A simulation study on the choice of transformations in Taguchi experiments

M. Erdbrügge and J. Kunert

University of Dortmund, Department of Statistics,
Vogelpothsweg 87, D-44221, Germany
{Erdbruegge, Kunert}@statistik.uni-dortmund.de

Abstract

In this paper we examine two widely used methods to obtain a transformation in Taguchi experiments, namely the lambda plot and the beta technique. We consider different situations with contrasts influencing the mean and / or the variance of the response. Our simulation study reveals that the variation of the β-Method proposed by Kunert and Lehmkuhl (1998) is a good compromise in terms of meeting the confidence level and identifying active effects.

Key Words: lambda plot, beta technique, adjustment effects, location effects, dispersion effects

1 Introduction

In many technical applications, off-line process control, i. e. design of experiments, is used in order to optimize processes. Taguchi (1986) suggests product array designs, which carry out an outer array (variation of noise factors) for every design factor combination of the design, also called inner array. The aim is to determine factor combinations which lead to a production on target while minimizing the variance of the response. Taguchi (1986) distinguishes among "dispersion effects", i. e. factors that influence the variance, "location effects", i. e. factors that influence the mean, and those which neither effect the mean nor the variance of the response. Effects influencing the mean but not the variance are called "adjustment effects".

Usually a data transformation is carried out to achieve maximum simplicity of the model and to meet the assumptions of parsimony and separation. This means we want to identify only a few factors (possibly none) with dispersion effects and a larger number of factors with location effects on the transformed response. TAGUCHI methods seek for a transformation accomplishing these assumptions.

We assume a data transformation belonging to the family of so-called BOX-COX-transformations (compare BOX AND COX, 1964) which is usually used in this situation

$$T_\lambda(y_{ij}) = \begin{cases} \dfrac{y_{ij}^\lambda - 1}{\lambda} & : \quad \lambda \neq 0 \\ \ln(y_{ij}) & : \quad \lambda = 0\,, \end{cases} \tag{1}$$

where y_{ij} is assumed to be positive, $i = 1\,, \ldots, n$ denotes the design point and $j = 1, \ldots, m$ the replication or the design point of the inner array, respectively.

If λ^* denotes the true underlying transformation parameter, then $z_{ij}^* = T_{\lambda^*}(y_{ij})$ is assumed to follow the underlying model

$$z_{ij}^* = \alpha_0 + \alpha_1 x_1 + \ldots + \alpha_s x_s + \alpha_{s+1} x_{s+1} + \ldots + \alpha_k x_k + (\,\gamma_1^{x_1} \cdot \ldots \cdot \gamma_s^{x_s}\,)\, e_{ij}\,, \tag{2}$$
$$\text{with} \quad e_{ij} \ \text{i.i.d.} \sim N(0, \sigma^2),\ \alpha_0 \neq 0\,,\ \gamma \neq 1\,,\ \gamma_q > 0,\ q = 1\,, \ldots, s\,,$$

(compare e. g. the discussion contribution of NAIR AND PREGIBON to the article by BOX, 1988), where $k = k^D + k^I$ with k^D denoting the number of design factors and k^I the number of considered factor interaction contrasts. In this equation the design factors $x_1, \ldots, x_{k^D}$ as well as the factor interactions $x_{k^D+1}, \ldots, x_k$ are assumed to take the two levels -1 (low level) and $+1$ (high level).

If the transformation parameter λ^* is known, model (2) can be used to estimate location and dispersion effects. Without loss of generality we assume the first s factors to be factors with dispersion effects. We expect s, the number of dispersion effects, to be very small, i. e. zero to two. For $s = 0$, the model simplifies to a linear model with an additive error term. Furthermore we do not expect all factors to influence the mean, therefore some of the $\alpha_p\,,\ p = 1\,, \ldots, k$ are likely to be zero. The number of adjustment effects will be denoted by r and can then be determined by $r = |\,\{\,\alpha_p\ :\ \alpha_p \neq 0\,,\ p > s\,\}_{p=1,\ldots,k}|$. We assume $r > s$ unless both are zero.

When using model (2) and for $k = k^D$, an appropriate design would be a screening design which is not used to optimize the process but to determine the most important variables among the design factors. If $k^I > 0$, a design to model the process could be used, e. g. a response surface design.

In the case of unknown transformation parameter λ^*, this parameter needs to be estimated first. For this situation BOX (1988) developed the λ-Plot, which is a graphical method used to achieve an appropriate data transformation. GRIZE (1991) extended the procedure to unreplicated designs by adapting the variance estimation for the coefficients. A description of the λ-Plot procedure and different variance estimation methods follows in Section 2.

On the other hand the mean-variance-plot (compare e. g. BOX, HUNTER AND HUNTER, 1978) has been used for TAGUCHI experiments by LOGOTHETIS (1990) resulting in the β-Method. A generalization suggested by ENGEL (1992), which as well as the original β-Method possibly leads to inconsistent estimates for the transformation parameter λ^*, has been adapted to model (2) by KUNERT AND LEHMKUHL (1998). Section 3 introduces the original β-Method by LOGOTHETIS and the generalized version by KUNERT AND LEHMKUHL.

In Section 4 the simulation study used to compare both methods is described. Section 5 contains a summary of the results, conclusions and further discussion.

2 Description of the λ-Plot

To use the λ-Plot as a graphical tool, we need to compute two sequences containing the estimated influences of all the design factors and considered interactions on the mean and standard deviation of the transformed response. Additionally, an estimate for the variance of the effects is calculated for each value of λ in order to standardize the estimated effects and obtain t-statistics. Two separate graphics for location and dispersion effects are created where a curve of these t-values is plotted against λ for each design factor or interaction.

The standard deviation of the coefficient estimates can be estimated as proposed by BOX (1988). If model (2) is fitted for every value of λ, we get

$$E\left(z_{ij}\right) = \alpha_0 + \sum_{p=1}^{k} \alpha_p \, x_p \,,$$

$$Var\left(z_{ij}\right) = \sigma^2 \left(\prod_{q=1}^{s} \gamma_q^{x_q} \right)^2 \iff \ln \sqrt{Var\left(z_{ij}\right)} = \ln \sigma + \sum_{q=1}^{s} x_q \ln \gamma_q \,. \tag{3}$$

To analyze the means we use a simplification of (2) that neglects the presence of dispersion effects

$$(\bar{z}_1, \ldots, \bar{z}_n)' = (\mathbf{1}, x_1, \ldots, x_k)\, \alpha(\lambda) + e_1\,, \quad e_1 \sim N(\mathbf{0}, \sigma_1^2\,\mathbf{I})\,,$$

where $\mathbf{1} = (1, \ldots, 1)'$, $x_i = (x_{i1}, \ldots, x_{in})'$, $\alpha(\lambda) = \big(\alpha_0(\lambda), \alpha_1(\lambda), \ldots, \alpha_k(\lambda)\big)'$ and $\sigma_1^2 = \frac{\sigma^2}{m}$. If the design is orthogonal, with $(\mathbf{1}, x_1, \ldots, x_k)'(\mathbf{1}, x_1, \ldots, x_k) = n\,\mathbf{I}$, the parameter vector $\alpha(\lambda)$ can then be estimated by

$$\widehat{\alpha}(\lambda) = \frac{1}{n}\,(\mathbf{1}, x_1, \ldots, x_k)'(\bar{z}_1, \ldots, \bar{z}_n)'\,. \tag{4}$$

The corresponding covariance matrix for this estimate is

$$\begin{aligned} Cov\,[\,\widehat{\alpha}(\lambda)\,] &= \frac{1}{n^2}\,(\mathbf{1}, x_1, \ldots, x_k)'\; Cov\,[\,(\bar{z}_1, \ldots, \bar{z}_n)'\,]\;(\mathbf{1}, x_1, \ldots, x_k) \\ &= \frac{1}{n^2}\,(\mathbf{1}, x_1, \ldots, x_k)'\;\frac{1}{m}\,\sigma^2\,\mathbf{I}\;\;(\mathbf{1}, x_1, \ldots, x_k) \\ &= \frac{1}{n\,m}\,\sigma^2\,\mathbf{I}\,. \end{aligned} \tag{5}$$

It can be estimated by

$$\widehat{Cov}\,[\,\widehat{\alpha}(\lambda)\,] = \frac{1}{n\,m}\,\bar{S}_z^2\,\mathbf{I}\,,$$

where $\bar{S}_z^2$ denotes the within-runs sums of squares based on $n(m-1)$ degrees of freedom

$$\bar{S}_z^2 = \frac{1}{n\,(m-1)}\sum_{i=1}^{n}\sum_{j=1}^{m}(z_{ij} - \bar{z}_i)^2\,.$$

The vector of t-values for the location effects can then be computed by

$$t_L^{\mathrm{Box}}(\lambda) = \frac{\sqrt{n\,m}}{\bar{S}_z}\;\;\widehat{\alpha}(\lambda)\,. \tag{6}$$

The estimation of dispersion effects is based on equation (3). To estimate $\ln\,\gamma_q\,,\; q = 1, \ldots, s\,,$ we use the following model

$$(\ln S_{z_1}, \ldots, \ln S_{z_n})' = (\mathbf{1}, x_1, \ldots, x_k)\,\ln\gamma(\lambda) + e_2\,, \quad e_2 \sim N(\mathbf{0}, \sigma_2^2\,\mathbf{I})$$

for the logarithm of standard deviations for n design points, where $\ln\gamma(\lambda) = \big(\ln\sigma(\lambda), \ln\gamma_1(\lambda), \ldots, \ln\gamma_k(\lambda)\big)'$ and most of the entries $\ln\gamma_1(\lambda), \ldots, \ln\gamma_k(\lambda)$ are assumed to be zero. The vector $\ln\gamma(\lambda)$ can be estimated by

$$\widehat{\ln\gamma}(\lambda) = \frac{1}{n}\,(\mathbf{1}, x_1, \ldots, x_k)'(\ln S_{z_1}, \ldots, \ln S_{z_n})'\,. \tag{7}$$

To achieve the variance of this estimate, we need to compute the variance of $\ln S$ first. It is well known that $Var\,(S^2) = 2\,\sigma^4/(m-1)$, because $(m-1)\,S^2/\sigma^2 \sim \chi^2_{m-1}$ and therefore by using the delta method

$$Var\,f(Y) \;\approx\; [f'(E\,Y)]^2 \; Var\,Y\,, \tag{8}$$

for $Y = S^2$, we get

$$Var\,(\ln S) \;\approx\; \left[\frac{1}{2\,E(S^2)}\right]^4 Var\,(S^2) \;=\; \frac{1}{2\,(m-1)}\,.$$

The covariance matrix of $\widehat{\ln\gamma}\,(\lambda)$ then becomes

$$\begin{aligned} Cov\,[\,\widehat{\ln\gamma}\,(\lambda)\,] \;&\approx\; \frac{1}{n^2}\,(\mathbf{1},x_1,\ldots,x_k)'\;Cov\,[\,(\ln S_{z_1},\ldots,\ln S_{z_n})'\,]\;(\,\mathbf{1},x_1,\ldots,x_k) \\ &=\; \frac{1}{n^2}\,(\mathbf{1},x_1,\ldots,x_k)'\;\frac{1}{2\,(m-1)}\;\mathbf{I}\;(\mathbf{1},x_1,\ldots,x_k) \\ &=\; \frac{1}{2\,n\,(m-1)}\;\mathbf{I}\,, \end{aligned} \tag{9}$$

compare Box (1988). Therefore the vector of t-values for dispersion effects can be computed according to

$$t_D^{\text{Box}}(\lambda) = \sqrt{2\,n\,(m-1)}\;\;\hat{\ln\gamma}\,(\lambda)\,. \tag{10}$$

To identify active effects for a given value of λ, we test the hypotheses

$$\begin{aligned} & H_{01}: \max_{p=1,\ldots,k} |\,E\,(\,t_L(\lambda)_p\,)\,| = 0 \qquad \text{vs.} \qquad H_{A1}: \max_{p=1,\ldots,k} |\,E\,(\,t_L(\lambda)_p\,)\,| > 0 \\ \text{and}\quad & H_{02}: \max_{p=1,\ldots,k} |\,E\,(\,t_D(\lambda)_p\,)\,| = 0 \qquad \text{vs.} \qquad H_{A2}: \max_{p=1,\ldots,k} |\,E\,(\,t_D(\lambda)_p\,)\,| > 0\,. \end{aligned}$$

Assuming $H_0 :=$ "H_{01} and H_{02} both hold", the vectors of t-values for location and dispersion effects given in (6) and (10) are both approximately multivariate standard normal. Heading for a confidence level CL, the critical value c_1 can then be achieved by

$$\begin{aligned} P_{H_0}\left(\max_{p=1,\ldots,k} |\,t^{\text{Box}}(\lambda)_p| > c_1\right) \;&\le\; 1 - \text{ CL} \\ \Longrightarrow\; c_1 \;&\ge\; \Phi^{-1}\left(\frac{1+\sqrt[k]{\text{CL}}}{2}\right)\,. \end{aligned}$$

Factors or interactions with corresponding values exceeding the critical value are called active effects. If more than one effect is present, this proceeding is conservative!

