Quadrature formulas for matrix measures — a geometric approach

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Abstract

A geometric approach to quadrature formulas for matrix measures is presented using the relations between the representations of the boundary points of the moment space (generated by all matrix measures) and quadrature formulas. Simple proofs of existence and uniqueness of quadrature formulas of maximal degree of precision are given. Several new quadrature formulas for matrix measures supported on a compact interval are presented and several examples are discussed. Additionally, a special construction of degenerated quadrature formulas is discussed and some results regarding the location of the zeros of polynomials orthogonal with respect to a matrix measure on a comapct interval are obtained as a by-product.

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1 Introduction

In recent years considerable interest has been shown in the construction of quadrature formulas to approximate matrix integrals using orthogonal matrix polynomials [see e.g. Basu and Bose (1983), Sinap and Van Assche (1994), Durán and Lopez-Rodriguez (1996) or Durán and Defez (2002) among many others]. A matrix measure μ is a $p \times p$ matrix $\mu = (\mu_{ij})$ of finite signed measures on the Borel field of the real line, (or an appropriate subset) such that $\mu(A)$ is symmetric and nonnegative definite for all Borel sets $A \subset \mathbb{R}$. A quadrature formula with degree of precision $n \in \mathbb{N}$

for a matrix measure μ consists of real numbers x_k ($k = 1, \dots, m$), called nodes, and corresponding $p \times p$ matrices Λ_k, called the quadrature coefficients or weights, such that

$$S_\ell = \int t^\ell d\mu(t) = \sum_{k=1}^m x_k^\ell \Lambda_k \quad \ell = 0, \dots, n\,. \tag{1.1}$$

It is well known that the nodes and quadrature coefficients in (1.1) are related to the matrix orthogonal polynomials which are obtained by orthogonalizing the matrix polynomials

$$I_p t^i \quad i = 0, \dots, m \tag{1.2}$$

(I_p denotes the $p \times p$ identity matrix) with respect to the right inner product

$$\langle P, Q \rangle := \int P^T(t) d\mu(t) Q(t), \tag{1.3}$$

where

$$P(t) = \sum_{i=0}^r A_i t^i, \qquad Q(t) = \sum_{i=0}^s B_i t^i$$

are matrix polynomials ($A_i, B_i \in \mathbb{R}^{p\times p}$). The nodes of a quadrature formula with exact degree of precision $n = 2m - 1$ are given by the roots of the mth matrix orthogonal polynomial $P_m(x)$ (i.e. the zeros of $\det P_m(x)$) and several formulas for the weights have been given in the literature [see e.g. Sinap and Van Assche (1994), Durán and Lopez-Rodriguez (1996) and Durán (1996)]. Recently Durán and Defez (2002) showed that, subject to conditions on the rank of Λ_i, there is in fact a unique quadrature formula of maximal degree of precision $n = 2m - 1$.

It is the purpose of the present paper to give an alternative approach to the derivation of quadrature formulas for matrix measures, which is based on geometric properties of the moment space generated by all matrix measures (with existing moments of appropriate order). The main idea is that the identities (1.1) show that a quadrature formula corresponds to a representation of the moment point $(S_0, \dots, S_n)$ in the moment space

$$M_{n+1} = \Big\{ \Big(\int d\eta(t), \dots, \int t^n d\eta(t) \Big) \Big| \eta \in \Xi_n \Big\}, \tag{1.4}$$

where η varies over the set Ξ_n of all matrix measures with existing moments up to the order n. It will be demonstrated that the quadrature formulas correspond to certain boundary points of the moment space M_{n+1} and that different types of boundary points can be used to characterize quadrature formulas with different degree of precision.

In the one dimensional case on the interval [0,1] boundary points in M_{n+1} have unique representations [see e.g. Karlin and Studden (1966)]. If S_n^- (S_n^+) denotes the smallest (largest) value of S_n among the measures with moments $S_0, \cdots, S_{n-1}$, then the points $(S_0, \dots, S_{n-1}, S_n^-)$ and $(S_0, \dots, S_{n-1}, S_n^+)$ are on the boundary of M_{n+1} and the unique representing measures yield quadrature formulas of slightly different form. The classical quadrature formula corresponds to the point $(S_0, \cdots, S_{2m-1}, S_{2m}^-)$ and is supported on the zeros of the mth orthogonal polynomial $P_m(x)$

which are in the interior of the interval. Other cases correspond to what are referred as Bouzitat quadrature formula of the first and second kind [see Ghizetti and Ossicini (1970)]. The quadrature formula or representation corresponding to the point $(S_0, \cdots, S_{2m-1}, S_{2m}^+)$ uses both endpoints and the zeros of the $(m-1)$th polynomial orthogonal with respect to the measure $x(1-x)d\mu$ and is a Bouzitat formula of the second kind. Corresponding to the point $(S_0, \cdots, S_{2m}, S_{2m+1}^-)$ there is a quadrature formula using the left end point and the zeros of the mth polynomial orthogonal with respect to the measure $xd\mu$. The remaining case with S_{2m+1}^+ uses the right endpoint and the polynomials orthogonal with respect to the measure $(1-x)d\mu$. Complete analogs of these results are given here for the case of matrix measures.

Properties of the moment space (1.4) for the matrix measures have been discussed in Dette and Studden (2002a) and the relevant results are briefly reviewed in Section 2 for the sake of completeness. In Section 3 we present a derivation of quadrature formulas from the viewpoint of moment theory. We give new (and in our opinion simple) proofs of the existence and uniqueness of a quadrature formula with precise degree of precision $n = 2m - 1$ such that the sum of ranks of the quadrature coefficients is equal to mp. As already indicated the main argument is that these formulas correspond to particular boundary points of the moment space M_{2m}, for which the representing matrix measure in (1.4) must be unique. For the sake of brevity we restrict ourselves to matrix measures supported in the interval $[0, 1]$, but an extension of the approach to the intervals $[0, \infty)$ or $(-\infty, \infty)$ is straightforward and will be indicated at several points of this paper. Section 4 discusses the matrix quadrature formulas corresponding to the Bouzitat formulas in one dimension [see Bouzitat (1949)]. Finally, in Section 5 we present some new quadrature formulas, which integrate polynomials of degree $n = 2m-1$ exactly and the sum of the ranks of quadrature coefficients is equal to $mp + h$, where $0 < h < p$. These quadrature formulas are of the same type as the formulas recently considered by Durán and Polo (2002). The paper contains also some new results regarding the location of the roots of orthogonal matrix polynomials, which are obtained as a by-product.

2 Preliminary results

For the sake of clarity we briefly review some results from the recent work of Dette and Studden (2002a) concerning the moment space defined in (1.1) and the orthogonal matrix polynomials with respect to matrix measures on the interval $[0, 1]$.

Lemma 2.1. *The set M_{n+1} defined in (1.4) is equal to the convex cone $\mathcal{C}(C_{n+1})$, where*

$$C_{n+1} = \{ \ (aa^T, taa^T, \cdots, t^n aa^T) \mid 0 \leq t \leq 1, \quad a \ \in \mathbb{R}^p \ \} \tag{2.1}$$

and $\mathcal{C}(C_{n+1})$ denotes the convex cone generated by this set.

It follows from Lemma 2.1 and Caratheodory's theorem [see Rockafellar (1970)] that every point $(S_0, S_1, \dots, S_n) \in M_{n+1}$ can be represented by a matrix measure with finite support. The following result gives a characterization of points belonging to the moment space M_{n+1} generated by the matrix measures on the interval $[0,1]$ and generalizes classical work on the Hausdorff moment problem [see Shohat and Tamarkin (1943)]. Defining the "Hankel" matrices

$$\underline{H}_{2m} = \begin{pmatrix} S_0 & \cdots & S_m \\ \vdots & & \vdots \\ S_m & \dots & S_{2m} \end{pmatrix}, \qquad \overline{H}_{2m} = \begin{pmatrix} S_1 - S_2 & \cdots & S_m - S_{m+1} \\ \vdots & & \vdots \\ S_m - S_{m+1} & \dots & S_{2m-1} - S_{2m} \end{pmatrix}, \tag{2.2}$$

and

$$\underline{H}_{2m+1} = \begin{pmatrix} S_1 & \cdots & S_{m+1} \\ \vdots & & \vdots \\ S_{m+1} & \dots & S_{2m+1} \end{pmatrix}, \qquad \overline{H}_{2m+1} = \begin{pmatrix} S_0 - S_1 & \cdots & S_m - S_{m+1} \\ \vdots & & \vdots \\ S_m - S_{m+1} & \dots & S_{2m} - S_{2m+1} \end{pmatrix}, \tag{2.3}$$

Dette and Studden (2002a) obtained the following characterization.

Theorem 2.2.
a) *The point $(S_0, \cdots, S_n)$ is in the moment space M_{n+1} generated by the matrix measures on the interval $[0,1]$, if and only if the matrices $\underline{H}_n$ and $\overline{H}_n$ are nonnegative definite.*

b) *The point $(S_0, \dots, S_n)$ is in the interior of the moment space M_{n+1} generated by the matrix measures on the interval $[0,1]$, if and only if the matrices $\underline{H}_n$ and $\bar{H}_n$ are positive definite.*

The nonnegativity of the matrices $\underline{H}_n$ and $\overline{H}_n$ imposes bounds on the moments S_k as in the one dimensional case [see Dette and Studden (1997), Chapter 1]. To be precise let

$$\underline{h}_{2m}^T = (S_{m+1}, \cdots, S_{2m})$$

$$\underline{h}_{2m-1}^T = (S_m, \cdots, S_{2m-1})$$

$$\bar{h}_{2m}^T = (S_m - S_{m+1}, \cdots, S_{2m-1} - S_{2m})$$

$$\bar{h}_{2m-1}^T = (S_m - S_{m+1}, \cdots, S_{2m-2} - S_{2m-1})$$

and define $S_1^- = 0$ and

$$S_{n+1}^- = \underline{h}_n^T \underline{H}_{n-1}^{-1} \underline{h}_n, \quad n \geq 1 \ , \tag{2.4}$$

and $S_1^+ = S_0$, $S_2^+ = S_1$ and

$$S_{n+1}^+ = S_n - \bar{h}_n^T \bar{H}_{n-1}^{-1} \bar{h}_n, \quad n \geq 2 \ , \tag{2.5}$$

whenever the inverses of the Hankel matrices exist. It is to be noted and stressed that S_n^- and S_n^+ depend on $(S_0, S_1, \cdots, S_{n-1})$ although this is not mentioned explicitly. It follows from Theorem 2.2 and a straightforward calculation with partitioned matrices that $(S_0, \dots, S_{n-1})$ is in the interior of the moment space M_n if and only if $S_n^- < S_n^+$ in the sense of Loewner ordering (note that a matrix is positive definite if and only if its main subblock and the corresponding Schur complement are positive definite). Moreover, for $(S_0, \dots, S_n) \in M_{n+1}$ we have

$$S_n^- \leq S_n \leq S_n^+ \tag{2.6}$$

in the sense of Loewner ordering. If $(S_0, S_1, \cdots, S_{n-1})$ is in the interior of the moment space M_n, then we define the kth matrix canonical moment as the matrix

$$U_k = D_k^{-1}(S_k - S_k^-) , \qquad 1 \le k \le n , \tag{2.7}$$

where

$$D_k = S_k^+ - S_k^- . \tag{2.8}$$

These quantities are the analog of the classical canonical moments p_k in the scalar case [see Skibinsky (1967, 1969, 1986) or Dette and Studden (1997)]. We will also make use of the quantities

$$V_k = I_p - U_k = D_k^{-1}(S_k^+ - S_k) , \qquad 1 \le k \le n . \tag{2.9}$$

The matrix canonical moments of low order can easily be calculated. From $S_1^+ = S_0$, $S_1^- = 0$ we have $D_1 = S_0$ and $U_1 = S_0^{-1}S_1$. Similarly, the definitions (2.4) and (2.5) imply

$$\begin{aligned} S_2^+ &= S_1 \\ S_2^- &= S_1 S_0^{-1} S_1 \end{aligned}$$

which gives

$$D_2 = S_1(I_p - S_0^{-1}S_1) = S_1(I_p - U_1) = S_1 V_1$$

and

$$U_2 = (I_p - S_0^{-1}S_1)^{-1} S_1^{-1} (S_2 - S_1 S_0^{-1} S_1) .$$

Two of the main results in Dette and Studden (2002a) are the following theorems. The first represents the width D_{n+1} of the moment space M_{n+1} in terms of the matrix canonical moments U_k and V_k and the second relates the canonical moments to quantities ζ_n and γ_n, which appear in the three term recurrence relation for the monic matrix orthogonal polynomials [see Lemma 2.6 below].

