HY-A-PARCH: A Stationary A-PARCH Model with Long Memory[1]

Olaf Schoffer

Fachbereich Statistik, Universität Dortmund, D-44221 Dortmund

December 2003

Abstract

The FI-A-PARCH process has been developed by Tse (1998) to model essential characteristics of financial market returns. However, due to the nonstationarity described by Níguez (2002) the process exhibits infinite conditional second moments and no statements about the autocovariance function can be derived. Thus, the new Hyperbolic A-PARCH model is considered, first introduced in Schoffer (2003). Subsequently the characteristics of this extension of the FI-A-PARCH process are inspected. It can be shown, that under certain parameter restrictions the intrinsic process as well as the process of conditional volatilities is stationary. Furthermore, for an asymmetric transformation of the conditional volatilities the presence of long memory is proven. Thus, the introduced model is able to reproduce the main characteristics of financial market returns such as volatility clustering, leptokurtosis, asymmetry and long memory.

[1]Research supported by Deutsche Forschungsgemeinschaft under SFB 475

1 Introduction and summary

This paper generalizes existing ARCH models introduced by Engle (1982) and Bollerslev (1986) to allow for long memory in conditional second moments. It combines the FI-A-PARCH model of Tse (1998) and the HYGARCH model of Davidson (2003) to produce a new model which exhibits all the features found in empirical investigations for financial market returns: volatility clustering, leptokurtosis, asymmetry and long memory.

The asymmetric behavior of certain financial market data as well as the persistence of shocks in conditional volatilities are investigated in recent studies. Asymmetry means that a decrease, e.g. for asset prices, is followed by an increase of the conditional volatility, as observed by Black (1976) and Christie (1982). Concerning persistence it is known that the empirical autocorrelations of the conditional volatilities decay only very slowly by rising order, more precisely with hyperbolic decay rate. That fact is called long memory. To model these characteristics various extensions of the GARCH model like EGARCH, A-PARCH and FIGARCH are introduced (for details see Andersen and Bollerslev, 1998).

Many of these extensions have an ARCH(∞) representation. Based on this a representation can be derived, which allows the derivation of moment and stationarity properties using the Volterra series expansion (see Priestley, 1988). In particular Giraitis et al. (2000) deduce sufficient conditions for strict and weak stationarity. Furthermore, they derive conditions for the existence of certain moments as well as statements about the form of the autocovariance function. Consequently the presence of long memory, based on the Volterra representation of ARCH processes is given.

The fractional integrated models FIGARCH and FI-A-PARCH introduced by Baillie et al. (1996) and Tse (1998) do not permit statements about the autocovariance function due to infinite conditional second moments. Thus, the new Hyperbolic A-PARCH model is considered, first introduced in Schoffer (2003). This model allows to reproduce all presented characteristics of returns of financial time series (volatility clustering, leptokurtosis, asymmetry, long memory). Furthermore, it considers that the conditional volatility is represented at best by non-integer power of the absolute value of the observations according to Ding et al. (1993). This extension of the A-PARCH approach is formulated analogous to the HYGARCH model presented of Davidson (2003). Finally, first characteristics of this model are derived based on its Volterra series expansion.

2 The Hyperbolic A-PARCH model

Consider first the above mentioned FI-A-PARCH model, which includes the GARCH, A-PARCH and FIGARCH models as special cases. Let $z_t = \big(|y_t| - \eta y_t\big)^\delta$ with $|\eta| < 1$ and $\nu_t = z_t - \sigma_t^\delta$. Then the FI-A-PARCH process solves the equation

$$\big(1 - \alpha(L) - \beta(L)\big)(1 - L)^d\big(\{z_t\}\big) = \alpha_0 + \big(1 - \beta(L)\big)\big(\{\nu_t\}\big) \ . \tag{2.1}$$

If $(1 - \beta(L))^{-1}$ exists, $\{\sigma_t^\delta\}$ can be represented as a function of $\{z_t\}$ as follows

$$\{\sigma_t^\delta\} = \big(1-\beta(L)\big)^{-1}\Big(\alpha_0 - \alpha(L) + \big(1-\alpha(L)-\beta(L)\big)\big(1-(1-L)^d\big)\Big)\big(\{z_t\}\big) \ .$$

Denote by $\varphi(L)$ the filter

$$\varphi(L) := \big(1-\beta(L)\big)^{-1}\Big(\alpha_0 - \alpha(L) + \big(1-\alpha(L)-\beta(L)\big)\big(1-(1-L)^d\big)\Big) \ .$$

