, any p-form is contained in its norm field, and this norm field corresponds to a quasi-Pfister form. Hence, any (now necessarily anisotropic) p-form φ is contained in an anisotropic quasi-Pfister form of dimension $\operatorname{ndeg}_F \varphi$; in particular $\dim \varphi \leq \operatorname{ndeg}_F \varphi$. It is also easy to see that if the norm form of a p-form is generated over F^p by n elements, then the dimension of this p-form must be at least $n+1$.

Proposition 2.13 ([Hof04, Prop. 4.8])**.** *Let φ be a nonzero p-form with $\operatorname{ndeg}_F \varphi = p^n$.*

(i) *It holds $n+1 \leq \dim \varphi_{\mathrm{an}} \leq p^n$.*

(ii) *Let $a_1, \dots, a_n \in F^*$ be such that $N_F(\varphi) = F^p(a_1, \dots, a_n)$, and let $c \in D_F^*(\varphi)$. Then $\varphi_{\mathrm{an}} \subseteq c\langle\langle a_1, \dots, a_n\rangle\rangle$. Furthermore, if there is $m \leq n$ and $b_1, \dots, b_m \in F^*$ such that $\varphi_{\mathrm{an}} \subseteq c\langle\langle b_1, \dots, b_m\rangle\rangle$, then $m = n$ and $\langle\langle a_1, \dots, a_n\rangle\rangle \cong \langle\langle b_1, \dots, b_m\rangle\rangle$; moreover, this quasi-Pfister form is anisotropic.*

Let φ be a p-form over F with $\operatorname{ndeg}_F \varphi = p^n$ and $N_F(\varphi) = F^p(a_1, \dots, a_n)$ (so, in particular, $\{a_1, \dots, a_n\}$ is p-independent over F). Then we define the *norm form* of φ over F, denoted by $\hat{\nu}_F(\varphi)$, as the quasi-Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$. Note that if φ is a quasi-Pfister form itself, then we have $\hat{\nu}_F(\varphi) \cong \varphi_{\mathrm{an}}$. In general, by Proposition 2.13, we have $\varphi_{\mathrm{an}} \subseteq c\,\hat{\nu}_F(\varphi)$ for any $c \in D_F^*(\varphi)$.

Lemma 2.14. *Let φ be a p-form.*

(i) *It holds that $\hat{\nu}_F(\varphi) \cong \hat{\nu}_F(\varphi_{\mathrm{an}})$.*

(ii) *Let E/F be a field extension. Then $\hat{\nu}_E(\varphi) \cong (\hat{\nu}_F(\varphi)_E)_{\mathrm{an}}$.*

Proof. Part (i) follows from the fact that $N_F(\varphi) = N_F(\varphi_{\mathrm{an}})$. For (ii), see [Scu16b, Lemma 3.2]. □

Norm fields and norm forms can be used to characterize anisotropic quasi-Pfister forms.

Proposition 2.15. *Let φ be a p-form over F. Then the following are equivalent:*

(i) *φ_{an} is a quasi-Pfister form;*

(ii) *$D_F(\varphi) = G_F(\varphi)$;*

(iii) *$D_F(\varphi) = N_F(\varphi)$;*

(iv) *$\hat{\nu}_F(\varphi) \cong \varphi_{\mathrm{an}}$;*

(v) $\operatorname{ndeg}_F \varphi = \dim \varphi_{\mathrm{an}}$ *and* $1 \in D_F^*(\varphi)$.

Proof. (i) ⇔ (ii) follows by Lemma 1.41, and (i) ⇔ (iv) is trivial.

(i) ⇒ (iii) follows from Proposition 2.11. For the opposite direction, let $\{a_1, \dots, a_n\}$ be p-independent over F such that $N_F(\varphi) = F^p(a_1, \dots, a_n)$. Then $a_1^{i_1} \cdots a_n^{i_n} \in D_F(\varphi)$ for any $(i_1, \dots, i_n) \in \{0, \dots, p-1\}^n$. Since these elements are linearly independent over F^p by Lemma 2.4, the form

$$\psi \cong \mathop{\perp}_{(i_1,\dots,i_n)\in\{0,\dots,p-1\}^n} \langle a_1^{i_1} \cdots a_n^{i_n}\rangle \cong \langle\langle a_1, \dots, a_n\rangle\rangle$$

is anisotropic over F, and hence $\psi \subseteq \varphi_{\mathrm{an}}$ by Lemma 2.1. Since we have $\dim \varphi_{\mathrm{an}} \leq \operatorname{ndeg}_F \varphi = \dim \psi$ by Proposition 2.13, we get $\psi \cong \varphi_{\mathrm{an}}$, and hence (iii) ⇒ (i).

The implication (iv) $\Rightarrow$ (v) follows from the facts $\dim \hat{\nu}_F(\varphi) = \mathrm{ndeg}_F\, \varphi$ and $1 \in D_F(\hat{\nu}_F(\varphi))$. To prove the opposite direction, recall that we have $\varphi_{\mathrm{an}} \subseteq c\hat{\nu}_F(\varphi)$ for any $c \in D_F^*(\varphi)$; hence, in particular, $\varphi_{\mathrm{an}} \subseteq \hat{\nu}_F(\varphi)$. Now we have $\dim \varphi_{\mathrm{an}} = \dim \hat{\nu}_F(\varphi)$ by the assumption, and so (iv) follows. $\square$

We can analogously characterize quasi-Pfister neighbors. To be able to say a bit more, we need to recall some notation from Subsection 1.1.9: Let φ be a p-form over F. Then F_k denotes the k-th field in the standard splitting tower of φ, and $\varphi_k \cong (\varphi_{F_k})_{\mathrm{an}}$ is the k-th kernel form. We write $\mathfrak{i}_k(\varphi)$ for $\mathfrak{i}_{\mathrm{t}}(\varphi_{F_k})$. Finally, $h(\varphi)$ is the height of the p-form φ, i.e., the smallest integer h for which $\dim(\varphi_{F_h})_{\mathrm{an}} \leq 1$.

Proposition 2.16 ([Scu16a, Cor. 3.11])**.** *Let φ be an anisotropic p-form of dimension ≥ 2 over F and let n be the smallest nonnegative integer such that $\dim \varphi \leq p^{n+1}$. Then the following are equivalent:*

(i) *φ is a quasi-Pfister neighbor,*
(ii) $\dim(\hat{\nu}_F(\varphi)) < p \dim \varphi$,
(iii) *$\varphi_{F(\hat{\nu}_F(\varphi))}$ is isotropic,*
(iv) $\mathrm{ndeg}\, \varphi = p^{n+1}$,
(v) $h(\varphi) = n + 1$,
(vi) *$i_1(\varphi) = \dim \varphi - p^n$ and $i_2(\varphi) = p^n - p^{n-1}$,*
(vii) $\mathrm{sSP}(\varphi) = (\dim \varphi, p^n, p^{n-1}, \dots, p, 1)$,
(viii) *φ_1 is similar to a quasi-Pfister form.*

Note in particular that if φ is a quasi-Pfister neighbor, then it is a quasi-Pfister neighbor of $\hat{\nu}_F(\varphi)$.

Let us now look at quasi-Pfister forms and norm forms over field extensions.

Proposition 2.17 ([Hof04, Prop. 5.2])**.** *Let φ be an anisotropic p-form over F with $\mathrm{ndeg}_F(\varphi) = p^n$, and let $N_F(\varphi) = F^p(a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in F^*$. Let E/F be any field extension such that the p-form φ_E is isotropic. Then $\mathrm{ndeg}_E(\varphi) = p^k < p^n$ and there exists a subset $\{a_{i_1}, \dots, a_{i_k}\} \subseteq \{a_1, \dots, a_n\}$ such that $N_E(\varphi) = E^p(a_{i_1}, \dots, a_{i_k})$. In particular, for each $c \in D_E^*(\varphi)$, we have $c(\varphi_E)_{\mathrm{an}} \subseteq \langle\langle a_{i_1}, \dots, a_{i_k}\rangle\rangle_E$.*

Combining Propositions 2.11 and 2.17, we get:

Lemma 2.18. *Let π be a quasi-Pfister form over F and E/F a field extension. Then $(\pi_E)_{\mathrm{an}}$ is a quasi-Pfister form.*

We can ask when an anisotropic quasi-Pfister form becomes isotropic after "adding a new slot".

Lemma 2.19. *Let φ be an anisotropic quasi-Pfister form or an anisotropic quasi-Pfister neighbor over F, and let $x \in F^*$. Then $\varphi \otimes \langle\langle x\rangle\rangle$ is isotropic if and only if $x \in N_F(\varphi)$.*

Proof. First, we prove the lemma in the case when $\varphi \cong \pi$ is a quasi-Pfister form. Assume that $\pi \cong \langle\langle a_1, \dots, a_n\rangle\rangle$, i.e., $N_F(\pi) = F^p(a_1, \dots, a_n)$. Then the form $\pi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $(\pi \otimes \langle\langle x \rangle\rangle)_{\mathrm{an}} \cong \pi$ if and only if $F^p(a_1, \dots, a_n) = F^p(a_1, \dots, a_n, x)$ if and only if $x \in N_F(\pi)$.

Now suppose that φ is a quasi-Pfister neighbor. Then, by Proposition 2.16, $c\varphi \subseteq \pi$ for $\pi \cong \hat{\nu}_F(\varphi)$ and some $c \in F^*$. It follows that $c\varphi \otimes \langle\langle x \rangle\rangle$ is a quasi-Pfister neighbor of $\pi \otimes \langle\langle x \rangle\rangle$, and hence $c\varphi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $\pi \otimes \langle\langle x \rangle\rangle$ is isotropic (Lemma 1.26). By the first part of the proof, the latter is equivalent to $x \in N_F(\pi)$. Since $N_F(\pi) = N_F(c\varphi) = N_F(\varphi)$, the claim follows. □

The above lemma cannot be generalized to a product of quasi-Pfister forms of higher norm degrees: Consider a p-independent set $\{a_1, a_2, x_1, x_2\}$ over F and the p-forms $\langle\langle a_1, a_2, a_1x_1 + a_2x_2 \rangle\rangle$ and $\langle\langle x_1, x_2 \rangle\rangle$. Then one can show that $F^p(a_1, a_2, a_1x_1 + a_2x_2) \cap F^p(x_1, x_2) = F^p$. But the form $\langle\langle a_1, a_2, a_1x_1 + a_2x_2 \rangle\rangle \otimes \langle\langle x_1, x_2 \rangle\rangle$ is isotropic by Corollary 2.6 since we have $F^p(a_1, a_2, a_1x_1 + a_2x_2, x_1, x_2) = F^p(a_1, a_2, x_1, x_2)$.

However, we can at least state a condition, under which a product of two anisotropic p-forms remains anisotropic.

Lemma 2.20. *Let φ, ψ be two anisotropic p-forms over F. If we have $\operatorname{ndeg}_F(\varphi \otimes \psi) = \operatorname{ndeg}_F \varphi \cdot \operatorname{ndeg}_F \psi$, then $\varphi \otimes \psi$ is anisotropic.*

Proof. We have $\varphi \otimes \psi \subseteq \hat{\nu}_F(\varphi) \otimes \hat{\nu}_F(\psi)$, where $\hat{\nu}_F(\varphi) \otimes \hat{\nu}_F(\psi)$ is a quasi-Pfister form. As $\hat{\nu}_F(\varphi \otimes \psi)$ is the smallest quasi-Pfister form containing $\varphi \otimes \psi$, we have $\hat{\nu}_F(\varphi \otimes \psi) \subseteq \hat{\nu}_F(\varphi) \otimes \hat{\nu}_F(\psi)$. By the assumption, we have $\dim \hat{\nu}_F(\varphi \otimes \psi) = \dim \hat{\nu}_F(\varphi) \cdot \dim \hat{\nu}_F(\psi)$; thus, $\hat{\nu}_F(\varphi \otimes \psi) \cong \hat{\nu}_F(\varphi) \otimes \hat{\nu}_F(\psi)$, and this form is in particular anisotropic. Therefore, its subform $\varphi \otimes \psi$ is anisotropic as well. □

Remark 2.21. The other implication in the previous lemma does not hold in general: Let $\{a, b, c\} \subseteq F^*$ be a p-independent set over F, and consider the p-forms $\varphi \cong \langle 1, a, b, c \rangle$ and $\psi \cong \langle 1, abc \rangle$. Then

$$\varphi \otimes \psi \cong \langle 1, a, b, c, abc, a^2bc, ab^2c, abc^2 \rangle \subseteq \langle\langle a, b, c \rangle\rangle,$$

and hence $\varphi \otimes \psi$ is anisotropic. But

$$\operatorname{ndeg}_F(\varphi \otimes \psi) = p^3 < p^3 \cdot p = \operatorname{ndeg}_F \varphi \cdot \operatorname{ndeg}_F \psi.$$

However, if φ is a quasi-Pfister form and ψ a quasi-Pfister neighbor, then the other implication holds as well: Since in this situation $\varphi \cong \hat{\nu}_F(\varphi)$ and $\dim \psi > \frac{1}{p} \dim \hat{\nu}_F(\psi)$, it follows that $\dim(\varphi \otimes \psi) > \frac{1}{p} \dim(\hat{\nu}_F(\varphi) \otimes \hat{\nu}_F(\psi))$, and hence $\varphi \otimes \psi$ is a quasi-Pfister neighbor of $\hat{\nu}_F(\varphi) \otimes \hat{\nu}_F(\psi)$. Then $\varphi \otimes \psi$ is anisotropic if and only if $\hat{\nu}_F(\varphi) \otimes \hat{\nu}_F(\psi)$ is anisotropic if and only if the union of the p-bases of $N_F(\varphi)$ and $N_F(\psi)$ over F is a p-independent set over F.

2.1.3 Minimal p-forms

In this subsection, we introduce a new class of p-forms, so-called minimal p-forms. These are the p-forms of minimal dimension with respect to their norm degree (cf. Proposition 2.13(i)):

Definition 2.22. Let φ be an anisotropic p-form over F. We call φ *minimal* over F if $\operatorname{ndeg}_F \varphi = p^{\dim \varphi - 1}$.

According to [Scu13, Th. 4.2], minimal p-forms of dimension at least two are exactly those p-forms φ, for which X_φ is a regular scheme.

First, we state some rather obvious properties of minimal p-forms.

Lemma 2.23. *Let φ, ψ be p-forms over F of dimension at least two.*

(i) *The p-form $\langle 1, a_1, \dots, a_n\rangle$ is minimal over F if and only if the set $\{a_1, \dots, a_n\}$ is p-independent over F.*
(ii) *Minimality is not invariant under field extensions.*
(iii) *If φ is minimal over F and $\psi \overset{\text{sim}}{\sim} \varphi$, then ψ is minimal over F.*
(iv) *If φ is minimal over F and $\psi \subseteq c\varphi$ for some $c \in F^*$, then ψ is minimal over F.*
(v) *If $\operatorname{ndeg}_F \varphi = p^k$ with $k \geq 1$, then φ contains a minimal subform of dimension $k+1$.*

Proof. (i) The p-form $\varphi = \langle 1, a_1, \dots, a_n\rangle$ is minimal over F if and only if $\operatorname{ndeg}_F(\varphi) = p^n$ if and only if $[F^p(a_1, \dots, a_n) : F^p] = p^n$ if and only if $\{a_1, \dots, a_n\}$ is p-independent over F (Corollary 2.6).

(ii) Let $\{a, b, c\} \subseteq F$ be a p-independent set over F. Let $E = F(\sqrt[p]{c})$ and consider $\varphi = \langle 1, a, b, abc\rangle$. Then $N_F(\varphi) = F^p(a, b, abc) = F^p(a, b, c)$, and hence φ is minimal over F. Nevertheless, $\varphi_E \cong \langle 1, a, b, ab\rangle$ is anisotropic but not minimal over E, because $\operatorname{ndeg}_E \varphi = p^2$.

(iii) This is a consequence of the fact that similar forms have the same dimension (Lemma 1.36) and the same norm field.

(iv) Invoking (iii), we can assume $\psi \subseteq \varphi$ and $1 \in D_F(\psi)$. So, there exist $a_1, \dots a_n \in F^*$ and some $m \leq n$ such that $\psi \cong \langle 1, a_1, \dots, a_m\rangle$ and $\varphi \cong \langle 1, a_1, \dots, a_n\rangle$; see Lemma 2.1. By (i), $\{a_1, \dots, a_n\}$ is p-independent over F; thus, $\{a_1, \dots, a_m\}$ has to be p-independent over F, and the claim follows by applying (i) again.

(v) Note that by Proposition 2.13, it follows from the assumption on k that $\dim \varphi_{\mathrm{an}} \geq 2$. There exists $c \in F^*$ such that $1 \in D_F^*(c\varphi_{\mathrm{an}})$; then, by Corollary 2.9, $c\varphi_{\mathrm{an}} \cong \langle 1, b_1, \dots, b_n\rangle$ for some $b_1, \dots, b_n \in F^*$ with $n \geq k$ and $\{b_1, \dots, b_k\}$ p-independent over F. It follows that $c^{-1}\langle 1, b_1, \dots, b_k\rangle$ is a subform of φ which is minimal over F by parts (i) and (iii). □

The following lemma characterizes elements represented by a given minimal p-form.

Lemma 2.24. *Let φ be a minimal p-form over F with $1 \in D_F(\varphi)$. Let $n \geq 2$ and $\{b_1, \dots, b_n\} \subseteq D_F(\varphi)$ be p-independent over F. Denote*

$$S = \Big\{\lambda : \{1, \dots, n\} \to \{0, \dots, p-1\}\Big\},$$

and consider

$$\beta = \sum_{\lambda \in S} x_\lambda^p \prod_{i=1}^{n} b_i^{\lambda(i)}$$

with $x_\lambda \in F$ for all $\lambda \in S$. Then the following are equivalent:

(i) *$\beta \in D_F(\varphi)$,*
(ii) *$x_\lambda = 0$ for all $\lambda \in S$ such that $\sum_{i=1}^n \lambda(i) \geq 2$, i.e., $\beta = y_0^p + \sum_{i=1}^n b_i y_i^p$ for some $y_i \in F$, $0 \leq i \leq n$.*

Proof. Write $\tau \cong \langle 1, b_1, \ldots, b_n \rangle$. Then (ii) holds if and only if $\beta \in D_F(\tau)$. Since $\tau \subseteq \varphi$ by the assumptions, the implication (ii) $\Rightarrow$ (i) is obvious.

To prove (i) $\Rightarrow$ (ii), suppose that (ii) does not hold (e.g., $\beta = b_1 b_2$). Then we can see that $\beta \notin D_F(\tau)$. Hence, the p-form $\tau \perp \langle \beta \rangle$ is anisotropic. On the other hand, $\beta \in F^p(b_1, \ldots, b_n)$, so $\tau \perp \langle \beta \rangle$ is not minimal over F by part (i) of Lemma 2.23. If $\beta \in D_F(\varphi)$, then $\tau \perp \langle \beta \rangle \subseteq \varphi$, in which case φ could not be minimal over F by part (iv) of the same lemma, which is a contradiction; therefore, $\beta \notin D_F(\varphi)$. $\square$

2.2 Isotropy and splitting patterns

The main goal of this section is to determine the full splitting pattern at least of some families of p-forms. On the way, we look at some properties of the isotropy of p-forms over field extensions in general.

2.2.1 Isotropy over purely inseparable field extensions

Recall that we define the *full splitting pattern* of a p-form φ over F as

$$\mathrm{fSP}(\varphi) = \{\dim(\varphi_E)_{\mathrm{an}} \mid E/F \text{ a field extension}\}.$$

Thus, the goal of this section is to look at isotropy properties of p-forms over different kinds of field extensions.

First of all, the purely transcendental extensions are not interesting at all from the point of view of isotropy by Lemma 1.13. Furthermore, we know from Lemma 1.14 that anisotropic p-forms also remain anisotropic over any separable field extension. But the situation gets more interesting when we consider purely inseparable field extensions. The easiest one is a simple purely inseparable extension of exponent one (that is, an extension E/F such that $E^p \subseteq F$); such an extension is of the form $F(\sqrt[p]{a})/F$ for some $a \in F \setminus F^p$. In the following lemma, we take a closer look at the behavior of p-forms over such extensions.

Lemma 2.25 ([Scu16a, Lemma 2.27])**.** *Let φ be a p-form over F and let $a \in F \setminus F^p$. Then:*

(i) $D_{F(\sqrt[p]{a})}(\varphi) = D_F(\langle\langle a\rangle\rangle \otimes \varphi) = \sum_{i=0}^{p-1} a^i D_F(\varphi)$.

(ii) $\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) = \frac{1}{p}\mathfrak{i}_{\mathrm{d}}(\langle\langle a\rangle\rangle \otimes \varphi)$.

(iii) $\operatorname{ndeg}(\varphi_{F(\sqrt[p]{a})}) = \begin{cases} \frac{1}{p}\operatorname{ndeg}\varphi & \text{if } a \in N_F(\varphi), \\ \operatorname{ndeg}\varphi & \text{if } a \notin N_F(\varphi). \end{cases}$

(iv) *If φ is anisotropic and $\varphi_{F(\sqrt[p]{a})}$ is isotropic, then $a \in N_F(\varphi)$.*

(v) $\dim(\varphi_{F(\sqrt[p]{a})})_{\mathrm{an}} \geq \frac{1}{p}\dim\varphi_{\mathrm{an}}$.

(vi) *Equality holds in (v) if and only if there exists a p-form γ over F such that $\varphi_{\mathrm{an}} \cong \langle\langle a\rangle\rangle \otimes \gamma$.*

We want to point out that $N_F(\varphi)$ is the smallest field extension of F^p with property (iv) of Lemma 2.25. More precisely, we have the following:

Proposition 2.26. *Let φ be an anisotropic p-form over F and E/F^p be a field extension such that the following holds for any $a \in F^*$:*

$$\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) > 0 \quad \Longrightarrow \quad a \in E. \tag{❋}$$

Then $N_F(\varphi) \subseteq E$.

Proof. Let $\operatorname{ndeg}_F\varphi = p^k$. Since neither the isotropy nor the norm field depends on the choice of the representative of the similarity class, we can assume $1 \in D_F^*(\varphi)$. Invoking Corollary 2.9, we find $b_1, \dots, b_n \in F$ with $n \geq k$ such that $\varphi \cong \langle 1, b_1, \dots, b_n\rangle$ and $\{b_1, \dots, b_k\}$ is a p-basis of $N_F(\varphi)$ over F. Since $\varphi_{F(\sqrt[p]{b_i})}$ is obviously isotropic for each $1 \leq i \leq k$, we get by the assumption (❋) that $b_1, \dots, b_k \in E$. Since E is a field containing F^p, it follows that $N_F(\varphi) = F^p(b_1, \dots, b_k) \subseteq E$. □

Non-simple purely inseparable extensions of exponent one, i.e., extensions of the form $F(\sqrt[p]{a_1},\dots,\sqrt[p]{a_r})/F$, are basically covered by repeated application of Lemma 2.25. So we are interested in fields of higher exponents, and how to compare them with an appropriate field of exponent one. The following is known.

Proposition 2.27 ([Hof04, Prop. 5.7]). *Let $r \geq 1$. For each $1 \leq i \leq r$, let $a_i \in F$, $n_i \geq 1$, and α_i, β_i be such that $\alpha_i^p = \beta_i^{p^{n_i}} = a_i$. Furthermore, set $K = F(\alpha_1,\dots,\alpha_r)$, $L = F(\beta_1,\dots,\beta_r)$, and assume that $[K:F] = p^r$. Then $[L:F] = p^{n_1+\dots+n_r}$ and the quasi-Pfister form $\pi = \langle\!\langle a_1,\dots,a_r\rangle\!\rangle$ is anisotropic.*

Let φ be an anisotropic p-form. Then the following statements are equivalent.

(i) *$\varphi \otimes \pi$ is isotropic,*
(ii) *φ_K is isotropic,*
(iii) *φ_L is isotropic.*

Note that the previous proposition does not cover all purely inseparable extensions of exponent higher than one; only those extensions L/F are considered, for which $L = F(\sqrt[p^{n_1}]{a_1},\dots,\sqrt[p^{n_r}]{a_r})$ with $\langle\!\langle a_1,\dots,a_r\rangle\!\rangle$ anisotropic (i.e., the set $\{a_1\dots,a_r\}$ is p-independent; this follows from the assumption $[K:F] = p^r$). Purely inseparable field extensions for which such elements $a_1,\dots,a_r$ can be found are called *modular*. A known example of a non-modular extension is E/F with $F = \mathbb{F}_2(a,b,c)$ for some algebraically independent elements a,b,c, and $E = F(\sqrt[4]{a},\sqrt[4]{b^2a+c^2})$, see [Wei65].

However, we can strengthen the results by comparing the values $\mathfrak{i}_{\mathrm{d}}(\varphi\otimes\pi)$, $\mathfrak{i}_{\mathrm{d}}(\varphi_K)$ and $\mathfrak{i}_{\mathrm{d}}(\varphi_L)$; it turns out that we can prove $\mathfrak{i}_{\mathrm{d}}(\varphi_K) = \mathfrak{i}_{\mathrm{d}}(\varphi_L)$ even if the extension L/F is not modular.

Theorem 2.28. *Let $r \geq 1$. For each $1 \leq i \leq r$, let $a_i \in F$, $n_i \geq 1$, and α_i, β_i be such that $\alpha_i^p = \beta_i^{p^{n_i}} = a_i$. Furthermore, set $K = F(\alpha_1,\dots,\alpha_r)$, $L = F(\beta_1,\dots,\beta_r)$ and $\pi = \langle\!\langle a_1,\dots,a_r\rangle\!\rangle$. Let φ be an anisotropic p-form. Then*

(i) $\mathfrak{i}_{\mathrm{d}}(\varphi_K) = \mathfrak{i}_{\mathrm{d}}(\varphi_L)$,
(ii) $\mathfrak{i}_{\mathrm{d}}(\varphi_K) = \frac{1}{p^r}\mathfrak{i}_{\mathrm{d}}(\varphi\otimes\pi)$ *if π is anisotropic.*

Proof. To prove (i), we proceed by induction on r. Let $r = 1$, and write $a = a_1$, $n = n_1$, $\alpha = \alpha_1$ and $\beta = \beta_1$. Let V be the vector space associated with φ, and write $V_K = K \otimes_F V$ and $V_L = L \otimes_F V \simeq L \otimes_K V_K$ as usual. Let $W_K = \{w \in V_K \mid \varphi_K(w) = 0\}$ (resp. $W_L = \{w \in V_L \mid \varphi_L(w) = 0\}$) be the maximal isotropic subspace of V_K (resp. V_L); then $\mathfrak{i}_{\mathrm{d}}(\varphi_K) = \dim_K W_K$ (resp. $\mathfrak{i}_{\mathrm{d}}(\varphi_L) = \dim_L W_L$). We will show that $W_L = L \otimes_K W_K$, which will not only justify the notation but also prove the claim because we have $\dim_L(L \otimes_K W_K) = \dim_K W_K$ by Lemma 1.9.

Let $w = \sum_{i=1}^k \gamma_i \otimes w_i \in L \otimes_K W_K$ for some $\gamma_i \in L$ and $w_i \in W_K$; then

$$\varphi_L(w) = \varphi_L\left(\sum_{i=1}^k \gamma_i \otimes w_i\right) = \sum_{i=1}^k \gamma_i^p \varphi_K(w_i) = 0.$$

Hence, $w \in W_L$, and so we get $L \otimes_K W_K \subseteq W_L$.

For a proof of the opposite inclusion, first note that $\varphi_K(v) \in F$ for any $v \in V_K$ by Lemma 2.25(i). Now, let $w \in W_L$. Since $\{1, \beta, \beta^2, \ldots, \beta^{p^{n-1}-1}\}$ is a basis of L over K, we can write

$$w = \sum_{i=0}^{p^{n-1}-1} \beta^i \otimes w_i$$

with $w_i \in V_K$. Then

$$0 = \varphi_L(w) = \sum_{i=0}^{p^{n-1}-1} \beta^{pi} \varphi_K(w_i).$$

Note that the set $B = \{1, \beta^p, \beta^{2p}, \ldots, \beta^{p^n-p}\}$ is a subset of $\{1, \beta, \ldots, \beta^{p^n-1}\}$, which is a basis of L over F; therefore, B is linearly independent over F. Since, as we noted, $\varphi_K(w_i) \in F$ for all the i's, it follows that $\varphi_K(w_i) = 0$. Thus, we have $w_i \in W_K$ for all i, and hence $w \in L \otimes_K W_K$. It follows that $W_L \subseteq L \otimes_K W_K$, which concludes the proof in the case $r = 1$.

Let $r > 1$; for $0 \leq i \leq r$, set

$$L_i = F(\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_r).$$

Then $L_0 = K$, $L_r = L$, and $\mathfrak{i}_{\mathrm{d}}(\varphi_{L_i}) = \mathfrak{i}_{\mathrm{d}}(\varphi_{L_{i+1}})$ for all $0 \leq i \leq r-1$ by the previous part of the proof. Therefore, $\mathfrak{i}_{\mathrm{d}}(\varphi_K) = \mathfrak{i}_{\mathrm{d}}(\varphi_L)$.

(ii) Set $K_i = F(\alpha_1, \ldots, \alpha_i)$ for $0 \leq i \leq r$. Since π is anisotropic by the assumption, we have $[K_{i+1} : K_i] = p$ for all $0 \leq i \leq r-1$. Now the equality $\mathfrak{i}_{\mathrm{d}}(\varphi \otimes \pi) = \frac{1}{p^r} \mathfrak{i}_{\mathrm{d}}(\varphi_K)$ follows by a repeated application of Lemma 2.25:

$$\begin{aligned}\mathfrak{i}_{\mathrm{d}}(\varphi_K) = \mathfrak{i}_{\mathrm{d}}(\varphi_{K_r}) = p\, \mathfrak{i}_{\mathrm{d}}((\langle\langle a_r \rangle\rangle \otimes \varphi)_{K_{r-1}}) = \ldots \\ = p^r \mathfrak{i}_{\mathrm{d}}((\langle\langle a_1, \ldots, a_r \rangle\rangle \otimes \varphi)_{K_0}) = p^r \mathfrak{i}_{\mathrm{d}}(\varphi \otimes \pi). \quad \square\end{aligned}$$

Next to the standard and full splitting pattern, we can define *purely inseparable splitting pattern* of a p-form φ over F as

$$\mathrm{piSP}(\varphi) = \{\dim(\varphi_L)_{\mathrm{an}} \mid L/F \text{ a purely inseparable extension}\}.$$

By the previous theorem, we only need to consider purely inseparable extensions of exponent one.

Corollary 2.29. *Let φ be a p-form over F. Then*

$$\mathrm{piSP}(\varphi) = \{\dim(\varphi_K)_{\mathrm{an}} \mid K/F \textit{ finite purely inseparable of exponent } 1\}.$$

Remark 2.30. We can restrict the field extensions necessary to determine $\mathrm{piSP}(\varphi)$ a bit more: Let φ be anisotropic with $\mathrm{ndeg}_F \varphi = p^n$ and $N_F(\varphi) = F^p(a_1, \ldots, a_n)$. Consider the field $K = F(\sqrt[p]{b_1}, \ldots, \sqrt[p]{b_r})$, write $\pi \cong \langle\langle b_1, \ldots, b_r \rangle\rangle$, and assume that π is anisotropic. Then we know from Proposition 2.27 that φ_K is isotropic if and only if $\varphi \otimes \pi$ is isotropic. By Lemma 2.20, this can happen only if $\mathrm{ndeg}_F(\varphi \otimes \pi) < \mathrm{ndeg}_F(\varphi) \cdot \mathrm{ndeg}_F(\pi)$, i.e., if the set $\{a_1, \ldots, a_n, b_1, \ldots, b_r\}$ is p-dependent. Thus, we have

$$\mathrm{piSP}(\varphi) = \{\dim(\varphi_K)_{\mathrm{an}} \mid [K^p(a_1, \ldots, a_n) : F^p] < p^n [K^p : F^p]\},$$

where K runs over purely inseparable extensions of F of exponent one.

This also indicates that a p-form may become isotropic even over a field E such that $E^p \cap N_F(\varphi) = F^p$. For example, consider a set $\{a_1, a_2, a_3, b_1\}$ p-independent over F, and write $\varphi \cong \langle 1, a_1, a_2, a_3\rangle$ and $E = F(\sqrt[p]{b_1}, \sqrt[p]{b_2})$ with $b_2 = \frac{a_1 b_1 + a_3}{a_2}$. It can be shown that indeed $E^p \cap N_F(\varphi) = F^p$. Moreover, $\langle a_1 b_1, a_2 b_2, a_3\rangle \subseteq \varphi \otimes \langle\langle b_1, b_2\rangle\rangle$, and since $a_3 = a_1 b_1 + a_2 b_2$, the form $\langle a_1 b_1, a_2 b_2, a_3\rangle$ is isotropic. Therefore, $\varphi \otimes \langle\langle b_1, b_2\rangle\rangle$ is isotropic, and so is φ_E.

Later in this section, we will compute the full splitting patterns of some families of p-forms using solely purely inseparable field extensions of exponent one. So, at this point, we would like to prove that $\mathrm{piSP}(\varphi) = \mathrm{fSP}(\varphi)$ for any p-form φ. However, that is not as straightforward as it may seem to be: It means that for any field extension E/F, we must find a purely inseparable extension K/F such that $\mathfrak{i}_\mathrm{d}(\varphi_E) = \mathfrak{i}_\mathrm{d}(\varphi_K)$. As a possible first step in this direction, we propose Conjecture 2.31.

For a finite extension K/F and $\alpha \in K$, we denote the norm of α over F by $\mathcal{N}_{K/F}(\alpha)$.

Conjecture 2.31. *Let φ be a p-form over F and E/F a separable field extension, $\alpha \in E \setminus E^p$ and $n \geq 1$. Denote $L = E(\sqrt[p^n]{\alpha})$ and $a = \mathcal{N}_{F(\alpha)/F}(\alpha)$. If φ_L is isotropic, then the following hold:*
(i) *$\langle\langle a\rangle\rangle \otimes \varphi$ is isotropic;*
(ii) *if $a \notin F^p$, then $\varphi_{F(\sqrt[p]{a})}$ is isotropic.*

2.2.2 Bounds on the size of $\mathrm{fSP}(\varphi)$

Putting aside the trivial case when φ_an is the zero form, it is obvious that $\mathrm{fSP}(\varphi) \subseteq \{1, 2, \ldots, \dim\varphi\}$, and hence $|\mathrm{fSP}(\varphi)| \leq \dim\varphi$. We will show that the bound cannot be improved. Moreover, we provide a lower bound for $|\mathrm{piSP}(\varphi)|$; since obviously $\mathrm{piSP}(\varphi) \subseteq \mathrm{fSP}(\varphi)$, we will also get a lower bound for $|\mathrm{fSP}(\varphi)|$.

For a given p-form φ, we construct a tower of fields over which the defect of φ is strictly increasing similarly as over the standard splitting tower. To do that, we put a seemingly strong assumption on the p-form. But by Corollary 2.9, all p-forms which represent one fulfill this assumption.

Lemma 2.32. *Let $\varphi = \langle 1, a_1, \ldots, a_n\rangle$ be a p-form over F, and assume that $\{a_1, \ldots, a_m\}$ is a p-basis of the field $N_F(\varphi)$ over F for some $m \leq n$. We denote $E_0 = F$ and $E_i = F\left(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_i}\right)$ for $1 \leq i \leq m$. Then we have $\mathfrak{i}_\mathrm{d}(\varphi_{E_{i+1}}) > \mathfrak{i}_\mathrm{d}(\varphi_{E_i})$ and $\langle 1, a_{i+1}, \ldots, a_m\rangle_{E_i} \subseteq (\varphi_{E_i})_\mathrm{an}$ for any $0 \leq i \leq m-1$.*

Proof. Let $i \in \{0, \ldots, m-1\}$ and set $\tau_i = (\varphi_{E_i})_\mathrm{an}$. Since $D_F(\varphi) \subseteq D_{E_i}(\tau_i)$, it follows from Lemma 2.1 that

$$(\langle 1, a_{i+1}, \ldots, a_m\rangle_{E_i})_\mathrm{an} \subseteq \tau_i,$$

and so we only need to prove that the form $\langle 1, a_{i+1}, \ldots, a_m\rangle_{E_i}$ is anisotropic over E_i. Suppose not; then we can find $k \in \{i+1, \ldots, m\}$, such that

$$a_k \in \mathrm{span}_{E_i^p}\{1, a_{i+1}, \ldots, \widehat{a_k}, \ldots, a_m\}.$$

But that implies $a_k \in F^p(a_1, \dots, \widehat{a_k}, \dots, a_m)$, which is a contradiction with the choice of $\{a_1, \dots, a_m\}$ as a p-basis over F by Lemma 2.4.

