A note on one-sided nonparametric analysis of covariance by ranking residuals

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Abstract

In a recent paper Speckman et al. (2002) introduced a technique for accounting covariates when their effects are nonlinear. They proposed a test for a one-sided analysis of covariance which is based on a rank test for the residuals obtained by smoothing the dependent variable on the covariate. In this paper we study some of the asymptotic properties of this test and a modifications of the test which try to take into account different sizes of the variances in both samples.

1 Introduction

Consider the classical two sample problem

$$Y_{ij} = m_i(X_{ij}) + \sigma_i(X_{ij})\varepsilon_{ij}, \quad j = 1, \dots, n_i; i = 1, 2 \tag{1.1}$$

where X_{ij} $(j = 1, \dots, n_i)$ are independent observations with positive density r_i $(i = 1, 2)$ on the interval $[0, 1]$ and ε_{ij} are independent identically distributed errors with mean 0 and variance 1 $(j = 1, \dots, n_i; i = 1, 2)$. The comparison of the two regression functions m_1 and m_2 is fundamental in applied statistic and much effort has been devoted to this problem. Many authors propose tests for the two-sided hypotheses

$$H_0 : m_1 \equiv m_2 \quad \text{vs.} \quad H_1 : m_1 \neq m_2$$

[see e.g. Delgado (1993), Hall and Hart (1990), King, Hart and Wehrly (1991), Kulasekera (1995), Kulasekera and Wang (1997), Young and Bowman (1995), Cabus (2000), Munk und

Dette (1998), Dette and Neumeyer (2001)]. In this paper we are interested in the problem of comparing two curves, where a one-sided analysis is appropriate, i.e.

$$H_0 : m_1 \equiv m_2 \quad \text{vs.} \quad H_1 : m_1(x) > m_2(x) \ \forall \ x. \tag{1.2}$$

Recently this problem has been studied by several authors [see Koul and Schick (1997), Hall, Huber and Speckman (1997) among others]. A very interesting method was proposed by Speckman, Chiu, Hewett and Bertelson (2002), which is based on the Wilcoxon-Mann-Whitney statistic applied to the residuals obtained from a nonparametric fit of the unknown regression function under the null hypothesis of equality. The corresponding test is simple to implement and has easily computed level probabilities, provided that the joint sample $\{(X_{ij}, Y_{ij}) \mid j = 1, \ldots, n_i; i = 1, 2\}$ is a sample of i.i.d. observations [see Speckman et al. (2002)]. This situation obviously occurs under the null hypothesis $m_1 = m_2$ and under the additional assumptions of equal design densities (i.e. $r_1 \equiv r_2$) and equal variance functions (i.e. $\sigma_1 \equiv \sigma_2$).
It is the purpose of the present paper to study the asymptotic properties of the test proposed by Speckman et al. (2002) in the case of (local) alternatives and in the case of different design densities and variance functions. We prove an asymptotic normal law in all cases, where the variance of the asymptotic distribution depends on several features of the data (design density, distribution of the error, variance function), which are not known by the experimenter [except in the case considered by Speckman et al. (2002)]. These results give further insight in the theoretical properties of the procedure proposed by Speckman et al. (2002) and explain certain deficiencies of this procedure if the basic assumption of equal design densities and variance functions is not satisfied. The paper will be organized as follows. In Section 2 we review the rank test of Speckman et al. (2002) and study some of its robustness properties if the random variables (X_{ij}, Y_{ij}) are not identically distributed. We also investigate a modification of the procedure which tries to deal with the problem of unequal variances in both samples. Section 3 contains our main theoretical results and some proofs are given in an appendix.

2 A rank test based on residuals

Consider the regression model (1.1) and let

$$\hat{m}(x) = \frac{\sum_{i=1}^{2} \sum_{j=1}^{n_i} K\left(\frac{X_{ij}-x}{h}\right) Y_{ij}}{\sum_{i=1}^{2} \sum_{j=1}^{n_i} K\left(\frac{X_{ij}-x}{h}\right)} \tag{2.1}$$

denote the Nadaraya-Watson estimator obtained from the total sample. If the null hypothesis of equal regression curves $m_1 \equiv m_2$ holds we expect the residuals

$$\hat{\varepsilon}_{ij} = Y_{ij} - \hat{m}(X_{ij}) \tag{2.2}$$

to be approximately centered. On the other hand note that in general the statistic $\hat{m}(x)$ estimates the function

$$\tilde{m}(x) = \frac{\kappa_1 r_1(x) m_1(x) + \kappa_2 r_2(x) m_2(x)}{r(x)}, \tag{2.3}$$

where

$$r(x) = \kappa_1 r_1(x) + \kappa_2 r_2(x), \tag{2.4}$$

and we assume that the individual sample sizes satisfy for $N = n_1 + n_2 \to \infty$

$$\frac{n_i}{N} = \kappa_i + O\Big(\frac{1}{N}\Big) \tag{2.5}$$

with $\kappa_i \in (0,1); i = 1,2$. Therefore it is easy to see that under the alternative $m_1 > m_2$ the residuals $\hat{\varepsilon}_{1j}$ from the first sample tend to larger values than the residuals $\hat{\varepsilon}_{2j}$ from the second sample. Based on these observations Speckman et al. (2002) suggested the following procedure for testing the hypotheses in (1.2). Let

$$\hat{R}_{ij} = \sum_{\ell=1}^{2} \sum_{k=1}^{n_\ell} I\{\hat{\varepsilon}_{\ell k} \le \hat{\varepsilon}_{ij}\} \tag{2.6}$$

denote the rank of the residual $\hat{\varepsilon}_{ij}$ among $\hat{\varepsilon}_{11}, \ldots, \hat{\varepsilon}_{2n_2}$ and

$$W_N = \sum_{j=1}^{n_1} \hat{R}_{1j} \tag{2.7}$$

the corresponding Wilcoxon statistic, then the null hypothesis in (1.2) is rejected for large values of W_N. A simple argument of exchangeability [see Speckman et al. (2002)] shows that under the null hypothesis $H_0 : m_1 \equiv m_2$ and the additional assumption of equal variance functions

$$\sigma_1^2 \equiv \sigma_2^2 \tag{2.8}$$

and equal design densities

$$r_1 \equiv r_2 \tag{2.9}$$

the statistic W_N has the classical Wilcoxon rank-sum distribution [see e.g. Randles and Wolfe (1979)] and consequently the corresponding level probabilities can easily be computed even in the finite sample case. In the following two tables we give some simulation results of the test which rejects the null hypothesis, whenever

$$Z_N = \frac{W_N - (N+1)n_1/2}{\sqrt{n_1 n_2 (N+1)/12}} > u_{1-\alpha}, \tag{2.10}$$

where $u_{1-\alpha}$ denotes the $(1-\alpha)$ quantile of the standard normal distribution. We considered the common regression function

$$m_1(x) = m_2(x) = \sin(5x) \tag{2.11}$$

but used different variances or design densities in our simulation in order to study the robustness properties of the test (2.10) with respect to violations of the assumptions (2.8) and (2.9) (note that in the case $m_1 \equiv m_2, r_1 \equiv r_2, \sigma_1^2 \equiv \sigma_2^2$ the random variable Z_N is asymptotically standard normal distributed). Table 2.1 shows the simulated level of the test (2.10) for a uniform design in both samples (i.e. $r_1 \equiv r_2 \equiv 1$) but different variances, i.e. $\sigma_1^2, \sigma_2^2 \in \{1, 2\}$. For the error distribution we assumed a $(\mathcal{X}_1^2 - 1)/\sqrt{2}$ distribution and the results are based on 10 000 runs for each scenario. In order to eliminate boundary effects we used a local linear estimator [see Fan and Gijbels (1996)] with bandwidth