Then we have from (2.1)

$$\{\sigma_t^\delta\} = \varphi(L)\big(\{z_t\}\big) \ .$$

Hence, the model equation of a FI-A-PARCH process is given as

$$\sigma_t^\delta = \varphi_0 + \sum_{j=1}^{\infty} \varphi_j z_{t-j} = \varphi_0 + \sum_{j=1}^{\infty} \varphi_j (|y_{t-j}| - \eta y_{t-j})^\delta \ , \tag{2.2}$$

where φ_j denotes the weight of the filter $\varphi(L)$ at j^{th} lag ($j \in \mathbb{N}$).

Equation (2.2) presents $\{y_t\}$ as an A-PARCH$(0,\infty)$ process, and $\{z_t\}$ as an ARCH(∞) process with innovations $(|\varepsilon_t| - \eta\varepsilon_t)^\delta$. Moreover, the filter $\varphi(L)$ is identical to that of the FIGARCH model using the appropriate ARCH(∞) representation. Note that the restrictions $\delta = 2$ and $\eta = 0$ in the FI-A-PARCH model do not affect the form of $\varphi(L)$.

In order to eliminate the nonstationarity of the FIGARCH process, Davidson (2003) introduces a Hyperbolic GARCH model. The modification in relation to the FIGARCH model consists of replacing the fractional difference $(1-L)^d$ with $\big((1-\tau)+\tau(1-L)^d\big)$ where $\tau \geq 0$. Under the condition $\tau < 1$ and further restrictions on the remaining parameters of the model the resulting stochastic process is weakly stationary. In addition the squared values of the series exhibit long memory. Thus, the HYGARCH model is able to reproduce the characteristics volatility clustering, leptokurtosis and long memory. As a result of the described modification the model equation for HYGARCH is given by

$$\big(1-\alpha(L)-\beta(L)\big)\big((1-\tau)+\tau(1-L)^d\big)\big(\{y_t^2\}\big)=\alpha_0+\big(1-\beta(L)\big)\big(\{y_t^2-\sigma_t^2\}\big)\,.$$

However, this model disregards asymmetry and the fact that the conditional volatility is best represented by non-integer powers of the absolute value of the observations. Therefore, Schoffer (2003) suggests a new approach to combine the features of the HYGARCH model with those of the A-PARCH model. The necessary modification in relation to the HYGARCH model consists of replacing y_t^2 with $z_t = \big(|y_t| - \eta y_t\big)^\delta$ and σ_t^2 with σ_t^δ.

Definition 2.1: *A stochastic process $\{y_t\}$ is called **Hyperbolic A-PARCH Process** of the orders p and q with **memory parameter** d or briefly HY-A-PARCH(p,d,q) process, if for $t \in \mathbb{Z}$, $\delta > 0$, $|\eta| < 1$, $0 \leq d \leq 1$ and $\tau \geq 0$ the following equations are satisfied*

$$y_t = \varepsilon_t \cdot \sigma_t, \qquad \varepsilon_t \overset{iid.}{\sim} \mathcal{P}_{0,1}$$
$$\big(1-\alpha(L)-\beta(L)\big)\big((1-\tau)+\tau(1-L)^d\big)\big(\{z_t\}\big)=\alpha_0+\big(1-\beta(L)\big)\big(\{\nu_t\}\big)\,, \qquad (2.3)$$

where $\alpha(L)=\sum_{j=1}^{q}\alpha_j L^j$, $\beta(L)=\sum_{j=1}^{p}\beta_j L^j$, $z_t=\big(|y_t|-\eta y_t\big)^\delta$ and $\nu_t = z_t - \sigma_t^\delta$.

Since model (2.3) is a direct generalization of the HYGARCH and FI-A-PARCH models it contains these as special cases. The HY-A-PARCH model corresponds to the HYGARCH model for $\delta = 2$, $\eta = 0$ and to the FI-A-PARCH model for $\tau = 1$.