Not only in unreplicated experimental designs, but also in replicated situations the standard deviation estimator can also be based on the estimated factor and interaction effects as done for the scaled λ-Plot introduced by GRIZE (1991). Again assuming $H_0 :=$ "H_{01} and H_{02} both hold", we get for $\widehat{\alpha}(\lambda)$ and $\widehat{\ln\gamma}(\lambda)$ given in (4) and (7):

$$\begin{aligned} \widehat{\alpha}(\lambda) &\sim N\left(\mathbf{0}, \frac{1}{n}\sigma_1^2\,\mathbf{I}\right) \\ \text{and} \quad \widehat{\ln\gamma}(\lambda) &\overset{app.}{\sim} N\left(\mathbf{0}, \frac{1}{n}\sigma_2^2\,\mathbf{I}\right), \quad \text{if } n > 1\,. \end{aligned}$$

Median-based variance estimation by

$$\begin{aligned} s_{0,L} &:= \frac{3}{2} \operatorname*{median}_{p=1,\ldots,k} \left(\left| \widehat{\alpha}_p(\lambda) \right| \right) \\ \text{and} \quad s_{0,D} &:= \frac{3}{2} \operatorname*{median}_{p=1,\ldots,k} \left(\left| \widehat{\ln\gamma_p}(\lambda) \right| \right), \end{aligned}$$

will be approximately unbiased because

$$\begin{aligned} E\left(s_{0,L}\right) &\approx \frac{3}{2}\,\Phi^{-1}\left(\frac{3}{4}\right)\frac{\sigma_1}{\sqrt{n}} \approx \frac{\sigma_1}{\sqrt{n}} \\ \text{and} \quad E\left(s_{0,D}\right) &\approx \frac{3}{2}\,\Phi^{-1}\left(\frac{3}{4}\right)\frac{\sigma_2}{\sqrt{n}} \approx \frac{\sigma_2}{\sqrt{n}}\,. \end{aligned}$$

This leads to vectors of t-values

$$t_L^{\text{Med}}(\lambda) = \frac{\widehat{\alpha}(\lambda)}{s_{0,L}} \tag{11}$$

$$\text{and} \quad t_D^{\text{Med}}(\lambda) = \frac{\widehat{\ln\gamma}(\lambda)}{s_{0,D}}\,. \tag{12}$$

When using these t-values, critical values c_2 for testing both hypotheses (H_{01} and H_{02}) can be achieved by numerical approximation. This has been done by KNUTH (1994), selected values are given in Table A.1.

The estimators $s_{0,L}$ and $s_{0,D}$ can be used even under the alternative of active effects as long as the parsimony assumption holds, i. e. most of the coefficients $\widehat{\alpha}_p(\lambda)$ and $\widehat{\ln\gamma_p}(\lambda)$, $p = 1, \ldots, k$, have expectation zero. But clearly in this case the variance estimation increases and will not be unbiased.

The median-based variance estimation has been improved by several authors. LENTH (1989) also suggested the pseudo standard error (PSE) which is defined as follows

$$\begin{aligned} \text{PSE}_L &:= \frac{3}{2} \operatorname*{median}_{\substack{|\widehat{\alpha}_p(\lambda)| \le 2.5\, s_{0,L} \\ p=1,\ldots,k}} \left(\left| \widehat{\alpha}_p(\lambda) \right| \right) \\ \text{and} \quad \text{PSE}_D &:= \frac{3}{2} \operatorname*{median}_{\substack{\left|\widehat{\ln\gamma_p}(\lambda)\right| \le 2.5\, s_{0,D} \\ p=1,\ldots,k}} \left(\left| \widehat{\ln\gamma_p}(\lambda) \right| \right). \end{aligned}$$

The results of the λ-Plot using the estimators suggested by Box (1988) and the PSE suggested by Lenth (1989) has been compared for some known examples by Grize (1991).

Another possibility of estimating the variance has been proposed by Dong (1993). These estimators are again based on s_0 and defined by

$$\begin{aligned} s_{1,L} &:= \sqrt{\frac{1.08}{m_L} \sum_{\substack{|\widehat{\alpha}_p| \le 2.56\, s_{0,L},\\ p=1,\dots,k}} \widehat{\alpha}_p^{\;2}} \\ \text{and} \quad s_{1,D} &:= \sqrt{\frac{1.08}{m_D} \sum_{\substack{|\widehat{\ln\gamma}_p| \le 2.56\, s_{0,L},\\ p=1,\dots,k}} \widehat{\ln\gamma}_p^{\;2}}\,, \end{aligned}$$

with

$$\begin{aligned} m_L &= \left| \{\widehat{\alpha}_p :\ \widehat{\alpha}_p \le 2.56\, s_{0,L}\}_{p=1,\dots,k} \right| \\ \text{and} \quad m_D &= \left| \{\widehat{\ln\gamma}_p :\ \widehat{\ln\gamma}_p \le 2.56\, s_{0,D}\}_{p=1,\dots,k} \right|\,, \end{aligned}$$

respectively. The estimators given here, slightly differ from the ones proposed by Dong (1993), see Kunert (1997).

The t-values therefore will be computed by

$$t_L^{\text{DONG}} = \frac{\widehat{\alpha}\,(\lambda)}{s_{1,L}} \tag{13}$$

$$\text{and} \quad t_D^{\text{DONG}} = \frac{\widehat{\ln\gamma}\,(\lambda)}{s_{1,D}}\,. \tag{14}$$

Simple critical values for identifying location and dispersion effects can be achieved by the $\left(\left[1 + \text{ CL}^{\,1/n}\right]/2\right) \cdot 100$ % Quantile of the t-distribution with $0.69 \cdot k$ degrees of freedom (for motivation of df, compare Kunert, 1997).

In this study we use the estimators suggested by Box and Dong as well as the Median-based estimators $s_{0,L}$ and $s_{0,D}$.

When using the λ-Plot procedure in practice, one decides by eye which transformation to choose. Usually a simple transformation like the logarithm ($\lambda = 0$), the square root ($\lambda = .5$) or the reciprocal ($\lambda = -1$) is used, as long as one of these leads to possibly none active dispersion effect and a few location effects. For the simulation study and especially to permit the comparison with the β-Method, the λ-Plot procedure needs to be formalized. The formalization scheme used in our study has been suggested by LEHMKUHL (1998).

First the values of λ that do not lead to any active dispersion effect are considered (**Step 1.0**) and ordered by the size of the sum of unsigned dispersion effect estimates. The first transformation which identifies active location effects, is chosen (**Step 2**). If there is no transformation to meet these criterions, we take step by step all values for λ which lead to one or more dispersion effects, into account (**Step 1.1** to **Step 1.k-1**). Then the order of consideration depends on the t-value of the smallest active dispersion effect, which will be maximized. This procedure also ends as soon as a transformation is found that leads to at least one location effect (again **Step 2**). If the λ-Plot does not suggest a transformation, the conclusion would be to use the original data and check only for dispersion effects. The formalized procedure is explained more precisely by the flow chart in Figure A.1.

Because the λ-Plot uses a sequence of transformations (i.e. more than one value for λ) and especially seeks for active location effects, we expect this procedure to identify location effects in more than $(1 - \mathrm{CL}) \cdot 100~\%$ of experiments without active effects.

3 Description of the β-Method

The β-Method as described by LOGOTHETIS (1990) uses a different viewpoint and leads to another procedure for the choice of transformation, but again corresponds to the model assumptions made in Section 1.

LOGOTHETIS (1990) considers the situation where a functional relationship between the mean μ_y and the standard deviation σ_y of the untransformed response of the following kind can be assumed

$$\sigma_y = g(\mu_y)\,.$$

Now the aim is to find a transformation T_c that leads to a constant variance, say $Var\ T_c\,(y) = c^2$. The delta method (8) yields to the rough approximation

$$Var\ (T_c\,(y)) \;\approx\; [T_c'\,(E\,(y))]^2\ Var\ (y)\ ,$$

which can be used to determine an appropriate data transformation:

$$\begin{aligned} [\,T_c'\,(\mu_y)\,]^2 \;&\approx\; \frac{Var\;\;(T_c(y))}{Var\;\;(y)} \;=\; \frac{c^2}{g(\mu_y)^2} \\ \Longrightarrow \quad T_c\,(\mu_y) \;&\approx\; \int \frac{c}{g(\mu_y)}\, d\mu\,. \end{aligned}$$

If we assume a special kind of functional relationship, namely

$$\sigma_y = g(\mu_y) = \delta\, \mu_y^{\beta}\,, \tag{15}$$

then the transformation attained by this procedure will be proportional to the Box-Cox-transformation stated in equation (1),

$$T_c\,(\mu_y) \approx \left\{ \begin{array}{lcl} \dfrac{c}{\delta(1-\beta)}\,\mu_y^{1-\beta} & : & \beta \neq 1 \\ \dfrac{c}{\delta}\,\ln(\mu_y) & : & \beta = 1 \end{array} \right\} \propto T_\lambda\,(\mu_y)\,,$$

with $\lambda = 1 - \beta$ (compare also Box, Hunter and Hunter, 1978).

This motivates another common way to identify an appropriate data transformation based on the estimation of the parameter β using equation (15). By taking the logarithm we get

$$\ln \sigma_y = \ln \delta + \beta\, \ln \mu_y$$

and fit the linear model

$$\ln\,(S_y) = \ln \delta + \beta\;\ln\,(\bar{y}) + e_3\;, \quad e_3 \sim (\mathbf{0}, \sigma_3^2 \mathbf{I})\;, \tag{16}$$

with $S_y = (\,S_{y_1}, \ldots, S_{y_n})'$ and $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)'$.

Engel (1992), however, pointed out that this procedure leads to inconsistent estimates if dispersion effects are present. In this case, equation (15) can be extended to

$$\sigma_y = \delta\;\mu_y^{\beta}\;\prod_{q=1}^{s} \nu_q\, x_q\,,$$

which leads to an analysis of covariance model with $\ln\,(\bar{y})$ being the covariate

$$\ln\,(S_y) = \ln \delta + \beta\;\ln\,(\bar{y}) + \sum_{q=1}^{s} x_q \ln \nu_q + e_4\;, \quad e_4 \sim (\mathbf{0}, \sigma_4^2 \mathbf{I})\;.$$

We expect the number of dispersion effects to be at most two, according to the parsimony assumption. For the case $s = 1$, the appropriate model becomes

$$\ln(S_y) = \ln \delta + \beta^{(q^*)}\, \ln\,(\bar{y}) + x_{q^*}\, \ln \nu_{q^*} + e_5\;, \quad e_5 \sim N(0, \sigma_5^2)\,. \tag{17}$$

This model is fitted for each x_i, $i = 1, \ldots, k$ The factor or interaction accomplishing the largest measure of fit R^2 is denoted by x_{q^*}. The use of the model containing x_{q^*} implies the consideration of the factor or interaction with the largest estimated effect on the variability. Therefore the estimation of β also depends on q^*, which is indicated by the notation $\beta^{(q^*)}$ (compare Kunert and Lehmkuhl, 1998). We do not extend the model to $s = 2$ here.