Theorem 2.3. *If the point $(S_0, \cdots, S_n)$ is in the interior of the moment space M_{n+1} generated by the matrix measures on the interval $[0, 1]$, then*

$$D_{n+1} = S_{n+1}^+ - S_{n+1}^- = S_0 U_1 V_1 U_2 V_2 \cdots U_n V_n. \tag{2.10}$$

Theorem 2.4. *If for some $n \ge 1$, the point $(S_0, S_1, \cdots, S_n)$ is in the interior of the moment space M_{n+1} generated by the matrix measures on the interval $[0, 1]$, then ($S_0^- = 0$, $V_0 = I_p$)*

$$(S_{n-1} - S_{n-1}^-)^{-1}(S_n - S_n^-) = V_{n-1}U_n =: \zeta_n \tag{2.11}$$

$$(S_{n-1}^+ - S_{n-1})^{-1}(S_n^+ - S_n) = U_{n-1}V_n =: \gamma_n.$$

For the following discussion let

$$\underline{D}_{2m} = \begin{vmatrix} S_0 & \cdots & S_m \\ \vdots & & \vdots \\ S_m & \cdots & S_{2m} \end{vmatrix} \tag{2.12}$$

denote the determinant of the Hankel matrix $\underline{H}_{2m}$ and define a matrix polynomial by

$$P_m(t) = \underline{H}_{2m-1}(t) = (H_{ij}(t))_{ij=1}^p ,$$

where the elements $H_{ij}(t)$ are determinants given by

$$H_{ij}(t) = \begin{vmatrix} S_0 & S_1 & \cdots & S_m \\ \vdots & & & \vdots \\ S_{m-1} & S_m & \cdots & S_{2m-1} \\ S_m^{ij}(t) & S_{m+1}^{ij}(t) & \cdots & S_{2m}^{ij}(t) \end{vmatrix} \qquad i,j = 1,\ldots,p$$

and the matrix $S_{m+k}^{ij}(t)$ is obtained from S_{m+k} by replacing the jth row by $e_i^T t^k$. Here $e_i \in \mathbb{R}^p$ denotes the vector with a one in the ith component and zero elsewhere.

Theorem 2.5. *The polynomial $P_m(t) = \underline{H}_{2m-1}(t)$ is an mth matrix orthogonal polynomial with respect to the inner product (1.3) and has leading coefficient*

$$L_m = \underline{D}_{2m}(S_{2m} - S_{2m}^-)^{-1},$$

where the determinant $\underline{D}_{2m}$ is defined in (2.12). Moreover the L^2-norm of the polynomial P_m is given by

$$\int P_m^T(t)d\mu(t)P_m(t) \;=\; L_m^T diag(\underline{D}_{2m}, \cdots, \underline{D}_{2m}) \;=\; L_m^T \underline{\Delta}_{2m}$$

where $\underline{\Delta}_{2m} = I_p \underline{D}_{2m}$.

It follows from Theorem 2.5 that the monic orthogonal polynomials with respect to the matrix measure μ are given by

$$\underline{P}_m(x) \;=\; P_m(x)L_m^{-1} \;=\; P_m(x)(S_{2m} - S_{2m}^-)/\underline{D}_{2m} , \tag{2.13}$$

where we have to multiply from the right, because we use the right inner product. It can also be easily seen that

$$\underline{P}_m(x) = x^m I_p - (I_p, xI_p, \cdots, x^{m-1}I_p)\underline{H}_{2m-2}^{-1}\underline{h}_{2m-1}, \tag{2.14}$$

and consequently the definition of the mth monic orthogonal polynomial with respect to a matrix measure μ only requires the moments up to the order $2m-1$. From this representation and (2.4) it is easy to see that the L^2-norm of the mth monic orthogonal polynomal is given by

$$\int \underline{P}_m^T(x)d\mu(x)\underline{P}_m(x) \;=\; S_{2m} - S_{2m}^-. \tag{2.15}$$

For a matrix measure μ on the interval $[0,1]$ matrix orthogonal polynomials with respect to the measures $td\mu(t)$, $(1-t)d\mu(t)$ and $t(1-t)d\mu(t)$ are obtained similarly [see Dette and Studden (2002a) for more details]. We finally mention a special form of the recurrence relation, for these orthogonal matrix polynomials, which expresses the coefficents of the orthogonal polynomials in terms of canonical moments or the quantities ζ_n and γ_n introduced in Theorem 2.4.

Lemma 2.6.

1) *The sequence of monic orthogonal polynomials* $\{\underline{P}_k(x)\}_{k\geq 0}$ *with respect to the matrix measure* μ *satisfies the recurrence formula* $\underline{P}_0(x) = I_p, \underline{P}_{-1}(x) = 0$ *and for* $m \geq 0$

$$x\underline{P}_m(x) = \underline{P}_{m+1}(x) + \underline{P}_m(x)(\zeta_{2m+1} + \zeta_{2m}) + \underline{P}_{m-1}(x)\zeta_{2m-1}\zeta_{2m}, \tag{2.16}$$

where the quantities $\zeta_j \in \mathbb{R}^{p\times p}$ *are defined by* $\zeta_0 = 0$, $\zeta_1 = U_1$, $\zeta_j = V_{j-1}U_j$ *if* $j \geq 2$ *and the sequences* $\{U_j\}$ *and* $\{V_j\}$ *are given in (2.7) and (2.9).*

2) *Let* μ *denote a matrix measure on the interval* $[0,\infty)$*. The sequence of monic orthogonal polynomials* $\{\underline{Q}_k(x)\}_{k\geq 0}$ *with respect to the matrix measure* $xd\mu(x)$ *satisfies the recurrence formula* $\underline{Q}_0(x) = I_p, \underline{Q}_{-1}(x) = 0$ *and for* $m \geq 0$

$$x\underline{Q}_m(x) = \underline{Q}_{m+1}(x) + \underline{Q}_m(x)(\zeta_{2m+1} + \zeta_{2m+2}) + \underline{Q}_{m-1}(x)\zeta_{2m}\zeta_{2m+1} \ . \tag{2.17}$$

3) *Let* μ *denote a matrix measure on the interval* $(-\infty, 1]$*. The sequence of monic orthogonal polynomials* $\{\overline{P}_k(x)\}_{k\geq 0}$ *with respect to the matrix measure* $(1-x)\mu(x)$ *satisfies the recurrence formula* $\overline{P}_0(x) = I_p, \overline{P}_1(x) = xI_p - \gamma_2$ *and for* $m \geq 1$

$$x\overline{P}_m(x) = \overline{P}_{m+1}(x) + \overline{P}_m(x)(\gamma_{2m+1} + \gamma_{2m+2}) + \overline{P}_{m-1}(x)\gamma_{2m}\gamma_{2m+1}, \tag{2.18}$$

where the quantities $\gamma_j \in \mathbb{R}^{p\times p}$ *are defined by* $\gamma_1 = V_1$, $\gamma_j = U_{j-1}V_j$ *if* $j \geq 2$ *and the sequences* $\{U_j\}$ *and* $\{V_j\}$ *are given in (2.7) and (2.9).*

4) *Let* μ *denote a matrix measure on the interval* $[0,1]$*. The sequence of monic orthogonal polynomials* $\{\overline{Q}_k(x)\}_{k\geq 0}$ *with respect to the matrix measure* $x(1-x)d\mu(x)$ *satisfies the recurrence formula* $\overline{Q}_0(x) = I_p, \overline{Q}_{-1}(x) = 0$ *and for* $m \geq 0$

$$x\overline{Q}_m(x) = \overline{Q}_{m+1}(x) + \overline{Q}_m(x)(\gamma_{2m+2} + \gamma_{2m+3}) + \overline{Q}_{m-1}(x)\gamma_{2m+1}\gamma_{2m+2} \ . \tag{2.19}$$

3 A geometric view of quadrature formulas

Let μ denote a matrix measure on the interval $[0,1]$ with moments

$$S_\ell = \int_0^1 t^\ell d\mu(t) \ , \quad \ell = 0, 1, \ldots, 2m-1 \ , \tag{3.1}$$

such that $(S_0, \ldots, S_{2m-2}) \in \text{Int}(M_{2m-1})$. Recall the definition of S_{2m}^- in (2.4) and consider the Hankel matrix $\underline{H}_{2m}$ corresponding to the moment point $(S_0, \ldots, S_{2m-1}, S_{2m}^-)$, then $\det\underline{H}_{2m} = 0$

and the point $(S_0, \ldots, S_{2m-1}, S_{2m}^-)$ is at the boundary of the moment space M_{2m+1}. Consequently, there exists a matrix measure, say μ_{mp}^-, on the interval $[0, 1]$ with finite support [see Lemma 2.1] representing the point $(S_0, \ldots, S_{2m-1}, S_{2m}^-)$, that is

$$
(3.2) \qquad \begin{aligned} S_\ell &= \int_0^1 t^\ell d\mu_{mp}^-(t) \ , \qquad \ell = 0, \ldots, 2m-1, \\ S_{2m}^- &= \int_0^1 t^{2m} d\mu_{mp}^-(t) \ , \end{aligned}
$$

where

$$
(3.3) \qquad \mu_{mp}^- = \sum_{j=1}^k \Lambda_j \delta_{x_j},
$$

and δ_x denotes the Dirac measure at the point $x \in [0, 1]$. Comparing (1.1) and (3.2) we see that μ_{mp}^- defines a quadrature formula which integrates polynomials of degree $2m - 1$ exactly. Thus the existence of a quadrature formula is a simple matter.

