A note on the matrix valued q-d algorithm and matrix orthogonal polynomials on $[0,1]$ and $[0,\infty)$

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Abstract

In this note a matrix version of the q-d algorithm is introduced. It is shown that the algorithm may be used to obtain the coefficients of the recurrence relations for matrix orthogonal polynomials on the interval $[0,\infty)$ and $[0,1]$ from its moment generating functional. The algorithm is illustrated by several examples, which generalize classical orthogonal polynomials on the real line.

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1 Introduction

The q-d algorithm is useful in numerical analysis and approximation theory. Some discussion of the algorithm is given in the monograph of Henrici (1977), Vol. 2. One of the applications of the algorithm is to turn a power series into a continued fraction. Formally we have

$$(1.1)\qquad c_0 + c_1 z + c_2 z^2 + \cdots \;=\; \frac{c_0|}{|\,1} - \frac{q_1^{(0)} z|}{|\;\;1} - \frac{e_1^{(0)} z|}{|\;\;1} - \frac{q_2^{(0)} z|}{|\;\;1} - \frac{e_2^{(0)} z|}{|\;\;1} - \;\cdots$$

The algorithm relates the quantities c_i in the power series with the coefficients $q_i^{(0)}, e_i^{(0)}$ in the continued fraction and proceeds by some simple quotient-difference operations. Given the quantities $c_0, c_1, \cdots$ we start by defining for $n \geq 0$

$$q_1^{(n)} = \frac{c_{n+1}}{c_n} \tag{1.2}$$

and successively define $e_m^{(n)}, q_{m+1}^{(n)}$ for $m \geq 1$ and $n \geq 0$ by the relations

$$\begin{aligned} e_m^{(n)} &= q_m^{(n+1)} - q_m^{(n)} + e_{m-1}^{(n+1)} \\ q_{m+1}^{(n)} &= (e_m^{(n)})^{-1} q_m^{(n+1)} e_m^{(n+1)}. \end{aligned} \tag{1.3}$$

This is usually displayed in an array of the form

$$\begin{array}{cccccc} 0 & & & & & \\ & q_1^{(0)} & & & & \\ 0 & & e_1^{(0)} & & & \\ & q_1^{(1)} & & q_2^{(0)} & & \\ 0 & & e_1^{(1)} & & e_2^{(0)} & \\ & q_1^{(2)} & & q_2^{(1)} & & \ddots \\ 0 & & e_1^{(2)} & & e_2^{(0)} & \\ & q_1^{(3)} & & q_2^{(2)} & & \\ & \vdots & \vdots & & & \end{array} \tag{1.4}$$

where the top diagonal produces the entries $q_i^{(0)}, e_i^{(0)}$ of the continued fraction expansion in (1.1). When the quantities c_k are the moments

$$c_k = \int_0^\infty t^k d\mu(t)$$

for some measure μ on the interval $[0, \infty)$, the series gives the Stieltjes transform of the measure μ, that is,

$$\int_0^\infty \frac{d\mu(t)}{1 - zt} = c_0 + c_1 z + c_2 z^2 + \cdots. \tag{1.5}$$

Truncating the continued fraction produces a rational approximation to the series. The coefficients in the continued fraction also generate the coefficients of the three term recurrence relation for the polynomials orthogonal with respect to the measure μ. Thus, if we set

$$\zeta_{2m-1} = q_m^{(0)} \qquad \zeta_{2m} = e_m^{(0)}, \qquad m \geq 1 \tag{1.6}$$

then the corresponding monic orthogonal polynomials satisfy the relations: $P_{-1}(x) \equiv 0, P_0(x) \equiv 1$ and

$$P_{n+1}(x) = (x - \alpha_{n+1})P_n(x) - \beta_{n+1}P_{n-1}(x) \ , \quad n \ \geq \ 1 \tag{1.7}$$

where

$$\beta_{k+1} = \zeta_{2k-1}\zeta_{2k} \quad \text{and} \quad \alpha_{k+1} = \zeta_{2k} + \zeta_{2k+1}.$$