Furthermore, the p-form $\langle 1, a_{i+1}, \dots, a_m \rangle$ becomes isotropic over the field E_{i+1}, and hence so does τ_i. It follows that $\mathfrak{i}_\mathrm{d}(\varphi_{E_{i+1}}) > \mathfrak{i}_\mathrm{d}(\varphi_{E_i})$. □

As a corollary to the previous lemma, we get a lower bound on the size of the sets $\mathrm{piSP}(\varphi)$ and $\mathrm{fSP}(\varphi)$.

Corollary 2.33. *Let φ be a p-form over F, and suppose that $\mathrm{ndeg}_F \varphi = p^m$. Then $|\mathrm{piSP}(\varphi)| \geq m + 1$ and $|\mathrm{fSP}(\varphi)| \geq m + 1$.*

Proof. Since the splitting patterns do not depend on the choice of the similarity class, we can assume without loss of generality that $1 \in D_F(\varphi)$; Then, invoking Corollary 2.9, we can suppose that φ is as in Lemma 2.32. Using the fields E_i for $0 \leq i \leq m$ from that lemma, it follows that we have $\dim(\varphi_{E_i})_\mathrm{an} \neq \dim(\varphi_{E_j})_\mathrm{an}$ for any $i \neq j$. The claim follows. □

Remark 2.34. (i) The same lower bound on $|\mathrm{fSP}(\varphi)|$ as in Corollary 2.33 could be obtained through the standard splitting tower: Indeed, by [Scu16b, Lemma 4.3], it holds that $\mathrm{ndeg}_{F(\varphi)}(\varphi_{F(\varphi)})_\mathrm{an} = \frac{1}{p}\,\mathrm{ndeg}_F \varphi$; therefore, it follows that $|\mathrm{sSP}(\varphi)| = m + 1$ where $\mathrm{ndeg}_F \varphi = p^m$. Since $\mathrm{sSP}(\varphi) \subseteq \mathrm{fSP}(\varphi)$, we get $|\mathrm{fSP}(\varphi)| \geq m + 1$.

(ii) The tower of fields $F = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$ in Lemma 2.32 might look like a purely inseparable analogy of the standard splitting tower (note that they are of the same length). But the values of $\mathfrak{i}_\mathrm{d}(\varphi_{E_i})$ depend on the ordering of $a_1, \dots, a_m$, so in particular we cannot expect that $\mathfrak{i}_\mathrm{d}(\varphi_{E_1}) = \mathfrak{i}_\mathrm{d}(\varphi_{F(\varphi)})$.

For example, if $\varphi \cong \langle\langle a_1, a_2 \rangle\rangle \perp a_3\langle 1, a_1 \rangle$ for some set $\{a_1, a_2, a_3\} \subseteq F^*$ p-independent over F, then $(\varphi_{F(\sqrt[p]{a_1})})_\mathrm{an} \cong (\langle\langle a_2 \rangle\rangle \perp a_3\langle 1 \rangle)_{F(\sqrt[p]{a_1})}$. The p-form $(\varphi_{F(\varphi)})_\mathrm{an} \cong \langle\langle a_1, a_2 \rangle\rangle_{F(\varphi)}$ corresponds rather to $(\varphi_{F(\sqrt[p]{a_3})})_\mathrm{an} \cong \langle\langle a_1, a_2 \rangle\rangle_{F(\sqrt[p]{a_3})}$.

The following example shows that the lower bound from Corollary 2.33 cannot be improved.

Example 2.35. Let $\varphi \cong \langle 1, a_1, \dots, a_m \rangle$ be a minimal p-form over F (i.e., the set $\{a_1, \dots, a_m\}$ is p-independent over F). Then $\mathrm{ndeg}_F \varphi = p^m$ and

$$\mathrm{fSP}(\varphi) = \{1, 2, \dots, m + 1\}.$$

2.2.3 Full splitting pattern of quasi-Pfister forms

To determine the full splitting pattern of quasi-Pfister forms, we start by looking at their behavior over purely inseparable field extensions.

Lemma 2.36. *Let π and π' be anisotropic quasi-Pfister forms over F such that $\pi \cong \langle\langle a \rangle\rangle \otimes \pi'$ for some $a \in F$, and let $E = F(\sqrt[p]{a})$. Then π'_E is anisotropic. Furthermore, if φ is a p-form over F such that $\hat{\nu}_F(\varphi) \cong \pi$, then $\hat{\nu}_E(\varphi) \cong \pi'_E$.*

Proof. Let $\pi' = \langle\langle b_1, \dots, b_m \rangle\rangle$. If π'_E is isotropic, then, by Corollary 2.6, $\{b_1, \dots, b_m\}$ is p-dependent over E, and hence we find $i \in \{1, \dots, m\}$ such that $b_i \in E^p(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_m)$ by Lemma 2.4. But it means that $b_i \in F^p(a, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_m)$, and hence π is isotropic.

The second statement follows from the first one and Lemma 2.14. □

Lemma 2.37. *Let $\pi = \langle\langle a_1, \dots, a_m\rangle\rangle$ be an anisotropic quasi-Pfister form over F, and let φ be a p-form over F with $\hat{\nu}_F(\varphi) \cong \pi$. We set $E_0 = F$ and $E_i = F\left(\sqrt[p]{a_1}, \dots, \sqrt[p]{a_i}\right)$ for $1 \le i \le m$. Then $(\pi_{E_i})_{\mathrm{an}} \cong \langle\langle a_{i+1}, \dots, a_m\rangle\rangle$ and $\hat{\nu}_{E_i}(\varphi) \cong \langle\langle a_{i+1}, \dots, a_m\rangle\rangle$ for any $0 \le i \le m$. (For $i = m$ we get the 0-fold Pfister form $\langle 1\rangle$.)*

Proof. The first claim follows from Lemma 2.36 by induction on i. The second claim is a consequence of the first one and Lemma 2.14. $\square$

Example 2.38. Let $\pi \cong \langle\langle a_1, \dots, a_n\rangle\rangle$ be an anisotropic quasi-Pfister form. Combining Lemmas 2.18 and 2.37, we get

$$\mathrm{fSP}(\pi) = \{1, p, \dots, p^n\}.$$

2.2.4 Full splitting pattern of quasi-Pfister neighbors

We look at anisotropic p-forms of the form $\varphi \cong \pi \perp d\sigma$, where π is a quasi-Pfister form and $\sigma \subseteq \pi$. Note that φ is a quasi-Pfister neighbor of $\hat{\nu}_F(\varphi) \cong \pi \otimes \langle\langle d\rangle\rangle$.

Lemma 2.39. *Let π be a quasi-Pfister form over F, $\sigma \subseteq \pi$ and $d \in F^*$ be such that the p-form $\varphi \cong \pi \perp d\sigma$ is anisotropic. Let E/F be a field extension.*

(i) *If $d \in D_E(\pi)$, then we have $(\varphi_E)_{\mathrm{an}} \cong (\pi_E)_{\mathrm{an}}$, and so, in particular, $\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\pi_E) + \dim \sigma$.*

(ii) *If $d \notin D_E(\pi)$, then $(\varphi_E)_{\mathrm{an}} \cong (\pi_E)_{\mathrm{an}} \perp d(\sigma_E)_{\mathrm{an}}$, and so, in particular, $\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\pi_E) + \mathfrak{i}_{\mathrm{d}}(\sigma_E)$.*

Proof. If $d \in D_E(\pi)$, then $\varphi_E \cong d\pi_E \perp d\sigma_E$, and hence $(\varphi_E)_{\mathrm{an}} \cong (\pi_E)_{\mathrm{an}}$.

Now assume $d \notin D_E(\pi)$; if $(\pi_E)_{\mathrm{an}} \perp d(\sigma_E)_{\mathrm{an}}$ were isotropic, then so would be $(\pi_E)_{\mathrm{an}} \otimes \langle 1, d\rangle$, which would imply $d \in D_E(\pi)$, a contradiction. Thus, $(\pi_E)_{\mathrm{an}} \perp d(\sigma_E)_{\mathrm{an}}$ is anisotropic. As clearly $(\varphi_E)_{\mathrm{an}} \cong ((\pi_E)_{\mathrm{an}} \perp d(\sigma_E)_{\mathrm{an}})_{\mathrm{an}}$, we get $(\varphi_E)_{\mathrm{an}} \cong (\pi_E)_{\mathrm{an}} \perp d(\sigma_E)_{\mathrm{an}}$. $\square$

Corollary 2.40. *Let π be an n-fold quasi-Pfister form over F, $\sigma \subseteq \pi$ and $d \in F^*$ be such that the p-form $\varphi \cong \pi \perp d\sigma$ is anisotropic. Then the following hold:*

(i) $\mathrm{fSP}(\varphi) \subseteq \{p^k + l \mid 0 \le k \le n, 0 \le l \le \dim \sigma\}$*;*

(ii) *if E/F is a field extension such that φ_E is isotropic, then we have $\mathfrak{i}_{\mathrm{d}}(\varphi_E) \ge \mathfrak{i}_{\mathrm{d}}(\varphi_{F(\varphi)})$.*

Proof. Part (i) is a direct consequence of Lemma 2.39.

To prove part (ii), note that φ is a quasi-Pfister neighbor of $\pi \otimes \langle\langle d\rangle\rangle$, and so we have $\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\varphi)}) = \dim \sigma$ by Proposition 2.16. Therefore, we want to show that $\mathfrak{i}_{\mathrm{d}}(\varphi_E) \ge \dim \sigma$.

Note that the p-form $\hat{\nu}_F(\varphi) \cong \pi \otimes \langle\langle d\rangle\rangle$ has the dimension p^{n+1}. Since φ_E is isotropic, the p-form $\hat{\nu}_F(\varphi)_E$ is isotropic too, and so $\dim(\hat{\nu}_F(\varphi)_E)_{\mathrm{an}} \le p^n$ by Lemma 2.18. As $(\varphi_E)_{\mathrm{an}} \subseteq (\hat{\nu}_F(\varphi)_E)_{\mathrm{an}}$, it follows that $\mathfrak{i}_{\mathrm{d}}(\varphi_E) \ge \dim \sigma$. $\square$

Lemma 2.41. *Let $\pi \cong \langle\langle a_1, \dots, a_n\rangle\rangle$ be an anisotropic quasi-Pfister form over F and $\sigma \cong \langle 1, a_1, \dots, a_s\rangle$ for some $s \le n$. Let E/F be a field extension.*

(i) *If* $\dim(\sigma_E)_{\mathrm{an}} = l$, *then* $\dim(\pi_E)_{\mathrm{an}} \geq p^{\lceil \log_p l \rceil}$.
(ii) *If* $\dim(\pi_E)_{\mathrm{an}} = p^k$, *then* $\dim(\sigma_E)_{\mathrm{an}} \geq k - n + s + 1$.

Proof. Part (i) follows from the facts that $(\sigma_E)_{\mathrm{an}} \subseteq (\pi_E)_{\mathrm{an}}$ and the dimension of $(\pi_E)_{\mathrm{an}}$ is a p-power.

To prove part (ii), we apply Proposition 2.17 on π_E to find a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $(\pi_E)_{\mathrm{an}} \cong \langle\langle a_{i_1}, \ldots, a_{i_k} \rangle\rangle_E$. Let

$$\{j_1, \ldots, j_l\} = \{i_1, \ldots, i_k\} \cap \{1, \ldots, s\};$$

since $a_{j_1}, \ldots, a_{j_l}$ are p-independent over E, the p-form $\langle 1, a_{j_1}, \ldots, a_{j_l} \rangle_E$ is an anisotropic subform of $(\sigma_E)_{\mathrm{an}}$. In particular, $\dim(\sigma_E)_{\mathrm{an}} \geq l + 1$. Since

$$l = |\{i_1, \ldots, i_k\} \cap \{1, \ldots, s\}| \geq k - (n - s),$$

the claim follows. $\square$

If, in the situation of Lemma 2.39, the form σ is minimal, then it is possible to describe the full splitting pattern of $\varphi \cong \pi \perp d\sigma$.

Theorem 2.42. *Let π be an n-fold quasi-Pfister form over F, σ be a minimal subform of π of dimension at least* 2 *and $d \in F^*$ be such that the p-form $\varphi \cong \pi \perp d\sigma$ is anisotropic. Then $m \in \mathrm{fSP}(\varphi)$ if and only if $m = p^k + l$ and one of the following holds:*
(i) $0 \leq k \leq n$ *and* $l = 0$*;*
(ii) $0 \leq k \leq n$ *and* $\max\{1, k - n + \dim\sigma\} \leq l \leq \min\{\dim\sigma, p^k\}$.

Proof. If $x \in D_F(\sigma)$, then $x^{-1} \in G_F(\pi)$, and hence $x^{-1}\varphi \cong \pi \perp d(x^{-1}\sigma)$ with $1 \in D_F(x^{-1}\sigma)$; therefore, we can assume without loss of generality that $1 \in D_F(\sigma)$. Let $\sigma \cong \langle 1, a_1, \ldots, a_s \rangle$ for some $1 \leq s \leq n$; since σ is assumed to be minimal, the set $\{a_1, \ldots, a_s\}$ is p-independent by part (i) of Lemma 2.23, and hence it can be extended to a p-independent set $\{a_1, \ldots, a_n\}$ such that $N_F(\pi) = F^p(a_1, \ldots, a_n)$. Then $\pi \cong \langle\langle a_1, \ldots, a_n \rangle\rangle$ and

$$\varphi \cong \langle\langle a_1 \ldots, a_n \rangle\rangle \perp d\langle 1, a_1, \ldots, a_s \rangle.$$

By Corollary 2.40, we have

$$\mathrm{fSP}(\varphi) \subseteq \{p^k + l \mid 0 \leq k \leq n, 0 \leq l \leq s + 1\}.$$

Furthermore, if $l \geq 1$ (i.e., if σ_E does not disappear completely in $(\varphi_E)_{\mathrm{an}}$), then by Lemma 2.41, it must hold $k - n + s + 1 \leq l \leq p^k$ for any tuple (k, l) such that $p^k + l \in \mathrm{fSP}(\varphi)$. Therefore, $\mathrm{fSP}(\varphi) \subseteq I$ where

$$I = \{p^k + l \mid 0 \leq k \leq n, \max\{1, k - n + s + 1\} \leq l \leq \min\{s + 1, p^k\}\} \\ \cup \{p^k \mid 0 \leq k \leq n\}.$$

So we only need to prove that all the values in I are realizable. We define

$$I_0 = \{(k, 0) \mid 0 \leq k \leq n\},$$
$$I_1 = \{(k, l) \mid 0 \leq k \leq n, \max\{1, k - n + s + 1\} \leq l \leq \min\{s + 1, k + 1\}\},$$
$$I_2 = \{(k, l) \mid 0 \leq k \leq n, k + 1 < l \leq \min\{s + 1, p^k\}\};$$

then $I = \{p^k + l \mid (k,l) \in I_0 \cup I_1 \cup I_2\}$.

If $(k,l) = (k,0) \in I_0$, then set $D_k = F(\sqrt[p]{a_{k+1}}, \ldots, \sqrt[p]{a_n}, \sqrt[p]{d})$. We get

$$(\varphi_{D_k})_{\mathrm{an}} \cong \langle\langle a_1, \ldots, a_k \rangle\rangle_{D_k},$$

i.e., $\dim(\varphi_{D_k})_{\mathrm{an}} = p^k$.

To cover the values in the set I_1, we define $E_{n,s+1} = F$ and

$$E_{k,l} = F(\sqrt[p]{a_l}, \ldots, \sqrt[p]{a_{n-(k-l)-1}})$$

for all tuples $(k,l) \in I_1$ such that $(k,l) \neq (n, s+1)$; note that since $k \leq n-1$ for such tuples, we have $l \leq n-(k-l)-1$. Moreover, the condition $l \leq k+1$ can be rewritten as $n-(k-l)-1 \leq n$. Therefore, $E_{k,l}$ is well-defined. Moreover, the inequality $k-n+s+1 \leq l$ ensures that $s \leq n-(k-l)-1$, and hence $(\sigma_{E_{k,l}})_{\mathrm{an}} \cong \langle 1, a_1, \ldots, a_{l-1} \rangle_{E_{k,l}}$. Thus, for any $(k,l) \in I_1$, we get

$$(\varphi_{E_{k,l}})_{\mathrm{an}} \cong (\langle\langle a_1, \ldots, a_{l-1} \rangle\rangle \otimes \langle\langle a_{n-(k-l)}, \ldots, a_n \rangle\rangle \perp d\langle 1, a_1, \ldots, a_{l-1} \rangle)_{E_{k,l}}$$

and so $\dim(\varphi_{E_{k,l}})_{\mathrm{an}} = p^k + l$.

Finally, let $(k,l) \in I_2$. We denote $S_k = \{\lambda : \{1, \ldots, k\} \to \{0, \ldots, p-1\}\}$ and write $a^\lambda = \prod_{i=1}^k a_i^{\lambda(i)}$ for any $\lambda \in S_k$. Furthermore, we pick a subset $\mathcal{L}_{k,l} \subseteq S_k$ such that $|\mathcal{L}_{k,l}| = l-k-1$ and $\sum_{i=1}^k \lambda(i) > 1$ for each $\lambda \in \mathcal{L}_{k,l}$; note that this is possible since $l-k-1 \leq p^k-k-1 = \left|\{\lambda \in S_k \mid \sum_{i=1}^k \lambda(i) > 1\}\right|$. Let $\mathcal{L}_{k,l} = \{\lambda_1, \ldots, \lambda_{l-k-1}\}$. We define

$$G_{k,l} = F\left(\sqrt[p]{\frac{a^{\lambda_1}}{a_{k+1}}}, \ldots, \sqrt[p]{\frac{a^{\lambda_{l-k-1}}}{a_{l-1}}}, \sqrt[p]{a_l}, \ldots, \sqrt[p]{a_n}\right)$$

(the field depends on the choice of $\mathcal{L}_{k,l}$ and the ordering of its elements). Then $[G_{k,l} : F] \leq p^{n-k}$, and

$$\begin{aligned} G_{k,l}^p(a_1, \ldots, a_k) = F^p\left(a_1, \ldots, a_k, \frac{a^{\lambda_1}}{a_{k+1}}, \ldots, \frac{a^{\lambda_{l-k-1}}}{a_{l-1}}, a_l, \ldots, a_n\right) \\ = F^p(a_1, \ldots, a_n) \end{aligned}$$

because by the definition $a^{\lambda_i} \in F^p(a_1, \ldots, a_k)$ for all $1 \leq i \leq l-k-1$. As $[G_{k,l}^p : F^p] = [G_{k,l} : F]$ by Lemma 2.3, we get

$$p^n = [F^p(a_1, \ldots, a_n) : F^p] = [G_{k,l}^p(a_1, \ldots, a_k) : G_{k,l}^p] \cdot [G_{k,l}^p : F^p] \leq p^k \cdot p^{n-k}.$$

It follows that $[G_{k,l} : F] = p^{n-k}$ and $[G_{k,l}^p(a_1, \ldots, a_k) : G_{k,l}^p] = p^k$; hence, in particular, the set $\{a_1, \ldots, a_k\}$ is p-independent over $G_{k,l}$, which means that the p-form $\langle\langle a_1, \ldots, a_k \rangle\rangle$ is anisotropic over $G_{k,l}$. Moreover, note that we have $a^{\lambda_i} \equiv a_{k+i} \mod G_{k,l}^{*p}$ for any $1 \leq i \leq l-k-1$; in particular $a_{k+1}, \ldots, a_{l-1} \in D_{G_{k,l}}(\langle\langle a_1, \ldots, a_k \rangle\rangle)$. Therefore,

$$\begin{aligned} (\pi_{G_{k,l}})_{\mathrm{an}} &\cong \langle\langle a_1, \ldots, a_k \rangle\rangle_{G_{k,l}}, \\ (\sigma_{G_{k,l}})_{\mathrm{an}} &\cong \langle 1, a_1, \ldots, a_{l-1} \rangle_{G_{k,l}}, \end{aligned}$$

where the anisotropy of $\langle 1, a_1, \ldots, a_{l-1}\rangle_{G_{k,l}}$ follows from

$$\langle 1, a_1, \ldots, a_{l-1}\rangle_{G_{k,l}} \cong \Big(\langle 1, a_1, \ldots, a_k\rangle \perp \mathop{\perp}_{i=1}^{l-k-1} \langle a^{\lambda_i}\rangle\Big)_{G_{k,l}} \subseteq \langle\langle a_1, \ldots, a_k\rangle\rangle_{G_{k,l}}.$$

Since $d \notin F^p(a_1, \ldots, a_n) = D_{G_{k,l}}(\pi)$, we get by Lemma 2.39 that

$$(\varphi_{G_{k,l}})_{\text{an}} \cong (\langle\langle a_1, \ldots, a_k\rangle\rangle \perp d\langle 1, a_1, \ldots, a_{l-1}\rangle)_{G_{k,l}};$$

in particular $\dim(\varphi_{G_{k,l}})_{\text{an}} = p^k + l$. $\square$

Remark 2.43. Note that we used in the proof of Theorem 2.42 only purely inseparable field extensions of exponent one. Thus, for any p-form φ satisfying the conditions of that theorem, we have $\operatorname{piSP}(\varphi) = \operatorname{fSP}(\varphi)$.

The following example illustrates both the result and the proof of Theorem 2.42.

Example 2.44. Let $a_1, a_2, a_3, a_4, d \in F^*$ be p-independent and

$$\varphi \cong \langle\langle a_1, a_2, a_3, a_4\rangle\rangle \perp d\langle 1, a_1, a_2, a_3\rangle.$$

Then we have by Theorem 2.42

$$\begin{aligned} \operatorname{fSP}(\varphi) =& \{p^0, p^1, p^2, p^3, p^4\} \\ & \cup \{p^0+1, p^1+1, p^1+2, p^2+2, p^2+3, p^2+4, p^3+3, p^3+4\} \\ & \cup \{p^1+3 \mid \text{if } p \geq 3\} \cup \{p^1+4 \mid \text{if } p \geq 4\}. \end{aligned}$$

Table 2.1 provides the fields used in the proof of Theorem 2.42 and the obtained p-forms for the case $p \geq 4$.

dim	0	1	2	3	4
p^0	$D_0 = F(\sqrt[p]{a_1}, \sqrt[p]{a_2}, \sqrt[p]{a_3}, \sqrt[p]{a_4}, \sqrt[p]{d})$ $\langle 1\rangle$	$E_{0,1} = F(\sqrt[p]{a_1}, \sqrt[p]{a_2}, \sqrt[p]{a_3}, \sqrt[p]{a_4})$ $\langle 1\rangle \perp d\langle 1\rangle$	X	X	X
p^1	$D_1 = F(\sqrt[p]{a_1}, \sqrt[p]{a_2}, \sqrt[p]{a_3}, \sqrt[p]{d})$ $\langle\langle a_4\rangle\rangle$	$E_{1,1} = F(\sqrt[p]{a_1}, \sqrt[p]{a_2}, \sqrt[p]{a_3})$ $\langle\langle a_4\rangle\rangle \perp d\langle 1\rangle$	$E_{1,2} = F(\sqrt[p]{a_2}, \sqrt[p]{a_3}, \sqrt[p]{a_4})$ $\langle\langle a_1\rangle\rangle \perp d\langle 1, a_1\rangle$	$G_{1,3} = F\left(\sqrt[p]{\frac{a_1^2}{a_2}}, \sqrt[p]{a_3}, \sqrt[p]{a_4}\right)$ $\langle\langle a_1\rangle\rangle \perp d\langle 1, a_1, a_1^2\rangle$	$G_{1,4} = F\left(\sqrt[p]{\frac{a_1^2}{a_2}}, \sqrt[p]{\frac{a_1^3}{a_3}}, \sqrt[p]{a_4}\right)$ $\langle\langle a_1\rangle\rangle \perp d\langle 1, a_1, a_1^2, a_1^3\rangle$
p^2	$D_2 = F(\sqrt[p]{a_1}, \sqrt[p]{a_2}, \sqrt[p]{d})$ $\langle\langle a_3, a_4\rangle\rangle$	X	$E_{2,2} = F(\sqrt[p]{a_2}, \sqrt[p]{a_3})$ $\langle\langle a_1, a_4\rangle\rangle \perp d\langle 1, a_1\rangle$	$E_{2,3} = F(\sqrt[p]{a_3}, \sqrt[p]{a_4})$ $\langle\langle a_1, a_2\rangle\rangle \perp d\langle 1, a_1, a_2\rangle$	$G_{2,4} = F\left(\sqrt[p]{\frac{a_1 a_2}{a_3}}, \sqrt[p]{a_4}\right)$ $\langle\langle a_1, a_2\rangle\rangle \perp d\langle 1, a_1, a_2, a_1 a_2\rangle$
p^3	$D_3 = F(\sqrt[p]{a_1}, \sqrt[p]{d})$ $\langle\langle a_2, a_3, a_4\rangle\rangle$	X	X	$E_{3,3} = F(\sqrt[p]{a_3})$ $\langle\langle a_1, a_2, a_3\rangle\rangle \perp d\langle 1, a_1, a_2\rangle$	$E_{3,4} = F(\sqrt[p]{a_4})$ $\langle\langle a_1, a_2, a_3\rangle\rangle \perp d\langle 1, a_1, a_2, a_3\rangle$
p^4	$D_4 = F(\sqrt[p]{d})$ $\langle\langle a_1, a_2, a_3, a_4\rangle\rangle$	X	X	X	$E_{4,4} = F$ $\langle\langle a_1, a_2, a_3, a_4\rangle\rangle \perp d\langle 1, a_1, a_2, a_3\rangle$

Table 2.1: Splitting of the p-form $\langle\langle a_1, a_2, a_3, a_4\rangle\rangle \perp d\langle 1, a_1, a_2, a_3\rangle$ in the case $p \geq 4$ (see Example 2.44).

2.2.5 Subforms of p-forms

The next Lemma has been proved in [Scu16b] (and in [Lag04] in the case of $p = 2$), but since it will be an important building block in the next chapter, we provide the proof here.

Proposition 2.45 ([Scu16b, Lemma 3.9])**.** *Let φ be an anisotropic p-form over F, $a \in F \setminus F^p$ and $m \in \mathbb{N}$. If $\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) \geq m$, then there exists a p-form $\tau \subseteq \varphi$ with $\dim \tau \leq pm$ such that $\mathfrak{i}_{\mathrm{d}}(\tau_{F(\sqrt[p]{a})}) \geq m$.*

Proof. We proceed by induction on m. Let V be the underlying vector space of φ and $V_a \cong F(\sqrt[p]{a}) \otimes_F V$. Let $0 \neq v \in V_a$ be such that $\varphi_{F(\sqrt[p]{a})}(v) = 0$. Note that v can be written as

$$v = \sum_{i=0}^{p-1} (\sqrt[p]{a})^i \otimes v_i \quad \text{with} \quad v_i \in V,\ 0 \leq i \leq p-1.$$

Let $U = \operatorname{span}_F\{v_0, \dots, v_{p-1}\}$ and $\sigma \cong \varphi|_U$; then $\dim \sigma \leq p$ and $\sigma_{F(\sqrt[p]{a})}$ is isotropic. Thus, if $m = 1$, we set $\tau \cong \sigma$.

Assume $m > 1$. Let $k = \mathfrak{i}_{\mathrm{d}}(\sigma_{F(\sqrt[p]{a})})$ and let $\varphi' \subseteq \varphi$ be such that $\varphi \cong \sigma \perp \varphi'$. Furthermore, by Lemma 2.2, we can find $\sigma' \subseteq \sigma$ such that $(\sigma_{F(\sqrt[p]{a})})_{\mathrm{an}} \cong \sigma'_{F(\sqrt[p]{a})}$. We have

$$\varphi_{F(\sqrt[p]{a})} \cong (\sigma \perp \varphi')_{F(\sqrt[p]{a})} \cong (\sigma' \perp \varphi')_{F(\sqrt[p]{a})} \perp k \times \langle 0 \rangle;$$

it follows that

$$\mathfrak{i}_{\mathrm{d}}((\sigma' \perp \varphi')_{F(\sqrt[p]{a})}) = \mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) - k \geq m - k.$$

Since $k \geq 1$ by the construction of σ, we can apply the induction hypothesis on the p-form $\sigma' \perp \varphi'$ to find its subform τ' with $\dim \tau' \leq p(m-k)$ such that $\mathfrak{i}_{\mathrm{d}}(\tau'_{F(\sqrt[p]{a})}) \geq m - k$. The form $\tau \cong \tau' \perp \sigma$ has the desired properties. $\square$

Note that in the special case of $p = 2$ and $m = 1$, the p-form τ has dimension two. It follows that there exists $c \in F^*$ such that $\tau \cong c\langle 1, a \rangle \subseteq \varphi$. However, for $p > 2$, there does not have to exist a two-dimensional subform of φ which becomes isotropic over $E = F(\sqrt[p]{a})$, as we will see in the following example.

Example 2.46. Let $p > 2$ and $\varphi \cong \langle 1, b, a(1+ab) \rangle$ be a p-form, where $\{a, b\} \subseteq F$ is a p-independent set over F. Note that φ is a subform of the anisotropic quasi-Pfister form $\langle\langle a, b \rangle\rangle$, and hence φ is anisotropic over F. On the other hand, over $E = F(\sqrt[p]{a})$, we have

$$\varphi_E \cong \langle 1, b, 1+ab \rangle_E \cong \langle 1, b, ab \rangle_E \cong \langle 1, b, b \rangle_E \cong \langle 1, b, 0 \rangle_E,$$

i.e., φ_E is isotropic.

Let σ be a two-dimensional subform of φ; then we can write

$$\sigma \cong \langle x_1^p + by_1^p + a(1+ab)z_1^p, x_2^p + by_2^p + a(1+ab)z_2^p \rangle$$

for some $x_i, y_i, z_i \in F$ such that $x_i^p + by_i^p + a(1+ab)z_i^p \neq 0$ for $i = 1, 2$. Now either at least one of z_1, z_2 is zero, or we can add the $(-z_1)^p$-multiple of the second coefficient of σ to the z_2^p-multiple of its first coefficient and reduce the $a(1+ab)$ term of the first coefficient; hence, we can assume without loss of generality that $z_1 = 0$. Then $N_F(\sigma) = F^p(\alpha)$ with

$$\alpha = \frac{x_2^p + by_2^p + a(1+ab)z_2^p}{x_1^p + by_1^p}.$$

Assume that σ is isotropic over $F(\sqrt[p]{a})$. Then $a \in F^p(\alpha)$ by Lemma 2.25, i.e., $F^p(a) = F^p(\alpha)$, which means that $\alpha = \sum_{k=0}^{p-1} a^k w_k^p$ for some $w_k \in F$. Therefore,

$$x_2^p + by_2^p + a(1+ab)z_2^p = (x_1^p + by_1^p)\sum_{k=0}^{p-1} a^k w_k^p.$$

This can be rewritten as

$$\begin{aligned}(x_2 - x_1w_0)^p + a(z_2 - x_1w_1)^p + b(y_2 - y_1w_0)^p + a^2(-x_1w_2)^p + ab(-y_1w_1)^p \\ + a^2b(z_2 - y_1w_2)^p + \sum_{k=3}^{p-1} a^k(-x_1w_k)^p + \sum_{k=3}^{p-1} a^k b(-y_1w_k)^p = 0.\end{aligned}$$

Recalling that a, b are p-independent over F by the assumption, all the coefficients (the p-powers) must be zero by Lemma 2.4; in particular:

$$\begin{gathered} x_2 = x_1w_0, \quad y_2 = y_1w_0, \quad z_2 = x_1w_1 = y_1w_2, \\ x_1w_2 = 0, \quad y_1w_1 = 0, \quad y_1w_k = 0 \ \forall k \geq 3. \end{gathered}$$

If $x_1 = 0$, then necessarily $x_2 = z_2 = 0$, and we get $\sigma \cong \langle b, b\rangle$, which is a contradiction to its anisotropy; thus, $x_1 \neq 0$. Since $x_1w_2 = 0$, this means that $w_2 = 0$, and hence $z_2 = 0$. Furthermore, if $y_1 = 0$, then $y_2 = 0$, too, and $\sigma \cong \langle 1, 1\rangle$, a contradiction. Thus, $y_1 \neq 0$, and so $w_1 = 0$ and $w_k = 0$ for all $k \geq 3$; since we already know $w_2 = 0$, it follows that $\alpha = w_0^p$. That contradicts the anisotropy of σ.

We have proven that σ, an arbitrary two-dimensional subform of φ, is anisotropic over the field $F(\sqrt[p]{a})$. Note that we needed to use the coefficient w_2 to do this, which can be done only under the assumption $p > 2$; indeed, for $p = 2$, we have $\varphi \cong a\langle 1, a, ab\rangle$, so φ contains a subform similar to $\langle 1, a\rangle$.

Moreover, note that if we apply Proposition 2.45 on φ, the "isotropic subform" τ would be the whole φ; since $\hat{\nu}_F(\varphi) \nsubseteq \langle\langle a\rangle\rangle$, it means that in general we cannot even expect the p-form τ to be a subform of $\langle c_1, \dots, c_m\rangle \otimes \langle\langle a\rangle\rangle$ for some $c_1, \dots, c_m \in F^*$.

Lemma 2.47. *Let $\pi = \langle\langle a_1, \dots, a_n\rangle\rangle$ be an anisotropic quasi-Pfister form over F and φ a p-form over F such that $x\pi \subseteq \varphi$ for some $x \in F^*$. Then for every $b \in F^p(a_1, \dots, a_n) \setminus F^p$, the following hold:*
(i) $\mathfrak{i}_{\mathrm{d}}(\pi_{F(\sqrt[p]{b})}) = p^n - p^{n-1}$,
(ii) $\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{b})}) \geq p^n - p^{n-1}$.

Proof. It is obvious that $\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{b})}) \geq \mathfrak{i}_{\mathrm{d}}(\pi_{F(\sqrt[p]{b})})$, so it is sufficient to prove part (i). As $b \notin F^p$, we can extend b to a p-basis of the field $F^p(a_1, \dots, a_n)$ by Lemma 2.7; so, $F^p(a_1, \dots, a_n) = F^p(b, c_2, \dots, c_n)$ for some $c_2, \dots, c_n \in F^*$. It follows from Proposition 2.11 that $\pi = \langle\langle a_1, \dots, a_n\rangle\rangle \cong \langle\langle b, c_2, \dots, c_n\rangle\rangle$. Therefore, we have $(\pi_{F(\sqrt[p]{b})})_{\mathrm{an}} \cong \langle\langle c_2, \dots, c_n\rangle\rangle$ by Lemma 2.36, and it follows that $\mathfrak{i}_{\mathrm{d}}(\pi_{F(\sqrt[p]{b})}) = p^n - p^{n-1}$. □

It would be great if a converse to the previous lemma held: Suppose φ is an anisotropic p-form and $\{a_1, \dots, a_n\}$ a p-independent set over F such that

$$\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{b})}) \geq p^n - p^{n-1} \text{ for any } b \in F^p(a_1, \dots, a_n) \setminus F^p. \tag{❃}$$

Does it follow that φ must contain a subform similar to the quasi-Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle$? The following example shows that this is not true in general.

Example 2.48. Let $\pi = \langle\langle a_1, \dots, a_n \rangle\rangle$ be an anisotropic quasi-Pfister form over F with $n \geq 2$. For $x \in F^*$ such that $x \notin D_F(\pi)$ and a p-form π' such that $\langle 1 \rangle \perp \pi' \cong \pi$ set $\varphi \cong \pi' \perp x\pi'$.

Let $b \in F^p(a_1, \dots, a_n) \setminus F^p$. First, note that

$$\mathfrak{i}_{\mathrm{d}}(\pi'_{F(\sqrt[p]{b})}) \geq p^n - p^{n-1} - 1,$$

since $(\pi'_{F(\sqrt[p]{b})})_{\mathrm{an}} \subseteq (\pi_{F(\sqrt[p]{b})})_{\mathrm{an}}$ and $\dim(\pi_{F(\sqrt[p]{b})})_{\mathrm{an}} = p^{n-1}$. It follows that

$$\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{b})}) \geq 2\mathfrak{i}_{\mathrm{d}}(\pi'_{F(\sqrt[p]{b})}) \geq 2(p^n - p^{n-1} - 1) \geq p^n - p^{n-1}$$

for any $p \geq 2$ and $n \geq 2$; hence, the condition (❃) is satisfied.