We now study some properties of the measure μ_{mp}^-. To this end we calculate the L^2-norm of the mth monic orthogonal polynomial $\underline{P}_m(x)$ with respect to the measure μ_{mp}. From Theorem 2.5, (2.15) and (3.2) we obtain

$$
0 \ = \ S_{2m}^- - S_{2m}^- = \int_0^1 \underline{P}_m^T(t) d\mu_{mp}^-(t) \underline{P}_m(t) \ = \ \sum_{j=1}^k \underline{P}_m^T(x_j) \Lambda_j \underline{P}_m(x_j) \ ,
$$

which implies

$$
\underline{P}_m^T(x_j) \Lambda_j \underline{P}_m(x_j) = 0 \quad j = 1, \ldots k.
$$

Consequently, we have for all roots of the representation (3.3)

$$
(3.4) \qquad \underline{P}_m^T(x_j) \Lambda_j = 0 \quad \text{and} \quad \det \underline{P}_m(x_j) = 0 \quad j = 1, \ldots, k.
$$

If ℓ_j denotes the multiplicity of x_j as a root of the polynomial $\underline{P}_m(x)$ $(j = 1, \ldots, k)$ then it is easy to see [see e.g. Sinap and Van Assche (1994), p. 81, or Lemma 2.2 in Durán and Lopez-Rodriguez (1996)] that

$$
(3.5) \qquad \operatorname{rank}(\Lambda_j) \leq \ell_j \quad j = 1, \ldots, k.
$$

Lemma 3.1. *Assume that $(S_0, \ldots, S_{2m-2}) \in Int(M_{2m-1})$ and that the measure μ_{mp}^- in (3.3) represents the point $(S_0, \ldots, S_{2m-1}, S_{2m}^-)$, i.e. (3.2) is satisfied, then (3.4) holds for all nodes in the representation (3.3) and*

$$
rank(\Lambda_j) = \ell_j \quad j = 1, \ldots, k.
$$

Moreover, $x_1, \ldots, x_k$ *are precisely the different roots of the mth monic orthogonal polynomial* $\underline{P}_m(x)$ *with respect to the measure* $d\mu$, *i.e.* $\{x \mid \det \underline{P}_m(x) = 0\} = \{x_1, \ldots, x_k\}$.

Proof. By the above discussion it is sufficient to show

$$\sum_{j=1}^{k} \operatorname{rank}(\Lambda_j) \geq mp \ , \tag{3.6}$$

because we have from (3.5)

$$\sum_{j=1}^{k} \operatorname{rank}\,(\Lambda_j) \ \leq \ \sum_{j=1}^{k} \ell_j \ \leq \ mp$$

(note that $\underline{P}_m(x)$ has mp zeros counting multiplicities). In order to prove (3.6) recall the definition of the Hankel matrix $\underline{H}_{2m-2}$ in (2.2). Observing (3.2) we obtain the representation

$$\underline{H}_{2m-2} = X_m D_k X_m^T \in \mathbb{R}^{mp \times mp}$$

with matrices $X_m \in \mathbb{R}^{mp \times kp}$, $D_k \in \mathbb{R}^{kp \times kp}$ defined by

$$X_m = \begin{bmatrix} I_p & \ldots & I_p \\ x_1 I_p & \ldots & x_k I_p \\ \vdots & & \vdots \\ x_1^{m-1} I_p & \ldots & x_k^{m-1} I_p \end{bmatrix} \ ; \quad D_k = \begin{bmatrix} \Lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \Lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \Lambda_k \end{bmatrix} . \tag{3.7}$$

Note that the second part of Theorem 2.2 implies $\operatorname{rank}(\underline{H}_{2m-2}) = mp$ [because $(S_0, \ldots, S_{2m-2}) \in \operatorname{Int}(M_{2m-1})$ by assumption] and it follows that

$$mp \ = \ \operatorname{rank}(\underline{H}_{2m-2}) \ \leq \ \min\{\operatorname{rank}(X_m), \operatorname{rank}(D_k)\} \ \leq \ \sum_{j=1}^{k} \operatorname{rank}(\Lambda_j)$$

which proves (3.6).

□

Remark 3.2. The arguments in the proof of Lemma 3.1 show that any representation

$$\sum_{i=1}^{\ell} \Gamma_i \delta_{y_i}$$

of a moment point $(S_0, \ldots, S_{2m-2}) \in \operatorname{Int}(M_{2m-1})$ satisfies

$$\sum_{i=1}^{\ell} \operatorname{rank}(\Gamma_i) \geq mp.$$

In order to identify the weights in the representation (3.3) we can solve the following system of equations

$$
\underbrace{\begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_{m-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} I_p & I_p & \dots & I_p \\ x_1 I_p & x_2 I_p & \dots & x_k I_p \\ \vdots & \vdots & & \vdots \\ x_1^{m-1} I_p & x_2^{m-1} I_p & \dots & x_k^{m-1} I_p \\ \underline{P}_m^T(x_1) & 0 & \dots & 0 \\ 0 & \underline{P}_m^T(x_2) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \underline{P}_m^T(x_k) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_k \end{bmatrix}}_{\Lambda}, \tag{3.8}
$$

where $B \in \mathbb{R}^{(m+k)p \times p}$, $A \in \mathbb{R}^{(m+k)p \times kp}$, $\Lambda \in \mathbb{R}^{kp \times p}$, and by the previous discussion there exists at least one solution of this system corresponding to a measure representing the point $(S_0, \dots, S_{2m-1}, S_{2m}^-)$. Our next result shows that the equation $B = A\Lambda$ has a unique solution.

Theorem 3.3. *Let $(S_0, \dots, S_{2m-2}) \in Int(M_{2m-1})$, then a representation (3.3) of the point $(S_0, \dots, S_{2m-1}, S_{2m}^-)$ such that (3.2) is satisfied is unique. The nodes $x_1, \dots, x_k$ are the different roots of the monic orthogonal polynomial $\underline{P}_m(x)$ with respect to the measure $d\mu$ and the weights $\Lambda_1, \dots, \Lambda_k$ are obtained as the unique solution of the system (3.8).*

Proof. Since the rank of Λ_i is ℓ_i we can write $\Lambda_i = \Delta_i \Delta_i^T$ where Δ_i is $p \times \ell_i$ with rank ℓ_i. The matrix D_k in (3.7) can then be written as $D_k = EE^T$ where

$$
E = \begin{pmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots 0 & \Delta_k \end{pmatrix} \in \mathbb{R}^{kp \times mp}
$$

and we obtain the representation

$$
\underline{H}_{2m-2} = X_m E E^T X_m^T.
$$

The matrix $X_m E$ is square of size $mp \times mp$ and is nonsingular since $\underline{H}_{2m-2}$ is nonsingular. This implies that the solution to (3.8) is unique. Specifically, we can consider a fixed solution to (3.8) or (3.2), say, $\Lambda_1^o, \cdots, \Lambda_k^o$. The columns of the corresponding decomposition Δ_i^o form a basis for the nullspace of the polynomial $\underline{P}_m^T(x_i)$ and hence for any other solution $\Delta_i = \Delta_i^o B_i$ where B_i is $\ell_i \times \ell_i$ and nonsingular. Then the upper part of the equations (3.8) can be written as

$$
\begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{m-1} \end{pmatrix} = X_m E^o F
$$

where the matrix E^o is defined in the same way as E with Δ_i replaced by Δ_i^o and F is given by

$$F = \begin{pmatrix} B_1 B_1^T \Delta_1^o \\ \vdots \\ B_k B_k^T \Delta_k^o \end{pmatrix}$$

The solution for F and hence for $\Delta_1, \cdots, \Delta_k$ is unique.

□

Example 3.4. It should be noted that in general the representation of a boundary point is not necessarily unique if S_n is not identically equal to either S_n^- or S_n^+ which is in contrast to the case $p = 1$ [see Dette and Studden (1997)]. Consider as a simple example the case $p = 2$ and the point (S_0, S_1, S_2) with

$$S_0 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}; \quad S_1 = \begin{pmatrix} 2 & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}; \quad S_2 = \begin{pmatrix} \frac{11}{8} & \frac{9}{16} \\ \frac{9}{16} & \frac{9}{16} \end{pmatrix}.$$

From (2.4) it follows that

$$S_2^- = \begin{pmatrix} \frac{43}{32} & \frac{9}{16} \\ \frac{9}{16} & \frac{9}{16} \end{pmatrix} \neq S_2$$

and Theorem 3.3 is not applicable for the point (S_0, S_1, S_2). However, from

$$|S_2 - S_2^-| = \begin{vmatrix} \frac{1}{32} & 0 \\ 0 & 0 \end{vmatrix} = 0$$

and Theorem 2.2 it follows that $(S_0, S_1, S_2) \in \partial M_3$ and it is easy to check that this point can be presented by the two measures

$$\begin{pmatrix} \frac{1}{13} & 0 \\ 0 & 0 \end{pmatrix} \delta_0 + \begin{pmatrix} \frac{25}{13} & 0 \\ 0 & 0 \end{pmatrix} \delta_{\frac{13}{20}} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_{\frac{3}{4}}$$

(3.9)

$$\begin{pmatrix} \frac{9}{5} & 0 \\ 0 & 0 \end{pmatrix} \delta_{\frac{7}{12}} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_{\frac{3}{4}} + \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{pmatrix} \delta_1 .$$

These two representations correspond to (S_0, S_1, S_2, S_3^-) and (S_0, S_1, S_2, S_3^+) respectively. Thus there exist at least two representations of the boundary point (S_0, S_1, S_2). There is in fact an infinite number of representing measures for boundary points of this type. Further discussion on this matter will be given in Section 5.

So far we have shown that a representation of a boundary point $(S_0, S_1, \ldots, S_{2m-1}, S_{2m}^-)$ is unique. Comparing (1.1) and (3.2) it follows that (3.3) defines a quadrature formula with degree of precision equal to $2m-1$ such that

$$\sum_{j=1}^{k} \operatorname{rank}(\Lambda_j) = mp \tag{3.10}$$

[see Lemma 3.1]. The following result shows that these properties actually determine the quadrature formula uniquely. We note that this statement was proved independently by Durán and Defez (2002) using a rather different technique.

Theorem 3.5. *Assume that μ is a matrix measure such that $(S_0, \ldots, S_{2m-2}) \in Int(M_{2m-1})$, and let*

$$\mu_Q = \sum_{j=1}^{k} \Lambda_j \delta_{x_j}$$

denote a quadrature formula with degree of precision $2m-2$ satisfying (3.10) with $(2m-1)$th moment

$$\tilde{S}_{2m-1} = \int_0^1 t^{2m-1} d\mu_Q(t).$$

The following assertions are valid.

(1) $k \geq m$

(2) $x_1, \ldots, x_k$ *are the different zeros of the polynomial*

$$P(x) = \underline{P}_m(x) + \underline{P}_{m-1}(x)(S^+_{2m-2} - S^-_{2m-2})^{-1}(S^+_{2m-2} - S_{2m-2})(S^+_{2m-1} - S^-_{2m-1})^{-1}(S_{2m-1} - \tilde{S}_{2m-1}),$$

where S^+_{2m-2}, S^+_{2m-1} and S^-_{2m-2}, S^-_{2m-1} are defined in (2.5) and (2.4), respectively, $S_{2m-1} = \int_0^1 t^{2m-1} d\mu(t)$ is the $(2m-1)$th moment of the measure μ and $\underline{P}_j(x)$ is the jth monic orthogonal polynomial with respect to the matrix measure μ $(j = 0, \ldots, m)$.

