Error Correction Models for

Fractionally Cointegrated Time Series[#]

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Abstract:

This note provides a proof of Granger's (1986) error correction model for fractionally cointegrated variables and points out a necessary assumption that has not been noted before. Moreover, a simpler, alternative error correction model is proposed which can be employed to estimate fractionally cointegrated systems in three steps.

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1. Introduction

Fractional cointegration has become an important and relevant topic in time series econometrics in recent years (see, among others, Cheung and Lai, 1993, Baillie and Bollerslev, 1994, Booth and Tse, 1995, and Dittmann, 1998, 1999). A vector x_t of I(1) time series is called fractionally cointegrated if there is a linear combination $a'x_t$ that is a long-memory process or, more formally, I(d) with $d \in (0, 1)$. In a side remark, Granger (1986) provided an error correction model for this case, which has been widely cited in the fractional cointegration literature. Interestingly enough, this error correction model has never been explicitly proven in the econometrics literature.

This note provides a proof for Granger's error correction model and points out a necessary assumption that seems to have been neglected in the literature so far, namely that all cointegrating relationships share the same long-memory parameter d. A detailed interpretation of Granger's error correction model concludes that this model is far too complicated to be useful in practice. Therefore, a simpler error correction model is proposed which can be employed to estimate fractionally cointegated systems in three steps.

2. Fractional Cointegration and Cointegation Rank

Definition (fractional cointegration): Let x_t be an n-dimensional vector of I(1) processes. x_t is called *fractionally cointegrated* if there is an $a \in IR^n$, $a \neq 0$, such that $a'x_t \sim I(d)$ with $0 < d < 1$. We call d the *equilibrium long-memory parameter* and write $x_t \sim FCI(d)$. □

Compared to classical cointegration, where d = 0, defining the cointegration rank is more difficult for fractionally cointegrated systems, because different cointegrating relationships need not have the same long-memory parameter.

Definition (cointegration rank): Let x_t be an n-dimensional vector of I(1) time series. If there are exactly r linearly independent cointegrating vectors $a_1, a_2, ..., a_r \in IR^n$ with $A = (a_1, a_2, ..., a_r)$ and $A'x_t \sim I(d)$, $d \in (0, 1)$, r is called the *cointegration rank with respect to the equilibrium long-memory parameter d*. We write $x_t \sim FCI(d, r)$. Moreover, if there is a fixed $\tilde{d} \in (0, 1)$ so that any linear combination of x_t is either $I(\tilde{d})$ or I(1), x_t is called *purely fractionally cointegrated with cointegration rank r*. □

Note that $x_t \sim FCI(d, r)$ implies that $r \leq (n - 1)$. However, if $x_t \sim FCI(d_1, r_1)$ and $x_t \sim FCI(d_2, r_2)$, $r_1 + r_2$ can be larger than $(n - 1)$. Hence, the number of independent cointegrating relationships can be larger than $(n - 1)$. Consider for instance $x_{1\,t} \sim I(1)$, $x_{2\,t} = x_{1\,t} + y_t$, $x_{3\,t} = x_{2\,t} + z_t$ with $y_t \sim I(d_1)$, $z_t \sim I(d_2)$ and $d_1 > d_2$. Then there are three cointegrating relationships: $(-1, 1, 0)\, x_t \sim I(d_1)$, $(-1, 0, 1)\, x_t \sim I(d_1)$ and $(0, -1, 1)\, x_t \sim I(d_2)$.

3. Granger's Error Correction Model

We consider the multivariate Wold representation of Δx_t:

$$(1 - B)\, x_t = \delta + C(B)\, \varepsilon_t \tag{1}$$

with the (possibly infinite order) moving average polynomial $C(B) = I_n + C_1 B + C_2 B^2 + C_3 B^3 + \ldots$ and a zero-mean white-noise vector process ε_t (see, e.g., Brockwell and Davis, 1991, p. 187).

Proposition 1 (Granger Representation Theorem): If x_t is purely FCI(d, r) with full-rank $n \times r$ cointegrating matrix A, then $A'\delta = 0$ and $A'C(1) = 0$. Furthermore, there exist linear filters A(B) and d(B) and an $n \times r$ matrix Γ of rank r such that

$$A(B)\,(1 - B)\, x_t = -\,[1 - (1 - B)^{1-d}](1 - B)^d\, \Gamma z_t + d(B)\varepsilon_t \tag{2}$$

with $z_t = A'x_t$ and $A(0) = I_n$, where d(B) is a scalar filter. □

Proof: As $z_t = A'x_t$ is I(d), $A'\delta = 0$ and $A'\Delta x_t = A'C(B)\varepsilon_t$ is I(d – 1). The sum of the Wold coefficients of a long-memory process with negative long-memory parameter is zero, so that $A'C(1) = 0$.

Note that C(B) can be rewritten as

$$C(B) = C(1) + (1 - B)^{1-d}\, C^*(B). \tag{3}$$

$C^*(B)$ does not contain any fractional differencing filters, so that the corresponding matrix polynomial $C^*(z)$ with complex argument z is well-defined on the unit-disk with $C^*(1) \neq 0$.

