Nonparametric Analysis of Covariance – the Case of Inhomogeneous and Heteroscedastic Noise

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Abstract

The purpose of this paper is to propose a procedure for testing the equality of several regression curves f_i in nonparametric regression models when the noise is inhomogeneous. This extends work of Dette and Neumeyer (2001) and it is shown that the new test is asymptotically uniformly more powerful. The presented approach is very natural because it transfers the maximum likelihood statistic from a heteroscedastic one way ANOVA to the context of nonparametric regression. The maximum likelihood estimators will be replaced by kernel estimators of the regression functions f_i. It is shown that the asymptotic distribution of the obtained test statistic is nuisance parameter free. Finally, for practical purposes a bootstrap variant is suggested. In a simulation study, level and power of this test will be briefly investigated. In summary, our theoretical findings are supported by this study.

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1 Introduction

A classical theme of econometric analysis is the comparison of two (or more) groups, which were measured under different experimental conditions. As an example consider for instance the comparison of wage functions in different groups defined by gender or location (see Lavergne, 2001, for more examples). In order to simplify notation we will restrict for the moment to the case of two groups, the extension to three and more groups will be presented later on. In the context of regression one observes independent real valued data Y_{ij}, which follow the model

$$Y_{ij} = f_i(t_{ij}) + \sigma_i(t_{ij})\varepsilon_{ij}\,, \qquad j = 1, \ldots, n_i \quad (i = 1, 2), \tag{1}$$

where t_{ij} are fixed locations of measurements, f_i denotes the unknown regression function,

$$f_i(t_{ij}) \;=\; E[Y_{ij}],$$

and σ_i^2 the unknown variance function,

$$\sigma_i^2(t_{ij}) \;=\; \mathrm{Var}(Y_{ij})$$

of the i-th group, respectively ($i = 1, 2$). The errors ε_{ij} are assumed to be independent identically distributed random variables with mean 0 and variance 1. Our aim is to test the equality of the regression functions f_1 and f_2.

Under a parametric assumption on the error ε_{ij} and the functions f_i and σ_i^2 this leads to the Analysis of Covariance (see Scheffé, 1959, or Chow, 1960). Without these assumptions, in particular when the functional form of f_i is not specified, this is denoted as nonparametric analysis of covariance (Young and Bowman, 1995) and has received much attention during the last years (see Hall and Hart, 1990; Delgado, 1993; Kulasekera, 1995; Munk and Dette, 1998; or Yatchew, 1999, among many others). As pointed out by Gørgens (2002) many tests in the literature for

$$H_0 : f_1 = f_2 \quad \text{versus} \quad H_1 : f_1 \neq f_2 \tag{2}$$

cannot be applied in the general model (1) because often it is assumed that sample sizes are equal, the regressors follow the same distribution between populations, or that there is a homoscedastic error, i. e. the variances σ_i^2 are independent of the regressor t. For the general setting (1) there are only a very few tests available, see Cabus (1998), Dette and Neumeyer (2001), Lavergne (2001), Gørgens (2002) and Neumeyer and Dette (2003). Whereas Lavergne (2001) and Gørgens (2002) consider a stochastic regressor, Cabus (1998) and Neumeyer and Dette (2003) use test statistics, which are based on the associated marked empirical process. The presented method is most similar in spirit to Dette and Neumeyer (2001). These authors compared theoretically as well as in Monte Carlo study their test with various tests from the literature and came to the conclusion that their test outperforms their competitors in terms of power. In this paper we present a test, which will be shown to be superior to Dette and Neumeyer's (2001) test with respect to power.

More specifically, our test is based on the idea to compare a weighted "least squares" estimator under the assumption of equal regression curves with an estimator, which is based on nonparametric estimators $\hat{f}_i$ for f_i, exactly as in a parametric analysis of covariance. To motivate the procedure assume for the moment the regression functions to be constant $f_i(t) \equiv \mu_i$, the variance functions to be constant and known $\sigma_i^2(t) \equiv \sigma_i^2$ and the errors ε_{ij} to be normally distributed. In other words consider testing the equality of the means $H_0 : \mu_1 = \mu_2$ in two samples

$$Y_{ij} \text{ i.i.d. } \sim N(\mu_i, \sigma_i^2), \quad j = 1, \ldots, n_i \ (i = 1, 2).$$

The maximum likelihood method leads to the estimates $\hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}$ in the individual samples $(i = 1, 2)$, respectively, and

$$\hat{\mu} = a\hat{\mu}_1 + (1-a)\hat{\mu}_2, \quad \text{where} \quad a = \frac{\sigma_1^{-2} n_1}{\sigma_1^{-2} n_1 + \sigma_2^{-2} n_2},$$

in the pooled sample (under H_0). The logarithm of the likelihood ratio has the form

$$\frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} (Y_{ij} - \hat{\mu})^2 \sigma_i^{-2} - \frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} (Y_{ij} - \hat{\mu}_i)^2 \sigma_i^{-2}, \tag{3}$$

where $N = n_1 + n_2$ denotes the total sample size. Now we transfer this statistic to a nonparametric set up and consider in the nonparametric regression model (1) the class of pooled estimators

$$\tilde{f}(x) \ = \ a(x)\hat{f}_1(x) + (1 - a(x))\hat{f}_2(x), \tag{4}$$

where $\hat{f}_i$ denote kernel based estimators of the regression functions f_i $(i = 1, 2)$. In this class, minimization of the asymptotic MSE

$$\text{AMISE}[\tilde{f}] \ = \ a^2(x) \int K^2(u)\, du \, \frac{\sigma_1^2(x)}{n_1 h r_1(x)} + (1 - a(x))^2 \int K^2(u)\, du \, \frac{\sigma_2^2(x)}{n_2 h r_2(x)},$$

where h denotes a smoothing parameter that fulfils conditions (11) stated in the next section, and K denotes a kernel function, gives the weight

$$a(x) = \frac{\sigma_1^{-2}(x) n_1 r_1(x)}{\sigma_1^{-2}(x) n_1 r_1(x) + \sigma_2^{-2}(x) n_2 r_2(x)}. \tag{5}$$

Now we replace σ_i^2 and r_i by appropriate kernel based estimators $\hat{\sigma}_i^2$, $\hat{r}_i$ $(i = 1, 2)$ and denote by $\hat{f}$ the resulting pooled estimator $\tilde{f}$ as in (4). As a test statistic for the hypotheses (2) we consider in analogy of (3),

$$T_N \ = \ \frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} (Y_{ij} - \hat{f}(t_{ij}))^2 \hat{\sigma}_i^{-2}(t_{ij}) - \frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} (Y_{ij} - \hat{f}_i(t_{ij}))^2 \hat{\sigma}_i^{-2}(t_{ij}). \tag{6}$$

We will show that under the null hypothesis the standardized test statistic

$$N\sqrt{h}\Big(T_N - \frac{C}{Nh}\Big)$$

is asymptotically centered normal with a variance, which only depends on the kernel function K, as well as C does. This might be particularly appealing because, hence, asymptotically the resulting test does not depend on any nuisance parameter, such as f_i, σ_i^2 or the distribution of the ε_{ij}, in contrast to most procedures suggested in the literature (a notable exception is Gørgens, 2002).
The rest of the paper is organized as follows. In section 2 we present the required theory. The asymptotic behaviour under fixed and local alternatives is discussed and it is shown that the test of Dette and Neumeyer (2001) is outperformed in general. Only in special cases asymptotically these tests achieve the same power. We show in particular, when the variances are inhomogeneous, i. e. unequal in both groups, or when they are heteroscedastic, i. e. dependent of the regressor, the new test gains significantly in power. We mention that from a practical point of view the case of inhomogeneous variances is very common in applications. For ANOVA models this is well known as the celebrated Behrens–Fisher problem (see for example Weerahandi, 1987), in our context of nonparametric analysis of covariance we refer to Gørgens (2002) for an econometric example. Hence our method may be regarded as an approach, which adapts automatically to inhomogeneous and heteroscedastic variability.
In section 3 the present setting is extended to random regressors and the k–sample case. In section 4 a wild bootstrap variant of the test is proposed, and a numerical study illustrates the performance of our method. Section 5 contains some concluding remarks. Proofs are postponed to an Appendix in order to keep the paper more readable.

2 Asymptotic Theory

2.1 Notation and Main Results

We start with various technical assumptions required throughout this section. We assume model (1), where the fixed design points t_{ij} can be modelled by a so called design density r_i on $[0, 1]$ such that

$$\int_0^{t_{ij}} r_i(t)dt = \frac{j}{n_i}, \qquad j = 1, \ldots, n_i \quad (i = 1, 2), \tag{7}$$

see Sacks and Ylvisaker (1970). We further assume the densities r_i and the variance functions σ_i^2 to be bounded away from zero, i. e.

$$\inf_{t\in[0,1]} r_i(t) > 0\,, \qquad \inf_{t\in[0,1]} \sigma_i^2(t) > 0 \quad (i = 1, 2). \tag{8}$$

The densities, regression and variance functions are assumed to be d–times continuously differentiable, i. e.

$$r_i, f_i, \sigma_i \in C^d(0, 1) \quad (i = 1, 2), \tag{9}$$

where $d \geq 2$. As mentioned in the Introduction our approach is based on kernel estimators of f_i and σ_i^2. To this end we require a symmetrical kernel $K : I\!R \to I\!R$, which is compactly supported and of order d (cf. Gasser, Müller and Mammitzsch, 1985), i. e.

$$\frac{(-1)^j}{j!}\int K(u)u^j\,du = \begin{cases} 1 & : \quad j=0 \\ 0 & : \quad 1\le j\le d-1 \\ k_d\neq 0 & : \quad j=d \end{cases}, \qquad \int K^2(u)\,du<\infty. \tag{10}$$

Let $h=h_N$ denote a sequence of bandwidths, such that

$$Nh^{2d}\to 0 \quad \text{and} \quad Nh^2\to\infty \quad \text{for} \quad N\to\infty, \tag{11}$$

where $N=n_1+n_2$ denotes the total sample size. Further we assume that the sample sizes in each group are of the same order, i. e.