(3) *rank $(\Lambda_j) = \ell_j$ $(j = 1, \ldots, k)$, where ℓ_j is the multiplicity of x_j as a root of the polynomial $P(x)$ defined in (2).*

(4) *The quadrature coefficients $\Lambda_1, \ldots, \Lambda_k$ are defined as the unique solution of the system (3.8).*

(5) *The quadrature formula has degree of precision $2m-1$ if and only if $S_{2m-1} = \tilde{S}_{2m-1}$. In this case the nodes are the roots of the mth monic orthogonal polynomial $\underline{P}_m(x)$ with respect to the measure μ and the weights are determined as the unique solution of the system (3.8). In other words a quadrature formula with degree of precision $(2m-1)$ is uniquely determined.*

Proof. Because $(S_0, \ldots, S_{2m-2}) \in \operatorname{Int}(M_{2m-1})$ it follows from Theorem 2.2 that $\underline{H}_{2m-2}$ is positive definite. Consider the moment point $(S_0, \ldots, S_{2m-2}, \tilde{S}_{2m-1}, \tilde{S}^-_{2m})$ where $\tilde{S}^-_{2m}$ is obtained from (2.4) and S_{2m-1} is replaced by $\tilde{S}_{2m-1}$, that is

$$\tilde{S}_{2m}^{-} = (S_m, \ldots, S_{2m-2}, \tilde{S}_{2m-1}) \underline{H}_{2m-2}^{-1} \begin{pmatrix} S_m \\ \vdots \\ S_{2m-2} \\ \tilde{S}_{2m-1} \end{pmatrix} .$$

We will prove below that

$$\tilde{S}_{2m} = \int_0^1 t^{2m} d\mu_Q(t) = \tilde{S}_{2m}^{-}, \tag{3.11}$$

which implies that μ_Q is a representation of the point $(S_0, \ldots, S_{2m-2}, \tilde{S}_{2m-1}, \tilde{S}_{2m}^{-})$. Therefore, if (3.11) is correct, it follows from Theorem 3.3 that $x_1, \ldots, x_k$ are the different roots of the monic orthogonal polynomial $\underline{P}_m(x, \mu_Q)$ with respect to the measure $d\mu_Q$ and that the weights are obtained as the unique solution of the system (3.8). Now let

$$\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_{2m-1}$$

denote the canonical moments of the matrix measure μ_Q and

$$U_1, \ldots, U_{2m-1}$$

be the canonical moments of the matrix measure μ. Observing that the moments of μ and μ_Q up to the order $2m-2$ coincide we obtain from the definitions (2.7), (2.8) and (2.4), (2.5) that

$$\begin{aligned} U_j &= \tilde{U}_j \quad j = 1, \ldots, 2m-2; \\ U_{2m-1} &= \tilde{U}_{2m-1} + (S_{2m-1}^{+} - S_{2m-1}^{-})^{-1}(S_{2m-1} - \tilde{S}_{2m-1}) \; . \end{aligned} \tag{3.12}$$

This implies for the corresponding quantities $\tilde{\zeta}_j = \tilde{V}_{j-1}\tilde{U}_j$ and $\zeta_j = V_{j-1}U_j$ the relation

$$\begin{aligned} \tilde{\zeta}_j &= \zeta_j \quad j = 1, \ldots, 2m-2 \\ \tilde{\zeta}_{2m-1} &= \zeta_{2m-1} - V_{2m-2}(S_{2m-1}^{+} - S_{2m-1}^{-})^{-1}(S_{2m-1} - \tilde{S}_{2m-1}) \\ &= \zeta_{2m-1} - A_{m-1} \end{aligned} \tag{3.13}$$

where the matrix A_{m-1} is defined by

$$A_{m-1} = (S_{2m-2}^{+} - S_{2m-2}^{-})^{-1}(S_{2m-2}^{+} - S_{2m-2})(S_{2m-1}^{+} - S_{2m-1}^{-})^{-1}(S_{2m-1} - \tilde{S}_{2m-1}) \tag{3.14}$$

and we have used the representaton (2.9). Observing the first part of Lemma 2.6 it therefore follows for the monic orthogonal polynomials $\underline{P}_\ell(x, \mu_Q)$ and $\underline{P}_\ell(x, \mu) = \underline{P}_\ell(x)$ with respect to the measures $d\mu_Q$ and $d\mu$ that

$$\underline{P}_\ell(x, \mu_Q) = \underline{P}_\ell(x, \mu) \;, \qquad \ell = 0, 1, \ldots, 2m-1, \tag{3.15}$$

and

$$
\begin{aligned}
\underline{P}_m(x,\mu_Q) &= \underline{P}_{m-1}(x,\mu_Q)(xI_p - \tilde{\zeta}_{2m-1} - \tilde{\zeta}_{2m-2}) - \underline{P}_{m-2}(x,\mu_Q)\tilde{\zeta}_{2m-3}\tilde{\zeta}_{2m-2} \\
&= \underline{P}_{m-1}(x,\mu)(xI_p - \zeta_{2m-1} - \zeta_{2m-2}) - \underline{P}_{m-2}(x,\mu)\zeta_{2m-3}\zeta_{2m-2} \\
&\quad + \underline{P}_{m-1}(x,\mu)(\zeta_{2m-1} - \tilde{\zeta}_{2m-1}) \\
&= \underline{P}_m(x,\mu) + \underline{P}_{m-1}(x,\mu)A_{m-1},
\end{aligned}
$$

where we have used the identities (3.13) - (3.15). This proves part (2) of Theorem 3.5 and part (3) and (4) are a direct consequence of Theorem 3.3. The quadrature formula has obviously degree of precision $2m-1$ if and only if $S_{2m-1} = \tilde{S}_{2m-1}$ and part (5) of the assertion follows from (2),(3) and (4) and the uniqueness statement in Theorem 3.3.
We finally prove the statement (3.11) and as a by-product the assertion (1). To this end recall the representations

$$
\begin{aligned}
&(3.16) \qquad & \underline{H}_{2m-2} &= X_m D_k X_m^T \\
&(3.17) \qquad & \tilde{\underline{H}}_{2m} &= X_{m+1} D_k X_{m+1}^T ,
\end{aligned}
$$

where the Hankel matrix $\tilde{\underline{H}}_{2m}$ is obtained from $\underline{H}_{2m}$ replacing S_{2m-1} and S_{2m} by the moments $\tilde{S}_{2m-1}$ and $\tilde{S}_{2m}$, respectively, and the matrices X_m, X_{m+1} and D_k are defined in (3.7). From $(S_0,\dots,S_{2m-2}) \in \mathrm{Int}(M_{2m-1})$ it follows that $\mathrm{rank}(\underline{H}_{2m-2}) = mp$ and the representation (3.16) implies $k \geq m$. On the other hand the equation (3.17) and the assumption (3.10) yield

$$\mathrm{rank}(\tilde{\underline{H}}_{2m}) \leq \mathrm{rank}(D_k) = mp.$$

However, the matrix $\tilde{\underline{H}}_{2m}$ has the same rank as the matrix

$$\underline{H}_{2m}^* = \left[\begin{array}{c|c} \underline{H}_{2m-2} & 0 \\ 0 & \tilde{S}_{2m} - \tilde{S}_{2m}^- \end{array}\right] \in \mathbb{R}^{(m+1)p\times(m+1)p} ,$$

which follows from the definition of $\tilde{S}_{2m}^-$ and well known results on partitioned matrices [see e.g. Muirhead (1982), p. 581-582]. The assertion (3.11) is now obtained from the identity

$$mp \geq \mathrm{rank}(\tilde{\underline{H}}_{2m}) = \mathrm{rank}(\underline{H}_{2m}^*) \geq \mathrm{rank}(\underline{H}_{2m-2}) = mp,$$

which completes the proof of Theorem 3.5.

□

Part (5) of Theorem 3.5 shows that a quadrature formula with degree of precision $2m-1$ and $\sum_{i=1}^k \mathrm{rank}(\Lambda_i) = mp$ is uniquely determined. This proves that several formulas for the weights given in the literature must be identical [see Sinap and Van Assche (1994), Durán and Lopez-Rodriguez (1996) and Durán (1996)]. The solution of the system (3.8) gives a further representation for these weights. Moreover, it follows from Remark 3.2 that any quadrature formula with degree of precision $2m-2$ necessarily satisfies $\sum_{i=1}^{k} \operatorname{rank}(\Lambda_i) \geq mp$. It is also worthwhile to mention that our representation of the polynomial $P(x)$ and the quadrature coefficients Λ_i differs from the corresponding formulas in Durán and Defez (2002).

Remark 3.6. Note that the results presented so far are actually correct for arbitrary matrix measures on the real line. This follows by a careful inspection of the proofs in Dette and Studden (2002a), which shows that all results regarding the quantities S_{2j}^- are actually correct for measures on the real line. For example, a point $(S_0, S_1, S_2, \ldots)$ corresponds to the moments of a matrix measure on the real line if and only if $\underline{H}_{2n} \geq 0$ for all $n \in \mathbb{N}$ and it is in the interior of the corresponding moment space if there is strict inequality [see also Defez, Jódar, Law and Ponsoda (2000) for a corresponding result of the existence of pseudo orthogonal polynomials]. We can now define S_{2m}^- exactly in the same way as in (2.4) and all arguments given in the proofs of this section are applicable [note that we cannot define $S_{2j-1}^-, S_{2j-1}^+, S_{2j}^+$ for matrix measures on the real line because Theorem 2.2 does not hold here].

Example 3.7. Consider the matrix measure

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}$$

where

$$\begin{aligned} \frac{d\mu_{11}}{dt} = \frac{d\mu_{22}}{dt} &= \frac{1}{\pi} \frac{1}{\sqrt{t(1-t)}} \\ \frac{d\mu_{12}}{dt} &= \frac{1}{\pi} \frac{2t-1}{\sqrt{t(1-t)}} \end{aligned}$$

(see also VanAssche (1993), who considered this example on the interval $[-1, 1]$). A straightforward calculation shows

$$S_k = \binom{2k}{k} \frac{1}{2^{2k}(k+1)} \begin{pmatrix} k+1 & k \\ k & k+1 \end{pmatrix} \quad , \quad k \geq 0 \, ,$$

which gives for the first canonical moments

$$U_1 = S_0^{-1} S_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \; , \; U_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \, .$$

It can be shown [see Dette and Studden (2002b)] that the canonical moments of the matrix measure μ are given by

$$U_{2k} = \frac{k}{2k+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; , \; U_{2k-1} = \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{2k} \\ \frac{1}{2k} & 1 \end{pmatrix} \; ,$$

for $k \in \mathbb{N}$. This gives for the quantities ζ_j in the recursion for the orthogonal matrix polynomials

$$\zeta_{2k-1} = \frac{k}{2(2k-1)} \begin{pmatrix} 1 & \frac{1}{2k} \\ \frac{1}{2k} & 1 \end{pmatrix} , \quad \zeta_{2k} = \frac{k}{2(2k+1)} \begin{pmatrix} 1 & -\frac{1}{2k} \\ -\frac{1}{2k} & 1 \end{pmatrix} .$$