$$h = \Big\{ \Big(\frac{n_1}{N}\hat{\sigma}_2^2 + \frac{n_2}{N}\hat{\sigma}_1^2 \Big) / N \Big\}^{1/5},$$

where $\hat{\sigma}_i^2$ denotes the estimator of Rice (1984) for the integrated variance function $\int_0^1 \sigma_i^2(x) r_i(x) dx$, $i = 1, 2$ [see Neumeyer and Dette (2003)]. For equal variances we observe an excellent approximation of the nominal level [see also Speckman et al. (2002), who obtained similar results with a different smoothing procedure]. However, in the case of different variances we observe that the test does not keep its level with sufficient accuracy. If $\sigma_1^2 < \sigma_2^2$ the level is substantially overestimated while in the opposite case the level is drastically underestimated. The situation with equal variances but different design densities is somehow similar and depicted in Table 2.2, where we investigated the densities

$$h(x) \equiv 1, \quad h(x) = \frac{1}{3} x^{-2/3} \tag{2.12}$$

as possible choices for r_1 and r_2. The reason for these deviations will be given in Section 3, where it is shown that the statistic Z_N is not asymptotically standard normal distributed, if one of the assumptions (2.8) or (2.9) is violated. More precisely we prove asymptotic normality of the statistic Z_N under fairly general conditions, where the mean and variance of the limit distribution is not necessarily equal to 0 and 1, respectively. Additionally, we show that a sufficient condition for a standard normal distribution as limit distribution of Z_N are the assumptions (2.8) and (2.9).

To understand these observations heuristically consider for a moment the classical situation of the Wilcoxon-Mann-Whitney test, where there are two samples

$$\begin{array}{cc} X_1, \ldots, X_{n_1} & \text{i.i.d.} \sim F \\ Y_1, \ldots, Y_{n_2} & \text{i.i.d} \sim G \end{array}$$

with continuous distribution functions F, G, and we are interested in the problem of testing the hypothesis $H_0 : F \equiv G$. If

$$R_i = \sum_{j=1}^{n_1} I\{X_j \leq X_i\} + \sum_{j=1}^{n_2} I\{Y_j \leq X_i\}$$

denotes the rank of X_i in the total sample, then it follows from the classical result of Chernoff and Savage (1958)

$$\frac{\sum_{j=1}^{n_1} R_j - \frac{(N+1)n_1}{2} + n_1 n_2 (\frac{1}{2} - w)}{n_1 n_2 / \sqrt{N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^2),$$

where $w = P(X_1 \leq Y_1)$ and the asymptotic variance is given by

$$s^2 = \frac{1}{\kappa_2} \operatorname{Var}(F(Y_1)) + \frac{1}{\kappa_1} \operatorname{Var}(G(X_1)).$$

A simple calculation now shows that the power of the test which rejects $H_0 : F = G$, whenever

$$\frac{\sum_{j=1}^{n_1} R_j - (N+1)n_1/2}{\sqrt{n_1 n_2 (N+1)/12}} > u_{1-\alpha},$$

is approximately given by

$$\Phi\left(\sqrt{\frac{N}{s^2}} \left\{ -u_{1-\alpha} \sqrt{\frac{N+1}{12 n_1 n_2}} + \left(\frac{1}{2} - w\right) \right\} \right), \tag{2.13}$$

where Φ denotes the distribution function of the standard normal distribution. Note that under the null hypothesis $H_0 : F = G$ we have $s^2 = 1/(12\kappa_1\kappa_2)$ and $w = 1/2$, which yields the asymptotic level α.

The test proposed by Speckman et al. (2002) replaces in this statistic the X_i and Y_j by the residuals $\hat{\varepsilon}_{ij}$ defined in (2.2). In order to understand the problem with this substitution heuristically we assume for the moment that we could replace the X_i and Y_j by the "true" residuals, that is

$$\begin{aligned} X_j &= Y_{1j} - m(X_{1j}) = \sigma_1(X_{ij})\varepsilon_{1i} =: e_{1j} \\ Y_j &= Y_{2j} - m(X_{2j}) = \sigma_2(X_{2j})\varepsilon_{2j} =: e_{2j}. \end{aligned} \tag{2.14}$$

If one of the assumptions (2.8) or (2.9) is not satisfied we can neither expect that the corresponding probability $w = P(e_{1j} \leq e_{2j})$ is equal 1/2 nor that the asymptotic variance s^2 is equal $1/(12\kappa_1\kappa_2)$ and this explains heuristically the loss of accuracy in the simulations presented in Table 2.1 and 2.2. We note that for $w \neq 1/2$ this loss will be substantial asymptotically, because the factor $w - 1/2 \neq 0$ is multiplied with the factor $\sqrt{N}$ in (2.13) and dominates the asymptotic power function. This is already indicated by the simulation results presented in Table 2.1, which show that in the case $\sigma_1^2 \neq \sigma_2^2$ the approximation of the level is not improved with increasing sample size. It will be demonstrated in Section 3 that the situation is even more complicated because the Wilcoxon Mann-Whitney test based on the "true" residuals e_{ij} behaves quite differently than the corresponding test based on the observable residuals $\hat{\varepsilon}_{ij}$.

$n_1 \backslash n_2$		10				20			
	$\sigma_1^2 \backslash \sigma_2^2$	1		2		1		2	
		5%	10%	5%	10%	5%	10%	5%	10%
10	1	0.054	0.110	0.132	0.229	0.050	0.107	0.131	0.227
	2	0.024	0.060	0.051	0.111	0.018	0.042	0.049	0.097
20	1	0.050	0.100	0.162	0.260	0.052	0.097	0.190	0.295
	2	0.013	0.029	0.050	0.097	0.011	0.026	0.053	0.101

Table 2.1 *Simulated level of the rank test (2.10) of Speckman et al. (2002) for various sample sizes and different variances in both groups. The common regression function is given by (2.11) and the designs in both samples are obtained from the uniform distribution.*

$n_1 \backslash n_2$		10				20			
	$r_1 \backslash r_2$	1		$\frac{1}{3}x^{-2/3}$		1		$\frac{1}{3}x^{-2/3}$	
		5%	10%	5%	10%	5%	10%	5%	10%
10	1	0.054	0.110	0.067	0.136	0.050	0.107	0.069	0.138
	$\frac{1}{3}x^{-2/3}$	0.027	0.063	0.053	0.110	0.026	0.058	0.048	0.099
20	1	0.050	0.100	0.080	0.147	0.052	0.097	0.094	0.167
	$\frac{1}{3}x^{-2/3}$	0.026	0.056	0.052	0.099	0.018	0.043	0.051	0.097

Table 2.2. *Simulated level of the rank test (2.10) of Speckman et al. (2002) for various sample sizes, equal variances $\sigma_1^2 = \sigma_2^2 = 0.5$ in both groups. The common regression function is given by (2.11), and the design densities are defined by (2.12).*

