Inf – sup theory for the quasi – static Biot´s equations in poroelasticity

Christian Kreuzer, Pietro Zanotti

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Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

INF-SUP THEORY FOR THE QUASI-STATIC BIOT'S EQUATIONS IN POROELASTICITY

CHRISTIAN KREUZER AND PIETRO ZANOTTI

Abstract. We analyze the two-field formulation of the quasi-static Biot's equations by means of the inf-sup theory. For this purpose, we exploit an equivalent four-field formulation of the equations, introducing the so-called total pressure and total fluid content as independent variables. We establish existence, uniqueness and stability of the solution. Our stability estimate is two-sided and robust, meaning that the regularity established for the solution matches the regularity requirements for the data and the involved constants are independent of all material parameters. We prove also that additional regularity in space of the data implies, in some cases, corresponding additional regularity in space of the solution. These results are instrumental to the design and the analysis of discretizations enjoying accurate stability and error estimates.

1. Introduction

The analysis and the discretization of the quasi-static Biot's equations have been the subject of several studies in recent years. The equations arise in the theory of poroelasticity and model the flow of a fluid inside an elastic medium. They fit into the abstract framework

$$\mathcal{B}y = \ell \tag{1.1}$$

for a linear operator $\mathcal{B}$, with y and ℓ denoting the solution and the load, respectively; see (2.1) for the specific definitions.

In this paper, we develop some analytical results to be used in a follow-up paper [17] regarding the discretization of the Biot's equations. More precisely, we establish existence, uniqueness, two-sided stability and regularity in space of the solution by means of the inf-sup theory, i.e., via the so-called Banach-Nečas theorem, cf. [23]. This approach and various aspects of our results are new to our best knowledge. The main advantage stemming from the use of the inf-sup theory is that we obtain a two-sided stability estimate in the form

$$\|y\|_1 \eqsim \|\ell\|_{2,*}. \tag{1.2}$$

In other words, differently from previous works, we are able to prove that the operator $\mathcal{B}$ in (1.1) establishes an isomorphism between the space for the solution and the one for the load. Hence, the regularity established for the solution matches the regularity requirements for the data. In passing, we relax also the assumption made on the regularity of the load in previous references.

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A major challenge in our analysis consists in selecting the norms in (1.2), i.e., in characterizing the regularity of the solution and of the load in the equations. The difficulty hinges on the action of the differential operator $\mathcal{B}$ in (1.1), which couples the two components of the solution (the displacement of the elastic medium and the pressure of the fluid) in a highly nontrivial way, cf. (2.1a) below. We deal with this issue by considering an equivalent four-field formulation of the Biot's equations from [16], that is obtained by introducing the so-called total pressure and total fluid content as independent variables. The new formulation motivates the definition of the norm $\|\cdot\|_{2,*}$ as a product norm. The definition of the norm $\|\cdot\|_1$ is more involved, as it still couples the regularity of two components of the solution. This coupling of the regularity is indeed necessary.

Once the norms are selected, we establish the above-mentioned results by the inf-sup theory. This technique, differently from other ones, always implies a two-sided stability estimate, like (1.2), when it can be applied. For this purpose, one has to verify that the bilinear form induced by the operator $\mathcal{B}$ in (1.1) fulfills three properties: boundedness, inf-sup stability and nondegeneracy. The use of the inf-sup theory was made popular in numerical analysis by the pioneering works of Babuška [2] and Brezzi [9], who proposed the technique to obtain accurate stability and a priori error estimates for a linear equation and its discretization. Later on inf-sup stability has been noticed to be important also for a posteriori error estimation [30] and for the convergence of adaptive discretizations [14, 20]. Still, for some reason, the use of the inf-sup theory has been mostly confined to stationary equations and only recently there have been attempts to apply it to evolutionary ones, see [13, 28, 29].

This paper aims at further contributing to the development of the infs-sup theory for evolution equations, developing tools to be later used for the numerical analysis in [17] as well. We intend also to highlight the benefits of our technique in comparison with other ones. In addtion to the early contribution [1] (restricted to a rather specific case), we are aware of two other approaches to the analysis of the Biot's equations.

Ženíšek [33] used the so-called Faedo-Galerkin (Rothe's) method, in combination with a ingenious way of testing the equations, in order to infer stability. This technique is quite popular also in numerical analysis for the derivation of error estimates, see e.g. [26]. Other results in this flavor can be found in [7, 18, 24]. Another possible technique is the one of Showalter [27], who studied both strong and weak solutions by the theory of implicit evolution equations. Both approaches assume more regular data than our one and do not establish an equivalence like (1.2). Extensions to nonlinear problems in poromechanics are found, e.g., in [3, 4, 5, 10, 11].

Finally, motivated by the numerical analysis in [17], we are interested also in shift theorems, i.e., in determining if more regular data give rise to more regular solutions. We give a positive answer for the regularity in space, under a set of relatively restrictive assumptions, by using again the inf-sup theory. The relaxation of our assumptions and the regularity in time are more challenging tasks and we do not discuss them here. For instance, it is known that the time derivative of the solution can be singular at the initial time, cf. [22, section 2] and [27, section 3]. We are aware of only few other regularity results, see [7, 32].

Contribution. Summarizing, we propose a new approach to the analysis of the Biot's equation, which is used to establish both well-posedness and additional regularity in space. Differently from previous contributions, we make a weaker regularity assumption on the data and we make sure that it matches with the regularity guaranteed for the solution by establishing two-sided stability estimates like (1.2). All constants in our bounds are robust with respect to the material parameters in the equations. In particular, we treat at the same time the critical case of vanishing and nonvanishing specific storage coefficient, also in combination with general boundary conditions.

Organization. In section 2 we recall the Biot's equations and introduce the setting for their analysis. In section 3 we establish existence, uniqueness and two-sided stability of the solution. In section 4 we investigate the stability estimate more extensively. In section 5 we establish additional regularity in space.

Notation. We denote by $L^2(\mathbb{X})$, $H^1(\mathbb{X})$ and $C^0(\mathbb{X})$ the spaces of all L^2, H^1 and C^0 functions mapping the time interval $[0,T]$ into a Banach space $\mathbb{X}$, equipped with the norm $\|\cdot\|_{\mathbb{X}}$. The symbol $\langle\cdot,\cdot\rangle_{\mathbb{X}}$ indicates the duality of $\mathbb{X}$ and $\mathbb{X}^*$. For $\mathbb{X} = L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^d$, we use the abbreviations $\|\cdot\|_\Omega$ for the norm and $(\cdot,\cdot)_\Omega$ for the corresponding scalar product. We write $a \lesssim b$ and $a \eqsim b$ when there are constants $0 < \underline{c} \leq \overline{c}$ such that $a \leq \overline{c}\, b$ and $\underline{c}\, b \leq a \leq \overline{c}\, b$, respectively. As a rule of thumb, the hidden constants are independent of the material parameters involved in the equations. The dependence on other relevant quantities is addressed case by case.

2. Biot's equations and abstract formulation

In this section we introduce the Biot's equations and propose a framework for analyzing them by the inf-sup theory.

2.1. **Two-field formulation.**

Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain (i.e. a bounded, open and connected set) whose boundary can be locally represented as the graph of a Lipschitz-continuous function. The quasi-static Biot's equations in $\Omega \times (0,T)$, $T > 0$, read as

$$\begin{aligned} -\mathrm{div}(2\mu\nabla_S u + (\lambda \mathrm{div} u - \alpha p)I) &= f_u && \text{in } \Omega \times (0,T) \\ \partial_t(\alpha \mathrm{div} u + \sigma p) - \mathrm{div}(\kappa \nabla p) &= f_p && \text{in } \Omega \times (0,T). \end{aligned} \tag{2.1a}$$

The equations model the flow of a Newtonian fluid inside a linear elastic porous medium. The first one states the momentum balance, the second one states the mass balance. The unknowns are the displacement $u : \Omega \times (0,T) \to \mathbb{R}^d$ of the medium and the pressure $p : \Omega \times (0,T) \to \mathbb{R}$ of the fluid. The symbols ∇_S and I denote the symmetric part of the gradient and the $d \times d$ identity tensor respectively. Moreover, the following material parameters are involved: the Lamé constants $\mu, \lambda > 0$, the Biot-Willis constant $\alpha > 0$, the constrained specific storage coefficient $\sigma \geq 0$ and the hydraulic conductivity $\kappa > 0$. For simplicity, we assume that

$$\text{all parameters are constant in } \Omega \times (0,T).$$

Remark 2.1 (Parameters). Our subsequent analysis could be applied, up to minor modifications, under the assumption that the material parameters are bounded by positive constants from above and from below in $\overline{\Omega} \times [0,T]$. Of course, the ratio of the upper and the lower bound would affect the constants in our estimates.

In addition, we should require that α is constant in space, because we sometimes commute it with space derivatives. Similarly, κ should be constant in time. In general, the case with time-dependent κ is known to be delicate, see [4, section 2.3]. Finally, although we can treat also the case $\sigma = 0$, it is unclear to us whether this parameter can be zero and nonzero in different regions of $\Omega \times (0, T)$.

We complement the Biot's equations (2.1a) by the initial condition

$$(\alpha \mathrm{div} u + \sigma p)_{|t=0} = \ell_0 \quad \text{in} \quad \Omega \tag{2.1b}$$

and by the boundary conditions

$$\begin{aligned} u &= 0 && \text{on} \quad \Gamma_{u,E} \times (0,T) \\ (2\mu \nabla_S u + (\lambda \mathrm{div} u - \alpha p) I)\mathsf{n} &= g_u && \text{on} \quad \Gamma_{u,N} \times (0,T) \\ p &= 0 && \text{on} \quad \Gamma_{p,E} \times (0,T) \\ \kappa \nabla p \cdot \mathsf{n} &= g_p && \text{on} \quad \Gamma_{p,N} \times (0,T) \end{aligned} \tag{2.1c}$$

where $\Gamma_{u,E} \cup \Gamma_{u,N} = \partial\Omega = \Gamma_{p,E} \cup \Gamma_{p,N}$ and $\Gamma_{u,E} \cap \Gamma_{u,N} = \emptyset = \Gamma_{p,E} \cap \Gamma_{p,N}$. The letter n denotes the outward unit normal vector on $\partial\Omega$. The subscripts 'E' and 'N' indicate essential and natural boundary conditions, respectively. Inhomogeneous essential boundary conditions can be treated as usual, by modifying the data in the equations (2.1a).

Remark 2.2 (Initial condition). The time derivative acts in (2.1a) only on a combination of u and p, namely $\alpha \mathrm{div} u + \sigma p$, thus suggesting that only the initial value of such auxiliary variable should be prescribed as done, e.g., in [27, sections 3-4]. Still, different formulations are sometimes encountered in numerical analysis. Phillips and Wheeler [26] suggest to set $p(0)$ equal to the hydrostatic pressure. Then, they evaluate the first equation in (2.1a) at $t = 0$ and solve a linear elasticity problem for $u(0)$. Other authors, see e.g. [21], assume $\sigma = 0$ and set $\mathrm{div} u(0) = 0$ (that is equivalent to (2.1b) with $\ell_0 = 0$). Then, they evaluate the first equation in (2.1a) at $t = 0$ and solve a Stokes problem for $u(0)$ and $p(0)$. In other references, the values of $u(0)$ and $p(0)$ are just prescribed, see e.g. [18]. In this case, the compatibility of the prescribed initial values with the other data can be problematic and give rise to irregular behaviors, see [31].

Remark 2.3 (Boundary conditions). The boundary conditions considered in [27] are slightly more sophisticated than (2.1c) in that they allow for a coupling on the intersection of the nonessential parts of the boundary $\Gamma_{u,N} \cap \Gamma_{p,N}$, which requires additional regularity of the data and compatibility conditions on the initial values. In all other references we are aware of (from both the analytical and the numerical side), the boundary conditions (2.1c) (see [26, 33]) or simplified versions thereof are considered.

2.2. **Four-field formulation.** Our starting point for the analysis of the initial-boundary value problem (2.1) are the results established in [16]. In that reference, the stationary equations obtained after a time semi-discretization of (2.1a) are considered. The inf-sup theory developed in [16, section 2] reveals that, in the stationary case, it is possible to control two auxiliary variables, in addition to the displacement u and the pressure p, namely the total pressure

$$p_{\mathrm{tot}} := \alpha \mathrm{div} u - \alpha \mathcal{P}_{\mathbb{D}} p \tag{2.2a}$$

and the total fluid content

$$m := \mathcal{P}_{\overline{\mathbb{P}}}(\alpha \mathrm{div} u) + \sigma p. \tag{2.2b}$$

The operators $\mathcal{P}_{\mathbb{D}}$ and $\mathcal{P}_{\overline{\mathbb{P}}}$ are $L^2(\Omega)$-orthogonal projections, whose specific definition can be found in section 2.3 below. Treating these variables as independent unknowns leads to the following formulation of the Biot's equations (2.1a)

$$\begin{aligned} -\mathrm{div}(2\mu\nabla_S u + p_{\mathrm{tot}} I) &= f_u && \text{in } \Omega \times (0,T) \\ \lambda \mathrm{div} u - p_{\mathrm{tot}} - \alpha \mathcal{P}_{\mathbb{D}} p &= 0 && \text{in } \Omega \times (0,T) \\ \alpha \mathcal{P}_{\overline{\mathbb{P}}} \mathrm{div} u + \sigma p - m &= 0 && \text{in } \Omega \times (0,T) \\ \partial_t m - \mathrm{div}(\kappa \nabla p) &= f_p && \text{in } \Omega \times (0,T). \end{aligned} \tag{2.3}$$

Our use of the projections $\mathcal{P}_{\mathbb{D}}$ and $\mathcal{P}_{\overline{\mathbb{P}}}$ is motivated by our subsequent choice of the test functions for the equations. Similarly to [16], the analysis in the next sections equivalently applies to the original two-field formulation (2.1a) or to the above four-field formulation of the Biot's equations. We find working with the latter one more convenient, because the definition of the trial space is less involved, cf. Remark 2.7 below.

Remark 2.4 (Auxiliary variables). Introducing auxiliary variables as independent unknowns is a common practice in numerical analysis, which can foster the construction of discretizations with specific stability and/or approximation properties. To our best knowledge, the idea of introducing the total pressure p_{tot} is relatively recent and dates back to [19, 25]. In contrast, we are not aware of any reference paying specific attention to the approximation of the total fluid content m, although this is a relevant variable, as pointed out by the formulation (2.3), the initial condition (2.1b) and also previous theoretical results like those in [27, sections 3-4].

2.3. **Weak formulation.** We propose the weak formulation of the equations (2.3), with the initial and boundary conditions (2.1b) and (2.1c), that is the subject of our analysis in the next sections. Some minor differences are possible in the definition of the function spaces involved in such formulation, depending on the boundary conditions and on whether the constrained specific storage coefficient σ vanishes or not. Therefore, we introduce an abstract setting in order to treat all possible cases simultaneously.

The second-order elliptic operators involved in (2.3) and the boundary conditions (2.1c) suggest that u and p should be functions with values in the spaces

$$\mathbb{U} := \begin{cases} H^1(\Omega)^d/\mathrm{RM} & \text{if } \Gamma_{u,N} = \partial\Omega \\ H^1_{\Gamma_{u,E}}(\Omega)^d & \text{otherwise} \end{cases} \tag{2.4a}$$

and

$$\mathbb{P} := \begin{cases} H^1(\Omega) \cap L^2_0(\Omega) & \text{if } \Gamma_{p,N} = \partial\Omega \\ H^1_{\Gamma_{p,E}}(\Omega) \cap L^2_0(\Omega) & \text{if } \Gamma_{p,N} \neq \partial\Omega, \Gamma_{u,E} = \partial\Omega, \sigma = 0 \\ H^1_{\Gamma_{p,E}}(\Omega) & \text{otherwise.} \end{cases} \tag{2.4b}$$

The quotient in the definition of $\mathbb{U}$ is taken with respect to rigid body motions. The definition of $\mathbb{P}$ involves also the space of the functions in $L^2(\Omega)$ with zero integral mean in Ω, i.e., $L^2_0(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q = 0\}$.