If a data transformation is needed, usually a plot of the logarithm of the mean responses versus the logarithm of the standard deviations for every design factor combination is drawn. The so-called mean-variance-plot can give visual help in deciding which of the models (16) and (17) to use, or whether even a model considering more than one dispersion effect is needed. If there is no active dispersion effect, the points are approximately falling on a straight line with slope $\beta = 1 - \lambda$. No transformation is necessary if $\beta = 0$.

It is also possible to estimate s, the number of dispersion effects. Lehmkuhl (1998) presents a formalized stepwise procedure for determining factors or interactions that effect the variability when using the β-Method. In this study we only consider the two models addressed above, where s is equal to zero or one.

Once the parameter β has been estimated, the estimate for the transformation parameter λ will be determined by $\hat{\lambda} = 1 - \hat{\beta}$ or $\hat{\lambda} = 1 - \hat{\beta}^{(q^*)}$. The data transformation is carried out according to the chosen value and the transformed data is examined relative to dispersion and location effects among the factors and considered interactions. Again, critical values are depending on the variance estimation method used to standardize the factor effects. To attain comparable results, we use the Median-based estimators (t-values given in equation (11) and (12)), the estimator proposed by Dong (1993, compare equations (13) and (14)), as well as the estimators used by Box (1988, compare (6) and (10)).

4 Design of the Simulation study

A simulation study has been done to compare the two procedures λ-Plot and β-Method specified in sections 2 and 3. In particular we focus on the case where neither dispersion nor location effects are present, to see whether the methods introduce artificial significances, i. e. we have random data following the normal distribution. In addition we examine a smaller number of data sets with effects on the mean and variance of the outcome. The seven scenarios that will be considered, are summarized in Table 1.

Table 1: Number of contrasts influencing the mean and variation of the outcome - Examined scenarios

	1	2	3	4	5	6	7
Dispersion Effects (DE)	0	0	0	0	1	1	2
Location Effects (LE)	0	1	3	5	3	5	5
Adjustment Effects	0	1	3	5	2	4	3

The design used for our simulation study is a 16-run fractional factorial design with 15 factors on two levels each, namely a 2^{15-11} design. No noise factor array (outer array) has been carried out, but the estimation of location and dispersion effects is based on four replicates for every design factor combination.

Values for the observed response y_{ij} are sampled from a normal distribution with mean 10 and variance 1 to achieve only positive outcomes, i. e. $y_{ij} \sim N(10, 1)$, with $1 \leq i \leq 16$ and $1 \leq j \leq 4$. Consequently λ is set to 1 in equation (1), which implies no transformation. Thus the underlying model for scenario 1 where neither dispersion nor location effects are present, is given by

$$z^*_{ij} \;=\; T_{\lambda^*}(y_{ij}) \;=\; y_{ij} - 1 \;=\; 9 + e_{ij}, \quad \text{with} \quad e_{ij} \sim N(0, 1)\,,$$

compare equation (2).

For the remaining six situations mentioned above, the data sets are simulated as follows: We assume that every factor with a dispersion effect will also influence the mean of the outcome. In order to attain comparable results, the transformation parameter λ^* is again set to 1 and $\alpha_0 = 9$.

Comparing equations (5) and (9), the coefficients $\alpha_1, \ldots, \alpha_k$ and $\ln\gamma_1, \ldots, \ln\gamma_s$ can be estimated with variance 1/64 or approximately 1/96, respectively. The values α_i used in the simulation study for factors assumed to have location effects, are chosen to start at five times the standard deviation of the estimate, increasing by one standard deviation for each additional effect. In analogy, values for $\ln\gamma_i$ will be taken as multiples of $\sqrt{1/96} \approx 1/10$. Furthermore, factors with dispersion effects are randomly selected among the factors with location effects.

As an example, the model for the fifth scenario, i. e. one dispersion effect which also influences the mean and two additional adjustment effects (compare Table 1), is given below:

$$z_{ij}^* = 9 + \frac{5}{8} \cdot x_1 + \frac{6}{8} \cdot x_2 + \frac{7}{8} \cdot x_3 + \exp\{\frac{5}{10} \cdot x_{ra}\}\; e_{ij}\;,$$

where $e_{ij} \sim N(0,1)$ and $x_{ra} \in \{x_1, x_2, x_3\}$ is chosen at random.

10 000 runs are simulated for the first scenario and 5 000 runs for each of the scenarios two to seven. A total of 13 of these data sets has been replaced because of negative entries, but this will not change the results by much. For every data set the transformation parameter λ is estimated according to the λ-Plot procedures and the two versions of the β-Method.

As a basis to test for active effects, three ways to estimate the variance of the coefficients have been presented in Section 2. The formalized λ-Plot procedure is carried out at 37 equidistanced points between -8 and 10 to estimate λ, compare Figure A.1. In general, different variance estimation methods do not yield the same estimate for the transformation parameter. On the other hand the transformation parameter achieved by using the β-Method does not depend on the variance estimation used. But the original version assuming the model stated in equation (16) and the improved procedure that assumes one dispersion effect as described in model equation (17) will lead to different results.

After the choice of transformation, vectors of t-values for dispersion and location effects are computed and active effects are determined to the confidence level CL$= 0.95$. To allow further comparison, we also compute the resulting vectors for the true transformation parameter $\lambda^* = 1$.

We expect the original β-Method to meet the given confidence level for both, dispersion and location effects, while the assumption of one dispersion effect in advance (s=1) could possibly lead to a higher proportion of identified dispersion effects. As mentioned earlier, the λ-Plot on the contrary is anticipated to identify more location effects because of the used formalization scheme.

We will see that there are great differences in proportions of identified effects, depending on the method and the variance estimators used.

5 Simulation Results

In this section the results of the simulation study are summarized. It is divided into three subsections that deal with the estimation of the transformation parameter λ (section 5.1), the percentage of contrasts identified to effect the mean or variation of the outcome (section 5.2) and some additional remarks, explanations and recommendations (section 5.3).

5.1 Estimation of λ

First we focus on the transformation parameter λ that indicates the transformation chosen by each of the procedures. The β-Method on one hand always suggests transformation parameters, but these are not restricted to a certain range. The results of the λ-Plot procedures on the other hand depend on the considered λ-sequence which is presumed to contain the 'real' transformation parameter. Therefore the λ-sequence needed for the λ-Plot procedure has been chosen as $\{-8, -\frac{15}{2}, \dots, -\frac{1}{2}, 0, \frac{1}{2}, 1, \dots, \frac{19}{2}, 10\}$ in accordance to the range of values achieved by the β-Method in former simulations.

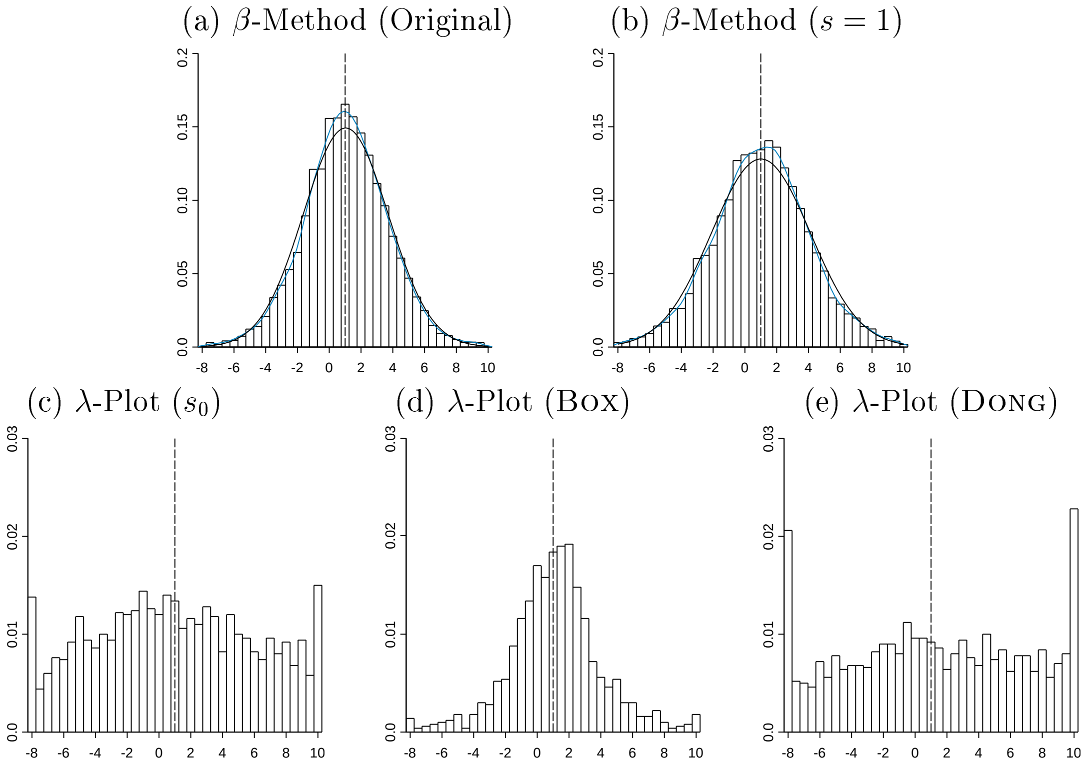
Histograms are drawn to visualize the empirical distributions for transformation parameter estimations suggested by each of the methods. Density estimations are added where suitable (dark: density of normal distribution with estimated parameters, bright: density estimation by smoothing).

Figure 1 compares suggested transformation parameters for data sets containing neither effects. The empirical distributions of values suggested by the β-Methods almost seem to be normal, with smaller variance for the original version (a).

The λ-Plot procedure does not necessarily suggest a transformation. If no transformation parameter leads to location effects, the original data set is used. This procedure coincides with using $\hat{\lambda} = 1$, which results in identifying the true transformation parameter for about 80 - 90 % of the simulated data sets.

In Figure 1 only the data sets for which a transformation has been suggested, are considered in the graphs for the λ-Plot, (c) to (e). We should keep in mind that these figures

Figure 1: Estimated λ-values for data sets containing neither dispersion nor location effect (Scenario 1)



only represent about 10 - 20 % of the data sets, and imagine very high peaks at $\lambda = 1$, i. e. add 1.62, 1.79 and 1.69 to the corresponding bars.

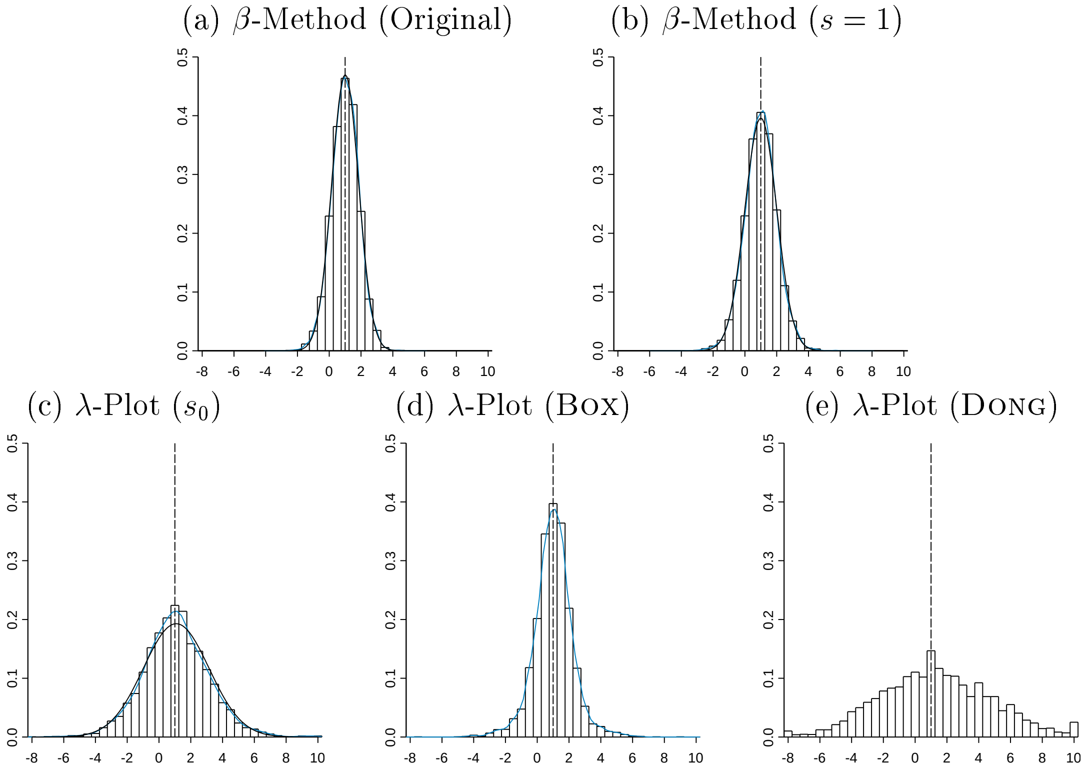
Consequently the λ-Plot yields very good results in identifying the true transformation parameter, especially the procedure based on the suggestion by Box.