$$\frac{n_i}{N} \;=\; \kappa_i+O(\frac{1}{N}) \quad (i=1,2), \tag{12}$$

where $\kappa_i\in(0,1)$. In the following we require various estimators for r_i, f_i and σ_i^2. In order to be concise, the theory will be presented for Nadaraya–Watson type estimators. However, we mention that local polynomial estimators of higher order will work as well, of course, and due to their better performance at the boundary of the regressor space even better performance is to be expected (Fan and Gijbels, 1996). However, because the suggested test statistic is an integrated quantity of these function estimators, the boundary behaviour will be of minor importance in the present context. In order to estimate the design densities r_i we use

$$\hat r_i(x) \;=\; \frac{1}{n_ih}\sum_{j=1}^{n_i}K\Big(\frac{x-t_{ij}}{h}\Big), \tag{13}$$

which yields an estimator for f_i,

$$\hat f_i(x) \;=\; \frac{1}{n_ih}\sum_{j=1}^{n_i}K\Big(\frac{x-t_{ij}}{h}\Big)Y_{ij}\,\frac{1}{\hat r_i(x)} \quad (i=1,2). \tag{14}$$

For the test statistic T_N defined in (6) a pooled estimator of f is required (when $f_1=f_2=f$), which is

$$\hat f(x) \;=\; \frac{\sum_{i=1}^2\sum_{j=1}^{n_i}K(\frac{x-t_{ij}}{h})Y_{ij}\,\hat\sigma_i^{-2}(x)}{\sum_{i=1}^2\sum_{j=1}^{n_i}K(\frac{x-t_{ij}}{h})\hat\sigma_i^{-2}(x)}. \tag{15}$$

Note that $\hat f$ equals $\tilde f$ defined in (4) using the weights (5) with estimators (13) and (14), that is

$$\hat f(x) \;=\; \hat a(x)\hat f_1(x)+(1-\hat a(x))\hat f_2(x), \text{ where } \hat a(x)=\frac{\hat\sigma_1^{-2}(x)n_1\hat r_1(x)}{\hat\sigma_1^{-2}(x)n_1\hat r_1(x)+\hat\sigma_2^{-2}(x)n_2\hat r_2(x)}.$$

To this end the variances σ_i^2 have to be estimated by a nonparametric estimator, which is similar in spirit to Ruppert, Wand, Holst and Hössler (1997), Fan and Yao (1998) or Härdle and Tsybakov (1998). In the present context we define

$$\hat\sigma_i^2(x) \;=\; \frac{1}{n_ih}\sum_{j=1}^{n_i}K\Big(\frac{x-t_{ij}}{h}\Big)(Y_{ij}-\hat f_i(t_{ij}))^2\frac{1}{\hat r_i(x)} \quad (i=1,2). \tag{16}$$

The following theorem gives the asymptotic distribution of the test statistic T_N.

Theorem 2.1 *Assume model (1), where the ε_{ij} are i.i.d. centered random variables with variance $\mathrm{Var}(\varepsilon_{ij}) = 1$ and $E[\varepsilon_{ij}^4] < \infty$. Then under the assumptions (7)–(12) and $H_0 : f_1 = f_2 = f$, for T_N defined in (6) it holds that*

$$N\sqrt{h}\Big(T_N - \frac{C}{Nh}\Big) \xrightarrow[N \to \infty]{\mathcal{D}} \mathcal{N}(0, \tau^2)\,,$$

where $\mathcal{N}(0, \tau^2)$ denotes a centered normal random variable with variance

$$\tau^2 \;=\; 2\int (2K - K * K)^2(u)\,du.$$

The constant C is defined as $C = 2K(0) - \int K^2(u)\,du$.

In order to test the hypotheses stated in (2), one rejects H_0 at nominal level α, whenever

$$\frac{N\sqrt{h}\,\big(T_N - \frac{C}{Nh}\big)}{\tau} \;>\; u_{1-\alpha} \tag{17}$$

where $u_{1-\alpha} = \Phi^{-1}(1 - \alpha)$ denotes the $(1 - \alpha)$-quantile of the standard normal distribution. Note, that C and τ are known constants. The consistency of the testing procedure (17) against any nonparametric alternative follows from the next result.

Theorem 2.2 *Assume that $f_1 \neq f_2$ on a set of positive Lebesgue measure. Under the assumptions of Theorem 2.1 we have*

$$\sqrt{N}\,(T_N - \mu) \xrightarrow[N \to \infty]{\mathcal{D}} \mathcal{N}(0, \gamma^2)\,,$$

where

$$\mu \;=\; \int (f_1 - f_2)^2(x) \frac{\kappa_1 r_1(x) \kappa_2 r_2(x)}{\sigma_2^2(x)\kappa_1 r_1(x) + \sigma_1^2(x)\kappa_2 r_2(x)}\,dx \tag{18}$$

and $\gamma^2 = 4\,\mu$.

Theorem 2.2 can be utilized in various ways. First a power approximation can be obtained via

$$\begin{aligned} P_{H_1}\left(\frac{N\sqrt{h}\,\big(T_N - \frac{C}{Nh}\big)}{\tau} > u_{1-\alpha}\right) \;&=\; \Phi\left(\frac{\mu\sqrt{N}}{\gamma} - \frac{\tau u_{1-\alpha}}{\gamma\sqrt{Nh}} - \frac{C}{\gamma\sqrt{Nh}}\right) + o(1) \\ &=\; \Phi\left(\frac{\mu}{\gamma}\sqrt{N}\right) + o(1). \end{aligned} \tag{19}$$

We will use this result in the next section in order to compare the presented test with a procedure of Dette and Neumeyer (2001) in terms of power, see Lemma 2.3.

Second a simple $1-\alpha$ confidence interval for the discrepancy measure μ between f_1 and f_2 in (18) is obtained as $(0 < \alpha < \frac{1}{2})$

$$CI_{1-\alpha} \;=\; \left[0, T_N + \sqrt{T_N c + \frac{c^2}{4}} + \frac{c}{2}\right] \tag{20}$$

where $c = 4u^2_{1-\frac{\alpha}{2}}/N$. To this end observe that $\mu \geq 0$ always and hence for $T_N < 0$ the inequality $(T_N - \mu)/\sqrt{\mu} > 2u_{1-\frac{\alpha}{2}}/\sqrt{N}$ has no solution. The confidence interval (20) might be of some practical appeal because it gives more accurate insight in *how much* the true regression functions f_1, f_2 deviate from equality in terms of the discrepancy measure μ. In contrast, a simple decision based on (17) leaves the experimenter in the difficult situation whether rejection of H_0 is based on a *significantly* relevant difference between f_1 and f_2, or in the case of acceptance, whether there is really evidence in favour of $f_1 = f_2$ or just a lack of power, e. g. due to too small sample sizes. For a careful discussion of these issues cf. Munk and Dette (1998). Similarly, Theorem 2.2 allows testing precise L^2-neighbourhoods

$$H_{\Delta_0} : \mu > \Delta_0 \quad \text{versus} \quad K_{\Delta_0} : \mu \leq \Delta_0$$

where Δ_0 is a preassigned discrepance the experimenter is willing to tolerate.

Finally, we mention that the test in (17) can detect local alternatives of the form

$$H_{1_N} \;:\; f_1 = f_2 + \frac{g}{(N\sqrt{h})^{1/2}}\,, \tag{21}$$

where $g \in C^d(0,1)$, that tend to the null hypothesis at a rate $1/(N\sqrt{h})^{1/2}$. Under the local alternatives H_{1_N} the test statistic $N\sqrt{h}(T_N - \frac{C}{Nh})$ converges in distribution to a normal distribution $\mathcal{N}(\Delta, \tau^2)$ with mean

$$\Delta = \int g^2(x) \frac{\kappa_1 r_1(x) \kappa_2 r_2(x)}{\sigma_2^2(x)\kappa_1 r_1(x) + \sigma_1^2(x)\kappa_2 r_2(x)}\, dx.$$

The constants C and τ^2 are defined in Theorem 2.1. Under (21) we obtain the following approximation of the power,

$$P_{H_{1_N}}\left(N\sqrt{h}\left(T_N - \frac{C}{Nh}\right) > \tau u_{1-\alpha}\right) \;=\; \Phi\left(\frac{\Delta}{\tau} - u_{1-\alpha}\right) + o(1). \tag{22}$$

2.2 Comparison with a procedure of Dette and Neumeyer (2001)

The presented test statistic T_N is an enhancement of Dette and Neumeyer's (2001) test statistic

$$T_N^{(1)} \;=\; \frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(Y_{ij} - \tilde{f}(t_{ij}))^2 - \frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(Y_{ij} - \hat{f}_i(t_{ij}))^2, \tag{23}$$

where the pooled regression estimator is defined as

$$\tilde{f}(x) \;=\; \frac{\sum_{i=1}^{2}\sum_{j=1}^{n_i} K(\frac{x-t_{ij}}{h})Y_{ij}}{\sum_{i=1}^{2}\sum_{j=1}^{n_i} K(\frac{x-t_{ij}}{h})}.$$

$T_N^{(1)}$ does not take into account the potentially different variance functions in the two samples. The combined regression estimator $\tilde{f}$ and the test statistic $T_N^{(1)}$ conform the definitions of $\hat{f}$ in (15) and T_N in (6) but with replacing the variance estimates $\hat{\sigma}_i^2(\cdot)$ by the constant value 1 ($i = 1, 2$). Under the assumptions of the Theorems 2.1 and 2.2 the statistic $T_N^{(1)}$ has an asymptotic normal law, similar to T_N, but with different constants, i. e.

$$N\sqrt{h}\Big(T_N^{(1)} - \frac{\widetilde{C}}{Nh}\Big) \;\xrightarrow[N\to\infty]{\mathcal{D}}\; \mathcal{N}(0, \widetilde{\tau}^2) \quad (\text{under } H_0)$$

$$\sqrt{N}\,\Big(T_N^{(1)} - \widetilde{\mu}\Big) \;\xrightarrow[N\to\infty]{\mathcal{D}}\; \mathcal{N}(0, \widetilde{\gamma}^2) \quad (\text{under } H_1),$$

where

$$\begin{aligned}
\widetilde{C} &= \Big[2K(0) - \int K^2(u)\,du\Big]\Big(\int \sigma_1^2(x)\,dx + \int \sigma_2^2(x)\,dx - \int \frac{\sigma_1^2(x)\kappa_1 r_1(x) + \sigma_2^2(x)\kappa_2 r_2(x)}{\kappa_1 r_1(x) + \kappa_2 r_2(x)}\,dx\Big)\\
\widetilde{\tau}^2 &= 2\int (2K - K * K)^2(u)\,du \int \frac{(\sigma_2^2(x)\kappa_1 r_1(x) + \sigma_1^2(x)\kappa_2 r_2(x))^2}{(\kappa_1 r_1(x) + \kappa_2 r_2(x))^2}\,dx\\
\widetilde{\mu} &= \int (f_1 - f_2)^2(x)\frac{\kappa_1 r_1(x)\kappa_2 r_2(x)}{\kappa_1 r_1(x) + \kappa_2 r_2(x)}\,dx\\
\widetilde{\gamma}^2 &= 4\int (f_1 - f_2)^2(x)\frac{\kappa_1 r_1(x)\kappa_2 r_2(x)(\sigma_2^2(x)\kappa_1 r_1(x) + \sigma_1^2(x)\kappa_2 r_2(x))}{(\kappa_1 r_1(x) + \kappa_2 r_2(x))^2}\,dx.
\end{aligned}$$

The power approximation (19) (which is analogously valid for $T_N^{(1)}$) motivates that a large value of the ratio of the mean to the asymptotic standard deviation under the alternative yields large power. This gives us the possibility to compare the two competing procedures and leads to the following result.