Consequently we obtain from Lemma 2.6 for the monic orthogonal polynomials with respect to the matrix measure $d\mu(x)$ the following recursions

$$\begin{aligned}
\underline{P}_0(x) &= I_2 , \\
\underline{P}_1(x) &= x\underline{P}_0(x) - \zeta_1 = \begin{pmatrix} x-\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & x-\frac{1}{2} \end{pmatrix} , \\
\underline{P}_{k+1}(x) &= (x-\frac{1}{2})\underline{P}_k(x) - \frac{1}{16}\underline{P}_{k-1}(x) , \; k \geq 1 .
\end{aligned}$$

A straightforward induction argument shows that the mth monic orthogonal polynomial is given by

$$\underline{P}_m(x) = \frac{1}{2^{2m}} \begin{pmatrix} U_m(2x-1) & -U_{m-1}(2x-1) \\ -U_{m-1}(2x-1) & U_m(2x-1) \end{pmatrix}$$

where $U_m(x)$ denotes the mth Chebyshev polynomial of the second kind. Consequently

$$\det \underline{P}_m(x) = \frac{1}{4^m} U_{2m}(2x-1)$$

and all roots of the corresponding quadrature formular are simple and given by

$$x_j = \frac{1}{2}\Big(1 + \cos(\pi \frac{j}{2m+1})\Big) \quad j = 1, \dots, 2m.$$

The corresponding weights Λ_j have rank 1 and are determined as the unique solution of the system (3.8). For example, if $m = 3$ we obtain

$$\begin{aligned}
x_1 &= 0.0495156, & x_2 &= 0.188255, & x_3 &= 0.388740 \\
x_4 &= 0.611260, & x_5 &= 0.811745, & x_6 &= 0.950484
\end{aligned}$$

with the corresponding weights

$$\begin{aligned}
\Lambda_1 &= 0.271567 \cdot E_1, & \Lambda_2 &= 0.053787 \cdot E_2, & \Lambda_3 &= 0.174646 \cdot E_1, \\
\Lambda_4 &= 0.174646 \cdot E_2, & \Lambda_5 &= 0.053787 \cdot E_1, & \Lambda_6 &= 0.271567 \cdot E_2
\end{aligned}$$

where the matrices E_1 and E_2 are given by

$$E_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.18}$$

respectively.

We conclude this section with a simple proof that the zeros of orthogonal matrix polynomials with respect to matrix measures on an interval, say $[a, b]$ are all located in the interior of this interval.

Corollary 3.8. *Let μ denote a matrix measure on the interval $[a, b]$ such that its moments satisfy $(S_0, \ldots, S_{2m-1}) \in Int(M_{2m})$ for some $m \in \mathbb{N}$, then the roots of the mth monic matrix orthogonal polynomial with respect to the measure $d\mu$ are located in the interior of the interval $[a, b]$.*

Proof. We assume without loss of generality that $[a, b] = [0, 1]$. The representation of the point $(S_0, \ldots, S_{2m-1}, S_{2m}^-)$ is supported at the roots of the polynomial $\underline{P}_m(x)$ orthogonal with respect to the measure $d\mu$ and consequently all zeros of the polynomial $\underline{P}_m(x)$ are located in the interval $[0, 1]$. Now assume that $x_0 = 0$ is a root of $\underline{P}_m(x)$; the other case $x_0 = 1$ is treated similary. If $x_1, \ldots, x_k$ denote the remaining distinct roots of the polynomial $\underline{P}_m(x)$, it follows that

$$S_j = \sum_{i=1}^{k} x_i^j \Lambda_i \quad 1 \leq j \leq 2m - 1,$$

where the nonnegative definite matrices Λ_j denote the corresponding weights of the representation of the point $(S_0, \ldots, S_{2m-1}, S_{2m}^-)$. Therefore we obtain the representation

$$\underline{H}_{2m-1} = X_m \tilde{D}_k X_m^T \tag{3.19}$$

where the matrix X_m is defined in (3.7) and

$$\tilde{D}_k = \begin{bmatrix} x_1\Lambda_1 & 0 & \ldots & 0 \\ 0 & x_2\Lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & x_k\Lambda_k \end{bmatrix}.$$

From $(S_0, \ldots, S_{2m-1}) \in \mathrm{Int}(M_{2m})$ we have $\underline{H}_{2m-1} > 0$ and the identity (3.19) implies

$$mp \leq \sum_{j=1}^{k} \mathrm{rank}(\Lambda_j) \leq \sum_{j=0}^{k} \mathrm{rank}(\Lambda_j) \leq mp,$$

where we used (3.5) for the last inequality. Consequently, we obtain from $\mathrm{rank}(\Lambda_0) = 0$, that $x_0 = 0$ is not a support point of the measure representing the point $(S_0, \ldots, S_{2m-1}, S_{2m}^-)$. On the other hand, by Lemma 3.1, the support points of this representation are precisely the zeros of the polynomial $\underline{P}_m(x)$, which proves the assertion of Corollary 3.8.

□

Remark 3.9. It might be noted that in Lemma 3.1 we assumed that $(S_0, \cdots, S_{2m-2})$ was in the interior of M_{2m-1} and no condition was placed on $(S_0, \cdots, S_{2m-1})$. By Corollary 3.8 all the roots of $\underline{P}_m(x)$ are in the interior of the interval $[0,1]$ if $(S_0, \cdots, S_{2m-1})$ is in the interior of the moment space M_{2m}, i.e. $S^-_{2m-1} < S_{2m-1} < S^+_{2m-1}$. It will be shown below in Theorem 4.3 that if $S_{2m-1} = S^-_{2m-1}$ then $x_0 = 0$ and the other zeros are in the interior of the interval $[0,1]$. A similar situation occurs at the upper endpoint if $S_{2m-1} = S^+_{2m-1}$

4 Quadrature formulas with boundary points as nodes

In this section we briefly discuss the development of quadrature formulas which also use the boundary points as nodes. This generalizes classical work of Bouzitat (1949) to the case of matrix measures on the interval $[0,1]$. Roughly speaking these quadrature formulas correspond to representations of the boundary points

$$\begin{aligned}(S_0, \ldots, S_{2m}, S^+_{2m+1}) &\in \partial M_{2m+2}\\ (S_0, \ldots, S_{2m}, S^-_{2m+1}) &\in \partial M_{2m+2}\\ (S_0, \ldots, S_{2m-1}, S^+_{2m}) &\in \partial M_{2m+1}\ .\end{aligned}$$

For the sake of brevity we concentrate on the last case, the resulting formulas for the other cases are briefly mentioned at the end of this section. Let

$$\mu^+_{mp} = \sum_{j=1}^{k} \Lambda_j \delta_{x_j}$$

denote a representing measure of the boundary point $(S_0, \ldots, S_{2m-1} S^+_{2m})$ with $(S_0, \ldots, S_{2m-2}) \in \mathrm{Int}(M_{2m-1})$, and define $\bar{Q}_{m-1}(t)$ as the $(m-1)$th monic orthogonal polynomial with respect to the measure $t(1-t)d\mu(t)$. It follows from the representation

$$\bar{Q}_{m-1}(x) = x^m I_p - (x^{m-1} I_p, \ldots, x I_p, I_p) \overline{H}^{-1}_{2m-2} \bar{h}_{2m-1}$$

that

$$\begin{aligned}0 \ = \ S^+_{2m} - S^+_{2m} &= \int \bar{Q}^T_{m-1}(t) t(1-t) d\mu^+_{mp}(t) \bar{Q}_{m-1}(t)\\ &= \sum_{j=1}^{k} x_j (1-x_j) \bar{Q}^T_{m-1}(x_j) \Lambda_j \bar{Q}_{m-1}(x_j),\end{aligned}$$

which gives

$$x_j (1-x_j) \bar{Q}^T_{m-1}(x_j) \Lambda_j = 0 \quad j = 1, \ldots, k.$$

If $x_j \notin \{0,1\}$, then the same arguments as given in the proof of Lemma 3.1 show $\det \bar{Q}_{m-1}(x_j) = 0$ and from Lemma 2.2 in Durán and Lopez Rodriguez (1996) we have

$$\mathrm{rank}(\Lambda_j) \le \ell_j, \tag{4.1}$$

where ℓ_j is the multiplicity of x_j as a root of the polynomial $\bar{Q}_{m-1}(x)$. Note that for $j \ge 1$

$$\begin{aligned}S_j - S_{j+1} \ &= \ \int_0^1 t(1-t) t^{j-1} d\mu^+_{mp}(t) = \sum_{i=1}^{k} x_i (1-x_i) x_i^{j-1} \Lambda_i\\ &= \sum_{i: x_i \notin \{0,1\}} x_i (1-x_i) x_i^{j-1} \Lambda_i\ .\end{aligned}$$

Let $y_1, \dots, y_h$ denote the nodes among $x_1, \dots, x_k$, which are different from 0 and 1. Note that rank $(\overline{H}_{2m-2}) = (m-1)p$ [because $(S_0, \dots, S_{2m-2}) \in \mathrm{Int}(M_{2m-1})$] and consider the representation

(4.2) $$\bar{H}_{2m-2} = Y_{m-2} G Y_{m-2}^T \ ,$$

where the matrices $Y_{m-2} \in \mathbb{R}^{(m-1)p \times hp}$ and $G \in \mathbb{R}^{hp \times hp}$ are given by

$$Y_{m-2} \ = \ \begin{bmatrix} I_p & I_p & \dots & I_p \\ y_1 I_p & y_2 I_p & \dots & y_h I_p \\ \vdots & \vdots & & \vdots \\ y_1^{m-2} I_p & y_2^{m-2} I_p & \dots & y_h^{m-2} I_p \end{bmatrix}$$

$$G \ = \ \begin{bmatrix} y_1(1-y_1)\Lambda_1 & 0 & \dots & 0 \\ 0 & y_2(1-y_2)\Lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & y_h(1-y_h)\Lambda_h \end{bmatrix} .$$

It then follows by the same arguments as given in the proof of Lemma 3.1 that

$$\sum_{i=1}^{h} \mathrm{rank}(\Lambda_i) \ \geq \ (m-1)p \ ,$$

which implies, observing (4.1), that

(4.3) $$\{x_1, \dots, x_k\} \backslash \{0,1\} = \{x \mid \det \bar{Q}_{m-1}(x) = 0\}$$

(4.4) $$\mathrm{rank}(\Lambda_i) = \ell_i \text{ for all } x_i \neq 0, 1$$

(4.5) $$\sum_{i: x_i \notin \{0,1\}} \mathrm{rank}(\Lambda_i) = (m-1)p.$$

If $(S_0, \dots, S_{2m-1}) \in \mathrm{Int}(M_{2m})$ we can use a similar argument for the Hankel matrices $\bar{H}_{2m-1}$ and $\bar{H}_{2m-1}$ and obtain

(4.6) $$\sum_{i: x_i \neq 1} \mathrm{rank}(\Lambda_i) = mp,$$

(4.7) $$\sum_{i: x_i \neq 0} \mathrm{rank}(\Lambda_i) = mp,$$

which gives for the weight $\tilde{\Lambda}$ at the point 0

$$\begin{aligned} p \ \geq \ \mathrm{rank}(\tilde{\Lambda}) &= \sum_{i: x_i \neq 1} \mathrm{rank}(\Lambda_i) - \sum_{i: x_i \notin \{0,1\}} \mathrm{rank}(\Lambda_i) \\ &\geq mp - (m-1)p \ = \ p \end{aligned}$$

and similary for the weight $\bar{\Lambda}$ at the point 1

$$p = \operatorname{rank}(\bar{\Lambda}),$$

where we have used the identities (4.5) - (4.7). Moreover, this argument also shows that 0 and 1 are support points of the measure μ^+_{mp}. Similary to the discussion in Section 3 the weights can be determined explicitly as the (unique) solution of the system

$$\underbrace{\begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} I_p & I_p & \dots & I_p & I_p \\ 0 & x_2 I_p & \dots & x_{k-1} I_p & I_p \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & x_2^m I_p & \dots & x_{k-1}^m I_p & I_p \\ 0 & \bar{Q}_{m-1}(x_2) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \bar{Q}_{m-1}(x_{k-1}) & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{k-1} \\ \Lambda_k \end{bmatrix}}_{\Lambda} \tag{4.8}$$

where $A \in \mathbb{R}^{(m+k-1)p \times kp}$, $B \in \mathbb{R}^{(m+k-1)p \times kp}$, $\Lambda \in \mathbb{R}^{kp \times p}$, $x_2, \dots, x_{k-1}$ are the different roots of the polynomial $\overline{Q}_{m-1}(x)$, and Λ_1 and Λ_k are the weights at the nodes $x_1 = 0$ and $x_k = 1$, respectively. We summarize these results in the following theorem.