Similar to the FI-A-PARCH model a HY-A-PARCH process has an A-PARCH$(0,\infty)$ representation using

$$\{\sigma_t^\delta\} = \varphi(L)\big(\{z_t\}\big)\ .$$

Then the required filter $\varphi(L)$ has the form

$$\begin{aligned}\varphi(L) \;=\; & \big(1-\beta(L)\big)^{-1}\big(\alpha_0+\alpha(L)\big) \\ & +\tau\big(1-\beta(L)\big)^{-1}\big(1-\alpha(L)-\beta(L)\big)\big(1-(1-L)^d\big).\end{aligned} \tag{2.4}$$

Therefore, the HY-A-PARCH model with the changed weights regarding FI-A-PARCH φ_j can be described as A-PARCH$(0,\infty)$ process by equation (2.2). Hence, in analogy to the nonnegativity constraint for A-PARCH models in Ding et al. (1993) the following restriction ensures that the conditional variances σ_t^2 remain nonnegative with probability 1:

$$\varphi_j \geq 0 \quad \forall\, j \in \mathbb{N}\ .$$

To simplify the investigation of the weak stationarity of HY-A-PARCH processes in section 4 let

$$\phi^{(1)}(L) := \big(1-\beta(L)\big)^{-1}\big(\alpha_0+\alpha(L)\big)$$

and

$$\phi^{(2)}(L) := \big(1-\beta(L)\big)^{-1}\big(1-\alpha(L)-\beta(L)\big)\,\big(1-(1-L)^d\big)\,.$$

Then for HY-A-PARCH processes the following relation for the weights of the filters $\varphi(L)$, $\phi^{(1)}(L)$ and $\phi^{(2)}(L)$ holds

$$\varphi_j = \phi_j^{(1)} + \tau\phi_j^{(2)}\ . \tag{2.5}$$

3 Volterra series expansion of Asymmetric Power GARCH models

The Volterra series expansion (see Priestley, 1988, p. 25) facilitates asymptotic statements about stationarity, moments and the autocorrelation structure of non-linear stochastic processes. It is exemplified here using the Asymmetric Power GARCH models. The HY-A-PARCH model and its special cases can be described using equation (2.2) with appropriate weights $\varphi_j \quad (j \in \mathbb{N})$. Thus, this equation is the initial point for Volterra series expansion.

Because $y_t = \varepsilon_t \cdot \sigma_t$ holds, it follows from (2.2) that

$$\begin{aligned}\sigma_t^\delta &= \varphi_0 + \sum_{j=1}^{\infty} \varphi_j (|\varepsilon_{t-j}\sigma_{t-j}| - \eta\varepsilon_{t-j}\sigma_{t-j})^\delta \\ &= \varphi_0 + \sum_{j=1}^{\infty} \varphi_j (|\varepsilon_{t-j}| - \eta\varepsilon_{t-j})^\delta \sigma_{t-j}^\delta \ .\end{aligned}$$

The following recursive representation, which is equivalent to equation (2.2), results using the notation $\zeta_t := \big(|\varepsilon_t| - \eta\varepsilon_t\big)^\delta$

$$\sigma_t^\delta = \varphi_0 + \sum_{j=1}^{\infty} \varphi_j \zeta_{t-j} \sigma_{t-j}^\delta \ . \tag{3.1}$$

Solving equation (3.1) iteratively produces

$$\begin{aligned}\sigma_t^\delta &= \varphi_0 + \sum_{j_1=1}^{\infty} \varphi_{j_1}\zeta_{t-j_1}\left(\varphi_0 + \sum_{j_2=1}^{\infty} \varphi_{j_2}\zeta_{t-j_1-j_2}\sigma_{t-j_1-j_2}^\delta\right) \\ &= \varphi_0 + \sum_{j_1=1}^{\infty} \varphi_0\varphi_{j_1}\zeta_{t-j_1} + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} \varphi_{j_1}\varphi_{j_2}\zeta_{t-j_1}\zeta_{t-j_1-j_2}\sigma_{t-j_1-j_2}^\delta \\ &\ \vdots \\ &= \varphi_0 \sum_{\ell=0}^{\infty} \sum_{j_1,\dots,j_\ell=1}^{\infty} \varphi_{j_1}\cdots\varphi_{j_\ell}\cdot\zeta_{t-j_1}\cdots\zeta_{t-j_1-\dots-j_\ell} \ .\end{aligned}$$

This expression is called the Volterra series expansion of Asymmetric Power GARCH models. Here $\sum_{j_1,\dots,j_\ell=1}^{\infty}$ denotes the ℓ-fold execution of the summation with the indices j_1 to j_ℓ.