Lemma 2.3 *Under the assumptions of Theorem 2.2 we obtain for the asymptotic signal to noise ratio of T_N and $T_N^{(1)}$ that*

$$\frac{\widetilde{\mu}}{\widetilde{\gamma}} \;\le\; \frac{\mu}{\gamma}. \tag{24}$$

Proof. From Cauchy–Schwarz's inequality we obtain

$$\begin{aligned}
\widetilde{\mu} &= \int (f_1 - f_2)^2(x) \frac{\kappa_1 r_1(x)\kappa_2 r_2(x)}{\kappa_1 r_1(x) + \kappa_2 r_2(x)}\, dx \\
&\leq \Big(4 \int (f_1 - f_2)^2(x) \frac{\kappa_1 r_1(x)\kappa_2 r_2(x)(\sigma_2^2(x)\kappa_1 r_1(x) + \sigma_1^2(x)\kappa_2 r_2(x))}{(\kappa_1 r_1(x) + \kappa_2 r_2(x))^2}\, dx\Big)^{1/2} \\
&\quad \times \Big(\frac{1}{4} \int (f_1 - f_2)^2(x) \frac{\kappa_1 r_1(x)\kappa_2 r_2(x)}{\sigma_2^2(x)\kappa_1 r_1(x) + \sigma_1^2(x)\kappa_2 r_2(x)}\, dx\Big)^{1/2} \\
&= \widetilde{\gamma}\, (\frac{1}{4}\mu)^{1/2} \;=\; \widetilde{\gamma}\, \frac{\mu}{\gamma}
\end{aligned}$$

■

It follows from the Cauchy–Schwarz inequality that one obtains equality in (24) if and only if there exists a constant c such that a. e.

$$\frac{\sigma_2^2 \kappa_1 r_1 + \sigma_1^2 \kappa_2 r_2}{\kappa_1 r_1 + \kappa_2 r_2} \;\equiv\; c.$$

Essentially this holds in the case of homoscedastic and equal variances in the two samples or in the case of equal design densities and homoscedastic variances.

From Lemma 2.3 we see also, that Dette and Neumeyer's (2001) statistic becomes inefficient compared to our approach, when μ/γ is large compared to $\tilde{\mu}/\tilde{\gamma}$. As an example assume that $\kappa_1 = \kappa_2 = \frac{1}{2}$ (equal sample sizes), $r_i \equiv 1$ (uniform designs) and let $f_1 - f_2 \equiv 1$. Then $\tilde{\mu} = \frac{1}{4}$, $\tilde{\gamma} = \frac{1}{\sqrt{2}}\{\int (\sigma_1^2(x) + \sigma_2^2(x))\, dx\}^{1/2}$, $\mu = \frac{1}{2}\int (\sigma_1^2(x) + \sigma_2^2(x))^{-1}\, dx$, $\mu/\gamma = \frac{1}{2}\sqrt{\mu}$. Hence inequality (24) in Lemma 2.3 becomes equivalent to

$$\Big(\int (\sigma_1^2(x) + \sigma_2^2(x))\, dx\Big)^{-1/2} \;\leq\; \Big(\int (\sigma_1^2(x) + \sigma_2^2(x))^{-1}\, dx\Big)^{1/2}.$$

For example, if $\sigma_1^2(x) + \sigma_2^2(x) = x$, the r. h. s. is infinity, and it is expected that in this case our test outperforms the test by Dette and Neumeyer (2001) significantly. We will investigate this more detailed in section 4 where a simulation study is presented.

Under the local alternatives H_{1_N} considered in (21) the statistic $T_N^{(1)}$ of Dette and Neumeyer (2001) shows a similar behaviour like T_N but with asymptotic variance $\widetilde{\tau}^2$ and mean

$$\widetilde{\Delta} = \int g^2(x) \frac{\kappa_1 r_1(x)\kappa_2 r_2(x)}{\kappa_1 r_1(x) + \kappa_2 r_2(x)}\, dx.$$

Due to the power approximation in (22) an inequality of the form

$$\frac{\widetilde{\Delta}}{\widetilde{\tau}} \;\leq\; \frac{\Delta}{\tau}$$

like in Lemma 2.3 for local alternatives would be desirable but is not valid in general.

3 Extensions

3.1 Random Design

In the random design case the design points t_{ij} $(j = 1, \ldots, n_i)$ are i.i.d. realisations of a random variable X_i with design density r_i $(i = 1, 2)$. In this setting the asymptotic distribution under the null hypothesis H_0 stated in Theorem 2.1 remains valid; but under the fixed alternative H_1 the asymptotic variance changes to

$$\gamma^2 + \sum_{i=1}^{2} \kappa_i \mathrm{Var}\Big((f_1 - f_2)^2(X_i) \frac{\kappa_{3-i}^2 r_{3-i}^2(X_i)\sigma_i^4(X_i) + 2\kappa_i r_i(X_i)\kappa_{3-i} r_{3-i}(X_i)\sigma_{3-i}^4(X_i)}{(\kappa_1 r_1(X_i)\sigma_2^2(X_i) + \kappa_2 r_2(X_i)\sigma_1^2(X_i))^2}\Big)$$

where γ^2 is defined in Theorem 2.2.

3.2 Bandwidths and additional prior information on the variances

All results can be generalized to the use of different bandwidths in the three regression estimates, i. e. a bandwidth h_i in $\hat{f}_i(\cdot)$ defined in (14), $i = 1, 2$, and a bandwidth h in the pooled estimator $\hat{f}(\cdot)$ defined in (15), cf. Remark 2.7 in Dette and Neumeyer (2001).

Note that the bandwidth conditions (11) required here are more restrictive than the bandwidth conditions used by Dette and Neumeyer (2001). They are due to the appearance of an additional bias that originates from the variance estimation (16). But the suggested test statistic can be modified in various ways due to prior knowledge on the variances in order to weaken these bandwidth conditions.
On the one hand, if homoscedasticity of the two variances can be assumed, respectively, i. e. $\sigma_i^2(\cdot) \equiv \sigma_i^2$, $i = 1, 2$, then for the estimation of the constant variance within the i-th sample every estimator that satisfies

$$\hat{\sigma}_i^2 - \sigma_i^2 \;=\; O_p(\frac{1}{\sqrt{N}}) \quad (i = 1, 2)$$

can be used, see for example Rice (1984) or Hall and Marron (1990). The bandwidth conditions (11) can then be weakened to the conditions used by Dette and Neumeyer (2001),

$$h = O(N^{-2/(4d+1)}) \quad \text{and} \quad Nh^2 \to \infty \quad \text{for} \quad N \to \infty \tag{25}$$

and under these conditions we obtain the following limit distributions. Under the null hypothesis H_0 of equal regression functions we have

$$N\sqrt{h}\left(T_N - Bh^{2d} - \frac{C}{Nh}\right) \xrightarrow[N \to \infty]{\mathcal{D}} \mathcal{N}(0, \tau^2)\,,$$

where the constant B is defined by

$$B \;=\; k_d^2\Big(\int \frac{\{\sigma_1^{-2}\kappa_1(f_1 r_1^{(d)} - (f_1 r_1)^{(d)})(x) + \sigma_2^{-2}\kappa_2(f_1 r_2^{(d)} - (f_1 r_2)^{(d)})(x)\}^2}{\sigma_1^{-2}\kappa_1 r_1(x) + \sigma_2^{-2}\kappa_2 r_2(x)}\, dx$$

$$
- \kappa_1 \int \{(f_1 r_1)^{(d)}(x) - (f_1 r_1^{(d)})(x)\}^2 \frac{1}{\sigma_1^2 r_1(x)}\, dx
$$
$$
- \kappa_2 \int \{(f_1 r_2)^{(d)}(x) - (f_1 r_2^{(d)})(x)\}^2 \frac{1}{\sigma_2^2 r_2(x)}\, dx \Big),
$$

k_d is defined in (10) and C and τ^2 are defined in Theorem 2.1. Under the fixed alternative H_1 the same limit distribution as in Theorem 2.2 holds. If additionally equality of the variances $\sigma_1^2 = \sigma_2^2 = \sigma_0^2$ can be assumed, σ_0^2 could be estimated from the pooled sample, of course. However, in this case weighting by the variances is not necessary at all and our test statistic essentially reduces to the statistic by Dette and Neumeyer (2001).
On the other hand the less restrictive bandwidth conditions (25) can also be sufficient in the case where we have extra information about the smoothness of the variance functions. We consider the following setting. Condition (9) is replaced by the assumption

$$
r_i, f_i \in C^d(0,1), \quad \sigma_i^2 \in C^s(0,1) \quad (i = 1,2),
$$

where $s > d$. Moreover, instead of K and h we use a kernel $\tilde{K}$ of order s and a bandwidth $b = b_N$ in the definition (16) of the variance estimate. In place of the bandwidth conditions (11) we assume

$$
Nb^{2s} \to 0, \quad Nb^2 \to \infty, \quad h^{2d+1/2} = o(b) \quad \text{and} \quad \frac{b}{\sqrt{h}} = O(1) \quad \text{for} \quad N \to \infty
$$

for the bandwidth b and the conditions (25) for the bandwidth h used for the regression estimators. Under these assumptions the same limit distributions for T_N under H_0 and H_1 as stated above for the homoscedastic case hold.