Forms of the q-d algorithm have appeared using vectors, see Van Iseghem (1987, 1989). Recently there has been considerable interest in matrix measures and matrix orthogonal polynomials, see for example Rodman (1990), Sinap and Van Assche (1994,1996), Duran and Van Assche (1995), Duran (1995, 1996, 1999) or Duran and Lopez-Rodriguez (1996, 1997) and the references therein. The purpose of the present note is describe a matrix version of the q-d algorithm and to illustrate its application in some examples of the matrix version of the recurrence formula (1.7). The remaining part of the paper is organized as follows. In the next section some background material is discussed and the matrix algorithm precisely stated. Section 3 contains a proof of the result and section 4 has some examples illustrating applications of the algorithm from different viewpoints.

2 Review and Statement of Theorem

In this section a number of preliminary results are discussed. Let $\mu = (\mu_{ij})$ denote a $p \times p$ matrix of measures on the Borel field of the interval $[0, \infty)$ such that for each Borel set A the matrix $(\mu_{ij}(A))$ is symmetric and positive semi-definite and such that the moments

$$S_k = \int_0^\infty t^k d\mu(t) \in \mathbb{R}^{p \times p} \tag{2.1}$$

exist for all $k \ \geq \ 0$. For $n \in \mathbb{N}_0$ let H_n denote the block Hankel determinants defined by

$$H_{2m} = \begin{pmatrix} S_0 & \cdots & S_m \\ \vdots & & \vdots \\ S_m & \cdots & S_{2m} \end{pmatrix} \qquad H_{2m+1} = \begin{pmatrix} S_1 & \cdots & S_{m+1} \\ \vdots & & \vdots \\ S_{m+1} & \cdots & S_{2m+1} \end{pmatrix}. \tag{2.2}$$

It has been shown in Dette and Studden (2001) that if

$$M_{n+1} = \{ \ (S_0, \cdots, S_n) \ \mid \ S_k = \int_0^\infty t^k d\mu(t) \ \text{ for some } \ \mu \ \} \tag{2.3}$$

denotes the n-th moment space generated by these measures, then $(S_0, \cdots, S_n) \in M_{n+1}$ if and only if the Hankel matrix H_n is semi-definite and $(S_0, \cdots, S_n) \in \text{int } M_{n+1}$ if and only if H_n is positive definite. If $(S_0, \cdots, S_{n-1}) \in \text{int } M_n$ let S_n^- be defined by

$$(2.4) \qquad S_{2m}^- = (S_m, \cdots, S_{2m-1}) H_{2m-2}^{-1} \begin{pmatrix} S_m \\ \vdots \\ S_{2m-1} \end{pmatrix}$$

if $n = 2m$ is even, and by

$$(2.5) \qquad S_{2m+1}^- = (S_{m+1}, \cdots, S_{2m}) H_{2m-1}^{-1} \begin{pmatrix} S_{m+1} \\ \vdots \\ S_{2m} \end{pmatrix}$$

if $n = 2m+1$ is odd. The matrix S_n^- gives the "smallest" value for S_n if $(S_0, \cdots, S_{n-1}) \in \text{int } M_n$ and

$$(S_0, \cdots, S_n) \in M_{n+1}.$$

It was shown in Dette and Studden (2001) that if $(S_0, \cdots, S_n) \in \text{int } M_{n+1}$ for all n and the $p \times p$-matrix ζ_k is defined by

$$(2.6) \qquad \zeta_k = (S_{k-1} - S_{k-1}^-)^{-1}(S_k - S_k^-)$$

for $k \geq 1$, then the monic matrix orthogonal polynomials corresponding to the matrix measure μ satisfy an analog of the recurrence relation (1.7) for the coefficients ζ_k defined in (2.6). More precisely, if the polynomials are orthogonalized with respect to a right inner product, we have for the monic orthogonal matrix polynomials the recursive relation $\underline{P}_0(x) = I_p, \underline{P}_{-1}(x) = 0$ and for $m \geq 0$

