Optimal designs for estimating individual coefficients in Fourier regression models

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Abstract

In the common trigonometric regression model we investigate the optimal design problem for the estimation of the individual coefficients, where the explanatory variable varies in the interval $[-a, a]; 0 < a \leq \pi$. It is demonstrated that the structure of the optimal design depends sensitively on the size of the design space. For many cases optimal designs can be found explicitly, where the complexity of the solution depends on the value of the parameter a and the order of the term, for which the corresponding coefficient has to be estimated.

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1 Introduction

Trigonometric regession models of the form

$$y = \beta_0 + \sum_{j=1}^{m} \beta_{2j-1} \sin(jt) + \sum_{j=1}^{m} \beta_{2j} \cos(jt) + \varepsilon, \quad x \in [-a, a]; \quad 0 < a \leq \pi \tag{1.1}$$

are widely used to describe periodic phenomena [see e.g. Mardia (1972), Graybill (1976) or Kitsos, Titterington and Torsney (1988)] and the problem of designing experiments for

Fourier regression models has been discussed by several authors [see e.g. Hoel (1965), Karlin and Studden (1966), page 347, Fedorov (1972), page 94, Hill (1978), Lau and Studden (1985), Riccomagno, Schwabe and Wynn (1997)]. While most authors concentrate on the design space $[-\pi, \pi]$ much less attention has been paid to the case of a smaller design space [see e.g. Hill (1978)]. This situation is of practical importance because in many applications it is impossible to take observations on the full circle $[-\pi, \pi]$. We refer for example to Kitsos, Titterington and Torsney (1988), who investigated a design problem in rhythmometry involving circadian rhythm exhibited by peak expiratory flow, for which the design region has to be restricted to a partial cycle of the complete 24-hour period.
It is the purpose of the present paper to study the optimal design problem for the estimation of the individual coefficients β_k in the trigonometric regression model (1.1) on the interval $[-a, a]$. In Section 2 we introduce the general notation and state several preliminary results, which give some lower bounds on the number of support points of the optimal design. Section 3 deals with the full circle $[-\pi, \pi]$, for which the solution of the optimal design problem is already difficult. Here we are able to find the optimal designs for estimating the individual coefficients β_k explicitly, whenever $k > \frac{2m}{3}$. In Section 4 we consider the optimal design problem for the estimation of the coefficients of the cosine terms, which is intimately related to the c-optimal design problem for the common polynomial regression on the interval $[\cos a, 1]$. It is demonstrated that the optimal design problem for the estimation of the parameter β_k can be solved analytically for any $k \in \{0, 2, \ldots, 2m\}$, provided that the design space $[-a, a]$ is sufficiently small. Section 5 considers the problem of estimating individual coefficients of the sine terms, for which the situation is completely different. We use the implicit function theorem to prove that the optimal design depends analytically on the parameter a and find the limiting design as $a \to 0$. From these results the optimal designs for estimating the coefficient of the highest sine term can be obtained numerically by a Taylor expansion with arbitrary precision. For the remaining coefficients of the sine terms it is shown that on a sufficiently small design space the corresponding optimal designs have the same support points as the optimal design for the estimation of the coefficient β_{2m-1} and an explicit formula for the corresponding weights is derived.
The results of this paper demonstrate that the optimal design problem for the estimation of the individual parameters in a trigonometric regression is substantially more difficult than the corresponding problem in the polynomial case, which was recently solved by Sahm (2000) and Dette, Melas and Pepelysheff (2000). Nevertheless, for many important cases the optimal designs can be found explicitly by the results and methods given in this paper.

2 Optimal designs for estimating individual coefficients

Consider the trigonometric regression model (1.1), define $\beta = (\beta_0, \beta_1, \ldots, \beta_{2m})^T$ and

$$f(t) = (1, \sin t, \cos t, \ldots, \sin(mt), \cos(mt))^T = (f_0(t), \ldots, f_{2m}(t))^T \tag{2.1}$$

as the vector of regression functions. An approximate design is a probability measure ξ on the design space $[-a, a]$ with finite support [see e.g. Kiefer (1974)]. The support points of the design ξ give the location where observations are taken, while the weights give the corresponding proportions of total observations to be taken at these points. For uncorrelated observations (obtained from an approximate design) the covariance matrix of the least squares estimator for the parameter β is approximately proportional to the matrix

$$M(\xi) = \int_{-a}^{a} f(t)f^T(t)d\xi(t) \in \mathbb{R}^{2m+1\times 2m+1} \tag{2.2}$$

which is called information matrix in the design literature. An optimal design minimizes (or maximizes) an appropriate convex (or concave) function of the information matrix and there are numerous criteria proposed in the literature, which can be used for the discrimination between competing designs [see e.g. Fedorov (1972), Silvey (1980) or Pukelsheim (1993)].