3.3 The k–sample case

In this section we indicate how the presented test can be extended to the case of k samples, i. e. we are concerned with the model

$$
Y_{ij} \;=\; f_i(t_{ij}) + \sigma_i(t_{ij})\varepsilon_{ij}\,, \qquad j = 1,\dots,n_i, \quad i = 1,\dots,k, \tag{26}
$$

and the testing problem is

$$
H_0 : f_1 = \cdots = f_k \quad \text{versus} \quad H_1 : f_i \neq f_j \text{ for some } i \neq j.
$$

Further assume for the sample sizes that

$$
\frac{n_i}{N} \;=\; \kappa_i + O(\frac{1}{N})\,, \quad i = 1,\dots,k,
$$

where $\kappa_i \in (0,1)$, and for the design densities r_i we require (7), $i = 1,\dots,k$. Following the same idea as in the introduction we end up with a k–sample generalization of the ANOVA–Welch statistic (Welch, 1937)

$$
T_N \;=\; \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \hat{f}(t_{ij}))^2 \hat{\sigma}_i^{-2}(t_{ij}) - \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \hat{f}_i(t_{ij}))^2 \hat{\sigma}_i^{-2}(t_{ij}),
$$

where now

$$\hat{f}(x) \;=\; \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i} K(\frac{x-t_{ij}}{h}) Y_{ij} \hat{\sigma}_i^{-2}(t_{ij})}{\sum_{i=1}^{k}\sum_{j=1}^{n_i} K(\frac{x-t_{ij}}{h}) \hat{\sigma}_i^{-2}(t_{ij})},$$

$\hat{f}_i$ and $\hat{\sigma}_i^2$ are defined in (14) and (16), respectively, for $i = 1, \ldots, k$.

Theorem 3.1 *Assume model (26) where the ε_{ij} are i.i.d. centered random variables with variance $\mathrm{Var}(\varepsilon_{ij}) = 1$ and $E[\varepsilon_{ij}^4] < \infty$, s. t. the assumptions stated in this section and (7)–(12) for $i = 1, \ldots, k$ are satisfied. Under the null hypothesis H_0 we have*

$$N\sqrt{h}\Big(T_N - \frac{C}{Nh}\Big) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \tau^2)$$

where the constants are defined as

$$\begin{aligned} C &= 2K(0) - \int K^2(u)\,du \\ \tau^2 &= 2(k-1)\int (2K - K*K)^2(u)\,du. \end{aligned}$$

Theorem 3.2 *Under the assumptions of Theorem 3.1 under the fixed alternative H_1 we have*

$$\sqrt{N}\,(T_N - \mu) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \gamma^2)$$

where the constants are defined as

$$\mu \;=\; \sum_{j=1}^{k}\sum_{\substack{l=1\\ l<j}}^{k} \int (f_j - f_l)^2(x) \frac{\sigma_l^{-2}(x)\kappa_l r_l(x)\sigma_j^{-2}(x)\kappa_j r_j(x)}{\sum_{l=1}^{k}\sigma_l^{-2}(x)\kappa_l r_l(x)}\,dx$$

and $\gamma^2 = 4\,\mu$.

4 Wild bootstrap and finite sample properties

Although the testing procedure (17) is distribution free and therefore applicable directly without any estimation of nuisance parameters, our simulations indicated that for small and moderate sample sizes the performance of the test can be improved by the bootstrap technique. Hence in this section we present the finite sample behaviour of a wild bootstrap version of the proposed testing procedure. We confine ourselves to a power comparison with the procedure of Dette and Neumeyer (2001), because these authors already compared their test to various procedures and we will show that the new test outperforms the testing procedure of the aforementioned authors in most cases. For the sake of brevity we do not present level simulations but our simulations show that the new procedure keeps the level just as well as Dette and Neumeyer's (2001) test.

We consider the following wild bootstrap approach based on the residuals

$$\hat{\varepsilon}_{ij} \;=\; Y_{ij} - \hat{f}(t_{ij}), \quad j = 1, \dots, n_i \quad (i = 1, 2),$$

where $\hat{f}$ is the pooled regression estimator defined in (15). Let V_{ij} denote i.i.d. random variables, independent of the sample $\{Y_{ij}\}$, with masses $\frac{\sqrt{5}+1}{2\sqrt{5}}$ and $\frac{\sqrt{5}-1}{2\sqrt{5}}$ at the points $\frac{1}{2}(1-\sqrt{5})$ and $\frac{1}{2}(1+\sqrt{5})$, respectively. We define bootstrap observations

$$Y^*_{ij} \;=\; \hat{f}(t_{ij}) + V_{ij}\hat{\varepsilon}_{ij}, \quad j = 1, \dots, n_i \quad (i = 1, 2),$$

and denote by T^*_N the test statistic defined in (6) but based on the bootstrap sample $\{Y^*_{ij}\}$. A test of asymptotic level α rejects the null hypothesis whenever the statistic T_N (based on the original sample $\{Y_{ij}\}$) is larger than the $(1-\alpha)$–quantile of the distribution of T^*_N conditioned on the sample $\{Y_{ij}\}$. The consistency of this bootstrap procedure can be shown in the same spirit as in the proof of Dette and Neumeyer (2001, section 4.4). In each of 1000 simulations we resampled $B = 200$ times and estimated the bootstrap quantile by $T^*_{N(\lfloor B(1-\alpha)\rfloor)}$, where $T^*_{N(\ell)}$ denotes the ℓ-th order statistic of the bootstrap sample $T^*_{N,1}, \dots, T^*_{N,B}$.
For all kernel based estimators we used the Epanechnikov kernel. The bandwidths are chosen according to the "rule of thumb" (cf. Dette and Neumeyer, 2001), $h_i = (s_i^2/n_i)^{0.3}$ in the estimators $\hat{f}_i$ and $\hat{\sigma}_i^2$ $(i = 1, 2)$ and $h = ((\kappa_1 s_1 + \kappa_2 s_2)/N)^{0.3}$ in the pooled regression estimator $\hat{f}$. Here s_i denotes Rice's estimator (Rice, 1984)

$$s_i = \frac{2}{n_i - 1}\sum_{j=1}^{n_i}(Y_{i\,j+1} - Y_{i\,j})^2$$

of the integrated variance $\int \sigma_i^2(t) r_i(t)\, dt$ in the i-th sample $(i = 1, 2)$.

The analogous bootstrap procedure was also simulated for Dette and Neumeyer's (2001) test statistic $T_N^{(1)}$ defined in (23). We restrict in the following our presentation to normal errors $\varepsilon_{ij} \sim \mathcal{N}(0, 1)$ (various other settings have been simulated and yielded similar results) and present the results for different combinations of sample sizes (n_1, n_2) and nominal levels α. First we consider the case of equidistant design points (i. e. $r_i \equiv 1$, $i = 1, 2$) in three settings corresponding to the cases of equal homoscedastic, equal heteroscedastic and inhomogeneous heteroscedastic variances. The results for the following regression functions and equal homoscedastic variances,

$$f_1(x) = \exp(x),\ f_2(x) = \exp(x) + \sin(4\pi x), \qquad \sigma_i^2 \equiv 0.5 \ \ (i = 1, 2), \tag{27}$$

can be depicted in Table 1 for the new test statistic T_N and in Table 2 for Dette and Neumeyer's (2001) procedure for the sake of comparison. The new procedure turns out to be uniformly more powerful in this case. The results for equal heteroscedastic variances according to the following setting,

$$f_1(x) = x^2,\ f_2(x) = x^2 + \sin(4\pi x), \qquad \sigma_i^2(x) = x \ \ (i = 1, 2), \tag{28}$$

are presented in Tables 3 and 4 for the test statistics T_N defined in (6) and $T_N^{(1)}$ defined in (23), respectively. In all cases we observe a better power of the new test. Results for the case of inhomogeneous and heteroscedastic variances,

$$f_1 \equiv 1,\ f_2 \equiv 0, \qquad \sigma_1^2(x) = x^2\ ,\ \ \sigma_2^2(x) = 5x - x^2 \tag{29}$$

are presented in Tables 5 and 6. In this case we observe slightly better power of Dette and Neumeyer's (2001) test for equal and nearly equal sample sizes, but the new procedure outperforms Dette and Neumeyer's (2001) test, when the sample sizes are rather different, e. g. when $n_1 = 10$, $n_2 = 50$. This phenomenon presumably originates from the interplay of sample size and variance in the weight $1 - a = \sigma_2^{-2} n_2 / (\sigma_1^{-2} n_1 + \sigma_2^{-2} n_2)$ from (5) that is assigned to the observations from the second sample in the pooled regression estimate in the definition of test statistic T_N. In contrast the corresponding weight used in test statistic $T_N^{(1)}$ is $1 - \tilde{a} = n_2/(n_1 + n_2)$.
Finally, we present simulations for the setting where both the design densities and the variances are different in the two samples,

$$r_1 \equiv 1,\ r_2(x) = 0.5 + x, \quad f_1 \equiv 1,\ f_2 \equiv 0, \quad \sigma_1^2 \equiv 2\ , \sigma_2^2 \equiv 3. \tag{30}$$

The results are shown in tables 7 and 8 and the new test turns out to be uniformly more powerful in this case, where for equal sample sizes the gain in power is remarkable. This is perfectly in accordance with our theoretical findings in Lemma 2.3 and the explanations given in Section 2.2.

The Tables 1–8 are positioned at the end of the paper.

5 Conclusion

In this paper we have suggested a new procedure for testing the equality of regression curves in different nonparametric regression models. The new test generalizes naturally the method of analysis of covariance to the setting of nonparametric regression. The asymptotic normal distribution of the proposed test statistic under the null hypothesis of equal regression functions as well as under fixed and local alternatives is shown. Under the null hypothesis the test turns out to be asymptotically distribution free. Our procedure is similar in spirit to a test based on a difference of variance estimators recommended by Dette and Neumeyer (2001). We have shown that the new test gains in power particularly in the case of inhomogeneous and heteroscedastic variances and for different sample sizes resp. design densities.