Theorem 4.1. *Assume that $(S_0, \dots, S_{2m-1}) \in Int(M_{2m})$, then the representation (4.1) of the point $(S_0, \dots, S_{2m-1}, S^+_{2m})$ is uniquely determined. The roots are the different zeros of the polynomial*

$$x(1-x) \det \bar{Q}_{m-1}(x),$$

where $\bar{Q}_{m-1}(x)$ is the $(m-1)$th monic orthogonal polynomial with respect to the matrix measure $x(1-x)d\mu(x)$. The weights are determined as the unique solution of the system (4.8) and satisfy

$$\begin{aligned} rank(\Lambda_i) &= p \quad \text{if } x_i \in \{0, 1\} \\ rank(\Lambda_i) &= \ell_i \quad \text{if } \det \bar{Q}_{m-1}(x_i) = 0, \end{aligned} \tag{4.9}$$

where ℓ_i is the multiplicity of x_i as a root of the polynomial $\det \bar{Q}_{m-1}(x)$.
Moreover, the matrix measure μ^+_{mp} defines a quadrature formula for the matrix measure μ on the interval $[0, 1]$ with degree of precision $2m-1$, which uses the boundary points 0 and 1 as nodes and satisfies (4.5) and (4.9).

Example 4.2. In the situation of Example 3.7 we determine the quadrature formula with degree of precision equal to 5 using the nodes 0 and 1. To this end we calculate

$$\gamma_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} \frac{1}{6} & -\frac{1}{24} \\ -\frac{1}{24} & \frac{1}{6} \end{pmatrix}$$

$$\gamma_4 = \begin{pmatrix} \frac{3}{10} & \frac{3}{40} \\ \frac{3}{40} & \frac{3}{10} \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} \frac{1}{5} & \frac{1}{30} \\ \frac{1}{30} & \frac{1}{5} \end{pmatrix}$$

and obtain from Lemma 2.6 for the quadratic monic orthogonal polynomial with respect to the matrix measure $x(1-x)d\mu(x)$

$$\overline{Q}_2(x) = \begin{pmatrix} x^2 - x + \frac{5}{24} & \frac{1}{6}(x - \frac{1}{2}) \\ \frac{1}{6}(x - \frac{1}{2}) & x^2 - x + \frac{5}{24} \end{pmatrix} ,$$

which yields for the nodes

$$x_1 = 0; \quad x_2 = 0.196187; \quad x_3 = 0.362854; \quad x_4 = 0.637146; \quad x_5 = 0.830813; \quad x_6 = 1.$$

The corresponding quadrature coefficients are calculated as the solution of the system (4.8) and given by

$$\Lambda_1 = \begin{pmatrix} 0.154762 & -0.136905 \\ -0.136905 & 0.154762 \end{pmatrix} ; \quad \Lambda_2 = 0.235613 \cdot E_1 ; \quad \Lambda_3 = 0.109625 \cdot E_2,$$

$$\Lambda_4 = 0.109625 \cdot E_1 ; \quad \Lambda_5 = 0.235613 \cdot E_2 ; \quad \Lambda_6 = \begin{pmatrix} 0.154762 & 0.136905 \\ 0.136905 & 0.154762 \end{pmatrix} ,$$

where the matrices E_1 and E_2 are defined in (3.18). Note that the coefficients of the nodes 0 and 1 have rank 2.

It is worthwhile to mention that there exists an analogue of Theorem 3.5 regarding the uniqueness of the quadrature formula μ^+_{mp} which is omitted for the sake of brevity. We also mention that the quadrature formula μ^+_{mp} specified in Theorem 4.1 has the advantage that it only requires the calculation of the roots of a matrix polynomial of degree $m-1$, which might be useful if the dimension p of the matrix measure is large.

We conclude this section with a brief discussion of quadrature formulas using only one boundary point. For the sake of brevity we restrict ourselves to the case of the left boundary, for which we have the following result.

Theorem 4.3. *Assume that $(S_0, \dots, S_{2m}) \in Int(M_{2m+1})$, then a representation*

$$\mu^-_{mp} = \sum_{j=1}^{k} \Lambda_j \delta_{x_j}$$

of the point $(S_0, \dots, S_{2m}, S^-_{2m+1})$ is uniquely determined. The roots are the different zeros of the polynomial $x \det \underline{Q}_m(x)$, where $\underline{Q}_m(x)$ is the mth monic orthogonal polynomial with respect to the *measure $xd\mu(x)$. The weights are the unique solution of the system*

$$
\begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} I_p & I_p & \dots & I_p \\ 0 & x_2 I_p & \dots & x_k I_p \\ \vdots & & & \\ 0 & x_2^m I_p & \dots & x_k^m I_p \\ 0 & \underline{Q}_m(x_2) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \underline{Q}_m(x_k) \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_k \end{bmatrix} \tag{4.10}
$$

(here we assume $x_1 = 0$ without loss of generality) and satisfy

$$
\begin{aligned} rank(\Lambda_1) &= p \\ rank(\Lambda_i) &= \ell_i \quad i = 2, \dots, k, \end{aligned}
$$

where ℓ_i is the multiplicity of x_i as a root of the polynomial $\det \underline{Q}_m(x)$. Moreover, the matrix measure μ_{mp}^- defines a quadrature formula for the matrix measure μ on the interval $[0,1]$ with degree of precision $2m$, which uses the boundary point 0 as node and satisfies

$$
\sum_{i=2}^{m} \operatorname{rank}(\Lambda_i) = mp \ , \quad \operatorname{rank}(\Lambda_1) = p.
$$

Remark 4.4. Note that Theorem 4.3 remains valid for matrix measures supported on the non-negative line $[0,\infty)$. This follows from the results in Dette and Studden (2002a) which show that μ is a matrix measure on the interval $[0,\infty)$ if and only if $\underline{H}_n \geq 0$ for all $n \in \mathbb{N}_0$. On the other hand, a generalization of Theorem 4.1 for measures with unbounded support is not available, because the definition of the matrix S_n^+ is not possible in such cases.

5 Degenerate quadrature formulas

In the previous part of the paper we have discussed quadrature formulas corresponding to boundary points

$$
S = (S_0, S_1, \dots, S_{n-1}, \tilde{S}_n) \in \partial M_{n+1}
$$

of the $(n+1)$th moment space M_{n+1} where $\tilde{S}_n$ is either S_n^- or S_n^+. From Theorem 2.2 it follows that S is also a boundary point of the moment space M_{n+1} if

$$
|\tilde{S}_n - S_n^+| = 0 \tag{5.1}
$$

or

$$
|\tilde{S}_n - S_n^-| = 0, \tag{5.2}
$$

and it is of some interest to investigate the structure of quadrature formulas corresponding to this situation. For the sake of brevity we restrict ourselves to the case (5.2) with $n = 2m$, but note that other cases can be treated similary. Let

$$s = \operatorname{rank}(\tilde{S}_{2m} - S_{2m}^-) \tag{5.3}$$

denote the rank of the matrix $\tilde{S}_{2m} - S_{2m}^-$, and assume that $(S_0, \dots, S_{2m-1}) \in \operatorname{Int}(M_{2m})$. Note that the case $s = 0$ (in other words $\tilde{S}_{2m} = S_{2m}^-$) was discussed in Section 3 and the unique representing measure of the point $(S_0, \dots, S_{2m-1}, \tilde{S}_{2m})$

$$\mu_Q = \sum_{i=1}^{k} \Lambda_i S_{x_i}$$

yields a quadrature formula satisfying

$$\sum_{i=1}^{k} \operatorname{rank}(\Lambda_i) = mp.$$

Example 3.4 shows that in the case $0 < s < p$ a representation of the boundary point $(S_0, \dots, S_{2m-1}, \tilde{S}_{2m})$ is not necessarily unique. In the following discussion we demonstrate that there are at least two representations of the boundary point $(S_0, \dots, S_{2m-1}, \tilde{S}_{2m})$ such that the sum of the ranks of the weights satisfies

$$\sum_{i} \operatorname{rank}(\Lambda_i) = mp + s. \tag{5.4}$$

Quadrature formulas for matrix measures with weights satisfying (5.4) have been independently studied by Durán and Polo (2002), who gave a rather explicit description of this type of formulas. The following result gives an extremely simple representation of two quadrature formulas which satisfy (5.4) and have one of the end points of the interval $[0, 1]$ as a node.

Theorem 5.1. *Let μ denote a matrix measure on the interval $[0, 1]$ with moments $S_j = \int_0^1 x^j d\mu(x)$, $(j \geq 0)$ such that*

$$\begin{aligned} (S_0, \dots, S_{2m-1}) &\in Int(M_{2m}) \\ (S_0, \dots, S_{2m-1}, \tilde{S}_{2m}) &\in \partial M_{2m+1}. \end{aligned}$$

(1) Let $x_1, \dots, x_k$ denote the different roots of the polynomial

$$x \det \underline{Q}_m(x) \tag{5.5}$$

where $\underline{Q}_m(x)$ is the mth monic orthogonal polynomial with respect to the matrix measure $x d\mu(x)$ (without loss of generality we put $x_1 = 0$), and define $\Lambda_1, \dots, \Lambda_k$ as the unique solution of the system (4.10) . The matrix measure

$$\mu_{mp+s}^- = \sum_{j=1}^{k} \Lambda_j \delta_{x_j} \tag{5.6}$$

is a representation of the point $(S_0, \ldots, S_{2m-1}, \tilde{S}_{2m})$*. Moreover,* μ^-_{mp+s} *defines a quadrature formula with degree of precision* $2m-1$ *and*

$$(5.7)\qquad \begin{aligned} rank(\Lambda_1) &= s \\ rank(\Lambda_j) &= \ell_j \ , \qquad j = 2, \ldots, k, \end{aligned}$$

where ℓ_j *denotes the multiplicity of* x_j *as a root of the polynomial* $\underline{Q}_m(x)$*. In particular* μ^-_{mp+s} *satisfies (5.4) and integrates all polynomials of the form*

$$(5.8)\qquad A_{2m}t^{2m} + A_{2m-1}t^{2m-1} + \ldots + A_1 t + A_0$$

with

$$A_{2m} \in \{A \in \mathbb{R}^{p\times p} \mid A^T(S_{2m} - \tilde{S}_{2m}) = 0\}$$

exactly.