Remark 2.5 (Pressure space). The second case in (2.4b) is nonstandard, because it prescribes both the essential condition on a portion of the boundary and the zero integral mean in Ω. In previous approaches, like e.g. [33], the boundedness of the pressure in the $L^2(H^1(\Omega))$-norm is established under a regularity assumption on the load. Thus, the $L^2(L^2(\Omega))$-norm can be controlled by a Poincaré inequality. In our approach, we assume less regularity of the load, therefore we do not control the $L^2(H^1(\Omega))$-norm of the pressure, cf. Proposition 4.4. Hence, it appears for $\Gamma_{u,E} = \partial\Omega$ and $\sigma = 0$, that we can only bound the $L^2(L^2(\Omega))$-norm of the pressure up to constant functions in space and thus the restriction to mean value free functions is needed. This informal guess is confirmed by Proposition 4.1.

Owing to (2.2), we regard p_{tot} and m as functions with values in the following closed subspaces of $L^2(\Omega)$:

$$\mathbb{D} := \operatorname{div}(\mathbb{U}) = \begin{cases} L^2_0(\Omega) & \text{if } \Gamma_{u,E} = \partial\Omega \\ L^2(\Omega) & \text{otherwise} \end{cases} \tag{2.5a}$$

and

$$\overline{\mathbb{P}} := \begin{cases} L^2_0(\Omega) & \text{if } \Gamma_{p,N} = \partial\Omega \text{ or } \Gamma_{u,E} = \partial\Omega, \sigma = 0 \\ L^2(\Omega) & \text{otherwise.} \end{cases} \tag{2.5b}$$

The closure in the definition of $\overline{\mathbb{P}}$ is taken with respect to the $L^2(\Omega)$-norm. We denote by $\mathcal{P}_{\mathbb{D}}$ and $\mathcal{P}_{\overline{\mathbb{P}}}$ the L^2-orthogonal projections onto $\mathbb{D}$ and $\overline{\mathbb{P}}$, respectively.

We equip the spaces $\mathbb{U}$ and $\mathbb{P}$ with $H^1(\Omega)$-like-norms scaled by $\sqrt{2\mu}$ and $\sqrt{\kappa}$, respectively, and the spaces $\mathbb{D}$ and $\overline{\mathbb{P}}$ with the $L^2(\Omega)$-norm. More precisely, we set

$$\|\cdot\|_{\mathbb{U}} := \|\sqrt{2\mu}\nabla_S \cdot\|_\Omega \qquad \text{and} \qquad \|\cdot\|_{\mathbb{P}} := \|\sqrt{\kappa}\nabla\cdot\|_\Omega. \tag{2.6}$$

Note that by Korn's and Poincarè-Friedrichs inequalities these are indeed norms on their respective spaces. By standard functional analysis arguments, we have that

$$\mathbb{P} \subseteq \overline{\mathbb{P}} \quad \text{is a dense compact subspace.}$$

We identify $\overline{\mathbb{P}}$ with $\overline{\mathbb{P}}^*$ via the $L^2(\Omega)$-scalar product. Hence, $\mathbb{P} \subseteq \overline{\mathbb{P}} \equiv \overline{\mathbb{P}}^* \subseteq \mathbb{P}^*$ is a Hilbert triplet and the duality $\langle\cdot,\cdot\rangle_{\mathbb{P}}$ coincides with $(\cdot,\cdot)_\Omega$ when both arguments are in $\overline{\mathbb{P}}$.

We introduce an abstract notation also for the (weak form of) the differential operators involved in the Biot's equations, so as to make many formulae more compact. We denote by $\mathcal{E} : \mathbb{U} \to \mathbb{U}^*$ and $\mathcal{L} : \mathbb{P} \to \mathbb{P}^*$ the elliptic operators acting on u and p, respectively, and by $\mathcal{D} : \mathbb{U} \to \mathbb{D}$ the divergence, namely

$$\mathcal{E} := -\operatorname{div}(2\mu\nabla_S) \qquad \mathcal{D} := \operatorname{div} \qquad \mathcal{L} := -\operatorname{div}(\kappa\nabla). \tag{2.7}$$

Note that we can regard the adjoint $\mathcal{D}^*$ of $\mathcal{D}$ as an operator acting on $\mathbb{D}$ upon identifying also this space with its dual $\mathbb{D}^*$ via the $L^2(\Omega)$-scalar product. Figure 1 summarizes the relation between the abstract spaces and operators.

By comparing (2.7) with the definition of the norms (2.6), we readily infer the identities

$$\|\cdot\|^2_{\mathbb{U}} = \langle\mathcal{E}\cdot,\cdot\rangle_{\mathbb{U}} \qquad \text{and} \qquad \|\cdot\|^2_{\mathbb{P}} = \langle\mathcal{L}\cdot,\cdot\rangle_{\mathbb{P}}. \tag{2.8}$$

Then, by duality, we obtain

$$\|\cdot\|^2_{\mathbb{U}^*} = \left\langle\cdot,\mathcal{E}^{-1}\cdot\right\rangle_{\mathbb{U}} \qquad \text{and} \qquad \|\cdot\|^2_{\mathbb{P}^*} = \left\langle\cdot,\mathcal{L}^{-1}\cdot\right\rangle_{\mathbb{P}}. \tag{2.9}$$

$$
\begin{tikzcd}
\mathbb{U}^* & \mathbb{U} \arrow[l, "\mathcal{E}"'] \arrow[d, "\mathcal{D}"'] & L^2(\Omega) \arrow[dl, "\mathcal{P}_{\mathbb{D}}"'] \arrow[dr, "\mathcal{P}_{\overline{\mathbb{P}}}"] & \mathbb{P} \arrow[r, "\mathcal{L}"] \arrow[d, "i"] & \mathbb{P}^* \\
\mathbb{D}^* \arrow[u, "\mathcal{D}^*"'] \arrow[r, equal] & \mathbb{D} & & \overline{\mathbb{P}} \arrow[r, equal] & \overline{\mathbb{P}}^* \arrow[u, "i^*"']
\end{tikzcd}
$$

FIGURE 1. Spaces and operators describing the regularity in space for the weak formulation (2.11) of the Biot's equations. The symbol '$\equiv$' denotes the identification via the $L^2(\Omega)$-scalar product and i is the embedding operator.

Furthermore, the surjectivity of the divergence and the open mapping theorem [12, Lemma 53.9] imply

$$
\frac{c}{\mu}\|\cdot\|_\Omega^2 \leq \|\mathcal{D}^*\cdot\|_{\mathbb{U}^*}^2 \leq \frac{C}{\mu}\|\cdot\|_\Omega^2 \qquad \text{in } \mathbb{D} \tag{2.10}
$$

with constants $0 < c \leq C$ depending only on Ω.

After this preparation, we are in position to state the abstract weak formulation of the equations (2.3), with (2.1b) and (2.1c), as follows

$$
\begin{aligned}
\mathcal{E}u + \mathcal{D}^* p_{\mathrm{tot}} &= \ell_u && \text{in } L^2(\mathbb{U}^*) \\
\lambda\mathcal{D}u - p_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}p &= 0 && \text{in } L^2(\mathbb{D}) \\
\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}u + \sigma p - m &= 0 && \text{in } L^2(\overline{\mathbb{P}}) \\
\partial_t m + \mathcal{L}p &= \ell_p && \text{in } L^2(\mathbb{P}^*) \\
m(0) &= \ell_0 && \text{in } \mathbb{P}^*.
\end{aligned} \tag{2.11}
$$

The loads ℓ_u and ℓ_p result from the data in the equations (2.3) and in the boundary conditions (2.1c). More precisely, they are obtained as

$$
\begin{aligned}
\ell_u(v) &= \int_0^T \Big(\langle f_u,\, v\rangle_{\mathbb{U}} + \langle g_u,\, v\rangle_{H^{1/2}(\Gamma_{u,N})} \Big) \\
\ell_p(n) &= \int_0^T \Big(\langle f_p,\, n\rangle_{\mathbb{P}} + \langle g_p,\, n\rangle_{H^{1/2}(\Gamma_{p,N})} \Big)
\end{aligned} \tag{2.12}
$$

for all $v \in L^2(\mathbb{U})$ and $n \in L^2(\mathbb{P})$.

We search for a solution of the equations (2.11) in the trial space $\overline{\mathbb{Y}}_1$, with

$$
\mathbb{Y}_1 := L^2(\mathbb{U}) \times L^2(\mathbb{D}) \times L^2(\mathbb{P}) \times \big(L^2(\overline{\mathbb{P}}) \cap H^1(\mathbb{P}^*) \big). \tag{2.13}
$$

The closure is taken with respect to the norm

$$
\begin{aligned}
&\|(\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m})\|_1^2 := \\
&\quad \int_0^T \left(\|\widetilde{u}\|_{\mathbb{U}}^2 + \frac{1}{\mu}\|\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 + \|\partial_t\widetilde{m} + \mathcal{L}\widetilde{p}\|_{\mathbb{P}^*}^2 \right) + \|\widetilde{m}(0)\|_{\mathbb{P}^*}^2 \\
&\quad + \int_0^T \left(\frac{1}{\mu+\lambda}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_\Omega^2 + \gamma\|\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}\|_\Omega^2 \right)
\end{aligned} \tag{2.14}
$$

where

$$
\gamma = \begin{cases} \min\left\{\dfrac{\mu+\lambda}{\alpha^2}, \dfrac{1}{\sigma}\right\} & \text{if } \sigma > 0 \text{ and } \overline{\mathbb{P}} \subseteq \mathbb{D} \\ \dfrac{\mu+\lambda}{\alpha^2} + \dfrac{1}{\sigma} & \text{if } \sigma > 0 \text{ and } \overline{\mathbb{P}} \not\subseteq \mathbb{D} \\ \dfrac{\mu+\lambda}{\alpha^2} & \text{if } \sigma = 0 \end{cases} \tag{2.15}
$$

We verify in Proposition 4.4 below that taking the closure in the definition of the trial space is indeed necessary. Hereafter, we use the superscript '$\sim$' to distinguish a general trial function in $\overline{\mathbb{Y}}_1$ from the solution of the Biot's equations.

Remark 2.6 (Definition of γ). The reason for distinguishing three different cases in the definition of γ is made clear in the proof of the inf-sup stability in Lemma 3.2 below. The first two cases are actually equivalent under the condition $\sigma \eqsim \frac{\alpha^2}{\mu+\lambda}$, which is justified in [19, section 2.2], arguing by physical principles. The condition characterizing the second case is equivalent to prescribing $\sigma > 0$, $\Gamma_{u,E} = \partial\Omega$ and $\Gamma_{p,N} \neq \partial\Omega$. This is, in a sense, the most critical case, because the space $\overline{\mathbb{P}}$ changes from $L^2(\Omega)$ to $L^2_0(\Omega)$ when passing from $\sigma > 0$ to $\sigma = 0$. The unboundedness of γ in the limit $\sigma \to 0$ reflects the lack of uniform control on the mean value of the pressure.

The corresponding test space, i.e., the space of all functions used to test the equations, is just the pre-dual of the product of the spaces on the rightmost column in (2.11), namely

$$
\mathbb{Y}_2 := L^2(\mathbb{U}) \times L^2(\mathbb{D}) \times L^2(\overline{\mathbb{P}}) \times L^2(\mathbb{P}) \times \mathbb{P}.
$$

We equip $\mathbb{Y}_2$ with the norm

$$
\begin{aligned} \|(v, q_{\mathrm{tot}}, q, n, n_0)\|_2^2 := &\int_0^T \left(\|v\|^2_{\mathbb{U}} + \|n\|^2_{\mathbb{P}}\right) + \|n_0\|^2_{\mathbb{P}} \\ &+ \int_0^T \left((\mu+\lambda)\|q_{\mathrm{tot}}\|^2_\Omega + \gamma^{-1}\|q\|^2_\Omega\right). \end{aligned} \tag{2.16}
$$

The dual of $\|\cdot\|_2$ is the norm $\|\cdot\|_{2,*}$ mentioned in (1.2) in the introduction, i.e. our measure of the regularity of the data. It is defined as

$$
\begin{aligned} \|(\widetilde{\ell}_u, \widetilde{\ell}_{p_{\mathrm{tot}}}, \widetilde{\ell}_m, \widetilde{\ell}_p, \widetilde{\ell}_0)\|^2_{2,*} := &\int_0^T \left(\|\widetilde{\ell}_u\|^2_{\mathbb{U}^*} + \|\widetilde{\ell}_p\|^2_{\mathbb{P}^*}\right) + \|\widetilde{\ell}_0\|^2_{\mathbb{P}^*} \\ &+ \int_0^T \left(\frac{1}{\mu+\lambda}\|\widetilde{\ell}_{p_{\mathrm{tot}}}\|^2_\Omega + \gamma\|\widetilde{\ell}_m\|^2_\Omega\right) \end{aligned} \tag{2.17}
$$

for $(\widetilde{\ell}_u, \widetilde{\ell}_{p_{\mathrm{tot}}}, \widetilde{\ell}_m, \widetilde{\ell}_p, \widetilde{\ell}_0) \in \mathbb{Y}_2^*$. In (2.11), the data components $\widetilde{\ell}_{p_{\mathrm{tot}}}$ and $\widetilde{\ell}_m$ of the second and third equation vanish, but they may be nonzero, e.g. in a discretization method.

Remark 2.7 (Two-field formulation). As mentioned before, we could equivalently consider the original two-field formulation (2.1a) of the Biot's equations in place of (2.3). In this case, the trial space to be used for the abstract weak formulation is the closure of

$$
\left\{(\widetilde{u}, \widetilde{p}) \in L^2(\mathbb{U}) \times L^2(\mathbb{P}) \mid (\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p}) \in H^1(\mathbb{P}^*)\right\}
$$

with respect to the norm

$$\int_0^T \left(\|\widetilde{u}\|_{\mathbb{U}}^2 + \frac{1}{\mu}\|\lambda\mathcal{D}\widetilde{u} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_{\Omega}^2 + \|\partial_t(\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p}) + \mathcal{L}\widetilde{p}\|_{\mathbb{P}^*}^2 \right) + \|(\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p})_{|t=0}\|_{\mathbb{P}^*}^2. \tag{2.18}$$

The corresponding test space is

$$L^2(\mathbb{U}) \times L^2(\mathbb{P}) \times \mathbb{P} \tag{2.19}$$

equipped with the norm

$$\int_0^T \left(\|v\|_{\mathbb{U}}^2 + \|n\|_{\mathbb{P}}^2\right) + \|n_0\|_{\mathbb{P}}^2. \tag{2.20}$$

The relatively involved definition of the trial space motivates our use of the four-field formulation, which appears to be especially convenient for the proof of the nondegeneracy in section 3.2.

Remark 2.8 (Norms). The definition of the trial and test norms deserves some justification. For this purpose, it is useful starting from the two-field formulation discussed in Remark 2.7. The test norm (2.20) is simply the product norm on the test space (2.19). Prescribing it corresponds to prescribing the regularity of the data. Then, the expression of the trial norm (2.18) is not at our disposal, because it is determined (up to norm equivalence) by requiring that the trial space is isomorphic to the dual of the test space through the weak formulation (2.11), cf. (1.2). When passing from the two- to the four-field formulation, two additional terms must be included in the definition of the norms, because inhomogeneous data $\ell_{p_{\text{tot}}}$ and ℓ_m are in principle allowed in the second and third equations of (2.11). The scaling of such terms is motivated by the inf-sup theory (see, in particular, the proof of Lemma 3.2) and it is not much relevant in a priori stability estimates since the solution of (2.11) annihilates those two terms, cf. Theorem 3.5. Still, this may play a role in the numerical analysis, if the solution is approximated by a function not satisfying the second and third equations exactly.