In this paper we are interested in optimal designs for the estimation of the individual coefficients β_k in the trigonometric regression model (1.1). To be precise let $e_k \in \mathbb{R}^{2m+1}$ denote the $(k+1)$th unit vector $(k = 0, \ldots, 2m)$ and A^- be a generalized inverse of the matrix $A \in \mathbb{R}^{2m+1\times 2m+1}$, then a design ξ is called e_k-optimal or optimal for estimating the coefficient β_k, if β_k is estimable by the design ξ [i.e. $e_k \in \text{Range}(M(\xi))$] and ξ minimizes the function

$$\Phi_k(\eta) = e_k^T M^-(\eta) e_k \tag{2.3}$$

in the set of all designs η such that β_k is estimable by the design η. e_k-optimal designs have been discussed by several authors, mainly for the case of polynomial regression on the interval $[-1, 1]$ [see e.g. Studden (1968), Kiefer and Wolfowitz (1965), Hoel and Levine (1964) and Sahm (2000)], but nothing is known for the trigonometric case. Our first result is an important tool for the determination of optimal designs and gives a slightly different formulation of the equivalence theorem for e_k-optimal designs as it is usually stated in the literature [see e.g. Pukelsheim (1993), Section 2, or Studden (1968)]. The result is stated here for general regression models and a proof can be found in Dette, Melas and Pepelysheff (2000).

Lemma 2.1 *For $k = 0, 1, \ldots, d$ let $\bar{f}_k(t) = (f_0(t), \ldots, f_{k-1}(t), f_{k+1}(t), \ldots, f_d(t))^T$ denote the vector obtained by omitting the component $f_k(t)$ in the vector $f(t) = (f_0(t), \ldots, f_d(t))^T$. A design ξ^* is optimal for estimating the parameter β_k in the model*

$$y = \sum_{j=0}^{d} \beta_j f_j(t) + \varepsilon \ ; \quad t \in \tau \subset \mathbb{R}$$

if and only if there exist a positive number h and a vector $q \in \mathbb{R}^d$ such that the function

$$\varphi(t) = f_k(t) - q^T \bar{f}_k(t)$$

satisfies

(1) $h\varphi^2(t) \leq 1$ for all $t \in \tau$

(2) $supp(\xi^*) \subset \{t \in \tau \mid h\varphi^2(t) = 1\}$

(3) $\int_T \varphi(t)\bar{f}_k(t)d\xi^*(t) = 0 \in \mathbb{R}^d$

Moreover, in this case $h = \Phi_k(\xi^*)$ *and the function* φ *is called extremal polynomial.*

It follows by standard arguments [see Pukelsheim (1993), Chapter 4,5] that Φ is a convex function on the set of designs on the interval $[-a, a]$, which is invariant with respect to a reflection of the design at the origin. Consequently there exists a symmetric e_k-optimal design (which is not necessarily unique) and we will restrict ourselves to the determination of optimal designs in the set Ξ_s of all symmetric designs on the interval $[-a, a]$. As pointed out by Dette and Haller (1998) this set can be mapped in a one to one manner onto the set of designs on the interval $[\alpha, 1]$ where $\alpha = \cos a$. More precisely, define for a symmetric design ξ on the interval $[-a, a]$ its projection η_ξ as the design on the interval $[\alpha, 1]$ given by

$$\eta_\xi(\cos x) = \begin{cases} \xi(x) + \xi(-x) & \text{if } 0 < x \le a \\ \xi(0) & \text{if } x = 0. \end{cases} \tag{2.4}$$