Using Engle and Granger's (1987) Lemma, which provides the determinant and adjoint of a singular matrix polynomial, we obtain:

$$\det(C(B)) = (1-B)^{r\,(1-d)}\, d(B)\, I_n\,, \tag{4}$$

and
$$\mathrm{Adj}(C(B)) = (1-B)^{(r-1)\,(1-d)}\, A^*(B). \tag{5}$$

Multiplying (1) by (5) gives:

$$(1-B)^{1+(r-1)\,(1-d)}\, A^*(B)\, x_t = (1-B)^{r\,(1-d)}\, d(B)\, \varepsilon_t + (1-B)^{(1-r)\,(1-d)} A^*(B)\, \delta \tag{6}$$

$$\Leftrightarrow \quad (1-B)^d\, A^*(B)\, x_t = d(B)\, \varepsilon_t + (1-B)^{d-1}\, A^*(B)\, \delta \tag{7}$$

I first show that $A^*(1) = \Gamma A'$. Then $(1-B)^{d-1}\, A^*(B)\, \delta = 0$, as $A^*(B)\, \delta = A^*(1)\, \delta = 0$.

Multiplication of (1) by $A^*(B)$ leads to

$$(1-B)\, A^*(B)\, x_t = A^*(B)\, C(B)\, \varepsilon_t + A^*(B)\, \delta\,. \tag{8}$$

Substituting with (7) gives

$$(1-B)^{1-d}\, [d(B)\, \varepsilon_t + (1-B)^{d-1}\, A^*(B)\, \delta] = A^*(B)\, C(B)\, \varepsilon_t + A^*(B)\, \delta \tag{9}$$

$$\Leftrightarrow \quad (1-B)^{1-d}\, d(B)\, \varepsilon_t = A^*(B)\, C(B)\, \varepsilon_t\,. \tag{10}$$

With $B = 1$ we obtain $A^*(1)C(1) = 0$. Hence, $A^*(1)$ is a linear combination of the cointegrating vectors A': $A^*(1) = \Gamma\, A'$.

Consequently, (7) reduces to

$$(1-B)^d\, A^*(B)\, x_t = d(B)\, \varepsilon_t\,. \tag{11}$$

Factorization of $A^*(B) = A^*(1) + (1-B)^{1-d}\, A^{**}(B)$ gives (together with $A^*(1) = \Gamma A'$):

$$(1-B)^d\, [(1-B)^{1-d}\, (A^*(1) + A^{**}(B)) + [1-(1-B)^{1-d}]\, A^*(1)]\, x_t = d(B)\, \varepsilon_t \tag{12}$$

$$\Leftrightarrow \quad [A^*(1) + A^{**}(B)]\, (1-B)\, x_t = -\,[1-(1-B)^{1-d}]\, (1-B)^d\, \Gamma A' x_t + d(B)\, \varepsilon_t \tag{13}$$

Choosing $A(B) := A^*(1) + A^{**}(B)$, we obtain (due to the definition of $A^*(B)$) $A(0) = A^*(1) + A^{**}(0) = A^*(0)$. It remains to show that $A^*(0) = I_n$. With $B = 0$ in (10) we obtain $A^*(0)\, C(0) = d(0)$. As $C(0) = I_n$ and $d(0)$ is the determinant of $C(0)$, we get $A^*(0) = I_n$. ■

Note that Proposition 1 only holds if x_t is *purely* fractionally cointegrated. If x_t is multiply fractionally cointegrated, i.e., $x_t \sim FCI(d, r)$ and $x_t \sim FCI(d_2, r_2)$ with $d_2 \neq d$, the error correction model (2) cannot be a representation of the system x_t. The reason is that (2) takes into account only the information contained in the r cointegrating relationships of order I(d), whereas the information contained in all the other cointegrating relationships is neglected. Formally, the above proof does not work any more, because $C^*(B)$ in (3) is no longer well-defined on the unit disk with $C^*(1) \neq 0$, so that the Lemma of Engle and Granger (1987) cannot be applied. In order to see this, let A and Ã be the two cointegrating matrices with $A'x_t \sim I(d)$ and $\tilde{A}'x_t \sim I(d_2)$ and consider

$$(1 - B)^{d_2} \tilde{A}'x_t = (1 - B)^{d_2 - 1} \tilde{A}'\Delta x_t = (1 - B)^{d_2 - 1} \tilde{A}'C(B)\varepsilon_t = (1 - B)^{d_2 - d} \tilde{A}'C^*(B)\varepsilon_t \qquad (14)$$

As $(1 - B)^{d_2} \tilde{A}'x_t \sim I(0)$, $C^*(B)$ contains the factor $(1 - B)^{d - d_2}$. If $d_2 < d$, $C^*(1) = 0$ and, if $d_2 > d$, $C^*(1)$ does not exist. Note that multiple fractional cointegration can only occur if the dimension n of the vector x_t is larger than two.