Thus, $\big\{|y_t|^\delta\big\}$ can be represented as a function of $\{\varepsilon_t\}$ using $\zeta_t = \big(|\varepsilon_t| - \eta\varepsilon_t\big)^\delta$:

$$|y_t|^\delta = \varphi_0 |\varepsilon_t|^\delta \sum_{\ell=0}^{\infty} \sum_{j_1,\dots,j_\ell=1}^{\infty} \varphi_{j_1}\cdots\varphi_{j_\ell}\cdot\zeta_{t-j_1}\cdots\zeta_{t-j_1-\dots-j_\ell} \ . \tag{3.2}$$

In order to be able to use the statements based on the Volterra series expansion of ARCH processes directly for the Asymmetric Power GARCH models the following representation is necessary

$$z_t = \varphi_0 \sum_{\ell=0}^{\infty} \sum_{j_1,\ldots,j_\ell=1}^{\infty} \varphi_{j_1} \ldots \varphi_{j_\ell} \cdot \zeta_t \zeta_{t-j_1} \ldots \zeta_{t-j_1-\ldots-j_\ell} \ . \tag{3.3}$$

Let $X_t = z_t$ and $\xi_t = \zeta_t$. Then the equation (3.3) is identical with the Volterra series expansion of ARCH processes described in equation (2.1) in Giraitis et al. (2000). However, by using this identity it is only possible to derive directly conclusions on z_t and not on $|y_t|^\delta$ or y_t, respectively.

Consider the relation

$$z_t^\delta = \big(|y_t| - \eta y_t\big)^\delta = \big(1-\eta\cdot\text{sign}(y_t)\big)^\delta \cdot |y_t|^\delta = \big(1-\eta\cdot\text{sign}(\varepsilon_t)\big)^\delta \cdot |y_t|^\delta ,$$

where $\text{sign}(x) = \left\{\begin{smallmatrix} 1, & x\geq 0 \\ -1, & x<0 \end{smallmatrix}\right.$.

According to Randles and Wolfe (1979), p. 49 $\text{sign}(\varepsilon_t)$ and $|\varepsilon_t|$ are stochastically independent if the distribution of ε_t is symmetric about zero. Since σ_t in addition depends only on the values ε_{t-k} with $k \geq 1$ also σ_t is independent of $\text{sign}(\varepsilon_t)$. Thus, $\text{sign}(\varepsilon_t)$ and the product $|y_t| = \sigma_t \cdot |\varepsilon_t|$ are stochastically independent as well as their monotonous transformations $\big(1-\eta\cdot\text{sign}(\varepsilon_t)\big)^\delta$ and $|y_t|^\delta$.

Therefore, if the distribution of ε_t is symmetric about zero the following equation holds for arbitrary moments of z_t

$$\begin{aligned} E\big(z_t^k\big) &= E\big((1-\eta\cdot\text{sign}(\varepsilon_t))^{k\delta} \cdot |y_t|^{k\delta}\big) \\ &= E\big((1-\eta\cdot\text{sign}(\varepsilon_t))^{k\delta}\big) \cdot E\big(|y_t|^{k\delta}\big) \ . \end{aligned} \tag{3.4}$$

Since $E\big((1-\eta\cdot\text{sign}(\varepsilon_t))^{k\delta}\big)$ is a constant, statements about moments of $|y_t|^\delta$ follow directly from statements about moments of z_t using equation (3.4).

Under the mentioned assumption of symmetry it also follows that

$$\begin{aligned}
E\big((1-\eta\cdot\text{sign}(\varepsilon_t))^{k\delta}\big) &= \int\limits_{-\infty}^{\infty}(1-\eta\cdot\text{sign}(x))^{k\delta}\, f_{\varepsilon_t}(x)\; dx \\
&= \int\limits_{-\infty}^{0}(1+\eta)^{k\delta}\, f_{\varepsilon_t}(x)\; dx + \int\limits_{0}^{\infty}(1-\eta)^{k\delta}\, f_{\varepsilon_t}(x)\; dx \\
&= (1+\eta)^{k\delta}\int\limits_{-\infty}^{0} f_{\varepsilon_t}(x)\; dx + (1-\eta)^{k\delta}\int\limits_{0}^{\infty} f_{\varepsilon_t}(x)\; dx \\
&= (1+\eta)^{k\delta}\cdot\frac{1}{2} + (1-\eta)^{k\delta}\cdot\frac{1}{2} \\
&= \frac{1}{2}\left((1+\eta)^{k\delta} + (1-\eta)^{k\delta}\right) \ ,
\end{aligned}$$

where f_{ε_t} is the unconditional density of ε_t.