We show $1 \notin D_F(\varphi)$: For a contradiction, suppose $1 \in D_F(\varphi)$. Then $\varphi \perp \langle 1 \rangle$ is isotropic. Since $\varphi \perp \langle 1 \rangle \subseteq \pi \perp x\pi \subseteq \pi \otimes \langle\langle x \rangle\rangle$, the quasi-Pfister form $\pi \otimes \langle\langle x \rangle\rangle$ is isotropic, too. But then $x \in N_F(\pi) = D_F(\pi)$ by Lemma 2.19, which contradicts the choice of x.

Finally, assume that $c\pi \subseteq \varphi$ for some $c \in F^*$. Consider the p-form $c\pi \perp \pi'$: If this form is isotropic, then $\pi \otimes \langle\langle c \rangle\rangle$ is isotropic as well, and hence $c \in D_F(\pi)$, again by Lemma 2.19. But then $c\pi \cong \pi$, so $\pi \subseteq \varphi$, and hence $1 \in D_F(\varphi)$, which is not the case. Thus, the p-form $c\pi \perp \pi'$ must be anisotropic. Since both $c\pi$ and π' are subforms of φ, we get $c\pi \perp \pi' \subseteq \varphi$. But that is absurd, because $\dim(c\pi \perp \pi') > \dim \varphi$. Therefore, φ does not contain a subform similar to π.

2.2.6 Isotropy over a function field

We present an extended version of a known proposition.

Proposition 2.49. *Let φ, ψ be anisotropic p-forms over F. Assume that $\psi \cong \langle 1 \rangle \perp \psi'$, and denote $\boldsymbol{X}' = (X_2, \dots, X_{\dim \psi})$. Then the following are equivalent:*

(i) *$\varphi_{F(\psi)}$ is isotropic,*

(ii) *$\varphi_{F(\boldsymbol{X}')} \otimes \langle\langle \psi'(\boldsymbol{X}') \rangle\rangle$ is isotropic.*

If, moreover, φ is a quasi-Pfister form or a quasi-Pfister neighbor, then the conditions above are also equivalent to

(iii) *$\psi'(\boldsymbol{X}') \in N_{F(\boldsymbol{X}')}(\varphi)$.*

Proof. For the equivalence of (i) and (ii), see [Hof04, Prop. 7.15]. The equivalence of (ii) and (iii) follows from Lemma 2.19. □

Recall that $\langle D_F^*(\varphi) \rangle$ denotes the multiplicative subgroup of F^* generated by $D_F^*(\varphi)$. In the following lemma, we give a sufficient condition for an anisotropic p-form to become isotropic over a function field of a polynomial.

Lemma 2.50. *Let φ be a p-form over F (or, more generally, any homogeneous polynomial of degree p). Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $f \in F[\boldsymbol{X}]$ be an irreducible polynomial. If there exists $a \in F^*$ such that $af \in \langle D^*_{F(\boldsymbol{X})}(\varphi)\rangle$, then $\varphi_{F(f)}$ is isotropic.*

Proof. Since the function fields $F(f)$ and $F(af)$ coincide, we can assume without loss of generality that $a = 1$. Set $n = \dim \varphi$. Let

$$f = \prod_{i=1}^{k} \varphi(\boldsymbol{\xi}'_i)$$

for some $k > 0$ and $\boldsymbol{\xi}'_i = (\xi'_{i1}, \ldots, \xi'_{in})$ with $\xi'_{ij} \in F(\boldsymbol{X})$ for all i, j. For each i, we can find $0 \neq h_i \in F[\boldsymbol{X}]$ such that $h_i\xi'_{ij} \in F[\boldsymbol{X}]$ for all j; we denote $\xi_{ij} = h_i\xi'_{ij}$ and $\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{in})$. Moreover, set $h = \prod_{i=1}^{k} h_i$. Then

$$h^p f = \prod_{i=1}^{k} \varphi(\boldsymbol{\xi}_i) \neq 0.$$

If there exists an $i \in \{1, \ldots, k\}$ such that $f \mid \xi_{ij}$ for all j, then $f^p \mid h^p f$; since f is irreducible, we get $f \mid h$. Hence, we can replace ξ_{ij} by $\frac{\xi_{ij}}{f}$ for each $1 \leq j \leq n$, and h by $\frac{h}{f}$. Repeating this step if necessary, we may assume that for each $i \in \{1, \ldots, k\}$, there exists at least one $j \in \{1, \ldots, n\}$ such that $f \nmid \xi_{ij}$.

Since $f \mid \prod_{i=1}^{k} \varphi(\boldsymbol{\xi}_i)$ and f is irreducible, there exists an $i \in \{1, \ldots, k\}$ such that $f \mid \varphi(\boldsymbol{\xi}_i)$, i.e., $\varphi(\boldsymbol{\xi}_i) = fg$ for some $g \in F[\boldsymbol{X}]$. Then $\varphi(\boldsymbol{\xi}_i) = 0$ over the domain $F[\boldsymbol{X}]/(f)$, and hence also over its quotient field $F(f)$. By the previous paragraph, $\boldsymbol{\xi}_i \mod (f)$ is a nonzero vector over $F(f)$. Therefore, $\varphi_{F(f)}$ is isotropic. □

We close this subsection with a known lemma, which we will use in the following section to determine some necessary conditions for stable birational p-forms.

Lemma 2.51 ([Hof04, Lemma 7.12])**.** *Let φ, ψ be anisotropic p-forms over F. If $\varphi_{F(\psi)}$ is isotropic, then $\hat{\nu}_F(\psi) \subseteq \hat{\nu}_F(\varphi)$, $N_F(\psi) \subseteq N_F(\varphi)$ and* $\operatorname{ndeg}_F(\psi) \leq \operatorname{ndeg}_F(\varphi)$*. In this case,* $\operatorname{ndeg}_{F(\psi)}(\varphi_{F(\psi)}) = \frac{1}{p} \operatorname{ndeg}_F(\varphi)$*.*

2.3 Birational and stable birational equivalence

We devote this section to a summary of results about the birational and the stable birational equivalence. These are all known results, but the original statements do not always talk about the (stable) birational equivalence explicitly; whenever this is the case, we provide a short proof, which translates the original statement into our notation.

Recall that by Lemma 1.53, if $\varphi \overset{\text{stb}}{\sim} \psi$, then $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic. By a double application of Lemma 2.51, we get the following:

Lemma 2.52. *Let φ, ψ be anisotropic p-forms over F such that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic (e.g., such that $\varphi \overset{\text{stb}}{\sim} \psi$). Then $\hat{\nu}_F(\varphi) \cong \hat{\nu}_F(\psi)$, $N_F(\varphi) = N_F(\psi)$ and $\operatorname{ndeg}_F(\varphi) = \operatorname{ndeg}_F(\psi)$.*

Under some strong assumptions on the p-forms, the stable birational equivalence coincides with the similarity.

Proposition 2.53. *Let φ, ψ be anisotropic p-forms over F of the same dimension such that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic (e.g., such that $\varphi \overset{\text{stb}}{\sim} \psi$).*

(i) *If φ is similar to a quasi-Pfister form, then $\varphi \overset{\text{sim}}{\sim} \psi$. In particular, if we have $1 \in D_F(\varphi) \cap D_F(\psi)$, then $\varphi \cong \psi$.*

(ii) *If φ is a quasi-Pfister neighbor of codimension one, then $\varphi \overset{\text{sim}}{\sim} \psi$.*

Proof. By Lemma 2.52, there exists a quasi-Pfister form π over F such that $\pi \cong \hat{\nu}_F(\varphi) \cong \hat{\nu}_F(\psi)$.

(i) We can find $c, d \in F^*$ such that $c\varphi \cong \pi \supseteq d\psi$. For dimension reasons, $\varphi \overset{\text{sim}}{\sim} \psi$. Assume $1 \in D_F(\varphi) \cap D_F(\psi)$. As $\varphi \cong c^{-1}\pi$ represents 1, it follows from Lemma 1.42 that $c^{-1}\pi \cong \pi$; thus, $\varphi \cong \pi$. Analogously, $\psi \cong \pi$; hence, $\varphi \cong \psi$.

(ii) Since φ is a quasi-Pfister neighbor of π of codimension 1, we have $\dim \psi = \dim \varphi = \dim \pi - 1$. Moreover, $c\psi \subseteq \hat{\nu}_F(\psi) \cong \pi$ for some $c \in F^*$. Therefore, ψ is a quasi-Pfister neighbor of π of codimension 1. Using [Hof04, Prop. 4.15], any two such p-forms are similar, i.e., $\varphi \overset{\text{sim}}{\sim} \psi$ as claimed. □

The following proposition reinterprets some results from [Scu13] in a way that they solve the analogue of the Quadratic Zariski Problem for some families of p-forms.

Proposition 2.54. *Let φ, ψ be anisotropic p-forms over F of the same dimension d such that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic (e.g., such that $\varphi \overset{\text{stb}}{\sim} \psi$), and assume that at least one of the following holds:*

(i) *φ is similar to a quasi-Pfister form,*

(ii) *φ is a quasi-Pfister neighbor,*

(iii) *$d \leq p^n + n + 2$ where n is the unique integer such that $p^n < d \leq p^{n+1}$,*

(iv) *$p = 2$ or $p = 3$.*

Then $\varphi \overset{\text{bir}}{\sim} \psi$.

Proof. If $d = 1$, then $\varphi \overset{\text{sim}}{\sim} \psi$, and so in particular $\varphi \overset{\text{bir}}{\sim} \psi$. Thus, assume $d \geq 2$.

Since similar forms are birational by Lemma 1.50, part (i) is an easy consequence of Proposition 2.53.

By Lemma 1.34, there exist rational maps $X_\psi \dashrightarrow X_\varphi$ and $X_\varphi \dashrightarrow X_\psi$. Now, part (iv) is covered by [Scu13, Theorem 7.7]. Recalling that $\dim X_\varphi = \dim \varphi - 2$, part (iii) follows from [Scu13, Proposition 7.11].

To prove part (ii), note that $\hat{\nu}_F(\varphi) \cong \hat{\nu}_F(\psi)$ by Lemma 2.52. Since $\dim \varphi = \dim \psi$ by the assumption, it follows that ψ is also a quasi-Pfister neighbor. Now the claim follows from [Scu13, Proposition 7.9]. $\square$

Corollary 2.55. *Any two anisotropic quasi-Pfister neighbors of the same quasi-Pfister form which have the same dimension are birationally equivalent.*

Proof. Let φ and ψ be quasi-Pfister neighbors of a quasi-Pfister form π over F. If $\dim \varphi = \dim \psi = 1$, then the claim is trivial. Thus, suppose $\dim \varphi = \dim \psi \geq 2$. For any extension E/F, we have by Lemma 1.26 that φ_E is isotropic if and only if π_E is isotropic if and only if ψ_E is isotropic. Applying this to $E = F(\varphi)$ and $E = F(\psi)$, we get that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic. The claim follows by Proposition 2.54(ii). $\square$

We conclude the section with a proposition, which connects the stable birational equivalence with the standard splitting pattern (see Subsection 1.1.9 for the definitions); roughly, it states that two stably birationally equivalent p-forms φ, ψ can have different dimensions over F, but each pair of the higher kernel forms φ_k, ψ_k must be of the same dimension.

Proposition 2.56 ([Scu16b, Prop. 5.7])**.** *Let φ, ψ be anisotropic p-forms of dimension at least two such that $\varphi \overset{\text{stb}}{\sim} \psi$. Then* $\mathrm{sSP}(\varphi_1) = \mathrm{sSP}(\psi_1)$*.*

2.4 Vishik equivalence

Recall that, by Lemma 1.61, if φ and ψ are two p-forms over F of dimension at least two such that $\varphi \overset{v}{\sim} \psi$, then $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic. In the case of $\dim\varphi = \dim\psi = 1$, the assumption $\varphi \overset{v}{\sim} \psi$ trivially implies $\varphi \overset{\text{sim}}{\sim} \psi$. Combining these with results from the previous section, we immediately get the following:

Corollary 2.57. *Let φ, ψ be anisotropic p-forms such that $\varphi \overset{v}{\sim} \psi$. Then the following hold:*

(i) $\hat{\nu}_F(\varphi) \cong \hat{\nu}_F(\psi)$, $N_F(\varphi) = N_F(\psi)$ *and* $\operatorname{ndeg}_F(\varphi) = \operatorname{ndeg}_F(\psi)$.

(ii) *If φ is a quasi-Pfister form or a quasi-Pfister neighbor of codimension one, then $\varphi \overset{\text{sim}}{\sim} \psi$.*

(iii) *Assume that at least one of the following holds:*
- *φ is a quasi-Pfister neighbor;*
- $p \in \{2,3\}$;
- $\dim\varphi \leq p^n + n + 2$ *where n is the unique integer such that* $p^n < \dim\varphi \leq p^{n+1}$.

Then $\varphi \overset{\text{bir}}{\sim} \psi$.

2.4.1 Weak Vishik equivalence

Proving that two p-forms are Vishik equivalent might be difficult. Moreover, we do not always need the full strength of the Vishik equivalence; therefore, we define a weaker version.

Definition 2.58. Let φ, ψ be p-forms over F. We say that they are *weakly Vishik equivalent* and write $\varphi \overset{v_0}{\sim} \psi$ if $\dim\varphi = \dim\psi$ and the following holds:

$$\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) = \mathfrak{i}_{\mathrm{d}}(\psi_{F(\sqrt[p]{a})}) \qquad \forall\, a \in F. \tag{v_0}$$

The weak Vishik equivalence has three (rather obvious) properties.

Lemma 2.59. *Let φ, ψ be p-forms over F. Then:*

(i) *If $\varphi \overset{v_0}{\sim} \psi$, then $\mathfrak{i}_{\mathrm{d}}(\varphi) = \mathfrak{i}_{\mathrm{d}}(\psi)$.*

(ii) *$\varphi \overset{v_0}{\sim} \psi$ if and only if $\varphi_{\mathrm{an}} \overset{v_0}{\sim} \psi_{\mathrm{an}}$.*

(iii) *If $\varphi \overset{v}{\sim} \psi$, then $\varphi \overset{v_0}{\sim} \psi$.*

Proof. Part (i) follows directly from the definition, because we include the equality of the defects over the field $F(\sqrt[p]{1}) \simeq F$. Part (ii) is then an easy consequence of part (i). Finally, part (iii) is trivial. □

To prove that Vishik equivalent forms have the same norm field, we used the isotropy over the function fields of each other. Another possibility to determine the norm field is to use Proposition 2.10, which only needs one particular purely inseparable field extension of exponent one. But for weakly Vishik equivalent forms, neither of these is at our disposal; nevertheless, thanks to Corollary 2.26, we are still able to prove that they have the same norm field.

Proposition 2.60. *Let φ, ψ be p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Then $\hat{\nu}_F(\varphi) \cong \hat{\nu}_F(\psi)$, $N_F(\varphi) = N_F(\psi)$ and $\mathrm{ndeg}_F(\varphi) = \mathrm{ndeg}_F(\psi)$.*

Proof. We prove only $N_F(\varphi) = N_F(\psi)$, then the rest follows. Invoking Lemma 2.59 and recalling that the norm field takes into account only the anisotropic part of a p-form, we can assume that φ and ψ are anisotropic. By the definition of the weak Vishik equivalence and by part (iv) of Lemma 2.25, we have for any $a \in F$:

$$\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) > 0 \quad \Longrightarrow \quad \mathfrak{i}_{\mathrm{d}}(\psi_{F(\sqrt[p]{a})}) > 0 \quad \Longrightarrow \quad a \in N_F(\psi).$$

Thus, we can apply Corollary 2.26 on the p-form φ and the field $E = N_F(\psi)$; we obtain $N_F(\varphi) \subseteq N_F(\psi)$. By the symmetry of the argument, we get $N_F(\varphi) = N_F(\psi)$. □

Since (weakly) Vishik equivalent forms are always of the same dimension, we immediately get:

Corollary 2.61. *Let φ, ψ be p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$.*
(i) *If φ is minimal over F, then ψ is also minimal over F.*
(ii) *If φ is a quasi-Pfister form or a quasi-Pfister neighbor of codimension one over F, then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. By Proposition 2.60, we have $\mathrm{ndeg}_F\,\varphi = \mathrm{ndeg}_F\,\psi$; since $\dim\varphi = \dim\psi$, part (i) follows immediately. To prove part (ii), note that we also have $\hat{\nu}_F(\varphi) \cong \hat{\nu}_F(\psi)$; if φ is a quasi-Pfister form, then $\varphi \cong \hat{\nu}_F(\psi) \supseteq c\psi$ for some $c \in F^*$, and hence $\varphi \overset{\mathrm{sim}}{\sim} \psi$ for dimensional reason. In the case when φ is a quasi-Pfister neighbor of codimension one, we proceed as in the proof of part (ii) of Proposition 2.53. □

2.4.2 Relation with similarity

We already know that for quasi-Pfister forms and quasi-Pfister neighbors of codimension one, Question Q has a positive answer (i.e., the Vishik equivalence is sufficient for the p-forms to be similar). We would like to prove that Question Q has a positive answer for all p-forms. Unfortunately, this is not true in general – as we will see, it is even not true for quasi-Pfister neighbors. We will provide two examples of pairs of p-forms that are Vishik equivalent but not similar. However, both of the counterexamples have two things in common: First, the considered p-forms are subforms of $\langle\langle a\rangle\rangle$, and so they have norm degree one. Second, they do not work for $p = 2$ and $p = 3$. We will focus on the characteristic two case in Chapter 3, but the case $p = 3$ remains open as well as p-forms of higher norm degrees.

Before we can get to the counterexamples, we need some lemmas.

Lemma 2.62. *Let $1 \leq k, l \leq p-1$, $a \in F \setminus F^p$, and let E/F be a field extension. Then the following are equivalent:*
(i) *$\langle 1, a^k\rangle$ is isotropic over E,*
(ii) *$\langle 1, a^l\rangle$ is isotropic over E,*
(iii) *$a^k \in E^p$,*
(iv) *$a^l \in E^p$.*

Proof. Assume that $\langle 1, a^k\rangle$ is isotropic over E. This is equivalent to the existence of $x, y \in E$, at least one (and hence both) of them nonzero, such that $x^p + a^k y^p = 0$, which is equivalent to $a^k = \frac{(-x)^p}{y^p} \in E^p$. This proves both (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv).

Now let $1 \le t \le p-1$ be such that $kt \equiv l \pmod p$. If $a^k \in E^p$, then $a^{kt} \in E^p$, and hence also $a^l \in E^p$. This proves (iii) $\Rightarrow$ (iv) and, by symmetry, also (iv) $\Rightarrow$ (iii). $\square$

Lemma 2.63. *Let $a \in F \setminus F^p$ and $k, l \in \mathbb{Z}$ with $k, l \not\equiv 0 \pmod p$. Then $\langle 1, a^k\rangle \overset{\text{sim}}{\sim} \langle 1, a^l\rangle$ if and only if $k \equiv \pm l \pmod p$.*

Proof. If $k \equiv l \pmod p$, then $k = pm + l$ for some $m \in \mathbb{Z}$, and so we have $\langle 1, a^k\rangle \cong \langle 1, a^{pm+l}\rangle \cong \langle 1, a^l\rangle$. If $k \equiv -l \pmod p$, then we can write $k = pn - l$ for some $n \in \mathbb{Z}$, and then we have

$$\langle 1, a^k\rangle = \langle 1, a^{pn-l}\rangle \cong \langle 1, a^{-l}\rangle \overset{\text{sim}}{\sim} a^l\langle 1, a^{-l}\rangle \cong \langle a^l, 1\rangle \cong \langle 1, a^l\rangle.$$

To prove the opposite direction, assume that $\langle 1, a^k\rangle \overset{\text{sim}}{\sim} \langle 1, a^l\rangle$, i.e., there exists $c \in F^*$ such that $c\langle 1, a^k\rangle \cong \langle 1, a^l\rangle$. Then necessarily $c \in D_F(\langle 1, a^l\rangle)$, so we can find $x, y \in F$ such that $c = x^p + a^l y^p$. On the other hand, $1 \in D_F(c\langle 1, a^k\rangle)$, and thus $1 = cu^p + ca^k v^p$ for some $u, v \in F$. Putting these together, we have

$$1 = (x^p + a^l y^p)u^p + (x^p + a^l y^p)a^k u^p = (xu)^p + a^l(yu)^p + a^k(xv)^p + a^{k+l}(yv)^p.$$

Suppose $k \not\equiv \pm l \pmod p$. Then $\tau = \langle 1, a^k, a^l, a^{k+l}\rangle$ is a subform of $\langle\langle a\rangle\rangle$, and hence anisotropic. Thus, τ represents every element of $D_F(\tau)$ uniquely, so we get $xu \neq 0$ and $yu = xv = yv = 0$. This implies $x, u \neq 0$ and $y = v = 0$; hence, $c = x^p$ and $\langle 1, a^k\rangle \cong \langle 1, a^l\rangle$. Thus, $a^l \in D_F(\langle 1, a^k\rangle)$, which is impossible for $l \not\equiv 0, k \pmod p$. $\square$

Lemma 2.64. *Let $a \in F \setminus F^p$ and φ, ψ be quasi-Pfister neighbors of $\langle\langle a\rangle\rangle$ of the same dimension. Then $\varphi \overset{v}{\sim} \psi$.*

Proof. Obviously, both φ and ψ are anisotropic. Let E/F be any field extension; it holds $(\varphi_E)_{\text{an}}, (\psi_E)_{\text{an}} \subseteq (\langle\langle a\rangle\rangle_E)_{\text{an}}$. Then either $\langle\langle a\rangle\rangle_E$ is anisotropic, and hence both φ_E and ψ_E are anisotropic, or we have $(\langle\langle a\rangle\rangle_E)_{\text{an}} \cong \langle 1\rangle_E$ by Lemma 2.18, in which case $(\varphi_E)_{\text{an}} \cong \langle 1\rangle_E \cong (\psi_E)_{\text{an}}$. Consequently, $\varphi \overset{v}{\sim} \psi$. $\square$

Example 2.65. Let $p > 3$ and $\varphi \cong \langle 1, a\rangle$, $\psi \cong \langle 1, a^2\rangle$ be p-forms with $a \in F \setminus F^p$. Note that φ and ψ are quasi-Pfister neighbors of $\langle\langle a\rangle\rangle$; therefore, $\varphi \overset{v}{\sim} \psi$ by Lemma 2.64. On the other hand, Lemma 2.63 ensures that φ and ψ are not similar for any $p > 3$.

Note that the problems with characteristics two and three are different. If $p = 2$, then φ is anisotropic while ψ is isotropic. If $p = 3$, it holds that $a^2\varphi \cong \psi$.

Example 2.66. Let $p = 5$ and $a \in F \setminus F^5$. Set $\pi \cong \langle 1, a, a^2, a^3, a^4 \rangle \cong \langle\langle a \rangle\rangle$, $\varphi \cong \langle 1, a, a^2 \rangle$ and $\psi \cong \langle 1, a, a^3 \rangle$. It follows $\varphi \overset{v}{\sim} \psi$ by Lemma 2.64.

Assume that $c \in F$ is such that $c\psi \cong \varphi$. Note that

$$D_F(\varphi) = \{x_0^5 + ax_1^5 + a^2x_2^5 \mid x_i \in F,\ 0 \leq i \leq 2\},$$
$$D_F(\pi) = \{x_0^5 + ax_1^5 + a^2x_2^5 + a^3x_3^5 + a^4x_4^5 \mid x_i \in F,\ 0 \leq i \leq 4\},$$

and the expression of any element of $D_F(\varphi)$ (resp. $D_F(\pi)$) is unique thanks to the anisotropy of φ (resp. π). In particular, φ does not represent any term of the form a^3x^5 or a^4x^5 with $x \in F$.

As $c \in D_F(\varphi)$, we can write $c = x^5 + ay^5 + a^2z^5$ for some $x, y, z \in F$. Since $ca = ax^5 + a^2y^5 + a^3z^5 \in D_F(\varphi)$, it follows that $z = 0$. Furthermore, $ca^3 = a^3x^5 + a^4y^5 + (az)^5 \in D_F(\varphi)$ implies that $x = y = 0$. But then $c = 0$, which is absurd. Therefore, φ and ψ are not similar.

2.4.3 Similarity factors

As we have seen, the Vishik equivalence is not as strong on p-forms as the similarity is, but it still does well when restricted to quasi-Pfister forms. In this subsection, we will show that this is also true for divisibility by quasi-Pfister forms. In particular, we will prove that if $\varphi \overset{v}{\sim} \psi$ for some p-forms φ, ψ with φ divisible by a quasi-Pfister form π, then ψ is divisible by π, too.

Let φ be a p-form defined over F. Recall that we write

$$G_F(\varphi) = \{x \in F^* \mid x\varphi \cong \varphi\} \cup \{0\},$$

and call the nonzero elements of this set the *similarity factors* of φ. As observed in [Hof04, Prop. 6.4], the set $G_F(\varphi)$ together with the usual operations is a finite field extension of F^p inside $N_F(\varphi)$; in particular, by Lemma 2.7, there exists a p-independent set $\{a_1, \dots, a_m\} \subseteq F^*$ such that $G_F(\varphi) = F^p(a_1, \dots, a_m)$. We denote $\hat{\sigma}_F(\varphi) \cong \langle\langle a_1, \dots, a_m \rangle\rangle$ and call it the *similarity form* of φ over F. Moreover, again by [Hof04, Prop. 6.4], there exists a p-form γ over F such that $\varphi_{\mathrm{an}} \cong \hat{\sigma}_F(\varphi) \otimes \gamma$. It holds that $D_F(\hat{\sigma}_F(\varphi)) = G_F(\varphi)$.

We will show that Vishik equivalent p-forms have the same similarity factors. But first, we need a simple lemma.

Lemma 2.67. *Let π_1, π_2 be anisotropic Pfister p-forms. Then $\pi_1 \subseteq \pi_2$ if and only if $\pi_2 \cong \pi_1 \otimes \gamma$ for some p-form γ over F. In that case, γ can be chosen to be a quasi-Pfister form.*

Proof. Write $\pi_1 \cong \langle\langle a_1, \dots, a_r \rangle\rangle$ and $\pi_2 \cong \langle\langle b_1, \dots, b_s \rangle\rangle$, which means that we have $N_F(\pi_1) = F^p(a_1, \dots, a_r)$ and $N_F(\pi_2) = F^p(b_1, \dots, b_s)$. Note that the bijection between finite field extensions of F^p inside F and Pfister p-forms over F described in Proposition 2.11 implies that $\pi_1 \subseteq \pi_2$ if and only if $N_F(\pi_1) \subseteq N_F(\pi_2)$.

If this holds, then (by Lemma 2.7) we extend $\{a_1, \dots, a_r\}$ to a p-basis of $F^p(b_1, \dots, b_s)$; thus, $F^p(b_1, \dots, b_s) = F^p(a_1, \dots, a_r, a_{r+1}, \dots, a_s)$ for some

$a_{r+1}, \ldots, a_s \in F^*$, and so $\langle\langle b_1, \ldots, b_s\rangle\rangle \cong \langle\langle a_1, \ldots, a_r\rangle\rangle \otimes \langle\langle a_{r+1}, \ldots, a_s\rangle\rangle$ by Corollary 2.6.

On the other hand, assume $\pi_2 \cong \pi_1 \otimes \gamma$ for a p-form γ. If $1 \in D_F^*(\gamma)$, then we can write $\gamma \cong \langle 1\rangle \perp \gamma'$ for a suitable p-form γ'; in that case, $\pi_2 \cong \pi_1 \perp \pi_1 \otimes \gamma'$ and we are done. More generally, if $c \in D_F^*(\gamma)$, then $c \in D_F^*(\pi_2) = G_F^*(\pi_2)$. Now $\pi_2 \cong c\pi_2 \cong \pi_1 \otimes c\gamma$ with $1 \in D_F^*(c\gamma)$, so we are done by the previous case. $\square$

Proposition 2.68. *Let φ, ψ be anisotropic p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Then $G_F(\varphi) = G_F(\psi)$.*

Proof. The inclusion $F^p \subseteq G_F(\psi)$ is obvious. Hence, pick $a \in G_F(\varphi) \setminus F^p$. Since $G_F(\varphi) = D_F(\hat{\sigma}_F(\varphi))$ is a field, we get that $a^2, \ldots, a^{p-1} \in D_F(\hat{\sigma}_F(\varphi))$ and $1 \in D_F(\hat{\sigma}_F(\varphi))$. As the p-form $\langle 1, a, \ldots, a^{p-1}\rangle \cong \langle\langle a\rangle\rangle$ is anisotropic, we get $\langle\langle a\rangle\rangle \subseteq \hat{\sigma}_F(\varphi)$ by Lemma 2.1; therefore, $\langle\langle a\rangle\rangle$ divides $\hat{\sigma}_F(\varphi)$ by Lemma 2.67. Since further $\hat{\sigma}_F(\varphi)$ divides φ, we can find a p-form φ' defined over F such that $\varphi \cong \langle\langle a\rangle\rangle \otimes \varphi'$.

Set $E = F(\sqrt[p]{a})$; it holds that $(\varphi_E)_{\mathrm{an}} \cong (\varphi'_E)_{\mathrm{an}}$. Therefore, we have $\dim(\varphi'_E)_{\mathrm{an}} \leq \frac{1}{p}\dim\varphi$. But the reverse inequality is true by Lemma 2.25; hence, φ'_E is anisotropic and $\dim(\varphi_E)_{\mathrm{an}} = \frac{1}{p}\dim\varphi$.

Since $\varphi \overset{v_0}{\sim} \psi$, we have $\dim(\psi_E)_{\mathrm{an}} = \frac{1}{p}\dim\psi$. Let ψ' be a p-form over F such that $\psi'_E \cong (\psi_E)_{\mathrm{an}}$. As it holds that

$$D_F(\psi) \subseteq D_E(\psi) = D_E(\psi') = D_F(\langle\langle a\rangle\rangle \otimes \psi'),$$

we get $\psi \subseteq \langle\langle a\rangle\rangle \otimes \psi'$ by Lemma 2.1. Comparing the dimensions implies that $\psi \cong \langle\langle a\rangle\rangle \otimes \psi'$, and hence $a \in G_F(\psi)$.

All in all, we have proved $G_F(\varphi) \subseteq G_F(\psi)$. The other inclusion follows by the symmetry of the argument. $\square$

Lemma 2.69. *Let τ be a p-form defined over F and K/F be a field extension such that $K^p \subseteq G_F(\tau)$. Let φ, ψ be p-forms defined over F, anisotropic over K and such that $\varphi_K \cong \psi_K$. Then $\varphi \otimes \tau \cong \psi \otimes \tau$ over F.*

Proof. Let $\varphi \cong \langle b_1, \ldots, b_m\rangle$ and $\psi \cong \langle c_1, \ldots, c_m\rangle$. It follows from the assumptions that $\{b_1, \ldots, b_m\}$ and $\{c_1, \ldots, c_m\}$ are two bases of the same K^p-vector space. Recall that we can get one basis from the other by a finite series of operations of the following type: an exchange of two basis elements, scalar multiplication of one basis element, and adding one basis element to another basis element. Thus, it is sufficient to show that the following hold for any $0 \leq i, j \leq m$, $i \neq j$, and $a \in K^p$:

(i) $\langle b_0, \ldots, b_i, \ldots, b_j, \ldots, b_m\rangle \otimes \tau \cong \langle b_0, \ldots, b_j, \ldots, b_i, \ldots, b_m\rangle \otimes \tau$,

(ii) $\langle b_0, \ldots, b_i, \ldots, b_m\rangle \otimes \tau \cong \langle b_0, \ldots, ab_i, \ldots, b_m\rangle \otimes \tau$,

(iii) $\langle b_0, \ldots, b_i, \ldots, b_j, \ldots, b_m\rangle \otimes \tau \cong \langle b_0, \ldots, b_i + b_j, \ldots, b_j, \ldots, b_m\rangle \otimes \tau$.

Part (i) is obvious. Part (ii) follows from the fact that $b_i\tau \cong ab_i\tau$ since $a \in G_F(\tau)$. To prove part (iii), note that $b_i\tau \perp b_j\tau \cong (b_i + b_j)\tau \perp b_j\tau$. $\square$

Proposition 2.70. *Let $\pi \cong \langle\langle a_1, \ldots, a_n\rangle\rangle$ be an anisotropic quasi-Pfister form over F and $K = F(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_n})$. Assume that $\varphi \cong \pi \otimes \varphi'$ and $\psi \cong \pi \otimes \psi'$ are anisotropic p-forms over F such that $\varphi'_K \overset{\mathrm{sim}}{\sim} \psi'_K$. Then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. Without loss of generality, assume that $1 \in D_F(\varphi') \cap D_F(\psi')$. Write $\varphi' \cong \langle 1, b_1, \dots, b_m \rangle$, and let $c \in K^*$ be such that $c\psi'_K \cong \varphi'_K$. Then

$$c \in D_K(\varphi') = \operatorname{span}_{K^p}\{1, b_1, \dots, b_m\} \subseteq F;$$

thus, $c\psi'$ is defined over F. By Theorem 2.28, we have $p^n \mathfrak{i}_\mathrm{d}(\varphi'_K) = \mathfrak{i}_\mathrm{d}(\varphi) = 0$, and hence φ'_K is anisotropic; analogously, we get that $c\psi'_K$ is anisotropic. Now Lemma 2.69 implies that $\pi \otimes \varphi' \cong \pi \otimes c\psi'$. As $\pi \otimes c\psi' \cong c(\pi \otimes \psi')$, the claim follows. $\square$

Remark 2.71. Propositions 2.68 and 2.70 can be used for a simplification of the problem, whether the Vishik equivalence implies the similarity. Namely, we can restrict ourselves to the case of forms with the similarity factors F^p: Let φ, ψ be two anisotropic p-forms over F such that $\varphi \overset{v}{\sim} \psi$. Since $G_F(\varphi) = G_F(\psi)$ by Proposition 2.68, we can write $\varphi \cong \pi \otimes \varphi'$ and $\psi \cong \pi \otimes \psi'$ with the quasi-Pfister form $\pi \cong \hat{\sigma}_F(\varphi) \cong \hat{\sigma}_F(\psi)$ and some anisotropic p-forms φ', ψ' defined over F. Let $\pi \cong \langle\langle a_1, \dots, a_n \rangle\rangle$, and put $K = F(\sqrt[p]{a_1}, \dots, \sqrt[p]{a_n})$. Then φ'_K, ψ'_K are anisotropic by Lemma 2.27. Hence $(\varphi_K)_\mathrm{an} \cong \varphi'_K$ and $(\psi_K)_\mathrm{an} \cong \psi'_K$, and we get $\varphi'_K \overset{v}{\sim} \psi'_K$ by Lemma 1.59 (note that to apply this lemma, we need the full strength of the Vishik equivalence, the weak Vishik equivalence does not have to be sufficient here). If we knew that $\varphi'_K \overset{\mathrm{sim}}{\sim} \psi'_K$, it would follow by Proposition 2.70 that $\varphi \overset{\mathrm{sim}}{\sim} \psi$.

2.4.4 Relation with birationality

Recall that we have already seen in Corollary 2.57 that, under some additional assumptions (e.g., $p = 2$), Vishik equivalent p-forms are also birationally equivalent. We are now interested in the opposite direction: Are birationally equivalent p-forms necessarily Vishik equivalent? The following example shows that it is not true in general, even for $p = 2$.

Example 2.72. Let $\{a, b, c\} \subseteq F$ be p-independent. Set $\varphi \cong \langle\langle a, b \rangle\rangle \perp c\langle 1, a \rangle$ and $\psi \cong \langle\langle a, b \rangle\rangle \perp c\langle 1, b \rangle$. Then φ and ψ are anisotropic quasi-Pfister neighbors of the p-form $\langle\langle a, b, c \rangle\rangle$, and hence $\varphi \overset{\mathrm{bir}}{\sim} \psi$ by Corollary 2.55.