(2) Let $x_1, \ldots, x_k$ *denote the different zeros of the polynomial*

$$(1-x)\det \bar{P}_m(x),$$

where $\bar{P}_m(x)$ *is the mth monic orthogonal polynomial with respect to the matrix measure* $(1-x)d\mu(x)$ *(without loss of generality we put* $x_k = 1$*) and define* $\Lambda_1, \ldots, \Lambda_k$ *as the unique solution of the system*

$$(5.9)\qquad \begin{bmatrix} S_0 \\ \vdots \\ S_{m-1} \\ S_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} I_p & \ldots & I_p & I_p \\ \vdots & & \vdots & \vdots \\ x_1^{m-1}I_p & \ldots & x_{k-1}^{m-1}I_p & I_p \\ x_1^m I_p & \ldots & x_{k-1}^m I_p & I_p \\ \underline{P}_m(x_1) & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & \overline{P}_m(x_{k-1}) & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \vdots \\ \Lambda_{k-1} \\ \Lambda_k \end{bmatrix}.$$

The measure

$$\mu^+_{mp+s} = \sum_{i=1}^k \Lambda_i \delta_{x_i}$$

is a representation of the point $(S_0, \ldots, S_{2m-1}, \tilde{S}_{2m})$*. Moreover,* μ^+_{mp+s} *defines a quadrature formula with degree of precision* $2m-1$ *and*

$$(5.10)\qquad \begin{aligned} rank(\Lambda_j) &= \ell_j \ , \qquad j = 1, \ldots, k-1 \\ rank(\Lambda_k) &= s \ , \end{aligned}$$

where ℓ_j *denotes the multiplicity of* x_j *as a root of the polynomial* $\bar{P}_m(x)$*. In particular* μ^+_{mp+s} *satisfies (5.4) and integrates all polynomials of the form (5.8) exactly.*

Proof. We only consider the first part of the theorem, the second part of the assertion is proved similary. Let S^{ε}_{2m} denote a continuous perturbation of the moment $\tilde{S}_{2m}$ such that

$$\operatorname{rank}(S^{\varepsilon}_{2m} - S^{-}_{2m}) = p$$

(5.11)

$$\lim_{\varepsilon \to 0} S^{\varepsilon}_{2m} = \tilde{S}_{2m}.$$

Note that $(S_0, \dots, S_{2m-1}, S^{\varepsilon}_{2m}) \in \operatorname{Int}(M_{2m+1})$ and define

$$\mu^{\varepsilon}_Q = \sum_{i=1}^{k_\varepsilon} \Lambda^{\varepsilon}_i \delta_{x^{\varepsilon}_i}$$

as the unique representation of the boundary point $(S_0, \dots, S_{2m-1}, S^{\varepsilon}_{2m}, S^{\varepsilon -}_{2m+1})$ defined in Theorem 4.3, where $S^{\varepsilon -}_{2m+1}$ is calculated by formula (2.4) for the sequence

$$(S_0, \dots, S_{2m-1}, S^{\varepsilon}_{2m}) \in \operatorname{Int}(M_{2m+1}) \ .$$

Note that according to Lemma 2.6 the mth monic orthogonal polynomial with respect to a matrix measure $x d\nu(x)$ is determined by the moments up to order $2m$. This follows because the recursive relation for the polynomial $\underline{Q}_m(x)$ involves only the canonical moments $U_1, \dots, U_{2m}$, which are in one-to-one correspondence with the moments up to the order $2m$ of the measure ν [see Dette and Studden (2002a)]. If $U^{\varepsilon}_1, \dots, U^{\varepsilon}_{2m}$ denote the canonical moments corresponding to the point $(S_0, \dots, S_{2m-1}, S^{\varepsilon}_{2m})$, $V^{\varepsilon}_j = I_p - U^{\varepsilon}_j$ $(1 \le j \le 2m, V^{\varepsilon}_0 = I_p)$ and $\zeta^{\varepsilon}_j = V^{\varepsilon}_{j-1} U^{\varepsilon}_j$, then it follows from Theorem 4.3 and Lemma 2.6 that $x^{\varepsilon}_1 = 0$ and $x^{\varepsilon}_2, \dots, x^{\varepsilon}_{k_\varepsilon}$ are the different roots of the polynomial $\underline{Q}^{\varepsilon}_m(x)$ defined recursively by $\underline{Q}^{\varepsilon}_0(x) = I_p$ $\underline{Q}^{\varepsilon}_{-1}(x) = 0$,

$$\underline{Q}^{\varepsilon}_{j+1}(x) = \underline{Q}^{\varepsilon}_j(x)\{x I_p - \zeta^{\varepsilon}_{2j+1} - \zeta^{\varepsilon}_{2j+2}\} - \underline{Q}_{j-1}(x) \zeta^{\varepsilon}_{2j} \zeta^{\varepsilon}_{2j+1}$$

$(j = 0, \dots, m-1)$. Note that the moments of the μ and μ^{ε}_Q up to the order $2m$ differ only in the $2m$th moment. Consequently, if $U_1, U_2, \dots$ denote the canonical moments of the matrix measure $\mu, V_j = I - U_j$ $(j \ge 1, V_0 = I_p)$ and $\zeta_j = V_{j-1} U_j$ $(j \ge 1)$ it follows from the definition of the canonical moments that $\zeta_j = \zeta^{\varepsilon}_j$ $(1 \le j \le 2m-1)$ and

$$\lim_{\varepsilon \to 0} \zeta^{\varepsilon}_{2m} = \zeta_{2m},$$

which implies

(5.12)
$$\lim_{\varepsilon \to 0} \underline{Q}^{\varepsilon}_m(x) = \underline{Q}_m(x),$$

where $\underline{Q}_m(x)$ is the mth monic orthogonal polynomial with respect to the measure $t d\mu(t)$ recursively defined by the equation (2.17). Observing that $k_\varepsilon \le mp$

$$x^{\varepsilon}_j \in [0, 1]; \ 0 \le \Lambda^{\varepsilon}_j \le S_0, \quad j = 1, \dots, k_\varepsilon,$$

(the last inequality is obtained from the representation of $S_0 = \sum_{i=1}^{k_\varepsilon} \Lambda_i$) it follows by a straightforward argument that there exists a weakly converging subsequence, say $\mu^{\varepsilon_n}_Q$, such that $(\varepsilon_n \to 0)$

(5.13)
$$\lim_{n \to \infty} \mu^{\varepsilon_n}_Q = \mu_Q = \sum_{i=1}^{k} \Lambda_i \delta_{x_i}.$$

Consequently, the matrix measure μ_Q is a representation of the point $(S_0, \ldots, S_{2m-1}, \tilde{S}_{2m})$ and from (5.12) and (5.13) it follows that

$$
\begin{aligned}
0 &= \lim_{\varepsilon_n \to 0} \int_0^1 x(\underline{Q}_m^{\varepsilon_n}(x))^T d\mu_Q^{\varepsilon_n}(x) \underline{Q}_m^{\varepsilon_n}(x) = \int_0^1 x \underline{Q}_m^T(x) d\mu_Q(x) \underline{Q}_m(x) \\
&= \sum_{i=1}^k x_i \underline{Q}_m^T(x_i) \Lambda_i \underline{Q}_m(x_i),
\end{aligned}
$$

which shows that $x_1, \ldots, x_k$ are the different roots of the polynomial $x \det \underline{Q}_m(x)$ (without loss of generality we put $x_1 = 0$). The weights can be found as a solution of the system (4.10) . Using a similar argument as given in the proof of Corollary 3.8 it follows that $x_2, \ldots, x_k$ are located in the interior of the interval $[0,1]$. From the representation

$$
\underline{H}_{2m-1} = ZWZ^T
$$

with

$$
Z = \begin{bmatrix} I_p & \ldots & I_p \\ x_2 I_p & \ldots & x_k I_p \\ \vdots & & \vdots \\ x_2^{m-1} I_p & \ldots & x_k^{m-1} I_p \end{bmatrix} \quad W = \begin{bmatrix} x_2 \Lambda_2 & 0 & \ldots & 0 \\ 0 & x_3 \Lambda_3 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & x_k \Lambda_k \end{bmatrix}
$$

we obtain by the same arguments as given in the proof of Lemma 3.1

$$
\begin{aligned}
&\operatorname{rank}(\Lambda_j) = \ell_j \quad j = 2, \ldots, k \\
&\sum_{j=2}^k \operatorname{rank}(\Lambda_j) = mp.
\end{aligned} \tag{5.14}
$$

For the calculation of the rank of the matrix Λ_1 we use the representation

$$
\underline{H}_{2m} = \begin{bmatrix} S_0 & \ldots & S_{m-1} & S_m \\ \vdots & & \vdots & \vdots \\ S_{m-1} & \ldots & S_{2m-2} & S_{2m-1} \\ S_m & \ldots & S_{2m-1} & \tilde{S}_{2m} \end{bmatrix} = X_{m+1} D_k X_{m+1}^T
$$

[here the matrices X_{m+1} and D_k are defined in (3.7)] and the fact that the rank of $\underline{H}_{2m}$ is $mp + s$ because it is equal to the rank of the matrix

$$
\begin{bmatrix} S_0 & \ldots & S_{m-1} & S_m \\ \vdots & & \vdots & \vdots \\ S_{m-1} & \ldots & S_{2m-2} & S_{2m-1} \\ 0 & \ldots & 0 & \tilde{S}_{2m} - S_{2m}^- \end{bmatrix}
$$

[note that $S_{2m}^- = (S_m, \dots, S_{2m-1})\underline{H}_{2m-2}^{-1}(S_m, \dots, S_{2m-1})^T$ by its definition (2.4)]. Now the polynomial $\underline{Q}_m(x)$ has at least m different zeros, and we obtain from (5.14) that

$$mp + s \;\leq\; \sum_{i=1}^{k} \operatorname{rank}(\Lambda_i) \;=\; mp + \operatorname{rank}(\Lambda_1) \;,$$

which implies

$$\operatorname{rank}(\Lambda_1) \;\geq\; s \;=\; \operatorname{rank}(\tilde{S}_{2m} - S_{2m}^-). \tag{5.15}$$

For the converse inequality we note that it follows from the formula (2.15) that

$$\tilde{S}_{2m} - S_{2m}^- = \int_0^1 \underline{P}_m^T(x) d\mu_Q(x) \underline{P}_m(x) = \sum_{i=1}^{k} \underline{P}_m^T(x_i)\Lambda_i \underline{P}_m(x_i)$$

where $\underline{P}_m(x)$ is the mth monic orthogonal polynomial with respect to the measure μ_Q. For nonnegative definite matrices A, B it is easy to see that

$$\operatorname{rank}(A+B) \;\geq\; \max\Big\{\operatorname{rank}(A), \operatorname{rank}(B)\Big\},$$

and thus we have

$$s \;\geq\; \operatorname{rank}(\underline{P}_m^T(x_1)\Lambda_1 \underline{P}_m(x_1)). \tag{5.16}$$