Remark 2.9 (Regularity of ℓ_u). Owing to the definition of the norm $\|\cdot\|_{2,*}$ in (2.17) we assume that the load in the first equation of the weak formulation (2.11) is such that $\ell_u \in L^2(\mathbb{U}^*)$, cf. Theorem 3.5 below. To our best knowledge, this relaxes the regularity assumption $\ell_u \in H^1(\mathbb{U}^*)$ made in all previous references, see e.g. [27, 33].

3. Well-posedness of the weak formulation

In this section we prove that the equations (2.11) are well-posed by means of the inf-sup theory. To this end, it is convenient rewriting the equations in the following variational form: find $y_1 \in \overline{\mathbb{Y}}_1$ such that

$$b(y_1, y_2) = \ell(y_2) \qquad \forall y_2 \in \mathbb{Y}_2. \tag{3.1}$$

The bilinear form $b : \overline{\mathbb{Y}}_1 \times \mathbb{Y}_2 \to \mathbb{R}$ and the load $\ell : \mathbb{Y}_2 \to \mathbb{R}$ are defined as

$$\begin{aligned} b(\widetilde{y}_1, y_2) = &\int_0^T \Big(\langle \mathcal{E}\widetilde{u} + \mathcal{D}^*\widetilde{p}_{\text{tot}},\, v\rangle_{\mathbb{U}} + \langle \partial_t\widetilde{m} + \mathcal{L}\widetilde{p},\, n\rangle_{\mathbb{P}} \Big) + \langle \widetilde{m}(0),\, n_0\rangle_{\mathbb{P}} \\ &+ \int_0^T \Big((\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\text{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p},\, q_{\text{tot}})_{\Omega} + \big(\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m},\, q\big)_{\Omega} \Big) \end{aligned} \tag{3.2}$$

and

$$\ell(y_2) = \ell_u(v) + \ell_p(n) + \langle \ell_0, n_0 \rangle_{\mathbb{P}} \tag{3.3}$$

for $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m}) \in \overline{\mathbb{Y}}_1$ and $y_2 = (v, q_{\mathrm{tot}}, q, n, n_0) \in \mathbb{Y}_2$. The projections $\mathcal{P}_{\mathbb{D}}$ and $\mathcal{P}_{\overline{\mathbb{P}}}$ could be neglected in the definition of b, because they are tested with functions from their respective range.

According to the so-called Banach-Nečas theorem, the well-posedness of the problem (3.1) is equivalent to the three properties verified in the next lemmas.

Lemma 3.1 (Boundedness)**.** *The bilinear form b in* (3.2) *is such that*

$$\sup_{y_2 \in \mathbb{Y}_2} \frac{b(\widetilde{y}_1, y_2)}{\|y_2\|_2} \lesssim \|\widetilde{y}_1\|_1 \tag{3.4}$$

for all $\widetilde{y}_1 \in \overline{\mathbb{Y}}_1$. The hidden constant depends only on the domain Ω.

Proof. The claimed bound readily follows from the Cauchy-Schwartz inequality, the identities in (2.8) and the upper bound in (2.10). □

Lemma 3.2 (Inf-sup stability)**.** *The bilinear form b in* (3.2) *is such that*

$$\begin{aligned} &\sup_{y_2 \in \mathbb{Y}_2} \frac{b(\widetilde{y}_1, y_2)}{\|y_2\|_2} \gtrsim \\ &\qquad (1+T)^{-\frac{1}{2}} \left(\|\widetilde{y}_1\|_1^2 + \|\widetilde{m}\|^2_{L^\infty(\mathbb{P}^*)} + \int_0^T \left(\lambda \|\mathcal{D}\widetilde{u}\|^2_\Omega + \sigma \|\widetilde{p}\|^2_\Omega \right) \right)^{\frac{1}{2}} \end{aligned} \tag{3.5}$$

for all $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{tot}, \widetilde{p}, \widetilde{m}) \in \overline{\mathbb{Y}}_1$. The hidden constant depends only on the domain Ω.

Proof. See section 3.1. □

The combination of the Lemmas 3.1 and 3.2 points out that the trial norm $\|\cdot\|_1$ is actually equivalent to the (stronger) norm on the right-hand side of (3.5). We deduce a first relevant consequence of this observation in the following remark.

Remark 3.3 (Continuity of the total fluid content)*.* The inclusion $H^1(\mathbb{P}^*) \subseteq C^0(\mathbb{P}^*)$ reveals that the linear operator

$$\mathbb{Y}_1 \ni \widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m}) \mapsto \widetilde{m} \in C^0(\mathbb{P}^*)$$

is well-defined. The combination of Lemmas 3.1 and 3.2 further implies that this operator is bounded with respect to the norm $\|\cdot\|_1$. Therefore we can extend it from $\mathbb{Y}_1$ to $\overline{\mathbb{Y}}_1$ by density. This observation is important to guarantee that the initial condition in (2.11) is meaningful.

Lemma 3.4 (Nondegeneracy)**.** *Let the bilinear form b be as in* (3.2) *and assume*

$$b(\widetilde{y}_1, y_2) = 0 \tag{3.6}$$

for all $\widetilde{y}_1 \in \overline{\mathbb{Y}}_1$ and for some $y_2 \in \mathbb{Y}_2$. Then, we have $y_2 = 0$.

Proof. See section 3.2. □

The combination of the above lemmas implies our main result, stating existence, uniqueness and two-sided stability of the solution of the equations (2.11), in the sense of (1.2). In particular, the latter property ensures that the regularity established for the solution matches the regularity requirements for the data.

Theorem 3.5 (Well-posedness)**.** *For all* $(\ell_u, \ell_p, \ell_0) \in L^2(\mathbb{U}^*) \times L^2(\mathbb{P}^*) \times \mathbb{P}^*$*, the equations* (2.11) *have a unique solution* $y_1 = (u, p_{tot}, p, m) \in \overline{\mathbb{Y}}_1$*, which fulfills the two-sided stability bound*

$$\begin{aligned}(3.7)\quad &\int_0^T \left(\|u\|_{\mathbb{U}}^2 + \lambda\|\mathcal{D}u\|_\Omega^2 + \frac{1}{\mu}\|p_{tot}\|_\Omega^2 + \sigma\|p\|_\Omega^2 + \|\partial_t m + \mathcal{L}p\|_{\mathbb{P}^*}^2 \right) + \|m\|_{L^\infty(\mathbb{P}^*)}^2 \\ &\qquad \eqsim \int_0^T \left(\|\ell_u\|_{\mathbb{U}^*}^2 + \|\ell_p\|_{\mathbb{P}^*}^2 \right) + \|\ell_0\|_{\mathbb{P}^*}^2.\end{aligned}$$

The hidden constants depend only on the domain Ω *and the final time* T*. Moreover, we have* $m \in C^0(\mathbb{P}^*)$*.*

Proof. The existence and the uniqueness of the solution result from the combination of the Banach-Nečas theorem [12, Theorem 25.9] with the Lemmas 3.1, 3.2 and 3.4. The same argument [12, Remark 25.12] yields

$$\|(u, p_{\mathrm{tot}}, p, m)\|_1 \eqsim \|(\ell_u, 0, 0, \ell_p, \ell_0)\|_{2,*}$$

where the hidden constants depend only on the constants in (3.4) and (3.5). Then, the claimed two-sided stability bound follows by the definitions (2.14) and (2.17) of the norms $\|\cdot\|_1$ and $\|\cdot\|_{2,*}$ and the equivalence of $\|\cdot\|_1$ with the stronger norm on the right-hand side of (3.5) and by recalling that $(u, p_{\mathrm{tot}}, p, m)$ solves (2.11). Finally, the continuity in time of the component m of the solution is guaranteed by Remark 3.3. $\square$

Before examining the proof of Lemmas 3.2-3.4, it is worth comparing Theorem 3.5 with related results in the literature.

Remark 3.6 (Comparison with [16]). The stability bound in Theorem 3.5 is consistent with the one established in [16, section 2] for the stationary equations obtained from (2.1a) by semi-discretization in time with the backward Euler scheme. Indeed, in that context, the stability estimate involves the trial norm

$$\tau(\|u\|_{\mathbb{U}}^2 + \lambda\|\mathcal{D}u\|_\Omega^2 + \frac{1}{\mu}\|p_{\mathrm{tot}}\|_\Omega^2 + \sigma\|p\|_\Omega^2 + \tau\|p\|_{\mathbb{P}}^2) + \|m\|_{\mathbb{P}^*}^2$$

where τ denotes the time step. By interpreting the multiplication by τ as a kind of time integration, we see that each term here has a corresponding one on the left-hand side of (3.7). The only exception is the $\mathbb{P}$-norm of p, which is multiplied by an additional factor τ. Hence, we cannot expect a uniform control on the $L^2(\mathbb{P})$-norm of p in terms of the left-hand side of (3.7) in the limit $\tau \to 0$, i.e., when passing from the time-semidiscretization to the original equations. In Proposition 4.4 below we verify our expectation by means of a counterexample.

Remark 3.7 (Comparison with [33]). The technique introduced by Ženíšek [33] consists in applying the so-called Feado-Galerkin scheme and in establishing a stability estimate that serves to infer the existence and the uniqueness of the solution by testing the equations in (2.1a) with $(\partial_t u, p)$. This ultimately provides a one-sided stability bound involving the following norm of the solution

$$\|u\|_{L^\infty(\mathbb{U})}^2 + \lambda\|\mathcal{D}u\|_{L^\infty(L^2(\Omega))}^2 + \sigma\|p\|_{L^\infty(L^2(\Omega))}^2 + \int_0^T \|p\|_{\mathbb{P}}^2;$$

see [33, Theorem 1] (the estimate is only established in the proof). The scaling with respect to the parameters is the same as in (3.7). Each term is measured in a stronger norm because of the higher regularity assumption $\ell_u \in H^1(\mathbb{U}^*)$, cf. Remark 2.9. Still, in contrast to (3.7), the bound in [33] (and similar ones) cannot be reversed, meaning that the regularity established for the solution does not match the regularity requirement for the data.

Remark 3.8 (Comparison with [18]). Li and Zikatanov [18] assume even more regular data than in [33], namely $\ell_u \in H^1(\mathbb{U}^*)$ and $\ell_p \in L^2(L^2(\Omega))$, as well as smooth and compatible initial data, cf. Remark 2.2. Then, by improving on the original technique of Ženíšek, they are able to establish the regularity

$$u \in H^1(\mathbb{U}) \qquad \text{and} \qquad p \in H^1(\mathbb{P}) \cap L^2(\mathbb{L})$$

where $\mathbb{L} = \{\widetilde{p} \in \mathbb{P} \mid \mathcal{L}\widetilde{p} \in \overline{\mathbb{P}}\}$. They establish also a corresponding stability estimate, where the scaling with respect to the material parameters is similar as in the above-mentioned results. Remarkably, such estimate is accurate, in the sense that it fulfills an equivalence like (1.2).

3.1. **Inf-sup stability.** This section is devoted to the proof of Lemma 3.2. For this purpose, let $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m}) \in \mathbb{Y}_1$. We shall construct a test function $y_2 \in \mathbb{Y}_2$ such that

$$b(\widetilde{y}_1, y_2) \gtrsim \|\widetilde{y}_1\|_1^2 + \|\widetilde{m}\|^2_{L^\infty(\mathbb{P}^*)} + \int_0^T \left(\lambda \|\mathcal{D}\widetilde{u}\|^2_\Omega + \sigma \|\widetilde{p}\|^2_\Omega\right) \tag{3.8}$$

as well as

$$\|y_2\|_2^2 \lesssim (1+T)\left(\|\widetilde{y}_1\|_1^2 + \|\widetilde{m}\|^2_{L^\infty(\mathbb{P}^*)}\right). \tag{3.9}$$

The combination of these inequalities with a density argument implies (3.5). Taking $\widetilde{y}_1$ in $\mathbb{Y}_1$ (and not directly in $\overline{\mathbb{Y}}_1$) simplifies the derivation of (3.8) and, in particular, the lower bound of the term $\mathfrak{I}_2$ below.

Let $s \in [0,T]$ be a value to be specified later. We denote by $\chi_s : [0,T] \to \mathbb{R}$ the indicator function on $[0,s]$. In other words, $\chi_s(t)$ equals 1 for $t \le s$ and it vanishes for $t > s$. We consider the test function

$$\begin{aligned} y_{2,s} := \Big(\ (\widetilde{u} + \mathcal{E}^{-1}\mathcal{D}^*\widetilde{p}_{\mathrm{tot}})\chi_s,\ & \frac{4\max\{1,C\}}{\mu+\lambda}(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p})\chi_s, \\ & \frac{4\gamma}{\min\{1,c\}}(\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m})\chi_s, \\ & \qquad \mathcal{L}^{-1}(2\widetilde{m} + \partial_t\widetilde{m} + \mathcal{L}\widetilde{p})\chi_s,\ 2\mathcal{L}^{-1}\widetilde{m}(0)\ \Big) \end{aligned}$$

with the constants c and C from (2.10) and γ as in (2.15).

Remark 3.9 (Motivating the test function). The definition (3.2) of the form b consists of five summands. Roughly speaking, the test function $y_{2,s}$ is designed so as to obtain a positive contribution (i.e., a squared norm) from each summand. The second and the third components of $y_{2,s}$ are additionally scaled by 'sufficiently large' constants, so as to compensate negative contributions arising from the analysis of the other terms. Furthermore, the function $\mathcal{L}^{-1}\widetilde{m}$ in the fourth component of $y_{2,s}$ is meant to control the point values of $\widetilde{m}$. The choice of the fourth and the fifth components is in line with the one that is typically made in the derivation of the inf-sup stability for scalar parabolic equations, see e.g. the proof of [13, Lemma 71.2].

First of all, it is worth noticing that we indeed have $y_{2,s} \in \mathbb{Y}_2$. This can be verified by recalling the setting in section 2.3. In particular, we mention that the operator $\mathcal{E}^{-1}\mathcal{D}^*$ maps $\mathbb{D}$ into $\mathbb{U}$ and that the inclusion $\widetilde{m}(0) \in \mathbb{P}^*$ follows from the inclusion $\widetilde{m} \in H^1(\mathbb{P}^*) \subseteq C^0(\mathbb{P}^*)$. Then, the multiplication of all components (except the last one) by χ_s is admissible, because the test space $\mathbb{Y}_2$ involves only L^2 regularity in time.