$$(2.7) \qquad x\underline{P}_m(x) = \underline{P}_{m+1}(x) + \underline{P}_m(x)(\zeta_{2m+1} + \zeta_{2m}) + \underline{P}_{m-1}(x)\zeta_{2m-1}\zeta_{2m},$$

where $I_p \in \mathbb{R}^{p\times p}$ denotes the identity matrix and the quantities $\zeta_j \in \mathbb{R}^{p\times p}$ are defined by (2.6) if $j \geq 1$ and $\zeta_0 = 0$. Our main result shows that these coefficients can be explicitly obtained by the q-d algorithm with a specific initialization.

Theorem 2.1 : *If the q-d algorithm in matrix form starts with the matrices*

$$q_1^{(n)} = S_n^{-1} S_{n+1} \qquad n \geq 0$$

and $e_m^{(n)}, q_{m+1}^{(n)}$, $m \geq 1$, $n \geq 0$ *are defined recursively by the equations (1.3) then*

$$e_m^{(0)} = \zeta_{2m} \ , \quad q_m^{(0)} = \zeta_{2m-1}$$

for $m \geq 1$ *where the matrices* $\zeta_k \in \mathbb{R}^{p\times p}$ *are defined by (2.6).*

Thus if we start with a matrix measure μ on the interval $[0,\infty)$ with moments $S_k \in \mathbb{R}^{p\times p}$ the coefficients in the recurrence formula for the orthogonal polynomials can be derived from the matrix version of the q-d algorithm. Conversely the moments of a matrix measure can be obtained by a reverse application of the q-d algorithm from the recurrence relation of the corresponding monic orthogonal polynomials. Note that in the 'quotient' or second part of the algorithm in (1.3) the matrices must be in the order presented, which is due to the fact that we used a right inner product for orthogonalization. It was shown in Dette and Studden (2001) that the point $(S_0, S_1, \ldots)$ corresponds to a matrix measure on the interval $[0,\infty)$ if and only if

$$S_0\zeta_1 \cdots \zeta_n \ > \ 0$$

for all $n \in \mathbb{N}_0$. Consequently the q-d-algorithm can be used to check if a given vector of matrices is in fact a vector of moments corresponding to a matrix measure on the interval $[0,\infty)$.

The matrix version of the q-d algorithm has some further information. It is known in the scalar case that the measure μ is supported on the interval $[0,1]$ if and only if the quantities ζ_k form a chain sequence, that is, there exists a further sequence $\{p_k\}$ with $0 < p_i < 1$ such that $\zeta_k = q_{k-1}p_k$ where $q_k = 1 - p_k$ and $q_0 = 1$ [see Chihara (1978) or Wall (1948)]. An analogous result for the matrix polynomials is described in Dette and Studden (2001). For a matrix measure supported on the interval $[0,1]$ there is a sequence of matrices $U_k, k \geq 1$ such that if $V_k = I_p - U_k$, then the quantities ζ_k defined in (2.6) are given by $\zeta_k = V_{k-1}U_k$ $(V_0 = I_p)$. The matrices U_k are called canonical moments and the conditions on the matrices U_k is that for all $n \geq 1$,

$$D_nU_n > 0 \quad \text{and} \quad D_nV_n > 0 \tag{2.8}$$

where

$$D_n = S_0U_1V_1 \cdots U_{n-1}V_{n-1}.$$

Thus the matrix version of the q-d algorithm in Theorem 2.1 provides us with a means of also calculating the matrix chain sequence from the moments for matrix measures on the interval $[0,1]$. Again the inequalities (2.8) can be used to check if a given vector of matrices is in fact a vector of moments corresponding to a matrix measure on the interval $[0,1]$. We will illustrate this application in the examples of Section 4.