Set $E = F(\sqrt[p]{a})$. Since $\langle\langle b, c \rangle\rangle$ is anisotropic over E by Lemma 2.36, its subforms $\langle\langle b \rangle\rangle \perp \langle c \rangle$ and $\langle\langle b \rangle\rangle \perp c\langle 1, b \rangle$ are also anisotropic over E; therefore, we have

$$(\varphi_E)_\mathrm{an} \cong (\langle\langle b \rangle\rangle \perp \langle c \rangle)_E \qquad \text{and} \qquad (\psi_E)_\mathrm{an} \cong (\langle\langle b \rangle\rangle \perp c\langle 1, b \rangle)_E.$$

Obviously, $\dim(\varphi_E)_\mathrm{an} \neq \dim(\psi_E)_\mathrm{an}$, and so $\varphi \overset{v}{\not\sim} \psi$.

Remark 2.73. We want to point out the following consequences of Example 2.72 for p-forms φ and ψ:

(i) $\varphi \overset{\mathrm{bir}}{\sim} \psi$ does not imply $\varphi \overset{\mathrm{sim}}{\sim} \psi$ (because otherwise we would have $\varphi \overset{\mathrm{bir}}{\sim} \psi \Rightarrow \varphi \overset{\mathrm{sim}}{\sim} \psi \Rightarrow \varphi \overset{v}{\sim} \psi$, a contradiction);

(ii) $\varphi \overset{\mathrm{stb}}{\sim} \psi$ does not imply $\varphi \overset{v}{\sim} \psi$ (since otherwise it would hold that $\varphi \overset{\mathrm{bir}}{\sim} \psi \Rightarrow \varphi \overset{\mathrm{stb}}{\sim} \psi \Rightarrow \varphi \overset{v}{\sim} \psi$, a contradiction again).

2.5 Summary

Now we are ready to complete the diagram from Figure 1.1 in the case of p-forms. Recall that in Examples 2.65 and 2.66, we provided pairs of p-forms which were Vishik equivalent but not similar. Example 2.72 and Remark 2.73 explain all the remaining "NO's" in the diagram.

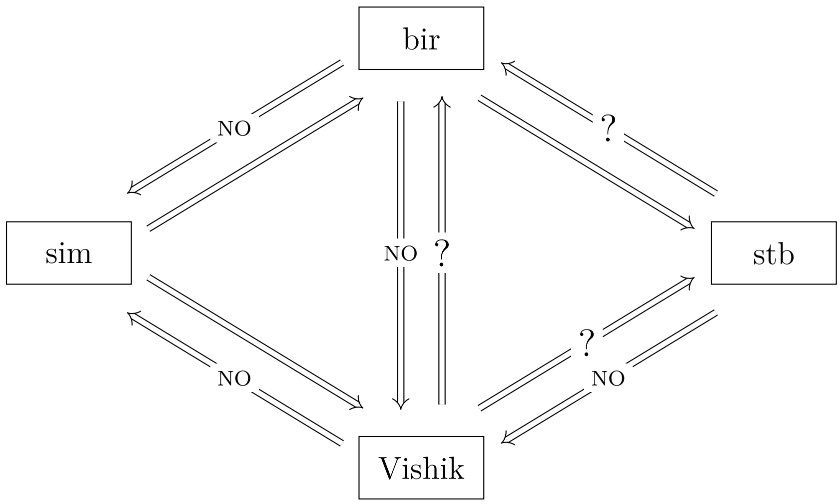

Figure 2.1: The equivalence relations for p-forms

Remark 2.74. We would like to emphasize that all the question marks are a "YES" in the case of $p \in \{2, 3\}$, and also for some families of p-forms. As one can see from the diagram in Figure 2.2, the bottleneck for all these implications is Proposition 2.54.

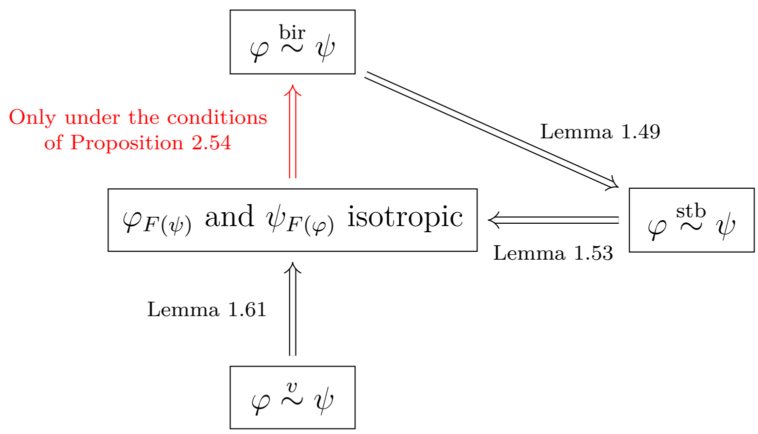

Figure 2.2: Relations of p-forms labeled with a question mark

Recall that Proposition 2.54 is just a reinterpretation of some theorems from [Scu13]. According to this paper, the actual obstacle is the following question:

Question A ([Scu13, Question 5.17]). *Let φ be an anisotropic p-form over F. Is it true that $\mathrm{i_d}(\varphi_{F(\varphi)})$ is minimal among the values of $\mathrm{i_d}(\varphi_E)$, where E runs over all field extensions of F over which φ becomes isotropic?*

This question has a positive answer in case of $p = 2$ and $p = 3$ (see [Scu13]), and it is expected that it is true for any prime p.

3. Totally singular quadratic forms

In this chapter, we consider the special case of quasilinear p-forms for $p = 2$, i.e., the totally singular quadratic forms. Thus, all fields in this chapter are of characteristic 2.

We start with a re-examination of Figure 2.1: First of all, recall that the unlabeled implications are true for all forms for trivial reasons (see Figure 1.1). By Remark 2.74, all the question marks have a positive answer in the case of totally singular quadratic forms. Moreover, note that Example 2.72 does not have any assumptions on p, and hence it is true for totally singular forms in particular. Together with this example, Remark 2.73 holds for totally singular forms, too. Contrary to that, Examples 2.65 and 2.66 exclude the case $p = 2$. Therefore, as it can be seen in Figure 3.1, we are left with Question Q, i.e., with the following question: *Are Vishik equivalent totally singular quadratic forms always similar?*

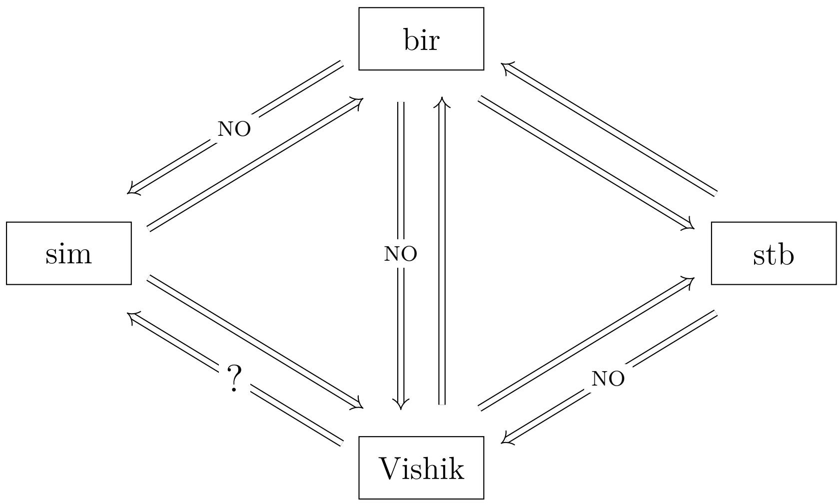

Figure 3.1: The equivalence relations for totally singular quadratic forms

Unfortunately, we are neither able to prove that the question has a positive answer nor find a counterexample. However, we show that it is true at least for some families of totally singular quadratic forms; see Theorems 3.6 and 3.16.

3.1 Preliminaries

Totally singular quadratic forms are the special case of p-forms with $p = 2$, so we will use many results of the previous chapter. One of them, Proposition 2.45, will be very important in our work, so we restate it here.

Proposition 3.1 ([Lag04, Prop. 2.11])**.** *Let $a \in F \setminus F^2$ and let φ be an anisotropic totally singular quadratic form over F with $\dim \varphi \geq 2$. If φ becomes isotropic over $E = F(\sqrt{a})$, then there exists a totally singular quadratic form τ over F such that $\dim \tau = \mathfrak{i}_{\mathrm{d}}(\varphi_E)$ and $\tau \otimes \langle 1, a \rangle \subseteq \varphi$.*

3.2 Minimal quadratic forms

Recall that a minimal quadratic form is an anisotropic totally singular quadratic form φ over F such that $\operatorname{ndeg}_F \varphi = 2^{\dim \varphi - 1}$. For some general properties on minimal forms, see Subsection 2.1.3.

3.2.1 Forms of dimensions two and three

Note that all anisotropic totally singular forms of dimensions two and three are minimal by Proposition 2.13. For now, we concentrate on such forms and prove some preparatory lemmas.

Lemma 3.2. *Let $b, c \in F \setminus F^2$. Then $c \in D_F(\langle 1, b\rangle)$ if and only if $b \in D_F(\langle 1, c\rangle)$ if and only if $\langle 1, b\rangle \cong \langle 1, c\rangle$.*

Proof. $c \in D_F(\langle 1, b\rangle)$ holds if we can find $x, y \in F$ such that $x^2 + by^2 = c$; as $c \notin F^2$, we have $y \neq 0$. It follows $\langle 1, b\rangle \cong \langle 1, by^2\rangle \cong \langle 1, x^2 + by^2\rangle \cong \langle 1, c\rangle$, and thus also $b \in D_F(\langle 1, c\rangle)$. $\square$

Lemma 3.3. *Let $a, b, x, y \in F$ and $y \neq 0$. Then*

$$\langle 1, a, bx^2 + aby^2\rangle \cong \left(a + \left(\tfrac{x}{y}\right)^2\right) \langle 1, a, b\rangle.$$

Proof. We have

$$\begin{aligned}\langle 1, a, bx^2 + aby^2\rangle &\cong \left\langle 1, a + \left(\tfrac{x}{y}\right)^2, b\left(a + \left(\tfrac{x}{y}\right)^2\right)\right\rangle \\ &\cong \left(a + \left(\tfrac{x}{y}\right)^2\right) \left\langle a + \left(\tfrac{x}{y}\right)^2, 1, b\right\rangle \cong \left(a + \left(\tfrac{x}{y}\right)^2\right) \langle 1, a, b\rangle. \quad \square\end{aligned}$$

Lemma 3.4. *Let $a, b, c \in F \setminus F^2$, and suppose that $a \in D_F(\langle 1, c, bc\rangle)$. Then there exists $s \in F$ such that $\langle 1, c, bc\rangle \cong (a + s^2)\langle 1, a, b\rangle$.*

Proof. Let $x, y, z \in F$ be such that

$$a = x^2 + cy^2 + bcz^2. \tag{3.1}$$

Suppose first $z = 0$; then $a \in D_F(\langle 1, c\rangle)$, and we have $\langle 1, c\rangle \cong \langle 1, a\rangle$ by Lemma 3.2. Moreover, since $a \notin F^2$, it must be $y \neq 0$, and (3.1) can be rewritten as $c = \left(\frac{x}{y}\right)^2 + a\left(\frac{1}{y}\right)^2$. Putting these together, we obtain

$$\langle 1, c, bc\rangle \cong \left\langle 1, a, b\Big(\left(\tfrac{x}{y}\right)^2 + a\left(\tfrac{1}{y}\right)^2\Big)\right\rangle \cong \langle 1, a, b(x^2 + a)\rangle;$$

hence, $\langle 1, c, bc\rangle \cong (a + x^2)\langle 1, a, b\rangle$ by Lemma 3.3.

Now let $z \neq 0$; then

$$\langle 1, c, bc\rangle \cong \langle 1, c, x^2 + cy^2 + bcz^2\rangle \cong \langle 1, a, c\rangle.$$

Moreover, note that $z \neq 0$ and $b \notin F^2$ imply $y^2 + bz^2 \neq 0$; hence, we can rewrite (3.1) as

$$c(y^2 + bz^2)^2 = (a + x^2)(y^2 + bz^2) = (xy)^2 + ay^2 + b(xz)^2 + abz^2.$$

Thus, we have

$$\langle 1, a, c\rangle \cong \langle 1, a, b(xz)^2 + abz^2\rangle \cong (a + x^2)\langle 1, a, b\rangle,$$

where the latter isometry follows from Lemma 3.3. Therefore, we get $\langle 1, c, bc\rangle \cong (a + x^2)\langle 1, a, b\rangle$ in this case, too. □

The following lemma mimics the situation we end up with after applying Proposition 3.1.

Lemma 3.5. *Let ψ be a totally singular quadratic form over F such that $1 \in D_F(\psi)$. Let $a, c \in F^*$ be such that $c\langle 1, a\rangle$ is anisotropic over F, and suppose $c\langle 1, a\rangle \subseteq \psi$. Then*
(i) *either $\langle 1, c, ac\rangle \subseteq \psi$ (this occurs if and only if $1 \notin D_F(c\langle 1, a\rangle)$),*
(ii) *or $\langle 1, a\rangle \subseteq \psi$ and $c \in D_F(\langle 1, a\rangle)$.*

Proof. If $1 \notin D_F(c\langle 1, a\rangle)$, then $\langle 1, a, ca\rangle$ is anisotropic, and so $\langle 1, c, ca\rangle \subseteq \psi$ by Lemma 2.1. On the other hand, if $1 \in D_F(c\langle 1, a\rangle)$, then it follows that $c \in D_F(\langle 1, a\rangle) = G_F(\langle 1, a\rangle)$; thus, $\langle 1, a\rangle \cong c\langle 1, a\rangle \subseteq \psi$. □

3.2.2 Vishik equivalence

Now we extend Subsection 2.1.3 and prove that (weakly) Vishik equivalent minimal quadratic forms are always similar. The proof is based mainly on the 2-independence of the coefficients as explored in Lemma 2.24.

Theorem 3.6. *Let φ, ψ be totally singular quadratic forms over F such that φ_{an} is minimal over F. If $\varphi \overset{v_0}{\sim} \psi$, then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. Invoking Lemmas 2.59 and 1.37, we can assume that the forms φ and ψ are anisotropic. By Lemma 2.23, we can suppose that $\varphi \cong \langle 1, a_1, \ldots, a_n\rangle$ for some $\{a_1, \ldots, a_n\} \subseteq F$ which is 2-independent over F; it follows that $N_F(\varphi) = F^2(a_1, \ldots, a_n)$, and $\mathcal{B}_{(0)} = \{a_1, \ldots, a_n\}$ is a 2-basis of $N_F(\varphi)$ over F. Moreover, $N_F(\varphi) = N_F(\psi)$ by Proposition 2.60, and ψ is minimal over F by Corollary 2.61.

We start with an observation:

> *For any $a \in D_F(\varphi)\setminus F^2$, the form $\varphi_{F(\sqrt{a})}$ is isotropic. As $\varphi \overset{v_0}{\sim} \psi$, ψ must be isotropic over $F(\sqrt{a})$ as well. By Proposition 3.1, we can find $c \in F^*$ such that $c\langle 1, a\rangle \subseteq \psi$.* (👁)

The main idea is to look at such ca for some $a \in D_F(\varphi)$, and express it with respect to an appropriate 2-basis of $N_F(\psi)$. Applying Lemma 2.24, we get that almost all coefficients must be zero. Via a combination of different values of a, we usually end up with the conclusion that c must be a square, which means that $a \in D_F(\psi)$. In particular, we will prove that there is a scalar multiple of ψ which represents all the values $1, a_1, \ldots, a_k$.

We divide the proof into several steps and cases. To simplify the notation and omit multiple indices, we use the same letters repeatedly – the meaning of s_i, u_i, x_i, y_i, z_i changes in each subcase (although they are usually used in similar situations). On the other hand, the meaning of a_i, c_i, d_i, e_i is "global", i.e., does not change during the proof. Moreover, we consider only 2-independence and 2-bases over F, and so we omit to repeat "over F" each time. We also would like to recall the notation from the end of Subsection 2.1.1: By a "unique expression with respect to a 2-basis" we actually mean the unique expression with respect to the corresponding F^2-linear basis.

(1) First of all, note that (◉) proves the claim completely if $\dim \psi = 2$. Therefore, suppose $\dim \psi \geq 3$.

As the first step, we will prove that ψ contains a subform similar to $\langle 1, a_1, a_2 \rangle$: By (◉), we find $c_1 \in F^*$ such that $c_1 \langle 1, a_1 \rangle \subseteq \psi$. Since we are interested in ψ only up to similarity, we can assume without loss of generality that $c_1 = 1$. Now, $c_2 \langle 1, a_2 \rangle \subseteq \psi$ for some $c_2 \in F^*$. By Lemma 3.5, there are two possibilities: Either $\langle 1, a_2 \rangle \subseteq \psi$ or $\langle 1, c_2, c_2 a_2 \rangle \subseteq \psi$.

(1A) If $\langle 1, a_2 \rangle \subseteq \psi$, we get $\langle 1, a_1, a_2 \rangle \subseteq \psi$ immediately.

(1B) Suppose that $\langle 1, c_2, c_2 a_2 \rangle \subseteq \psi$ holds. We have to further distinguish two cases, depending on whether a_1 is represented by the form $\langle 1, c_2, c_2 a_2 \rangle$ or not.

(1Bi) Assume $a_1 \in D_F(\langle 1, c_2, c_2 a_2 \rangle)$. Then $\langle 1, c_2, c_2 a_2 \rangle \overset{\text{sim}}{\sim} \langle 1, a_1, a_2 \rangle$ by Lemma 3.4.

(1Bii) Let $a_1 \notin D_F(\langle 1, c_2, c_2 a_2 \rangle)$; then $\langle 1, a_1, c_2, c_2 a_2 \rangle$ is an anisotropic subform of ψ, and hence $\psi \cong \langle 1, a_1, c_2, c_2 a_2, s_4, \dots, s_n \rangle$ for some suitable $s_4, \dots, s_n \in F^*$. Since ψ is minimal, the set

$$\mathcal{B}_{(1\text{Bii})} = \{a_1, c_2, c_2 a_2, s_4, \dots, s_n\}$$

is 2-independent, and hence it is a 2-basis of $N_F(\psi)$. Since $a_2 \in N_F(\psi)$, the element a_2 has a unique expression with respect to $\mathcal{B}_{(1\text{Bii})}$:

$$a_2 = c_2 \cdot c_2 a_2 \cdot \left(\frac{1}{c_2} \right)^2 . \tag{3.2}$$

It follows by Lemma 2.24 that $a_2 \notin D_F(\psi)$.

Furthermore, as $a_1 + a_2 \in D_F(\varphi) \setminus F^2$, we use (◉) to find $d_2 \in F^*$ such that $d_2 \langle 1, a_1 + a_2 \rangle \subseteq \psi$. In particular, $d_2 \in D_F(\psi)$; hence,

$$d_2 = x_0^2 + a_1 \cdot x_1^2 + c_2 \cdot x_2^2 + c_2 a_2 \cdot x_3^2 + \sum_{i=4}^{n} s_i \cdot x_i^2$$

for some suitable $x_0, \dots, x_n \in F$. Multiplying d_2 by $(a_1 + a_2)$ and using (3.2) (i.e., expressing $d_2(a_1 + a_2)$ with respect to the basis $\mathcal{B}_{(1\text{Bii})}$), we get

$$\begin{aligned} d_2(a_1 + a_2) = {} & (a_1 x_1)^2 + a_1 \cdot x_0^2 + c_2 \cdot (a_2 x_3)^2 + c_2 a_2 \cdot x_2^2 \\ & + a_1 \cdot c_2 \cdot x_2^2 + a_1 \cdot c_2 a_2 \cdot x_3^2 + \sum_{i=4}^{n} a_1 \cdot s_i \cdot x_i^2 + c_2 \cdot c_2 a_2 \cdot \left(\frac{x_0}{c_2} \right)^2 \\ & + a_1 \cdot c_2 \cdot c_2 a_2 \cdot \left(\frac{x_1}{c_2} \right)^2 + \sum_{i=4}^{n} c_2 \cdot c_2 a_2 \cdot s_i \cdot \left(\frac{x_i}{c_2} \right)^2 . \end{aligned}$$

We know that $d_2(a_1 + a_2)$ is represented by ψ; but that is impossible by Lemma 2.24 unless all the terms composed from at least two elements of

$\mathcal{B}_{(1\text{Bii})}$ (i.e., all but the first four) are zero. It follows $x_i = 0$ for all $0 \le i \le n$; hence, $d_2 = 0$ which is absurd. Therefore, this case cannot happen at all.

We can conclude step (1): We have proved that ψ has a subform similar to $\langle 1, a_1, a_2 \rangle$. If $\dim \psi = 3$, then there is nothing more to prove. From now on, we will assume that $\dim \psi \ge 4$ and $\langle 1, a_1, a_2 \rangle \subseteq \psi$.

(2) Now let $k \in \{3, \ldots, n\}$. By (◉), we find $c_k, d_k, e_k \in F^*$ such that $c_k\langle 1, a_k\rangle$, $d_k\langle 1, a_1 + a_k\rangle$ and $e_k\langle 1, a_2 + a_k\rangle$ are subforms of ψ. The case distinction is slightly different than in (1); here it depends on whether c_k is represented by $D_F(\langle 1, a_1, a_2\rangle)$ or not.

(2A) Assume $c_k \in D_F(\langle 1, a_1, a_2\rangle)$. Then

$$c_k = u_0^2 + a_1 \cdot u_1^2 + a_2 \cdot u_2^2 \tag{3.3}$$

for some $u_0, u_1, u_2 \in F$ (here we slightly abuse the notation; technically, u_0, u_1, u_2 depend on k), and so

$$c_k a_k = a_k \cdot u_0^2 + a_1 \cdot a_k \cdot u_1^2 + a_2 \cdot a_k \cdot u_2^2.$$

By the uniqueness of the expression of $c_k a_k$ with respect to $\mathcal{B}_{(0)}$, it follows by Lemma 2.24 that $c_k a_k \notin D_F(\langle 1, a_1, a_2\rangle)$. On the other hand, we know that $c_k a_k \in D_F(\psi)$; therefore, $\psi \cong \langle 1, a_1, a_2, c_k a_k, s_4, \ldots, s_n\rangle$ for some $s_4, \ldots, s_n \in F^*$ (possibly different from the s_i's in case (1Bii) above), and

$$\mathcal{B}_{(2\text{A})} = \{a_1, a_2, c_k a_k, s_4, \ldots, s_n\}$$

is a 2-basis of $N_F(\psi)$ by the minimality of ψ. Obviously, $a_k = c_k \cdot c_k a_k \cdot c_k^{-2}$; rewriting c_k via (3.3), we get the unique expression of a_k with respect to $\mathcal{B}_{(2\text{A})}$:

$$a_k = c_k a_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k}\right)^2 + a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k}\right)^2. \tag{3.4}$$

Furthermore, we have $d_k \in D_F(\psi)$, and hence

$$d_k = x_0^2 + a_1 \cdot x_1^2 + a_2 \cdot x_2^2 + c_k a_k \cdot x_3^2 + \sum_{i=4}^{n} s_i \cdot x_i^2$$

for suitable $x_0, \ldots, x_n \in F$ (again, we omit to express the dependence on k). Multiplying d_k by $(a_1 + a_k)$ and using (3.4), we can express $d_k(a_1 + a_k)$ with respect to $\mathcal{B}_{(2\text{A})}$ as follows:

$$\begin{aligned} d_k(a_1 + a_k) = {} & (a_1 x_1 + a_k u_0 x_3)^2 + a_1 \cdot (x_0 + a_k u_1 x_3)^2 + a_2 \cdot (a_k u_2 x_3)^2 \\ & + c_k a_k \cdot \left(\frac{u_0}{c_k} x_0 + \frac{a_1 u_1}{c_k} x_1 + \frac{a_2 u_2}{c_k} x_2\right)^2 + a_1 \cdot a_2 \cdot x_2^2 \\ & + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} x_0 + \frac{u_0}{c_k} x_1 + x_3\right)^2 + \sum_{i=4}^{n} a_1 \cdot s_i \cdot x_i^2 \\ & + a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k} x_0 + \frac{u_0}{c_k} x_2\right)^2 + \sum_{i=4}^{n} c_k a_k \cdot s_i \cdot \left(\frac{u_0}{c_k} x_i\right)^2 \\ & + a_1 \cdot a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k} x_1 + \frac{u_1}{c_k} x_2\right)^2 + \sum_{i=4}^{n} a_1 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_1}{c_k} x_i\right)^2 \\ & + \sum_{i=4}^{n} a_2 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_2}{c_k} x_i\right)^2 \end{aligned}$$

Similarly as before, since $d_k(a_1 + a_k) \in D_F(\psi)$ and ψ is minimal, it follows from Lemma 2.24 that all the "composed" terms, i.e., all the terms except for the first four, must be zero. In particular:

- The coefficient by $c_k a_k \cdot s_i$ is zero for each $4 \leq i \leq n$, and hence $x_i = 0$ for each $4 \leq i \leq n$.
- The coefficient by $a_1 \cdot a_2 \cdot$ equals zero; thus, $x_2 = 0$.
- The coefficient by $a_2 \cdot c_k a_k$ must be zero, i.e., $\frac{u_2}{c_k} x_0 + \frac{u_0}{c_k} x_2 = 0$. As $x_2 = 0$, we get $u_2 x_0 = 0$.
- The coefficient by $a_1 \cdot a_2 \cdot c_k a_k$ is zero, so $\frac{u_2}{c_k} x_1 + \frac{u_1}{c_k} x_2 = 0$. Again, as $x_2 = 0$, it must hold $u_2 x_1 = 0$.
- The coefficient by $a_1 \cdot c_k a_k$ must be zero, thus $x_3 + \frac{u_1}{c_k} x_0 + \frac{u_0}{c_k} x_1 = 0$.

If $u_2 \neq 0$, then $x_0 = x_1 = 0$, and in that case also $x_3 = 0$ (by the last bullet point); but then $d_k = 0$, which is absurd. Therefore, $u_2 = 0$.

We proceed analogously for e_k and $e_k(a_2+a_k)$, only now we know $u_2 = 0$, which means that

$$c_k = u_0^2 + a_1 \cdot u_1^2 \qquad \text{and} \qquad a_k = c_k a_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k}\right)^2.$$

Since $e_k \in D_F(\psi)$, we have

$$e_k = y_0^2 + a_1 \cdot y_1^2 + a_2 \cdot y_2^2 + c_k a_k \cdot y_3^2 + \sum_{i=4}^{n} s_i \cdot y_i^2$$

for some $y_o, \ldots, y_n \in F$, and

$$\begin{aligned} e_k(a_2 + a_k) = &\left(a_2 y_2 + c_k a_k \frac{u_0}{c_k} y_3\right)^2 + a_1 \cdot \left(c_k a_k \frac{u_1}{c_k} y_3\right)^2 + a_2 \cdot y_0^2 \\ &+ c_k a_k \cdot \left(\frac{u_0}{c_k} y_0 + a_1 \frac{u_1}{c_k} y_1\right)^2 + a_1 \cdot a_2 \cdot y_1^2 \\ &+ a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} y_0 + \frac{u_0}{c_k} y_1\right)^2 + a_2 \cdot c_k a_k \cdot \left(y_3 + \frac{u_0}{c_k} y_2\right)^2 \\ &+ \sum_{i=4}^{n} a_2 \cdot s_i \cdot y_i^2 + \sum_{i=4}^{n} c_k a_k \cdot s_i \cdot \left(\frac{u_0}{c_k} y_i\right)^2 \\ &+ a_1 \cdot a_2 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} y_2\right)^2 + \sum_{i=4}^{n} a_1 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_1}{c_k} y_i\right)^2. \end{aligned}$$

Again, all the coefficients by the composed terms must be zero, so in particular:

- $y_i = 0$ for all $4 \leq i \leq n$ because of the coefficients by $a_2 \cdot s_i$.
- $y_1 = 0$ because of the coefficient by $a_1 \cdot a_2$.
- $u_1 y_0 = 0$ because of the coefficient by $a_1 \cdot c_k a_k$.
- $u_1 y_2 = 0$ because of the coefficient by $a_1 \cdot a_2 \cdot c_k a_k$.
- $y_3 + \frac{u_0}{c_k} y_2 = 0$ because of the coefficient by $a_2 \cdot c_k a_k$.

If $u_1 \neq 0$, then necessarily $y_0 = 0$ and $y_2 = 0$, which implies $y_3 = 0$; it would follow $e_k = 0$, which is absurd. Thus, we have $u_1 = 0$. Therefore, $c_k = u_0^2$, and we have $\langle 1, a_k \rangle \subseteq \psi$; in particular, $a_k \in D_F(\psi)$.

(2B) Suppose $c_k \notin D_F(\langle 1, a_1, a_2 \rangle)$; then $\langle 1, a_1, a_2, c_k \rangle \subseteq \psi$. Here we distinguish two subcases:

(2Bi) Let $c_k a_k \in D_F(\langle 1, a_1, a_2, c_k \rangle)$; then

$$c_k a_k = u_0^2 + a_1 \cdot u_1^2 + a_2 \cdot u_2^2 + c_k \cdot u_3^2$$

for some $u_0, \ldots, u_3 \in F$, and hence

$$a_k = u_3^2 + c_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k \cdot \left(\frac{u_1}{c_k}\right)^2 + a_2 \cdot c_k \cdot \left(\frac{u_2}{c_k}\right)^2.$$

We have

$$\psi \cong \langle 1, a_1, a_2, c_k, s_4, \ldots, s_n \rangle$$

for some $s_4, \ldots, s_n \in F$; then

$$\mathcal{B}_{(2\mathrm{Bi})} = \{a_1, a_2, c_k, s_4, \ldots, s_n\}$$

is a 2-basis of $N_F(\psi)$. As before, we consider the unique representations of d_k and e_k by ψ, and express $d_k(a_1 + a_k)$ and $e_k(a_2 + a_k)$ with respect to $\mathcal{B}_{(2\mathrm{Bi})}$. We obtain that $u_1 = u_2 = 0$; therefore, $a_k = u_3^2 + c_k \left(\frac{u_0}{c_k}\right)^2$, so in particular $a_k \in D_F(\psi)$.

(2Bii) If $c_k a_k \notin D_F(\langle 1, a_1, a_2, c_k \rangle)$, then we have $\dim \psi \geq 5$ and

$$\psi \cong \langle 1, a_1, a_2, c_k, c_k a_k, s_5, \ldots, s_n \rangle$$

for some $s_5, \ldots, s_n \in F$, and the corresponding 2-basis of $N_F(\psi)$ is

$$\mathcal{B}_{(2\mathrm{Bii})} = \{a_1, a_2, c_k, c_k a_k, s_5, \ldots, s_n\}.$$

Again, we consider the unique representation of d_k by ψ. In this case, we express a_k with respect to $\mathcal{B}_{(2\mathrm{Bii})}$ as

$$a_k = c_k \cdot c_k a_k \cdot \left(\frac{1}{c_k}\right)^2.$$

This time, the consideration of the element $d_k(a_1 + a_k)$ with respect to $\mathcal{B}_{(2\mathrm{Bii})}$ already implies $d_k = 0$; that is absurd, and hence this case cannot happen.

(3) We have proved that, up to multiplying ψ by a constant from F^*, we have $\langle 1, a_1, a_2 \rangle \subseteq \psi$ (step (1)), and $a_k \in D_F(\psi)$ for all $3 \leq k \leq n$ (step (2)). Invoking Lemma 2.1, we get $\varphi \overset{\mathrm{sim}}{\sim} \psi$. □

3.3 Special quasi-Pfister neighbors

The definition of a special quasi-Pfister neighbor is motivated by its counterpart in characteristic not 2, which appeared in the paper [AO95].

Definition 3.7. We call a totally singular quadratic form φ over F a *special quasi-Pfister neighbour* if $\varphi \overset{\text{sim}}{\sim} \pi \perp b\sigma$ with π a quasi-Pfister form over F, $b \in F^*$ and $\sigma \subseteq \pi$. In such situation, we also say that φ is given by the triple (π, b, σ).

3.3.1 Quasi-Pfister neighbors of small dimensions

We will describe all special quasi-Pfister neighbors of a dimension up to eleven and their full splitting pattern. In particular, we will see that all quasi-Pfister neighbors up to dimension eight are special. First, we recall a proposition from [HL04].

Lemma 3.8 ([HL04, Prop. 8.12])**.** *Let φ be an anisotropic totally singular quadratic form over F with $\dim\varphi \leq 8$. Then φ is a quasi-Pfister neighbor if and only if*

(i) $\dim\varphi \leq 3$, *or*

(ii) $\dim\varphi = 2^n$ *for some $n \geq 1$ and φ is similar to an n-fold quasi-Pfister form, or*

(iii) *there exist $x, y, z \in F^*$ such that*

  (a) $\varphi \overset{\text{sim}}{\sim} \langle 1, x, y, xy, z\rangle$ *in the case of* $\dim\varphi = 5$,

  (b) $\varphi \overset{\text{sim}}{\sim} \langle 1, x, y, xy, z, xz\rangle$ *in the case of* $\dim\varphi = 6$,

  (c) $\varphi \overset{\text{sim}}{\sim} \langle x, y, z, xy, xz, yz, xyz\rangle$ *in the case of* $\dim\varphi = 7$.

With the previous lemma, it is easy to classify all special quasi-Pfister neighbors of dimensions up to 8: Any quasi-Pfister neighbor of dimension 2^n for some $n \geq 0$ is similar to an n-fold quasi-Pfister form, and hence special. For the other small dimensions, we have

- $\langle 1, x, y\rangle \cong \langle\langle x\rangle\rangle \perp z\langle 1\rangle$,
- $\langle 1, x, y, xy, z\rangle \cong \langle\langle x, y\rangle\rangle \perp z\langle 1\rangle$,
- $\langle 1, x, y, xy, z, xz\rangle \cong \langle\langle x, y\rangle\rangle \perp z\langle 1, x\rangle$,
- $\langle x, y, z, xy, xz, yz, xyz\rangle \cong xyz(\langle\langle x, y\rangle\rangle \perp z\langle 1, x, y\rangle)$;

therefore, we get the following corollary.

Corollary 3.9. *All anisotropic quasi-Pfister neighbors of dimensions up to 8 are special.*

Proposition 3.10. *Let φ be an anisotropic special quasi-Pfister neighbor.*

(i) *If* $\dim\varphi = 3$, *then we have* $\varphi \overset{\text{sim}}{\sim} \langle\langle a\rangle\rangle \perp d\langle 1\rangle$ *for some $a, d \in F^*$ and* $\mathrm{fSP}(\varphi) = \{1, 2, 3\}$.

(ii) *If* $\dim\varphi = 5$, *then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b\rangle\rangle \perp d\langle 1\rangle$ *for some $a, b, d \in F^*$ and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 5\}$.

(iii) *If* $\dim\varphi = 6$, *then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b\rangle\rangle \perp d\langle 1, a\rangle$ *for some $a, b, d \in F^*$ and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 6\}$.

(iv) *If* $\dim\varphi = 7$, *then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b\rangle\rangle \perp d\langle 1, a, b\rangle$ *for some $a, b, d \in F^*$ and* $\mathrm{fSP}(\varphi) = \{1, 2, 4, 7\}$.

(v) *If* $\dim\varphi = 9$*, then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a,b,c\rangle\rangle \perp d\langle 1\rangle$ *for some* $a,b,c,d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1,2,3,4,5,8,9\}$.

(vi) *If* $\dim\varphi = 10$*, then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a,b,c\rangle\rangle \perp d\langle 1,a\rangle$ *for some* $a,b,c,d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1,2,3,4,5,6,8,10\}$.

(vii) *If* $\dim\varphi = 11$*, then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a,b,c\rangle\rangle \perp d\langle 1,a,b\rangle$ *for some* $a,b,c,d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1,2,3,4,6,7,8,11\}$.

Proof. All special quasi-Pfister neighbors up to dimension 8 have been described by Lemma 3.8 and Corollary 3.9. If $\varphi \overset{\text{sim}}{\sim} \pi \perp d\sigma$ is a special quasi-Pfister neighbor and $9 \leq \dim\varphi \leq 11$, then necessarily $\dim\pi = 2^3$ and $1 \leq \dim\sigma \leq 3$. Recall that all totally singular quadratic forms of dimensions two or three are minimal; thus, without loss of generality, $\langle 1\rangle \subseteq \sigma \subseteq \langle 1,a,b\rangle$ for some 2-independent set $\{a,b\} \subseteq F$. By Lemma 2.7, we can find $c \in F^*$ such that $\{a,b,c\}$ is a 2-basis of $N_F(\pi)$ over F. Then we have $\pi \cong \langle\langle a,b,c\rangle\rangle$.