In order to establish (3.8) for a suitable test function y_2, we investigate the action of the form b onto the pair $(\widetilde{y}_1, y_{2,s})$. By recalling the definition (3.2) of b, we infer

$$
\begin{aligned}
b(\widetilde{y}_1, y_{2,s}) = &\int_0^s \left\langle \mathcal{E}\widetilde{u} + \mathcal{D}^*\widetilde{p}_{\mathrm{tot}},\, \widetilde{u} + \mathcal{E}^{-1}\mathcal{D}^*\widetilde{p}_{\mathrm{tot}} \right\rangle_{\mathbb{U}} && (=: \mathfrak{I}_1)\\
&+ 2\int_0^s \left\langle \partial_t\widetilde{m} + \mathcal{L}\widetilde{p},\, \mathcal{L}^{-1}\widetilde{m} \right\rangle_{\mathbb{P}} && (=: \mathfrak{I}_2)\\
&+ \int_0^s \left\langle \partial_t\widetilde{m} + \mathcal{L}\widetilde{p},\, \mathcal{L}^{-1}(\partial_t\widetilde{m} + \mathcal{L}\widetilde{p}) \right\rangle_{\mathbb{P}} && (=: \mathfrak{I}_3)\\
&+ 2\left\langle \widetilde{m}(0),\, \mathcal{L}^{-1}\widetilde{m}(0) \right\rangle_{\mathbb{P}} && (=: \mathfrak{I}_4)\\
&+ \frac{4\max\{1,C\}}{\mu+\lambda}\int_0^s \|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_\Omega^2\\
&+ \frac{4\gamma}{\min\{1,c\}}\int_0^s \|\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}\|_\Omega^2.
\end{aligned}
\tag{3.10}
$$

We further investigate the four terms $\mathfrak{I}_1, \dots, \mathfrak{I}_4$ on the right-hand side. According to the second identity in (2.9), we rewrite the third and the fourth terms as

$$
\mathfrak{I}_3 = \int_0^s \|\partial_t\widetilde{m} + \mathcal{L}\widetilde{p}\|_{\mathbb{P}^*}^2 \qquad \text{and} \qquad \mathfrak{I}_4 = 2\|\widetilde{m}(0)\|_{\mathbb{P}^*}^2.
$$

The first parts of (2.8) and (2.9) imply

$$
\mathfrak{I}_1 = \int_0^s \Big(\|\widetilde{u}\|_{\mathbb{U}}^2 + 2\left\langle \mathcal{D}^*\widetilde{p}_{\mathrm{tot}},\, \widetilde{u} \right\rangle_{\mathbb{U}} + \|\mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_{\mathbb{U}^*}^2 \Big).
$$

Regarding the second summand on the right-hand side, we have

$$
\begin{aligned}
\left\langle \mathcal{D}^*\widetilde{p}_{\mathrm{tot}},\, \widetilde{u} \right\rangle_{\mathbb{U}} &= (\mathcal{D}\widetilde{u},\, \widetilde{p}_{\mathrm{tot}} - \lambda\mathcal{D}\widetilde{u} + \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p})_\Omega + \lambda\|\mathcal{D}\widetilde{u}\|_\Omega^2 - \alpha(\mathcal{D}\widetilde{u}, \mathcal{P}_{\mathbb{D}}\widetilde{p})_\Omega\\
&\geq \frac{3\lambda}{4}\|\mathcal{D}\widetilde{u}\|_\Omega^2 - \frac{1}{4}\|\widetilde{u}\|_{\mathbb{U}}^2 - \frac{\max\{1,C\}}{\mu+\lambda}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_\Omega^2 - \alpha(\mathcal{D}\widetilde{u}, \widetilde{p})_\Omega
\end{aligned}
$$

according to the upper bound in (2.10) and the Young's inequality. Notice that we could neglect the projection $\mathcal{P}_{\mathbb{D}}$ in the last summand, because of the inclusion $\mathcal{D}\widetilde{u} \in \mathbb{D}$. We insert this lower bound into the previous identity. By invoking also the lower bound in (2.10), we obtain

$$
\begin{aligned}
\mathfrak{I}_1 \geq \int_0^s \Bigg(&\frac{1}{2}\|\widetilde{u}\|_{\mathbb{U}}^2 + \frac{3\lambda}{2}\|\mathcal{D}\widetilde{u}\|_\Omega^2 + \frac{c}{\mu}\|\widetilde{p}_{\mathrm{tot}}\|_\Omega^2\\
&- \frac{2\max\{1,C\}}{\mu+\lambda}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_\Omega^2 - 2\alpha(\mathcal{D}\widetilde{u}, \widetilde{p})_\Omega \Bigg).
\end{aligned}
$$

The last term to be considered is

$$
\mathfrak{I}_2 = 2\int_0^s \Big(\left\langle \partial_t\widetilde{m},\, \mathcal{L}^{-1}\widetilde{m} \right\rangle_{\mathbb{P}} + (\widetilde{m},\, \widetilde{p})_\Omega \Big).
$$

The second part of (2.9) and an integration by parts [13, Lemma 64.40] reveal

$$\int_0^s \left\langle \partial_t \widetilde{m},\, \mathcal{L}^{-1}\widetilde{m}\right\rangle_{\mathbb{P}^*} = \frac{1}{2}\|\widetilde{m}(s)\|^2_{\mathbb{P}^*} - \frac{1}{2}\|\widetilde{m}(0)\|^2_{\mathbb{P}^*}.$$

To investigate the other summand in $\mathfrak{I}_2$, we set $\widetilde{h} := \widetilde{m} - \alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} - \sigma\widetilde{p} \in L^2(\overline{\mathbb{P}})$ for shortness. For $\sigma > 0$, elementary manipulations and Young's inequality reveal

$$\begin{aligned}(\widetilde{m},\, \widetilde{p})_\Omega &= (\widetilde{h},\, \widetilde{p})_\Omega + \alpha(\mathcal{D}\widetilde{u},\, \widetilde{p})_\Omega + \sigma\|\widetilde{p}\|^2_\Omega \\ &\geq \alpha(\mathcal{D}\widetilde{u},\, \widetilde{p})_\Omega + \frac{\sigma}{2}\|\widetilde{p}\|^2_\Omega - \frac{1}{2\sigma}\|\widetilde{h}\|^2_\Omega.\end{aligned}$$

Alternatively, for general $\sigma \geq 0$, it holds that

$$\begin{aligned}(\widetilde{m},\, \widetilde{p})_\Omega &= (\widetilde{h},\, \widetilde{p})_\Omega + \alpha(\mathcal{D}\widetilde{u},\, \widetilde{p})_\Omega + \sigma\|\widetilde{p}\|^2_\Omega \\ &= \frac{1}{\alpha}(\widetilde{h},\, \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p} - \lambda\mathcal{D}\widetilde{u} + \widetilde{p}_{\mathrm{tot}})_\Omega + \frac{\lambda}{\alpha}(\widetilde{h},\, \widetilde{\mathcal{D}u})_\Omega - \frac{1}{\alpha}(\widetilde{h},\, \widetilde{p}_{\mathrm{tot}})_\Omega \\ &\quad + (\widetilde{h},\, \widetilde{p} - \mathcal{P}_{\mathbb{D}}\widetilde{p})_\Omega + \alpha(\mathcal{D}\widetilde{u},\, \widetilde{p})_\Omega + \sigma\|\widetilde{p}\|^2_\Omega.\end{aligned}$$

The term $H := (\widetilde{h},\, \widetilde{p} - \mathcal{P}_{\mathbb{D}}\widetilde{p})_\Omega$ deserves some specialized comments. For $\overline{\mathbb{P}} \subseteq \mathbb{D}$, we have $\mathcal{P}_{\mathbb{D}}\widetilde{p} = \widetilde{p}$, hence H vanishes. This covers, in particular, the case $\sigma = 0$, cf. (2.5). When the above inclusion fails, we have $\sigma > 0$. Then, we apply Young's inequality to obtain $H \leq \|\widetilde{h}\|^2_\Omega/(2\sigma) + \sigma\|\widetilde{p}\|^2_\Omega/2$. We combine this observation with (2.10) and other applications of Young's inequality. It results

$$\begin{aligned}(\widetilde{m},\, \widetilde{p})_\Omega \geq &-\frac{\max\{1, C\}}{2(\mu+\lambda)}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|^2_\Omega - \frac{c}{4\mu}\|\widetilde{p}_{\mathrm{tot}}\|^2_\Omega - \frac{\lambda}{2}\|\mathcal{D}\widetilde{u}\|^2_\Omega \\ &+ \frac{\sigma}{2}\|\widetilde{p}\|^2_\Omega - \frac{3\gamma}{2\min\{1, c\}}\|\widetilde{h}\|^2_\Omega + \alpha(\mathcal{D}\widetilde{u},\, \widetilde{p})_\Omega.\end{aligned} \tag{3.11}$$

Thus, recalling $\widetilde{h} = \widetilde{m} - \alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} - \sigma\widetilde{p}$, we infer

$$\begin{aligned}\mathfrak{I}_2 \geq\ &\|\widetilde{m}(s)\|^2_{\mathbb{P}^*} - \|\widetilde{m}(0)\|^2_{\mathbb{P}^*} \\ &+ \int_0^s \Big(-\frac{\max\{1, C\}}{(\mu+\lambda)}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|^2_\Omega - \frac{c}{2\mu}\|\widetilde{p}_{\mathrm{tot}}\|^2_\Omega - \frac{\lambda}{2}\|\mathcal{D}\widetilde{u}\|^2_\Omega \\ &\qquad + \sigma\|\widetilde{p}\|^2_\Omega - \frac{3\gamma}{\min\{1, c\}}\|\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}\|^2_\Omega + 2\alpha(\mathcal{D}\widetilde{u},\, \widetilde{p})_\Omega\Big).\end{aligned}$$

Remark 3.10 (Critical terms). It is worth noticing that the (non necessarily positive) term $2\alpha\left\langle \mathcal{D}\widetilde{u}, \widetilde{p}\right\rangle_\Omega$ in the lower bound of $\mathfrak{I}_2$ is compensated by the corresponding term $-2\alpha\left\langle \mathcal{D}\widetilde{u}, \widetilde{p}\right\rangle_\Omega$ in the lower bound of $\mathfrak{I}_1$. A similar compensation, obtained by a different test function, underlines the proof of the stability estimate established by Ženíšek [33] and later used by several other authors.

We insert the identities for $\mathfrak{I}_3$ and $\mathfrak{I}_4$ and the lower bounds for $\mathfrak{I}_1$ and $\mathfrak{I}_2$ into (3.10) and obtain

$$\begin{aligned}b(\widetilde{y}_1, y_{2,s}) \geq &\int_0^s \left(\frac{1}{2}\|\widetilde{u}\|^2_{\mathbb{U}} + \frac{\lambda}{2}\|\mathcal{D}\widetilde{u}\|^2_\Omega + \frac{c}{2\mu}\|\widetilde{p}_{\mathrm{tot}}\|^2_\Omega + \sigma\|\widetilde{p}\|^2_\Omega + \|\partial_t\widetilde{m} + \mathcal{L}\widetilde{p}\|^2_{\mathbb{P}^*}\right) \\ &+ \|\widetilde{m}(s)\|^2_{\mathbb{P}^*} + \|\widetilde{m}(0)\|^2_{\mathbb{P}^*} \\ &+ \int_0^s \Big(\frac{1}{\mu+\lambda}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|^2_\Omega + \gamma\|\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}\|^2_\Omega\Big).\end{aligned}$$

We conclude that the lower bound (3.8) holds true and the hidden constant depends only on Ω, provided that we select the test function

$$y_2 = y_{2,T} + y_{2,\overline{s}} \tag{3.12}$$

where $\overline{s} \in [0,T]$ is chosen so that $\|\widetilde{m}(\overline{s})\|_{\mathbb{P}^*} = \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)}$.

The last step of the proof consists in bounding the norm of the test function y_2 in (3.12), in order to verify (3.9). To this end, we establish a corresponding upper bound for the norm of any function $y_{2,s}$, that is uniform with respect to the parameter $s \in [0,T]$. According to the definition (2.16) of the test norm, we have

$$\|y_{2,s}\|_2^2 = \int_0^s \Big(\|\widetilde{u} + \mathcal{E}^{-1}\mathcal{D}^*\widetilde{p}_{\text{tot}}\|_{\mathbb{U}}^2 + \|\mathcal{L}^{-1}(2\widetilde{m} + \partial_t \widetilde{m} + \mathcal{L}\widetilde{p})\|_{\mathbb{P}}^2 \Big) + \|\mathcal{L}^{-1}\widetilde{m}(0)\|_{\mathbb{P}}^2$$
$$+ \int_0^s \Big(\frac{16\max\{1,C^2\}}{\mu+\lambda} \|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\text{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_{\Omega}^2 + \frac{16\gamma}{\min\{1,c^2\}} \|\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}\|_{\Omega}^2 \Big).$$

We first exploit the identities (2.8) and (2.9) and the second part of (2.10). Then, we extend the integration from $(0,s)$ to $(0,T)$. This results in

$$\begin{aligned}
\|y_{2,s}\|_2^2 &\lesssim \int_0^T \Big(\|\widetilde{u}\|_{\mathbb{U}}^2 + \frac{1}{\mu}\|\widetilde{p}_{\text{tot}}\|_{\Omega}^2 + \|\widetilde{m}\|_{\mathbb{P}^*}^2 + \|\partial_t\widetilde{m} + \mathcal{L}\widetilde{p}\|_{\mathbb{P}^*}^2 \Big) + \|\widetilde{m}(0)\|_{\mathbb{P}^*}^2 \\
&\quad + \int_0^T \Big(\frac{1}{\mu+\lambda}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\text{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_{\Omega}^2 + \gamma\|\alpha\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}\|_{\Omega}^2 \Big) \\
&= \|\widetilde{y}_1\|_1^2 + \int_0^T \|\widetilde{m}\|_{\mathbb{P}^*}^2.
\end{aligned}$$

Notice that the hidden constant depends only on Ω. Estimating the $L^2(\mathbb{P}^*)$-norm of $\widetilde{m}$ in terms of the $L^\infty(\mathbb{P}^*)$-norm yields the desired bound

$$\|y_{2,s}\|^2 \lesssim (1+T)\left(\|\widetilde{y}_1\|_1^2 + \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)}^2 \right).$$

We conclude that (3.9) holds true by combining this with (3.12).