Thus, equation (3.4) can be further simplified to

$$E\big(z_t^k\big) = \frac{1}{2}\left((1+\eta)^{k\delta} + (1-\eta)^{k\delta}\right)\cdot E\big(|y_t|^{k\delta}\big) \ .$$

In section 4 the characteristics of HY-A-PARCH processes are examined. Some findings are based on Giraitis et al. (2000) using the Volterra series expansion of ARCH processes.

According to theorem 2.1 in Giraitis et al. (2000) and under the conditions

$$E(\zeta_t) < \infty \qquad \text{and} \qquad E(\zeta_t)\sum_{j=1}^{\infty}\varphi_j < 1 \tag{3.5}$$

the equation (3.3) is a strict stationary solution to $z_t = \sigma_t^\delta\cdot\zeta_t$ and (2.2) with the finite first moment $E(z_t)$. If in addition the conditions

$$E(\zeta_t^2) < \infty \qquad \text{and} \qquad \big(E(\zeta_t^2)\big)^{\frac{1}{2}}\sum_{j=1}^{\infty}\varphi_j < 1 \tag{3.6}$$

hold, the stochastic process described by equation (3.3) is a unique and weak stationary solution to $z_t = \sigma_t^\delta\cdot\zeta_t$ and (2.2).

Consider that the weights φ_j fulfill the condition

$$c_1 j^{-\gamma} \le \varphi_j \le c_2 j^{-\gamma} \tag{3.7}$$

for sufficiently large $j \in \mathbb{N}$ and for fixed values $\gamma > 1$, $c_1 > 0$, $c_2 > 0$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that for sufficiently large $k \in \mathbb{N}$

$$C_1 k^{-\gamma} \leq \text{Cov}(z_t, z_{t-k}) \leq C_2 k^{-\gamma} \ . \tag{3.8}$$

Thus, under assumption (3.6) $\{z_t\}$ is a stationary process with long memory if the weights φ_j decay hyperbolically in j according to proposition 3.2 in Giraitis et al. (2000). Note that inequality (3.8) is equivalent to the widely used definition of long memory for a stochastic process $\{z_t\}$ given in Campbell et al. (1997), p.61:

$$\text{Cov}(z_t, z_{t-k}) \asymp \begin{cases} k^\nu f(k) \ , \ \nu \in (-1,0) \ \text{ or} \\ -k^\nu f(k) \ , \ \nu \in (-2,-1) \end{cases} \quad \text{as} \ \ k \to \infty \ ,$$

where $\lim_{k\to\infty} \frac{f(tk)}{f(k)} = 1$ holds for any $t \geq 1$. $\alpha_k \asymp \beta_k$ means that there are positive constants C_1 and C_2 such that $C_1\beta_k < \alpha_k < C_2\beta_k, \ k > k_0$ for some $k_0 > 0$.

4 Properties of HY-A-PARCH processes

The HY-A-PARCH approach enables the modelling of many characteristics of financial market returns in the framework of stationary processes. To see this parameter restrictions are presented, which ensure the weak stationarity of these processes. Then follows the derivation of stationarity constraints for the transformed process $\{z_t\} = \big\{(|y_t| - \eta y_t)^\delta\big\}$ using the findings of Giraitis et al. (2000). Based on this the presence of long memory in $\{z_t\}$ concludes.

Let $\varphi(L)$ the ARCH(∞) filter belonging to the HYGARCH model (see Davidson, 2003). Then equation (2.2) describes a HY-A-PARCH process. In accordance to the law of iterated expectations

$$\begin{aligned} E\big(\sigma_t^\delta\big) &= \varphi_0 + \sum_{j=1}^{\infty} \varphi_j E\big((|y_{t-j}| - \eta y_{t-j})^\delta\big) \\ &= \varphi_0 + \sum_{j=1}^{\infty} \varphi_j E\big((|\varepsilon_{t-j}| - \eta \varepsilon_{t-j})^\delta\big) E(\sigma_{t-j}^\delta) \\ &\overset{\varepsilon_t \ iid.}{=} \varphi_0 + \sum_{j=1}^{\infty} \varphi_j E\big((|\varepsilon_t| - \eta \varepsilon_t)^\delta\big) E(\sigma_t^\delta) \end{aligned}$$

under the condition

$$\sum_{j=1}^{\infty} \varphi_j E\big((|\varepsilon_t| - \eta\varepsilon_t)^{\delta}\big) < 1 \tag{4.1}$$

the following equation holds

$$E\big(\sigma_t^{\delta}\big) = \frac{\varphi_0}{1 - \sum_{j=1}^{\infty} \varphi_j E\big((|\varepsilon_t| - \eta\varepsilon_t)^{\delta}\big)} \; .$$