3 Proof of Theorem

Notice first in processing down each column of (1.4) that the entries are formally the same except that the indices are shifted up by one, for example $q_1^{(n)} = S_n^{-1}S_{n+1}$ (by definition),

$$e_1^{(n)} = q_1^{(n+1)} - q_1^{(n)} + e_0^{(n+1)} = S_{n+1}^{-1}(S_{n+2} - S_{n+1}S_n^{-1}S_{n+1})$$

$$q_2^{(n)} = (e_1^{(n)})^{-1}q_1^{(n+1)}e_1^{(n+1)} = (S_{n+2} - S_{n+1}S_n^{-1}S_{n+1})^{-1}(S_{n+3} - S_{n+2}S_{n+1}^{-1}S_{n+2}) \ .$$

Therefore, without loss of generality we can concern ourselves with only the first two diagonal rows $e_n^{(0)}, q_n^{(0)}$ and $e_n^{(1)}, q_n^{(1)}$. Observing the representation (2.6) the entries in the first diagonal row in the array (1.4) should be

$$
\begin{aligned}
e_m^{(0)} &= (S_{2m-1} - S_{2m-1}^{-})^{-1}(S_{2m} - S_{2m}^{-}) \\
q_{m+1}^{(0)} &= (S_{2m} - S_{2m}^{-})^{-1}(S_{2m+1} - S_{2m+1}^{-}).
\end{aligned} \tag{3.1}
$$

Note that by the above discussion the entries $q_m^{(1)}$ and $e_m^{(1)}$ in the second diagonal are obtained from $q_m^{(0)}$ and $e_m^{(0)}$ by replacing in these expressions the matrices S_n by S_{n+1}. Therefore, introducing the notation

$$
S_{2m}^{*} = (S_{m+1}, \cdots, S_{2m-1}) \begin{pmatrix} S_2 & \cdots & S_m \\ \vdots & & \vdots \\ S_m & \cdots & S_{2m-2} \end{pmatrix}^{-1} \begin{pmatrix} S_{m+1} \\ \vdots \\ S_{2m-1} \end{pmatrix} \tag{3.2}
$$

and observing (2.4) and (2.5) the entries in the second diagonal row should be

$$
\begin{aligned}
q_m^{(1)} &= (S_{2m-1} - S_{2m-1}^{-})^{-1}(S_{2m} - S_{2m}^{*}) \\
e_m^{(1)} &= (S_{2m} - S_{2m}^{*})^{-1}(S_{2m+1} - S_{2m+1}^{-}).
\end{aligned} \tag{3.3}
$$

Using equations (3.1) and (3.3) it is then required to show that

$$
e_m^{(0)} = q_m^{(1)} - q_m^{(0)} + e_{m-1}^{(1)} \tag{3.4}
$$

and

$$
q_{m+1}^{(0)} = (e_m^{(0)})^{-1} q_m^{(1)} e_m^{(1)}. \tag{3.5}
$$

Equation (3.5) is very simple and follows easily by inserting the appropriate quantities from (3.1) and (3.3).

Equation (3.4) requires more work. By inserting the expressions from (3.1) and (3.3) into (3.4) and multiplying the resulting equation by $(S_{2m-1} - S_{2m-1}^{-})$ it follows that we are required to show the identity

$$
S_{2m}^{*} - S_{2m}^{-} = (S_{2m-1} - S_{2m-1}^{-})\ Q\ (S_{2m-1} - S_{2m-1}^{-}) \tag{3.6}
$$

where the matrix Q is defined by

$$
Q = (S_{2m-2} - S_{2m-2}^{*})^{-1} - (S_{2m-2} - S_{2m-2}^{-})^{-1}. \tag{3.7}
$$