3.2. **Nondegeneracy.** This section is devoted to the proof of Lemma 3.4. To this end, let $y_2 = (v, q_{\text{tot}}, q, n, n_0) \in \mathbb{Y}_2$ be such that (3.6) holds true for all $\widetilde{y}_1 \in \overline{\mathbb{Y}}_1$. We aim at showing

$$y_2 = 0. \tag{3.13}$$

First, we use trial functions in the form $\widetilde{y}_1 = (0, \widetilde{p}_{\text{tot}}, 0, 0)$ in (3.6). We obtain

$$\int_0^T (\widetilde{p}_{\text{tot}},\, \mathcal{D}v - q_{\text{tot}})_\Omega = 0$$

for all $\widetilde{p}_{\text{tot}} \in L^2(\mathbb{D})$. Since both $\mathcal{D}v$ and q_{tot} are in $L^2(\mathbb{D})$, we infer

$$q_{\text{tot}} = \mathcal{D}v \qquad \text{in } L^2(\mathbb{D}). \tag{3.14}$$

Second, we use trial functions in the form $\widetilde{y}_1 = (\widetilde{u}, 0, 0, 0)$ in (3.6). We obtain

$$\int_0^T \left(\langle \mathcal{E}\widetilde{u},\, v\rangle_{\mathbb{U}} + \lambda(\mathcal{D}\widetilde{u},\, q_{\text{tot}})_\Omega + \alpha(\mathcal{D}\widetilde{u},\, \mathcal{P}_{\mathbb{D}}q)_\Omega \right) = 0$$

for all $u \in L^2(\mathbb{U})$. We rearrange terms and use (3.14). It results

$$\int_0^T \langle \widetilde{u},\, \mathcal{E}v + \lambda\mathcal{D}^*\mathcal{D}v + \alpha\mathcal{D}^*\mathcal{P}_{\mathbb{D}}q\rangle_{\mathbb{U}} = 0.$$

The operator $\mathcal{Q} : \mathbb{U} \to \mathbb{U}^*$ defined as $\mathcal{Q} = \mathcal{E} + \lambda \mathcal{D}^*\mathcal{D}$ is the one involved in the displacement formulation of the linear elasticity equations. In particular, it is self-adjoint and invertible. Since both $\mathcal{Q}v$ and $\mathcal{D}^*\mathcal{P}_{\mathbb{D}}q$ are in $L^2(\mathbb{U}^*)$, we infer

$$v = -\alpha \mathcal{Q}^{-1}\mathcal{D}^*\mathcal{P}_{\mathbb{D}}q \qquad \text{in } L^2(\mathbb{U}). \tag{3.15}$$

Third, we use trial functions in the form $\widetilde{y}_1 = (0, 0, \widetilde{p}, 0)$ in (3.6) to obtain

$$\int_0^T \big(-\alpha(\widetilde{p},\, \mathcal{P}_{\overline{\mathbb{P}}}q_{\text{tot}})_\Omega + \sigma(\widetilde{p},\, q)_\Omega + \langle \mathcal{L}\widetilde{p},\, n\rangle_{\mathbb{P}}\big) = 0$$

for all $\widetilde{p} \in L^2(\mathbb{P})$. Rearranging terms and using (3.14) and (3.15) results in

$$\int_0^T (\widetilde{p},\, \alpha^2 \mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\mathcal{Q}^{-1}\mathcal{D}^*\mathcal{P}_{\mathbb{D}}q + \sigma q)_\Omega = -\int_0^T \langle \widetilde{p},\, \mathcal{L}n\rangle_{\mathbb{P}}\,.$$

Recall that $\mathbb{P} \subseteq \overline{\mathbb{P}} \equiv \overline{\mathbb{P}}^* \subseteq \mathbb{P}^*$ is a Hilbert triplet. Since both $\mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\mathcal{Q}^{-1}\mathcal{D}^*\mathcal{P}_{\mathbb{D}}q$ and q are in $L^2(\overline{\mathbb{P}})$, we infer the inclusion $\mathcal{L}n \in L^2(\overline{\mathbb{P}})$ with

$$\mathcal{L}n = -\alpha^2 \mathcal{P}_{\overline{\mathbb{P}}}\mathcal{D}\mathcal{Q}^{-1}\mathcal{D}^*\mathcal{P}_{\mathbb{D}}q - \sigma q \qquad \text{in } L^2(\overline{\mathbb{P}}). \tag{3.16}$$

Finally, we use trial functions in the form $\widetilde{y}_1 = (0, 0, 0, \widetilde{m})$ in (3.6). We obtain

$$\int_0^T \Big(-(\widetilde{m},\, q)_\Omega + \langle \partial_t \widetilde{m},\, n\rangle_{\mathbb{P}}\Big) + \langle \widetilde{m}(0),\, n_0\rangle_{\mathbb{P}} = 0$$

for all $\widetilde{m} \in L^2(\overline{\mathbb{P}}) \cap H^1(\mathbb{P}^*)$. Considering first $\widetilde{m} = \phi\widetilde{w}$ with $\phi \in C_0^\infty(0, T)$ and $\widetilde{w} \in \overline{\mathbb{P}}$ reveals

$$\int_0^T \phi(\widetilde{w},\, q)_\Omega = \int_0^T \partial_t\phi\, \langle \widetilde{w},\, n\rangle_{\mathbb{P}}\,.$$

By [13, Proposition 64.33], we infer $n \in L^2(\mathbb{P}) \cap H^1(\overline{\mathbb{P}})$ with

$$\partial_t n = -q \qquad \text{in } L^2(\overline{\mathbb{P}}). \tag{3.17}$$

Then, assuming $\phi \in C^\infty(0, T)$ with $\phi(0) = 1$ and $\phi(T) = 0$ or, respectively, $\phi(1) = 0$ and $\phi(T) = 1$, it follows that $n(0), n(T) \in \mathbb{P}$ with

$$n(0) = n_0 \quad \text{and} \quad n(T) = 0 \qquad \text{in } \mathbb{P}. \tag{3.18}$$

According to (3.16), (3.17) and (3.18), it holds that

$$\begin{aligned} -\frac{1}{2}\|n_0\|_{\mathbb{P}}^2 = \frac{1}{2}\|n(T)\|_{\mathbb{P}}^2 - \frac{1}{2}\|n(0)\|_{\mathbb{P}}^2 &= \int_0^T (\partial_t n,\, \mathcal{L}n)_\Omega \\ &= \alpha^2 \int_0^T \langle \mathcal{Q}^{-1}\mathcal{D}^*\mathcal{P}_{\mathbb{D}}q,\, \mathcal{D}^*\mathcal{P}_{\mathbb{D}}q\rangle_{\mathbb{U}} + \sigma \int_0^T \|q\|_\Omega^2. \end{aligned}$$

Therefore, we have $n_0 = 0$, $\mathcal{P}_{\mathbb{D}}q = 0$ and $\sigma q = 0$ in the respective spaces. This implies $q = 0$ because we have $\overline{\mathbb{P}} \subseteq \mathbb{D}$ (hence $\mathcal{P}_{\mathbb{D}}q = q$) for $\sigma = 0$, cf. (2.5). Then, the identities (3.15), (3.16) and (3.14) imply $v = 0$, $n = 0$ and $q_{\text{tot}} = 0$, respectively. This verifies (3.13) and completes the proof.

4. Further stability estimates

The definition of the trial norm $\|\cdot\|_1$ in (2.14) is, in a sense, minimal, because it includes only the terms that are necessary for an immediate proof of the boundedness of the form b in (3.2), cf. Lemma 3.1. Lemma 3.2 reveals that we could equivalently augment $\|\cdot\|_1$ with other terms, namely the $L^2(L^2(\Omega))$-norm of the total pressure and of the pressure as well as the $L^\infty(\mathbb{P}^*)$-norm of the total fluid content. This section aims at determining if we can control other relevant terms by the trial norm. In particular, we are interested in understanding whether $\mathbb{Y}_1$ is closed with respect to $\|\cdot\|_1$. Indeed, for $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{\text{tot}}, \widetilde{p}, \widetilde{m}) \in \overline{\mathbb{Y}}_1$, we have

$$\|\widetilde{y}_1\|_1^2 \geq \int_0^T \|\partial_t \widetilde{m} + \mathcal{L}\widetilde{p}\|_{\mathbb{P}^*}^2$$

but it is not clear whether $\|\widetilde{y}_1\|_1$ controls $\|\partial_t \widetilde{m}\|_{\mathbb{P}^*}^2 + \|\widetilde{p}\|_{\mathbb{P}}^2$. Thus, the question arises if $\overline{\mathbb{Y}}_1$ is indeed larger than $\mathbb{Y}_1$ and, if yes, how much larger it is. We establish some results, showing that the two spaces are actually different but the difference is somehow subtle.

4.1. **Positive results.** Let us first recall that the combination of Lemmas 3.1 and 3.2 with the definition of the norm $\|\cdot\|_1$ in (2.14) implies some control on the single components of a trial function $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{\text{tot}}, \widetilde{p}, \widetilde{m}) \in \overline{\mathbb{Y}}_1$. Indeed, we have

$$(1+T)\|\widetilde{y}_1\|_1^2 \gtrsim \int_0^T \left(\|\widetilde{u}\|_{\mathbb{U}}^2 + \lambda \|\mathcal{D}\widetilde{u}\|_\Omega^2 + \frac{1}{\mu}\|\widetilde{p}_{\text{tot}}\|_\Omega^2 + \sigma\|\widetilde{p}\|_\Omega^2 \right) + \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)}^2 \tag{4.1}$$

and the hidden constant depends only on Ω. The control on the $L^2(L^2(\Omega))$-norm of the third component $\widetilde{p}$ can be improved as follows.

Proposition 4.1 ($L^2(L^2(\Omega))$-norm of the pressure)**.** *For $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{tot}, \widetilde{p}, \widetilde{m}) \in \overline{\mathbb{Y}}_1$, we have*

$$(1+T)\|\widetilde{y}_1\|_1^2 \gtrsim \int_0^T \left(\frac{\alpha^2}{\mu+\lambda}\|\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_\Omega^2 + \sigma\|\widetilde{p}\|_\Omega^2 \right) \geq \gamma^{-1}\int_0^T \|\widetilde{p}\|_\Omega^2,$$

where the hidden constant depends only on the domain Ω and γ is as (2.15).

Proof. The combination of Lemmas 3.1 and 3.2 implies

$$(1+T)\|\widetilde{y}_1\|_1^2 \gtrsim \int_0^T \left(\frac{C}{\mu+\lambda}\|\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\text{tot}} - \alpha\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_\Omega^2 + \frac{1}{\mu}\|\widetilde{p}_{\text{tot}}\|_\Omega^2 + \lambda\|\mathcal{D}\widetilde{u}\|_\Omega^2 + \sigma\|\widetilde{p}\|_\Omega^2 \right).$$

for all constants $C \in [0,1]$. The first summand on the right-hand side gives rise to six terms. By applying Young's inequality to the ones that are not necessarily positive, we obtain

$$(1+T)\|\widetilde{y}_1\|_1^2 \gtrsim \int_0^T \left(\frac{1-4C}{\mu}\|\widetilde{p}_{\text{tot}}\|_\Omega^2 + (1-4C)\lambda\|\mathcal{D}\widetilde{u}\|_\Omega^2 + \frac{C\alpha^2}{2(\mu+\lambda)}\|\mathcal{P}_{\mathbb{D}}\widetilde{p}\|_\Omega + \sigma\|\widetilde{p}\|_\Omega \right).$$

We choose $C = 1/4$ and notice that $\mathcal{P}_{\mathbb{D}}\widetilde{p} = \widetilde{p}$ whenever $\overline{\mathbb{P}} \subseteq \mathbb{D}$. This holds true, in particular, for $\sigma = 0$. Then, we conclude by recalling the definition of γ. □

Next, we show that we can control the $L^2(\mathbb{P})$-norm of an approximation of the the third component $\widetilde{p}$ of a trial function, namely the L^2-orthogonal projection onto $\mathbb{P}$-valued polynomials of any degree $r \in \mathbb{N}_0$. Unfortunately, the control is not uniform with respect to r.

The space of $\mathbb{P}$-valued polynomials of degree r is defined as

$$P_r(\mathbb{P}) := \left\{ q \in L^2(\mathbb{P}) \mid q(t) = \sum_{j=0}^{r} w_j t^j \quad \text{with} \quad (w_j)_{j=0}^r \subseteq \mathbb{P} \right\}.$$

The L^2-orthogonal projection $\mathcal{P}_r : L^2(\mathbb{P}) \to P_r(\mathbb{P})$ is obtained via the condition

$$\int_0^T \langle \mathcal{L}\mathcal{P}_r\widetilde{p},\, q\rangle_{\mathbb{P}} = \int_0^T \langle \mathcal{L}\widetilde{p},\, q\rangle_{\mathbb{P}} \tag{4.2}$$

for $\widetilde{p} \in L^2(\mathbb{P})$ and for all $q \in P_r(\mathbb{P})$. Recall that $\langle \mathcal{L}\cdot,\, \cdot\rangle_{\mathbb{P}}$ is the scalar product inducing the norm $\|\cdot\|_{\mathbb{P}}$ on $\mathbb{P}$, cf. (2.8).

The next proposition shows that the mapping

$$(\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m}) \mapsto \mathcal{P}_r\widetilde{p}$$

defines a bounded linear operator on $\mathbb{Y}_1$ with respect to the norm $\|\cdot\|_1$. Hence, we can extend such operator to the trial space $\overline{\mathbb{Y}}_1$ by density. In other words, the L^2-orthogonal projection of the fluid pressure $\widetilde{p}$ onto $P_r(\mathbb{P})$ is well-defined and bounded on $\overline{\mathbb{Y}}_1$.

Proposition 4.2 (Polynomial-in-time projection of the pressure)**.** *Let $r \in \mathbb{N}_0$. For $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{tot}, \widetilde{p}, \widetilde{m}) \in \mathbb{Y}_1$, it holds that*

$$\int_0^T \|\mathcal{P}_r\widetilde{p}\|_{\mathbb{P}}^2 \lesssim \|\widetilde{y}_1\|_1^2.$$

The hidden constant depends only on the domain Ω, the final time T and the degree r.

Proof. Recall the boundedness of the form b stated in Lemma 3.1. Using the test function $y_2 = (0, 0, 0, \mathcal{P}_r\widetilde{p}, 0) \in \mathbb{Y}_2$ reveals

$$\int_0^T \langle \partial_t\widetilde{m} + \mathcal{L}\widetilde{p}, \mathcal{P}_r\widetilde{p}\rangle_{\mathbb{P}} \lesssim \|\widetilde{y}_1\|_1 \left(\int_0^T \|\mathcal{P}_r\widetilde{p}\|_{\mathbb{P}}^2 \right)^{\frac{1}{2}} \tag{4.3}$$

where the hidden constant depends only on Ω. The inclusion $P_r(\mathbb{P}) \subseteq H^1(\mathbb{P})$, the second identity in (2.8) and (4.2) entail that we have

$$\begin{aligned}\int_0^T \langle \partial_t\widetilde{m} + \mathcal{L}\widetilde{p}, \mathcal{P}_r\widetilde{p}\rangle_{\mathbb{P}} = \int_0^T \|\mathcal{P}_r\widetilde{p}\|_{\mathbb{P}}^2 &- \int_0^T \langle \widetilde{m}, \partial_t\mathcal{P}_r\widetilde{p}\rangle_{\mathbb{P}} \\ &+ \langle \widetilde{m}(T), \mathcal{P}_r\widetilde{p}(T)\rangle_{\mathbb{P}} - \langle \widetilde{m}(0), \mathcal{P}_r\widetilde{p}(0)\rangle_{\mathbb{P}}\,.\end{aligned}$$

Note that the space $P_r(\mathbb{P})$ is complete with respect to both the $L^\infty(\mathbb{P})$- and the $L^2(\mathbb{P})$-norm. Therefore, the open mapping theorem implies

$$\langle \widetilde{m}(T), \mathcal{P}_r\widetilde{p}(T)\rangle_{\mathbb{P}} - \langle \widetilde{m}(0), \mathcal{P}_r\widetilde{p}(0)\rangle_{\mathbb{P}} \lesssim \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)} \left(\int_0^T \|\mathcal{P}_r\widetilde{p}\|_{\mathbb{P}}^2 \right)^{\frac{1}{2}}$$

with hidden constants depending on $P_r(\mathbb{P})$ itself, hence on Ω, T and r. A Hölder estimate and a similar argument entail also

$$\int_0^T \langle \widetilde{m}, \partial_t \mathcal{P}_r \widetilde{p} \rangle_{\mathbb{P}} \leq \sqrt{T} \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)} \left(\int_0^T \|\partial_t \mathcal{P}_r \widetilde{p}\|_{\mathbb{P}}^2 \right)^{\frac{1}{2}}$$
$$\lesssim \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)} \left(\int_0^T \|\mathcal{P}_r \widetilde{p}\|_{\mathbb{P}}^2 \right)^{\frac{1}{2}}.$$

We combine these bounds and the previous identity with (4.3) to obtain

$$\int_0^T \|\mathcal{P}_r \widetilde{p}\|_{\mathbb{P}}^2 \lesssim \|\widetilde{y}_1\|_1^2 + \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)}^2.$$

We conclude by recalling the estimate $\|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)} \lesssim \|\widetilde{y}_1\|_1$ from (4.1), with the hidden constant depending only on Ω and T. $\square$

Finally, the estimate (4.1) controls the $L^\infty(\mathbb{P}^*)$-norm of $\widetilde{m}$, that is the anti-derivative of $\partial_t \widetilde{m}$. This readily implies that we can control the $L^\infty(\mathbb{P})$-norm of the anti-derivative of $\widetilde{p}$.

Proposition 4.3 (Anti-derivative of the pressure)**.** *For $\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{tot}, \widetilde{p}, \widetilde{m}) \in \mathbb{Y}_1$, we have*

$$\sup_{t \in [0,T]} \left\| \int_0^t \widetilde{p} \right\|_{\mathbb{P}} \lesssim \|\widetilde{y}_1\|_1.$$

The hidden constant depends only on the domain Ω and the final time T.