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A Appendix: Proofs

A.1 Proof of Theorems 2.1 and 2.2

The strategy of the proof is in principle similar to the proof of Theorem 2.1 of Dette and Neumeyer (2001). However, technical it becomes quite delicate due to the additional variance estimators involved. By the ease of brevity we will only state the main differences due to the additional variance estimation. With the definition of weights

$$w_{jk}^{(i)} = \frac{K(\frac{t_{ij}-t_{ik}}{h})}{\sum_{l=1}^{n_i} K(\frac{t_{ij}-t_{il}}{h})}$$

the individual regression estimates defined in (14) have the form

$$\hat{f}_i(t_{ij}) = \sum_{k=1}^{n_i} w_{jk}^{(i)} Y_{ik} \quad (i = 1, 2).$$

An analogous form can be achieved for the combined estimator defined in (15),

$$\hat{f}(t_{ij}) = \sum_{l=1}^{2} \sum_{k=1}^{n_i} w_{lk,ij} Y_{lk}$$

with the weights

$$w_{lk,ij} = \frac{K(\frac{t_{lk}-t_{ij}}{h})\hat{\sigma}_l^{-2}(t_{ij})}{\sum_{l'=1}^{2}\sum_{k'=1}^{n_{l'}} K(\frac{t_{l'k'}-t_{ij}}{h})\hat{\sigma}_{l'}^{-2}(t_{ij})} = \frac{1}{Nh} K(\frac{t_{lk}-t_{ij}}{h})\hat{\sigma}_{3-l}^2(t_{ij})\frac{1}{\hat{R}(t_{ij})}, \tag{31}$$

where

$$\hat{R}(t) = \frac{1}{Nh}\sum_{l=1}^{2}\sum_{k=1}^{n_l} K(\frac{t_{lk}-t}{h})\hat{\sigma}_{3-l}^2(t) = \frac{n_1}{N}\hat{r}_1(t)\hat{\sigma}_2^2(t) + \frac{n_2}{N}\hat{r}_2(t)\hat{\sigma}_1^2(t)$$

is an estimator for

$$R(t) = \kappa_1 r_1(t)\sigma_2^2(t) + \kappa_2 r_2(t)\sigma_1^2(t). \tag{32}$$

Now with the notations ($j = 1, \ldots, n_i, i = 1, 2$)

$$\Delta_{ij} = f_i(t_{ij}) - \sum_{l=1}^{2}\sum_{k=1}^{n_l} w_{lk,ij} f_l(t_{lk}) = \sum_{l=1}^{2}\sum_{k=1}^{n_l} w_{lk,ij}(f_i(t_{ij}) - f_l(t_{lk})) \tag{33}$$

$$\delta_{ij} = f_i(t_{ij}) - \sum_{k=1}^{n_i} w_{jk}^{(i)} f_i(t_{ik}) = \sum_{l=1}^{2}\sum_{k=1}^{n_l} w_{jk}^{(i)}(f_i(t_{ij}) - f_i(t_{ik})) \tag{34}$$

we decompose T_N in (6) as

$$T_N = \frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\hat\sigma_i^{-2}(t_{ij})\Big\{\Delta_{ij}^2 - \delta_{ij}^2 - 2\Delta_{ij}\sum_{l=1}^{2}\sum_{k=1}^{n_l} w_{lk,ij}\sigma_l(t_{lk})\varepsilon_{lk} + 2\delta_{ij}\sum_{k=1}^{n_i} w_{jk}^{(i)}\sigma_i(t_{ik})\varepsilon_{ik}$$
$$+\Big(\sum_{l=1}^{2}\sum_{k=1}^{n_l} w_{lk,ij}\sigma_l(t_{lk})\varepsilon_{lk}\Big)^2 - \Big(\sum_{k=1}^{n_i} w_{jk}^{(i)}\sigma_i(t_{ik})\varepsilon_{ik}\Big)^2 + 2\sigma_i(t_{ij})\varepsilon_{ij}(\Delta_{ij}-\delta_{ij})$$
$$- 2\sigma_i(t_{ij})\varepsilon_{ij}\sum_{l=1}^{2}\sum_{k=1}^{n_l} w_{lk,ij}\sigma_l(t_{lk})\varepsilon_{lk} + 2\sigma_i(t_{ij})\varepsilon_{ij}\sum_{k=1}^{n_i} w_{jk}^{(i)}\sigma_i(t_{ik})\varepsilon_{ik}\Big\}. \tag{35}$$

Lemma A.1 *Under the assumptions of Theorem 2.1 we obtain the following expansion of the expectation of the test statistic under the null hypothesis H_0,*

$$E[T_N] = \frac{C}{Nh} + o(\frac{1}{N\sqrt{h}}),$$

and under the alternative H_1,

$$E[T_N] = \mu + o(\frac{1}{\sqrt{N}}),$$

where the constants C and μ are defined in the Theorems 2.1 and 2.2.

Proof. We use the above definitions and the decomposition (35) of the test statistic T_N. A Taylor expansion together with (31) and (33) gives

$$\begin{aligned}
\Delta_{ij} &= \sum_{l=1}^{2}\frac{\hat\sigma_{3-l}^2(t_{ij})}{\hat R(t_{ij})}\frac{1}{Nh}\sum_{k=1}^{n_l} K(\frac{t_{ij}-t_{lk}}{h})(f_i(t_{ij}) - f_l(t_{lk}))\\
&= \sum_{l=1}^{2}\frac{\hat\sigma_{3-l}^2(t_{ij})}{\hat R(t_{ij})}\Big\{\int K(\frac{t_{ij}-t}{h})(f_i(t_{ij}) - f_l(t))\kappa_l r_l(t)\,dt + O(\frac{1}{Nh})\Big\}\\
&= \sum_{l=1}^{2}\frac{\hat\sigma_{3-l}^2(t_{ij})}{\hat R(t_{ij})}\Big\{(f_i(t_{ij}) - f_l(t_{ij}))\kappa_l r_l(t_{ij}) + O(h^d) + O(\frac{1}{Nh})\Big\} && (36)\\
&= \frac{\hat\sigma_i^2(t_{ij})}{\hat R(t_{ij})}(f_i(t_{ij}) - f_{3-i}(t_{ij}))\kappa_{3-i}r_{3-i}(t_{ij}) + \Big\{O(h^d) + O(\frac{1}{Nh})\Big\}O_p(1) && (37)
\end{aligned}$$

where the last line only holds under the alternative H_1. For the expectation of the first term on the r. h. s. in (35) we obtain under the alternative H_1 with an application of Proposition A.4 in section A.2

$$\frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} E[\hat\sigma_i^{-2}(t_{ij})\Delta_{ij}^2] = \frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} E\Big[\frac{\hat\sigma_i^2(t_{ij})}{\hat R^2(t_{ij})}\Big](f_i(t_{ij}) - f_{3-i}(t_{ij}))^2\kappa_{3-i}^2 r_{3-i}^2(t_{ij})$$

$$
\begin{aligned}
& + O(h^d) + O(\frac{1}{Nh}) \\
= & \sum_{i=1}^{2} \int \frac{\sigma_i^2(t)}{R^2(t)} (f_i(t) - f_{3-i}(t))^2 \kappa_{3-i}^2 r_{3-i}^2(t) \kappa_i r_i(t)\, dt + o(\frac{1}{\sqrt{N}}) \\
= & \int \frac{1}{R(t)} (f_1(t) - f_2(t))^2 \kappa_1 r_1(t) \kappa_2 r_2(t)\, dt + o(\frac{1}{\sqrt{N}}) \\
= & \ \mu + o(\frac{1}{\sqrt{N}}).
\end{aligned}
$$

Under the null hypothesis H_0 we directly obtain from (36)

$$
\frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} E[\hat{\sigma}_i^{-2}(t_{ij}) \Delta_{ij}^2] \ = \ \Big\{ O(h^d) + O(\frac{1}{Nh}) \Big\}^2 \ = \ o(\frac{1}{N\sqrt{h}}).
$$

Similar calculations give

$$
\frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} E[\hat{\sigma}_i^{-2}(t_{ij})] \delta_{ij}^2 \ = \ o(\frac{1}{N\sqrt{h}})
$$

and analogously we obtain for the terms

$$
\frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} E\Big[\hat{\sigma}_i^{-2}(t_{ij}) \Delta_{ij} \sum_{l=1}^{2} \sum_{k=1}^{n_l} w_{lk,ij} \sigma_l(t_{lk}) \varepsilon_{lk} \Big] \text{ and } \frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} E\Big[\hat{\sigma}_i^{-2}(t_{ij}) \delta_{ij} \sum_{k=1}^{n_i} w_{jk}^{(i)} \sigma_i(t_{ik}) \varepsilon_{ik} \Big]
$$

the rate of convergence $O(1/(Nh)) = o(1/\sqrt{N})$ under H_1 and $O(1/(Nh))(O(h^d) + O(\frac{1}{Nh})) = o(\frac{1}{N\sqrt{h}})$ under H_0, respectively. With (31) and Proposition A.4 we further obtain

$$
\begin{aligned}
& \frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{n_i} E\Big[\hat{\sigma}_i^{-2}(t_{ij}) \Big(\sum_{l=1}^{2} \sum_{k=1}^{n_l} w_{lk,ij} \sigma_l(t_{lk}) \varepsilon_{lk} \Big)^2 \Big] \\
= & \frac{1}{N^3 h^2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} \sum_{l=1}^{2} \sum_{k=1}^{n_l} E\Big[\frac{\hat{\sigma}_i^{-2}(t_{ij})}{\hat{R}^2(t_{ij})} \hat{\sigma}_{3-l}^4(t_{ij}) K^2(\frac{t_{ij} - t_{lk}}{h}) \sigma_l^2(t_{lk}) \varepsilon_{lk}^2 \Big] \\
& + \frac{1}{N^3 h^2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} \sum_{l=1}^{2} \sum_{k=1}^{n_l} \sum_{l'=1}^{2} \sum_{\substack{k'=1 \\ (l,k) \neq (l',k')}}^{n_{l'}} E\Big[\frac{\hat{\sigma}_i^{-2}(t_{ij})}{\hat{R}^2(t_{ij})} \hat{\sigma}_{3-l}^2(t_{ij}) \hat{\sigma}_{3-l'}^2(t_{ij}) \\
& \qquad K(\frac{t_{ij} - t_{lk}}{h}) K(\frac{t_{ij} - t_{l'k'}}{h}) \sigma_l(t_{lk}) \sigma_{l'}(t_{l'k'}) \varepsilon_{lk} \varepsilon_{l'k'} \Big] \\
= & \frac{1}{Nh^2} \sum_{i=1}^{2} \sum_{l=1}^{2} \int \int \frac{\sigma_i^{-2}(t)}{R^2(t)} \sigma_{3-l}^4(t) K^2(\frac{t - x}{h}) \sigma_l^2(x) \kappa_i r_i(t) \kappa_l r_l(t)\, dt\, dx \\
& + \frac{1}{Nh} (O(h^d) + O(\frac{1}{Nh})) \\
= & \frac{1}{Nh} \int K^2(u)\, du + o(\frac{1}{N\sqrt{h}}).
\end{aligned}
$$

