Sheppard's Correction for Variances and the "Quantization Noise Model"

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Abstract

In this note we examine the relevance of Sheppard's correction for variances and (both the original and a valid weak form of) the so-called "quantization noise model" to understanding the effects of integer-rounding on continuous random variables. We further consider whether there is any real relationship between the two. We observe that the strong form of the model is not really relevant to describing rounding effects, demonstrate using simple cases the substantial limitations of the Sheppard correction, and use simple versions of a weak form of the model to establish that there is no real connection between the correction and the model.

I. Introduction

The famous analysis of Sheppard (see [1] and [2]) provides approximate relationships between the moments of a continuous distribution and those of a corresponding approximating discrete distribution. For example, if X has probability density $f(x)$ and

$$[X] = X \text{ rounded to the nearest integer}$$

the Sheppard analysis suggests that under smoothness conditions on $f(x)$ and the negligibility of a remainder term,

$$\text{Var}\,X \approx \text{Var}[X] - \frac{1}{12} \quad . \tag{1}$$

This suggests in turn that integer rounding/quantization "typically" increases variance. Additionally, the (actually completely fortuitous) fact that the Uniform $(-.5,.5)$ distribution has variance $1/12$ makes it tempting to (incorrectly) suspect that somehow independence and a variable with such distribution are lurking in the background.

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Engineers regularly deal with the issue of quantization. In the present context, one might call

$$Q = [X] - X \tag{2}$$

a quantization error. With this notation, it is trivially true that

$$[X] = X + Q \tag{3}$$

Now it would be consistent with (1) if Q could be treated as Uniform $(-.5,.5)$ and independent of X. In fact, (3) under such assumptions is sometimes called the "quantization noise model" (see [3],[4], and [5]) and there is motivation in the electrical engineering literature and folklore apart from the Sheppard result to perhaps consider such a model.

The folklore in the measurement community seems to be that Sheppard's correction and the quantization noise model are closely connected and widely applicable. But that folklore is fraught with confusing internal contradictions. It is our purpose here to spell out in simple terms what really does and does not follow from a careful analysis of the nature of Q, models like the quantization noise model, and the Sheppard analysis.

II. The Nature of the Joint Distribution of X and Q and "Histogram Densities"

It is obvious from (2) (and well-recognized by electrical engineers) that Q is a deterministic function of X and is certainly *not* independent of it. It is perhaps not so well-recognized that Sheppard's approximation really has *nothing* to do with the quantization noise model, and can further be grossly inappropriate for some very simple continuous distributions. Let us elaborate.

The function

$$q(x) = [x] - x \tag{4}$$

that transforms X to $Q = q(X)$ is pictured in Figure 1. It is clear that the joint distribution of X and Q is singular and in fact concentrated on set of line segments that make up $\{(x, q(x)) \in \Re^2 \mid x \in \Re\}$.

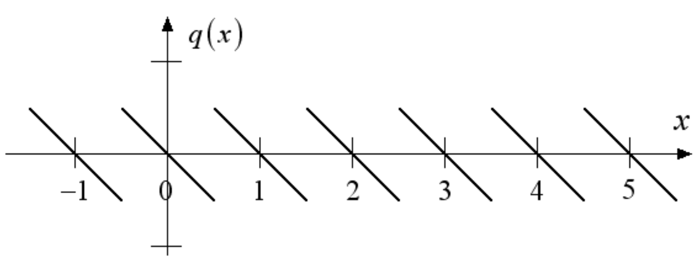

Fig. 1. Plot of $q(x)$ (and thus the support of the joint distribution of X and Q).

One condition sufficient to make $Q \sim \mathrm{U}(-.5,.5)$ is that X have what we will here call a "histogram density," that is for $f(x)$ to be constant on each interval $(i-.5, i+.5)$ for integer i. In fact, more can be said about such simple cases. Though Q and X are of necessity dependent (and can not be described by the quantization noise model), it is very easy to see that Q is $\mathrm{U}(-.5,.5)$ and independent of $[X]$ exactly when X has a histogram density. (The conditional distributions of $Q\,|\,[X]=[x]$ are all $\mathrm{U}(-.5,.5)$ exactly when X has a histogram density.) Notice that independence of $[X]$ and Q is something quite different from independence of X and Q.