Proof. We observe first that $\int_0^t \partial_t \widetilde{m} = \widetilde{m}(t) - \widetilde{m}(0)$ on $\mathbb{P}^*$. Therefore, by applying a triangle and a Hölder inequality, we obtain

$$\left\| \int_0^t \widetilde{p}\, dt \right\|_{\mathbb{P}}^2 \lesssim \left\| \int_0^t \left(\partial_t \widetilde{m} + \mathcal{L}\widetilde{p} \right) \right\|_{\mathbb{P}^*}^2 + \|\widetilde{m}(t)\|_{\mathbb{P}^*}^2 + \|\widetilde{m}(0)\|_{\mathbb{P}^*}^2$$
$$\lesssim \int_0^T \|\partial_t \widetilde{m} + \mathcal{L}\widetilde{p}\|_{\mathbb{P}^*}^2 + \|\widetilde{m}\|_{L^\infty(\mathbb{P}^*)}^2.$$

The claimed estimate follows from (4.1) and the definition of $\|\cdot\|_1$. $\square$

4.2. **Negative results.** Roughly speaking, the results in section 4.1 state that, if the space $\mathbb{Y}_1$ differs from its closure $\overline{\mathbb{Y}}_1$, then the difference is somehow subtle. On the other hand, the next result confirms that indeed the two spaces are different. In view of Proposition 4.2, the proof builds upon the construction of a function in $\overline{\mathbb{Y}}_1$, so that the third and the fourth components are polynomials in time of arbitrarily high degree.

Proposition 4.4 (Existence of 'rough' trial functions)**.** *It holds that*

$$\sup_{\widetilde{y}_1 = (\widetilde{u}, \widetilde{p}_{tot}, \widetilde{p}, \widetilde{m}) \in \mathbb{Y}_1} \frac{\int_0^T \left(\|\partial_t \widetilde{m}\|_{\mathbb{P}^*}^2 + \|\widetilde{p}\|_{\mathbb{P}}^2 \right)}{\|\widetilde{y}_1\|_1^2} = +\infty.$$

Proof. Recall that $\mathbb{P} \subseteq \overline{\mathbb{P}} \equiv \overline{\mathbb{P}}^* \subseteq \mathbb{P}^*$ is a Hilbert triplet. Then by the theory of self-adjoint coercive operators we have for the eigenvalues $(\lambda_k)_{k \geq 1} \subseteq (0, +\infty)$ of the operator $\mathcal{L}$ that $\lambda_k \nearrow +\infty$ as $k \to +\infty$. Let $(w_k)_{k\geq 1} \subseteq \mathbb{P}$ be the associated eigenfunctions. Hence, we have $\mathcal{L}w_k = \lambda_k w_k$ as well as

$$\|w_k\|_\Omega^2 = 1 \qquad \text{and} \qquad \|w_k\|_{\mathbb{P}}^2 = \lambda_k, \qquad k \geq 1. \tag{4.4}$$

Denote by $r_k := \lceil \lambda_k \rceil$ the ceiling function applied to λ_k, i.e., the smallest integer larger than or equal to λ_k. We define $\widetilde{p}^{(k)} \in L^2(\mathbb{P})$ and $\widetilde{m}^{(k)} \in L^2(\overline{\mathbb{P}}) \cap H^1(\mathbb{P}^*)$ by

$$\widetilde{p}^{(k)}(t) := \frac{w_k}{T^{r_k}} t^{r_k} \qquad \text{and} \qquad \widetilde{m}^{(k)}(t) := -\frac{\lambda_k w_k}{T^{r_k}(r_k+1)} t^{r_k+1}.$$

By construction, we have

$$\partial_t \widetilde{m}^{(k)} + \mathcal{L}\widetilde{p}^{(k)} = 0 \qquad \text{and} \qquad \widetilde{m}^{(k)}(0) = 0. \tag{4.5}$$

Moreover, elementary computations reveal

$$\int_0^T \|\widetilde{p}^{(k)}\|_\Omega^2 = \frac{T}{2r_k+1} \qquad \text{and} \qquad \int_0^T \|\widetilde{m}^{(k)}\|_\Omega^2 = \frac{T^3\lambda_k^2}{(r_k+1)^2(2r_k+3)}.$$

Thus, for $\widetilde{y}_1^{(k)} = (0, 0, \widetilde{p}^{(k)}, \widetilde{m}^{(k)}) \in \mathbb{Y}_1$, the definition (2.14) of the trial norm implies

$$\|\widetilde{y}_1^{(k)}\|_1^2 \to 0 \quad \text{as } k \to +\infty.$$

On the other hand, it holds that

$$\int_0^T \|\partial_t \widetilde{m}^{(k)}\|_{\mathbb{P}^*}^2 = \int_0^T \|\widetilde{p}^{(k)}\|_{\mathbb{P}}^2 = \frac{\lambda_k T}{2r_k+1} \to \frac{T}{2} \quad \text{as } k \to +\infty.$$

Comparing this limit with the previous one concludes the proof. □

Remark 4.5 ('Rough' trial functions)*.* The elements of the sequence $(\widetilde{y}_1^{(k)})_{k\geq 1}$ in the poof of Proposition 4.4 are in $\mathbb{Y}_1$ but do not solve the weak formulation (2.11) of the Biot's equations. Indeed, they do not fulfill the constraints in the second and in the third lines of (2.11). Still, we might modify the first component of $\widetilde{y}_1$ by choosing it (more precisely, its divergence) so as to fulfill the constraint in the third line. Then, we might modify also the second component according to the second line in (2.11). This observation reveals a remarkable difference between our analysis and former ones: we do not control the $L^2(\mathbb{P})$-norm of the pressure, due to the weaker regularity assumption on the load ℓ_u in (2.11), cf. Remark 2.9.

Remark 4.6 (Time regularity)*.* Some results in the spirit of Propositions 4.3-4.4 are proved by Murad, Thomée and Loula [22, section 2] under the assumption $\ell_u \in H^1(\mathbb{U}^*)$ in (2.11). Indeed, a bound on the $L^\infty(\mathbb{P})$-norm of the anti-derivative of the pressure is established and it is observed that the same bound does not hold true for the pressure itself, because of a singularity at $t = 0$. Numerical evidence of the latter observation can be found also in [3]. As in Remark 3.7, the higher integrability in time, compared to our approach, follows from the higher regularity of ℓ_u, cf. Remark 2.9.

We conclude this section by recalling that, in the analysis of scalar parabolic equations, the L^∞ control in time over the point values of the solution is obtained by combining some L^2 control over the solution itself and on its time derivative. Here, in contrast, Proposition 4.4 states that we do not have a L^2 control over the time derivative of the total fluid content. Hence, it is remarkable that we could nevertheless establish the embedding of such variable into $C^0(\mathbb{P}^*)$, cf. Remark 3.3.

5. Shift of the regularity in space

In addition to the well-posedness of the weak formulation (2.11), we are interested also in shift theorems, i.e. the question, whether more regular data than in Theorem 3.5 give rise to correspondingly more regular solutions. In fact, the regularity of the solution is a necessary ingredient to justify the error decay for the discretization we propose and analyze in [17]. Still, it is known that the regularity theory for the Biot's equations is subtle. For instance, Murad, Thomée and Loula [22] observed that the regularity in time is limited at $t = 0$ and also Showalter [27] came to a similar conclusion.

Due to the complexity of the subject, we do not attempt at establishing a comprehensive result here. We confine our discussion to a rather specific case, where no singularity occurs. More precisely, we investigate only the regularity in space under the set of assumptions detailed below. What is more relevant for us is that we can use inf-sup theory once more to this end. This appears to be an innovative technique not only for the Biot's equations but also in the general framework of coupled problems. In fact, we introduce another variational formulation of the initial-boundary value problem (2.1) for the Biot's equations. Compared to the weak formulation (2.11), we prescribe additional regularity in space of the trial functions. Therefore, any solution of the new 'strong' formulation solves also the weak one. We verify the well-posedness again by applying the Banach-Nečas theorem. In this way, we establish a two-sided estimate, in the vein of (3.7), ensuring that the regularity guaranteed for the solution matches the regularity requirement for the data.

Our first assumption concerns the domain Ω. We require

$$\Omega \subseteq \mathbb{R}^2 \text{ is a convex polygon.} \tag{5.1a}$$

Alternatively, we could work with $\partial\Omega$ smooth, but (5.1a) is more relevant for the discretization analyzed in [17]. Second, we assume pure essential boundary conditions for the displacement and pure natural boundary conditions for the pressure

$$\Gamma_{u,E} = \partial\Omega = \Gamma_{p,N}. \tag{5.1b}$$

Third, we restrict ourselves to the usually more critical case for the Lamé constants

$$\mu \ll \lambda \tag{5.1c}$$

meaning that we have $C\mu \leq \lambda$ for some constant C (depending only on the domain Ω) that is as large as necessary for the arguments in sections 5.3 and 5.4.

Remark 5.1 (Justification of the assumptions). Linear elasticity is one of the building blocks in the Biot's equations. Therefore, we use arguments introduced by Brenner and Sung [8, section 2] for that problem. This motivates the assumptions (5.1a) and (5.1c), as well as the first part of (5.1b). Note that [8] covers also pure natural boundary conditions. The second part of (5.1b) implies that we can use the same space for the total pressure and the total fluid content (cf. Figure 2), and an important relation between the differential operators, see (5.3). Our proof heavily exploits both properties.

5.1. **Strong formulation in space.**

First of all, we introduce dedicated symbols for the most frequently used spaces, in the vein of section 2.3. According to the assumptions (5.1), we use

$$\mathbb{E} := H^2(\Omega)^2 \cap \mathbb{U} = H^2(\Omega)^2 \cap H^1_0(\Omega)^2$$

for the regularity in space of the displacement. For the total pressure and the total fluid content, we recall the space $\mathbb{P}$, which reads

$$\mathbb{P} = H^1(\Omega) \cap L^2_0(\Omega)$$

in this case. For the pressure, we recall the space $\mathbb{L}$ from Remark 3.8, namely

$$\mathbb{L} = \{\widetilde{p} \in \mathbb{P} \mid \mathcal{L}\widetilde{p} \in \overline{\mathbb{P}}\} = \{\widetilde{p} \in \mathbb{P} \mid \mathcal{L}\widetilde{p} \in L^2_0(\Omega)\}$$

where the closure of $\mathbb{P}$ is taken with respect to the $L^2(\Omega)$-norm.

The interplay between these spaces and (the restriction of) the differential operators in (2.7) is summarized in Figure 2. Note the different structure compared to the weak formulation (Figure 1) and that, in the left part of the diagram, $\mathbb{P}$ and $L^2(\Omega)^2$ are identified with subspaces of $\mathbb{D}^*$ and $\mathbb{U}^*$ via the $L^2(\Omega)$-scalar product. Upon this identification, we have

$$\mathcal{D}^* = -\nabla \quad \text{in } \mathbb{P} \tag{5.2}$$

as well as

$$\langle \mathcal{L}\cdot, \cdot\rangle_{\mathbb{P}} = \kappa(\mathcal{D}^*\cdot, \mathcal{D}^*\cdot)_\Omega \quad \text{in } \mathbb{P}\times\mathbb{P}. \tag{5.3}$$

$$\begin{tikzcd}
\mathbb{E} \arrow[d, "\mathcal{E}"'] \arrow[dr, "\mathcal{D}"] & & \mathbb{L} \arrow[dl, "i"'] \arrow[d, "\mathcal{L}"] \\
L^2(\Omega)^2 & \mathbb{P} \arrow[l, "\mathcal{D}^*"] \arrow[r, "i"'] & \overline{\mathbb{P}}
\end{tikzcd}$$

FIGURE 2. Spaces and operators describing the regularity in space for the strong formulation (5.6) of the Biot's equations. Note that i denotes the embedding operator.

Following [6, section 2.1], we define $H^{\frac{1}{2}}(\partial\Omega)$ as the image of $\mathbb{P}$ via the trace operator on $\partial\Omega$. We equip this space with the quotient norm

$$\|g\|^2_{H^{\frac{1}{2}}(\partial\Omega)} := \frac{1}{\kappa} \inf_{\widetilde{p}\in\mathbb{P},\, \widetilde{p}_{|\partial\Omega}=g} \|\widetilde{p}\|^2_{\mathbb{P}} = \inf_{\widetilde{p}\in\mathbb{P},\, \widetilde{p}_{|\partial\Omega}=g} \|\mathcal{D}^*\widetilde{p}\|^2_\Omega. \tag{5.4}$$

Let $H^{-\frac{1}{2}}(\partial\Omega)$ be the dual of $H^{\frac{1}{2}}(\partial\Omega)$. For simplicity, we denote by $\langle\cdot, \cdot\rangle_{1/2}$ the corresponding duality. By comparing (5.2) with the definitions (2.7) and (5.1) of $\mathcal{L}$ and $\mathbb{L}$, respectively, we infer that the normal derivative $\partial_{\mathsf{n}} : \mathbb{L} \to H^{-\frac{1}{2}}(\partial\Omega)$ is well-defined and we have

$$\langle \mathcal{L}\cdot, \cdot\rangle_{\mathbb{P}} = (\mathcal{L}\cdot, \cdot)_\Omega + \kappa\,\langle \partial_{\mathsf{n}}\cdot, \cdot\rangle_{1/2} \quad \text{in } \mathbb{L}\times\mathbb{P}. \tag{5.5}$$

After this preparation, we are in position to state the announced strong formulation of (2.3), with (2.1b) and (2.1c), as follows

$$
\begin{aligned}
\mathcal{E}u + \mathcal{D}^* p_{\mathrm{tot}} &= f_u && \text{in } L^2(L^2(\Omega)^2)\\
\lambda \mathcal{D}u - p_{\mathrm{tot}} - \alpha p &= 0 && \text{in } L^2(\mathbb{P})\\
\alpha \mathcal{D}u + \sigma p - m &= 0 && \text{in } L^2(\mathbb{P})\\
\partial_t m + \mathcal{L}p &= f_p && \text{in } L^2(\overline{\mathbb{P}})\\
\partial_{\mathsf{n}} p &= g_p && \text{in } L^2(H^{-\frac{1}{2}}(\partial\Omega))\\
m(0) &= \ell_0 && \text{in } \overline{\mathbb{P}}.
\end{aligned}
\tag{5.6}
$$

Note that, compared to the weak formulation (2.11), the data are assumed to be more regular in space and the boundary condition for the pressure is satisfied in the sense of traces. We look for a solution of (5.6) in the trial space $\overline{\mathbb{X}}_1$, with

$$\mathbb{X}_1 := L^2(\mathbb{E}) \times L^2(\mathbb{P}) \times L^2(\mathbb{L}) \times \big(L^2(\mathbb{P}) \cap H^1(\overline{\mathbb{P}})\big)$$

where the closure is taken with respect to the norm

$$
\begin{aligned}
\|(\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m})\|_1^2 := &\int_0^T \left(\mu\|D^2\widetilde{u}\|_\Omega^2 + \frac{1}{\mu}\|\mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 + \frac{1}{\kappa}\|\partial_t\widetilde{m} + \mathcal{L}\widetilde{p}\|_\Omega^2\right)\\
&+ \int_0^T \left(\frac{1}{\lambda}\|\mathcal{D}^*(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p})\|_\Omega^2 + \gamma\|\mathcal{D}^*(\alpha\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m})\|_\Omega^2\right)\\
&+ \int_0^T \frac{1}{\gamma}\|\partial_{\mathsf{n}}\widetilde{p}\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \frac{1}{\kappa}\|\widetilde{m}(0)\|_\Omega^2.
\end{aligned}
$$

According to the additional regularity of the data, each component of $\mathbb{X}_1$ is more regular in space compared to the corresponding one of the space $\mathbb{Y}_1$ for the weak formulation (2.13).