For the case of three effects influencing the mean of the response, the suggested transformation parameters are compared in Figure 2. Obviously the Original β-Method attains best results, while the β-Method assuming one dispersion effect as well as the λ-Plot by Box also yield reasonable results. (Here the proportion of data sets with $\hat{\lambda} = 1$ is comparable.)

Histograms covering Scenarios 2 and 4 are given in the appendix (Figure B.1 and Figure B.2). Altogether, in the situation of data without dispersion effects, the estimation of λ evidently improves with increasing number of location effects.

Different outcomes are attained in the case of one effect influencing the variation (and also the mean) and additional adjustment effects. Figure 3 summarizes the results for

Figure 2: Estimated λ-values for data sets containing three location effects (Scenario 3)



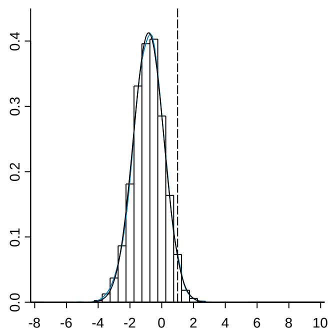
Scenario 5 with two additional adjustment effects. The case of four additional effects on the mean is covered in Figure B.3.

Both figures reveal that only the β-Method that assumes one dispersion effect in advance, attains consistent estimates for the true parameter $\lambda^* = 1$. Estimates resulting from the original β-Method are biased, and the empirical distributions realized when using the λ-Plot procedures with Median-based variance estimator or the one proposed by Dong, have their modus at $\lambda = 1$ due to the formalization scheme. This implies that, for about half of these cases, we identify none of the location effects.

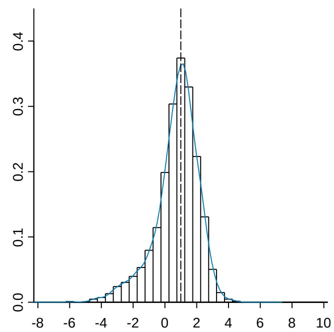
The results for the last examined scenario, the situation of two dispersion effects (also influencing the mean) and three additional adjustment effects are shown in Figure 4. Both β-Methods lead to biased estimates, but the version assuming one dispersion effect yields much better results. None of the λ-Plot procedures leads to satisfying outcomes. Again, for about 3/4 of the data sets forming the peaks in figures (c) and (e), no location effect is found.

Figure 3: Estimated λ-values for data sets containing one dispersion effect and two additional adjustment effects (Scenario 5)

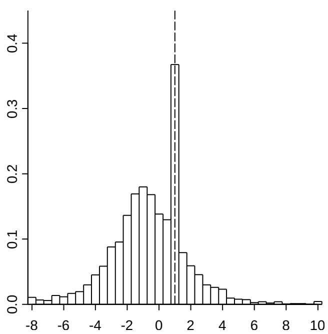
(a) β-Method (Original)



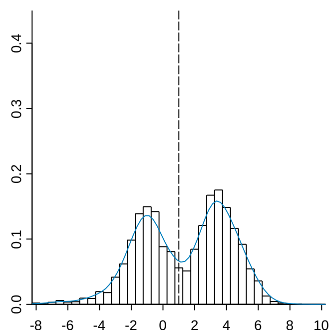
(b) β-Method ($s = 1$)



(c) λ-Plot (s_0)



(d) λ-Plot (Box)



(e) λ-Plot (Dong)

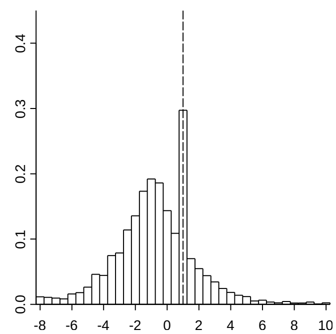
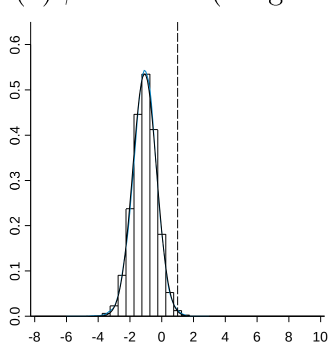
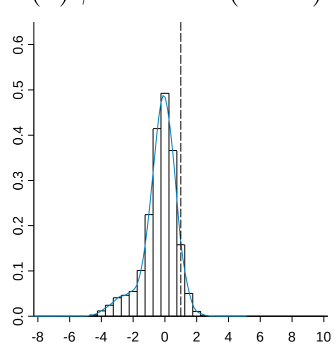


Figure 4: Estimated λ-values for data sets containing two dispersion effects and three additional adjustment effects (Scenario 7)

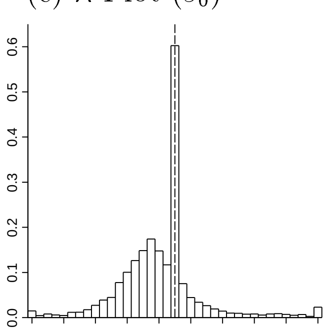
(a) β-Method (Original)



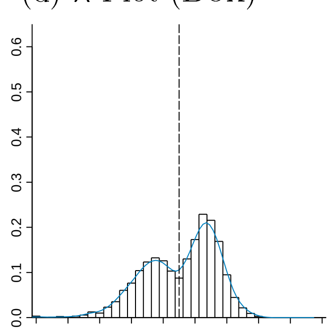
(b) β-Method ($s = 1$)



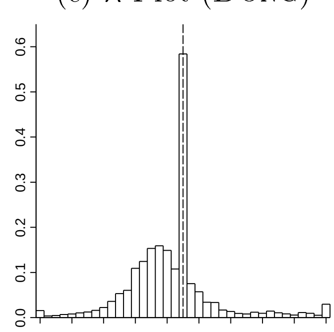
(c) λ-Plot (s_0)



(d) λ-Plot (Box)



(e) λ-Plot (Dong)



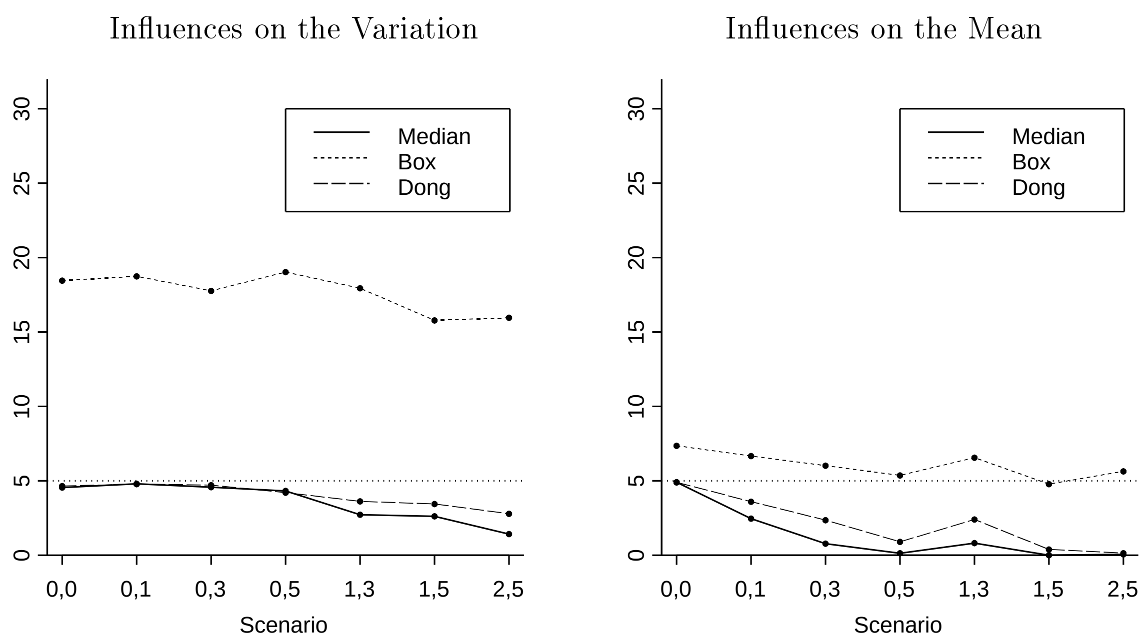
To allow further comparison among the suggested transformations, summary statistics of the empirical distributions are given in Table B.2 of the appendix.

5.2 Identification of active contrasts

We now want to concentrate on determining design factor effects on the mean and variation of the response after carrying out the suggested transformation.

Percentages of data sets with wrongly identified effects on mean and variation are represented in Figure 5 to Figure 8. We distinguish between results determined when using the true underlying transformation parameter $\lambda^* = 1$ (reference) and the three methods Original β-Method, β-Method with one dispersion effect and λ-Plot. All possible combinations of these four procedures (including the reference) with the three variance estimation methods explained in Section 2 and the seven scenarios of Table 1, are considered.

Figure 5: Percentage of data sets with wrongly identified effects for the true transformation parameter $\lambda^* = 1$, depending on the variance estimation method



For Figure 5 no transformation has been carried out. When using the Median-based variance estimation or the estimation proposed by Dong, the percentage of data sets for which effects have been wrongly identified to be active, does not exceed 5 % for any scenario. In contrary the estimation method suggested by Box leads to very high proportions of contrasts wrongly ascribed to influence the variation and still proportions mostly above 5 % with wrongly identified effects on the mean.

Figure 6: Percentage of data sets with wrongly identified effects, transformation parameter λ estimated by Original β-Method, depending on the variance estimation method

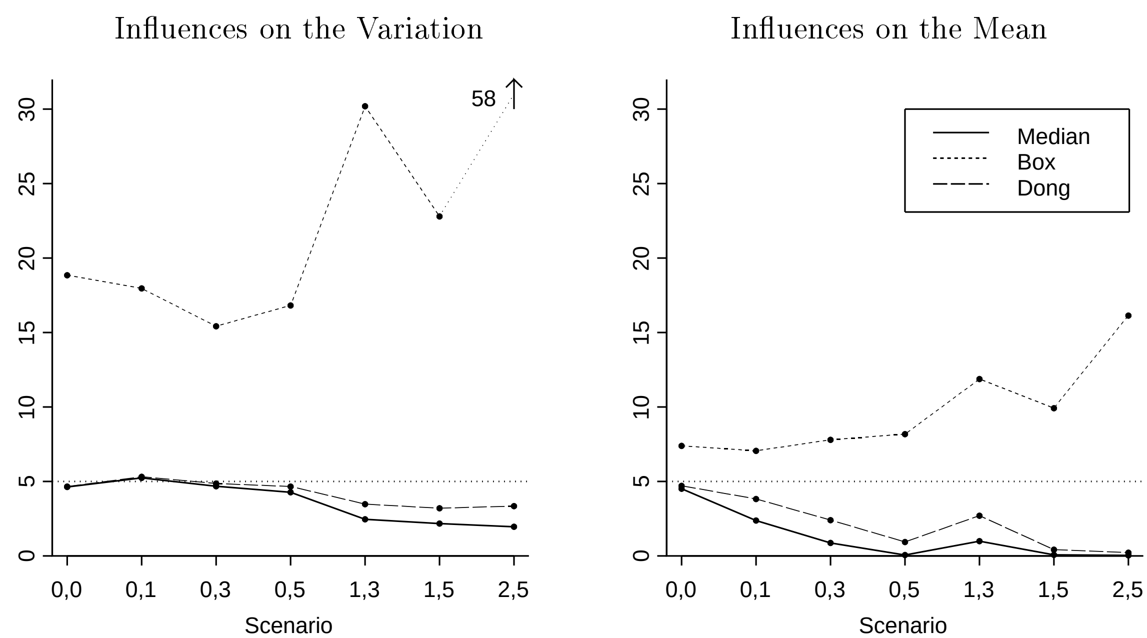


Figure 6 illustrates the according proportions in case of λ being estimated by the Original β-Method. Again, the variance estimation method based on the Median and the suggestion by Dong attain results meeting the required level. The estimation used by Box performs even worse than before, especially for Scenarios 5 to 7, that include effects on the variation. (Note that the proportion for scenario 7 using the method by Box is 58 %.)