The full splitting pattern of φ follows directly from Theorem 2.42. □

In the following example, we will show that not all quasi-Pfister neighbors are special.

Example 3.11. Let $\varphi \cong \langle 1,a,b,c,d,ab,ac,ad,bc\rangle$ with $\{a,b,c,d\} \subseteq F$ 2-independent over F. Then φ is a quasi-Pfister neighbor of $\langle\langle a,b,c,d\rangle\rangle$.

On the other hand, we have

$$\varphi_{F(\sqrt{b})} \cong \langle 1,a,1,c,d,a,ac,ad,c\rangle_{F(\sqrt{b})} \cong \langle 1,a,c,d,ac,ad,0,0,0\rangle_{F(\sqrt{b})};$$

since $\langle 1,a,c,d,ac,ad\rangle \subseteq \langle\langle a,c,d\rangle\rangle$ and $\langle\langle a,c,d\rangle\rangle$ is anisotropic over $F(\sqrt{b})$ (e.g., by Lemma 2.36), we get

$$(\varphi_{F(\sqrt{b})})_{\mathrm{an}} \cong \langle 1,a,c,d,ac,ad\rangle_{F(\sqrt{b})},$$

so we have in particular $6 \in \mathrm{fSP}(\varphi)$. As the full splitting pattern of any 9-dimensional special quasi-Pfister neighbor equals to $\{1,2,3,4,5,8,9\}$ by Proposition 3.10, it follows that φ cannot be any special quasi-Pfister neighbor.

3.3.2 Vishik equivalence

Ideally, we would like to prove that if two totally singular forms are Vishik equivalent and at least one of them is a special quasi-Pfister neighbor, then they are similar. Unfortunately, we will need some additional assumptions. We start with a few lemmas.

First, we prove that if a quasi-Pfister neighbor of norm degree 2^n contains a quasi-Pfister form of dimension 2^{n-1}, then it must be a special quasi-Pfister neighbor.

Lemma 3.12. *Let* π *be an anisotropic quasi-Pfister form over* F*. Moreover, let* ψ *be an anisotropic totally singular quadratic form over* F *such that* $\pi \subseteq \psi$ *and* $\hat{\nu}_F(\psi) \cong \pi \otimes \langle\langle b\rangle\rangle$ *for some* $b \in F^*$*. Then there exists a totally singular form* $\rho \subseteq \pi$ *such that* $\psi \cong \pi \perp b\rho$*.*

Proof. First, recall that a norm form is always anisotropic; hence, $\pi \otimes \langle\langle b \rangle\rangle$ is anisotropic, and we get by Lemma 2.19 that $b \notin D_F(\pi)$.

Let ρ' be a totally singular quadratic form over F such that $\psi \cong \pi \perp b\rho'$. Denote $s = \dim \rho'$ and $\rho' = \langle d'_1, \dots, d'_s \rangle$. For each $k \in \{1, \dots, s\}$, we proceed as follows: Since $bD_F(\rho') \subseteq D_F(\psi) \subseteq D_F(\pi \otimes \langle\langle b \rangle\rangle)$, we can write

$$bd'_k = \pi(\boldsymbol{\xi}_k) + b\pi(\boldsymbol{\zeta}_k)$$

for some appropriate vectors $\boldsymbol{\xi}_k, \boldsymbol{\zeta}_k$. If $\boldsymbol{\zeta}_k = 0$, then $bd'_k = \pi(\boldsymbol{\xi}_k) \in D_F(\pi)$, which contradicts the anisotropy of ψ; thus, the vector $\boldsymbol{\zeta}_k$ must be nonzero. It follows that $\pi(\boldsymbol{\zeta}_k) \neq 0$, and so we set $d_k = \pi(\boldsymbol{\zeta}_k)$; then

$$\pi \perp b\langle d'_k \rangle \cong \pi \perp \langle \pi(\boldsymbol{\xi}_k) + bd_k \rangle \cong \pi \perp b\langle d_k \rangle.$$

It follows that

$$\pi \perp b\langle d'_1, \dots, d'_k \rangle \perp b\langle d_{k+1}, \dots, d_s \rangle \cong \pi \perp b\langle d'_1, \dots, d'_{k-1} \rangle \perp b\langle d_k, \dots, d_s \rangle$$

for any $1 \leq k \leq s$. Therefore,

$$\pi \perp b\langle d'_1, \dots, d'_s \rangle \cong \pi \perp b\langle d_1, \dots, d_s \rangle,$$

where $\rho \cong \langle d_1, \dots, d_s \rangle$ is a subform of π, because $d_k \in D_F(\pi)$ for each k. $\square$

Proposition 3.13. *Let φ, ψ be anisotropic totally singular quadratic forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Assume that φ is a special quasi-Pfister neighbor given by a triple (π, b, σ). Moreover, suppose that $c\pi \subseteq \psi$ for some $c \in F^*$. Then ψ is a special quasi-Pfister neighbor given by a triple (π, b, ρ) for some form ρ over F such that $\rho \overset{v_0}{\sim} \sigma$.*

Proof. Let $d \in D_F(\sigma)$. Then $d \in D_F(\pi) = G_F(\pi)$; hence, $d\varphi \cong \pi \perp b(d\sigma)$ and $1 \in D_F(d\sigma)$. Therefore, we can assume that $1 \in D_F(\sigma)$. Then, since φ is anisotropic, we must have $b \notin D_F(\pi)$. It follows that $\hat{\nu}_F(\varphi) \cong \pi \otimes \langle\langle b \rangle\rangle$. Since $\hat{\nu}_F(\psi) \cong \hat{\nu}_F(\varphi)$ by Proposition 2.60, we also have $\hat{\nu}_F(\psi) \cong \pi \otimes \langle\langle b \rangle\rangle$. Finally, we can suppose that $\pi \subseteq \psi$. Therefore, we can apply Lemma 3.12 to find a totally singular quadratic form $\rho \subseteq \pi$ such that $\psi \cong \pi \perp b\rho$.

It remains to show that $\sigma \overset{v_0}{\sim} \rho$: The equality $\dim \sigma = \dim \rho$ follows directly from $\dim \varphi = \dim \psi$. So, consider the field $E = F(\sqrt{a})$ for some $a \in F^*$. If $a \notin N_F(\pi)$, then π_E is anisotropic by Lemma 2.25, and so are its subforms σ_E and ρ_E; in particular, $\mathfrak{i}_{\mathrm{d}}(\sigma_E) = \mathfrak{i}_{\mathrm{d}}(\rho_E)$. On the other hand, if $a \in N_F(\pi)$, then $(\pi \otimes \langle\langle a \rangle\rangle)_{\mathrm{an}} \cong \pi$; together with Lemma 2.25, we obtain

$$D_E(\pi_E) = D_F(\pi \otimes \langle\langle a \rangle\rangle) = D_F(\pi).$$

It follows that $b \notin D_E(\pi_E)$, and so we get by Lemma 2.39 that

$$\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\pi_E) + \mathfrak{i}_{\mathrm{d}}(\sigma_E) \qquad \text{and} \qquad \mathfrak{i}_{\mathrm{d}}(\psi_E) = \mathfrak{i}_{\mathrm{d}}(\pi_E) + \mathfrak{i}_{\mathrm{d}}(\rho_E).$$

Since $\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E)$ by the assumption, we get $\mathfrak{i}_{\mathrm{d}}(\sigma_E) = \mathfrak{i}_{\mathrm{d}}(\rho_E)$. $\square$

Remark 3.14. With the same notation as in Proposition 3.13, we can consider a stronger assumption $\varphi \overset{v}{\sim} \psi$ and ask whether it implies $\sigma \overset{v}{\sim} \rho$.

First, note that any field E with $b \in D_E(\pi)$ is problematic: In this case, we have

$$b\varphi_E \cong b\pi_E \perp \sigma_E \cong (\pi \perp \sigma)_E.$$

Hence, $(\varphi_E)_{\mathrm{an}} \cong (\pi_E)_{\mathrm{an}}$, so we do not get any information about $\mathfrak{i}_{\mathrm{d}}(\sigma_E)$.

We can still give a more specific characterization of the problematic fields: As in the proof of Proposition 3.13, we can assume that $1 \in D_F(\sigma)$. First, let T and S be fields such that $F \subseteq T \subseteq S \subseteq E$, T/F is purely transcendental, S/T is separable and E/S is purely inseparable.

If $b \in D_S(\pi)$, then $(\pi \perp \langle b\rangle)_S$ is isotropic. Then $\pi \perp \langle b\rangle$ is isotropic over F by Lemmas 1.13 and 1.14. But that is impossible because $\pi \perp \langle b\rangle \subseteq \pi \perp b\sigma$ and $\pi \perp b\sigma$ is anisotropic. Thus, $b \notin D_S(\pi)$.

Furthermore, isotropy is a finite problem; thus, we can construct a field L with $S \subseteq L \subseteq E$ such that L/S is finite, and we have $\mathfrak{i}_{\mathrm{d}}(\sigma_L) = \mathfrak{i}_{\mathrm{d}}(\sigma_E)$ and $\mathfrak{i}_{\mathrm{d}}(\rho_L) = \mathfrak{i}_{\mathrm{d}}(\rho_E)$. Then, by Lemma 1.15, $L = S\left(\sqrt[2^{n_1}]{s_1}, \dots, \sqrt[2^{n_k}]{s_k}\right)$ for some $k \geq 0$, $s_i \in S$ and $n_i \geq 1$. Set $K = S(\sqrt{s_1}, \dots, \sqrt{s_k})$; then $\mathfrak{i}_{\mathrm{d}}(\sigma_K) = \mathfrak{i}_{\mathrm{d}}(\sigma_L)$ and $\mathfrak{i}_{\mathrm{d}}(\rho_K) = \mathfrak{i}_{\mathrm{d}}(\rho_L)$ by Theorem 2.28.

If $b \notin D_K(\pi)$, then we can apply Lemma 2.39 to get $\mathfrak{i}_{\mathrm{d}}(\sigma_K) = \mathfrak{i}_{\mathrm{d}}(\rho_K)$; then $\mathfrak{i}_{\mathrm{d}}(\sigma_E) = \mathfrak{i}_{\mathrm{d}}(\rho_E)$ by the construction of K, and we are done. Therefore, assume $b \in D_K(\pi)$.

Now we can construct a field M such that $S \subseteq M \subseteq K$, it holds that $b \notin D_M(\pi)$, and for each $\varepsilon \in K \setminus M$, we have $b \in D_{M(\varepsilon)}(\pi)$. Then we have $(\varphi_M)_{\mathrm{an}} \cong (\pi_M)_{\mathrm{an}} \perp b(\sigma_M)_{\mathrm{an}}$ and $(\psi_M)_{\mathrm{an}} \cong (\pi_M)_{\mathrm{an}} \perp b(\rho_M)_{\mathrm{an}}$ and $\mathfrak{i}_{\mathrm{d}}(\sigma_M) = \mathfrak{i}_{\mathrm{d}}(\rho_M)$ by Lemma 2.39. Thus, let φ', ψ', π', σ', ρ' be forms over F such that $\varphi'_M \cong (\varphi_M)_{\mathrm{an}}$, $\psi'_M \cong (\psi_M)_{\mathrm{an}}$, etc. In particular, π'_M is a quasi-Pfister form.

Let $\varepsilon \in K \setminus M$, and $e \in S$ be such that $\varepsilon^2 = e$. Then we know

$$b \in D_{M(\sqrt{e})}(\pi') \setminus D_M(\pi') = D_M(\pi' \otimes \langle\langle e\rangle\rangle) \setminus D_M(\pi');$$

in particular, $D_M(\pi') \subsetneq D_M(\pi' \otimes \langle\langle e\rangle\rangle)$, which is only possible if $(\pi' \otimes \langle\langle e\rangle\rangle)_M$ is anisotropic. It follows by Lemma 2.25 that $\pi'_{M(\sqrt{e})}$ is anisotropic. In that case, its subforms $\sigma'_{M(\sqrt{e})}$, $\rho'_{M(\sqrt{e})}$ must be anisotropic, too. It follows that

$$\mathfrak{i}_{\mathrm{d}}(\sigma_{M(\sqrt{e})}) = \mathfrak{i}_{\mathrm{d}}(\sigma_M) = \mathfrak{i}_{\mathrm{d}}(\rho_M) = \mathfrak{i}_{\mathrm{d}}(\rho_{M(\sqrt{e})}).$$

However, the field extension K/M does not have to be simple, as we show in the following example: Let $\pi'_M \cong \langle\langle a_1, \dots, a_n\rangle\rangle_M$ for some $a_1, \dots, a_n \in F^*$, and assume that $n \geq 2$. Set $K = M(\sqrt{b}, \sqrt{a_1 + a_2 b})$. It is easy to see that $\{b, a_1 + a_2 b\}$ is 2-independent over M, and hence $[K : M] = 4$. Since K does not contain any element of degree four over M, it follows that K/M is not simple. Now consider $\varepsilon \in K \setminus M$; then

$$\varepsilon = w + x\sqrt{b} + y\sqrt{b}\sqrt{a_1 + a_2 b} + z\sqrt{a_1 + a_2 b}$$

with $w, x, y, z \in M$, at least one of x, y, z nonzero. Then

$$D_{M(\varepsilon)}(\pi') = D_M(\pi' \otimes \langle\langle \varepsilon^2\rangle\rangle) = M^2(a_1, \dots, a_n, \varepsilon^2)$$

Considering ε^2 as an element of $K^2/M^2(a_1,\ldots,a_n)$, we get:

$$\begin{aligned}\varepsilon^2 = w^2 + bx^2 + a_1by^2 + a_2(by)^2 + a_1z^2 + a_2bz^2 &\equiv bx^2 + a_1by^2 + a_2bz^2 \\ = b(x^2 + a_1y^2 + a_2z^2) &\equiv b \mod M^2(a_1,\ldots,a_n),\end{aligned}$$

where we used that $x^2 + a_1y^2 + a_2z^2 \neq 0$ (this holds because $\{a_1, a_2\}$ is 2-independent over M and at least one of x, y, z is nonzero by the assumption). Therefore, we have $D_{M(\varepsilon)}(\pi') = M^2(a_1,\ldots,a_n,b)$. In particular, $b \in D_{M(\varepsilon)}(\pi')$ for any $\varepsilon \in K \setminus M$, and so we cannot find any field M' with $M \subsetneq M' \subseteq K$ such that $b \notin D_{M'}(\pi')$.

With the notation and assumptions of Proposition 3.13, we know that $\sigma \overset{v_0}{\approx} \rho$, and hence $N_F(\sigma) \simeq N_F(\rho)$ by Proposition 2.60. To conclude this section, we prove $N_F(\sigma) \simeq N_F(\rho)$ by a different approach.

Lemma 3.15. *Let $\varphi = \pi \perp b\sigma$, $\psi = \pi \perp b\rho$ be anisotropic totally singular quadratic forms over F with π a quasi-Pfister form, $b \in F^*$ and $\sigma, \rho \subseteq \pi$. Moreover, suppose that $1 \in D_F(\sigma) \cap D_F(\rho)$. If $\varphi \overset{v}{\sim} \psi$, then $N_F(\sigma) = N_F(\rho)$.*

Proof. First, note that the anisotropy of φ together with the assumption $1 \in D_F(\sigma)$ implies that $b \notin D_F(\pi)$.

By Corollary 2.9, we can find $c_1,\ldots,c_s \in F^*$ and $0 \leq k \leq s$ such that $\sigma = \langle 1, c_1,\ldots,c_s\rangle$ and $\{c_1,\ldots,c_k\}$ is a 2-basis of $N_F(\sigma)$ over F. Set $K = F(\sqrt{c_1},\ldots,\sqrt{c_k})$. Then $(\sigma_K)_{\mathrm{an}} \cong \langle 1\rangle$ by Proposition 2.10. Write $\pi \cong \langle\langle a_1,\ldots,a_n\rangle\rangle$ and $\pi' \cong (\pi_K)_{\mathrm{an}}$. Then

$$\begin{aligned}D_K(\pi') = K^2(a_1,\ldots,a_n) &= F^2(c_1,\ldots,c_k,a_1,\ldots,a_n) \\ &\overset{N_F(\sigma)\subseteq N_F(\pi)}{=} F^2(a_1,\ldots,a_n) = D_F(\pi);\end{aligned}$$

hence, $b \notin D_K(\pi')$, and so $(\varphi_K)_{\mathrm{an}} \cong \pi' \perp b\langle 1\rangle$. We will show that this is isometric to $(\psi_K)_{\mathrm{an}}$: Obviously, $\pi' \subseteq (\psi_K)_{\mathrm{an}}$. From the Vishik equivalence of φ and ψ we know $\dim(\psi_K)_{\mathrm{an}} = \dim \pi' + 1$. Set $\rho' \cong (\rho_K)_{\mathrm{an}}$. Since $b \notin D_K(\pi')$, we get by Lemma 2.39 that $\pi' \perp b\rho'$ is anisotropic. It means that $\pi' \perp b\rho' \cong (\psi_K)_{\mathrm{an}}$, and hence $\dim \rho' = 1$. As $1 \in D_F(\rho) \subseteq D_K(\rho')$, it follows that $\rho' \cong \langle 1\rangle$. Consequently, again by Proposition 2.10, we get $N_F(\rho) \subseteq F^2(c_1,\ldots,c_k) = N_F(\sigma)$. The other inclusion can be proved analogously. Therefore, $N_F(\sigma) = N_F(\rho)$. □

3.4 Vishik equivalence

In this final section, we put together all our results on the Vishik equivalence of totally singular quadratic forms.

Before stating the main theorem of this section, note that any anisotropic totally singular quadratic form of dimension less or equal to four is minimal or it is similar to a quasi-Pfister form.

Theorem 3.16. *Let φ, ψ be totally singular quadratic forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Assume that φ is a special quasi-Pfister neighbor given by the triple (π, b, σ), and $c\pi \subseteq \psi$ for some $c \in F^*$. Moreover, suppose that σ is either a quasi-Pfister form, a quasi-Pfister neighbor of codimension one, or a minimal form. Then $\varphi \overset{\text{sim}}{\sim} \psi$.*

Proof. By Proposition 3.13, there exists a totally singular quadratic form $\rho \subseteq \pi$ and $c' \in F^*$ such that $c'\psi \cong \pi \perp b\rho$ and $\sigma \overset{v_0}{\sim} \rho$. Then, by Corollary 2.61, resp. by Theorem 3.6, we have $\sigma \overset{\text{sim}}{\sim} \rho$.

Let $d \in F^*$ be such that $\rho \cong d\sigma$, and let $a \in D_F^*(\sigma)$. Then

$$da \in D_F^*(d\sigma) = D_F^*(\rho) \subseteq D_F^*(\pi) = G_F^*(\pi).$$

As $a \in G_F^*(\pi)$ and $G_F^*(\pi)$ is a group, it follows that $d \in G_F^*(\pi)$. Hence,

$$d\varphi \cong d\pi \perp b(d\sigma) \cong \pi \perp b\rho \cong c'\psi,$$

i.e., $\varphi \overset{\text{sim}}{\sim} \psi$. $\square$

Corollary 3.17. *Let φ, ψ be totally singular quadratic forms over F such that $\varphi \overset{v}{\sim} \psi$. Let K/F be an extension such that $K^2 \simeq G_F(\varphi)$, and let φ' be a form over F such that $\varphi'_K \cong (\varphi_K)_{\text{an}}$. Assume that φ'_K is a special quasi-Pfister neighbor given by the triple (π, b, σ), and that $c\pi \subseteq \psi_K$ for some $c \in K^*$. Moreover, suppose that σ is either a quasi-Pfister form, a quasi-Pfister neighbor of codimension one, or a minimal form (over K). Then $\varphi \overset{\text{sim}}{\sim} \psi$.*

Proof. Let $\tau \cong \hat{\sigma}_F(\varphi)$, i.e., let τ be an anisotropic quasi-Pfister form over F such that $D_F(\tau) = G_F(\varphi)$; then there exists γ over F such that $\varphi \cong \tau \otimes \gamma$. Set $\tau \cong \langle\langle a_1, \dots, a_n \rangle\rangle$ for some $a_1, \dots, a_n \in F^*$; then $K \simeq F(\sqrt{a_1}, \dots, \sqrt{a_n})$, and we have $(\varphi_K)_{\text{an}} \cong \gamma_K$. In particular, we can assume $\gamma \cong \varphi'$. Since $G_F(\varphi) \simeq G_F(\psi)$ by Proposition 2.68, we can use analogous arguments to find a form ψ' over F such that $\psi \cong \tau \otimes \psi'$ and $\psi'_K \cong (\psi_K)_{\text{an}}$.

Since $\varphi \overset{v}{\sim} \psi$, we get by Lemma 1.59 that $\varphi'_K \overset{v}{\sim} \psi'_K$. Since $c\pi \subseteq \psi_K$ and π is anisotropic over K, it follows that $c\pi \subseteq \psi'_K$. By Theorem 3.16, we have $\varphi'_K \overset{\text{sim}}{\sim} \psi'_K$. Finally, Proposition 2.70 implies that $\varphi \overset{\text{sim}}{\sim} \psi$. $\square$

Remark 3.18. Note that we need in the proof of Corollary 3.17 that $\varphi'_K \overset{v_0}{\sim} \psi'_K$. To be able to conclude that, we need the full strength of $\varphi \overset{v}{\sim} \psi$, it would not be sufficient to assume $\varphi \overset{v_0}{\sim} \psi$.

We conclude the chapter with an observation about the splitting patterns. It is obvious that Vishik equivalent forms have the same full splitting pattern. But what about the standard splitting pattern?

Proposition 3.19. *Let φ, ψ be totally singular quadratic forms over F. If $\varphi \overset{v}{\sim} \psi$, then* $\mathrm{sSP}(\varphi) = \mathrm{sSP}(\psi)$.

Proof. Note that Vishik equivalent forms have in particular the same dimension; thus, without loss of generality, assume that φ and ψ are anisotropic. If $\dim\varphi = \dim\psi = 1$, then the statement is obvious. Thus, assume $\dim\varphi = \dim\psi \geq 2$; then φ and ψ are irreducible.

By Corollary 2.57, we know that $\varphi \overset{v}{\sim} \psi$ implies $\varphi \overset{\mathrm{bir}}{\sim} \psi$; hence, we have $F(\varphi) \simeq F(\psi)$ by Proposition 1.44. Therefore, we have

$$\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\varphi)}) = \mathfrak{i}_{\mathrm{d}}(\varphi_{F(\psi)}) = \mathfrak{i}_{\mathrm{d}}(\psi_{F(\psi)}),$$

and it follows that $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} = \dim(\psi_{F(\psi)})_{\mathrm{an}}$. Setting $\varphi_1 \cong (\varphi_{F(\varphi)})_{\mathrm{an}}$ and $\psi_1 \cong (\psi_{F(\varphi)})_{\mathrm{an}}$, we note that $\varphi_1 \overset{v}{\sim} \psi_1$ by Lemma 1.59. Thus, we can proceed by induction. □

4. Nonsingular quadratic forms

Unlike in the other chapters, we do not put any general assumptions on the characteristic of the fields, because many of the statements hold in any characteristic. However, our main emphasis still lies on characteristic two. Therefore we often omit to provide separate references for the case of characteristic other than two.

Similarly as in the previous chapter, our main goal is to examine Question Q, that is, whether Vishik equivalent quadratic forms are necessarily similar. This is known to be true for some quadratic forms of odd dimension (see Theorems 4.9 and 4.10). In Theorem 4.13, we prove that this also holds for excellent nonsingular quadratic forms. Furthermore, we provide an example of nonsingular quadratic forms which are stably birationally equivalent, but not Vishik equivalent; see Example 4.19. We will also see in Proposition 4.15 and Theorem 4.17 that quadratic Pfister forms and quadratic Pfister neighbors of codimension one are stably birationally equivalent if and only if they are similar.

4.1 Preliminaries

Let $\operatorname{char} F = 2$; recall that the Artin–Schreier map $\wp : F \to F$ is given by $x \mapsto x^2 + x$. Let

$$\varphi \cong [a_1, b_1] \perp \ldots \perp [a_n, b_n]$$

be a nonsingular quadratic form over F. We define its *Arf invariant* $\Delta(\varphi)$ as an element of $F/\wp(F)$ given by

$$\Delta(\varphi) = \sum_{i=1}^{n} a_i b_i.$$

If ψ is a nonsingular quadratic form such that $\varphi \cong \psi$, then we have $\Delta(\varphi) = \Delta(\psi) \in F/\wp(F)$; see [Arf41, Th. 5].

Lemma 4.1. *Let* $\operatorname{char} F = 2$ *and* φ *be an anisotropic nonsingular quadratic form over* F *such that* $\dim \varphi = 4$*. If* $\Delta(\varphi) \in \wp(F)$*, then* φ *is similar to a quadratic Pfister form.*

Proof. Let $\varphi \cong a_1[1, b_1] \perp a_2[1, b_2]$ for some $a_i, b_i \in F^*$. Since we have $b_1 + b_2 = \Delta(\varphi) \in \wp(F)$ by the assumption, there exists $x \in F$ such that $b_1 = b_2 + x^2 + x$. We get $[1, b_2] \cong [1, b_2 + x^2 + x] \cong [1, b_1]$; therefore,

$$\varphi \cong \langle a_1, a_2 \rangle_b \otimes [1, b_1] \cong a_1^{-1} \langle\!\langle a_1 a_2; b_1]\!].$$ □

We recall some statements about (not necessarily nonsingular) quadratic forms. We start with a proposition analogous to Proposition 3.1. Since the original proof of the statement is in a slightly different setting, we provide a proof here.

Proposition 4.2 ([Bae78, Ch. IV., Th. 4.2])**.** *Let* $\operatorname{char} F = 2$*. Let* φ *be an anisotropic quadratic form over* F*, and let* $d \in F \setminus \wp(F)$*. Then* $\varphi_{F(\wp^{-1}(d))}$ *is isotropic if and only if there exists* $c \in F^*$ *such that* $c\,[1, d] \subseteq \varphi$*.*

Proof. Let $\delta \in \overline{F}$ be such that $\delta^2 + \delta = d$; then $[1, d]_{F(\delta)}$ is isotropic. Thus, if $c\,[1, d] \subseteq \varphi$, then $\varphi_{F(\delta)}$ is isotropic as well.

Now let V be the vector space associated to φ, and let $V_{F(\delta)} = F(\delta) \otimes_F V$. Assume that $\varphi_{F(\delta)}$ is isotropic. Then we can find a nonzero vector $w \in V_{F(\delta)}$ such that $\varphi_{F(\delta)}(w) = 0$. Since we can write $w = 1 \otimes u + \delta \otimes v$ for some $u, v \in V$, we get

$$0 = \varphi_{F(\delta)}(w) = \varphi(u) + \delta^2 \varphi(v) + \delta \mathfrak{b}_\varphi(u, v) = \varphi(u) + d\varphi(v) + \delta(\varphi(v) + \mathfrak{b}_\varphi(u, v)),$$

and it follows

$$\varphi(u) = d\varphi(v) \qquad \text{and} \qquad \varphi(v) = \mathfrak{b}_\varphi(u, v).$$

Since φ is anisotropic, we have $v \neq 0$ and $\varphi(v) \neq 0$; set $c = \varphi(v)$. Then we have $c = \mathfrak{b}_\varphi(u, v) \neq 0$, which means that u, v are linearly independent. For $W = \operatorname{span}_F\{u, c^{-1}v\}$, we get

$$\varphi|_W \cong \left[cd, c^{-2}c\right] \cong \left[c^{-1}, cd\right] \cong c\,[1, d]\,.$$

Therefore, $c\,[1, d] \subseteq \varphi$. □

Theorem 4.3 (Domination Theorem, [HL04, Th. 4.2(i)] and [EKM08, Th. 22.5]). *Let* $\operatorname{char} F$ *be arbitrary. Let* φ, ψ *be quadratic forms over* F *with* φ *anisotropic and* ψ *nondefective. Assume that the form* $\varphi_{F(\psi)}$ *is hyperbolic. Let* $a \in D_F^*(\varphi)$ *and* $b \in D_F^*(\psi)$*. Then* $ab\psi \preccurlyeq \varphi$*; in particular,* $\dim \psi \leq \dim \varphi$*.*

We call a quadratic form *nondegenerate* if it is nondefective over any field extension. If the characteristic is not two, then "nondegenerate" coincides with "regular". In characteristic two, a quadratic form φ is nondegenerate if and only if $\dim \mathrm{ql}(\varphi) \leq 1$ and $\mathrm{ql}(\varphi) \ncong \langle 0 \rangle$.

Proposition 4.4 ([EKM08, Cor. 23.4]). *Let* $\operatorname{char} F$ *be arbitrary. Let* φ *be a nondegenerate anisotropic quadratic form of dimension at least two over* F*. Then the following are equivalent:*

(i) φ *is similar to some* n*-fold quadratic Pfister form with* $n \geq 1$*;*
(ii) $\varphi_{F(\varphi)}$ *is hyperbolic.*

Proposition 4.5 ([EKM08, Cor. 23.6]). *Let* $\operatorname{char} F$ *be arbitrary. Let* φ *be an anisotropic* n*-fold quadratic Pfister form over* F *with* $n \geq 1$ *and* ψ *an anisotropic quadratic form over* F *of even dimension. Then the following are equivalent:*

(i) $\psi \cong \mathfrak{b} \otimes \varphi$ *for some symmetric bilinear form* $\mathfrak{b}$ *over* F*,*
(ii) $\psi_{F(\varphi)}$ *is hyperbolic.*

Theorem 4.6 (Separation Theorem, [EKM08, Th. 26.5.]). *Let* $\operatorname{char} F$ *be arbitrary. Let* φ*,* ψ *be anisotropic quadratic forms over* F*, and suppose that* $\dim \varphi \leq 2^n < \dim \psi$ *for some* $n \geq 0$*. Then* $\varphi_{F(\psi)}$ *is anisotropic.*

Let φ be a quadratic form on a vector space V over F (not necessarily nonsingular), and let σ be another quadratic form over F such that $\sigma \preccurlyeq \varphi$. By the definition, there exists a subspace $U \subseteq V$ such that $\sigma \cong \varphi|_U$. Let $W = \{v \in V \mid \mathfrak{b}_\varphi(u, v) = 0 \ \forall\, u \in U\}$. Then we denote the quadratic form $\varphi|_W$ by $\sigma_\varphi^{\mathrm{c}}$ and call it the *complement* of σ in φ. It holds that $\sigma \perp \varphi \overset{\mathrm{Witt}}{\sim} \sigma_\varphi^{\mathrm{c}}$ (see [HL04, Lemma 3.7]). With the notation from Proposition 1.7, we have $\sigma_\varphi^{\mathrm{c}} \cong \tau \perp \langle c_1, \dots, c_{s''} \rangle$ (see [HL04, Lemma 3.1]). In particular, if σ is nonsingular, then we have $\sigma \subseteq \varphi$ and $\varphi \cong \sigma \perp \sigma_\varphi^{\mathrm{c}}$.

Theorem 4.7 (Fitzgerald's Theorem, [EKM08, Th. 27.1]). *Let* $\operatorname{char} F$ *be arbitrary. Let* ρ *be a nonhyperbolic quadratic form over* F *and* $\varphi \preccurlyeq \rho$ *with* $\dim \varphi \geq 2$*. Suppose that:*

(i) φ *and* $\varphi_\rho^{\mathrm{c}}$ *are anisotropic,*
(ii) $\rho_{F(\varphi)}$ *is hyperbolic,*
(iii) $2 \dim \varphi > \dim \rho - \dim \rho_{h(\rho)-1}$ *(where* $\rho_{h(\rho)-1}$ *is the penultimate form in the standard splitting tower of* ρ*).*

Then ρ *is similar to an anisotropic quadratic Pfister form.*

Recall that a quadratic form φ over F is called *excellent* if, for any field extension E/F, the form $(\varphi_E)_{\mathrm{an}}$ is defined over F. It is easy to see that if φ is an excellent form over F and E/F a field extension, then $(\varphi_E)_{\mathrm{an}}$ is excellent as well.

The following theorem characterizes nondegenerate excellent quadratic forms.

Theorem 4.8 ([EKM08, Th. 28.3])**.** *Let* $\operatorname{char} F$ *be arbitrary. Let* φ *be a nondegenerate quadratic form over* F*. Then the following are equivalent:*

(i) φ *is excellent.*

(ii) *There exists a sequence* $\varphi_{\mathrm{an}} \cong \zeta_0, \zeta_1, \ldots, \zeta_{t-1}$ *of anisotropic quadratic Pfister neighbors over* F *with associated quadratic Pfister forms* ρ_0*,* $\rho_1, \ldots, \rho_{t-1}$ *and a form* ζ_t *with* $\dim \zeta_t \leq 1$ *such that* $\zeta_r \cong (\rho_r \perp \zeta_{r+1})_{\mathrm{an}}$ *for all* $0 \leq r-1 \leq t$*,*

In other words, for each $0 \leq r < t$, the form ζ_r is a quadratic Pfister neighbor with complementary form $-\zeta_{r+1}$. Moreover, the length t of the sequence coincides with the height $h(\varphi)$, and $(\zeta_r)_{F_r} \cong \varphi_r$, the r-th kernel form of φ, for every $0 \leq r \leq t$. It follows that all the kernel forms of an excellent form are excellent and in particular defined over the base field F. Furthermore, it follows that the standard splitting pattern of an anisotropic nondegenerate excellent form is uniquely given by its dimension.

Note that, in the case of even-dimensional forms, the form φ_{h-1} must be similar to a quadratic Pfister form; this is the only one among the higher kernel forms, whose dimension is a power of two.

4.2 Vishik equivalence

First, we state some known results about the Vishik equivalence on nondegenerate quadratic forms of odd dimension.

Theorem 4.9 ([Izh98, Th. 2.5.]). *Let* $\operatorname{char} F \neq 2$. *Let* φ, ψ *be quadratic forms over* F *of the same odd dimension. Then* $\varphi \overset{v}{\sim} \psi$ *if and only if* $\varphi \overset{\text{sim}}{\sim} \psi$.

Theorem 4.10 ([EKM08, Th. 27.3.]). *Let* F *be a field of an arbitrary characteristic. Let* φ, ψ *be nondegenerate quadratic forms over* F *of the same odd dimension. If* $\varphi \overset{v}{\sim} \psi$, *then* $\varphi \overset{\text{sim}}{\sim} \psi$.

As we can see, the question of whether Vishik equivalent quadratic forms are similar is settled for quadratic forms of odd dimension (although in characteristic two only for the nondegenerate ones). What about even-dimensional forms? It does not hold in general; see counterexamples in [Izh98] for fields of characteristic other than two. However, we will prove that it does hold for excellent nonsingular quadratic forms. But first, we need some preparation.

The following proposition is a generalization of [Izh98, Prop. 1.1] to fields of an arbitrary characteristic. The proof remains almost the same, but one has to make sure that the used known results are available in characteristic two (at least for nonsingular forms).

Proposition 4.11. *Let* $\operatorname{char} F$ *be arbitrary. Let* φ *and* ψ *be anisotropic nonsingular quadratic forms over* F *such that* $\dim \varphi \geq \dim \psi$. *Suppose that the form* $\tau \cong \varphi \perp \psi$ *is hyperbolic over* $F(\varphi)$.

(i) *If* τ *is isotropic, then* τ *is hyperbolic.*

(ii) *If* τ *is anisotropic, then* τ *is similar to a quadratic Pfister form.*

Proof. (i) Since

$$\tau_{\text{an}} \perp -\varphi \overset{\text{Witt}}{\sim} \tau \perp -\varphi \cong \varphi \perp \psi \perp -\varphi \cong (\dim \varphi) \times \mathbb{H} \perp \psi \overset{\text{Witt}}{\sim} \psi,$$

we have $\tau_{\text{an}} \perp -\varphi \overset{\text{Witt}}{\sim} \psi$, which means $(\tau_{\text{an}} \perp -\varphi)_{\text{an}} \cong \psi$, as ψ is assumed to be anisotropic. In particular, $\dim(\tau_{\text{an}} \perp -\varphi)_{\text{an}} = \dim \psi$.