An analogous calculation yields

$$
\begin{aligned}
&-\frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} E\Big[\hat\sigma_i^{-2}(t_{ij})\Big(\sum_{k=1}^{n_i} w_{jk}^{(i)}\sigma_i(t_{ik})\varepsilon_{ik}\Big)^2\Big]\\
&= -\frac{1}{Nh^2}\sum_{i=1}^{2}\int\int \frac{\sigma_i^{-2}(t)}{r_i^2(t)}K^2(\frac{t-x}{h})\sigma_i^2(x)r_i(t)r_i(x)\,dt\,dx + o(\frac{1}{N\sqrt{h}})\\
&= -\frac{2}{Nh}\int K^2(u)\,du + o(\frac{1}{N\sqrt{h}}).
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
&-\frac{2}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} E\Big[\hat\sigma_i^{-2}(t_{ij})\sigma_i(t_{ij})\varepsilon_{ij}\sum_{l=1}^{2}\sum_{k=1}^{n_l} w_{lk,ij}\sigma_l(t_{lk})\varepsilon_{lk}\Big]\\
&= -\frac{2}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} E\Big[\hat\sigma_i^{-2}(t_{ij})\sigma_i^2(t_{ij})\varepsilon_{ij}^2\frac{K(0)}{Nh}\hat\sigma_{3-i}^2(t_{ij})\frac{1}{\hat R(t_{ij})}\Big]\\
&\quad -\frac{2}{N^2h}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\sum_{l=1}^{2}\sum_{\substack{k=1\\(i,j)\neq(l,k)}}^{n_l} E\Big[\hat\sigma_i^{-2}(t_{ij})\sigma_i(t_{ij})\varepsilon_{ij}\sigma_l(t_{lk})\varepsilon_{lk}K(\frac{t_{ij}-t_{lk}}{h})\hat\sigma_{3-l}^2(t_{ij})\frac{1}{\hat R(t_{ij})}\Big]\\
&= -\frac{2K(0)}{Nh}\int\frac{\sigma_2^2(t)\kappa_1 r_1(t)+\sigma_1^2(t)\kappa_2 r_2(t)}{R(t)}\,dt + \frac{1}{Nh}(O(h^d)+O(\frac{1}{Nh}))\\
&= -\frac{2K(0)}{Nh} + o(\frac{1}{N\sqrt{h}})
\end{aligned}
$$

and

$$
\frac{2}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} E\Big[\hat\sigma_i^{-2}(t_{ij})\sigma_i(t_{ij})\varepsilon_{ij}\sum_{k=1}^{n_i} w_{jk}^{(i)}\sigma_i(t_{ik})\varepsilon_{ik}\Big] \;=\; \frac{4K(0)}{Nh} + o(\frac{1}{N\sqrt{h}}).
$$

Analogous to the previous calculations we obtain that

$$
\frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} E\Big[\hat\sigma_i^{-2}(t_{ij})\sigma_i(t_{ij})\varepsilon_{ij}(\Delta_{ij}-\delta_{ij})\Big]
$$

is of order $O(1/(Nh)) = o(1/\sqrt{N})$ under H_1 and of order $O(1/(Nh))(O(h^d)+O(\frac{1}{Nh})) = o(\frac{1}{N\sqrt{h}})$ under H_0. From the decomposition (35) of T_N and the above calculation the assertion follows. ■

A.1.1 Proof of Theorem 2.2

Analogous to the proof of Theorem 2.1, Dette and Neumeyer (2001), the following expansion of the test statistic holds under the alternative H_1:

$$
T_N - E[T_N] \;=\; T_N^{(1)} + T_N^{(2)} + o_p(\frac{1}{\sqrt{N}})
$$

where

$$T_N^{(i)} \;=\; \frac{1}{N}\sum_{j=1}^{n_i}\alpha_{ij}\varepsilon_{ij} \quad (i=1,2)$$

and the coefficients are defined by

$$\alpha_{ij} \;=\; 2\Delta_{ij}\sigma_i(t_{ij})/\hat{\sigma_i}^2(t_{ij}), \quad j=1,\ldots,n_i, i=1,2.$$

Lemma A.2 *Under the assumptions of Theorem 2.1 under the alternative H_1 it holds that*

$$\mathrm{Var}(T_N^{(i)}) = \frac{4}{N}\int (f_1-f_2)^2(x)\frac{\kappa_i r_i(x)\kappa_{3-i}^2 r_{3-i}^2(x)\sigma_i^2(x)}{(\kappa_1 r_1(x)\sigma_2^2(x)+\kappa_2 r_2(x)\sigma_1^2(x))^2}\,dx + o(\frac{1}{N}) \quad (i=1,2).$$

Proof. We only consider the case $i=1$. With Δ_{1j} from (37) we obtain

$$T_N^{(1)} \;=\; \frac{2}{N}\sum_{j=1}^{n_1}(f_1(t_{1j})-f_2(t_{1j}))\kappa_2 r_2(t_{1j})\frac{\sigma_1(t_{1j})}{\hat{R}(t_{1j})}\varepsilon_{1j} + o_p(\frac{1}{\sqrt{N}}).$$

Now for calculating the variance $\mathrm{Var}(T_N^{(1)})$ we can substitute $\hat{R}(t)$ by $R(t)$ defined in (32). The remainder of the expansion

$$\frac{1}{\hat{R}(t)} \;=\; \frac{1}{R(t)} + \Big\{\frac{1}{\hat{R}(t)} - \frac{1}{R(t)}\Big\}$$

is equal to

$$\begin{aligned}\frac{R(t)-\hat{R}(t)}{\hat{R}(t)R(t)} \;&=\; \frac{1}{R^2(t)}(R(t)-\hat{R}(t))(1+o_p(1)) \\ &=\; -\frac{1}{R^2(t)}\sum_{i=1}^{2}\Big\{\hat{r}_{3-i}(t)(\hat{\sigma}_i^2(t)-\sigma_i^2(t))+\sigma_i^2(t)(\hat{r}_{3-i}(t)-r_{3-i}(t))\Big\}(1+o_p(1)).\end{aligned}$$

This yields remainder terms $T_N^{(1,i)}$ $(i=1,2)$ in the expansion

$$T_N^{(1)} \;=\; \tilde{T}_N^{(1)} + T_N^{(1,1)} + T_N^{(1,2)} + o_p(\frac{1}{\sqrt{N}})$$

where

$$\tilde{T}_N^{(1)} \;=\; \frac{2}{N}\sum_{j=1}^{n_1}(f_1(t_{1j})-f_2(t_{1j}))\kappa_2 r_2(t_{1j})\frac{\sigma_1(t_{1j})}{R(t_{1j})}\varepsilon_{1j}$$

and the remainders are of the form

$$
\begin{aligned}
T_N^{(1,i)} &= \frac{1}{N}\sum_{j=1}^{n_1}\Delta(t_{1j})\varepsilon_{1j}\Big\{\,(r_{3-i}(t_{1j})+o(1))\,(\hat{\sigma}_i^2(t_{1j})-\sigma_i^2(t_{1j}))+o(1)\Big\}\\
&= o_p(\frac{1}{\sqrt{N}}).
\end{aligned}
$$

The last equality can be obtained by inserting the decomposition of the variance estimator $\hat{\sigma}_i^2(t)$ from Proposition A.4 (see section A.2) and a tedious calculation of the variance

$$
\mathrm{Var}(T_N^{(1,i)}) \;=\; o(\frac{1}{N}) \quad (i=1,2).
$$

From the negligibility of the remainder terms we obtain for the variance

$$
\begin{aligned}
\mathrm{Var}(T_N^{(1)}) &= \mathrm{Var}(\tilde{T}_N^{(1)}) + o(\frac{1}{N})\\
&= \frac{4}{N}\int (f_1-f_2)^2(x)\frac{\kappa_1 r_1(x)\kappa_2^2 r_2^2(x)\sigma_1^2(x)}{R^2(x)}\,dx + o(\frac{1}{N}).
\end{aligned}
$$

■ (Lemma A.2)

From the proof of the last lemma we additionally obtain under the alternative H_1:

$$
\begin{aligned}
\sqrt{N}(T_N - E[T_N]) &= \sqrt{N}(T_N^{(1)}+T_N^{(2)}) + o_p(1)\\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\varepsilon_{ij}(f_i(t_{ij})-f_{3-i}(t_{ij}))\kappa_{3-i}r_{3-i}(t_{ij})\frac{\sigma_i(t_{ij})}{R(t_{ij})} + o_p(1)
\end{aligned}
$$

with the asymptotic variance

$$
\begin{aligned}
&4\int (f_1-f_2)^2(x)\frac{\kappa_1 r_1(x)\kappa_2^2 r_2^2(x)\sigma_1^2(x)}{R^2(x)}\,dx + 4\int (f_1-f_2)^2(x)\frac{\kappa_2 r_2(x)\kappa_1^2 r_1^2(x)\sigma_2^2(x)}{R^2(x)}\,dx\\
=\;& 4\int (f_1-f_2)^2(x)\frac{\kappa_1 r_1(x)\kappa_2 r_2(x)}{\sigma_2^2(x)\kappa_1 r_1(x)+\sigma_1^2(x)\kappa_2 r_2(x)}\,dx\\
=\;& \gamma^2.
\end{aligned}
$$

An application of the central limit theorem using Lyapunov's condition yields the asymptotic normality and completes the proof of Theorem 2.2. ■