Equation (2) is a special case of the formula given in Granger (1986). Granger (1986) considers the more general case that x_t is not I(1), but I(D) with some $D > d$. The present paper uses $D = 1$, because this is the special case commonly associated with fractional cointegration. Nevertheless, the proof given above also works for the more general case. The error correction model (2) then becomes

$$A(B)\,(1 - B)^D x_t = - [1 - (1 - B)^{D - d}](1 - B)^d \Gamma z_t + d(B)\varepsilon_t \qquad (2')$$

Note that (2) becomes the "traditional" error correction model (see, e.g., Engle and Granger, 1987) for $d = 0$, i.e., in the case of classical cointegration. In this case, only yesterday's equilibrium error z_{t-1} enters the model (2), indicating that today's changes Δx_t are influenced by yesterday's equilibrium error z_{t-1} but not by older equilibrium errors. If $d = 0$ and $r = 1$, (2) can be used for estimation of A(B) and d(B) after Γ has been estimated by an ordinary least squares regression.

If $d > 0$, (2) becomes more complicated. Let us first rewrite the filter in front of the equilibrium error Γz_t, by expanding the fractional differencing filter (see Granger and Joyeux, 1980, or Hosking, 1981):

$$[1-(1-B)^{1-d}](1-B)^d = (1-B)^d - 1 + B = (1-d)B + \sum_{j=2}^{\infty} d_j B^j \qquad (15)$$

with $d_1 = -d$ and $d_j = \frac{j - 1 - d}{j} d_{j-1}$. Note that all d_j with $j \geq 1$ are negative with $|d_j|$ decreasing in j and $\sum_{j=2}^{\infty} |d_j| = 1 - d$.

As all the terms in (15) contain the factor B, only past equilibrium errors z_{t-1}, z_{t-2}, z_{t-3},... appear in the error correction model (2). Hence, today's change Δx_t can be explained by past changes Δx_{t-1}, Δx_{t-2}, ... , past equilibrium errors z_{t-1}, z_{t-2}, ... and a stationary error $d(B)\varepsilon_t$. With increasing equilibrium long-memory parameter d, the influence of yesterday's equilibrium error z_{t-1} decreases. If d converges to 1, the error correction term vanishes (as (15) converges to zero), so that there is no error correction in the limit.

If $d \in (0, 1)$, the error correction model (2) contains infinitely many past values not only of the equilibrium error z_t, but also of the changes Δx_t and of the errors ε_t, because, in general, A(B) and d(B) also contain fractional differencing filters. Hence, (2) is unsuitable for estimation.

4. An Alternative Error Correction Model

If x_t is purely FCI(d, r), the cointegrating matrix A can be partitioned as $A = (A_1, A_2)$ with regular (r, r)-matrix A_1 and (r, n – r)-matrix A_2. Moreover, we can normalize $A_1 = I_r$ without loss of generality. Thus, with $z_t = A'x_t \sim I(d)$ and correspondingly partitioned x_t, we obtain

$$x_{1t} = -A_2 x_{2t} + z_t . \tag{16}$$

This is the Phillips's (1991) triangular representation of the system. It shows that the system contains only n – r random walks x_{2t}.

Proposition 2: If x_t is purely FCI(d, r) with cointegrating matrix $A = (I_r \mid A_2)'$, then there exists an n-dimensional nondeterministic I(0)-process u_t such that

$$\Delta x_t = \begin{pmatrix} I_r & -A_2 \\ 0 & I_{n-r} \end{pmatrix} u_t - \begin{pmatrix} I_r \\ 0 \end{pmatrix} \left[1 - (1 - B)^{1-d}\right] (1 - B)^d A' x_t \tag{17}$$

□

Proof: By taking differences in (16), we obtain

$$\Delta x_t = \begin{pmatrix} \Delta x_{1t} \\ \Delta x_{2t} \end{pmatrix} = \begin{pmatrix} -A_2 \Delta x_{2t} + (1-B) z_t \\ \Delta x_{2t} \end{pmatrix} \tag{18}$$

Let $u_{1t} = (1 - B)^d z_t$ and $u_{2t} = \Delta x_{2t}$. Then

$$\Delta x_t = \begin{pmatrix} I_r & -A_2 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} - \begin{pmatrix} I_r \\ 0 \end{pmatrix} u_{1t} + \begin{pmatrix} I_r \\ 0 \end{pmatrix} (1-B) z_t \tag{19}$$

and (17) follows immediately. ■

As in Proposition 1, the assumption that x_t is *purely* fractionally cointegrated is necessary. If x_t is multiply fractionally cointegrated as described above, u_t in (17) is not I(0) but instead some elements of u_t are antipersistent (i.e., I(d') with d' < 0).

In contrast to Granger's error correction model, (17) can be used for estimation of the system in three steps if r = 1: In the first step, A_2 is estimated by an ordinary least squares regression of x_{1t} on x_{2t}. Cheung and Lai (1993) show that this estimator is consistent if n = 2 and r = 1. In the second step, the long-memory parameter d of the cointegrating equilibrium $A'x_t$ is estimated, employing a periodogram regression (cf. Geweke and Porter-Hudak, 1983). In the third step, $\hat{u}_t$ is calculated from equation (17) and the estimates of the first two stages and modeled as an ARMA-process.

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