For a contradiction, assume τ to be isotropic but not hyperbolic; thus, $0 < \dim \tau_{\text{an}} < \dim \tau$. Then we have

$$\dim(\tau_{\text{an}} \perp -\varphi)_{\text{an}} = \dim \psi \leq \dim \varphi < \dim \tau_{\text{an}} + \dim \varphi = \dim(\tau_{\text{an}} \perp -\varphi);$$

thus, $\tau_{\text{an}} \perp -\varphi$ is isotropic. Hence the set $D_F^*(\tau_{\text{an}}) \cap D_F^*(\varphi)$ is nonempty; let a be an element of this set. Note that the hyperbolicity of $\tau_{F(\varphi)}$ implies that $(\tau_{\text{an}})_{F(\varphi)}$ is hyperbolic as well. Now Domination Theorem 4.3 implies that $a^2\varphi$, and hence φ as well, is dominated by τ_{an}. Since all the forms are nonsingular, we get $\tau_{\text{an}} \cong \varphi \perp \varphi^c_{\tau_{\text{an}}}$. Therefore,

$$\tau_{\text{an}} \perp -\varphi \cong \varphi \perp \varphi^c_{\tau_{\text{an}}} \perp -\varphi \cong (\dim \varphi) \times \mathbb{H} \perp \varphi^c_{\tau_{\text{an}}},$$

which implies

$$\dim(\tau_{\text{an}} \perp -\varphi)_{\text{an}} \leq \dim \varphi^c_{\tau_{\text{an}}} = \dim \tau_{\text{an}} - \dim \varphi.$$

But using the assumption $\tau_{\mathrm{an}} \ncong \tau$, we get

$$\dim \tau_{\mathrm{an}} - \dim \varphi < \dim \tau - \dim \varphi = \dim \psi,$$

and hence $\dim(\tau_{\mathrm{an}} \perp -\varphi)_{\mathrm{an}} < \dim \psi$; that is a contradiction to the equality $\dim(\tau_{\mathrm{an}} \perp -\varphi)_{\mathrm{an}} = \dim \psi$ proved above.

(ii) Instead of proving that τ is similar to a quadratic Pfister form, we will use the characterization of quadratic Pfister forms and prove that $\tau_{F(\tau)}$ is hyperbolic (see Proposition 4.4). As $2 \dim \psi \leq \dim \tau$, the Separation Theorem 4.6 yields the anisotropy of the form $\psi_{F(\tau)}$.

First, assume that $\varphi_{F(\tau)}$ is anisotropic as well. In such a case, we have $\tau_{F(\tau)} \cong \varphi_{F(\tau)} \perp \psi_{F(\tau)}$ isotropic and $\tau_{F(\tau,\varphi)}$ hyperbolic by the assumption; then using part (i) for the field $F(\tau)$ yields the hyperbolicity of $\tau_{F(\tau)}$.

Now, suppose that $\varphi_{F(\tau)}$ is isotropic; in this case, the field extension $F(\varphi,\tau)/F(\tau)$ is purely transcendental by Lemma 1.18. Moreover, as $\tau_{F(\varphi)}$ is hyperbolic, the form $\tau_{F(\varphi,\tau)}$ is hyperbolic as well. It follows from Lemma 1.13 that $\tau_{F(\tau)}$ is hyperbolic. $\square$

The following proposition is known in the case of a field of characteristic other than two, see [Izh98, Cor. 1.3]. We will prove it here for fields of characteristic two.

Proposition 4.12. *Let* $\operatorname{char} F$ *be arbitrary. Let* φ *and* ψ *be anisotropic nonsingular quadratic forms over* F *such that* $(\varphi_{F(\varphi)})_{\mathrm{an}} \cong (\psi_{F(\varphi)})_{\mathrm{an}}$ *and* $\dim \varphi \geq \dim \psi \geq 2$. *Then either* $\varphi \cong \psi$ *or* $\varphi \perp -\psi$ *is similar to an anisotropic quadratic Pfister form over* F.

Proof. Set $\tau = \varphi \perp -\psi$; since $(\varphi_{F(\varphi)})_{\mathrm{an}} \cong (\psi_{F(\varphi)})_{\mathrm{an}}$ by the assumption, it clearly holds that $\tau_{F(\varphi)}$ is hyperbolic.

First, suppose that τ is anisotropic over F; since φ is a subform of τ with a complementary form ψ, both φ and ψ are anisotropic. Moreover, as we assumed $\dim \psi \leq \dim \varphi$, we have $\dim \tau \leq 2 \dim \varphi$. It follows from Fitzgerald's Theorem 4.7 that τ is similar to an anisotropic quadratic Pfister form.

If τ is isotropic over F, then τ is hyperbolic over F by part (i) of Proposition 4.11. Since we assume φ and ψ to be nonsingular, it follows that $\varphi_{\mathrm{an}} \cong \psi_{\mathrm{an}}$. As both forms are anisotropic, we obtain $\varphi \cong \psi$ as desired. $\square$

Now we are ready to prove that Question Q has a positive answer for excellent nonsingular quadratic forms.

Theorem 4.13. *Let* F *be a field of an arbitrary characteristic, and let* φ, ψ *be nonsingular excellent forms over* F. *Then* $\varphi \overset{v}{\sim} \psi$ *if and only if* $\varphi \overset{\mathrm{sim}}{\sim} \psi$.

Proof. We know that the similarity implies the Vishik equivalence (see Lemma 1.60). Thus, suppose that $\varphi \overset{v}{\sim} \psi$. Invoking Lemmas 1.37 and 1.59, we may assume that φ and ψ are anisotropic.

Since φ and ψ are excellent forms of the same dimension, they have the same height h. Let $F = F_0 \subset F_1 \subset \cdots \subset F_h$ be the fields in the standard splitting tower of the form φ. Note that for any $0 \leq r \leq h$, the forms $\varphi_r \cong (\varphi_{F_r})_{\mathrm{an}}$ and $(\psi_{F_r})_{\mathrm{an}}$ are anisotropic excellent forms of the same dimension and $(\varphi_{F_r})_{\mathrm{an}} \overset{v}{\sim} (\psi_{F_r})_{\mathrm{an}}$ by Lemma 1.59.

First, let us consider the quadratic forms φ' and ψ' over F_{h-1} defined by

$$\varphi' \cong \varphi_{h-1} \cong (\varphi_{F_{h-1}})_{\mathrm{an}} \quad \text{and} \quad \psi' \cong (\psi_{F_{h-1}})_{\mathrm{an}}.$$

Then we know $\varphi' \overset{v}{\sim} \psi'$. We claim that we may replace the forms φ and ψ by forms similar over F in such a way that both φ' and ψ' represent 1: Let α be the form over F such that $\varphi' \cong \alpha_{F_{h-1}}$, and let $a \in D_F^*(\alpha)$. Then $a\varphi' \cong a\alpha_{F_{h-1}}$ represents 1, and it is the $(h-1)$th kernel form of $a\varphi$. In the same manner, we can replace ψ' by $b\psi'$ with $b \in F^*$ such that $b\psi'$ represents 1. Thus, let us assume that $1 \in D^*_{F_{h-1}}(\varphi') \cap D^*_{F_{h-1}}(\psi')$.

We proceed by induction by going down from the top of the tower to the bottom.

As we know, φ' is similar to an n-fold quadratic Pfister form for some $n \geq 1$. Since $\varphi'_{F_{h-1}(\varphi')}$ is hyperbolic by Corollary 1.25 and $\varphi' \overset{v}{\sim} \psi'$, it follows that $\psi'_{F_{h-1}(\varphi')}$ must be hyperbolic. Invoking Proposition 4.5, we get that there exists a symmetric bilinear form $\mathfrak{b}$ over F_{h-1} such that $\psi' \cong \mathfrak{b} \otimes \varphi'$; comparing the dimensions, we get $\dim \mathfrak{b} = 1$, i.e., $\varphi' \overset{\mathrm{sim}}{\sim} \psi'$. In particular, ψ' is similar to an n-fold quadratic Pfister form over F_{h-1}. Therefore, φ' and ψ' are two quadratic forms representing one and similar to the same quadratic Pfister form; by Lemma 1.42, it follows that $\varphi' \cong \psi'$.

As for the induction step, let us assume that $r \in \{0, \dots, h-2\}$ is such that we have already proved

$$(\varphi_{F_{r+1}})_{\mathrm{an}} \cong (\psi_{F_{r+1}})_{\mathrm{an}}.$$

Consider the forms

$$\varphi'' \cong (\varphi_{F_r})_{\mathrm{an}} \quad \text{and} \quad \psi'' \cong (\psi_{F_r})_{\mathrm{an}}$$

(i.e., we are going one step lower in the splitting tower). Since we have $(\varphi''_{F_r(\varphi'')})_{\mathrm{an}} \cong (\varphi_{F_{r+1}})_{\mathrm{an}}$ and $(\psi''_{F_r(\varphi'')})_{\mathrm{an}} \cong (\psi_{F_{r+1}})_{\mathrm{an}}$, the induction hypotheses assures the isometry $(\varphi''_{F_r(\varphi'')})_{\mathrm{an}} \cong (\psi''_{F_r(\varphi'')})_{\mathrm{an}}$. Invoking Proposition 4.12, we see that either $\varphi'' \perp -\psi''$ is similar to a quadratic Pfister form, or $\varphi'' \cong \psi''$. But the former is absurd since $\dim \varphi'' = \dim \psi''$ and from the excellence of φ, it follows that the dimension of φ'' is not a power of 2. Therefore, $\varphi'' \cong \psi''$, i.e., $(\varphi_{F_r})_{\mathrm{an}} \cong (\psi_{F_r})_{\mathrm{an}}$, which completes the proof. □

Note that any nonsingular quadratic form of dimension two is excellent: Over any field extension, it either remains anisotropic or becomes hyperbolic. Another explanation can be given through Lemma 1.28, because any two-dimensional nonsingular quadratic form is similar to a quadratic Pfister form.

In the next possible dimension, that is, in dimension four, not all nonsingular quadratic forms have to be excellent, as the following example shows.

Example 4.14. Recall that by Theorem 4.8, any anisotropic nonsingular quadratic form which is to be excellent must be a quadratic Pfister neighbor. In dimension four, all quadratic Pfister neighbors are similar to quadratic

Pfister forms. In particular, any nonsingular excellent form of dimension four is of the form $z([1,x]\perp y[1,x])$ for some $x,y,z \in F$, and so it has trivial Arf invariant.

Consider now $a,b,c \in F^*$ such that $a+bc \notin \wp(F)$ (e.g., let $F = \mathbb{F}_2(a,b,c)$ with a,b,c algebraically independent). Then the four-dimensional quadratic form $[1,a] \perp [b,c]$ has a nontrivial Arf invariant, and hence it cannot be excellent.

4.3 Stable birational equivalence

In this section, we provide a few results about the stable birational equivalence of nonsingular quadratic forms. We start with quadratic Pfister forms.

Proposition 4.15. *Let* char F *be arbitrary. Let* φ *and* ψ *be quadratic forms over* F *of the same dimension, and assume that* φ *is an anisotropic quadratic Pfister form. If* $\varphi \overset{\text{stb}}{\sim} \psi$, *then* $\varphi \overset{\text{sim}}{\sim} \psi$.

Proof. The form $\varphi_{F(\psi)}$ is isotropic by Lemma 1.53. Since φ is a quadratic Pfister form, it follows by Corollary 1.25 that $\varphi_{F(\psi)}$ is hyperbolic. Thus, we obtain from Domination Theorem 4.3 that $c\psi \preccurlyeq \varphi$ for some $c \in F^*$. For dimensional reasons, we must have $c\psi \cong \varphi$. □

Let π be a quadratic Pfister form over F. Then there exists a unique (up to isometry) quadratic form π' such that $\pi \perp \langle 1 \rangle \cong \pi' \perp \mathbb{H}$; we call such form π' the *pure part of* π.

Let us be a bit more specific: Let $\pi \cong \tau \perp [1, a]$ for a nonsingular quadratic form τ and $a \in F$. Then

$$\pi \perp \langle 1 \rangle \cong \tau \perp [1, a] \perp \langle 1 \rangle \cong \tau \perp \mathbb{H} \perp \langle 1 \rangle,$$

and hence $\pi' \cong \tau \perp \langle 1 \rangle$. In particular, it follows that $\pi' \preccurlyeq \pi$, and so π' is a Pfister neighbor of π of codimension one.

Lemma 4.16. *Let* char F *be arbitrary. Let* π *be a quadratic Pfister form over* F. *Then any quadratic Pfister neighbor of* π *of codimension one is similar to the pure part of* π.

Proof. If char $F \neq 2$, then the claim is trivial. Thus, assume char $F = 2$. Let φ be a quadratic Pfister neighbor of π of codimension one, i.e., let $c\varphi \preccurlyeq \pi$ for some $c \in F^*$ and $\dim \varphi = \dim \pi - 1$. Then $\varphi \cong \varphi_r \perp \langle a \rangle$ for a nonsingular quadratic form φ_r and $a \in F^*$. As $c\varphi_r \perp \langle ca \rangle \cong c\varphi \preccurlyeq \pi$, it follows that $ca \in D_F^*(\pi) = G_F^*(\pi)$ (we use Lemma 1.41); thus, we have

$$a\varphi_r \perp \langle 1 \rangle \cong (ca)c\varphi \preccurlyeq ca\pi \cong \pi.$$

Then there exists $b \in F$ such that $a\varphi_r \perp [1, b] \cong \pi$; it follows that

$$a\varphi \perp \mathbb{H} \cong a\varphi_r \perp [1, b] \perp \langle 1 \rangle \cong \pi \perp \langle 1 \rangle.$$

Thus, $a\varphi$ is the pure part of π. □

We extend Proposition 4.15 to quadratic Pfister neighbors of codimension one.

Theorem 4.17. *Let* F *be a field of arbitrary characteristic. Let* π *be a quadratic Pfister form over* F *and let* φ, ψ *be anisotropic quadratic forms over* F *such that* $\dim \varphi = \dim \psi$. *Assume that* φ *is a quadratic Pfister neighbor of* π *of codimension one. If* $\varphi \overset{\text{stb}}{\sim} \psi$, *then* $\varphi \overset{\text{sim}}{\sim} \psi$.

Proof. Assume $\operatorname{char} F = 2$; the proof in the other case is analogous and mostly follows the lines of the proof of [Wad75, Th. 4].

Similarly as in the proof of Proposition 4.15, we get that $c\psi \preccurlyeq \pi$ for some $c \in F^*$. Since $\dim \psi = \dim \varphi = \dim \pi - 1$, we must have $\psi \cong d\psi_r \perp \langle d \rangle$ for a nonsingular quadratic form ψ_r and $d \in F^*$. Since $(cd^{-1})d\psi \preccurlyeq \pi$ and $1 \in D_F^*(d\psi)$, it follows that $cd^{-1} \in D_F^*(\pi) = G_F^*(\pi)$ (we use Lemma 1.41), and hence $d\psi \preccurlyeq (cd^{-1})\pi \cong \pi$.

By Lemma 4.16, there exists $e \in F^*$ such that $e\varphi$ is a pure part of π, i.e., $e\varphi \cong \varphi_r \perp \langle 1 \rangle$ for a nonsingular quadratic form φ_r.

Since $\varphi_r \perp \langle 1 \rangle \preccurlyeq \pi$ and $\psi_r \perp \langle 1 \rangle \preccurlyeq \pi$, there exist $a, b \in F$ such that

$$\varphi_r \perp [1, a] \cong \pi \cong \psi_r \perp [1, b].$$

Then

$$\varphi_r \perp \mathbb{H} \perp \langle 1 \rangle \cong \varphi_r \perp [1, a] \perp \langle 1 \rangle \cong \psi_r \perp [1, b] \perp \langle 1 \rangle \cong \psi_r \perp \mathbb{H} \perp \langle 1 \rangle.$$

Cancelling the hyperbolic planes on both sides (we apply Theorem 1.4), we get $\varphi_r \perp \langle 1 \rangle \cong \psi_r \perp \langle 1 \rangle$, i.e., $e\varphi \cong d\psi$. $\square$

For general quadratic Pfister neighbors, we get a weaker statement.

Proposition 4.18. *Let* $\operatorname{char} F$ *be arbitrary. Let* φ *and* ψ *be irreducible quadratic Pfister neighbors of a quadratic Pfister form* π *over* F. *Then* $\varphi \overset{\text{stb}}{\sim} \psi$.

Proof. By Lemma 1.26, we know that for any extension E/F, it holds that φ_E is isotropic if and only if π_E is isotropic if and only if ψ_E is isotropic. Therefore, choosing $E = F(\psi)$ and $E = F(\varphi)$, we get that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic. It follows by Proposition 1.54 that $\varphi \overset{\text{stb}}{\sim} \psi$. $\square$

In the following example, we provide nonsingular quadratic forms which are stably birationally equivalent but not Vishik equivalent.

Example 4.19. Let F_0 be a field with $\operatorname{char} F_0 = 2$, and set $F = F_0(a, b, c)$ for some elements a, b, c algebraically independent over F_0. Consider the quadratic Pfister form $\pi \cong \langle\langle a, b; c]]$ over F. Then

$$\pi \cong [1, c] \perp a[1, c] \perp b[1, c] \perp ab[1, c].$$

Note that, using the standard relations (1.2) and (1.3), we have

$$b[1, c] \perp ab[1, c] \cong b[1, c] \perp b[a^{-1}, ac] \cong b[1, (a + 1)c] \perp b[a^{-1} + 1, ac];$$

denoting

$$\varphi \cong [1, c] \perp a[1, c] \perp b[1, c] \quad \text{and} \quad \psi \cong [1, c] \perp a[1, c] \perp b[1, (a + 1)c],$$

it follows that both φ and ψ are quadratic Pfister neighbors of π. Hence, $\varphi \overset{\text{stb}}{\sim} \psi$ by Proposition 4.18.

Consider the field $K = F(\wp^{-1}(c))$. Then obviously

$$\begin{aligned} \varphi_K &\cong 3 \times \mathbb{H}, \\ \psi_K &\cong 2 \times \mathbb{H} \perp b[1, (a+1)c]_K. \end{aligned}$$

Assume that the form $[1, (a+1)c]$ is isotropic over K. By Proposition 4.2, that is equivalent to $d[1, c] \cong [1, (a+1)c]$ for some $d \in F^*$. Comparing the Arf invariants of these two binary forms, we get $c = (a+1)c \in F/\wp(F)$, which implies $ac \in \wp(F)$. That contradicts the choice of a and c. Therefore, we have $\mathfrak{i}_{\mathrm{W}}(\psi_K) = 2$, and hence $\varphi \overset{v}{\not\sim} \psi$.

4.4 Summary

Although we have achieved some interesting results on nonsingular and nondegenerate quadratic forms in this chapter, none of them provided us with a decisive answer to whether one equivalence implies another one – with the exception of Example 4.19. This example shows that stable birationally equivalent nonsingular quadratic forms are in general not Vishik equivalent.

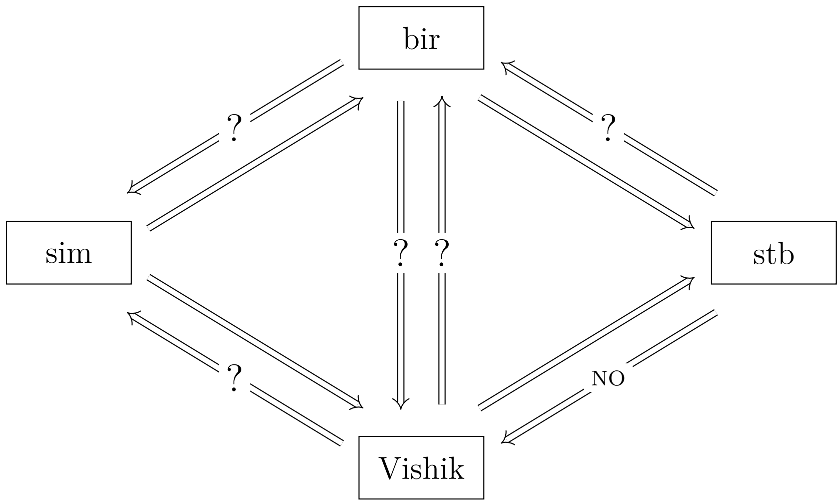

Figure 4.1: The equivalence relations for nonsingular quadratic forms

5. Singular quadratic forms

In this chapter, we come back to the assumption that all the fields are of characteristic 2.

Until now, we put restrictions on the forms we worked with. Not so in this chapter; we develop theorems that hold regardless of the type of the quadratic form. Let us recall a known theorem:

Theorem 5.1 ([EKM08, Th. 18.3]). *Let φ be a quadratic form over F, $f \in F[X]$ a nonzero monic polynomial in one variable, and $a \in F^*$. Then the following conditions are equivalent:*

(i) $af \in \langle D^*_{F(X)}(\varphi)\rangle$,

(ii) *$\varphi_{F(g)}$ is isotropic for each irreducible divisor g occurring to an odd power in the factorization of f.*

This theorem gives a characterization when an anisotropic quadratic form becomes isotropic over a function field of a polynomial. Unfortunately, it considers only polynomials in one variable. Inspired by this theorem and by the recent work of Roussey [Rou22], which treated the case of quadratic forms over fields of characteristic other than two, we focus on the question of when an anisotropic quadratic form becomes isotropic over a function field – first generally over a function field of a polynomial (Subsection 5.2.1), later more specifically over a function field of another quadratic form (Subsection 5.2.2). Recalling Proposition 1.54, the latter clearly has interesting consequences on when two quadratic forms are stably birational, and hence this topic also fits well into the general frame of this thesis.

At the end of the chapter, we shortly look at the Vishik equivalence; we apply some of the results from previous chapters to singular quadratic forms.

5.1 Preliminaries

In this chapter, we will use many statements from other chapters, including the appendix. However, there is still one important proposition that we have not mentioned yet. Basically, it is well known but we weaken some of the assumptions of the original statement here.

Proposition 5.2. *Let φ, ψ, σ be quadratic forms over F with ψ nondefective. If $\varphi_{F(\psi)}$ and $\psi_{F(\sigma)}$ are isotropic, then $\varphi_{F(\sigma)}$ is isotropic as well.*

Proof. If all the forms φ, ψ, and σ are anisotropic, then the claim coincides with [EKM08, Prop. 22.16].

If φ is isotropic, then $\varphi_{F(\sigma)}$ is also isotropic for trivial reasons.

If ψ is isotropic and nondefective, then either $\psi \cong \mathbb{H}$ and $F(\psi) \simeq F$, or $F(\psi)/F$ is purely transcendental by Lemma 1.18; anyway, φ must be isotropic over F, and we are in the previous case.

Now assume that σ is isotropic but nondefective. By an analogous argument as above, we obtain that ψ must be isotropic over F (but still nondefective by the assumption). Then again, φ is isotropic over F by Lemma 1.13, so $\varphi_{F(\sigma)}$ is isotropic.

Finally, if σ is defective, then $F(\sigma)/F(\sigma_{\mathrm{nd}})$ is purely transcendental by Lemma 1.17, and so $\psi_{F(\sigma_{\mathrm{nd}})}$ is isotropic by Lemma 1.13. Then $\varphi_{F(\sigma_{\mathrm{nd}})}$ is isotropic by the previous part of the proof. Applying Lemma 1.13 again, we get that $\varphi_{F(\sigma)}$ is isotropic. □

5.2 Isotropy over function fields and stable birational equivalence

The content of this section is based on the paper [Zem22].

As a preparation for later proofs, we compare some of the sets and multiplicative groups (see Subsection 1.1.6 for the definitions).

Lemma 5.3. *Let φ be a quadratic form.*
(i) $\langle D_F^*(\varphi)^2\rangle = \langle D_F^*(c\varphi)^2\rangle$ *for any* $c \in F^*$,
(ii) $F^{*2} \subseteq D_F^*(\varphi)^2 \subseteq \langle D_F^*(\varphi)^2\rangle \subseteq \langle D_F^*(\varphi)\rangle \subseteq F^*$,
(iii) $\langle D_F^*(\varphi)^2\rangle = \langle D_F^*(c\varphi)\rangle$ *for any* $c \in D_F^*(\varphi)$.

Proof. (i) Since $D_F(\varphi) = D_F(c^{-2}\varphi)$, the group $\langle D_F^*(c\varphi)^2\rangle$ is generated by the elements $ca \cdot c(c^{-2}b) = ab$, $a, b \in D_F^*(\varphi)$.

(ii) For the first inclusion, let $x \in F^*$; then for any $a \in D_F^*(\varphi)$, we have $a^{-1}, ax^2 \in D_F^*(\varphi)$. Hence, $x^2 = a^{-1} \cdot ax^2 \in D_F^*(\varphi)^2$. The second inclusion, $D_F^*(\varphi)^2 \subseteq \langle D_F^*(\varphi)^2\rangle$, follows directly from the definition. To see the third inclusion, let $y \in \langle D_F^*(\varphi)^2\rangle$; then $y = \prod_{i=1}^n a_ib_i$ for some $n > 0$ and $a_i, b_i \in D_F^*(\varphi)$, and hence $y \in \langle D_F^*(\varphi)\rangle$. Finally, the inclusion $\langle D_F^*(\varphi)\rangle \subseteq F^*$ is clear by the definition.

(iii) We have

$$\langle D_F^*(\varphi)^2\rangle \overset{\text{(i)}}{=} \langle D_F^*(c\varphi)^2\rangle \overset{(ii)}{\subseteq} \langle D_F^*(c\varphi)\rangle = \langle cD_F^*(\varphi)\rangle \subseteq \langle D_F^*(\varphi)^2\rangle,$$

where the last inclusion holds, because $c \in D_F^*(\varphi)$. □

5.2.1 Function fields of polynomials

Throughout this subsection, we denote $\boldsymbol{X} = (X_1, \dots, X_l)$ with $l > 0$. For a polynomial $f \in F[\boldsymbol{X}]$, we denote by $\operatorname{lc}(f)$ its leading coefficient with respect to the lexicographical ordering. We say that f is *monic* if $\operatorname{lc}(f) = 1$. By $\deg_{X_i} f$ we mean the maximal degree of the variable X_i appearing in f and by $\deg f$ the maximal total degree of all terms in f.

We start with a lemma which we have already seen in the context of p-forms, see Lemma 2.50. Since the proof does not change for the case of quadratic forms in any other way except for setting $p = 2$, we do not repeat it here.

Lemma 5.4. *Let φ be a quadratic form over F. Let $f \in F[\boldsymbol{X}]$ be an irreducible polynomial. If there exists $a \in F^*$ such that $af \in \langle D_{F(\boldsymbol{X})}^*(\varphi)\rangle$, then $\varphi_{F(f)}$ is isotropic.*

In the following, we want to prove the opposite of the previous lemma: namely, to prove that if a quadratic form φ becomes isotropic over a function field of a polynomial f, then f is represented by a product of some elements of $D_{F(\boldsymbol{X})}^*(\varphi)$. But first, we deal with the leading coefficient.

Lemma 5.5. *Let φ be a quadratic form over F. If $f \in F[\boldsymbol{X}]$ is such that $f \in D_{F(\boldsymbol{X})}^*(\varphi)^m$ for some $m > 0$, then $\operatorname{lc}(f) \in D_F^*(\varphi)^m$.*

Proof. Let $n = \dim\varphi$. By the assumption,

$$f = \prod_{i=1}^{m} \varphi(\boldsymbol{\xi}'_i)$$

for some $\boldsymbol{\xi}'_i = (\xi'_{i1}, \dots, \xi'_{in})$ with $\xi'_{ij} \in F(\boldsymbol{X})$. For each i, we can find a monic polynomial $h_i \in F[\boldsymbol{X}]$ such that $h_i\xi'_{ij} \in F[\boldsymbol{X}]$ for all j; denote $h = \prod_{i=1}^{m} h_i$, $\xi_{ij} = h_i\xi'_{ij}$ and $\boldsymbol{\xi}_i = (\xi_{i1}, \dots, \xi_{in})$. Then

$$fh^2 = \prod_{i=1}^{m} \varphi(\boldsymbol{\xi}_i).$$

For each i, we set $d_i = \max\{\deg \xi_{ij} \mid 1 \le j \le n\}$, and

$$\alpha_{ij} = \begin{cases} \mathrm{lc}(\xi_{ij}) & \text{if } \deg \xi_{ij} = d_i, \\ 0 & \text{otherwise.} \end{cases}$$

Writing $\boldsymbol{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{in})$, it follows that $\varphi(\boldsymbol{\alpha}_i)$ is the leading coefficient of $\varphi(\boldsymbol{\xi}_i)$. Therefore, $\prod_{i=1}^{m} \varphi(\boldsymbol{\alpha}_i)$ is the leading coefficient of $\prod_{i=1}^{m} \varphi(\boldsymbol{\xi}_i) = fh^2$. Since h is monic, we have $\mathrm{lc}(f) = \mathrm{lc}(fh^2)$, and the claim follows. $\square$

Lemma 5.6. *Let $f \in F[X]$ be a monic irreducible polynomial in one variable and φ a nondefective quadratic form over F of dimension n. Suppose that f divides $\varphi(\xi_1, \dots, \xi_n)$ for some $\xi_i \in F[X]$, $1 \le i \le n$, not all of them divisible by f. Then $f \in D^*_{F(X)}(\varphi)^m$ for some $m \le \deg f$.*

Proof. First assume that $\dim\varphi = 1$; then $\varphi \cong \langle a\rangle$ for some $a \in F^*$, and $f \mid \varphi(\xi_1) = a\xi_1^2$, i.e., f divides ξ_1, a contradiction to our assumption. Hence, this case cannot happen.

If φ is isotropic, then, since it is nondefective by the assumption, we have $\mathbb{H} \subseteq \varphi$. Hence, $D^*_{F(X)}(\varphi) = F(X)^*$, and $f \in D^*_{F(X)}(\varphi)$ trivially.

From now on, suppose that φ is anisotropic and $\dim\varphi \ge 2$. If $\deg f = 1$, then $f(X) = X - b$ for some $b \in F$. It means that $\varphi(\xi_1(X), \dots, \xi_n(X)) = (X - b)g(X)$ for some $g \in F[X]$, and hence $\varphi(\xi_1(b), \dots, \xi_n(b)) = 0$. Since φ is anisotropic, we get $\xi_i(b) = 0$ for every $1 \le i \le n$; but in that case $f(X) = X - b$ divides all ξ_i's, a contradiction.

Assume $\deg f = 2$; then, up to a linear substitution, there are only two possibilities: either $f(X) = X^2 + c$ or $f(X) = X^2 + X + c$ for some $c \in F^*$. Pick a root $\gamma \in \overline{F}$ of f (i.e., $\gamma^2 = c$, resp. $\gamma^2 + \gamma = c$). Since we can find $g(X) \in F[X]$ such that $f(X)g(X) = \varphi(\xi_1(X), \dots, \xi_n(X))$, it follows that over $F(\gamma)$, we have $\varphi(\xi_1(\gamma), \dots, \xi_n(\gamma)) = 0$. Note that if $\xi_i(\gamma) = 0$ for all $1 \le i \le n$, then $X + \gamma$ divides each ξ_i over $F(\gamma)$; as f is the minimal polynomial of γ over F, it implies that f divides each ξ_i over F, which is a contradiction. Therefore, not all $\xi_i(\gamma) = 0$, and so $\varphi_{F(\gamma)}$ is isotropic. Let us apply Proposition 3.1, resp. Proposition 4.2: We can find some $c' \in F^*$ such that in the case of $f(X) = X^2 + c$, we have $c'\langle 1, c\rangle \preccurlyeq \varphi$, and in the case of $f(X) = X^2 + X + c$, we have $c'\,[1, c] \subseteq \varphi$. In both cases we get $\frac{1}{c'} \in D^*_F(\varphi)$ and $c'f(X) \in D^*_{F(X)}(\varphi)$; thus, $f(X) \in D^*_F(\varphi)D^*_{F(X)}(\varphi) \subseteq D^*_{F(X)}(\varphi)^2$.

Now we will assume $\deg f \ge 2$ and proceed by induction on $\deg f$. For each i, we divide ξ_i by f with a remainder ξ'_i:

$$\xi_i = f\zeta_i + \xi'_i \quad \text{for some } \zeta_i, \xi'_i \in F[X],\ \deg \xi'_i < \deg f.$$

We denote

$$\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n), \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_n), \quad \boldsymbol{\xi}' = (\xi_1', \ldots, \xi_n');$$

then $\boldsymbol{\xi} = f\boldsymbol{\zeta} + \boldsymbol{\xi}'$, and by the assumption there exists $g \in F[X]$ such that

$$fg = \varphi(\boldsymbol{\xi}) = \varphi(\boldsymbol{\xi}') + f^2\varphi(\boldsymbol{\zeta}) + f\mathfrak{b}_\varphi(\boldsymbol{\zeta}, \boldsymbol{\xi}').$$

Let $g + f\varphi(\boldsymbol{\zeta}) + \mathfrak{b}_\varphi(\boldsymbol{\zeta}, \boldsymbol{\xi}') = ah$ with $a \in F^*$ and $h \in F[X]$ monic; then $\varphi(\boldsymbol{\xi}') = afh$, and we have

$$\deg f + \deg h = \deg \varphi(\boldsymbol{\xi}') \leq 2 \max\{\deg \xi_i' \mid i = 1, \ldots, n\} < 2 \deg f,$$

i.e., $\deg h < \deg f$. Moreover, since there is at least one ξ_i non-divisible by f, we have $\boldsymbol{\xi}' \neq 0$. As φ is anisotropic, it follows that $\varphi(\boldsymbol{\xi}') \neq 0$; hence, $\deg \varphi(\boldsymbol{\xi}')$ is even, so $\deg f$ and $\deg h$ have necessarily the same parity. Therefore, $\deg h \leq \deg f - 2$.

Write $h = \prod_{j=1}^r h_j$, where $h_j \in F[X]$ is a monic irreducible polynomial for each j. If there is a $k \in \{1, \ldots, r\}$ such that $h_k \mid \xi_i'$ for every i, then $h_k^2 \mid \varphi(\boldsymbol{\xi}') = afh$. Since both h_k and f are irreducible and $\deg h_k < \deg f$, it follows that $h_k^2 \mid h$. In that case we replace $\boldsymbol{\xi}'$ by $\left(\frac{\xi_1'}{h_k}, \ldots, \frac{\xi_n'}{h_k}\right)$ and h by $\frac{h}{h_k^2}$. Repeating this process if necessary, we end up with $afh'' = \varphi(\boldsymbol{\xi}'')$, $\boldsymbol{\xi}'' = (\xi_1'', \ldots, \xi_n'') \in F[X]^n$ and $h'' = \prod_{j=1}^s h_j$, where, for each $1 \leq j \leq s$, the polynomial h_j is monic irreducible and does not divide all ξ_i'''s.

As both f and h'' are monic, a is the leading coefficient of $afh'' = \varphi(\boldsymbol{\xi}'')$. Therefore, $a \in D_F^*(\varphi)$ by Lemma 5.5.

If $h'' = 1$, then

$$f = \frac{1}{a}\varphi(\boldsymbol{\xi}'') \in D_F^*(\varphi)D_{F(X)}^*(\varphi) \subseteq D_{F(X)}^*(\varphi)^2,$$

and we are done since we assumed $\deg f \geq 2$. If $h'' \neq 1$, then we know for each $1 \leq j \leq s$ that $\deg h_j < \deg f$, and h_j is a monic irreducible polynomial dividing $\varphi(\boldsymbol{\xi}'')$ but not dividing all ξ_i''. Hence, by the induction hypothesis, $h_j \in D_{F(X)}^*(\varphi)^{m_j}$ for some $m_j \leq \deg h_j$. It also follows that $\frac{1}{h_j} \in D_{F(X)}^*(\varphi)^{m_j}$. Consequently,

$$\frac{1}{h''} = \prod_{j=1}^s \frac{1}{h_j} \in D_{F(X)}^*(\varphi)^m \quad \text{where } m = \sum_{j=1}^s m_j,$$

and thus

$$f = \frac{1}{ah''}\varphi(\boldsymbol{\xi}'') \in D_F^*(\varphi)D_{F(X)}^*(\varphi)^m D_{F(X)}^*(\varphi) \subseteq D_{F(X)}^*(\varphi)^{m+2},$$

where

$$m + 2 \leq \deg h'' + 2 \leq \deg h + 2 \leq \deg f. \qquad \square$$

With Lemmas 5.4 and 5.6 at hand, we can prove our first characterization of the isotropy of $\varphi_{F(f)}$. Note its similarity to Proposition 5.2 but we provide an upper bound on the power of $D_{F(X)}^*(\varphi)$ necessary.