From Lemma 2.6 it follows by an induction argument that

$$\underline{P}_m(x_1) \;=\; \underline{P}_m(0) \;=\; (-1)^m U_1 V_2 U_3 V_4 U_5 \dots V_{2m-2} U_{2m-1}$$

and this matrix is nonsingular, because $(S_0, \dots, S_{2m-1}) \in \operatorname{Int}(M_{2m})$. Therefore we obtain from (5.16) that

$$s \;\geq\; \operatorname{rank}(\Lambda_1) \;,$$

which proves in combination with (5.15) the first part of the assertion (1). The statement regarding the uniqueness of the weights is obtained by the same arguments as given in the proof of Theorem 3.3. By construction μ_Q integrates the monomial s up the order $2m-1$ exactly and we obtain

$$\begin{aligned}\int_0^1 A_{2m}^T x^{2m} d\mu_Q(x) \;&=\; A_{2m}^T \tilde{S}_{2m} \\ &=\; A_{2m}^T(\tilde{S}_{2m} - S_{2m}) + A_{2m}^T S_{2m} = \int_0^1 A_{2m}^T x^{2m} d\mu(x)\end{aligned}$$

for all matrices A_{2m} in the set $\{A \in \mathbb{R}^{p\times p} \mid A^T(\tilde{S}_{2m} - S_{2m}) = 0\}$, which completes the proof of assertion (1) of Theorem 5.1. The second part of the assertion is obtained similary and its proof therefore omitted. □

Example 5.2. Consider the situation in Example 3.4, where $(S_0, S_1, S_2) \in \partial M_3$ and

$$\operatorname{rank}(S_2 - S_2^-) = 1.$$

From the definition (2.7) we have

$$U_1 = \begin{pmatrix} 5 & 0 \\ 1 & 6 \end{pmatrix} \; ; \; U_2 = \frac{1}{15} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

and Lemma 2.6 yields for the monic orthogonal polynomials with respect to the measures $xd\mu(x)$ and $(1-x)d\mu(x)$

$$\begin{aligned} \underline{Q}_1(x) &= I_p x - U_1 - V_1 U_2 = \begin{pmatrix} x - \frac{13}{20} & 0 \\ -\frac{1}{10} & x - \frac{3}{4} \end{pmatrix}, \\ \overline{P}_1(x) &= I_p x - U_1 V_1 = \begin{pmatrix} x - \frac{7}{12} & 0 \\ -\frac{1}{6} & x - \frac{3}{4} \end{pmatrix}, \end{aligned}$$

respectively. This gives for the nodes of the quadrature formulas μ_3^- and μ_3^+

$$x_0 = 0; \; x_1 = \frac{13}{30}; \; x_2 = \frac{3}{4}$$

and

$$x_0 = \frac{7}{12}; \; x_2 = \frac{3}{4}; \; x_3 = 1,$$

respectively. Solving the system of equations in (4.10) (for the measure μ_3^-) and (5.9) (for the measure μ_3^+) yields the two representations of the boundary point (S_0, S_1, S_2) given in formula (3.9). Theorem 5.1 shows that μ_3^- and μ_3^+ define two quadrature formulas which integrate linear polynomials exactly and satisfy $\sum_{i=0}^{2} \operatorname{rank}(\Lambda_i) = 3$.

Example 5.3. Consider the matrix measure

$$\begin{aligned} \frac{dW(t)}{dt} &= \frac{1}{2\pi} \frac{\sqrt{5\sqrt{16t^2 + 9t^4} - 17t^2}}{\sqrt{16 + 9t^2}} \cdot I_{[-\frac{5}{2}, \frac{5}{2}]}(t) \\ &\quad \cdot \begin{pmatrix} \frac{3|t| + \sqrt{16+9t^2}}{2\sqrt{2}} & 2\sqrt{2}\,\mathrm{sign}(t) \\ 2\sqrt{2}\,\mathrm{sign}(t) & \frac{-6|t| + 2\sqrt{16+9t^2}}{2\sqrt{2}} \end{pmatrix}, \end{aligned}$$

which was recently discussed by Durán and Polo (2002). A quadrature formula with degree of precision $3 = 1 \cdot 2 + 1$ has been given by these authors. The corresponding measure μ_{Q_1} has weights

$$\Gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; \Gamma_2 = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}; \Gamma_3 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}$$

at the points $0, -\sqrt{2}$ and $\sqrt{2}$, respectively, and also integrates polynomials of the form

$$\begin{pmatrix} f_1 & f_2 \\ 0 & 0 \end{pmatrix} t^2$$

exactly ($f_1, f_2 \in \mathbb{R}$). Note that in contrast to Durán and Polo (2002) we use the right inner product here. We now discuss the corresponding quadrature formulas given in Theorem 5.1. To this end we transform the problem onto the interval $[0,1]$ and consider the measure

$$\mu_{Q_1} = \delta_{\frac{1}{2}} \cdot \Gamma_1 + \delta_{\frac{1}{2}-\frac{1}{5}\sqrt{2}}\Gamma_2 + \delta_{\frac{1}{2}+\frac{1}{5}\sqrt{2}}\Gamma_3 \ ,$$

which has moments

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ S_1 = \frac{1}{10}\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \ ; \ \tilde{S}_2 = \frac{1}{100}\begin{pmatrix} 33 & 20 \\ 20 & 29 \end{pmatrix} \ ;$$

and satisfies

$$\tilde{S}_2 - S_2^- = \tilde{S}_2 - S_1 S_0^{-1} S_1 = \frac{4}{25}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e. $s = \operatorname{rank}(\tilde{S}_2 - S_2^-) = 1$. The quantities ζ_1, ζ_2 in the recursion (2.17) are easily obtained from their definition and given by

$$\zeta_1 = \frac{1}{10}\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}; \quad \zeta_2 = \frac{2}{105}\begin{pmatrix} 5 & 0 \\ -2 & 0 \end{pmatrix} \ ,$$

which yields for the polynomial $\underline{Q}_1(x)$ in Theorem 5.1

$$\underline{Q}_1(x) = xI_2 - \zeta_1 - \zeta_2 = \begin{pmatrix} x - \frac{25}{42} & -\frac{1}{5} \\ -\frac{17}{105} & x - \frac{1}{2} \end{pmatrix} \ ,$$

and consequently the nodes of the quadrature formula (on the interval $[0,1]$) determined by the first part of Theorem 5.1 are given by $x_1 = 0, x_2 = (115 - 2\sqrt{382})/210, x_3 = (115 + 2\sqrt{382})$. Solving the system (4.10) (with $m = 1$) and a transformation onto the interval $[-\frac{5}{2}, \frac{5}{2}]$ yields the quadrature formula

$$\mu_3^- = \delta_{-\frac{5}{2}}\Lambda_1 + \delta_{(5-\sqrt{382})/21}\Lambda_2 + \delta_{(5+\sqrt{382})/21}\Lambda_3$$

where the quadrature coefficients are given by

$$\begin{aligned} \Lambda_1 &= \frac{4}{557}\begin{pmatrix} 25 & -10 \\ -10 & 4 \end{pmatrix}; \Lambda_2 = \frac{1}{1114\sqrt{382}}\begin{pmatrix} 457\sqrt{382} - 2965 & 40\sqrt{382} - 9397 \\ 40\sqrt{382} - 9397 & 541\sqrt{382} + 1865 \end{pmatrix} \\ \Lambda_3 &= \frac{1}{1114\sqrt{382}}\begin{pmatrix} 457\sqrt{382} + 2965 & 40\sqrt{382} + 9397 \\ 40\sqrt{382} + 9397 & 541\sqrt{382} - 1865 \end{pmatrix}. \end{aligned} \tag{5.17}$$

Note that this formula integrates linear polynomials exactly, satisfies $\sum_{i=1}^{3} \operatorname{rank}(\Lambda_i) = 3$ and also integrates the quadratic polynomials

$$\begin{pmatrix} f_1 & f_2 \\ 0 & 0 \end{pmatrix} t^2 + Bt + C$$

($f_1, f_2 \in \mathbb{R}, B, C \in \mathbb{R}^{2\times 2}$) exactly. Finally, we mention that the quadrature formula generated by the second part of Theorem 5.1 is

$$\mu_3^+ = \delta_{(-5-\sqrt{382})/21}\tilde{\Lambda}_1 + \delta_{(-5+\sqrt{382})/21}\tilde{\Lambda}_2 + \delta_{\frac{5}{2}}\tilde{\Lambda}_3$$

with quadrature coefficients given by

$$\begin{aligned}
\tilde{\Lambda}_1 &= \frac{1}{1114\sqrt{382}}\begin{pmatrix} 457\sqrt{382}+2965 & -40\sqrt{382}-9397 \\ -40\sqrt{382}-9397 & 541\sqrt{382}-1865 \end{pmatrix} \\
\tilde{\Lambda}_2 &= \frac{1}{1114\sqrt{382}}\begin{pmatrix} 457\sqrt{382}-2965 & -40\sqrt{382}+9397 \\ -40\sqrt{382}+9397 & 541\sqrt{382}+1865 \end{pmatrix} \\
\tilde{\Lambda}_3 &= \frac{4}{557}\begin{pmatrix} 25 & 10 \\ 10 & 4 \end{pmatrix}.
\end{aligned}$$

We conclude this paper with a nice corollary regarding the zeros of the polynomials $\underline{Q}_m(x)$ and $\underline{P}_m(x)$. It is well known [see e.g. Karlin and Studden (1966)] that in the one-dimensional case $p = 1$ these zeros strictly interlace. The following result shows that this is not necessarily the case if $p > 1$.

Corollary 5.4. *Let μ denote a matrix measure on the interval $[0, 1]$ such that $(S_0, \dots, S_{2m-1}) \in Int(M_{2m})$. If x_j is a zero of the monic orthogonal polynomial $\underline{Q}_m(x)$ with respect to the measure $xd\mu(x)$ and x_j has multiplicity*

$$\ell_j > \; rank(S_{2m} - S_{2m}^-) = s, \tag{5.18}$$

then x_j is also a root of the polynomial $\underline{P}_m(x)$ orthogonal with respect to the measure $d\mu(x)$. Similary, any zero of the monic orthogonal polynomial $\bar{P}_m(x)$ with respect to the measure $(1-x)d\mu(x)$ is also a root of the polynomial $\underline{P}_m(x)$, if its multiplicity satisfies (5.18).

Proof. We only consider the first case, the statement regarding $\bar{P}_m(x)$ is treated exactly in the same way. The case $s = 0$ is trivial, because in this case we have $U_{2m} = 0$ and Lemma 2.6 and an induction argument show that $\underline{Q}_m(x) = \underline{P}_m(x)$. If $s > 0$ we obtain by the same argument as used in the derivation of (5.16) the inequality

$$\ell_j > s = \; \mathrm{rank}(S_{2m} - S_{2m}^-) \geq \; \mathrm{rank}(\underline{P}_m^T(x_j)\Lambda_j\underline{P}_m(x_j))$$

where Λ_j is the weight of the point x_j in the quadrature formula specified in part (1) of Theorem 5.1. Now if x_j was not a root of the polynomial $\underline{P}_m(x)$ we obtain from (5.7) the contradiction

$$\ell_j > s \geq \mathrm{rank}(\Lambda_j) \;=\; \ell_j \;.$$

$\square$

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