The proof now uses the idea that the right hand side of the equation (3.6) is a "quadratic form" in the matrix $(S_{2m-1} - S^{-}_{2m-1})$. We will therefore replace S_{2m-1} on the left hand by $(S_{2m-1} - S^{-}_{2m-1}) + S^{-}_{2m-1}$ and on expanding things out, show that everything cancels except for the two terms on the right hand side. To be precise we write the row and column vector in the definition of the matrix S^{-}_{2m} in (2.4) as

$$(S_m, \cdots, S_{2m-1}) \;=\; (S_m, \cdots, S^{-}_{2m-1}) + (0, \cdots, 0, S_{2m-1} - S^{-}_{2m-1}) \;=\; A' + B' \;,$$

where the last equality defines the matrices A and B. Similarly, for S^{*}_{2m} let

$$(S_{m+1}, \cdots, S_{2m-1}) \;=\; (S_{m+1}, \cdots, S^{-}_{2m-1}) + (0, \cdots, 0, S_{2m-1} - S^{-}_{2m-1}) \;=\; C' + D' \;,$$

and further let

$$G_{2m-2} = \begin{pmatrix} S_2 & \cdots & S_m \\ \vdots & & \vdots \\ S_m & \cdots & S_{2m-2} \end{pmatrix} .$$

Observing the equations in (2.4), (2.5) and (3.2) the right hand side of equation (3.6) is then

$$(C' + D')G^{-1}_{2m-2}(C + D) - (A' + B')H^{-1}_{2m-2}(A + B)$$

Note also that

$$D'G^{-1}_{2m-2}D - B'H^{-1}_{2m-2}B$$

is equal to the right hand side of (3.6), which also follows from (2.4), (2.5) and (3.2) by taking Schur complements in the matrices H_{2m} and G_{2m}. The other terms will pair up and cancel, that is

$$C'G^{-1}_{2m-2}C = A'H^{-1}_{2m-2}A \quad \text{and} \quad C'G^{-1}_{2m-2}D = A'H^{-1}_{2m-2}B \;.$$

These identities follow in essentially the same manner and for the sake of brevity we consider only the first one

$$C'G^{-1}_{2m-2}C = A'H^{-1}_{2m-2}A \;. \tag{3.8}$$

To accomplish this it is convenient to write

$$\begin{pmatrix} S_m \\ S_{m+1} \\ \vdots \\ S_{2m-2} \\ S^{-}_{2m-1} \end{pmatrix} = \begin{pmatrix} S_1 & \cdots & S_{m-1} \\ S_2 & & S_m \\ \vdots & & \vdots \\ S_{m-1} & \cdots & S_{2m-3} \\ S_m & \cdots & S_{2m-2} \end{pmatrix} \begin{pmatrix} S_1 & \cdots & S_{m-1} \\ \vdots & & \vdots \\ S_{m-1} & \cdots & S_{2m-3} \end{pmatrix}^{-1} \begin{pmatrix} S_m \\ \vdots \\ S_{2m-2} \end{pmatrix} , \tag{3.9}$$

where the equality for the last component is the definition of S^{-}_{2m-1} and the other identities follow very easily. Now the identity (3.8) is obtained by inserting the expressions for the matrices C and A from equation (3.9) into the above and using the fact that

$$\begin{pmatrix} S_1 & \cdots & S_m \\ \vdots & & \vdots \\ S_{m-1} & \cdots & S_{2m-2} \end{pmatrix} \begin{pmatrix} S_0 & \cdots & S_{m-1} \\ \vdots & & \vdots \\ S_{m-1} & \cdots & S_{2m-2} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_p \end{pmatrix}$$

□

4 Examples

4.1 Random walk measures

Consider the matrix measure

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}$$

where

$$\frac{d\mu_{11}}{dt} = \frac{d\mu_{22}}{dt} = \frac{1}{\pi} \frac{1}{\sqrt{t(1-t)}}$$

$$\frac{d\mu_{12}}{dt} = \frac{1}{\pi} \frac{2t-1}{\sqrt{t(1-t)}}$$

(see also VanAssche (1993), who considered this example on the interval $[-1,1]$). Note that this measure corresponds to the spectral measure of a symmetric random walk on the integers [see Karlin and Mcgregor (1959), Section 4]. A straightforward calculation gives for the moments

$$S_k = \binom{2k}{k} \frac{1}{2^{2k}(k+1)} \begin{pmatrix} k+1 & k \\ k & k+1 \end{pmatrix} , \quad k \geq 0 ,$$