A.1.2 Proof of Theorem 2.1

Under the hypothesis H_0 of equal regression functions in the two models we obtain similar to the proof of Theorem 2.1 of Dette and Neumeyer (2001) the decomposition

$$
T_N - E[T_N] \;=\; \sum_{j=3}^{5} T_N^{(j)} + o_p(\frac{1}{N\sqrt{h}})
$$

where

$$T_N^{(2+k)} = \frac{1}{N} \sum_{i=1}^{n_k} \sum_{\substack{j=1 \\ j \neq i}}^{n_k} \beta_{ij}^{(k)} \varepsilon_{ki} \varepsilon_{kj} \quad (k = 1, 2)$$

$$T_N^{(5)} = \frac{1}{N} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \gamma_{ij} \varepsilon_{1i} \varepsilon_{2j}$$

and the coefficients are defined by

$$\begin{aligned}
\beta_{ij}^{(1)} &= \Big\{ \sum_{l=1}^{2} \sum_{k=1}^{n_l} \frac{w_{1i,lk} w_{1j,lk}}{\hat{\sigma}_l^2(t_{lk})} - \frac{2w_{1j,1i}}{\hat{\sigma}_1^2(t_{1i})} - \sum_{k=1}^{n_1} \frac{w_{ki}^{(1)} w_{kj}^{(1)}}{\hat{\sigma}_1^2(t_{1k})} + \frac{2w_{ij}^{(1)}}{\hat{\sigma}_1^2(t_{1i})} \Big\} \sigma_1(t_{1i}) \sigma_1(t_{1j}) \\
\beta_{ij}^{(2)} &= \Big\{ \sum_{l=1}^{2} \sum_{k=1}^{n_l} \frac{w_{2i,lk} w_{2j,lk}}{\hat{\sigma}_l^2(t_{lk})} - \frac{2w_{2j,2i}}{\hat{\sigma}_2^2(t_{2i})} - \sum_{k=1}^{n_2} \frac{w_{ki}^{(2)} w_{kj}^{(2)}}{\hat{\sigma}_2^2(t_{2k})} + \frac{2w_{ij}^{(2)}}{\hat{\sigma}_2^2(t_{2i})} \Big\} \sigma_2(t_{2i}) \sigma_2(t_{2j}) \\
\gamma_{ij} &= \Big\{ 2 \sum_{l=1}^{2} \sum_{k=1}^{n_l} \frac{w_{1i,lk} w_{2j,lk}}{\hat{\sigma}_l^2(t_{lk})} - \frac{2w_{2j,1i}}{\hat{\sigma}_1^2(t_{1i})} - \frac{2w_{1i,2j}}{\hat{\sigma}_2^2(t_{2j})} \Big\} \sigma_1(t_{1i}) \sigma_2(t_{2j}).
\end{aligned}$$

Lemma A.3 *Under the assumptions of Theorem 2.1 under the null hypothesis H_0 it holds*

$$\begin{aligned}
\mathrm{Var}(T_N^{(2+k)}) &= \frac{2}{N^2 h} \int (2K - K * K)^2(u)\, du \\
&\quad \times \Big[1 + \int_0^1 \frac{\kappa_k^2 r_k^2(x) \sigma_{3-k}^4(x)}{R^2(x)}\, dx - 2 \int_0^1 \frac{\kappa_k r_k(x) \sigma_{3-k}^2(x)}{R(x)}\, dx \Big] + o(\frac{1}{N^2 h}) \quad (k = 1, 2), \\
\mathrm{Var}(T_N^{(5)}) &= \frac{4}{N^2 h} \int (2K - K * K)^2(u)\, du \int_0^1 \frac{\sigma_1^2(x) \sigma_2^2(x) \kappa_1 r_1(x) \kappa_2 r_2(x)}{R^2(x)}\, dx + o(\frac{1}{N^2 h}).
\end{aligned}$$

Proof. For simplicity we only consider $T_N^{(5)}$, the other two terms are treated similarly. By the definition of the weights in (31) the coefficients γ_{ij} can be rewritten as

$$\begin{aligned}
\gamma_{ij} &= \Big\{ \frac{2}{N^2 h^2} \sum_{l=1}^{2} \sum_{k=1}^{n_l} K(\frac{t_{1i} - t_{lk}}{h}) K(\frac{t_{2j} - t_{lk}}{h}) \frac{1}{\hat{R}^2(t_{lk})} \hat{\sigma}_{3-l}^2(t_{lk}) \\
&\quad - \frac{2}{Nh} K(\frac{t_{2j} - t_{1i}}{h}) \frac{1}{\hat{R}(t_{1i})} - \frac{2}{Nh} K(\frac{t_{2j} - t_{1i}}{h}) \frac{1}{\hat{R}(t_{2j})} \Big\} \sigma_1(t_{1i}) \sigma_2(t_{2j}) \\
&= \tilde{\gamma}_{ij} + \bar{\gamma}_{ij}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\gamma}_{ij} &= \Big\{ \frac{2}{N^2h^2} \sum_{l=1}^{2} \sum_{k=1}^{n_l} K(\frac{t_{1i}-t_{lk}}{h}) K(\frac{t_{2j}-t_{lk}}{h}) \frac{1}{\hat{R}^2(t_{lk})} \sigma^2_{3-l}(t_{lk}) \\
&\quad - \frac{2}{Nh} K(\frac{t_{2j}-t_{1i}}{h}) \frac{1}{\hat{R}(t_{1i})} - \frac{2}{Nh} K(\frac{t_{2j}-t_{1i}}{h}) \frac{1}{\hat{R}(t_{2j})} \Big\} \sigma_1(t_{1i}) \sigma_2(t_{2j}) \\
\bar{\gamma}_{ij} &= \frac{2}{N^2h^2} \sum_{l=1}^{2} \sum_{k=1}^{n_l} K(\frac{t_{1i}-t_{lk}}{h}) K(\frac{t_{2j}-t_{lk}}{h}) \frac{1}{\hat{R}^2(t_{lk})} \sigma_1(t_{1i}) \sigma_2(t_{2j}) \left(\hat{\sigma}^2_{3-l}(t_{lk}) - \sigma^2_{3-l}(t_{lk}) \right).
\end{aligned}$$

First, we consider the term

$$\tilde{T}_N^{(5)} = \frac{1}{N} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \tilde{\gamma}_{ij} \varepsilon_{1i} \varepsilon_{2j}.$$

Using the same argument as in the proof of Lemma A.2, we find that asymptotically the estimator $\hat{R}(t)$ can be replaced by the true $R(t)$ in order to calculate the variance of $\tilde{T}_N^{(5)}$. We then obtain

$$\begin{aligned}
\mathrm{Var}(\tilde{T}_N^{(5)}) &= \frac{1}{N^2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E[\tilde{\gamma}_{ij}^2 \varepsilon_{1i}^2 \varepsilon_{2j}^2] \\
&= \frac{1}{N^2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \Big\{ \frac{2}{N^2h^2} \sum_{l=1}^{2} \sum_{k=1}^{n_l} K(\frac{t_{1i}-t_{lk}}{h}) K(\frac{t_{2j}-t_{lk}}{h}) \frac{1}{R^2(t_{lk})} \sigma^2_{3-l}(t_{lk}) \\
&\quad - \frac{2}{Nh} K(\frac{t_{2j}-t_{1i}}{h}) \frac{1}{R(t_{1i})} - \frac{2}{Nh} K(\frac{t_{2j}-t_{1i}}{h}) \frac{1}{R(t_{2j})} \Big\}^2 \sigma_1^2(t_{1i}) \sigma_2^2(t_{2j}) \\
&\quad + o(\frac{1}{N^2h}) \\
&= \frac{4}{N^2} \int\int \Big\{ \frac{1}{h^2} \int K(\frac{x-z}{h}) K(\frac{y-z}{h}) \frac{1}{R^2(z)} \left(\kappa_1 r_1(z) \sigma_2^2(z) + \kappa_2 r_2(z) \sigma_1^2(z) \right) dz \\
&\quad - \frac{1}{h} K(\frac{y-x}{h}) \frac{1}{R(x)} - \frac{1}{h} K(\frac{y-x}{h}) \frac{1}{R(y)} \Big\}^2 \sigma_1^2(x) \sigma_2^2(y) \kappa_1 r_1(x) \kappa_2 r_2(y)\, dx\, dy \\
&\quad + o(\frac{1}{N^2h}) \\
&= \frac{4}{N^2h} \int (2K - K*K)^2(u)\, du \int_0^1 \frac{\sigma_1^2(x)\sigma_2^2(x)\kappa_1 r_1(x) \kappa_2 r_2(x)}{R^2(x)}\, dx \\
&\quad + o(\frac{1}{N^2h}).
\end{aligned}$$

Finally, we indicate the asymptotic negligibility of the second term

$$\bar{T}_N^{(5)} = \frac{1}{N} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \bar{\gamma}_{ij} \varepsilon_{1i} \varepsilon_{2j}.$$

In a first step we replace the estimate $\hat{R}(t)$ in the denominator by $R(t)$ without changing the asymptotic order. Then we insert the asymptotically dominating part of the expansion of the variance estimator from Proposition A.4 and obtain $\text{Var}(\bar{T}_N^{(5)}) = o(\frac{1}{N^2 h})$ with some tedious calculations. ■ (Lemma A.3)

With similar calculations as in the proof of Lemma A.3 we can rewrite $\tilde{T}_N^{(5)}$ as

$$\tilde{T}_N^{(5)} = \frac{2}{N^3}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\varepsilon_{1i}\varepsilon_{2j}\sigma_1(t_{1i})\sigma_2(t_{2j})\Big\{\frac{1}{h^2}\int K(\frac{t_{1i}-z}{h})K(\frac{t_{2j}-z}{h})\frac{1}{R(z)}\,dz$$
$$-\frac{1}{h}K(\frac{t_{2j}-t_{1i}}{h})\frac{1}{R(t_{1i})} - \frac{1}{h}K(\frac{t_{2j}-t_{1i}}{h})\frac{1}{R(t_{2j})}\Big\} + o(\frac{1}{N\sqrt{h}}).$$

Applying the same arguments to the terms $\tilde{T}_N^{(3)}$ and $\tilde{T}_N^{(4)}$ we obtain

$$N\sqrt{h}(T_N - E[T_N]) = N\sqrt{h}\left(\tilde{T}_N^{(3)} + \tilde{T}_N^{(4)} + \tilde{T}_N^{(5)}\right) + o_p(1),$$

which can be written as a quadratic form

$$W_N = \varepsilon_N^T A_N \varepsilon_N$$

of the random variable $\varepsilon_N = (\varepsilon_{11},\dots,\varepsilon_{1n_1},\varepsilon_{21},\dots,\varepsilon_{2n_2})^T$ with a symmetric matrix A_N with vanishing diagonal elements. From Lemma A.3 we obtain for the asymptotic variance

$$\begin{aligned}
\text{Var}(W_N) &= N^2 h\Big\{\text{Var}(T_N^{(3)}) + \text{Var}(T_N^{(4)}) + \text{Var}(T_N^{(5)})\Big\} + o(1)\\
&= 2\int (2K - K*K)^2(u)\,du\Big[2 + \int_0^1 \frac{\kappa_1^2 r_1^2(x)\sigma_2^4(x)}{R^2(x)}\,dx - 2\int_0^1 \frac{\kappa_1 r_1(x)\sigma_2^2(x)}{R(x)}\,dx\\
&\quad + \int_0^1 \frac{\kappa_2^2 r_2^2(x)\sigma_1^4(x)}{R^2(x)}\,dx - 2\int_0^1 \frac{\kappa_2 r_2(x)\sigma_1^2(x)}{R(x)}\,dx + 2\int_0^1 \frac{\sigma_1^2(x)\sigma_2^2(x)\kappa_1 r_1(x)\kappa_2 r_2(x)}{R^2(x)}\,dx\Big]\\
&\quad + o(1)\\
&= 2\int (2K - K*K)^2(u)\,du = \tau^2.
\end{aligned}$$