Using the β-Method that assumes one dispersion effect in advance to estimate the transformation parameter λ, the proportion of data sets with wrongly identified effects on the variation increases for all scenarios that do not contain such effects and tends to the 5 % level for the remaining. Only the results attained by the estimation method used by Box exceed this level by much (compare Figure 7).

Figure 8 shows the results attained by estimating the transformation parameter via λ-Plot. In this case the proportion of wrongly identified effects on the variation almost reduces to zero for all scenarios but the first, when using the variance estimators "M" and "D". The procedure suggested by Box does again not meet the required 5 % level, neither for dispersion nor for location effects. Focusing on the wrongly identified mean effects, both other methods lead to satisfying results for at least one location effect.

Figure 7: Percentage of data sets with wrongly identified effects, transformation parameter λ estimated by β-Method assuming one dispersion effect (s=1), depending on the variance estimation method

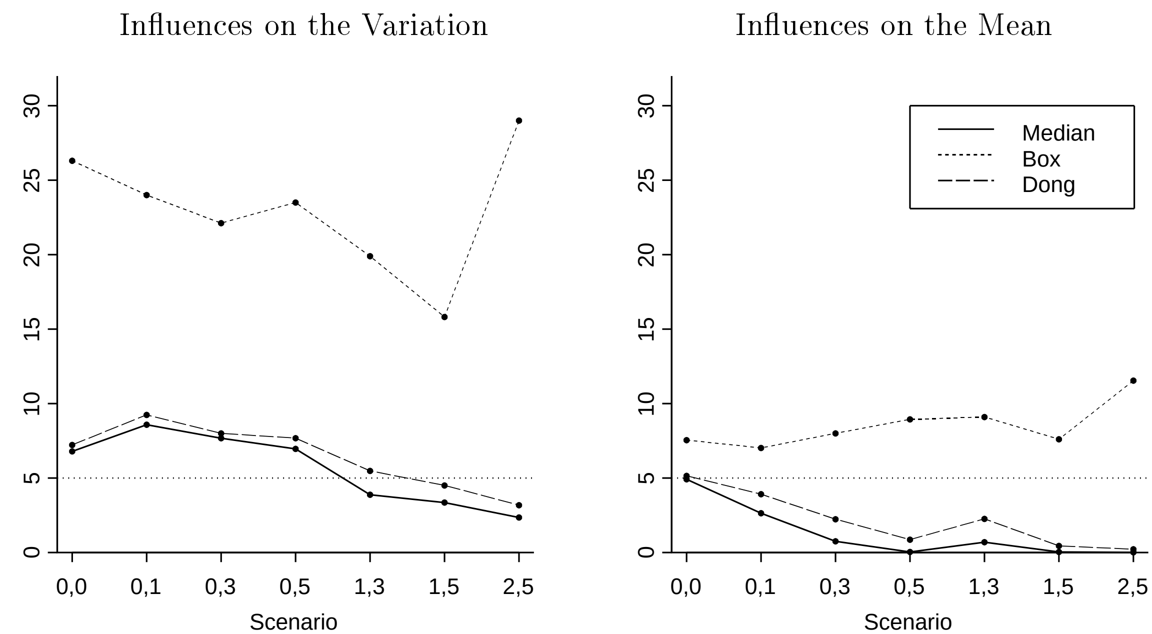
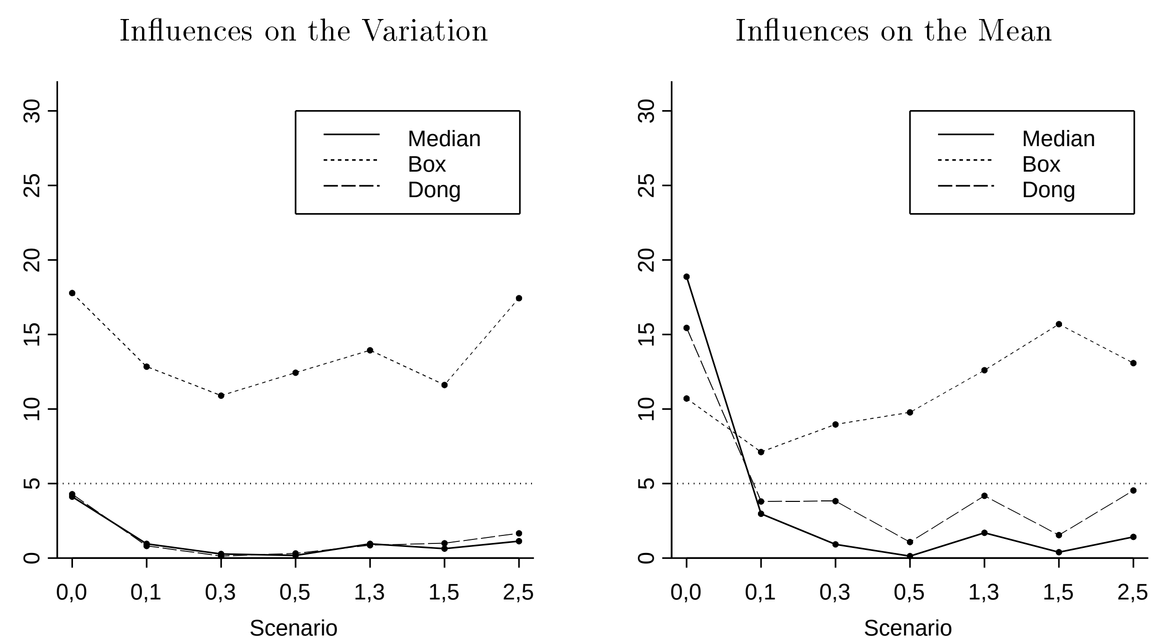


Figure 8: Percentage of data sets with wrongly identified effects, transformation parameter λ estimated by λ-Plot, depending on the variance estimation method



Up to this point we only focused on wrongly identified effects. In the following we will concentrate on the number of data sets for which effects on the mean and variation have been identified correctly, i. e. we consider the power of the underlying test procedures. We distinguish among data sets for which no effects are identified, denoted by “zero”, those for which at least one, but not all of the underlying effects are identified, denoted by “not all”, and those for which the underlying effects are all identified correctly, denoted by “correct”. Of course these proportions and the proportion of data sets for which at least one effect has been wrongly identified, add up to 100 %.

Figure 9: Identified effects on the variation, (i) $\lambda = 1$ (reference); (ii) $\hat{\lambda}$ from Original β-Method; (iii) $\hat{\lambda}$ from β-Method with $s = 1$; (iv) $\hat{\lambda}$ from λ-Plot