and an induction argument yields the entries in the array (1.4), that is

$$e_j^{(k)} = \frac{j^2}{(k+2j)(k+2j+1)} \begin{pmatrix} 1 & -\frac{1}{2j} \\ -\frac{1}{2j} & 1 \end{pmatrix} ,$$

$$q_j^{(k)} = \frac{(k+j)}{2(k+2j-1)(k+2j)} \begin{pmatrix} 2(k+j) & 1 \\ 1 & 2(k+j) \end{pmatrix} , \quad k,j \geq 0 .$$

Now Theorem 2.1 gives

$$\zeta_{2k} = e_k^{(0)} = V_{2k-1}U_{2k} = \frac{k}{k+1} \begin{pmatrix} 1 & -\frac{1}{2k} \\ -\frac{1}{2k} & 1 \end{pmatrix} ,$$

$$\zeta_{2k-1} = q_k^{(0)} = V_{2k-2}U_{2k-1} = \frac{1}{2(k-1)} \begin{pmatrix} 2k & 1 \\ 1 & 2k \end{pmatrix} ,$$

and it follows for the canonical moments of the matrix measure μ

$$U_{2k} = \frac{k}{2k+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$U_{2k-1} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{2k} \\ \frac{1}{2k} & 1 \end{pmatrix} ,$$

for $k \in \mathbb{N}$. Note that this result was stated in Dette and Studden (2001) without an explicit proof. In a similar way it can be shown that the canonical moments of the matrix measure

$$\frac{d\nu(t)}{dt} = c_{\alpha,\beta} x^{\alpha}(1-x)^{\beta}\begin{pmatrix} 1 & 2x-1 \\ 2x-1 & 1 \end{pmatrix} \tag{4.1}$$

are given by

$$U_{2k} = \frac{k}{2k+\alpha+\beta+2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$U_{2k-1} = \frac{1}{2(2k+\alpha+\beta+1)}\begin{pmatrix} 2k+2\alpha+1 & 1 \\ 1 & 2k+2\alpha+1 \end{pmatrix} ,$$

where the constant $c_{\alpha,\beta}$ in (4.1) is defined such that the diagonals integrate to 1.

4.2 Matrix Jacobi polynomials

During the past few years some important results in the theory of classical orthogonal polynomials have been extended to classical orthogonal matrix polynomials [see for example Jódar, Company and Navarro (1994), Jódar and Cortéz (1998a,b), Jódar and Sastre (2000)]. In the following example we discuss some properties of the generalized Jacobi polynomials corresponding to the measure determined by the moments of the Beta matrix distribution

$$B(A, B : t) = t^B(1-t)^A \tag{4.2}$$

where A and B are real $p \times p$ matrices such $A + I_p$ and $B + I_p$ are positive definite. The matrix form of the Beta-integral is given by

$$B(A+I_p, B+I_p) = \int_0^1 t^B(1-t)^A dt ,$$

where I_p denotes the $p \times p$ identity matrix. For some general discussion of this function we refer to Jódar and Cortéz (1998b) and we assume in this example that the matrices A and B are simultaneously diagonalizable, that is

$$A = SD_AS^{-1} , \quad B = SD_BS^{-1}$$

where S is a non-singular $p \times p$ matrix and D_A and D_B are diagonal matrices containing the (real) eigenvalues $a_1, \ldots, a_p$ and $b_1, \ldots, b_p$ of the matrices A and B, respectively. Now formula (6) in Jódar and Cortéz (1998b) gives for the "moments" of the (normalized) Beta distribution $B^{-1}(A + I_p, B + I_p)B(A, B : t)$

$$
\begin{aligned}
(4.3) \qquad S_k &= \int_0^1 t^k B^{-1}(A + I_p, B + I_p)B(A, B : t)dt \\
&= S \cdot B^{-1}(D_A + I_p, D_B + I_p) \int_0^1 t^k B(D_A, D_B : t)dt \cdot S^{-1} \; = \; SR_kS^{-1} \; ,
\end{aligned}
$$