The corresponding test space is

$$\mathbb{X}_2 := L^2(L^2(\Omega)^2) \times L^2(\mathbb{P}) \times L^2(\mathbb{P}) \times L^2(\overline{\mathbb{P}}) \times L^2(H^{\frac{1}{2}}(\partial\Omega)) \times \overline{\mathbb{P}}$$

equipped with the norm

$$
\begin{aligned}
\|(v, q_{\mathrm{tot}}, q, n, n_\partial, n_0)\|_2^2 := &\int_0^T \left(\mu\|v\|_\Omega^2 + \kappa\|n\|_\Omega^2 + \gamma\|n_\partial\|^2_{H^{\frac{1}{2}}(\partial\Omega)}\right) + \kappa\|n_0\|_\Omega^2\\
&+ \int_0^T \left(\lambda\|\mathcal{D}^* q_{\mathrm{tot}}\|_\Omega^2 + \gamma^{-1}\|\mathcal{D}^* q\|_\Omega^2\right).
\end{aligned}
\tag{5.7}
$$

Remark 5.2 (Closure of $\mathbb{X}_1$). Like the trial space $\mathbb{Y}_1$ for the weak formulation, the space $\mathbb{X}_1$ is not closed with respect to the norm defined on it. This can be verified by exactly the same argument as in the proof of Proposition 4.4. Indeed, the eigenfunctions of $\mathcal{L}$ are actually in $\mathbb{L}$. Note that, in analogy with the observation in Remark 3.7, the $L^2(\mathbb{L})$-norm of the pressure can be controlled upon assuming that the load f_u in the first equation of (5.6) is weakly differentiable in time, cf. Remark 3.8. This approach is widely used in connection with mixed formulations of the Biot's equations, introducing the Darcy velocity as an independent variable, see e.g. [18].

Before discussing further properties of (5.6), it is worth noticing that this is indeed a stronger formulation of the Biot's equations than (2.11).

Lemma 5.3 (Strong vs weak formulation)**.** *Assume $x_1 \in \overline{\mathbb{X}}_1$ solves* (5.6) *with data*

$$f_u \in L^2(L^2(\Omega)^2), \qquad f_p \in L^2(\overline{\mathbb{P}}), \qquad g_p \in L^2(H^{-\frac{1}{2}}(\partial\Omega)), \qquad \ell_0 \in \overline{\mathbb{P}}. \tag{5.8}$$

Then x_1 solves also (2.11) *with the data defined by* (2.12)*.*

Proof. The inclusion $\mathbb{X}_1 \subseteq \mathbb{Y}_1$ and the bound $\|\cdot\|_1 \lesssim |\!|\!|\cdot|\!|\!|_1$ imply $\overline{\mathbb{X}}_1 \subseteq \overline{\mathbb{Y}}_1$. Thus, if $x_1 = (u, p_{\mathrm{tot}}, p, m) \in \overline{\mathbb{X}}_1$ solves (5.6), it is an admissible trial function also for the weak formulation. The assumptions (5.1) entail that, in this case, we have $\mathbb{D} = L^2_0(\Omega) = \overline{\mathbb{P}}$, cf. (2.5). Then x_1 fulfills the second, third and fifth equations in (2.11). The first equation is fulfilled as well because of the inclusion $L^2(\Omega)^2 \subseteq \mathbb{U}^*$ and the identity

$$\int_0^T (\mathcal{E}u + \mathcal{D}^* p_{\mathrm{tot}},\, v)_\Omega = \int_0^T \langle \mathcal{E}u + \mathcal{D}^* p_{\mathrm{tot}},\, v\rangle_{\mathbb{U}}$$

where $v \in L^2(\mathbb{U})$ is arbitrary. Finally, for the fourth equation, we notice that (5.5) implies

$$\int_0^T \Big((\partial_t m + \mathcal{L}p,\, n)_\Omega + \kappa \left\langle \partial_{\mathsf{n}} p,\, n\right\rangle_{1/2} \Big) = \int_0^T \left\langle \partial_t m + \mathcal{L}p,\, n\right\rangle_{\mathbb{P}}$$

for all $n \in L^2(\mathbb{P})$. □

5.2. **Well-posedness and regularity.** As mentioned before, the assumptions in (5.1) prevent from any unexpected singularity of the solution of (5.6). Therefore, the well-posedness can be verified by mimicking the argument in section 3. Also in this case, we postpone most of the details of the proof to the next sections.

Theorem 5.4 (Well-posedness)**.** *For (f_u, f_p, g_p, ℓ_0) as in* (5.8)*, the equations* (5.6) *have a unique solution $x_1 = (u, p_{tot}, p, m) \in \overline{\mathbb{X}}_1$, which fulfills the two-sided stability bound*

$$\begin{aligned}
&\int_0^T \Big(\mu \|D^2 u\|^2_\Omega + \frac{1}{\mu}\|\mathcal{D}^* p_{tot}\|^2_\Omega + \frac{1}{\kappa}\|\partial_t m + \mathcal{L}p\|^2_\Omega \Big) \\
&+ \int_0^T \Big(\lambda \|\mathcal{D}^*\mathcal{D}u\|^2_\Omega + \frac{1}{\gamma}\|\mathcal{D}^* p\|^2_\Omega \Big) + \frac{1}{\kappa}\|m\|^2_{L^\infty(\overline{\mathbb{P}})} \\
&\qquad \eqsim \int_0^T \Big(\frac{1}{\mu}\|f_u\|^2_\Omega + \frac{1}{\kappa}\|f_p\|^2_\Omega + \frac{1}{\gamma}\|g_p\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \Big) + \frac{1}{\kappa}\|\ell_0\|^2_\Omega.
\end{aligned} \tag{5.9}$$

The hidden constants depend only on the domain Ω, the final time T and the constant in (5.1c)*. Moreover, we have $m \in C^0(\overline{\mathbb{P}})$.*

Proof. The equations (5.6) are equivalent to a linear variational problem like (3.1) with the bilinear form $b : \overline{\mathbb{X}}_1 \times \mathbb{X}_2 \to \mathbb{R}$

$$\begin{aligned}
b(\widetilde{x}_1, x_2) = &\int_0^T \Big((\mathcal{E}\widetilde{u} + \mathcal{D}^*\widetilde{p}_{\mathrm{tot}},\, v)_\Omega + (\partial_t \widetilde{m} + \mathcal{L}\widetilde{p},\, n)_\Omega \Big) \\
+ &\int_0^T \Big((\mathcal{D}^*(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p}),\, \mathcal{D}^* q_{\mathrm{tot}})_\Omega + (\mathcal{D}^*(\alpha\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}),\, \mathcal{D}^* q)_\Omega \Big) \\
+ &\int_0^T \langle \partial_{\mathsf{n}}\widetilde{p},\, n_\partial\rangle_{1/2} + (\widetilde{m}(0),\, n_0)_\Omega
\end{aligned} \tag{5.10}$$

and the load $\ell : \mathbb{X}_2 \to \mathbb{R}$

$$\ell(x_2) = \int_0^T \Big((f_u,\, v)_\Omega + (f_p,\, n)_\Omega + \langle g_p,\, n_\partial \rangle_{1/2} \Big) + (\ell_0,\, n_0)_\Omega \tag{5.11}$$

for $\widetilde{x}_1 = (\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m}) \in \overline{\mathbb{X}}_1$ and $x_2 = (v, q_{\mathrm{tot}}, q, n, n_\partial, n_0) \in \mathbb{X}_2$. The boundedness of the bilinear form with respect to the norm $\|\!|\cdot|\!\|_1$ and $\|\!|\cdot|\!\|_2$ follows from Cauchy-Schwartz inequalities. Section 5.3 establishes the (strengthened) inf-sup stability

$$\begin{aligned} &\sup_{x_2 \in \mathbb{X}_2} \frac{b(\widetilde{x}_1, x_2)}{\|\!|x_2|\!\|_2} \gtrsim \\ &\quad (1+T)^{-\frac{1}{2}} \left(\|\!|\widetilde{x}_1|\!\|_1^2 + \frac{1}{\kappa} \|\widetilde{m}\|^2_{L^\infty(\overline{\mathbb{P}})} + \int_0^T \left(\lambda \|\mathcal{D}^*\mathcal{D}\widetilde{u}\|^2_\Omega + \gamma^{-1} \|\mathcal{D}^*\widetilde{p}\|^2_\Omega \right) \right)^{\frac{1}{2}} \end{aligned} \tag{5.12}$$

for $\widetilde{x}_1 \in \overline{\mathbb{X}}_1$. Section 5.4 further verifies the nondegeneracy of b. The combination of these properties implies the well-posedness of the equations (5.6) by the Banach-Nečas theorem [12, theorem 25.9]. The estimate (5.4) then follows by combining boundedness and inf-sup stability with the definitions (5.11) and (5.7) of the load and of the test norm $\|\!|\cdot|\!\|_2$. Finally, the continuity in time of m can be verified by arguing as in Remark 3.3. □

The combination of Theorem 5.4 with Lemma 5.3 implies the main result in this section, which establishes additional regularity in space of the solution of the weak formulation (2.11) with correspondingly more regular data.

Corollary 5.5 (Regularity in space)**.** *Let* (f_u, f_p, g_p, ℓ_0) *be as in* (5.8)*. Denote by* $y_1 = (u, p_{tot}, p, m) \in \overline{\mathbb{Y}}_1$ *the solution of the equations* (2.11) *with the data* ℓ_u *and* ℓ_p *defined by* (2.12)*. Then we have* $y_1 \in \overline{\mathbb{X}}_1$ *and* y_1 *fulfills* (5.9)*.*

Proof. Owing to Theorem 5.4, the equations (5.6), with the given data, admit a unique solution $x_1 \in \overline{\mathbb{X}}_1$. By Lemma 5.3, x_1 solves also the equations (2.11). Then, we have $x_1 = y_1$ according to Theorem 3.5. This confirms that y_1 is in $\overline{\mathbb{X}}_1$ and fulfills the estimate (5.9) in Theorem 5.4. □

5.3. **Inf-sup stability.** This section establishes the inf-sup stability (5.12) of the bilinear form b in (5.10). For this purpose, let $\widetilde{x}_1 = (\widetilde{u}, \widetilde{p}_{\mathrm{tot}}, \widetilde{p}, \widetilde{m}) \in \mathbb{X}_1$. We proceed analogously to section 3.1, therefore, we mention only the main aspects of the argument.

Let $s \in [0,T]$ be arbitrary and denote by $\chi_s : [0,T] \to \mathbb{R}$ the indicator function on $[0,s]$. We consider the test function $x_{2,s} = (v, q_{\mathrm{tot}}, q, n, n_\partial, n_0) \in \mathbb{X}_2$ defined by

$$\begin{aligned} v &= \frac{C}{\mu}\Big(\mathcal{E}\widetilde{u} + \mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\Big)\chi_s && \in L^2(L^2(\Omega)^2) \\ q_{\mathrm{tot}} &= \frac{3}{\lambda}\Big(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p}\Big)\chi_s && \in L^2(\mathbb{P}) \\ q &= 9\gamma\Big(\alpha\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m}\Big)\chi_s && \in L^2(\mathbb{P}) \\ n &= \frac{1}{\kappa}\Big(\partial_t\widetilde{m} + \mathcal{L}\widetilde{p} + 2\widetilde{m}\Big)\chi_s && \in L^2(\overline{\mathbb{P}}) \\ n_{\partial\Omega} &= \frac{9}{\gamma}\Big(\mathcal{R}_\partial^{-1}\partial_{\mathsf{n}}\widetilde{p}\Big)\chi_s && \in L^2(H^{-\frac{1}{2}}(\partial\Omega)) \\ n_0 &= \frac{2}{\kappa}\widetilde{m}(0) && \in \overline{\mathbb{P}} \end{aligned} \tag{5.13}$$

with the constant C to be determined later and $\mathcal{R}_\partial : H^{\frac12}(\partial\Omega) \to H^{-\frac12}(\partial\Omega)$ denoting the Riesz isometry.

Using this test function in (5.10) yields

$$\begin{aligned}
b(\widetilde{x}_1, x_{2,s}) = {} & \frac{C}{\mu}\int_0^s \|\mathcal{E}\widetilde{u} + \mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 && =: \mathfrak{I}_1\\
& + \frac{3}{\lambda}\int_0^s \|\mathcal{D}^*(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p})\|_\Omega^2\\
& + 9\gamma\int_0^s \|\mathcal{D}^*(\alpha\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m})\|_\Omega^2\\
& + \frac{1}{\kappa}\int_0^s \|\partial_t\widetilde{m} + \mathcal{L}\widetilde{p}\|_\Omega^2\\
& + \frac{2}{\kappa}\int_0^s (\partial_t\widetilde{m} + \mathcal{L}\widetilde{p},\, \widetilde{m})_\Omega && =: \mathfrak{I}_2\\
& + \frac{9}{\gamma}\int_0^s \|\partial_{\mathfrak{n}}\widetilde{p}\|^2_{H^{-\frac12}(\partial\Omega)}\\
& + \frac{2}{\kappa}\|\widetilde{m}(0)\|_\Omega^2.
\end{aligned} \tag{5.14}$$

We aim bounding $\mathfrak{I}_1$ and $\mathfrak{I}_2$ from below. For the first one, we mimic the argument in the proof of [8, Lemma 2.2]. By [8, Lemma 2.1], there is $v' \in L^2(\mathbb{E})$ such that

$$\mathcal{D}v' = \mathcal{D}\widetilde{u} \quad \text{in } L^2(\mathbb{P}) \qquad \text{and} \qquad \|D^2 v'\|_\Omega \lesssim \|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega \tag{5.15}$$

with the hidden constant depending only on Ω. The identity $\mathcal{E} = -\mu(\Delta + \nabla\mathrm{div})$ in $\mathbb{E}$ and simple manipulations reveal

$$\begin{aligned}
-\mu\Delta(\widetilde{u} - v') + \mathcal{D}^*(\widetilde{p}_{\mathrm{tot}} + \mu\mathcal{D}\widetilde{u}) &= \widetilde{f} + \mu\Delta v' && \text{in } L^2([0,s],\, L^2(\Omega)^2)\\
\mathcal{D}(\widetilde{u} - v') &= 0 && \text{in } L^2([0,s],\, \mathbb{P}).
\end{aligned}$$

with $\widetilde{f} := \mathcal{E}\widetilde{u} + \mathcal{D}^*\widetilde{p}_{\mathrm{tot}}$. By the regularity theory for the Stokes equations [15, Theorem 2] a triangle inequality and (5.15), we have

$$\int_0^s \Big(\mu^2\|D^2\widetilde{u}\|_\Omega^2 + \|\mathcal{D}^*(\widetilde{p}_{\mathrm{tot}} + \mu\mathcal{D}\widetilde{u})\|_\Omega^2\Big) \lesssim \int_0^s \Big(\|\widetilde{f}\|_\Omega^2 + \mu^2\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2\Big).$$

Again, the hidden constant depends only on Ω. For the second term on the left-hand side, we use triangle and Young's inequalities

$$\begin{aligned}
\|\mathcal{D}^*(\widetilde{p}_{\mathrm{tot}} + \mu\mathcal{D}\widetilde{u})\|_\Omega^2 &= \|\mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 + \mu^2\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2 + 2\mu(\mathcal{D}^*\widetilde{p}_{\mathrm{tot}},\, \mathcal{D}^*\mathcal{D}\widetilde{u})_\Omega\\
&\geq \|\mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 + \mu(\mu+\lambda)\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2 - 2\alpha\mu(\mathcal{D}^*\widetilde{p},\, \mathcal{D}^*\mathcal{D}\widetilde{u})_\Omega\\
&\quad - \frac{\mu}{\lambda}\|\mathcal{D}^*(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p})\|_\Omega^2.
\end{aligned}$$