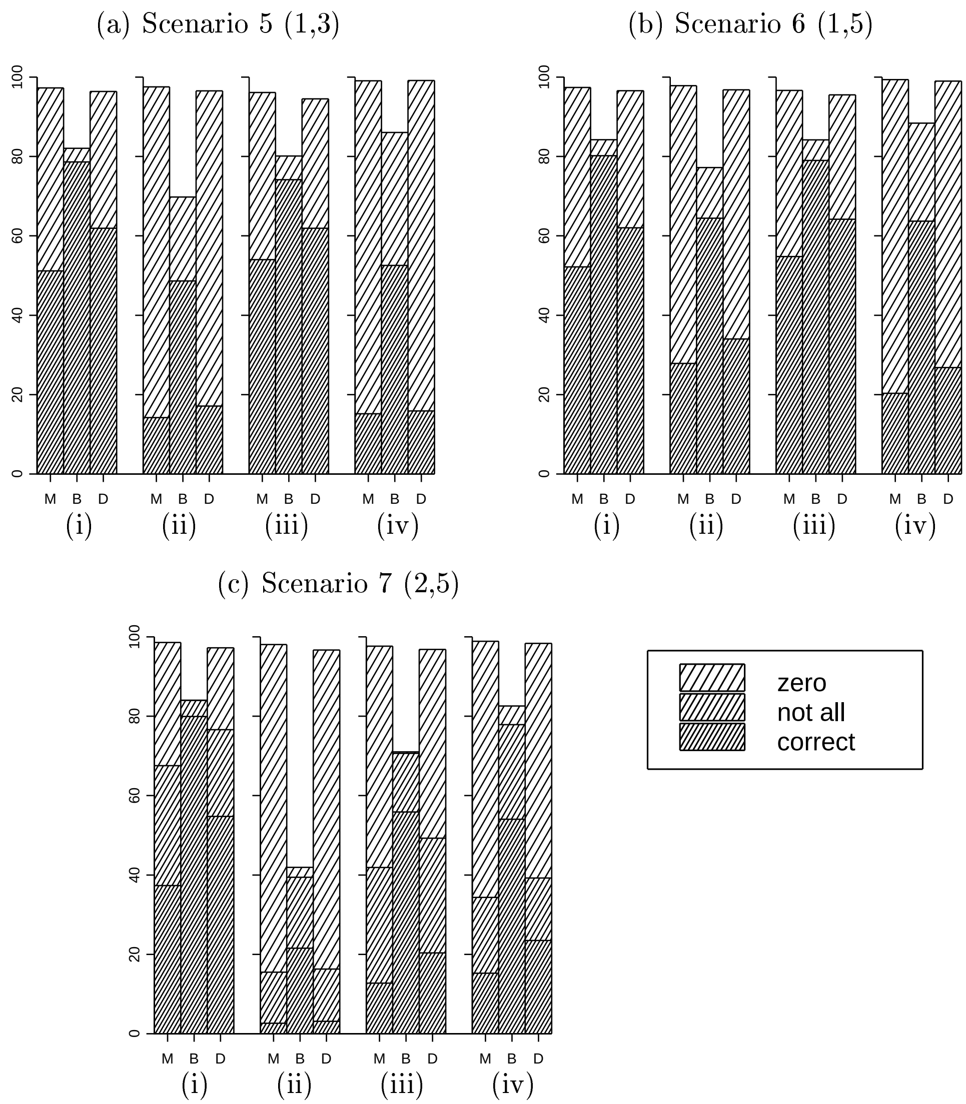


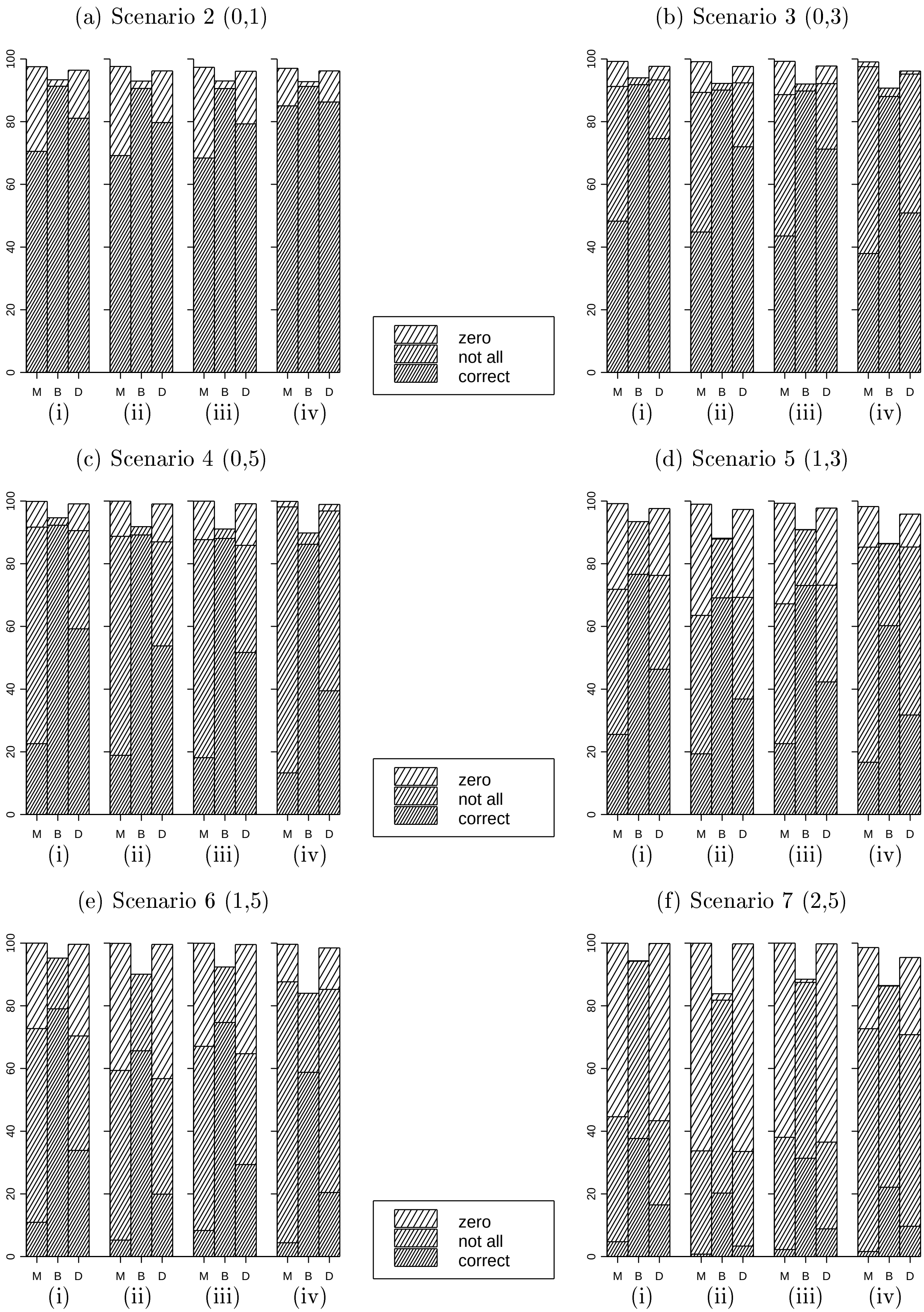
Figure 9 covers identified effects on the variation. Therefore only the last three scenarios that include such influences, are considered. In (a) and (b) it is only possible to identify one or none effect ("correct" or "zero"), and the β-Method assuming one dispersion effect clearly yields the best results. In (c) only the Original β-Method performs much worse than the other two methods.

All these graphics reveal that the variance estimation method suggested by Box does always lead to higher proportions of correctly (as well as wrongly) identified dispersion effects. The other two methods in comparison tend to identify zero or not all effects rather than all of the underlying or even wrong ones.

Finally Figure 10 summarizes the proportions of data sets for which factors with location effects are identified correctly. All scenarios which include influences on the mean, are considered. For the first scenario (a), the λ-Plot performs best, but for all other scenarios both of the β-Method procedures attain larger proportions of correctly identified effects, regardless of the variance estimation method used. If dispersion effects are present, the β-Method with $s = 1$ exceeds the Original β-Method in terms of correctly identified location effects. We observe again that the variance estimation method used by Box attains higher proportions of correctly identified effects than the other two methods. The proportion of data sets for which at least one of the underlying effects on the mean has been revealed, is largest for the λ-Plot procedure in almost all of the considered cases.

In addition, tables containing the proportions visualized in this section are given in the appendix (Table B.3 to Table B.7).

Figure 10: Identified effects on the mean, (i) $\lambda = 1$ (reference) (ii) $\hat{\lambda}$ from Original β-Method (iii) $\hat{\lambda}$ from β-Method with $s = 1$ (iv) $\hat{\lambda}$ from λ-Plot



5.3 Remarks, Explanations and Conclusions

In conclusion, the expectations expressed earlier are partly met. When using the Original β-Method, active dispersion as well as location effects are identified for about $(1 - \text{CL}) \cdot 100 = 5\,\%$ of the data sets as long as the variance estimation suggested by Dong or the Median-based estimation is used. The β-Method procedure proposed by Kunert and Lehmkuhl (1998), which assumes the presence of one dispersion effect, attains a higher proportion of active dispersion effects, but still almost meets the required 5 % level for location effects when using "M"or "D". Still focusing only on these two variance estimation methods, the λ-Plot procedures on the contrary leads to higher proportions of data sets for which location effects are identified for Scenario 1.

5.3.1 Variance estimation suggested by Box

Surprisingly, the variance estimators used by Box lead to higher proportions of data sets with wrongly identified contrasts in almost all of the considered combinations of determined values for λ, scenarios and location or dispersion effects. To further examine and possibly explain this result, the t-values achieved when using the true transformation parameter $\lambda^* = 1$ have been evaluated with regard to active dispersion and location effects. To do so, all three considered variance estimation methods have been used as described in Section 2. To allow visual impressions, histograms of t-values are drawn.

Figure 11: Empirical Distribution of Standardized Effects, t-values computed by using the Median-based variance estimator, λ set to 1

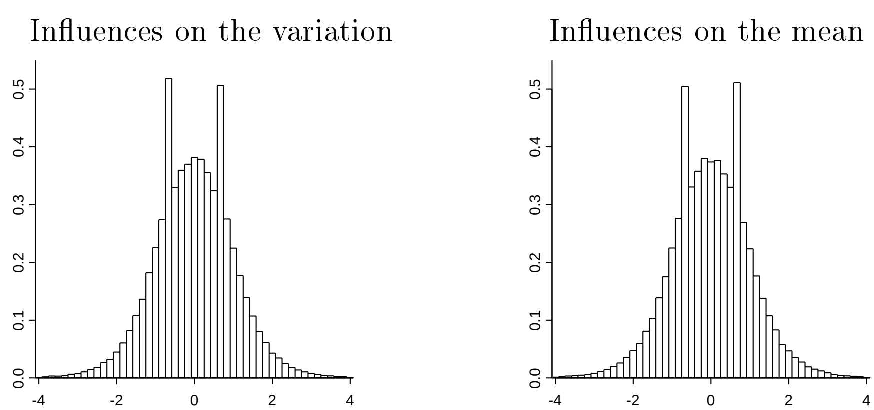
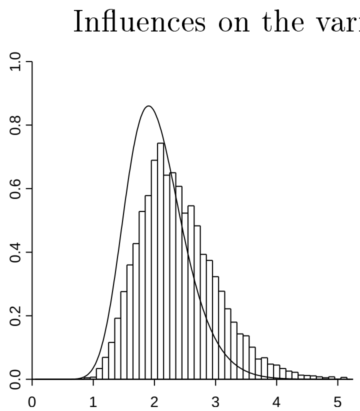


Figure 11 summarizes the t-values attained when using the Median-based variance estimators. The higher proportions of entries in the classes containing the values $-\frac{2}{3}$ and $\frac{2}{3}$, is caused by the fact that one entry of the vector of t-values will always be standardized to either $-\frac{2}{3}$ or $\frac{2}{3}$. This coincides with equations (11) and (12).

Figure 12: Empirical Distribution of unsigned maximum Standardized t-values, computed with variance estimators suggested by Box, λ set to 1



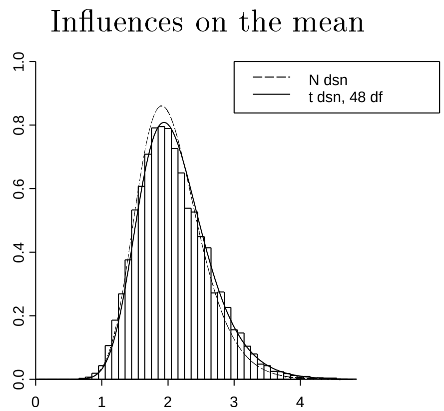
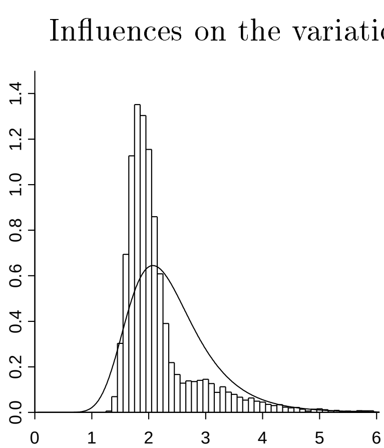
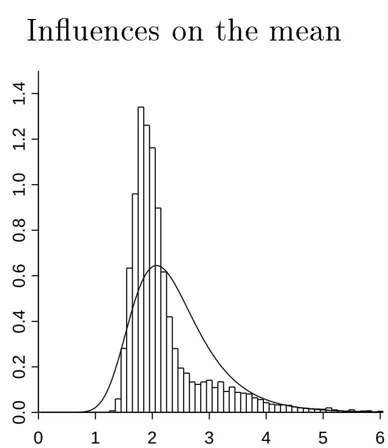


Figure 12 summarizes the empirical distribution of the unsigned maximum of t-values standardized by the variance estimators computed according to the suggestion given by Box. These t-values have been approximated by the standard normal distribution, therefore the corresponding density function for the unsigned maximum of 15 standard normal variables is added to the plots. In addition, the density function resulting from the more appropriate t-distribution with 48 degrees of freedom is presented. Evidently the t-values for dispersion as well as location effects differ from standard normal. This implies that the critical values that have been used for the estimation suggested by Box, are not appropriate. For location effects we should use the t-distribution instead of the approximated normal distribution.

Figure 13: Empirical Distribution of Standardized Effects, t-values computed with variance estimator proposed by Dong, λ set to 1





In Figure 13 the unsigned maximum of t-values standardized by using the method proposed by Dong are presented. The t distribution with $0.69 \cdot k$ degrees of freedom is assumed to yield to conservative critical values, therefore the corresponding density function for the unsigned maximum of 15 variables is added to the plots. The empirical distribution obviously differs from the density function achieved by the t distribution. For the most interesting right tail, the density in fact approximates the empirical distribution.

To allow further comparison between the assumed critical values and empirical values resulting from our study, the empirical 95 % Quantile of the distribution of unsigned maximum t-value is computed for each of the three methods. Table 2 opposes the critical values computed respectively given in Table A.1 (see Section 2), and the empirical values.

Table 2: Critical Values (c) and Quantiles of unsigned maximum t-values (95 %)

	$t_L^{\mathrm{Med}}(\lambda)$	$t_D^{\mathrm{Med}}(\lambda)$	$t_L^{\mathrm{Box}}(\lambda)$	$t_D^{\mathrm{Box}}(\lambda)$	$t_L^{\mathrm{Dong}}(\lambda)$	$t_D^{\mathrm{Dong}}(\lambda)$
c	3.669	3.669	2.928	2.928	3.776	3.776
95 %	3.662	3.597	3.071	3.532	3.767	3.715

When analyzing Table 2 we should recall that the numerators of t-values needed to identify location effects are normal, while the numerators of t-values for dispersion effects are only approximately normal. For the methods “M”and “D”, the empirical 95 % quantiles of unsigned maximum t-values are not far below the critical values. For the variance estimation method used by Box both the empirical values exceed the critical value.

We presume that the application of the delta method is the reason for the large difference between critical value and empirical quantile for dispersion effects. The variance of these effects is underestimated systematically, which can also be noticed in Figure 12. The discrepancy for location effects is due to the normal approximation and can be avoided by using the t-distribution.

5.3.2 Conclusions

When analyzing data and searching for an appropriate transformation, the advantages of using the λ-Plot as suggested by Box are obvious. It is much more convenient to examine a plot and choose a transformation by eye than just using a computed transformation.

In addition, the λ-Plot gives more information about the behaviour of design factor and interaction effects for various values of λ. On the other hand the variance approximation for dispersion effects made by Box (1988) appears not to be appropriate, and therefore this procedure might lead to wrongly identified effects. Furthermore, the fact that a sequence of λ-values is considered, might introduce artificial location effects.

A good alternative to the λ-Plot is given by the β-Method as used by Kunert and Lehmkuhl (1998). This method allows for one dispersion effect to estimate the transformation parameter. Consequently, it leads to better results if such effects are present, but still yields satisfying outcomes for situations without influences on the variation of the response.