where R_k is a diagonal matrix containing the moments of ordinary $B(a_j, b_j)$-distributions, that is

$$
(4.4) \qquad R_k \; = \; \mathrm{diag}\Big(\frac{\Gamma(a_1 + b_1 + 2)\Gamma(b_1 + k + 1)}{\Gamma(b_1 + 1)\Gamma(a_1 + b_1 + k + 2)}, \ldots, \frac{\Gamma(a_p + b_p + 2)\Gamma(b_p + k + 1)}{\Gamma(b_p + 1)\Gamma(a_p + b_p + k + 2)}\Big)
$$

(note that $R_0 = I_p$). When we apply the q-d algorithm to the moments S_k defined by (4.3), the matrices S and S^{-1} will factor out in the difference operation and a little bit of cancelling occurs with the quotient operation, such that the basic operations of the q-d algorithm have only to be performed for diagonal matrices. Consequently we obtain from Example 1.5.3 in Dette and Studden (1997) that

$$
(4.5) \qquad
\begin{aligned}
\zeta_{2m} &= S \cdot \zeta_{2m}^D \cdot S^{-1} \\
&= m(mI_p + A)(2mI_p + A + B)^{-1}((2m + 1)I_p + A + B)^{-1} \\
\zeta_{2m-1} &= S \cdot \zeta_{2m-1}^D \cdot S^{-1} \\
&= (mI_p + A + B)(mI_p + B)((2m - 1)I_p + A + B)^{-1}(2mI_p + A + B)^{-1}
\end{aligned}
$$

where

$$
\zeta_{2m}^D \; = \; \mathrm{diag}\Big(\frac{(a_1 + m)m}{(2m + a_1 + b_1)(2m + a_1 + b_1 + 1)}, \ldots, \frac{(a_p + m)m}{(2m + a_p + b_p)(2m + a_p + b_p + 1)}\Big)
$$

$$
\zeta_{2m-1}^D \; = \; \mathrm{diag}\Big(\frac{(a_1 + b_1 + m)(b_1 + m)}{(2m - 1 + a_1 + b_1)(2m + a_1 + b_1)}, \ldots, \frac{(a_p + b_p + m)(b_p + m)}{(2m - 1 + a_p + b_p)(2m + a_p + b_p)}\Big)
$$

The corresponding chain sequence is obtained by a straightforward calculation, solving the equations $V_{j-1} \cdot U_j = \zeta_j$, i.e.

$$
(4.6) \qquad
\begin{aligned}
U_{2m} &= S \cdot \mathrm{diag}\Big(\frac{m}{(2m + a_1 + b_1 + 1)}, \ldots, \frac{m}{(2m + a_p + b_p + 1)}\Big) \cdot S^{-1} \\
&= m((2m + 1)I_p + A + B)^{-1} \\
U_{2m-1} &= S \cdot \mathrm{diag}\Big(\frac{(b_1 + m)}{(2m + a_1 + b_1)}, \ldots, \frac{(b_p + m)}{(2m + a_p + b_p)}\Big) \cdot S^{-1} \\
&= (mI_p + B)(2mI_p + A + B)^{-1} \; .
\end{aligned}
$$

Note that the order of the factors in (4.5) and (4.6) is arbitrary, because the matrices A and B commute. The monic orthogonal polynomials with respect to the right inner matrix product induced by the beta distribution are defined by the recursive relation (2.7). For the sake of comparison we transfer this recursion to the interval $[-1,1]$ and obtain the recursive relation

$$\underline{P}_{m+1}(x) = \underline{P}_m(x)[(x+1)I_p - \zeta_{2m+1} - \zeta_{2m}] - 4\underline{P}_{m-1}(x)\zeta_{2m-1}\zeta_{2m}, \tag{4.7}$$

for the monic matrix orthogonal polynomials with respect to the right inner product induced by the beta-distribution on the interval $[-1,1]$. Now let $H_0 = I_p$,