Before we explain this asymptotic behaviour we introduce a modification of the statistic proposed by Speckman et al. (2002), which - on a first glance - seems to be able to deal with violations of the assumption (2.8). To be precise, we define standardized residuals

$$\tilde{\varepsilon}_{ij} = \frac{Y_{ij} - \hat{m}(X_{ij})}{\hat{\sigma}_i(X_{ij})}, \quad j = 1, \ldots, n_i; i = 1, 2, \tag{2.15}$$

where

$$\hat{\sigma}_i^2(x) = \sum_{j=1}^{n_i} \frac{K\left(\frac{X_{ij}-x}{h}\right)}{\sum_{\ell=1}^{n_i} K\left(\frac{X_{i\ell}-x}{h}\right)} (Y_{ij} - \hat{m}_i(X_{ij}))^2 \tag{2.16}$$

is an estimator for the variance function of the ith sample ($i = 1, 2$) and

$$\hat{m}_i(x) = \frac{\sum_{j=1}^{n_i} K\left(\frac{X_{ij}-x}{h}\right) Y_{ij}}{\sum_{j=1}^{n_i} K\left(\frac{X_{ij}-x}{h}\right)} \tag{2.17}$$

the corresponding estimator of the regression function. Let

$$\tilde{R}_{ij} = \sum_{\ell=1}^{2} \sum_{k=1}^{n_\ell} I\{\tilde{\varepsilon}_{\ell k} \le \tilde{\varepsilon}_{ij}\} \tag{2.18}$$

denote the rank of $\tilde{\varepsilon}_{ij}$ in the total sample of standardized residuals, $\tilde{W}_N = \sum_{j=1}^{n_1} \tilde{R}_{1j}$ the sum of the ranks of the first sample, then following Speckman et al. (2002) we propose to reject the null hypothesis, whenever

$$\tilde{Z}_N = \frac{\tilde{W}_N - (N+1)n_1/2}{\sqrt{n_1 n_2 (N+1)/12}} > u_{1-\alpha}. \tag{2.19}$$

Tabel 2.3 shows the simulated level of the test (2.19) for the scenario used in Table 2.1 (i.e. $r_1 \equiv r_2$; $\sigma_1^2, \sigma_2^2 \in \{1, 2\}$; $\varepsilon_{ij} \sim (\mathcal{X}_1^2 - 1)/\sqrt{2}$). Now we observe substantial differences between the nominal and simulated level in all cases. Even in the case where the variances are equal [which gives a sequence of exchangeable random variables (X_{ij}, Y_{ij})] the differences are substantial and increase with the sample size. Table 2.4 shows corresponding results with a standard normal distribution for the errors ε_{ij}. Here the nominal level is approximated with sufficient accuracy in the case of equal variances, but in the case of different variances the test (2.19) is not an improvement of the test (2.10). Our simulations indicate that the replacement of the true residuals $Y_{ij} - m(X_{ij}) = \sigma_i(X_{ij})\varepsilon_{ij}$ by their corresponding estimates $\hat{\varepsilon}_{ij}$ or $\tilde{\varepsilon}_{ij}$ has a non-trivial effect on the performance of the resulting testing procedure. In the following section we study the asymptotic behaviour of the tests defined in (2.10) and (2.19) in more detail and by this analysis we are able to explain the observations described in this section.

$n_1 \backslash n_2$		10				20			
	$\sigma_1^2 \backslash \sigma_2^2$	1		2		1		2	
		5%	10%	5%	10%	5%	10%	5%	10%
10	1	0.110	0.178	0.146	0.219	0.085	0.149	0.116	0.183
	2	0.115	0.173	0.131	0.200	0.100	0.156	0.103	0.163
20	1	0.180	0.238	0.196	0.257	0.152	0.216	0.168	0.228
	2	0.184	0.237	0.202	0.261	0.169	0.231	0.175	0.236

Table 2.3. *Simulated level of the rank test (2.19) for various sample sizes and variances. The design in both groups is uniform, while the error distribution is a* $(\mathcal{X}_1^2 - 1)/\sqrt{2}$ *distribution.*

$n_1 \backslash n_2$		10				20			
	$\sigma_1^2 \backslash \sigma_2^2$	1		2		1		2	
		5%	10%	5%	10%	5%	10%	5%	10%
10	1	0.061	0.127	0.094	0.162	0.071	0.126	0.115	0.179
	2	0.057	0.107	0.069	0.121	0.054	0.097	0.077	0.130
20	1	0.058	0.106	0.077	0.139	0.057	0.112	0.099	0.170
	2	0.055	0.098	0.056	0.107	0.044	0.085	0.055	0.108

Table 2.4. *Simulated level of the rank test (2.19) for various sample sizes and variances. The design in both groups is uniform, while the error distribution is standard normal.*

3 Asymptotic analysis

In order to explain the properties described in the simulation study from an asymptotic point of view we require various additional assumptions on the data generating process. For the bandwidth of the kernel estimators we assume

(3.1) $$Nh \to \infty,\ h \to 0,\ Nh^4 \to 0,\ Nh^{3+2\delta}(\log h^{-1})^{-1} \to \infty$$

for some $0 < \delta < 1/2$, while the design densities should satisfy

$$r_1, r_2, \in C^2([0,1]); \inf_{x \in [0,1]} r_i(x) > 0; \quad i = 1, 2. \tag{3.2}$$

Further we assume

$$m, \sigma_1, \sigma_2 \in C^2([0,1]); \quad \inf_{x \in [0,1]} \sigma_i^2(x) > 0; \quad i = 1, 2, \tag{3.3}$$

(3.4) K has compact support, is twice continuously differentiable, $\int_{\mathbb{R}} uK(u)du = 0$.

Let

$$F_i(y \mid x) = P(Y_{ij} \le y \mid X_{ij} = x)$$

$(i = 1, 2)$ denote the conditional distribution function of Y_{ij} given X_{ij}, then we assume that $F_i(y \mid x)$ is continuous in (y, x) and has a density, say

$$f_i(y \mid x) = \frac{\partial}{\partial y} F_i(y \mid x)$$

(also continuous in (y, x)) satisfying

$$\inf_{x \in [0,1]} \inf_{s \in [0,1]} f_i(F_i^{-1}(s \mid x) \mid x) > 0; \quad i = 1, 2 \tag{3.5}$$

$$\sup_{x,y} |y f_i(y \mid x)| < \infty; \quad i = 1, 2. \tag{3.6}$$

Finally, we require the following assumptions $(i = 1, 2)$

(3.7) $\frac{\partial}{\partial y} f_i(y \mid x)$ exists, is continuous in (y, x) and satisfies $\sup_{x,y} |y^2 \frac{\partial}{\partial y} f_i(y \mid x)| < \infty$

(3.8) $\frac{\partial}{\partial x} F_i(y \mid x)$ exists, is continuous in (y, x) and satisfies $\sup_{x,y} |x \frac{\partial}{\partial x} F_i(y \mid x)| < \infty$

(3.9) $\frac{\partial^2}{\partial^2 x} F_i(y \mid x)$ exists, is continuous in (y, x) and satisfies $\sup_{x,y} |x^2 \frac{\partial^2}{\partial^2 x} F_i(y \mid x)| < \infty$.