We conclude that using this method combined with the use of the variance estimator proposed by Dong (1993) to estimate the variance of effects on the variation, and the estimator suggested by Box (1988) to estimate the variance of influences on the mean, yields a good compromise in terms of transformation parameter estimation and the number of data sets with wrongly as well as correctly identified effects.

Acknowledgement

Financial support from the 'Deutsche Forschungsgemeinschaft' (SFB 475, *'Reduction of complexity in multivariate data structures'*) is gratefully acknowledged.

Appendix A: Description of Methods

Critical Values

Table A.1: Critical values for identifying active effects when using the Median-based variance estimation

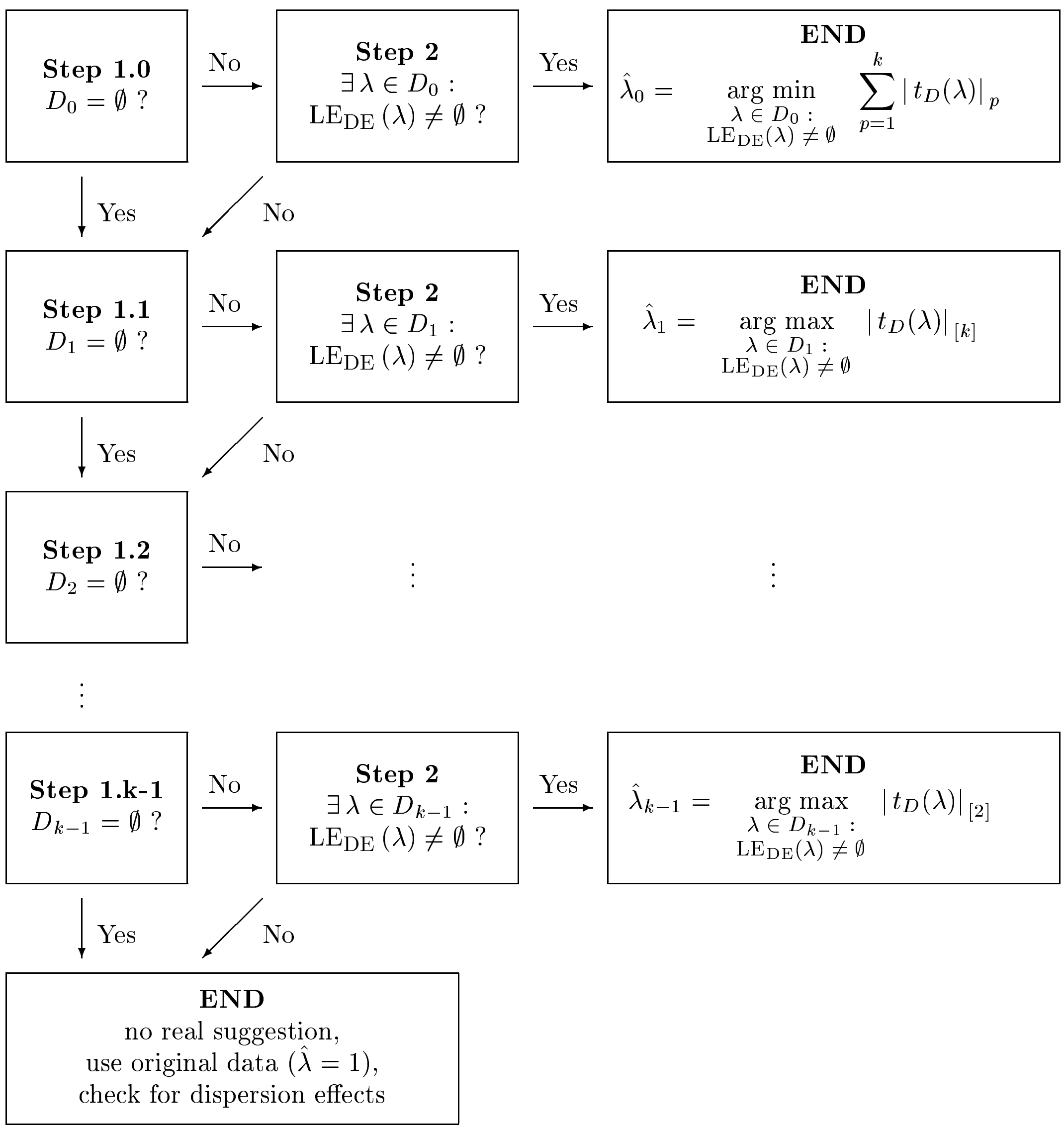
k	CL = 0 .9	CL = 0 .95	CL = 0.99
7	3.09933	3.87517	6.21262
15	3.15836	3.66889	4.96019
31	3.22513	3.59241	4.43574
63	3.31978	3.60575	4.23010

λ-Plot Formalization

For the description of the formalized λ-Plot in figure A.1 the following sets are required:

$$
\begin{aligned}
D_0 &= \{\, \lambda : t_D(\lambda) \text{ does not lead to active dispersion effects} \,\}, \\
D_1 &= \{\, \lambda : t_D(\lambda) \text{ leads to exactly one dispersion effect} \,\}, \\
&\ldots \\
D_{k-1} &= \{\, \lambda : t_D(\lambda) \text{ leads to exactly } k-1 \text{ dispersion effects} \,\} \\
\text{and} \quad \mathrm{DE}\,(\lambda) &= \{\, p : t_D(\lambda)_p \text{ implies Dispersion Effect} \,\}_{p=1,\ldots,k}\,, \\
\mathrm{LE_{DE}}\,(\lambda) &= \{\, p : t_L(\lambda)_p \text{ implies Location Effect} \,\}_{p=1,\ldots,k} \setminus \mathrm{DE}\,(\lambda)\,.
\end{aligned}
$$

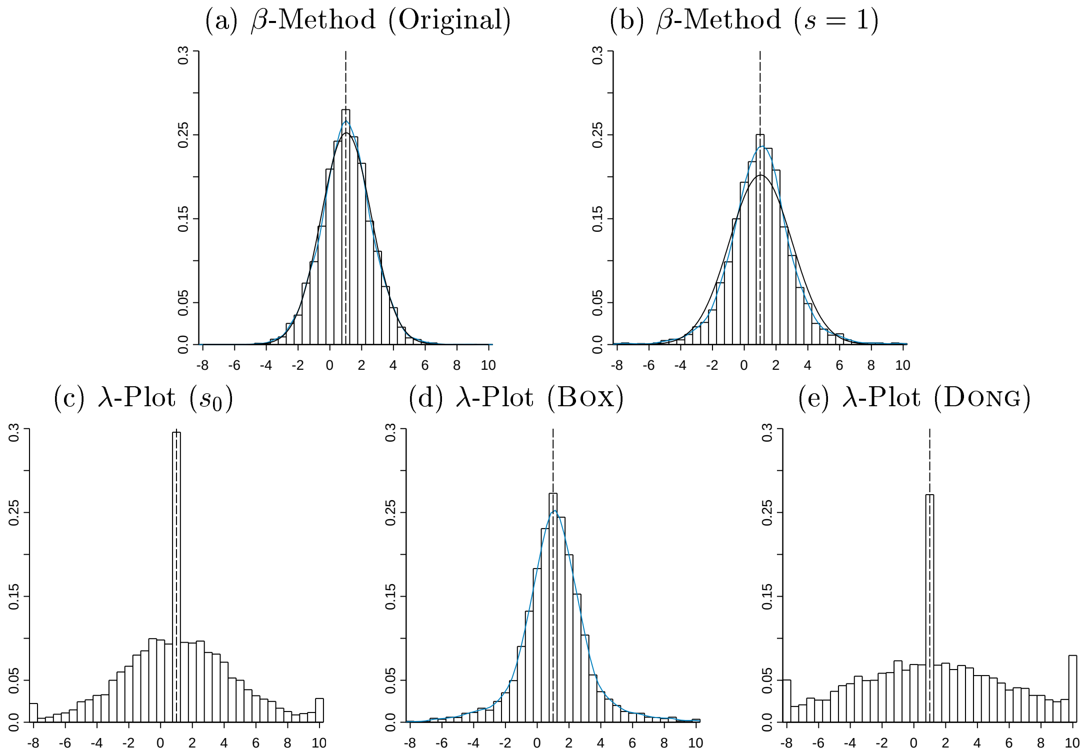
Figure A.1: Flow-chart for the Formalized λ-Plot procedure



Appendix B: Tables and Figures

Estimation of λ

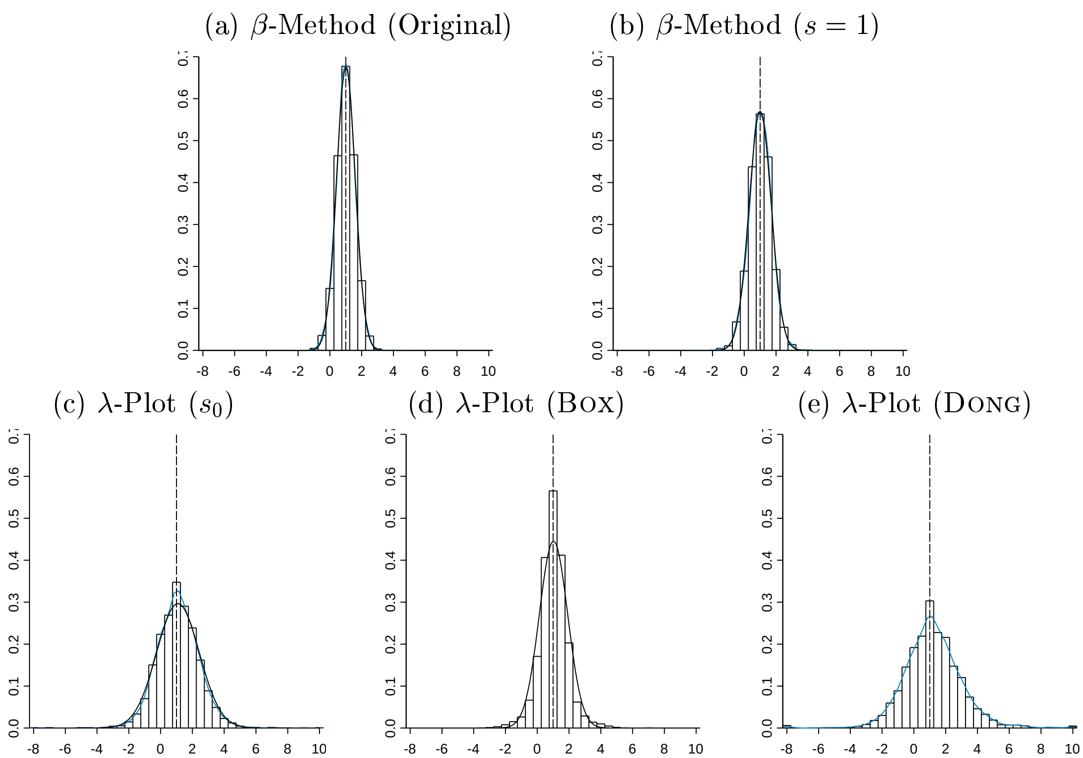
Figure B.1: Estimated λ-values for data sets containing one location effect (Scenario 2)



(a) β-Method (Original) (b) β-Method ($s = 1$)

(c) λ-Plot (s_0) (d) λ-Plot (Box) (e) λ-Plot (Dong)

Figure B.2: Estimated λ-values for data sets containing five location effects (Scenario 4)



(a) β-Method (Original) (b) β-Method ($s = 1$)

(c) λ-Plot (s_0) (d) λ-Plot (Box) (e) λ-Plot (Dong)

Figure B.3: Estimated λ-values for data sets containing one dispersion effect and four additional adjustment effects (Scenario 6)