We insert this estimate into the previous one and exploit the assumption (5.1c). Then, assuming that C in (5.13) is sufficiently large, we conclude

$$\begin{aligned}
\mathfrak{I}_1 \geq {} & \int_0^s \Big(- 2\alpha(\mathcal{D}^*\mathcal{D}\widetilde{u},\, \mathcal{D}^*\widetilde{p})_\Omega + \mu\|D^2\widetilde{u}\|_\Omega^2 + (\mu+\lambda)\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2\Big)\\
& + \int_0^s \Big(\frac{1}{\mu}\|\mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 - \frac{1}{\lambda}\|\mathcal{D}^*(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p})\|_\Omega^2\Big).
\end{aligned}$$

Regarding $\mathfrak{I}_2$, the integration by parts rule [13, Lemma 64.40] and the identities (5.3) and (5.5) reveal

$$\mathfrak{I}_2 = \frac{1}{\kappa}\|\widetilde{m}(s)\|_\Omega^2 - \frac{1}{\kappa}\|\widetilde{m}(0)\|_\Omega^2 + 2\int_0^s \Big((\mathcal{D}^*\widetilde{p},\, \mathcal{D}^*\widetilde{m})_\Omega - \langle\partial_{\mathsf{n}}\widetilde{p},\, \widetilde{m}\rangle_{1/2}\Big).$$

We bound the last term on the right-hand side, using (2.15), (5.4) and various Young's inequalities, by

$$\begin{aligned}\langle\partial_{\mathsf{n}}\widetilde{p},\, \widetilde{m}\rangle_{1/2} \le{}& \frac{\sigma}{4}\|\mathcal{D}^*\widetilde{p}\|_\Omega^2 + \frac{\mu+\lambda}{8}\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2 \\ &+ \frac{4}{\gamma}\|\partial_{\mathsf{n}}\widetilde{p}\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \frac{\gamma}{4}\|\mathcal{D}^*(\alpha\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m})\|_\Omega^2.\end{aligned}$$

Next, we perform the same steps as for the derivation of (3.11) in section 3.1

$$\begin{aligned}(\mathcal{D}^*\widetilde{p},\, \mathcal{D}^*\widetilde{m})_\Omega \ge{}& \frac{\sigma}{2}\|\mathcal{D}^*\widetilde{p}\|_\Omega^2 + \alpha(\mathcal{D}^*\mathcal{D}\widetilde{u},\, \mathcal{D}^*\widetilde{p})_\Omega - \frac{\mu+\lambda}{8}\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2 \\ &- \frac{1}{4\mu}\|\mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 - \frac{1}{2\lambda}\|\mathcal{D}^*(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p})\|_\Omega^2 - \frac{7\gamma}{2}\|\mathcal{D}^*(\alpha\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m})\|_\Omega^2.\end{aligned}$$

By inserting these bounds into the previous identity, it results

$$\begin{aligned}\mathfrak{I}_2 \ge{}& \frac{1}{\kappa}\|\widetilde{m}(s)\|_\Omega^2 - \frac{1}{\kappa}\|\widetilde{m}(0)\|_\Omega^2 + \int_0^s \Big(\frac{\sigma}{2}\|\mathcal{D}^*\widetilde{p}\|_\Omega^2 + 2\alpha(\mathcal{D}^*\mathcal{D}\widetilde{u},\, \mathcal{D}^*\widetilde{p})_\Omega\Big) \\ &- \int_0^s \Big(\frac{\mu+\lambda}{2}\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2 + \frac{1}{2\mu}\|\mathcal{D}^*\widetilde{p}_{\mathrm{tot}}\|_\Omega^2 + \frac{8}{\gamma}\|\partial_{\mathsf{n}}\widetilde{p}\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}\Big) \\ &- \int_0^s \Big(\frac{1}{\lambda}\|\mathcal{D}^*(\lambda\mathcal{D}\widetilde{u} - \widetilde{p}_{\mathrm{tot}} - \alpha\widetilde{p})\|_\Omega^2 + 8\gamma\|\mathcal{D}^*(\alpha\mathcal{D}\widetilde{u} + \sigma\widetilde{p} - \widetilde{m})\|_\Omega^2\Big).\end{aligned}$$

The combination of the lower bounds for $\mathfrak{I}_1$ and $\mathfrak{I}_2$ with (5.14) yields

$$2b(\widetilde{x}_1, x_{2,s}) \ge \|\!|\widetilde{x}_1|\!\|_1^2 + \frac{1}{\kappa}\|\widetilde{m}(s)\|_\Omega^2 + \int_0^s \Big(\lambda\|\mathcal{D}^*\mathcal{D}\widetilde{u}\|_\Omega^2 + \sigma\|\mathcal{D}^*\widetilde{p}\|_\Omega^2\Big).$$

Replacing σ by γ^{-1} can be done as in the proof of Proposition 4.1. This establishes (5.12) upon taking the test function $x_2 = x_{2,\overline{s}} + x_{2,T}$, where $\overline{s}$ is such that $\|\widetilde{m}(\overline{s})\|_\Omega = \|\widetilde{m}\|_{L^\infty(\mathbb{P})}$.

Finally, for all $s \in [0, T]$, the norm of $x_{2,s}$ is bounded by

$$\|\!|x_{2,s}|\!\|_2^2 \lesssim \|\!|\widetilde{x}_1|\!\|_1^2 + T\|\widetilde{m}\|^2_{L^\infty(\mathbb{P})}$$

and the hidden constant does not depend on s. The proof is identical to the corresponding one at the end of section 3.1. The combination of this estimate with the previous lower bound establishes (5.12).

5.4. **Nondegeneracy.** This section establishes the nondegeneracy of the bilinear form b in (5.10). For this purpose, assume

$$b(\widetilde{x}_1, x_2) = 0 \tag{5.16}$$

for all $\widetilde{x}_1 \in \overline{\mathbb{X}}_1$ and for some $x_2 = (v, q_{\mathrm{tot}}, q, n, n_\partial, n_0) \in \mathbb{X}_2$. We proceed in analogy with section 3.2 in order to verify $x_2 = 0$. Therefore, we mention only the main aspects of the argument.

Taking $\widetilde{x}_1 = (0, \widetilde{p}_{\mathrm{tot}}, 0, 0)$, the identity (5.16) implies

$$\int_0^T (\mathcal{D}^*\widetilde{p}_{\mathrm{tot}},\, v)_\Omega = \int_0^T (\mathcal{D}^*\widetilde{p}_{\mathrm{tot}},\, \mathcal{D}^*q_{\mathrm{tot}})_\Omega \tag{5.17}$$

for all $\widetilde{p}_{\mathrm{tot}} \in L^2(\mathbb{P})$.

Taking $\widetilde{x}_1 = (\widetilde{u}, 0, 0, 0)$, the identity (5.16), the inclusion $\mathcal{D}\widetilde{u} \in L^2(\mathbb{P})$ and (5.17) imply

$$\int_0^T (\mathcal{Q}\widetilde{u},\, v)_\Omega = -\int_0^T \alpha(\mathcal{D}^*\mathcal{D}\widetilde{u},\, \mathcal{D}^* q)_\Omega$$

for all $\widetilde{u} \in L^2(\mathbb{E})$, with $\mathcal{Q} = \mathcal{E} + \lambda\mathcal{D}^*\mathcal{D}$ the operator involved in the displacement formulation of the linear elasticity equations. According to [8, Lemma 2.2], $\mathcal{Q}$ is a one-to-one mapping from $\mathbb{E}$ to $L^2(\Omega)^2$. Thus, the adjoint $\mathcal{Q}^\star : L^2(\Omega)^2 \to \mathbb{E}^*$ is invertible. Here '$\star$' denotes the ajoint with respect to the $L^2(\Omega)$-scalar product, while '$*$' indicates the one with respect to the duality $\langle\cdot,\cdot\rangle_{\mathbb{U}}$. Thus,

$$v = -\alpha\mathcal{Q}^{-\star}(\mathcal{D}^*\mathcal{D})^\star\mathcal{D}^* q \qquad \text{in } L^2(L^2(\Omega)^2). \tag{5.18}$$

Taking $\widetilde{x}_1 = (0, 0, \widetilde{p}, 0)$, the identities (5.3), (5.5) and (5.16) imply

$$\int_0^T \Big((\mathcal{L}\widetilde{p},\, \kappa n - \alpha q_{\mathrm{tot}} + \sigma q)_\Omega + \kappa \left\langle \partial_{\mathsf{n}}\widetilde{p},\, n_\partial - \alpha q_{\mathrm{tot}} + \sigma q\right\rangle_{1/2} \Big) = 0$$

for all $\widetilde{p} \in L^2(\mathbb{L})$. We consider, in particular, the solution $\widetilde{p}$ of the Neumann problem

$$\begin{aligned} \mathcal{L}\widetilde{p} &= \kappa n - \alpha q_{\mathrm{tot}} + \sigma q && \text{in } L^2(\overline{\mathbb{P}}) \\ \partial_{\mathsf{n}}\widetilde{p} &= \mathcal{R}_\partial(n_\partial - \alpha q_{\mathrm{tot}} + \sigma q) && \text{in } L^2(H^{-\frac{1}{2}}(\partial\Omega)) \end{aligned}$$

where $\mathcal{R}_\partial : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is the Riesz isometry. We infer $n \in L^2(\mathbb{P})$ with

$$\kappa n = \alpha q_{\mathrm{tot}} - \sigma q \quad \text{in } L^2(\mathbb{P}) \qquad \text{and} \qquad \kappa n_{|\partial\Omega} = n_\partial \quad \text{in } L^2(H^{\frac{1}{2}}(\Omega)). \tag{5.19}$$

Finally, taking $\widetilde{x}_1 = (0, 0, 0, \widetilde{m})$, the identities (5.3) and (5.16) imply

$$\int_0^T \Big(-\frac{1}{\kappa} \left\langle \widetilde{m},\, \mathcal{L}q\right\rangle_{\mathbb{P}} + \left\langle \partial_t\widetilde{m},\, n\right\rangle_{\mathbb{P}} \Big) + (\widetilde{m}(0),\, n_0)_\Omega = 0$$

for all $\widetilde{m} \in L^2(\mathbb{P}) \cap H^1(\overline{\mathbb{P}})$. Thus, arguing as in the derivation of (3.17)-(3.18), we infer $n \in H^1(\mathbb{P}^*)$ with

$$\partial_t n = -\frac{1}{\kappa}\mathcal{L}q \quad \text{in } L^2(\mathbb{P}^*) \qquad \text{and} \qquad n(0) = n_0,\ n(T) = 0 \quad \text{in } \overline{\mathbb{P}}. \tag{5.20}$$

We combine (5.19) with (5.20), and exploit also (5.3), (5.17) as well as (5.18)

$$\begin{aligned} -\frac{\kappa}{2}\|n_0\|_\Omega^2 &= \kappa \int_0^T \left\langle \partial_t n,\, n\right\rangle_{\mathbb{P}} = \frac{1}{\kappa}\int_0^T \left\langle \mathcal{L}q,\, \sigma q - \alpha q_{\mathrm{tot}}\right\rangle_{\mathbb{P}} \\ &= \int_0^T \Big(\sigma\|\mathcal{D}^* q\|_\Omega^2 + \alpha^2(\mathcal{D}^*\mathcal{D}\mathcal{Q}^{-1}\mathcal{D}^* q,\, \mathcal{D}^* q)_\Omega \Big). \end{aligned} \tag{5.21}$$

We deal with the second term on the right-hand side by following once again the proof of [8, Lemma 2.2]. Recalling $\mathcal{Q} = -\mu\Delta + (\mu + \lambda)\mathcal{D}^*\mathcal{D}$ in $\mathbb{E}$, it follows that

$$\int_0^T (\mathcal{D}^*\mathcal{D}\mathcal{Q}^{-1}\mathcal{D}^* q,\, \mathcal{D}^* q)_\Omega = \frac{1}{\mu + \lambda}\int_0^T \Big(\|\mathcal{D}^* q\|_\Omega^2 + \mu(\Delta u',\, \mathcal{D}^* q)_\Omega \Big) \tag{5.22}$$

with $u' := \mathcal{Q}^{-1}\mathcal{D}^* q \in L^2(\mathbb{E})$. By [8, Lemma 2.1], there is $v' \in L^2(\mathbb{E})$ such that

$$\mathcal{D}v' = \mathcal{D}u' \quad \text{in } L^2(\mathbb{P}) \qquad \text{and} \qquad \|D^2 v'\|_\Omega \lesssim \|\mathcal{D}^*\mathcal{D}u'\|_\Omega$$

with the hidden constant depending only on Ω. Then, the expression of $\mathcal{Q}$ recalled above and simple manipulations reveal

$$\begin{aligned} -\mu\Delta(u'-v') + \mathcal{D}^*((\mu+\lambda)\mathcal{D}u' - q) &= \mu\Delta v' && \text{in } L^2(L^2(\Omega)^2) \\ \mathcal{D}(u'-v') &= 0 && \text{in } L^2(\mathbb{P}). \end{aligned}$$

By the regularity theory for the Stokes equations [15, Theorem 2] and a triangle inequality, we have

$$\begin{aligned} \mu\|D^2 u'\|_\Omega + \|\mathcal{D}^*((\mu+\lambda)\mathcal{D}u' - q)\|_\Omega &\lesssim \mu\|\mathcal{D}^*\mathcal{D}u'\|_\Omega \\ &\leq \frac{\mu}{\mu+\lambda}\Big(\|\mathcal{D}^* q\| + \|\mathcal{D}^*((\mu+\lambda)\mathcal{D}u' - q)\|_\Omega\Big). \end{aligned}$$

Again, the hidden constant depends only on Ω. By virtue of (5.1c), this actually implies $\mu\|\Delta u'\|_\Omega \lesssim \frac{\mu}{\mu+\lambda}\|\mathcal{D}^* q\|_\Omega$. We insert this bound and (5.22) into (5.21). Hence, recalling again assumption (5.1c), we obtain

$$\Big(\sigma + \frac{C\alpha^2}{\mu+\lambda}\Big)\int_0^T \|\mathcal{D}^* q\|_\Omega^2 + \frac{\kappa}{2}\|n_0\|_\Omega^2 = 0.$$

for some positive constant $C > 0$. We conclude $x_2 = 0$ in (5.16) by this identity and (5.18)-(5.20).

6. Conclusions and outlook

We have proposed a new approach and a corresponding setting for the analysis of the quasi-static Biot's equations in poroelasticity. In passing, we have relaxed the regularity assumptions on the data formulated in previous references. The results here are instrumental and tailored to our main goals, that are the design and the analysis of discretizations enjoying accurate and robust error bounds. To this end, we propose in [17] a class of discretizations inspired by the four-field formulation (2.3) and prove its stability by mimicking the technique introduced in section 3. The stability estimates in Theorem 3.5 and Proposition 4.1 provide a possible starting point for the a posteriori error analysis. The regularity result in section 5 is instrumental to the a priori error analysis in [17], as it establishes the regularity that is needed to infer first-order convergence in space. Both the a posteriori analysis and the derivation of further regularity results, e.g. relaxing the assumptions (5.1) or addressing the regularity in time, may be the subject of future investigation.

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TU Dortmund, Fakultät für Mathematik, D-44221 Dortmund, Germany
Email address: `christian.kreuzer@tu-dortmund.de`

Università degli Studi di Pavia, Dipartimento di Matematica, 27100, Pavia, Italy
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