Throughout this paper we denote with F_ε and F_{e_i} the distribution functions of the random variables ε_{ij} and $e_{ij} = \sigma_i(x_{ij})\varepsilon_{ij}$, respectively, while the corresponding densities are denoted by f_ε and $f_{e_i} (i = 1, 2)$. Note that F_{e_i} and f_{e_i} are not necessarily equal for both samples and are related to F_ε and f_ε by the equation

$$F_{e_i}(y) = P(\sigma_i(X_{ij})\varepsilon_{ij} \le y) = \int_0^1 F_\varepsilon\Big(\frac{y}{\sigma_i(x)}\Big) r_i(x) dx \tag{3.10}$$

$$f_{e_i}(y) = \int_0^1 \frac{r_i(x)}{\sigma_i(x)} f_\varepsilon\Big(\frac{y}{\sigma_i(x)}\Big) dx; \quad i = 1, 2. \tag{3.11}$$

In the following let

$$\hat{F}_{i,n_i}(y) = \frac{1}{n_i} \sum_{j=1}^{n_i} I\{\hat{\varepsilon}_{ij} \le y\}; \quad i = 1, 2,$$

$$\tilde{F}_{i,n_i}(y) = \frac{1}{n_i} \sum_{j=1}^{n_i} I\{\tilde{\varepsilon}_{ij} \le y\}; \quad i = 1, 2,$$

denote the empirical distribution functions of the residuals $\hat{\varepsilon}_{ij}$ and $\tilde{\varepsilon}_{ij}$ defined by (2.2) and (2.15), respectively, then it is easy to see that the statistics W_N defined in (2.7) and $\tilde{W}_N$ can be represented as

$$
\begin{aligned}
W_N &= n_1 N\Big\{\frac{n_1}{N}\int_{\mathbb{R}} \hat{F}_{1,n_1}(y)d\hat{F}_{1,n_1}(y) + \frac{n_2}{N}\int_{\mathbb{R}} \hat{F}_{2,n_2}(y)d\hat{F}_{1,n_1}(y)\Big\} \\
\tilde{W}_N &= n_1 N\Big\{\frac{n_1}{N}\int_{\mathbb{R}} \tilde{F}_{1,n_1}(y)d\tilde{F}_{1,n_1}(y) + \frac{n_2}{N}\int_{\mathbb{R}} \tilde{F}_{2,n_2}(y)d\tilde{F}_{1,n_1}(y)\Big\}.
\end{aligned} \tag{3.12}
$$

Our main results essentially specify the asymptotic distribution of the random variables W_N and $\tilde{W}_N$. A proof is complicated and given in the Appendix.

Theorem 3.1. *Assume that (2.5), (3.1) - (3.9) are satisfied, then under the null hypothesis of equal regression curves we have*

$$
U_N = \sqrt{N}\Big\{\int_{\mathbb{R}}\Big(\frac{n_1}{N}\hat{F}_{1,n_1}(y) + \frac{n_2}{N}\hat{F}_{2,n_2}(y)\Big)d\hat{F}_{1,n_1}(y) - \int_{\mathbb{R}}(\kappa_1 F_{e_1}(y) + \kappa_2 F_{e_2}(y))dF_{e_1}(y)\Big\} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\beta^2),
$$

where the asymptotic variance is given by

$$
\begin{aligned}
\beta^2 &= \frac{\kappa_2}{\kappa_1}\Big\{\sum_{i=1}^{2}\kappa_i\int_{\mathbb{R}^2}(F_{e_{3-i}}(u\wedge v) - F_{e_{3-i}}(u)F_{e_{3-i}}(v))dF_{e_i}(u)dF_{e_i}(v) \\
&+ \kappa_2\kappa_1\int_0^1(\kappa_1\sigma_1^2(t)r_1(t) + \kappa_2\sigma_2^2(t)r_2(t))\frac{1}{r^2(t)} \\
&\times \Big(\int_{\mathbb{R}}\int_0^1\Big[f_\varepsilon\Big(\frac{z\sigma_1(x)}{\sigma_2(t)}\Big)\frac{r_1(x)r_2(t)}{\sigma_2(t)} - f_\varepsilon\Big(\frac{z\sigma_2(x)}{\sigma_1(t)}\Big)\frac{r_1(t)r_2(x)}{\sigma_1(t)}\Big]\,dx\,dF_\varepsilon(z)\Big)^2 dt \\
&- 2\kappa_1\kappa_2\int_0^1\Big(\int_{\mathbb{R}}\int_0^1\Big[f_\varepsilon\Big(\frac{z\sigma_1(x)}{\sigma_2(t)}\Big)\frac{r_1(x)r_2(t)}{\sigma_2(t)} - f_\varepsilon\Big(\frac{z\sigma_2(x)}{\sigma_1(t)}\Big)\frac{r_1(t)r_2(x)}{\sigma_1(t)}\Big]\,dx\,dF_\varepsilon(z)\Big) \\
&\times \Big(\frac{\sigma_1(t)r_1(t)}{r(t)}\int_{\mathbb{R}}\int_0^1 E[\varepsilon I\{\varepsilon\sigma_1(t)\le y\sigma_2(x)\}]r_2(x)\,dx\,dF_\varepsilon(y) \\
&- \frac{\sigma_2(t)r_2(t)}{r(t)}\int_{\mathbb{R}}\int_0^1 E[\varepsilon I\{\varepsilon\sigma_2(t)\le y\sigma_1(x)\}]r_1(x)\,dx\,dF_\varepsilon(y)\Big)dt\Big\}.
\end{aligned}
$$

Moreover, under local alternatives $m_2(x) = m_1(x) + \Delta(x)/\sqrt{N}$ we have

$$
U_N \xrightarrow{\mathcal{D}} \mathcal{N}(\mu,\beta^2),
$$

where the asymptotic mean is given by

$$
\mu = -\kappa_2\int_0^1\frac{\Delta(x)r_1(x)r_2(x)}{r(x)}\Big\{\frac{\kappa_1}{\sigma_2(x)}\int_{\mathbb{R}} f_\varepsilon\Big(\frac{y}{\sigma_2(x)}\Big)dF_{e_1}(y) + \frac{\kappa_2}{\sigma_1(x)}\int_{\mathbb{R}} f_\varepsilon\Big(\frac{y}{\sigma_1(x)}\Big)dF_{e_2}(y)\Big\}\,dx
$$

Theorem 3.2. *Assume that (2.5), (3.1) - (3.9) are satisfied, then under the null hypothesis of equal regression curves we have*

$$
\tilde{U}_N = \sqrt{N}\Big\{\int_{\mathbb{R}}\Big(\frac{n_1}{N}\tilde{F}_{1,n_1}(y) + \frac{n_2}{N}\tilde{F}_{2,n_2}(y)\Big)d\tilde{F}_{1,n_1}(y) - \frac{1}{2}\Big\} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\tilde{\beta}^2),
$$

where the asymptotic variance is given by $\tilde{\beta}^2 = \frac{\kappa_2}{\kappa_1}(\frac{1}{12} + b^2)$ *and*

$$\begin{aligned} b^2 &= \int_{\mathbb{R}^2} \Big\{ f_\varepsilon(y) f_\varepsilon(z) \kappa_1 \kappa_2 \int_0^1 \Big(\frac{\kappa_1 r_1(x)}{\sigma_2^2(x)} + \frac{\kappa_2 r_2(x)}{\sigma_1^2(x)} \Big) \Big(\frac{r_1(x)\sigma_2(x) - r_2(x)\sigma_1(x)}{r(x)} \Big)^2 dx \\ &\quad + yz f_\varepsilon(y) f_\varepsilon(z) E\Big[\Big(\frac{\varepsilon^2 - 1}{2} \Big)^2 \Big] + y f_\varepsilon(y) E\Big[(\varepsilon^2 - 1) I\{\varepsilon \le z\} \Big] \\ &\quad + \kappa_1 \kappa_2 \int_0^1 \frac{r_2(x)\sigma_1(x) - r_1(x)\sigma_2(x)}{r(x)\sigma_1(x)\sigma_2(x)} (r_2(x)\sigma_2(x) - r_1(x)\sigma_1(x)) dx \\ &\quad \times f_\varepsilon(y) \Big(z f_\varepsilon(z) E[\varepsilon^3] + 2E[\varepsilon I\{\varepsilon \le z\}] \Big) \Big\} dF_\varepsilon(y) dF_\varepsilon(z). \end{aligned}$$