$$H_k \;=\; \frac{1}{2^k k!}(2kI_p + A + B)\cdots((k+1)I_p + A + B) \qquad (k \geq 1)$$

then it is easy to see that the matrix polynomials $P_k^{(A,B)}(x) = P_k(x)H_k$ are orthogonal polynomials with respect to the right inner product induced by the beta distribution and satisfy the recursive relation $P_0^{(A,B)}(x) = I_p$

$$P_1^{(A,B)}(x) = \frac{1}{2}[A+B+2I_p]x + \frac{1}{2}[A-B];$$

and for $k \geq 1$

$$\begin{aligned} 2(k+1)[(k+1)I_p + A + B][2kI_p + A + B]P_{k+1}^{(A,B)}(x) \;=& \\ [(2k+1)I_p + A + B]\Big\{[2kI_p + A + B][2(k+1)I_p + A + B]x + A^2 - B^2\Big\}P_k^{(A,B)}(x) & \\ - 2[kI_p + A][kI_p + B][2(k+1)I_p + A + B]P_{k-1}^{(A,B)}(x)(x)\ . & \end{aligned}$$

We finally note that in the case where A and B do not commute the situation is substantially more difficult. For a very interesting example considering the Nevai class $M(A,B)$ of orthogonal matrix polynomials with constant coefficients in the recurrence relations we refer to Duran (1999).

4.3 A recurrence relation with constant coefficients

In this example we illustrate how the q-d algorithm can be used to identify the measure of orthogonality corresponding to a sequence of (monic) orthogonal polynomials given by a recurrence relation. To this end consider a sequence of polynomials recursively defined by $P_1(x) = xI_p - A$, $P_0(x) = 1$

$$x\underline{P}_m(x) = \underline{P}_{m+1}(x) + 2\underline{P}_m(x)A + \underline{P}_{m-1}(x)A^2,$$

for some positive definite matrix $A \in \mathbb{R}^{p\times p}$. In this case we have from a comparison with the general recursive relation (2.7) $\zeta_j = A$ for all $j \geq 1$. We can now use the q-d-algorithm to find the moments of the corresponding measure. It can be proved by induction that these moment are given by

$$S_j = 2^{2j+1} \binom{1/2}{j+1} (-1)^j A^j \quad (j \geq 0) \ , \tag{4.8}$$

which yields for the Stieltjes transform of the corresponding measure of orthogonality

$$\begin{aligned} \int \frac{dW(t)}{z-t} &= \frac{S_0}{z} + \frac{S_1}{z^2} + \frac{S_2}{z^3} + \dots \quad = A^{-1} \sum_{j=0}^{\infty} 2^{2j+1} \binom{1/2}{j+1} \frac{A^{j+1}}{z^{j+1}} (-1)^j \\ &= A^{-1} \Big\{ \frac{I_p}{2} - \frac{1}{2} \sum_{j=0}^{\infty} \binom{1/2}{j} \Big(\frac{4}{z}\Big)^j (-1)^j A^j \Big\} \; = \; \frac{A^{-1}}{2} \Big\{ I_p - \sqrt{\Big(I_p - \frac{4A}{z}\Big)} \Big\} \ , \end{aligned}$$

where the square root is defined in the usual way, i.e. using the diagonal form of the matrix and applying the appropriate square root to the eigenvalues. The corresponding measure is now easily obtained using the arguments in Duran (1999), and we obtain the density

$$\frac{dW(t)}{dt} = \sum_{i=1}^{p} z_i z_i^T \frac{1}{2\pi\lambda_i} \sqrt{\frac{4\lambda_i - x}{x}} I_{[0,4\lambda_i]}(x) \ ,$$

where

$$A \;=\; \sum_{j=1}^{p} \lambda_j z_j z_j^T$$

is an eigenvalue decomposition of the matrix A with orthonormal vectors $z_j \in \mathbb{R}^p$ and eigenvalues λ_j counted with their respective multiplicities.

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