According to Ding et al. (1993) the inequality (4.1) is a sufficient and necessary condition for the existence of $E(\sigma_t^{\delta})$ and $E(|y_t|^{\delta})$. Under the restrictions $\delta = 2$ and $\eta = 0$ using the relation

$$\sum_{j=1}^{\infty} \varphi_j E\big((|\varepsilon_t| - 0\varepsilon_t)^2\big) = \sum_{j=1}^{\infty} \varphi_j E(\varepsilon_t^2) \overset{\varepsilon_t \overset{iid}{\sim} \mathcal{P}(0,1)}{=} \sum_{j=1}^{\infty} \varphi_j$$

the inequality (4.1), which ensures the existence of the moments $E(\sigma_t^2)$ and $E(|y_t|^2)$, reduces to

$$\sum_{j=1}^{\infty} \varphi_j < 1 \; .$$

According to Davidson (2003) a HYGARCH process with the restrictions $0 \leq \tau < 1$ and $\big(1 - \beta(1)\big)^{-1}\big(\alpha_0 - \alpha(1)\big) > 0$ is weakly stationary. However, due to infinite second moments this process is nonstationary if $\tau = 1$, i.e. for the special case FIGARCH. Since (4.1) is a sufficient and necessary condition for the existence of these moments, one has for stationary HYGARCH processes that

$$\sum_{j=1}^{\infty} \varphi_j < 1$$

and for FIGARCH processes that

$$\sum_{j=1}^{\infty} \varphi_j \geq 1 \; .$$

Thus, with (2.5) for $0 \leq \tilde{\tau} < 1$ holds

$$\sum_{j=1}^{\infty} \phi_j^{(1)} + \tilde{\tau}\phi_j^{(2)} < 1 \qquad \text{as well as} \qquad \sum_{j=1}^{\infty} \phi_j^{(1)} + \phi_j^{(2)} \geq 1 \; . \tag{4.2}$$

The inequalities correspond to the stationary HYGARCH process and the non-stationary FIGARCH process, respectively. Using the relation

$$\sum_{j=1}^{\infty} \phi_j^{(1)} + \tau\phi_j^{(2)} = \sum_{j=1}^{\infty} \phi_j^{(1)} + \tau \sum_{j=1}^{\infty} \phi_j^{(2)} \qquad \forall \tau \in [0,1]$$

follows from (4.2) that

$$\sum_{j=1}^{\infty} \phi_j^{(1)} + \phi_j^{(2)} = 1 \ . \tag{4.3}$$

Note that the condition (4.1) for the HY-A-PARCH model in the notation of equation (2.5) is given by

$$\sum_{j=1}^{\infty} \left(\phi_j^{(1)} + \tau\phi_j^{(2)}\right) E\big((|\varepsilon_t| - \eta\varepsilon_t)^\delta\big) < 1 \ .$$

Thus, using equation (4.3) this condition is equivalent to

$$\tau < 1 + \frac{1}{E\big((|\varepsilon_t| - \eta\varepsilon_t)^\delta\big) \sum\limits_{j=1}^{\infty} \phi_j^{(2)}} - \frac{1}{\sum\limits_{j=1}^{\infty} \phi_j^{(2)}} \ . \tag{4.4}$$

Thus, the moments $E(\sigma_t^\delta)$ and $E(|y_t|^\delta)$ exist under the constraints (4.4) and $\big(1-\beta(1)\big)^{-1}\big(\alpha_0-\alpha(1)\big) > 0$. If in addition $\delta \geq 2$ is fulfilled it can be concluded that the respective HY-A-PARCH process $\{y_t\}$ is weakly stationary on the analogy to appendix B in Ding et al. (1993).

In condition (4.4) as well as in theorem 2.1 from Giraitis et al. (2000)

$$E(\zeta_t) = E\left((|\varepsilon_t| - \eta\varepsilon_t)^\delta\right) \tag{4.5}$$

is considered. Since ε_t are *iid* this value is a constant depending only on the distribution of ε_t. This constant is specified for the distributions $\mathcal{N}(0,1)$, $GED(0,1)$ and $t(0,1)$ in Ding et al. (1993), Laurent and Peters (2002) and Lambert and Laurent (2001), respectively.