Moreover, under local alternatives $m_2(x) = m_1(x) + \Delta(x)/\sqrt{N}$ *we have*

$$\tilde{U}_N \xrightarrow{\mathcal{D}} \mathcal{N}(\tilde{\mu}, \tilde{\beta}^2),$$

where the asymptotic mean is given by

$$\tilde{\mu} = - \int_{\mathbb{R}} f_\varepsilon(y) dF_\varepsilon(y) \cdot \int_0^1 \Delta(x) \frac{\kappa_2 r_1(x) r_2(x)}{r(x)} \Big(\frac{\kappa_1}{\sigma_2(x)} + \frac{\kappa_2}{\sigma_1(x)} \Big) dx.$$

Remark 3.3. Note that for a symmetric error distribution the asymptotic variance of Theorem 3.2 reduces to

$$\begin{aligned} \tilde{\beta}^2 &= \frac{\kappa_2}{\kappa_1} \Big[\frac{1}{12} + \kappa_1 \kappa_2 \Big\{ \int_{\mathbb{R}} f_\varepsilon^2(y) dy \Big\}^2 \int_0^1 \frac{(\sigma_1(x) r_2(x) - \sigma_2(x) r_1(x))^2}{\sigma_1^2(x) \sigma_2^2(x) r^2(x)} (\kappa_1 r_1(x) \sigma_1^2(x) + \kappa_2 r_2(x) \sigma_2^2(x)) dx \\ &\quad + 2\kappa_1 \kappa_2 \int_{\mathbb{R}} E[I\{\varepsilon \le z\}\varepsilon] dF_\varepsilon(z) \int_{\mathbb{R}} f_\varepsilon^2(y) dy \\ &\quad \times \int_0^1 \frac{\sigma_1(x) r_2(x) - \sigma_2(x) r_1(x)}{r(x)\sigma_1(x)\sigma_2(x)} (\sigma_2(x) r_2(x) - \sigma_1(x) r_1(x)) dx \Big]. \end{aligned}$$

Moreover, studentizing the residuals as proposed in (2.15) does not simplify the asymptotitic variances substantially, because the estimation of the variance function yields some additional terms in the asymptotic variance of the corresponding rank statistic. In general the asymptotic null distribution of U_N and $\tilde{U}_N$ depend on certain features of the data generating process.

Remark 3.4. In the case of equal variances $\sigma_1 \equiv \sigma_2$ the variance estimators defined in (2.16) can be replaced by a combined estimator

$$\hat{\sigma}^2(x) = \sum_{i=1}^2 \sum_{j=1}^{n_i} \frac{K\Big(\frac{X_{ij}-x}{h}\Big)(Y_{ij} - \hat{m}_i(X_{ij}))^2}{\sum_{i=1}^2 \sum_{\ell=1}^{n_i} K\Big(\frac{X_{i\ell}-x}{h}\Big)}.$$

A careful inspection of the proof given in the Appendix shows that under the additional assumption of equal design densities $r_1 \equiv r_2$ the bandwidth conditions stated in (3.1) can be relaxed to

$$Nh \to \infty, \; h \to 0, \; Nh^5 = O(\log h^{-1}), \; Nh^{3+2\delta} (\log h^{-1})^{-1} \to \infty \tag{3.13}$$

and the optimal bandwidths for estimation of the regression and variance functions can be used. In this case the asymptotic variance of the test statistic $\tilde{U}_N$ defined in Theorem 3.1 and 3.2 reduces to

$$\beta^2 = \tilde{\beta}^2 = \frac{\kappa_2}{12\kappa_1}.$$

Remark 3.5. Theorem 3.1 and 3.2 can now easily be used for analyzing the asymptotic properties of the corresponding rank tests proposed in (2.10) and (2.19). We begin with the test (2.10) proposed by Speckman et al. (2002). A straightforward calculation gives under local alternatives $m_2(x) = m_1(x) + \Delta(x)/\sqrt{N}$ for the probability of rejection

$$P(Z_N > u_{1-\alpha}) \;=\; P\Big(\frac{W_N - n_1(N+1)/2}{\sqrt{n_1 n_2 (N+1)/12}} > u_{1-\alpha}\Big) \qquad (3.14)$$

$$\approx \; \Phi\Big(\frac{\mu + (a-\frac{1}{2})\sqrt{N} - u_{1-\alpha}\sqrt{\frac{\kappa_2}{12\kappa_1}}}{\beta}\Big),$$

where μ and β^2 are defined in Theorem 3.1 and the constant a is given by

$$a = \int_{\mathbb{R}} (\kappa_1 F_{e_1}(y) + \kappa_2 F_{e_2}(y)) dF_{e_1}(y). \qquad (3.15)$$

Now assume that the null hypothesis is valid, that is $\Delta(x) \equiv 0$, then we observe that the probability of falsely rejecting the null hypothesis by the test (2.10) converges to 1 whenever $a > \frac{1}{2}$. On the other hand, if $a < \frac{1}{2}$ this probability converges 0. Consequently the test does not keep its level for larger sample sizes whenever $a \neq \frac{1}{2}$. Recalling the definition (3.15) and (3.10) we see that in general the quantity a is not equal to 1/2. For example, in the situation depicted in Table 2.1 we have $r_1 = r_2 = 1; \sigma_1^2 = 1, \sigma_2^2 = 2$ which gives by (3.10)

$$F_{e_2}(y) = F_\varepsilon(\frac{y}{\sqrt{2}}) = F_{e_1}(\frac{y}{\sqrt{2}}).$$

Observing that $\varepsilon \sim (\mathcal{X}_1^2 - 1)/\sqrt{2}$ we obtain

$$\int_{\mathbb{R}} F_{e_2}(y) dF_{e_1}(y) = P(\sqrt{2}\varepsilon_1 < \varepsilon_2) \approx 0.605$$

and consequently $a \approx 0.553, 0.535, 0.570$ corresponding to the cases $\kappa_1 = \frac{1}{2}, \frac{2}{3}, \frac{1}{3}$, respectively. Therefore the test (2.10) rejects the null hypothesis too often, which explains the empirical results in Table 2.1 in the case $\sigma_1^2 < \sigma_2^2$. Moreover, the probability of rejection converges to 1 for increasing sample sizes. Similary, if $\sigma_1^2 = 2, \sigma_2^2 = 1$, then we obtain $a \approx 0.447, 0.429$ and 0.464 corresponding to the cases $\kappa_1 = \frac{1}{2}, \frac{2}{3}, \frac{1}{3}$, respectively, and the probability of a type I error is underestimated and converges to 0 for an increasing sample size. This was also observed in our simulation study in Section 2 [see Table 2.1]. Finally, if $a = 1/2$, then the test proposed by Speckman et al. (2002) keeps its asymptotic level if and only if the asymptotic variance in Theorem 3.1 satisfies

$$\beta^2 = \frac{\kappa_2}{12\kappa_1}. \qquad (3.16)$$

From Theorem 3.1 it follows that sufficient conditions for this property are given by (2.8) and (2.9) which explains our empirical findings in Table 2.1 and 2.2. On the other hand we obtain by a similar calculation for the probability of rejection by the test defined in (2.19)

$$P(\tilde{Z}_N > u_{1-\alpha}) \;\approx\; \Phi\Big(\frac{\tilde{\mu} - u_{1-\alpha}\sqrt{\frac{\kappa_2}{12\kappa_1}}}{\tilde{\beta}}\Big) \tag{3.17}$$

where $\tilde{\mu}$ and $\tilde{\beta}^2$ are defined in Theorem 3.2. Under the null hypothesis $m_1 \equiv m_2$ we have $\tilde{\mu} = 0$ and as a consequence this probability does not converge to 0 or 1 for an increasing sample size. However, it is asymptotically equal to α if and only if

$$\tilde{\beta}^2 = \frac{\kappa_2}{12\kappa_1}. \tag{3.18}$$

From Theorem 3.2 we see that the conditions (2.8) and (2.9) of equal design densities and variance functions are not sufficient for this property. In this case we need the additional assumption of a symmetric error distribution to guarantee (3.18), which explains our empirical observations in Table 2.3 and 2.4.