Proposition 5.7. *Let φ be a nondefective quadratic form over F such that $1 \in D_F^*(\varphi)$, and let $f \in F[X]$ be a monic irreducible polynomial in one variable. Then the following are equivalent:*

(i) *$f \in D_{F(X)}^*(\varphi)^m$, $m \le \deg f$;*

(ii) *$\varphi_{F(f)}$ is isotropic.*

Proof. The implication (i) $\Rightarrow$ (ii) is covered by Lemma 5.4. To prove the converse, assume that the form $\varphi_{F(f)}$ is isotropic and denote $n = \dim \varphi$. Then we can find $\overline{\xi}_1, \ldots, \overline{\xi}_n \in F(f) = F[X]/(f)$, not all zero, such that $\varphi_{F(f)}(\overline{\xi}_1, \ldots, \overline{\xi}_n) = 0$. For each $1 \le i \le n$, let $\xi_i \in F[X]$ be such that the image of ξ_i in $F[X]/(f)$ is precisely $\overline{\xi}_i$. Then $\varphi(\xi_1, \ldots, \xi_n) = fh$ for some $h \in F[X]$. Note that since not all of the $\overline{\xi}_i$'s were zero, not all of the ξ_i's are divisible by f. Hence, the claim follows by Lemma 5.6. $\square$

Now we extend the previous proposition to polynomials in more variables.

Theorem 5.8. *Let φ be a nondefective quadratic form over F such that $1 \in D_F^*(\varphi)$. Let $f \in F[\boldsymbol{X}]$ be a monic irreducible polynomial such that $\deg f \ge 1$. Then the following are equivalent:*

(i) *$f \in D_{F(\boldsymbol{X})}^*(\varphi)^m$ with $m \le \deg f$,*

(ii) *$\varphi_{F(f)}$ is isotropic.*

Proof. The implication (i) $\Rightarrow$ (ii) is covered by Lemma 5.4. Let us prove the converse. If φ is isotropic, then necessarily $\mathbb{H} \subseteq \varphi$, and $f \in D_{F(\boldsymbol{X})}^*(\varphi)$ for trivial reasons. Thus, assume that φ is anisotropic.

Note that we can assume $\deg_{X_i} f > 0$ for all $1 \le i \le l$; otherwise, we pick the maximal subset $\{i_1, \ldots, i_{\tilde{l}}\}$ of $\{1, \ldots, l\}$ such that $\deg_{X_{i_j}} f > 0$ for all $1 \le j \le \tilde{l}$ (where $\tilde{l} \ge 1$ by the assumption), and for $\widetilde{\boldsymbol{X}} = (X_{i_1}, \ldots, X_{i_{\tilde{l}}})$ prove that $f \in D_{F(\widetilde{\boldsymbol{X}})}^*(\varphi)^m$ for some m. Since $D_{F(\widetilde{\boldsymbol{X}})}^*(\varphi)^m \subseteq D_{F(\boldsymbol{X})}^*(\varphi)^m$, the claim follows.

First, we suppose that the field F is infinite, and we proceed by induction on the number of variables l. For $l = 1$, we apply Lemma 5.6.

Assume $l \ge 2$. Denote $\boldsymbol{X}' = (X_2, \ldots, X_l)$, $d = \deg_{X_1} f$ and $n = \dim \varphi$. Then we can find polynomials $f_d \in F[\boldsymbol{X}']$ and $\widetilde{f} \in F[\boldsymbol{X}]$ such that

$$f = f_d X_1^d + \widetilde{f}$$

and $\deg_{X_1} \widetilde{f} < d$; then $\deg f_d \le \deg f - d$. Note that, since f is monic, f_d is monic as well. Furthermore, consider $g = \frac{f}{f_d}$ as an element of $F(\boldsymbol{X}')[X_1]$; then $\deg_{X_1} g = d$ and by Gauss' lemma, g is a monic irreducible polynomial in X_1 over $F(\boldsymbol{X}')$. Since $F(\boldsymbol{X}')(g) = F(f)$, the quadratic form φ is isotropic over $F(\boldsymbol{X}')(g)$, and, by Proposition 5.7, we get

$$g \in D_{F(\boldsymbol{X}')(X_1)}^*(\varphi)^{d'} = D_{F(\boldsymbol{X})}^*(\varphi)^{d'}$$

for some $d' \le d$. Since $1 \in D_F^*(\varphi)$, we can assume $d' = d$.

Now let $h \in F[\boldsymbol{X}]$ be such that $gh^2 \in D_{F[\boldsymbol{X}]}^*(\varphi)^d$, i.e.,

$$gh^2 = \varphi(\xi_{11}, \ldots, \xi_{1n}) \cdots \varphi(\xi_{d1}, \ldots, \xi_{dn})$$

for some $\xi_{ij} \in F[\boldsymbol{X}]$; this can be rewritten as

$$fh^2 = f_d\,\varphi(\xi_{11}, \ldots, \xi_{1n}) \cdots \varphi(\xi_{d1}, \ldots, \xi_{dn}). \tag{5.1}$$

Moreover, for each i, we can assume that the polynomials $\xi_{i1}, \ldots, \xi_{in}$ have no common divisor in $F[\boldsymbol{X}]$: If $q \mid \xi_{ij}$ for all j, then necessarily $q^2 \mid fh^2$, and hence $q \mid h$ since f is irreducible. In that case, we can consider $\frac{\xi_{i1}}{q}, \ldots, \frac{\xi_{in}}{q}$ and $\frac{h}{q}$ instead.

Write $f_d = r^2 s$ with $r, s \in F[\boldsymbol{X}']$ monic polynomials (recall that f_d is monic) such that s has no square factors. If $s = 1$, then clearly $f_d = r^2 \in D^*_{F(\boldsymbol{X}')}(\varphi)$. If $s \neq 1$, then proving $s \in D^*_{F(\boldsymbol{X}')}(\varphi)^{m'}$ for some $m' > 0$ will imply that also $f_d = r^2 s \in D^*_{F(\boldsymbol{X}')}(\varphi)^{m'}$.

Suppose $s \neq 1$. For each monic irreducible polynomial $t \in F[\boldsymbol{X}']$ such that $t \mid s$, we proceed as follows: First, note that $t \nmid f$, because f is irreducible, and by the assumption $f \notin F[\boldsymbol{X}']$. Since F is infinite, we can find $c \in F$ such that $t(\boldsymbol{X}') \nmid f(c, \boldsymbol{X}')$. Denote $\xi'_{ij} = \xi_{ij}(c, \boldsymbol{X}')$, $f' = f(c, \boldsymbol{X}')$ and $h' = h(c, \boldsymbol{X}')$; then we rewrite (5.1) as

$$f'(h')^2 = f_d\,\varphi(\xi'_{11}, \ldots, \xi'_{1n}) \cdots \varphi(\xi'_{d1}, \ldots, \xi'_{dn}). \tag{5.2}$$

Note that $\xi'_{i1} = \cdots = \xi'_{in} = 0$ for some i would mean that $X_1 - c \mid \xi_{ij}$ for all j, which would contradict our assumption. Since φ is anisotropic over F and hence also over $F(\boldsymbol{X}')$ by Lemma 1.13, we have $\varphi(\xi'_{i1}, \ldots, \xi'_{in}) \neq 0$ for each i; hence, $f', h' \neq 0$. Let us compare the t-adic valuation v_t on the left and right hand side of (5.2): Since $v_t(f') = 0$, the value $v_t(f'(h')^2)$ must be even. On the other hand, $v_t(f_d)$ is odd by the assumption, and hence there exists a $k \in \{1, \ldots, d\}$ such that $v_t(\varphi(\xi'_{k1}, \ldots, \xi'_{kn}))$ is odd. Denote $u = \min\{v_t(\xi'_{kj}) \mid 1 \leq j \leq n\}$ and $\xi''_{kj} = t^{-u}\xi'_{kj}$ for $1 \leq j \leq n$. Then not all $\xi''_{k1}, \ldots, \xi''_{kn}$ are divisible by t, and we have

$$v_t(\varphi(\xi''_{k1}, \ldots, \xi''_{kn})) = v_t(\varphi(\xi'_{k1}, \ldots, \xi'_{kn})) - 2u > 0,$$

i.e., t divides $\varphi(\xi''_{k1}, \ldots, \xi''_{kn})$. Therefore, $\varphi(\xi''_{k1}, \ldots, \xi''_{kn}) = 0$ over $F[\boldsymbol{X}']/(t)$ and hence also over $F(t)$. Thus, φ is isotropic over $F(t)$. By the induction hypothesis, $t \in D^*_{F(\boldsymbol{X}')}(\varphi)^{\deg t}$.

All in all, we get

$$f_d = r^2 \prod_{t \mid s \text{ irred.}} t \ \in D^*_{F(\boldsymbol{X}')}(\varphi)^{m'}$$

where

$$m' = \begin{cases} 1 & \text{if } s = 1, \\ \deg s & \text{otherwise,} \end{cases}$$

and we have $m' \leq \deg f_d \leq \deg f - d$. Hence,

$$f = f_d\, g \in D^*_{F(\boldsymbol{X})}(\varphi)^m$$

with $m = m' + d \leq \deg f$ as claimed.

Now assume that F is finite; then the field $F(\!(Y)\!)$ is infinite, and $\varphi_{F(\!(Y)\!)(f)}$ is isotropic; by the previous result, we have $f \in D^*_{F(\!(Y)\!)(\boldsymbol{X})}(\varphi)^m$ for some

$m \leq \deg f$. Since $F(\!(Y)\!)(\boldsymbol{X}) \subseteq F(\boldsymbol{X})(\!(Y)\!)$, we also have $f \in D^*_{F(\boldsymbol{X})(\!(Y)\!)}(\varphi)^m$. Thus, we have

$$f = \prod_{i=1}^{m} \varphi(\boldsymbol{\xi}_i)$$

for some $\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{in})$ with $\xi_{ij} \in F(\boldsymbol{X})(\!(Y)\!)$. As the form φ is anisotropic, the value $v_Y(\varphi(\boldsymbol{\xi}_i))$ is even for each i by Lemma A.5; write $v_Y(\varphi(\boldsymbol{\xi}_i)) = 2k_i$, and set $\boldsymbol{\xi}_i' = Y^{-k_i}\boldsymbol{\xi}_i$. By Lemma A.6, it follows that $k_i = \min\{v_Y(\xi_{ij}) \mid 1 \leq j \leq n\}$, and hence $\boldsymbol{\xi}_i' \in F(\boldsymbol{X})[\![Y]\!]^n$.

We have

$$Y^{2k} f = \prod_{i=1}^{m} \varphi(\boldsymbol{\xi}_i'), \qquad \text{where} \qquad k = \sum_{i=1}^{m} k_i. \tag{5.3}$$

Since $v_Y(f) = 0$ and $v_Y(\varphi(\boldsymbol{\xi}_i')) = 0$ for each i, it follows that $k = 0$. Hence, passing the equation (5.3) to the residue field $F(\boldsymbol{X})$, we get

$$f = \prod_{i=1}^{m} \varphi\left(\overline{\boldsymbol{\xi}_i'}\right),$$

and thus $f \in D^*_{F(\boldsymbol{X})}(\varphi)^m$. $\square$

As the final step in reaching the goal of this subsection, we consider reducible polynomials in more variables. We obtain a characteristic two version of [BF95, Th. 1].

Theorem 5.9. *Let φ be a nondefective quadratic form over F such that $1 \in D_F(\varphi)$. Let $a \in F^*$ and $f_1, \ldots, f_r, g \in F[\boldsymbol{X}]$ be monic polynomials with $f_1, \ldots, f_r$ distinct and irreducible. Suppose that $f = af_1 \cdots f_r g^2$. Then the following are equivalent:*

(i) *$f \in \langle D^*_{F(\boldsymbol{X})}(\varphi)\rangle$,*

(ii) *$a \in \langle D^*_F(\varphi)\rangle$ and $f_k \in \langle D^*_{F(\boldsymbol{X})}(\varphi)\rangle$ for every $1 \leq k \leq r$,*

(iii) *$a \in \langle D^*_F(\varphi)\rangle$ and $\varphi_{F(f_k)}$ is isotropic for every $1 \leq k \leq r$.*

Proof. Applying Theorem 5.8 to each f_k, $1 \leq k \leq r$, we get the implication (iii) $\Rightarrow$ (ii). As $g^2 \in D^*_{F(\boldsymbol{X})}(\varphi)$, the implication (ii) $\Rightarrow$ (i) is trivial. Hence, we only need to prove (i) $\Rightarrow$ (iii).

Suppose $f \in \langle D^*_{F(\boldsymbol{X})}(\varphi)\rangle$. By Lemma 5.5, $a = \mathrm{lc}(f) \in \langle D^*_F(\varphi)\rangle$. By clearing denominators, we may assume

$$\prod_{i=1}^{m} \varphi(\boldsymbol{\xi}_i) = fh^2 = af_1 \cdots f_r g^2 h^2$$

for some $m > 0$, $\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{in}) \in F[\boldsymbol{X}]^n$ and a monic polynomial $h \in F[\boldsymbol{X}]$. Let $\gamma_i = \gcd(\xi_{i1}, \ldots, \xi_{in})$. Without loss of generality, we can assume $\gamma_i = 1$ for all $1 \leq i \leq m$; otherwise we replace ξ_{ij} by $\frac{\xi_{ij}}{\gamma_i}$ and gh by $\frac{gh}{\gamma_i}$. For any $1 \leq k \leq r$, we know that f_k is irreducible, and hence f_k divides $\varphi(\boldsymbol{\xi}_i)$ for some i; thus $\varphi(\boldsymbol{\xi}_i) = 0$ over $F(f_k)$. By the assumption above, $\boldsymbol{\xi}_i$ is a nonzero vector over $F[\boldsymbol{X}]/(f_k)$, and hence also over $F(f_k)$; it follows that $\varphi_{F(f_k)}$ is isotropic. $\square$

Remark 5.10. Note that by the transition to reducible polynomials, we lost the upper bound for the necessary power of $D^*_{F(\boldsymbol{X})}(\varphi)$. But if we keep in Theorem 5.9 the assumption that the polynomial f is monic, then we can also keep the upper bounds. In particular, with the notation from the theorem, the following are equivalent for monic f (i.e., $a = 1$):

(i) $f \in D^*_{F(\boldsymbol{X})}(\varphi)^m$ with $m \leq \deg f$,
(ii) $f_k \in D^*_{F(\boldsymbol{X})}(\varphi)^{m_k}$ with $m_k \leq \deg f_k$ for every $1 \leq k \leq r$,
(iii) $\varphi_{F(f_k)}$ is isotropic for every $1 \leq k \leq r$.

5.2.2 Function fields of quadratic forms

In this subsection, we concentrate on the isotropy over a function field of a quadratic form. Most of the proofs in this subsection must deal with the obstacle of "two types" of isotropy: Unlike in the case of characteristic other than two, the isotropy of a quadratic form φ does not necessarily mean $\mathbb{H} \subseteq \varphi$, because φ may be just defective. So an isotropic form does not have to be universal, which makes both the assertions and the proofs more elaborate.

For a start, we need to prepare some lemmas. In particular, the first lemma basically says that discarding the defect does not affect the Witt index.

Lemma 5.11. *Let φ be a semisingular and ψ an arbitrary quadratic form over F, and let E/F be a field extension. Assume that $\mathfrak{i}_W(\varphi_{F(\psi)}) > 0$ and $\mathfrak{i}_W(\varphi_E) = 0$. Let φ' be a quadratic form over E such that $\varphi' \cong (\varphi_E)_{\mathrm{an}}$. Then $\mathfrak{i}_W(\varphi'_{E(\psi)}) > 0$.*

Proof. As $\mathfrak{i}_W(\varphi_E) = 0$, we have $\varphi' \cong (\varphi_E)_{\mathrm{nd}}$; hence,

$$\varphi_{E(\psi)} \cong \varphi'_{E(\psi)} \perp \mathfrak{i}_{\mathrm{d}}(\varphi_E) \times \langle 0 \rangle,$$

and it follows that $\mathfrak{i}_W(\varphi'_{E(\psi)}) = \mathfrak{i}_W(\varphi_{E(\psi)})$. Therefore, $\mathfrak{i}_W(\varphi'_{E(\psi)}) > 0$. □

Lemma 5.12. *Let φ, ψ be quadratic forms over F with $\dim \varphi, \dim \psi \geq 2$, and let $\varphi_{F(\psi)}$ be isotropic. If E/F is a field extension such that both φ_E and ψ_E are anisotropic, then $D^*_E(\psi)^2 \subseteq D^*_E(\varphi)^2$.*

Proof. Let $c \in D^*_E(\psi)$; then $1 \in D^*_E(c\psi)$. Let $a \in D^*_E(c\psi)$. It suffices to show that $a \in D^*_E(\varphi)^2$.

First, assume $a = x^2$ for some $x \in E^*$; then $a = x^2 = bx^2 \cdot \frac{b}{b^2} \in D^*_E(\varphi)^2$ for any $b \in D^*_E(\varphi)$. Now suppose $a \notin E^2$. Let V_{ψ_E} be the underlying vector space of ψ_E, and let $u, v \in V_{\psi_E}$ be such that $c\psi_E(u) = 1$ and $c\psi_E(v) = a$. Note that u, v are linearly independent: if $v = tu$ for some $t \in E^*$, then $a = c\psi_E(v) = c\psi_E(tu) = t^2 \in E^2$, a contradiction.

If $\mathfrak{b}_{\psi_E}(u, v) = 0$, then $\langle 1, a \rangle \preccurlyeq c\psi_E$, and hence $\psi_{E(\sqrt{a})}$ is isotropic. By Proposition 5.2, $\varphi_{E(\sqrt{a})}$ is also isotropic, and by Proposition 3.1 we have $b\langle 1, a \rangle \preccurlyeq \varphi_E$ for some $b \in E^*$. In particular, $b, ba \in D^*_E(\varphi)$, and hence $a = ba \cdot \frac{b}{b^2} \in D^*_E(\varphi)^2$.

Let $\mathfrak{b}_{\psi_E}(u, v) \neq 0$. Set $s = \mathfrak{b}_{\psi_E}(u, v)$; then $[1, s^{-2}a] \subseteq c\psi_E$. As ψ_E is anisotropic by the assumption, it follows that $s^{-2}a \notin \wp(E)$. So, considering

$\alpha = \wp^{-1}(s^{-2}a) \in \overline{F}$, we get $[E(\alpha) : E] = 2$ and $\psi_{E(\alpha)}$ is isotropic. Similarly as above, it follows that $\varphi_{E(\alpha)}$ is isotropic by Proposition 5.2. Invoking Proposition 4.2, we get $b\,[1, s^{-2}a] \subseteq \varphi_E$ for some $b \in D_E^*(\varphi)$. In particular, $b, bs^{-2}a \in D_E^*(\varphi)$, hence $a = \frac{ab}{s^2} \cdot \frac{bs^2}{b^2} \in D_E^*(\varphi)^2$. $\square$

Proposition 5.13. *Let φ, ψ be nondefective quadratic forms over F. Set*

$$\mathcal{E} = \begin{cases} \{E \mid E/F \text{ an extension s.t. } \mathfrak{i}_{\mathrm{d}}(\varphi_E) = 0\} & \text{if } \psi \text{ is totally singular,} \\ \{E \mid E/F \text{ an extension}\} & \text{otherwise.} \end{cases}$$

If $\varphi_{F(\psi)}$ is isotropic, then $D_E^*(\psi)^2 \subseteq D_E^*(\varphi)^2$ for every $E \in \mathcal{E}$.

Proof. We will prove the proposition in several steps, starting with some trivial cases.

(0) Let E_0/F be a field extension such that $\mathfrak{i}_{\mathrm{W}}(\varphi_{E_0}) > 0$, i.e., $\mathbb{H}_{E_0} \subseteq \varphi_{E_0}$. Then $D_{E_0}^*(\varphi) = E_0^*$, and trivially, $D_{E_0}^*(\psi)^2 \subseteq D_{E_0}^*(\varphi)^2$.

(1) Since a nondefective one-dimensional quadratic form cannot become isotropic, we may assume $\dim \varphi \geq 2$.

If $\dim \psi = 1$, then $F(\psi) = F$. If ψ is isotropic (but nondefective by the assumption), then the field extension $F(\psi)/F$ is purely transcendental by Lemma 1.18. In both cases, we get that φ is isotropic over F (in the latter case by Lemma 1.13).

If φ is isotropic over F, then (since it is nondefective), $\mathbb{H} \subseteq \varphi$. Then also $\mathbb{H}_E \subseteq \varphi_E$ for any field extension E/F, and we are done by (0) with $E_0 = E$.

(2) Suppose that φ, ψ are anisotropic, and let $E \in \mathcal{E}$. We claim that there exists a form $\psi' \subseteq \psi$ (over F) such that $\psi'_E \cong (\psi_E)_{\mathrm{nd}}$: Write $\psi \cong \psi_r \perp \mathrm{ql}(\psi)$. Then, by Lemma 2.2, there exists an anisotropic form $\sigma \subseteq \mathrm{ql}(\psi)$ over F such that $\sigma_E \cong (\mathrm{ql}(\psi)_E)_{\mathrm{an}}$. Therefore,

$$(\psi_r \perp \sigma)_E \cong (\psi_r)_E \perp (\mathrm{ql}(\psi)_E)_{\mathrm{an}} \cong (\psi_E)_{\mathrm{nd}}.$$

Setting $\psi' = \psi_r \perp \sigma$, the claim follows. Furthermore, note that we have $D_E^*(\psi) = D_E^*(\psi')$.

If $\dim \psi' = 1$, then $D_E^*(\psi')^2 = E^{*2} \subseteq D_E^*(\varphi)^2$, and thus the inclusion $D_E^*(\psi)^2 \subseteq D_E^*(\varphi)^2$ holds for trivial reasons.

Suppose $\dim \psi' \geq 2$; then $\psi' \subseteq \psi$ implies that $\psi_{F(\psi')}$ is isotropic. Together with the assumptions that $\varphi_{F(\psi)}$ is isotropic and ψ is nondefective, we get that $\varphi_{F(\psi')}$ is isotropic by Proposition 5.2.

(3) Now fix a field $E \in \mathcal{E}$. By (1) we can assume that φ, ψ are anisotropic over F and $\dim \varphi, \dim \psi \geq 2$. Invoking (2) we can suppose (by replacing ψ with ψ') that ψ_E is nondefective. We treat different kinds of quadratic forms separately.

First, assume that φ_E is anisotropic. If ψ_E is isotropic, then (since we assume ψ_E to be nondefective), the extension $E(\psi)/E$ is purely transcendental by Lemma 1.18, and so, by Lemma 1.13, the isotropy of $\varphi_{E(\psi)}$ implies the isotropy of φ_E, a contradiction. Therefore, ψ_E must be anisotropic in this case, and the claim follows from Lemma 5.12.

Now suppose that φ_E is isotropic. If $\mathfrak{i}_W(\varphi_E) > 0$, then we are done by (0). Hence, assume $\mathfrak{i}_t(\varphi_E) = \mathfrak{i}_d(\varphi_E) > 0$. Note that $\mathfrak{i}_d(\varphi_{F(\psi)}) > 0$ means that $\mathrm{ql}(\varphi)_{F(\psi)}$ is isotropic, which is possible only if ψ is totally singular by Lemma 1.21; but in that case we have $\mathfrak{i}_d(\varphi_E) = 0$ by the assumption. Therefore, it must be $\mathfrak{i}_d(\varphi_{F(\psi)}) = 0$, and hence $\mathfrak{i}_W(\varphi_{F(\psi)}) > 0$. If ψ_E were isotropic, then (since ψ_E is nondefective) $E(\psi)/E$ were a purely transcendental extension by Lemma 1.18, and hence, by Lemma 1.13,

$$0 = \mathfrak{i}_W(\varphi_E) = \mathfrak{i}_W(\varphi_{E(\psi)}) \geq \mathfrak{i}_W(\varphi_{F(\psi)}) > 0,$$

a contradiction; therefore, ψ_E must be anisotropic. Moreover, note that the assumptions $\mathfrak{i}_d(\varphi_E) > 0$ and $\mathfrak{i}_W(\varphi_{F(\psi)}) > 0$ imply that φ is semisingular. Set $\varphi' = (\varphi_E)_{\mathrm{an}}$; by Lemma 5.11, we have $\mathfrak{i}_W(\varphi'_{E(\psi)}) > 0$. Applying Lemma 5.12 to the forms φ' and ψ, we get $D^*_E(\psi)^2 \subseteq D^*_E(\varphi')^2$. Since $D^*_E(\varphi') = D^*_E(\varphi)$, the claim follows. $\square$

Combining Proposition 5.13, Lemma 5.3 and Lemma 5.4 applied to the case $f = \psi$, we get a full characterisation of the isotropy of $\varphi_{F(\psi)}$.

Theorem 5.14. *Let φ, ψ be nondefective quadratic forms over F, and suppose that $\dim\psi \geq 2$. Denote $\boldsymbol{X} = (X_1, \dots, X_{\dim\psi})$, and set*

$$\mathcal{E} = \begin{cases} \{E \mid E/F \text{ an extension s.t. } \mathfrak{i}_d(\varphi_E) = 0\} & \text{if } \psi \text{ is totally singular,} \\ \{E \mid E/F \text{ an extension}\} & \text{otherwise.} \end{cases}$$

Then the following assertions are equivalent:
(i) $\varphi_{F(\psi)}$ *is isotropic,*
(ii) $D^*_E(\psi)^2 \subseteq D^*_E(\varphi)^2$ *for every* $E \in \mathcal{E}$,
(iii) $a\psi(\boldsymbol{X}) \in D^*_{F(\boldsymbol{X})}(\varphi)^2$ *for every* $a \in D^*_F(\psi)$,
(iv) $\langle D^*_E(\psi)^2\rangle \subseteq \langle D^*_E(\varphi)^2\rangle$ *for every* $E \in \mathcal{E}$,
(v) $a\psi(\boldsymbol{X}) \in \langle D^*_{F(\boldsymbol{X})}(\varphi)^2\rangle$ *for every* $a \in D^*_F(\psi)$,
(vi) $\langle D^*_E(a\psi)\rangle \subseteq \langle D^*_E(\varphi)\rangle$ *for every* $E \in \mathcal{E}$ *and every* $a \in D^*_F(\psi)$,
(vii) $a\psi(\boldsymbol{X}) \in \langle D^*_{F(\boldsymbol{X})}(\varphi)\rangle$ *for every* $a \in D^*_F(\psi)$.

Proof. (i) $\Rightarrow$ (ii) is covered by Proposition 5.13. The implications (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) follow trivially by setting $E = F(\boldsymbol{X})$; note that $F(\boldsymbol{X}) \in \mathcal{E}$ by Lemma 1.13. By Lemma 5.3, we get (iii) $\Rightarrow$ (v) $\Rightarrow$ (vii).

To prove (ii) $\Rightarrow$ (iv), let $x \in \langle D^*_E(\psi)^2\rangle$. Then $x = \prod_{i=1}^k a_i b_i$ for some $a_i, b_i \in D^*_E(\psi)$. Since $a_i b_i \in D^*_E(\varphi)^2$ by the assumption of (ii), we get $x \in \langle D^*_E(\psi)^2\rangle$.

By combining the assumption of (iv) with Lemma 5.3, we obtain that $\langle D^*_E(a\psi)\rangle = \langle D^*_E(\psi)^2\rangle \subseteq \langle D^*_E(\varphi)^2\rangle \subseteq \langle D^*_E(\varphi)\rangle$; therefore, (iv) $\Rightarrow$ (vi).

Finally, Lemma 5.4 gives (vii) $\Rightarrow$ (i) for all but one possible ψ; if $\psi \cong \mathbb{H}$, then the polynomial $\psi(\boldsymbol{X})$ is reducible. But by assuming (vii), we have $X_1X_2 \in \langle D^*_{F(X_1,X_2)}(\varphi)\rangle$, and it follows by Theorem 5.9 that φ_F must be isotropic (because for $f_i = X_i$, we have $F(f_i) \simeq F$). Since $F(\mathbb{H}) = F$ by definition, the claim follows. $\square$

Remark 5.15. The assumption that $\dim\psi \geq 2$ is only necessary for the proof of (vii) $\Rightarrow$ (i), but it is crucial there: Let $\psi \cong \langle 1\rangle$ and $\varphi \cong [1, b]$

for some $b \in F \setminus \wp(F)$; then φ is anisotropic over F. Here we have $a^2X_1^2 \cdot 1 \in \langle D^*_{F(X_1)}(\varphi)\rangle$ for any $a \in F^*$, i.e., (vii) is fulfilled. But $F(\psi) = F$, so (i) does not hold.

In the proof of Theorem 5.14, to show (ii) $\Rightarrow$ (i), we used the chain of implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (i), in which we needed only the field $F(\boldsymbol{X})$. Moreover, we could as well use any field $F(X_1, \dots, X_n)$ with $n \geq \dim\psi$. Hence, we have actually proved the following:

Corollary 5.16. *Let φ, ψ be nondefective quadratic forms over F with $\dim\psi \geq 2$, and let $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ with $n \geq \dim\psi$. Then the following are equivalent:*
(i) *$\varphi_{F(\psi)}$ is isotropic,*
(ii) *$D^*_{F(\boldsymbol{Y})}(\psi)^2 \subseteq D^*_{F(\boldsymbol{Y})}(\varphi)^2$.*

Let φ, ψ be quadratic forms over F. We can ask when both $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic, i.e., when

$$\mathfrak{i}_{\mathrm{t}}(\varphi_{F(\psi)}) > 0 \quad \& \quad \mathfrak{i}_{\mathrm{t}}(\psi_{F(\varphi)}) > 0. \tag{❃}$$

Recall that if the forms φ and ψ are nondefective, then (❃) is an equivalent condition to φ and ψ being stably birationally equivalent (see Proposition 1.54). We apply Theorem 5.14 together with Corollary 5.16 to get a characterization of this phenomenon.

Corollary 5.17. *Let φ and ψ be nondefective quadratic forms of dimension at least two, and let $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ with $n \geq \max\{\dim\varphi, \dim\psi\}$. Moreover, set*

$$\mathcal{E} = \begin{cases} \{E \mid E/F \text{ an extension}\} & \text{if neither } \varphi \text{ nor } \psi \text{ is totally singular,} \\ \{E \mid E/F \text{ an extension s.t. } \mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E) = 0\} & \text{otherwise.} \end{cases}$$

Then the following are equivalent:
(a) $\varphi \overset{\mathrm{stb}}{\sim} \psi$,
(b) *$D^*_E(\psi)^2 = D^*_E(\varphi)^2$ for every $E \in \mathcal{E}$,*
(c) *$\langle D^*_E(\psi)^2\rangle = \langle D^*_E(\varphi)^2\rangle$ for every $E \in \mathcal{E}$,*
(d) *$D^*_{F(\boldsymbol{Y})}(\psi)^2 = D^*_{F(\boldsymbol{Y})}(\varphi)^2$.*

If $1 \in D^*_F(\varphi) \cap D^*_F(\psi)$, then the conditions above are also equivalent to
(e) *$\langle D^*_E(\psi)\rangle = \langle D^*_E(\varphi)\rangle$ for every $E \in \mathcal{E}$.*

Proof. The proof can basically be obtained through two-sided applications of Theorem 5.14 and Corollary 5.16, but note that our current $\mathcal{E}$ is slightly different from the one in that theorem.

First, the implication (a) $\Rightarrow$ (b) follows directly from the theorem, and (b) $\Rightarrow$ (c) can be done exactly as the proof of (ii) $\Rightarrow$ (iv) of the theorem. For (c) $\Rightarrow$ (a), note that to prove (iv) $\Rightarrow$ (i) in the theorem, we have actually shown (iv) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (i), and we only used the field $F(\boldsymbol{X})$, which is an element of (our current) $\mathcal{E}$; therefore, (c) $\Rightarrow$ (a) holds.

The implication (b) $\Rightarrow$ (d) is obvious, as $F(\boldsymbol{Y}) \in \mathcal{E}$ by Lemma 1.13. Furthermore, (d) $\Rightarrow$ (a) follows directly from the corollary.

Finally, (a) $\Leftrightarrow$ (e) follows from the theorem if we note that in the proof of (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i), only the existence of one $a \in D^*_F(\psi)$ with the required property was necessary. □

5.2.3 Sums and Pfister multiples of quadratic forms

The goal of this subsection is to extend the results from the previous one by comparing the isotropy of $\varphi_{F(\psi)}$ with the isotropy of $(\pi \otimes \varphi)_{F(\pi\otimes\psi)}$ for a bilinear Pfister form π.

Lemma 5.18. *Let φ_0, φ_1 and τ be quadratic forms over F with φ_0 and φ_1 nondefective, and let $\varphi \cong \varphi_0 \perp X\varphi_1$ over $F(X)$. Then $\mathfrak{i}_W(\tau_{F(X)(\varphi)}) = \mathfrak{i}_W(\tau)$ and $\mathfrak{i}_d(\tau_{F(X)(\varphi)}) = \mathfrak{i}_d(\tau)$. In particular, if τ is nondefective, then $\tau_{F(X)(\varphi)}$ is also nondefective.*

Proof. If φ_i is isotropic for some i, then necessarily $\mathbb{H} \subseteq \varphi$, and hence $F(X)(\varphi)/F$ is purely transcendental by Lemma 1.18 (because φ is nondefective by Example A.3). Then the claim follows by Lemma 1.13. Hence, we can suppose that φ_0 and φ_1 are anisotropic.

Assume that τ is anisotropic. Let $\dim\tau = d$, $\dim\varphi = n$, and denote $\boldsymbol{Y} = (Y_1, \dots, Y_n)$, an n-tuple of variables. Without loss of generality, suppose that $1 \in D^*_{F(X)}(\varphi)$. For a contradiction, assume that $\tau_{F(X)(\varphi)}$ is isotropic; then, by Theorem 5.14, $\varphi(\boldsymbol{Y}) \in D^*_{F(X,\boldsymbol{Y})}(\tau)^2$. Thus, after the usual multiplication by a common denominator, we get

$$h^2(X,\boldsymbol{Y})\varphi(\boldsymbol{Y}) = \prod_{i=1}^{2} \tau(\xi_{i1}(X,\boldsymbol{Y}), \dots, \xi_{id}(X,\boldsymbol{Y})) \tag{5.4}$$

for some $h \in F[X,\boldsymbol{Y}]$ and $\xi_{ij} \in F[X,\boldsymbol{Y}]$, $1 \le i \le 2$, $1 \le j \le d$. Clearly, $\deg_X(\tau(\xi_{i1}(X,\boldsymbol{Y}), \dots, \xi_{id}(X,\boldsymbol{Y})))$ is even for each i, and hence the degree in X of the polynomial on the right side of (5.4) is even. On the other hand, we know $\deg_X(\varphi(\boldsymbol{Y})) = 1$, so the degree in X of the polynomial on the left side of (5.4) is odd. That is absurd; therefore, $\tau_{F(X)(\varphi)}$ is anisotropic.

If τ is isotropic, then we know by the previous part of the proof that τ_{an} remains anisotropic over $F(X)(\varphi)$. The claim follows. $\square$

Note that the previous lemma is a consequence of Lemmas A.4 and 1.13.

Proposition 5.19. *Let φ_0, φ_1, ψ_0, ψ_1 be nondefective quadratic forms over F. Write $F' = F(X)$ and $F'' = F(\!(X)\!)$. Let $\varphi \cong \varphi_0 \perp X\varphi_1$, $\psi \cong \psi_0 \perp X\psi_1$ be quadratic forms over F', and set*

$$\mathcal{E} = \{E \mid E/F \text{ s.t. } \mathfrak{i}_d((\varphi_0)_E) = \mathfrak{i}_d((\varphi_1)_E) = \mathfrak{i}_d((\psi_0)_E) = \mathfrak{i}_d((\psi_1)_E) = 0\}.$$

(1) *The following are equivalent:*
  - (i) *$\varphi_{F'(\psi)}$ is isotropic,*
  - (ii) *$\varphi_{F''(\psi)}$ is isotropic,*
  - (iii) *$D^*_E(\psi_0)D^*_E(\psi_1) \subseteq D^*_E(\varphi_0)D^*_E(\varphi_1)$ for each $E \in \mathcal{E}$.*

(2) *Let the form $\varphi_{F'(\psi)}$ (or the form $\varphi_{F''(\psi)}$) be isotropic. If $\dim\psi_i \ge 2$ for an $i \in \{0,1\}$, then at least one of the forms φ_0 and φ_1 is isotropic over $F(\psi_i)$.*

Proof. We start by proving (1). The implication (i) $\Rightarrow$ (ii) is obvious.