In order to prove the presence of long memory for the transformed process $\{z_t\}$ it is sufficient to examine whether the condition (3.6) is fulfilled and wether the weights of the associated filter $\varphi(L)$ decay hyperbolically (see Giraitis et al., 2000).

In condition (3.6) the expectation $E(\zeta_t^2)$ is considered. However, $E(\zeta_t^2) = E\big((|\varepsilon_t| - \eta\varepsilon_t)^{2\delta}\big)$ result from $E(\zeta_t) = E\big((|\varepsilon_t| - \eta\varepsilon_t)^{\tilde{\delta}}\big)$ using $\tilde{\delta} = 2\delta$. Thus, this constant can be computed using the expectation value (4.5), too. Therefore, the condition $E(\zeta_t^2) < \infty$ is fulfilled for the distributions considered in this work. The second inequality from (3.6) can be transformed equivalently to

$$\tau < 1 + \frac{1}{\big(E(\zeta_t^2)\big)^{\frac{1}{2}} \sum\limits_{j=1}^{\infty} \phi_j^{(2)}} - \frac{1}{\sum\limits_{j=1}^{\infty} \phi_j^{(2)}} \ . \tag{4.6}$$

Thus, for values of τ, which satisfy this inequality the condition (3.6) as first assumption for proposition 3.2 in Giraitis et al. (2000) is fulfilled. Then $\{z_t\}$ represents a weak stationary process.

The second condition for proposition 3.2 in Giraitis et al. (2000), i.e. the hyperbolic decay of the weights φ_j for the filter belonging to a HY-A-PARCH process can be traced back to the behavior of the weights for the appropriate HYGARCH filter, since the transformation $g(x) = \big(|x| - \eta x\big)^{\delta}$ does not change this filter.

In Davidson (2003) the filter $\varphi(L)$ of the HYGARCH model is approximated by

$$\varphi(L) \approx \big(1-\beta(L)\big)^{-1}\big(\alpha_0+\alpha(L)\big)+\tau\big(1-\beta(L)\big)^{-1}\big(1-\alpha(L)-\beta(L)\big)\psi(L)$$

and

$$\psi(L) = \zeta(1+d)^{-1} \sum_{j=1}^{\infty} j^{-1-d} L^j \ ,$$

where $\zeta(.)$ denotes the Riemann zeta function. From this the hyperbolic descent behavior of the weights can be derived. This holds for the HYGARCH model as well as for the HY-A-PARCH model. With this characteristic and a value for τ, which satisfies inequality (4.6) the presence of long memory in $\{z_t\} = \big\{(|y_t| - \eta y_t)^{\delta}\big\}$ according to proposition 3.2 in Giraitis et al. (2000) is ensured. The process $\{z_t\}$ can be construed as representation of the *asymmetric* conditional volatilities, so that HY-A-PARCH can be regarded as a stationary process which reproduces all characteristics presented above.

However, it is not yet possible to derive a statement for the process $\{|y_t|^\delta\}$ of the conditional volatilities in Asymmetric Power GARCH models. Properties of the autocovariances of $\{|y_t|^\delta\}$ do not result directly from properties of the autocovariances of $\{z_t\}$ since by construction of the A-PARCH models $\big(1-\eta\text{sign}(\varepsilon_{t-k})\big)^\delta$ and $|y_t|^\delta$ are not independent for $k > 0$ and $\eta \neq 0$. Thus, it holds that

$$\begin{aligned}
\text{Cov}(z_t, z_{t-k}) &= \text{Cov}\Big(\big(1-\eta\cdot\text{sign}(\varepsilon_t)\big)^\delta |y_t|^\delta, \big(1-\eta\cdot\text{sign}(\varepsilon_{t-k})\big)^\delta |y_{t-k}|^\delta\Big) \\
&= E\Big(\big(1-\eta\cdot\text{sign}(\varepsilon_t)\big)^\delta\Big)\cdot\text{Cov}\Big(|y_t|^\delta, \big(1-\eta\cdot\text{sign}(\varepsilon_{t-k})\big)^\delta |y_{t-k}|^\delta\Big) \\
&\neq E\Big(\big(1-\eta\cdot\text{sign}(\varepsilon_t)\big)^\delta\Big)^2\cdot\text{Cov}\big(|y_t|^\delta, |y_{t-k}|^\delta\big) \quad .
\end{aligned}$$

It is supposed that the generic form of the autocovariance function for lags of higher order is not changed by the transformation of $\big(|y_t| - \eta y_t\big)^\delta$ to $|y_t|^\delta$. I.e. the hyperbolic decay rate remains intact. However, the final answer to the question about the memory of $\{|y_t|^\delta\}$ remains for future research. First it can be examined whether the empirical autocorrelation function of $\{|y_t|^\delta\}$ exhibits behavior appropriate to long memory using simulations of different HY-A-PARCH processes. In order to examine the behavior of the autocovariance function under above transformation analytically the method of the Appell Polynomials is recommended (see Avram and Taqqu, 1987).