Remark 3.6. Note that Theorem 3.1 allows an asymptotic analysis of the statistic Z_N under local (and fixed) alternatives in the situation considered by Speckman et al. (2002). These authors assumed $r_1 \equiv r_2; \sigma_1 \equiv \sigma_2$ and showed that under the null hypothesis Z_N has the same distribution as the Wilcoxon statistic. In the case of homoscedasticity $\sigma_1^2(x) = \sigma_2^2(x) = \sigma^2 \; \forall\, x$ and $r_1 \not\equiv r_2$ this is still true asymptotically, because we obtain in this case $F_{e_1}(y) = F_{e_2}(y) = F_\varepsilon(y/\sigma)$ which implies $a = \frac{1}{2}$ and $\beta^2 = \frac{\kappa_2}{12\kappa_1}$. Under local alternatives it follows in the case $r_1 \equiv r_2; \sigma_1 \equiv \sigma_2$ that

$$\lim_{N\to\infty} P(Z_N > u_{1-\alpha}) \;=\; \Phi\Big(\mu\sqrt{\frac{12\kappa_1}{\kappa_2}} - u_{1-\alpha}\Big). \tag{3.19}$$

Observing the definition of μ in Theorem 3.1 we obtain for the shift

$$\mu\sqrt{\frac{12\kappa_1}{\kappa_2}} = \sqrt{12\kappa_1\kappa_2}\int_0^1 \frac{(-\Delta)(x)r_1(x)}{\sigma_1(x)}dx \int_{\mathbb{R}} f_\varepsilon\Big(\frac{y}{\sigma_1(x)}\Big)dF_{e_1}(y) \tag{3.20}$$

(note that $\Delta(x) < 0$ and that $\sigma_1 = \sigma_2, r_1 = r_2$).
Some conclusions can be drawn from this representation. For example, it follows from (3.20) that the best allocation of the observations to the treatment is obtained for $n_1 = n_2$, i.e. $\kappa_1 = \kappa_2 = \frac{1}{2}$. Similarly, different designs can be compared by its effect on the asymptotic power with respect to local alternatives. For example, if $\sigma_1 = \sigma_2 = 1$ and $\Delta = -x$, a design with density $r_1(x) = 2x = r_2(x)$ should be preferred to a design with density $2(1-x)$, while this design has a worse performance than the uniform design. In Table 3.1 we present some simulation results for different designs and the regression function m_1 in (2.11) which indicate that our asymptotic findings are already applicable for very small sample sizes. The power of the test of Speckman et al. (2002) can be substantially increased by an appropriate design of the experiment.
In principle such considerations are also possible in the general situation considered in Theorem 3.1. However, due to the complicated dependency of the term μ/β on the design densities, variance functions and κ_1, κ_2 such calculations definitively have to be done numerically.

	$r_1(x) = r_2(x) = 2(1-x)$				$r_1(x) = r_2(x) = 1$			
n_2	10		20		10		20	
n_1	5%	10%	5%	10%	5%	10%	5%	10%
10	0.294	0.439	0.354	0.499	0.455	0.606	0.569	0.709
20	0.369	0.505	0.499	0.635	0.536	0.661	0.725	0.827
	$r_1(x) = r_2(x) = e^x/(e-1)$				$r_1(x) = r_2(x) = 2x$			
n_2	10		20		10		20	
n_1	5%	10%	5%	10%	5%	10%	5%	10%
10	0.544	0.688	0.685	0.804	0.650	0.775	0.791	0.896
20	0.627	0.742	0.816	0.897	0.732	0.826	0.903	0.952

Table 3.1. *Simulated power of the test of Speckman et al. (2002) for various designs. The variances are $\sigma_1^2 = \sigma_2^2 = 1$, the errors are $(\mathcal{X}_1^2 - 1)/\sqrt{2}$ distributed and the alternative is $\Delta(x) = -x$. The factor $\int_0^1 -\Delta(x)r_1(x)/\sigma_1(x)dx$ in formula (3.20) is given by 0.133, 0.5, 0.582 and 0.667, respectively.*

Remark 3.7. Finally, we discuss the behaviour of a slightly modified test statistic where we use of a different regression estimator. Instead of $\hat{m}(x)$ defined in (2.1) we consider the estimator

$$\bar{m}(x) = \frac{\sum_{i=1}^2 \sum_{j=1}^{n_i} K\Big(\frac{X_{ij}-x}{h}\Big)\hat{\sigma}_i^{-2}(x)Y_{ij}}{\sum_{i=1}^2 \sum_{j=1}^{n_i} K\Big(\frac{X_{ij}-x}{h}\Big)\hat{\sigma}_i^{-2}(x)}$$

which can be shown to have better efficiency properties than $\hat{m}$. The variance estimators $\hat{\sigma}_i^2(x)$ are defined in (2.16). The residuals are now defined as

$$\bar{\varepsilon}_{ij} = \frac{Y_{ij} - \bar{m}(X_{ij})}{\hat{\sigma}_i(X_{ij})} \ ,$$

and the test statistic is the sum of the ranks of the residuals from the first sample, that is

$$\bar{W}_N \ = \ \sum_{j=1}^{n_1}\sum_{\ell=1}^{2}\sum_{k=1}^{n_\ell} I\{\bar{\varepsilon}_{\ell k} \le \bar{\varepsilon}_{1j}\} \ = \ n_1 N\Big\{\frac{n_1}{N}\int_{\mathbb{R}} \bar{F}_{1,n_1}(y)d\bar{F}_{1,n_1}(y) + \frac{n_2}{N}\int_{\mathbb{R}} \bar{F}_{2,n_2}(y)d\bar{F}_{1,n_1}(y)\Big\},$$

where $(i = 1, 2)$

$$\bar{F}_{i,n_i}(y) \ = \ \frac{1}{n_i}\sum_{j=1}^{n_i} I\{\bar{\varepsilon}_{ij} \le y\}.$$

Then under the conditions of Theorem 3.1 and local alternatives $m_2(x) = m_1(x) + \Delta(x)/\sqrt{N}$ (where the null hypothesis of equality of the regression functions corresponds to the case $\Delta \equiv 0$) it can be shown by similar arguments as given in the Appendix that

$$\bar{U}_N = \sqrt{N}\Big\{\int_{\mathbb{R}}\Big(\frac{n_1}{N}\bar{F}_{1,n_1}(y) + \frac{n_2}{N}\bar{F}_{2,n_2}(y)\Big)d\bar{F}_{1,n_1}(y) - \frac{1}{2}\Big\} \xrightarrow{\mathcal{D}} \mathcal{N}(\bar{\mu}, \bar{\beta}^2),$$