For (ii) $\Rightarrow$ (iii), let $E \in \mathcal{E}$, and denote $E' = E(X)$ and $E'' = E(\!(X)\!)$. If $(\varphi_i)_E$ is isotropic for some i, then, since $\mathfrak{i}_d((\varphi_i)_E) = 0$ by the assumption, we have $\mathbb{H} \subseteq \varphi_i$; in that case, $D^*_E(\varphi_0)D^*_E(\varphi_1) = E^*$, and the claim follows trivially. Thus, assume that both φ_0 and φ_1 are anisotropic over E. Let $a \in D^*_E(\psi_0)D^*_E(\psi_1)$; then $(\psi_0 \perp a\psi_1)_E$ is isotropic. Obviously, we have $\psi_{E''(\sqrt{aX})} \cong (\psi_0 \perp a\psi_1)_{E''(\sqrt{aX})}$, and hence $\psi_{E''(\sqrt{aX})}$ is isotropic. As $\varphi_{F''(\psi)}$ is isotropic by the assumption, $\varphi_{E''(\psi)}$ must be isotropic as well. Furthermore, $\varphi_{E''}$ and $\psi_{E''}$ are nondefective (see Example A.3). It follows from Proposition 5.2 that $\varphi_{E''(\sqrt{aX})}$ is isotropic. Moreover, we have $\varphi_{E''(\sqrt{aX})} \cong (\varphi_0 \perp a\varphi_1)_{E''(\sqrt{aX})}$. Since $E''(\sqrt{aX})$ is a complete discrete valuation field with residue field E, the form $(\varphi_0 \perp a\varphi_1)_E$ is isotropic by Lemma A.1. Since both φ_0 and φ_1 are anisotropic over E, we get that $a \in D^*_E(\varphi_0)D^*_E(\varphi_1)$.

Now we prove (iii) $\Rightarrow$ (i): First, note that $F'(\psi) \in \mathcal{E}$ by Lemma 5.18. Since $\psi_{F'(\psi)} \cong (\psi_0 \perp X\psi_1)_{F'(\psi)}$ is isotropic, we get $X \in D^*_{F'(\psi)}(\psi_0)D^*_{F'(\psi)}(\psi_1)$. Therefore, $X \in D^*_{F'(\psi)}(\varphi_0)D^*_{F'(\psi)}(\varphi_1)$, which implies that the quadratic form $(\varphi_0 \perp X\varphi_1)_{F'(\psi)} \cong \varphi_{F'(\psi)}$ is isotropic.

To prove (2), note that it follows from the assumptions that $\varphi_{F''(\psi)}$ is isotropic in any case. Observe that if ψ_0 or ψ_1 is isotropic, then φ must be isotropic over F'' by Lemma 1.13, and in that case φ_0 or φ_1 is isotropic over F by Lemma A.2. But if φ_0 or φ_1 is isotropic over F, then the claim is trivial. Hence, suppose that φ_0, φ_1, ψ_0, ψ_1 are all anisotropic.

Assume that $i \in \{0, 1\}$ such that $\dim \psi_i \geq 2$; then $\psi_{F''(\psi_i)}$ is isotropic. As $\varphi_{F''}$, $\psi_{F''}$ and $(\psi_i)_{F''}$ are anisotropic, it follows by Proposition 5.2 that $\varphi_{F''(\psi_i)}$ is isotropic, too. Since $F''(\psi_i) \subseteq F(\psi_i)(\!(X)\!)$, it follows that φ is isotropic over $F(\psi_i)(\!(X)\!)$. By Lemma A.2, φ_0 or φ_1 must be isotropic over $F(\psi_i)$. $\square$

Corollary 5.20. *Let φ_0, φ_1, ψ_0, ψ_1, φ, ψ, $\mathcal{E}$ be as in Proposition 5.19. Then the following are equivalent:*
 (i) $\varphi \overset{\text{stb}}{\sim} \psi$,
 (ii) $D^*_E(\psi_0)D^*_E(\psi_1) = D^*_E(\varphi_0)D^*_E(\varphi_1)$ *for each* $E \in \mathcal{E}$.

Lemma 5.21. *Let φ, ψ be nondefective quadratic forms over F, $\dim\psi \geq 2$, and let*

$$\mathcal{E} = \{E \mid E/F \textit{ an extension s.t. } \mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E) = 0\}.$$

Then the following are equivalent:
 (i) $\varphi_{F(\psi)}$ *is isotropic,*
 (ii) $D_E(\psi)^2 \subseteq D_E(\varphi)^2$ *for each* $E \in \mathcal{E}$,
 (iii) $\varphi \perp X\varphi$ *is isotropic over* $F(X)(\psi \perp X\psi)$,
 (iv) $\varphi \perp X\varphi$ *is isotropic over* $F(\!(X)\!)(\psi \perp X\psi)$,
 (v) *for every* $n \geq 0$, $(\langle\!\langle X_1, \dots, X_n\rangle\!\rangle_b \otimes \varphi)_{F(X_1,\dots,X_n)(\langle\!\langle X_1,\dots,X_n\rangle\!\rangle_b\otimes\psi)}$ *is isotropic,*
 (vi) *for every* $n \geq 0$, $(\langle\!\langle X_1, \dots, X_n\rangle\!\rangle_b \otimes \varphi)_{F(\!(X_1)\!)\dots(\!(X_n)\!)(\langle\!\langle X_1,\dots,X_n\rangle\!\rangle_b\otimes\psi)}$ *is isotropic.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 5.14, and, as $F(X_1, \dots, X_{\dim\psi}) \in \mathcal{E}$ by Lemma 1.13, (ii) $\Rightarrow$ (i) follows from Corollary 5.16. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are a consequence of Proposition 5.19. Furthermore, (iii) is a special case of (v). For the other way around, note that (v) coincides with (i) for $n = 0$; for $n = 1$, (v) coincides with (iii); and for $n > 1$, (v) can be obtained by a repeated application of (i) $\Rightarrow$ (iii). Thus, (iii) $\Leftrightarrow$ (v). Analogously, we get (iv) $\Leftrightarrow$ (vi). $\square$

Theorem 5.22. *Let φ, ψ be quadratic forms over F, and let π be a bilinear Pfister form over F. If $\varphi_{F(\psi)}$ is isotropic, then $(\pi \otimes \varphi)_{F(\pi\otimes\psi)}$ is isotropic.*

Proof. Suppose without loss of generality that φ, ψ, π are anisotropic over F. Let $\pi \cong \langle\langle a_1, \dots, a_n \rangle\rangle_b$. If $n = 0$, i.e., $\pi \cong \langle 1 \rangle$, then there is nothing to prove; thus, assume $n \geq 1$.

Since $\varphi_{F(\psi)}$ is isotropic, we get by Lemma 5.21 that the quadratic form $\langle\langle X_1, \dots, X_n \rangle\rangle_b \otimes \varphi$ is isotropic over $F(X_1, \dots, X_n)(\langle\langle X_1, \dots, X_n \rangle\rangle_b \otimes \psi)$. Set $T = X_n + a_n$; then we have $F(X_1, \dots, X_n) = F(X_1, \dots, X_{n-1}, T)$, and it follows that $F(X_1, \dots, X_{n-1}, T) \subseteq F(X_1, \dots, X_{n-1})(\!(T)\!)$. Let us write $K = F(X_1, \dots, X_{n-1})(\!(T)\!)(\sqrt{a_n T})$; then $1 + \frac{T}{a_n} \in K^{*2}$, and so

$$X_n = a_n + T \equiv (a_n + T)\left(1 + \frac{T}{a_n}\right) = \frac{(a_n + T)^2}{a_n} \equiv \frac{1}{a_n} \equiv a_n \mod K^{*2}.$$

Therefore, over K, we have

$$(\langle\langle X_1, \dots, X_n \rangle\rangle_b \otimes \varphi)_K \cong (\langle\langle X_1, \dots, X_{n-1}, a_n \rangle\rangle_b \otimes \varphi)_K,$$
$$(\langle\langle X_1, \dots, X_n \rangle\rangle_b \otimes \psi)_K \cong (\langle\langle X_1, \dots, X_{n-1}, a_n \rangle\rangle_b \otimes \psi)_K.$$

Write $\varphi' \cong \langle\langle X_1, \dots, X_{n-1}, a_n \rangle\rangle_b \otimes \varphi$ and $\psi' \cong \langle\langle X_1, \dots, X_{n-1}, a_n \rangle\rangle_b \otimes \psi$; we have $\varphi_{K(\psi)} \cong \varphi'_{K(\psi')}$, and it follows that $\varphi'_{K(\psi')}$ is isotropic. Note that the forms φ', ψ' are defined over $F(X_1, \dots, X_{n-1})$, so in particular, we have $K(\psi') \subseteq F(X_1, \dots, X_{n-1})(\psi')(\!(T)\!)(\sqrt{a_n T})$. Hence, φ' is isotropic over $F(X_1, \dots, X_{n-1})(\psi')(\!(T)\!)(\sqrt{a_n T})$, which is a complete discrete valuation field with the residue field $F(X_1, \dots, X_n)(\psi')$. Therefore, by Lemma A.1, $\varphi'_{F(X_1,\dots,X_{n-1})(\psi')}$ is isotropic. We proceed by induction. $\square$

Corollary 5.23. *Let φ, ψ be quadratic forms over F. Then the following are equivalent:*

(i) *$\varphi_{F(\psi)}$ is isotropic,*
(ii) *for any field extension E/F, any $n \geq 0$ and any n-fold bilinear Pfister form π over E, the form $(\pi \otimes \varphi)_{E(\pi\otimes\psi)}$ is isotropic.*

Remark 5.24. One could hope that, given anisotropic quadratic forms φ, ψ and a bilinear Pfister form π over F such that $(\pi \otimes \varphi)_{F(\pi\otimes\psi)}$ is isotropic, then $\varphi_{F(\psi)}$ must be isotropic. But that is not true in general: Let $a, b \in F$ be 2-independent over F, and set $\varphi \cong \langle 1, a \rangle$, $\psi \cong \langle 1, a, b \rangle$, and $\pi \cong \langle\langle a \rangle\rangle_b$. Then $\varphi_{F(\psi)}$ is anisotropic by Separation Theorem 4.6, while $\pi \otimes \varphi$ is obviously isotropic over F already.

Another example shows that the claim cannot hold even under the assumption that $\pi \otimes \varphi$ is anisotropic over F: Let a, b be as above, and set $\varphi' \cong \langle 1, a \rangle$, $\psi' \cong \langle 1, ab \rangle$ and $\pi' \cong \langle\langle b \rangle\rangle_b$. Then $\pi' \otimes \varphi' \cong \langle 1, a, b, ab \rangle \cong \pi' \otimes \psi'$, so $(\pi' \otimes \varphi')_{F(\pi'\otimes\psi')}$ is obviously isotropic. On the other hand, if $\varphi'_{F(\psi')}$ were isotropic, then, by Proposition 3.1, it must hold $\varphi' \overset{\text{sim}}{\sim} \psi'$, which is not true.

As the final result, we combine Corollary 5.17, Lemma 5.21 and Corollary 5.23.

Corollary 5.25. *Let φ, ψ be nondefective quadratic forms over F of dimension at least two, and let*

$$\mathcal{E} = \{E \mid E/F \text{ an extension s.t. } \mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E) = 0\}.$$

Then the following are equivalent:

(i) $\varphi \overset{\mathrm{stb}}{\sim} \psi$,

(ii) $D_E(\psi)^2 = D_E(\varphi)^2$ *for each* $E \in \mathcal{E}$,

(iii) *for every* $n \geq 1$, $\langle\langle X_1, \ldots, X_n \rangle\rangle_b \otimes \varphi \overset{\mathrm{stb}}{\sim} \langle\langle X_1, \ldots, X_n \rangle\rangle_b \otimes \psi$ *over* $F(X_1, \ldots, X_n)$,

(iv) *for any field extension* E/F, *any* $n \geq 0$ *and any* n*-fold bilinear Pfister form* π *over* E, *we have* $(\pi \otimes \varphi)_E \overset{\mathrm{stb}}{\sim} (\pi \otimes \psi)_E$.

5.3 Relation between equivalences

Since totally singular quadratic forms are a special case of singular quadratic forms, all the "no"'s from Figure 3.1 still apply. Combining with Figure 1.1, we can decide on all but three relations; see Figure 5.1.

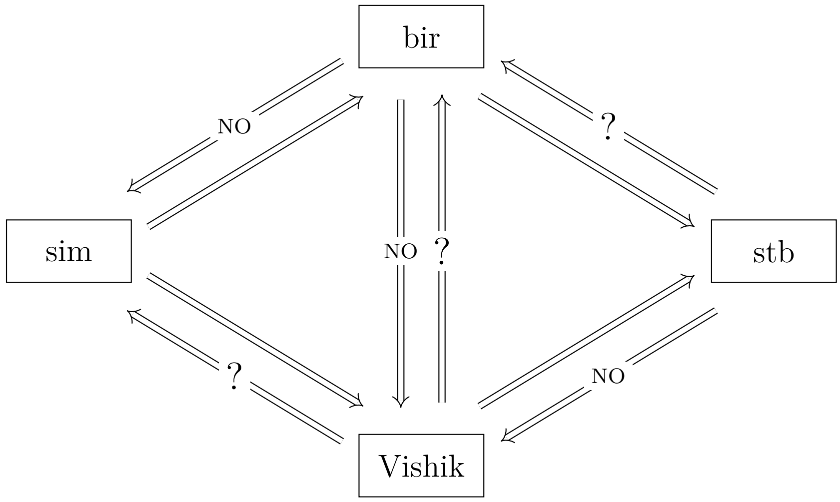

Figure 5.1: The equivalence relations for singular quadratic forms

In the rest of this section, we look at Question Q and provide some consequences of our previous results on singular quadratic forms.

Recall that the quasilinear part of a singular quadratic form is unique up to isometry. Therefore, for any quadratic form φ over F and any field extension E/F, we have $\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\mathrm{ql}(\varphi)_E)$. That proves the following lemma:

Lemma 5.26. *Let φ, ψ be singular quadratic forms over F. If $\varphi \overset{v}{\sim} \psi$, then $\mathrm{ql}(\varphi) \overset{v}{\sim} \mathrm{ql}(\psi)$.*

In Chapters 2 and 3, we gave a positive answer to Question Q for some families of totally singular quadratic forms. Thanks to the previous lemma, we can apply them to quasilinear parts of singular quadratic forms. In particular, invoking Corollary 2.57(ii) and Theorems 3.6 and 3.16, we get the following:

Corollary 5.27. *Let φ, ψ be singular quadratic forms over F, and assume that $\mathrm{ql}(\varphi)$ satisfies one of the following:*
- (i) *it is a quasi-Pfister form or a quasi-Pfister neighbor of codimension one,*
- (ii) *it is minimal over F,*
- (iii) *it is a special quasi-Pfister neighbor given by the triple (π, b, σ) with σ as in* (i) *or* (ii)*.*

If $\varphi \overset{v}{\sim} \psi$, then $\mathrm{ql}(\varphi) \overset{\mathrm{sim}}{\sim} \mathrm{ql}(\psi)$.

Since a "nonsingular part" of a semisingular quadratic form is not unique, it is more difficult to prove results about it. One possibility, how to extract a “nonsingular part” is to look at the quadratic form over an appropriate field extension. However, we do not know whether this “nonsingular part” remains anisotropic over such a field extension, or how to transfer the obtained similarity to the base field.

Proposition 5.28. *Let φ, ψ be semisingular quadratic forms over F and $\varphi \overset{v}{\sim} \psi$. Let $\{b_1, \dots, b_n\}$ be a 2-basis of the field $N_F(\mathrm{ql}(\varphi))$ over F. Assume that $n \geq 1$, and set $K = F(\sqrt{b_1}, \dots, \sqrt{b_n})$. Then $\varphi_K \cong c\psi_K$ for some $c \in F^*$.*

Proof. By Lemma 5.26, we have $\mathrm{ql}(\varphi) \overset{v}{\sim} \mathrm{ql}(\psi)$; thus, we get from Corollary 2.57 that $N_F(\mathrm{ql}(\varphi)) = N_F(\mathrm{ql}(\psi))$. By Proposition 2.10, it follows that $(\mathrm{ql}(\varphi)_K)_{\mathrm{an}} \cong \langle 1 \rangle_K \cong (\mathrm{ql}(\psi)_K)_{\mathrm{an}}$. Therefore, $(\varphi_K)_{\mathrm{an}}$ and $(\psi_K)_{\mathrm{an}}$ are nondegenerate quadratic forms of odd dimension. Since they are Vishik equivalent by Lemma 1.59, we get from Theorem 4.10 that $(\varphi_K)_{\mathrm{an}} \overset{\mathrm{sim}}{\sim} (\psi_K)_{\mathrm{an}}$. Let $c \in K^*$ be such that $(\varphi_K)_{\mathrm{an}} \cong c(\psi_K)_{\mathrm{an}}$; then we have in particular $(\mathrm{ql}(\varphi)_K)_{\mathrm{an}} \cong c(\mathrm{ql}(\psi)_K)_{\mathrm{an}}$, i.e., $\langle 1 \rangle_K \cong c\langle 1 \rangle_K$. It follows that $c \in K^{*2}$, where $K^{*2} = F^2(b_1, \dots, b_n)^* \subseteq F^*$; hence, $c \in F^*$. Finally, we have $\varphi_K \cong c\psi_K$ by Lemma 1.37. $\square$

Conclusion

The main goal of this thesis was to complete Figure 1.1 – to decide whether the missing implications are true or not, with a special focus on Question Q, that is, whether Vishik equivalent forms are always similar. Looking at Figures 2.1 (p-forms), 3.1 (totally singular forms), 4.1 (nonsingular forms) and 5.1 (singular forms), it may seem that I have not really succeeded. However, I believe that I provided some interesting results while trying. Moreover, in the beginning, I thought that giving a positive answer to Question Q for all quadratic forms should be possible; in particular, I expected the case of totally singular quadratic forms to be quite easy. After proving Theorem 3.6 and Proposition 3.13, I realized that the general case might be a bit more tricky than I thought.

Of course, it is in my nature as a mathematician not to be satisfied with my work until I find answers to all my questions. I also must say that I rather exhausted the available time than the ideas for this thesis. Therefore, I would like to conclude this thesis with a list of concrete tasks I would like to finish at some point in the future:

1. In [Izh98], there is a counterexample to Question Q for the case of characteristic other than two. Could that counterexample be somehow modified for nonsingular quadratic forms over fields of characteristic two?

2. In characteristic other than two, the implication "bir $\Rightarrow$ sim" is true for four-dimensional quadratic forms by [Wad75]. Is it also true for nonsingular quadratic forms in characteristic two? (Here I have already checked that the proof of Wadsworth cannot be straightforwardly rewritten into characteristic two.)

3. In the case of characteristic other than two, the counterexample for the implication "bir $\Rightarrow$ sim" is based on concrete constructions of rational maps ("transposition maps") between forms $\varphi \otimes \pi \perp a\sigma \perp b\tau$ and $\varphi \otimes \pi \perp b\sigma \perp a\tau$, where π is a Pfister form, φ is arbitrary, $\sigma, \tau \subseteq \pi$ and $a, b \in F^*$; see [Tot09, Lemma 5.1]. Could this result be translated into the characteristic two case?

A. Forms over discrete valuation fields

To the best of our knowledge, there is no comprehensive text about quadratic forms over discrete valuation fields of characteristic two, rather only isolated lemmas in different papers. For p-forms, there are probably no published results at all. Therefore, we devote this appendix at least to some basic results. These results, restricted to the case of quadratic forms, can also be found in [Zem22, Subsection 2.3].

Recall that we write *form* as a common term for quadratic forms and p-forms. We denote by A a discrete valuation ring, (t) its maximal ideal with a uniformizing element t, K the quotient field of A, v_t the valuation on K, and F the residue field $A/(t)$. If φ is a form defined over A, we denote by $\overline{\varphi}$ its image under the homomorphism $A \to F$. Moreover, if φ is a form defined over F, then we denote by φ_l some lifting to K, i.e., a form over A which satisfies $\overline{\varphi_l} \cong \varphi$ (it does not have to be unique). Then $(\varphi_l)_K$ is a form over K; by abuse of notation, we will denote this form φ_l, too.

We call a vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ with $\xi_i \in A$ *primitive* if $\xi_i \notin (t)$ for some i, i.e., if the image $\overline{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$ under $A \to F$ is nonzero. Note that for any vector $\boldsymbol{\xi}'$ over A, there exists an F-multiple of $\boldsymbol{\xi}'$ which is primitive: Let $\boldsymbol{\xi}' = (\xi_1', \dots, \xi_n')$ and set $k = \max\{v_t(\xi_i') \mid 1 \leq i \leq n\}$; then the vector $\boldsymbol{\xi} = t^{-k}\boldsymbol{\xi}'$ is primitive. In particular, if φ is an isotropic form over K, then it is also isotropic over A, and there exists a primitive vector $\boldsymbol{\xi}$ such that $\varphi(\boldsymbol{\xi}) = 0$.

Lemma A.1. *Let φ be a form over F. If φ_l is isotropic over K, then φ is isotropic over F.*

Proof. Let $\dim \varphi = n$. If φ_l is isotropic over K, then there exists a primitive vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ with all $\xi_i \in A$ such that $\varphi_l(\boldsymbol{\xi}) = 0$. Then $\overline{\boldsymbol{\xi}}$ is a nonzero vector in F for which $\overline{\varphi_l}(\overline{\boldsymbol{\xi}}) = 0$. Therefore, $\overline{\varphi_l} \cong \varphi$ is isotropic over F. $\square$

Lemma A.2. *Let $1 \leq k \leq p-1$ (where $p = \operatorname{char} F$), and $\varphi_0, \dots, \varphi_k$ be forms over A. If the forms $\overline{\varphi_0}, \dots, \overline{\varphi_k}$ are anisotropic over F, then $\varphi_0 \perp t\varphi_1 \perp \dots \perp t^k\varphi_k$ is anisotropic over K.*

Proof. For the case of quadratic forms, see [EKM08, Lemma 19.5]. In the case of p-forms, the proof is completely analogous: Assume that the p-form $\varphi_0 \perp t\varphi_1 \perp \dots \perp t^k\varphi_k$ is isotropic over K; then there exist some vectors $\boldsymbol{\xi}_0, \dots, \boldsymbol{\xi}_k$ over A, at least one of them primitive, such that

$$t^0\varphi_0(\boldsymbol{\xi}_0) \perp t\varphi_1(\boldsymbol{\xi}_1) \perp \dots \perp t^{k-1}\varphi_{k-1}(\boldsymbol{\xi}_{k-1}) \perp t^k\varphi_k(\boldsymbol{\xi}_k) = 0.$$

Reducing modulo t, we get $\overline{\varphi_0}(\overline{\boldsymbol{\xi}_0}) = 0$, and since $\overline{\varphi_0}$ is anisotropic over F, it follows that $\overline{\boldsymbol{\xi}_0}$ is the zero vector over F, i.e., $\boldsymbol{\xi}_0$ is divisible by t over A. Writing $\boldsymbol{\xi}_0 = t\boldsymbol{\xi}_0'$, we get

$$t^{p-1}\varphi_0(\boldsymbol{\xi}_0') \perp t^0\varphi_1(\boldsymbol{\xi}_1) \perp \dots \perp t^{k-2}\varphi_{k-1}(\boldsymbol{\xi}_{k-1}) \perp t^{k-1}\varphi_k(\boldsymbol{\xi}_k) = 0.$$

Reducing modulo t again, we obtain $\overline{\varphi_1}(\overline{\boldsymbol{\xi}_1}) = 0$. Again, $\boldsymbol{\xi}_1$ must be also divisible by t, and we write $\boldsymbol{\xi}_1 = t\boldsymbol{\xi}_1'$. Proceeding inductively, we finally get

$$t^{p-k}\varphi_0(\boldsymbol{\xi}_0') \perp t^{p-k+1}\varphi_1(\boldsymbol{\xi}_1') \perp \ldots \perp t^{p-1}\varphi_{k-1}(\boldsymbol{\xi}_{k-1}') \perp t^0\varphi_k(\boldsymbol{\xi}_k) = 0,$$

and reducing modulo t gives again $\overline{\varphi_k}(\overline{\boldsymbol{\xi}_k}) = 0$. Thus, all the vectors $\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_k$ are divisible by t, a contradiction. $\square$

Example A.3. For a field F, let $K = F(X)$ (resp. $K = F(\!(X)\!)$) be the rational function field (resp. the field of formal power series) in one variable. Then K is a discrete valuation field with respect to the X-adic valuation, the valuation ring is $\mathcal{O}_X = \left\{ \frac{f}{g} \;\middle|\; f, g \in F[X], X \nmid g \right\}$ (resp. $F[\![X]\!]$), and the residue field is F.

Let φ_0, φ_1 be anisotropic forms over F. By Lemma A.1, φ_0 and φ_1 are anisotropic over K. By Lemma A.2, the form $\varphi_0 \perp X\varphi_1$ is also anisotropic over K.

Analogously, by considering the anisotropic part, one can show that if φ_0, φ_1 are nondefective over F, then $\varphi_0 \perp X\varphi_1$ is nondefective over K.

Lemma A.4. *Let φ_0, φ_1 be nonzero forms over F, and let $\varphi \cong \varphi_0 \perp X\varphi_1$ over $F(X)$. Then $F(X)(\varphi)/F$ is a purely transcendental extension.*

Proof. By Lemma 1.17, we can assume φ_0 and φ_1 to be nondefective. If $\mathbb{H} \subseteq \varphi_i$ for some $i \in \{0, 1\}$, then $F(X)(\varphi)/F(X)$ is purely transcendental by Lemma 1.18, and the claim follows. Therefore, assume that both φ_0 and φ_1 are anisotropic. Then φ is anisotropic over $F(X)$ by Lemma A.2.

Let T, $\boldsymbol{Y} = (Y_1, \ldots, Y_{\dim\varphi_0 - 2})$ and $\boldsymbol{Z} = (Z_1, \ldots, Z_{\dim\varphi_1})$ be (tuples of) variables.

First, assume that $\dim \mathrm{ql}(\varphi_0) \geq 1$. Then there exists an $a \in F^*$ and a form φ_0' over F (possibly the zero form) such that $\varphi_0 \cong \langle a \rangle \perp \varphi_0'$. Without loss of generality, let $a = 1$; otherwise, consider the form $a^{-1}\varphi$. Then

$$F(X)(\varphi) \simeq \mathrm{Quot}\Big(F(X, T, \boldsymbol{Y}, \boldsymbol{Z})/(\varphi(T, 1, \boldsymbol{Y}, \boldsymbol{Z})\Big). \tag{A.1}$$

Over the quotient field, we have

$$T^p + \varphi_0'(1, \boldsymbol{Y}) + X\varphi_1(\boldsymbol{Z}) = 0.$$

Let

$$\alpha = \sqrt[p]{-\varphi_0'(1, \boldsymbol{Y}) - X\varphi_1(\boldsymbol{Z})};$$

then $F(X)(\varphi) \simeq F(X, \boldsymbol{Y}, \boldsymbol{Z})(\alpha)$. Moreover, as

$$X = -\frac{\alpha^p + \varphi_0'(1, \boldsymbol{Y})}{\varphi_1(\boldsymbol{Z})},$$

we get $X \in F(\boldsymbol{Y}, \boldsymbol{Z}, \alpha)$. It follows that $F(X)(\varphi) \simeq F(\boldsymbol{Y}, \boldsymbol{Z}, \alpha)$. Recall that, by Lemma 1.20, we have $\mathrm{trdeg}_{F(X)} F(X)(\varphi) = \dim\varphi - 2$; thus,

$$\mathrm{trdeg}_F F(\boldsymbol{Y}, \boldsymbol{Z}, \alpha) = \mathrm{trdeg}_F F(X)(\varphi) = \dim\varphi - 1.$$

Since

$$\mathrm{trdeg}_F F(\boldsymbol{Y}, \boldsymbol{Z}) = \dim\varphi_0 - 2 + \dim\varphi_1 = \dim\varphi - 2,$$

it follows that $\{\boldsymbol{Y}, \boldsymbol{Z}, \alpha\}$ must be a transcendence basis of $F(\boldsymbol{Y}, \boldsymbol{Z}, \alpha)$ over F, i.e., the field extension $F(\boldsymbol{Y}, \boldsymbol{Z}, \alpha)/F$ is purely transcendental.

Now, suppose that φ_0 is a nonsingular quadratic form. Multiplying the form φ by an element of F^* if necessary, we may assume that there exists an $a \in F^*$ and a nonsingular quadratic form φ_0' over F such that $\varphi_0 \cong [1, a] \perp \varphi_0'$. Then we can express the function field $F(X)(\varphi)$ as in (A.1), and over that quotient field, we have

$$T^2 + T + a + \varphi_0'(\boldsymbol{Y}) + X\varphi_1(\boldsymbol{Z}) = 0.$$

Denote

$$\beta = \wp^{-1}(a + \varphi_0'(\boldsymbol{Y}) + X\varphi_1(\boldsymbol{Z}));$$

then $F(X)(\varphi) \simeq F(X, \boldsymbol{Y}, \boldsymbol{Z})(\beta)$. We have

$$X = \frac{\beta^2 + \beta + \varphi_0'(\boldsymbol{Y})}{\varphi_1(\boldsymbol{Z})} \in F(\boldsymbol{Y}, \boldsymbol{Z}, \beta),$$

and hence $F(X)(\varphi) \simeq F(\boldsymbol{Y}, \boldsymbol{Z}, \beta)$. Comparing the transcendence degrees analogously as in the previous case, we get that $F(\boldsymbol{Y}, \boldsymbol{Z}, \beta)$ is purely transcendental over F. $\square$

Lemma A.5. *Let φ be an anisotropic form over F and $\dim \varphi = n$. Then $v_t(\varphi_l(\boldsymbol{\xi})) \equiv 0 \pmod p$ for any nonzero vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K^n$.*

Proof. Let $k = \min\{v_t(\xi_i) \mid 1 \leq i \leq n\}$, and set $\xi_i' = t^{-k}\xi_i$ for each i. Then all $\xi_i' \in A$, but at least one of them does not lie in (t). Write $\boldsymbol{\xi}' = (\xi_1', \ldots, \xi_n')$; then $\boldsymbol{\xi}' = t^{-k}\boldsymbol{\xi}$, and so $\varphi_l(\boldsymbol{\xi}') = t^{-pk}\varphi_l(\boldsymbol{\xi})$.

Suppose that $v_t(\varphi_l(\boldsymbol{\xi})) \not\equiv 0 \pmod p$; then also $v_t(\varphi_l(\boldsymbol{\xi}')) \not\equiv 0 \pmod p$, and in particular $v_t(\varphi_l(\boldsymbol{\xi}')) \geq 1$. Hence, $0 = \overline{\varphi_l(\boldsymbol{\xi}')} = \overline{\varphi_l}(\overline{\boldsymbol{\xi}'})$ in F. Since $\overline{\boldsymbol{\xi}'}$ is a nonzero vector over F, we get that $\overline{\varphi_l} \cong \varphi$ is isotropic over F. $\square$

Lemma A.6. *Let φ be an anisotropic form over F of dimension n and assume that $K = F(\!(X)\!)$. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K^n$ and $a = \varphi_K(\boldsymbol{\xi})$. Then $v_X(a) = p\min\{v_X(\xi_i) \mid 1 \leq i \leq n\}$.*

Proof. Write

$$\xi_i = \sum_{j=d_i}^{\infty} \xi_{ij}X^j \quad \text{and} \quad a = \sum_{j=d_a}^{\infty} a_jX^j$$

with $\xi_{ij}, a_j \in F$ and $\xi_{id_i}, a_{d_a} \neq 0$, i.e., $v_X(\xi_i) = d_i$ and $v_X(a) = d_a$. Moreover, set $D = \min\{d_i \mid 1 \leq i \leq n\}$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with

$$\alpha_i = \begin{cases} \xi_{id_i} & \text{if } d_i = D, \\ 0 & \text{otherwise.} \end{cases}$$

Note that there exists a lifting φ_l of φ to A such that $(\varphi_l)_K \cong \varphi_K$; therefore, as φ is anisotropic, φ_K is also anisotropic by Lemma A.1. Thus, comparing the terms of the lowest degree in $a = \varphi_K(\boldsymbol{\xi})$, we get that $\varphi(X^D\boldsymbol{\alpha}) = a_{d_a}X^{d_a}$. Hence, $d_a = pD$. $\square$

Bibliography

[Ahm97] Hamza Ahmad. The algebraic closure in function fields of quadratic forms in characteristic 2. *Bull. Aust. Math. Soc.*, 55(2):293–297, 1997.

[AO95] Hamza Ahmad and Jack Ohm. Function Fields of Pfister Neighbors. *J. Algebra*, 178:653–664, 1995.

[Arf41] Cahit Arf. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. (Teil I.). *J. Reine Angew. Math.*, 1941(183):148–167, 1941.

[Bae78] Ricardo Baeza. *Quadratic Forms Over Semilocal Rings.* Lecture Notes in Math. Springer-Verlag, 1978.

[BF95] Eva Bayer-Fluckiger. Formes quadratiques devenant isotropes sur une extension. *Enseign. Math. (2)*, 41:111–122, 1995.

[Bos20] Siegfried Bosch. *Algebra.* Springer-Lehrbuch. Springer, 2020.

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The Algebraic and Geometric Theory of Quadratic Forms.* Amer. Math. Soc. Colloq. Publ. American Mathematical Soc., 2008.

[GS06] Philippe Gille and Tamás Szamuely. *Central Simple Algebras and Galois Cohomology.* Cambridge studies in advanced mathematics. Cambridge University Press, 2006.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HL04] Detlev W. Hoffmann and Ahmed Laghribi. Quadratic Forms and Pfister Neighbors in Characteristic 2. *Trans. Amer. Math. Soc.*, 356(10):4019–4053, 02 2004.

[Hof04] Detlev W. Hoffmann. Diagonal forms of degree p in characteristic p. *Algebraic and Arithmetic Theory of Quadratic Forms. Contemp. Math.*, 344:135–183, 2004.

[Hof15] Detlev W. Hoffmann. Motivic equivalence and similarity of quadratic forms. *Doc. Math.*, Extra Volume Merkurjev: 265–275, 2015.

[Izh98] Oleg T. Izhboldin. Motivic equivalence of quadratic forms. *Doc. Math.*, 3:341–351, 1998.

[Lag02] Ahmed Laghribi. Certaines formes quadratiques de dimension au plus 6 et corps des fonctions en caractéristique 2. *Isr. J. Math.*, 129:317–361, 2002.

[Lag04] Ahmed Laghribi. On splitting of totally singular quadratic forms. *Rend. Circ. Mat. Palermo*, 53:325–336, 2004.

[Pic39] Günter Pickert. Neue Methoden in der Strukturtheorie der kommutativ-assoziativen Algebren. *Math. Ann.*, 116:217–280, 1939.

[Pic50] Günter Pickert. Inseparable Körpererweiterungen. *Math. Z.*, 52:81–136, 1950.

[Rou22] Sylvain Roussey. Some criteria for stably birational equivalence of quadratic forms. *preprint*, arXiv:2205.09819, 2022.

[Scu13] Stephen Scully. Rational maps between quasilinear hypersurfaces. *Compos. Math.*, 149(3):333–355, 2013.

[Scu16a] Stephen Scully. Hoffmann's conjecture for totally singular forms of prime degree. *Algebra Number Theory*, 10(5):1091 – 1132, 2016.

[Scu16b] Stephen Scully. On the splitting of quasilinear p-forms. *J. Reine Angew. Math.*, 2016(713):49 – 83, 2016.

[Tot08] Burt Totaro. Birational geometry of quadrics in characteristic 2. *J. Algebraic Geom.*, 17:577–597, 2008.

[Tot09] Burt Totaro. Birational geometry of quadrics. *Bull. Soc. Math. France*, 137(2):253–276, 2009.

[Vak17] Ravi Vakil. *Foundations of Algebraic Geometry*. 2017.

[Wad75] Adrian R. Wadsworth. Similarity of quadratic forms and isomorphism of their function fields. *Trans. Amer. Math. Soc.*, 208:352–358, 1975.

[Wei65] Morris Weisfeld. Purely inseparable extensions and higher derivations. *Trans. Amer. Math. Soc.*, 116:435–449, 1965.

[Zem22] Kristýna Zemková. Isotropy of quadratic forms over function fields in characteristic 2. *preprint*, 2022.