Asymptotic normality of W_N can be proved by an application of Theorem 5.2 of de Jong (1987) and this gives the conclusion of Theorem 2.1. ■

A.2 Auxiliary result

Proposition A.4 *Assume model (1) where the ε_{ij} are i. i. d. centered random variables with variance 1, such that assumptions (7)–(12) hold. For the heteroscedastic variance estimators defined in (16) we obtain the expansion ($i = 1, 2$)*

$$\hat{\sigma}_i^2(t) - \sigma_i^2(t) = \sum_{k=1}^{6} S_{n_i}^{(k)}(t)$$

where the dominating part is

$$\begin{aligned} S_{n_i}^{(1)}(t) &= \frac{1}{n_i h}\frac{1}{\hat{r}_i(t)}\sum_{l=1}^{n_i} K(\frac{t-t_{il}}{h})\sigma_i^2(t_{il})(\varepsilon_{il}^2-1) \\ &= O_p(\frac{1}{\sqrt{n_i h}}) \end{aligned}$$

with expectation zero. The second term $S_{n_i}^{(2)}(t)$ *is deterministic and satisfies*

$$\begin{aligned} S_{n_i}^{(2)}(t) &= \frac{1}{n_i h}\frac{1}{\hat{r}_i(t)}\sum_{l=1}^{n_i} K(\frac{t-t_{il}}{h})(\sigma_i^2(t_{il})-\sigma_i^2(t)) \\ &= O(h^d)+O(\frac{1}{n_i h}). \end{aligned}$$

Furthermore we have

$$\begin{aligned} S_{n_i}^{(3)}(t) &= \frac{2}{n_i h}\frac{1}{\hat{r}_i(t)}\sum_{l=1}^{n_i} K(\frac{t-t_{il}}{h})\sigma_i(t_{il})\varepsilon_{il}\Big(\frac{1}{n_i h}\sum_{k=1}^{n_i} K(\frac{t_{il}-t_{ik}}{h})\frac{f_i(t_{il})-f_i(t_{ik})}{\hat{r}_i(t_{il})}\Big) \\ &= \frac{2}{n_i h}\frac{1}{\hat{r}_i(t)}\sum_{l=1}^{n_i} K(\frac{t-t_{il}}{h})\sigma_i(t_{il})\varepsilon_{il}\Big(O(h^d)+O(\frac{1}{n_i h})\Big) \\ &= O_p(\frac{h^d}{\sqrt{n_i h}})+O_p(\frac{1}{(n_i h)^{3/2}}) \end{aligned}$$

with expectation zero,

$$\begin{aligned} S_{n_i}^{(4)}(t) &= -\frac{2}{n_i h}\frac{1}{\hat{r}_i(t)}\sum_{l=1}^{n_i} K(\frac{t-t_{il}}{h})\sigma_i(t_{il})\varepsilon_{il}\Big(\frac{1}{n_i h}\sum_{\substack{k=1\\k\neq l}}^{n_i} K(\frac{t_{il}-t_{ik}}{h})\frac{\sigma_i(t_{ik})\varepsilon_{ik}}{\hat{r}_i(t_{il})}\Big) \\ &= O_p(\frac{1}{n_i h}) \end{aligned}$$

with expectation zero, and

$$\begin{aligned} S_{n_i}^{(5)}(t) &= -\frac{2}{(n_i h)^2}\frac{1}{\hat{r}_i(t)}\sum_{l=1}^{n_i} K(\frac{t-t_{il}}{h})\frac{K(0)}{\hat{r}_i(t_{il})}\sigma_i^2(t_{il})\varepsilon_{il}^2 \\ &= O_p(\frac{1}{n_i h}) \end{aligned}$$

with asymptotic expectation $E[S_{n_i}^{(5)}(t)] = O(\frac{1}{n_i h})$. *Finally, we have*

$$\begin{aligned} S_{n_i}^{(6)}(t) &= \frac{1}{n_i h}\frac{1}{\hat{r}_i(t)}\sum_{l=1}^{n_i} K(\frac{t-t_{il}}{h})(f_i(t_{il})-\hat{f}_i(t_{il}))^2 \\ &= O_p(\frac{1}{n_i h})+O_p(h^{2d}) \end{aligned}$$

with asymptotic expectation $E|S_{n_i}^{(6)}(t)| = O(\frac{1}{n_i h})+O(h^{2d})$.

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(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.020	0.106	0.166	0.201	0.295	0.109	0.197	0.344
$\alpha = 5\%$	0.043	0.158	0.237	0.291	0.373	0.165	0.285	0.427
$\alpha = 10\%$	0.083	0.232	0.340	0.402	0.510	0.262	0.399	0.545
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.433	0.272	0.416	0.532	0.458	0.607	0.663	0.989
$\alpha = 5\%$	0.533	0.364	0.501	0.639	0.564	0.708	0.750	0.997
$\alpha = 10\%$	0.645	0.484	0.624	0.739	0.663	0.797	0.822	0.997

Table 1: Simulated power of the wild bootstrap version of the new test statistic (6) according to setting (27).

(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.017	0.046	0.075	0.119	0.157	0.058	0.138	0.278
$\alpha = 5\%$	0.028	0.077	0.132	0.194	0.247	0.109	0.210	0.354
$\alpha = 10\%$	0.054	0.126	0.211	0.301	0.357	0.162	0.298	0.459
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.349	0.189	0.326	0.465	0.370	0.525	0.592	0.984
$\alpha = 5\%$	0.447	0.267	0.419	0.550	0.470	0.633	0.664	0.989
$\alpha = 10\%$	0.543	0.377	0.530	0.644	0.567	0.728	0.755	0.993

Table 2: Simulated power of the wild bootstrap version of Dette and Neumeyer's (2001) test statistic (23) according to setting (27).

(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.030	0.102	0.175	0.248	0.296	0.099	0.197	0.292
$\alpha = 5\%$	0.054	0.149	0.234	0.328	0.379	0.152	0.267	0.401
$\alpha = 10\%$	0.084	0.214	0.322	0.430	0.493	0.225	0.366	0.518
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.355	0.252	0.373	0.461	0.401	0.521	0.651	0.991
$\alpha = 5\%$	0.449	0.328	0.473	0.590	0.513	0.648	0.734	0.995
$\alpha = 10\%$	0.573	0.430	0.579	0.717	0.620	0.754	0.832	1.000

Table 3: Simulated power of the wild bootstrap version of the new test statistic (6) according to setting (28).

(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.019	0.033	0.071	0.112	0.162	0.061	0.130	0.214
$\alpha = 5\%$	0.033	0.059	0.132	0.183	0.239	0.089	0.181	0.301
$\alpha = 10\%$	0.059	0.100	0.198	0.282	0.341	0.131	0.279	0.383
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.288	0.190	0.313	0.418	0.336	0.475	0.567	0.984
$\alpha = 5\%$	0.376	0.257	0.407	0.522	0.416	0.563	0.662	0.990
$\alpha = 10\%$	0.476	0.340	0.503	0.626	0.528	0.673	0.751	0.996

Table 4: Simulated power of the wild bootstrap version of Dette and Neumeyer's (2001) test statistic (23) according to setting (28).

(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.254	0.381	0.501	0.585	0.660	0.402	0.522	0.684
$\alpha = 5\%$	0.314	0.483	0.603	0.692	0.764	0.511	0.664	0.784
$\alpha = 10\%$	0.396	0.604	0.724	0.801	0.849	0.637	0.780	0.873
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.741	0.614	0.704	0.826	0.762	0.873	0.867	0.998
$\alpha = 5\%$	0.837	0.727	0.803	0.892	0.848	0.922	0.923	0.999
$\alpha = 10\%$	0.921	0.845	0.899	0.956	0.913	0.966	0.962	1.000

Table 5: Simulated power of the wild bootstrap version of the new test statistic (6) according to setting (29).

(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.302	0.313	0.325	0.350	0.354	0.524	0.628	0.707
$\alpha = 5\%$	0.366	0.427	0.446	0.482	0.501	0.611	0.722	0.795
$\alpha = 10\%$	0.457	0.543	0.576	0.613	0.635	0.704	0.810	0.872
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.724	0.761	0.784	0.866	0.852	0.892	0.929	0.998
$\alpha = 5\%$	0.807	0.829	0.858	0.918	0.909	0.935	0.955	0.999
$\alpha = 10\%$	0.868	0.890	0.904	0.948	0.938	0.963	0.981	0.999

Table 6: Simulated power of the wild bootstrap version of Dette and Neumeyer's (2001) test statistic (23) according to setting (29).

(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.071	0.175	0.217	0.281	0.259	0.082	0.244	0.315
$\alpha = 5\%$	0.109	0.234	0.287	0.355	0.346	0.139	0.311	0.421
$\alpha = 10\%$	0.175	0.347	0.410	0.466	0.452	0.220	0.398	0.532
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.391	0.103	0.278	0.393	0.125	0.286	0.131	0.162
$\alpha = 5\%$	0.496	0.157	0.366	0.500	0.195	0.378	0.193	0.243
$\alpha = 10\%$	0.615	0.246	0.472	0.611	0.288	0.476	0.287	0.348

Table 7: Simulated power of the wild bootstrap version of the new test statistic (6) according to setting (30).

(n_1, n_2)	(10,10)	(10,20)	(10,30)	(10,40)	(10,50)	(20,20)	(20,30)	(20,40)
$\alpha = 2.5\%$	0.005	0.134	0.205	0.244	0.226	0.010	0.182	0.302
$\alpha = 5\%$	0.008	0.225	0.282	0.341	0.323	0.021	0.239	0.386
$\alpha = 10\%$	0.029	0.310	0.371	0.446	0.451	0.056	0.314	0.494
(n_1, n_2)	(20,50)	(30,30)	(30,40)	(30,50)	(40,40)	(40,50)	(50,50)	(100,100)
$\alpha = 2.5\%$	0.384	0.026	0.191	0.337	0.030	0.188	0.035	0.062
$\alpha = 5\%$	0.477	0.051	0.257	0.432	0.050	0.266	0.067	0.111
$\alpha = 10\%$	0.584	0.083	0.346	0.540	0.099	0.352	0.113	0.173

Table 8: Simulated power of the wild bootstrap version of Dette and Neumeyer's (2001) test statistic (23) according to setting (30).