where the asymptotic variance is given by

$$\begin{aligned}\bar\beta^2 \;=\; & \frac{\kappa_2}{12\kappa_1} + \frac{\kappa_2}{4\kappa_1}\Big(\int_{\mathbb{R}} y f_\varepsilon^2(y)\,dy\Big)^2 E[(\varepsilon^2-1)^2] + \frac{\kappa_2}{\kappa_1}\int_{\mathbb{R}} y f_\varepsilon^2(y)\,dy \int_{\mathbb{R}} E[I\{\varepsilon\le z\}(\varepsilon^2-1)]\,dz \\ & + \kappa_2^2\Big[\Big(\int_{\mathbb{R}} f_\varepsilon^2(y)\,dy\Big)^2 + \int_{\mathbb{R}} f_\varepsilon^2(y)\,dy\Big\{2\int_{\mathbb{R}} E[I\{\varepsilon\le z\}\varepsilon]\,dF_\varepsilon(z) + \int_{\mathbb{R}} z f_\varepsilon^2(z)\,dz E[\varepsilon^3]\Big\}\Big] \\ & \times \int_0^1 \frac{(r_1(x)\sigma_1^{-1}(x) - r_2(x)\sigma_2^{-1}(x))^2}{\kappa_1 r_1(x)\sigma_1^{-2}(x) + \kappa_2 r_2(x)\sigma_2^{-2}(x)} dx\end{aligned}$$

and the asymptotic mean is obtained as

$$\bar\mu = -\int_{\mathbb{R}} f_\varepsilon(y)dF_\varepsilon(y)\cdot\int_0^1 \Delta(x)\frac{\kappa_2 r_1(x)r_2(x)\sigma_1^{-1}(x)\sigma_2^{-1}(x)}{\kappa_1 r_1(x)\sigma_1^{-2}(x)+\kappa_2 r_2(x)\sigma_2^{-2}(x)}\Big(\frac{\kappa_1}{\sigma_1(x)}+\frac{\kappa_2}{\sigma_2(x)}\Big)dx.$$

For a symmetric error distribution the asymptotic variance reduces to

$$\begin{aligned}\bar\beta^2 \;=\; & \frac{\kappa_2}{12\kappa_1} + \kappa_2^2\Big[\Big(\int_{\mathbb{R}} f_\varepsilon^2(y)\,dy\Big)^2 + 2\int_{\mathbb{R}} f_\varepsilon^2(y)\,dy\int_{\mathbb{R}} E[I\{\varepsilon\le z\}\varepsilon]\,dF_\varepsilon(z)\Big] \\ & \times\int_0^1 \frac{(r_1(x)\sigma_1^{-1}(x) - r_2(x)\sigma_2^{-1}(x))^2}{\kappa_1 r_1(x)\sigma_1^{-2}(x) + \kappa_2 r_2(x)\sigma_2^{-2}(x)} dx.\end{aligned}$$

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4 Appendix: proofs

For the sake of brevity we will restrict ourselves to a proof of Theorem 3.2. The proof of Theorem 3.1 follows in a similar manner. Recall the definition of the statistic $\tilde U_N$ in Theorem 3.2, then a straightforward calculation shows that

$$\begin{aligned}\tilde U_N \;=\; & \sqrt{N}\Big\{\int_{\mathbb{R}}\Big(\frac{n_1}{N}(\tilde F_{1,n_1}(y)-F_\varepsilon(y)) + \frac{n_2}{N}(\tilde F_{2,n_2}(y)-F_\varepsilon(y))\Big)d\tilde F_{1,n_1}(y) \\ & - \int_{\mathbb{R}}(\tilde F_{1,n_1}(y)-F_\varepsilon(y))dF_\varepsilon(y)\Big\},\end{aligned} \tag{5.1}$$

where we have used integration by parts. This expression can be estimated further by

$$\tilde U_N = \sqrt{N}\kappa_2\Big\{\int_{\mathbb{R}}(\tilde F_{2,n_2}(y)-F_\varepsilon(y))dF_\varepsilon(y) - \int_{\mathbb{R}}(\tilde F_{1,n_1}(y)-F_\varepsilon(y))dF_\varepsilon(y)\Big\} + o_p(1)\ , \tag{5.2}$$

where we have used the fact that

$$\int_{\mathbb{R}}(\tilde F_{1,n_1}(y)-F_\varepsilon(y))d(\tilde F_{1,n_1}(y)-F_\varepsilon(y)) = o_p\Big(\frac{1}{\sqrt{N}}\Big). \tag{5.3}$$

This estimate follows from the following Lemma, which gives an asymptotic equivalent expression for the integrands and will be proved at the end of this section.

Lemma 5.1. *Under the assumptions of Theorem 3.2 we have uniformly with respect to y*

$$\begin{aligned}\tilde{F}_{i,n_i}(y) - F_\varepsilon(y) \;=\; & \frac{1}{n_i}\sum_{j=1}^{n_i}\Big\{I\{\varepsilon_{ij} \le y\} - F_\varepsilon(y) + f_\varepsilon(y)\frac{\kappa_i r_i(X_{ij})}{r(X_{ij})}\varepsilon_{ij} + \frac{1}{2}yf_\varepsilon(y)(\varepsilon_{ij}^2 - 1)\Big\} \\ & + \frac{\kappa_{3-i}f_\varepsilon(y)}{n_{3-i}}\sum_{j=1}^{n_{3-i}}\frac{r_i(X_{3-i,j})\sigma_{3-i}(X_{3-i,j})}{r(X_{3-i,j})\sigma_i(X_{3-i,j})}\varepsilon_{3-i,j} \\ & - (-1)^i\frac{n_{3-i}}{\sqrt{N}}f_\varepsilon(y)\int_0^1 \Delta(x)\frac{r_1(x)r_2(x)}{r(x)\sigma_i(x)}dx + o_p\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Inserting the expression of Lemma 5.1 in (5.2) yields

$$\begin{aligned}\tilde{U}_N \;=\; & \kappa_2\sqrt{N}\Big\{\int_{\mathbb{R}}\tilde{G}_{1,n_1}(y)dF_\varepsilon(y) + \int_{\mathbb{R}}\tilde{G}_{2,n_2}(y)dF_\varepsilon(y)\Big\} + o_p(1) \\ & -\kappa_2\int_{\mathbb{R}}f_\varepsilon(y)dF_\varepsilon(y)\cdot\int_0^1 \Delta(x)\frac{r_1(x)r_2(x)}{r(x)}\Big\{\frac{\kappa_2}{\sigma_1(x)} + \frac{\kappa_1}{\sigma_2(x)}\Big\}dx\end{aligned} \tag{5.4}$$

where $\tilde{G}_{1,N}$ and $\tilde{G}_{2,N}$ are independent empirical processes defined by

$$\begin{aligned}\tilde{G}_{i,N}(y) \;=\; & (-1)^i\int_{\mathbb{R}}\Big\{\frac{1}{n_i}\sum_{j=1}^{n_i}(I\{\varepsilon_{ij} \le y\} - F_\varepsilon(y)) + \frac{1}{2}yf_\varepsilon(y)(\varepsilon_{ij}^2 - 1) \\ & + f_\varepsilon(y)\varepsilon_{ij}\kappa_i\Big(\frac{r_i(X_{ij})}{r(X_{ij})} - \frac{r_{3-i}(X_{ij})\sigma_i(X_{ij})}{r(X_{ij})\sigma_{3-i}(X_{ij})}\Big)\Big\}dF_\varepsilon(y).\end{aligned} \tag{5.5}$$

Now a standard calculation shows that these processes converge weakly to independent Gaussian processes, i.e.