5 Conclusion

The Hyperbolic GARCH model introduced by Davidson (2003) allows for long memory in the process of conditional volatilities. However, using this model no asymmetry can be described. Therefore, in this work the extension of A-PARCH processes to the Hyperbolic A-PARCH process is considered.

Characteristics of the new model are derived using the Volterra series expansion of Asymmetric Power GARCH models. Thus, necessary conditions for stationarity and the existence of second moments are derived. Furthermore, the presence of long memory in $\{(|y_t| - \eta y_t)^\delta\}$, which can be construed as representation of the *asymmetric* conditional volatilities, is proven. However, for the process $\{|y_t|^\delta\}$, which represents the conditional volatilities, it is not yet possible to derive such a statement about the autocorrelation structure. Thus, the HY-A-PARCH process enables to model the main characteristics of financial market returns such as volatility clustering, leptokurtosis and asymmetry as well as the presence of long memory at least for a transformation of the conditional volatilities.

It should be mentioned at last that the Hyperbolic A-PARCH model is implemented in the package 'Time Series Modelling' by James Davidson. The package is free for download under `http://www.cf.ac.uk/carbs/econ/davidsonje/software.html`.

References

Andersen, T. G. and T. Bollerslev (1998). ARCH and GARCH models. In S. Kotz, C. B. Read, and D. L. Banks (Eds.), *Encyclopedia of Statistical Sciences Vol. II*, pp. 6–16. New York: Wiley.

Avram, F. and M. S. Taqqu (1987). Noncentral limit theorems and Appell polynomials. *Annals of Probability 15*, 767–775.

Baillie, R. T., T. Bollerslev, and H. O. Mikkelsen (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics 74*, 3–30.

Black, F. (1976). Studies of stock price volatility changes. In *1976 Proceedings of the American Statistical Association, Business and Economic Statistics Section*, pp. 177–181.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics 31*, 307–327.

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay (1997). *The Econometrics of Financial Markets.* Princeton: University Press.

Christie, A. (1982). The stochastic behaviour of common stock variances: Values, leverage and interest rate effects. *Journal of Financial Economics 10*, 15–36.

Davidson, J. (2003). Moment and memory properties of linear conditional heteroscedasticity models. Forthcoming in *Journal of Business and Economics Statistics*.

Ding, Z., C. W. J. Granger, and R. F. Engle (1993). A long memory property of stock market returns and a new model. *Journal of Empirical Finance 1*, 83–106.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of U.K. inflation. *Econometrica 50*, 987–1008.

Giraitis, L., P. Kokoszka, and R. Leipus (2000). Stationary arch models: Dependence structure and central limit theorem. *Econometric Theory 16*, 3–22.

Lambert, P. and S. Laurent (2001). Using GARCH-type models with a skewed student distribution for the innovations. Discussion Paper 0125, Université Catholique de Louvain - Institut de Statistique.

Laurent, S. and J. P. Peters (2002). G@RCH 2.2: an Ox package for estimating and forecasting various ARCH models. *Journal of Economic Surveys 16*(3), 447–485.

Níguez, T.-M. (2002). Modelling daily value-at-risk using FIGARCH type models. Working paper, University of Alicante - Department of Quantitative Economics.

Priestley, M. B. (1988). *Non-Linear and Non-Stationary Time Series Analysis*. New York: Academic.

Randles, R. H. and D. A. Wolfe (1979). *Introduction to the Theory of Nonparametric Statistics*. New York: Wiley.

Schoffer, O. (2003). *Modellierung von Kapitalmarktrenditen mittels asymmetrischer GARCH-Modelle*. PhD thesis, Universität Dortmund, Dortmund.

Tse, Y. K. (1998). The conditional heteroscedasticity of the Yen-Dollar exchange rate. *Journal of Applied Econometrics 13*, 49–55.