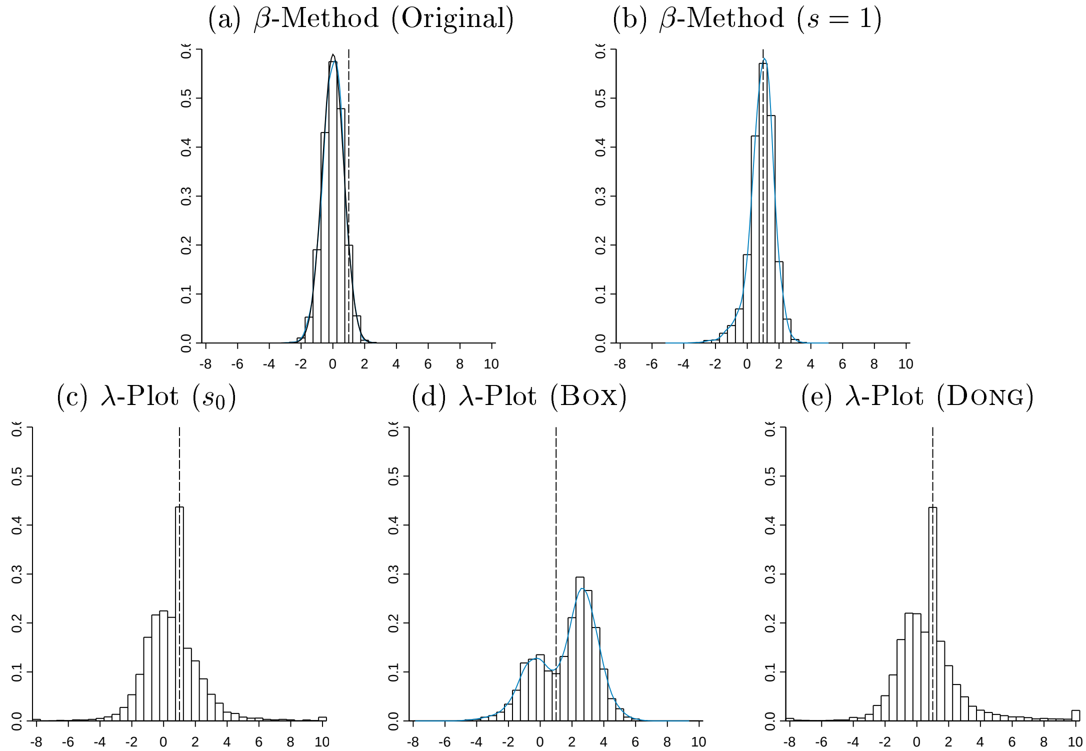


Table B.2: Parameters characterizing the empirical distribution of estimates for the transformation parameter λ

estimation of λ		(0,0)	(0,0)	(0,1)	(0,3)	(0,5)	(1,3)	(1,5)	(2,5)
Original β-Method	Mean	1.054	1.017	1.033	1.001	1.008	-0.826	0.015	-1.080
	Median	1.051	1.016	1.037	1.006	1.010	-0.808	0.033	-1.071
	Std. Dev.	2.682	2.667	1.581	0.849	0.591	0.966	0.677	0.746
β-Method with $s = 1$	Mean	1.017	0.994	1.032	0.998	0.998	0.736	0.914	-0.317
	Median	1.081	1.004	1.031	1.007	1.001	0.931	0.984	-0.164
	Std. Dev.	3.137	3.094	1.975	1.007	0.700	1.373	0.784	1.047
λ-Plot (s_0)	Mean	1.005	0.997	1.089	1.081	1.089	-0.500	0.591	0.119
	Median	1.000	1.000	1.000	1.000	1.000	-0.500	0.500	0.500
	Std. Dev.	2.199	2.190	3.386	2.069	1.345	2.390	1.804	2.609
λ-Plot (Box)	Mean	1.013	1.020	1.080	1.014	1.023	1.364	1.660	1.106
	Median	1.000	1.000	1.000	1.000	1.000	2.000	2.000	1.500
	Std. Dev.	0.972	0.928	2.103	1.238	0.896	2.792	1.847	2.254
λ-Plot (Dong)	Mean	1.046	1.038	1.194	1.291	1.191	-0.654	0.745	0.270
	Median	1.000	1.000	1.000	1.000	1.000	-0.500	0.500	0.500
	Std. Dev.	2.169	2.195	4.494	3.477	1.823	2.500	2.161	2.776

Identified Location and Dispersion Effects

In Table B.3 and Table B.4 the data sets for Scenario 1 are divided into two parts with 5 000 data sets each. The small differences between the results for both parts justify the number of data sets used in the simulation for all other scenarios.

We should keep in mind that the number of correctly identified effects in scenarios with at least two present effects on either the mean or the variation would be higher, if descending critical values for more than one effect would have been used. The critical values are conservative in such situations.

Table B.3: Percentage of data sets with wrongly identified dispersion effects

λ / $\hat{\lambda}$		(0,0)	(0,0)	(0,1)	(0,3)	(0,5)	(1,3)	(1,5)	(2,5)
$\lambda = 1$	s_0	4.42	4.68	4.80	4.58	4.32	2.72	2.62	0.78
	Box	18.30	18.60	18.74	17.76	19.02	17.94	15.78	7.86
	Dong	4.48	4.82	4.78	4.70	4.20	3.62	3.44	1.12
$\hat{\lambda}$ from Original β-Method	s_0	4.56	4.70	5.24	4.68	4.28	2.46	2.18	1.82
	Box	18.62	19.08	17.96	15.42	16.82	30.20	22.80	42.06
	Dong	4.6	4.72	5.32	4.86	4.66	3.48	3.20	2.92
$\hat{\lambda}$ from β-Method ($s = 1$)	s_0	6.72	6.86	8.58	7.68	6.96	3.88	3.36	2.08
	Box	26.36	26.24	24.00	22.12	23.50	19.9	15.82	16.92
	Dong	6.98	7.46	9.24	8.00	7.68	5.48	4.50	2.40
$\hat{\lambda}$ from λ-Plot	s_0	3.98	4.26	0.96	0.28	0.18	0.96	0.64	0.82
	Box	17.72	17.86	12.84	10.90	12.44	13.94	11.62	11.48
	Dong	4.12	4.46	0.82	0.14	0.32	0.86	1.00	1.06

Table B.4: Percentage of data sets with wrongly identified location effects

λ / $\hat{\lambda}$		(0,0)	(0,0)	(0,1)	(0,3)	(0,5)	(1,3)	(1,5)	(2,5)
$\lambda = 1$	s_0	5.00	4.82	2.46	0.36	0.06	0.48	0.00	0.04
	Box	7.40	7.30	6.66	2.98	1.90	3.26	1.72	1.86
	Dong	5.22	4.58	3.60	0.94	0.36	1.24	0.28	0.06
$\hat{\lambda}$ from Original β-Method	s_0	4.44	4.6	2.38	0.44	0.00	0.74	0.08	0.02
	Box	7.36	7.42	7.06	4.40	3.18	8.32	4.58	6.62
	Dong	4.68	4.74	3.82	1.22	0.38	1.92	0.34	0.08
$\hat{\lambda}$ from β-Method ($s = 1$)	s_0	4.90	4.94	2.64	0.38	0.02	0.40	0.04	0.02
	Box	7.62	7.46	7.02	4.56	3.62	5.38	2.90	4.72
	Dong	5.20	5.1	3.92	1.18	0.40	1.46	0.34	0.12
$\hat{\lambda}$ from λ-Plot	s_0	19.12	18.64	2.98	0.72	0.08	1.32	0.10	0.08
	Box	10.68	10.74	7.12	5.24	3.84	8.80	6.36	4.48
	Dong	15.80	15.08	3.80	2.48	0.28	3.34	0.26	0.18

Table B.5: Percentage of data sets with identified dispersion effects: no effects ("zero"), at least one, but not all ("not all") or all effects correctly ("correct")

λ / $\hat{\lambda}$	Scenario	(1,3) zero	(1,3) correct	(1,5) zero	(1,5) correct	(2,5) zero	(2,5) not all	(2,5) correct
$\lambda = 1$	s_0	51.16	46.12	52.18	45.20	37.34	30.20	31.04
	Box	78.62	3.44	80.20	4.02	79.94	4.06	0.04
	Dong	61.90	34.48	62.02	34.54	54.76	21.84	20.62
$\hat{\lambda}$ from Original β-Method	s_0	14.22	83.32	27.86	69.96	2.66	12.88	82.50
	Box	48.64	21.16	64.44	12.76	21.56	17.86	2.52
	Dong	17.12	79.40	34.04	62.76	3.16	13.14	80.36
$\hat{\lambda}$ from β-Method ($s = 1$)	s_0	54.02	42.10	54.78	41.86	12.76	29.12	55.76
	Box	74.16	5.94	79.00	5.18	55.88	14.76	0.36
	Dong	61.90	32.62	64.20	31.30	20.36	28.90	47.56
$\hat{\lambda}$ from λ-Plot	s_0	15.18	83.86	20.32	79.04	15.26	19.08	64.52
	Box	52.54	33.52	63.70	24.68	54.04	23.82	4.70
	Dong	15.90	83.24	26.84	72.16	23.52	15.72	59.10

Table B.6: Percentage of data sets with identified location effects: no effects ("zero"), at least one, but not all ("not all") or all effects correctly ("correct")

λ / $\hat{\lambda}$	Scenario	(0,1) zero	(0,1) correct	(0,3) zero	(0,3) not all	(0,3) correct	(0,5) zero	(0,5) not all	(0,5) correct
$\lambda = 1$	s_0	70.54	27.00	48.28	42.96	7.98	22.62	69.04	8.2
	Box	91.30	2.04	91.80	2.18	0.00	92.26	2.38	0.00
	Dong	81.1	15.30	74.6	18.72	4.32	59.22	31.32	8.56
$\hat{\lambda}$ from Original β-Method	s_0	69.18	28.44	44.80	44.52	9.80	18.9	69.82	11.22
	Box	90.58	2.36	90.06	2.14	0.00	89.18	2.64	0.00
	Dong	79.70	16.48	72.00	20.4	5.20	53.80	33.18	12.08
$\hat{\lambda}$ from β-Method ($s = 1$)	s_0	68.40	28.96	43.54	45.06	10.64	18.12	69.56	12.28
	Box	90.54	2.44	89.76	2.24	0.00	88.04	3.02	0.00
	Dong	79.30	16.78	71.24	20.90	5.62	51.70	34.14	13.30
$\hat{\lambda}$ from λ-Plot	s_0	85.04	11.98	37.92	59.62	1.52	13.28	84.84	1.74
	Box	91.24	1.52	88.02	2.70	0.00	86.18	3.62	0.00
	Dong	86.26	9.92	50.88	44.30	0.98	39.46	57.36	2.08

Table B.7: Percentage of data sets with identified location effects: no effects ("zero"), at least one, but not all ("not all") or all effects correctly ("correct")

λ / $\hat{\lambda}$	Scenario	(1,3) zero	(1,3) not all	(1,3) correct	(1,5) zero	(1,5) not all	(1,5) correct	(2,5) zero	(2,5) not all	(2,5) correct
$\lambda = 1$	s_0	25.56	46.28	27.34	10.94	61.78	27.28	4.76	39.86	55.32
	Box	76.62	16.80	0.02	79.04	16.18	0.00	37.66	56.56	0.14
	Dong	46.32	29.96	21.32	33.86	36.52	29.24	16.50	26.86	56.50
$\hat{\lambda}$ from Original β-Method	s_0	19.36	44.16	35.48	5.28	54.10	40.54	0.80	32.94	66.22
	Box	69.1	18.76	0.26	65.64	24.42	0.02	20.28	61.52	2.06
	Dong	36.84	32.42	28.04	19.92	36.84	42.82	3.38	30.14	66.26
$\hat{\lambda}$ from β-Method ($s = 1$)	s_0	22.60	44.62	32.08	8.32	58.74	32.90	2.24	35.8	61.94
	Box	73.06	17.76	0.08	74.68	17.72	0.00	31.38	56.08	1.00
	Dong	42.30	30.84	24.60	29.36	35.34	34.86	8.84	27.68	63.26
$\hat{\lambda}$ from λ-Plot	s_0	16.70	68.58	12.96	4.42	83.24	11.94	1.62	71.08	25.88
	Box	60.24	26.12	0.08	58.80	25.20	0.00	22.12	64.18	0.16
	Dong	31.74	53.60	10.46	20.44	64.78	13.24	9.66	61.12	24.64

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