$$\sqrt{N}\tilde{G}_{i,N} \Rightarrow G_i, \quad i = 1, 2;$$

where the covariance kernel of the process G_i is given by

$$\begin{aligned}k_i(y,z) \;=\; & \mathrm{Cov}(G_i(y), G_i(z)) = \frac{1}{\kappa_i}\Big\{F_\varepsilon(y \wedge z) - F_\varepsilon(y)F_\varepsilon(z) \\ & + f_\varepsilon(y)f_\varepsilon(z)\int_0^1\Big(\frac{r_{3-i}(x)\sigma_i(x)}{r(x)\sigma_{3-i}(x)} - \frac{r_i(x)}{r(x)}\Big)^2\kappa_i^2 r_i(x)dx \\ & + \frac{1}{4}yzf_\varepsilon(y)f_\varepsilon(z)E[(\varepsilon^2 - 1)^2] \\ & + \frac{1}{2}(yf_\varepsilon(y)E[I\{\varepsilon \le z\}(\varepsilon^2 - 1)] + zf_\varepsilon(z)E[I\{\varepsilon \le z\}(\varepsilon^2 - 1)]) \\ & - \int_0^1\Big(\frac{r_{3-i}(x)\sigma_i(x)}{r(x)\sigma_{3-i}(x)} - \frac{r_i(x)}{r(x)}\Big)\kappa_i r_i(x)dx \\ & \times\Big(f_\varepsilon(y)E[I\{\varepsilon \le z\}\varepsilon] + f_\varepsilon(z)E[I\{\varepsilon \le y\}\varepsilon] + \frac{1}{2}E[\varepsilon^3]f_\varepsilon(y)f_\varepsilon(z)(z + y)\Big)\Big\}.\end{aligned}$$

Now it follows from Proposition 1 in Shorack and Wellner (1986, p. 42) that

$$\tilde{U}_N \xrightarrow{\mathcal{D}} \mathcal{N}(\tilde{\mu}, \tilde{\beta}^2),$$

where

$$\tilde{\beta}^2 = \kappa_2^2 \int_{\mathbb{R}^2} (k_1(y,z) + k_2(y,z)) dF_\varepsilon(y) dF_\varepsilon(z).$$

The specific form of the variance in Theorem 3.2 now follows by a straightforward but tedious calculation.

□

Proof of Lemma 5.1. We will only consider the case $i = 1$; the assertion of the second sample follows precisely by the same arguments. From Lemma B1 in Akritas and Keilegom (2000) we have

$$\tilde{F}_{1,n_1}(y) - F_\varepsilon(y) = \frac{1}{n_1}\sum_{j=1}^{n_1}(I\{\varepsilon_{1j} \le y\} - F_\varepsilon(y)) + A_{n_1}(y) + o_p\Big(\frac{1}{\sqrt{n_1}}\Big) \tag{5.6}$$

uniformly in y, where

$$A_{n_1}(y) = f_\varepsilon(y) \int_0^1 \frac{y(\hat{\sigma}_1(x) - \sigma_1(x)) + \hat{m}(x) - m_1(x)}{\sigma_1(x)} r_1(x) dx. \tag{5.7}$$

Observing the definition of the estimate $\hat{m}$ in (2.1) we obtain

$$\begin{aligned} \hat{m}(x) - m_1(x) &= \frac{1}{Nhr(x)} \sum_{i=1}^{2}\sum_{j=1}^{n_i} K\Big(\frac{X_{ij}-x}{h}\Big)(\sigma_i(X_{ij})\varepsilon_{ij} + m_i(X_{ij}) - m_1(x)) + o_p\Big(\frac{1}{\sqrt{N}}\Big) \\ &= \frac{1}{Nhr(x)} \sum_{i=1}^{2}\sum_{j=1}^{n_i} K\Big(\frac{X_{ij}-x}{h}\Big)\sigma_i(X_{ij})\varepsilon_{ij} + \frac{1}{Nhr(x)}\sum_{j=1}^{n_2} K\Big(\frac{X_{2j}-x}{h}\Big)\frac{\Delta(X_{2j})}{\sqrt{N}} \\ &\quad + o_p\Big(\frac{1}{\sqrt{N}}\Big)\ . \end{aligned} \tag{5.8}$$

Now Proposition 7 in Akritas und Keilegom (2000) gives for the difference $\hat{\sigma}_1 - \sigma_1$ uniformly with respect to $x \in [0,1]$

$$\begin{aligned} \hat{\sigma}_1(x) - \sigma_1(x) &= -\frac{1}{n_1 h}\sum_{j=1}^{n_1} K\Big(\frac{x - X_{1j}}{h}\Big)\int_{\mathbb{R}} (I(Y_{1j} \le y) - F_1(y \mid x))\frac{y - m_1(x)}{r_1(x)\sigma_1(x)} dy \\ &\quad + o_p\Big(\frac{1}{\sqrt{n_1}}\Big), \end{aligned} \tag{5.9}$$

where the integral can be evaluated as follows :

$$\begin{aligned} \int_{\mathbb{R}} & (I\{z \le y\} - F_1(y \mid x))(y - m_1(x)) dy \\ &= m_1(x)\Big(\int_{-\infty}^{z} F_1(y \mid x) dy - \int_z^\infty (1 - F_1(y \mid x)) dy\Big) \\ &\quad + \int_z^\infty y(1 - F_1(y \mid x)) dy - \int_{-\infty}^z y F_1(y \mid x) dy \\ &= m_1(x)[z - m_1(x)] + \frac{1}{2}[m_1^2(x) + \sigma_1^2(x) - z^2]. \end{aligned}$$

Inserting this expression into (5.9) yields

$$(5.10) \qquad \hat{\sigma}_1(x) - \sigma_1(x) = \frac{1}{n_1 h r_1(x)\sigma_1(x)} \sum_{j=1}^{n_1} K\Big(\frac{x - X_{1j}}{h}\Big)\Big\{\frac{1}{2}(m_1(X_{1j}) - m_1(x))^2 + \varepsilon_{1j}\sigma_1(X_{1j})(m_1(X_{1j}) - m_1(x)) + \frac{1}{2}\sigma_1^2(X_{1j})(\varepsilon_{1j}^2 - 1) + \frac{1}{2}(\sigma_1^2(X_{1j}) - \sigma_1^2(x))\Big\} + o_p(\frac{1}{\sqrt{n_1}})$$

uniformly with respect to $x \in [0,1]$, which gives

$$(5.11) \qquad \int_0^1 \frac{\hat{\sigma}_1(x) - \sigma_1(x)}{\sigma_1(x)} r_1(x)dx = \frac{1}{2n_1}\sum_{j=1}^{n_1}(\varepsilon_{1j}^2 - 1) + o_p\Big(\frac{1}{\sqrt{n_1}}\Big).$$

Now the definition of A_{n_1} in (5.7), (5.9) and (5.11) yield

$$\begin{aligned} \tilde{F}_{1,n_1}(y) - F_\varepsilon(y) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\Big(I\{\varepsilon_{1j} \le y\} - F_\varepsilon(y) + \frac{1}{2}(\varepsilon_{1j}^2 - 1)y f_\varepsilon(y) + f_\varepsilon(y)\frac{\kappa_1 r_1(X_{1j})}{r(X_{1j})}\varepsilon_{1j}\Big) \\ &+ \frac{\kappa_2}{n_2} f_\varepsilon(y) \sum_{j=1}^{n_2} \frac{r_1(X_{2j})\sigma_2(X_{2j})}{r(X_{2j})\sigma_1(X_{2j})}\varepsilon_{2j} \\ &+ \frac{1}{\sqrt{N}}\frac{\kappa_2}{n_2} f_\varepsilon(y) \sum_{j=1}^{n_2} \frac{\Delta(X_{2j}) r_1(X_{2j})}{r(X_{2j})\sigma_1(X_{2j})} + o_p\Big(\frac{1}{\sqrt{n_1}}\Big) \end{aligned}$$

and the assertion of Lemma 5.1 follows from the strong law of large numbers.

□

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