For X with histogram density it is then the case that

$$\mathrm{Var}\,X = \mathrm{Var}\left([X]-Q\right) = \mathrm{Var}[X] + \mathrm{Var}\,Q = \mathrm{Var}[X] + \frac{1}{12} \quad . \tag{5}$$

Although the number $1/12$ appears in both equations (1) and (5), they are quite different! There is, of course, no contradiction between them, as a histogram density does not satisfy the conditions under which Sheppard derived (1). But (5) shows that one should not assume that quantization necessarily increases variance. For a histogram density, making the Sheppard correction to variance of the quantized variable is exactly the *wrong* thing to do. Quantized observations from a histogram density are *less* variable than their unquantized counterparts, in complete contradiction to any intuition drawn from (1) and naively taken to be general.

III. Weak Forms of the Quantization Noise Model?

Notice that (3) and (5) imply that for X with a histogram density

$$\begin{aligned} \mathrm{Var}[X] &= \mathrm{Var}(X+Q) \\ &= \mathrm{Var}\,X + \mathrm{Var}Q + 2\mathrm{Cov}(X,Q) \\ &= \left(\mathrm{Var}[X] + \frac{1}{12}\right) + \frac{1}{12} + 2\mathrm{Cov}(X,Q) \end{aligned} \tag{6}$$

so that

$$\mathrm{Cov}(X,Q) = -\frac{1}{12} \tag{7}$$

and X and its quantization error are negatively correlated. The question then arises as to whether there are simple densities for X that produce a $\mathrm{U}(-.5,.5)$ distribution for Q and (not negative, but rather) 0 correlation between X and Q. Such a distribution would produce a valid "weak" form of the quantization noise model and suffice to make (1) exact.

One simple positive answer to this existence question can be motivated as follows. It's obvious from Figure 1 that if one begins with $X \sim \mathrm{U}(-.5,.5)$, Q will also be $\mathrm{U}(-.5,.5)$, but will be perfectly negatively correlated with X. However, if one were to take some part of the $\mathrm{U}(-.5,.5)$ probability for X just below $x=.5$ and move it to just above $x=.5$, and simultaneously move the corresponding part of the probability just above $x=-.5$ and to just below $x=-.5$, it should be possible to reduce the correlation between X and Q. So for $0 \le c \le .5$ consider the density for X

$$f(x) = \begin{cases} 1 & \text{if } x \in (c-1,-.5) \cup (-c,c) \cup (.5,1-c) \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

It is clear from Figure 1 that Q is $\mathrm{U}(-.5,.5)$, and since Q has mean 0

$$\begin{aligned}\operatorname{Cov}(X,Q) &= \mathrm{E}(XQ) \\ &= \int_{c-1}^{-.5} x(-1-x)\,dx + \int_{-c}^{c} x(-x)\,dx + \int_{.5}^{1-c} x(1-x)\,dx \qquad (9) \\ &= \frac{1}{6} - c^2\end{aligned}$$

So the choice $c = 1/\sqrt{6}$ is one that produces a valid weak version of the quantization noise model, and for which (1) consequently holds. This is a model in which quantization *increases* variance by exactly the amount suggested by the Sheppard correction.

But note in this last regard, that any $0 \le c < 1/\sqrt{6}$ produces a *positive* correlation between X and Q and a distribution for X under which (1) represents an *under-correction* of $\operatorname{Var}[X]$ as an approximation for $\operatorname{Var} X$. And on the other hand, $c = \sqrt{5/24}$ gives a case where X and Q are negatively correlated, but $\operatorname{Var}[X] = \operatorname{Var} X$ exactly, quantization doesn't change variance, and no "correction" is called for. The point here is that these simple cases show that the Sheppard correction is not really related to even a weak form of the quantization noise model. While there are simple valid versions of that model for which it is appropriate, there are others for which it is wildly inappropriate (in *both* directions).

IV. Conclusion

In view of the considerations raised in this note, it would seem wise to be very careful before assuming that either the Sheppard correction or any form of a quantization noise model is appropriate for describing a particular case of integer rounding of a continuous variable. The two are not related, and neither need be particularly helpful in even simple situations involving such rounding.

References

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