It is now easy to see that after an appropriate permutation $P \in \mathbb{R}^{2m+1 \times 2m+1}$ of the order of the regression functions the information matrix (2.2) of a symmetric design is block diagonal, that is

$$\tilde{M}(\xi) = PM(\xi)P = \begin{pmatrix} M_c(\xi) & 0 \\ 0 & M_s(\xi) \end{pmatrix} \tag{2.5}$$

with diagonal blocks given by

$$\begin{aligned} M_c(\xi) &= \Big(\int_{-a}^{a} \cos(it)\cos(jt)d\xi(t)\Big)_{i,j=0}^{m} \\ &= 2 \cdot \Big(\int_{\alpha}^{1} T_i(x)T_j(x)d\eta_\xi(x)\Big)_{i,j=0}^{m} \end{aligned} \tag{2.6}$$

$$\begin{aligned} M_s(\xi) &= \Big(\int_{-a}^{a} \sin(it)\sin(jt)d\xi(t)\Big)_{i,j=1}^{m} \\ &= 2 \cdot \Big(\int_{\alpha}^{1} (1-x^2)U_{i-1}(x)U_{j-1}(x)d\eta_\xi(x)\Big)_{i,j=1}^{m} \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} T_i(x) &= \cos(i \arccos x) \\ U_i(x) &= \frac{\sin((i+1)\arccos x)}{\sin(\arccos x)} \end{aligned} \tag{2.8}$$

denote the Chebyshev polynomials of the first and second kind, respectively [see e.g. Rivlin (1974)]. Note that this transformation transfers the optimal design problem for the estimation of the individual coefficients in a trigonometric regression model to design problems for the estimation of the coefficients in the weighted polynomial regression models

$$y = \sum_{j=0}^{m} \delta_j T_j(x) + \varepsilon; \quad x \in [\alpha, 1] \tag{2.9}$$

$$y = \sqrt{1-x^2} \cdot \sum_{j=0}^{m-1} \delta_j U_j(x) + \varepsilon; \quad x \in [\alpha, 1]. \tag{2.10}$$

The proof of the following result is now straightforward and therefore omitted.

Lemma 2.2 *A symmetric design ξ^* on the interval $[-a, a]$ is optimal for estimating the coefficient $\beta_{2\ell}$ $(0 \le \ell \le m)$ in the trigonometric regression (1.1) if and only if the design η_{ξ^*} obtained by the transformation (2.4) is optimal for estimating the parameter δ_ℓ in the Chebyshev regression model (2.9).*

Similary, a symmetric design ξ^* on the interval $[-a, a]$ is optimal for estimating the coefficient $\beta_{2\ell-1}$ $(1 \le \ell \le m)$ in the trigonometric regression model (1.1) if and only if the design η_{ξ^*} obtained by the transformation (2.4) is optimal for estimating the coefficient $\delta_{\ell-1}$ in the weighted Chebyshev regression model (2.10).

Note that there is an alternative formulation of Lemma 2.2 in terms of c-optimality in the ordinary polynomial regression model. A c-optimal design minimizes the variance of the least squares estimator for the linear combination $\sum_{j=0}^{d} \delta_j c_j$, where $c = (c_0, \dots, c_d)^T \in \mathbb{R}^d$ is a given vector and $d \in \{m-1, m\}$ corresponding to the cases (2.10) and (2.9), respectively [see Pukelsheim (1993)]. To be precise let $T \in \mathbb{R}^{m+1 \times m+1}$ and $U \in \mathbb{R}^{m \times m}$ denote the matrix of the coefficients of the Chebyshev polynomials of the first and second kind, respectively, i.e.

$$\begin{aligned} (T_0(x), \dots, T_m(x))^T &= T \cdot (1, x, \dots, x^m)^T \\ (U_0(x), \dots, U_{m-1}(x))^T &= U \cdot (1, x, \dots, x^{m-1})^T. \end{aligned} \tag{2.11}$$

Defining $t(\ell) = T^{-1} e_\ell$ $(\ell = 0, \dots, m)$ and $u(\ell) = U^{-1} e_\ell$ $(\ell = 0, \dots, m-1)$, then we obtain the following auxiliary result.

Lemma 2.3 *A symmetric design ξ^* on the interval $[-a, a]$ is optimal for estimating the individual coefficient $\beta_{2\ell}$ $(0 \le \ell \le m)$ in the trigonometric regression (1.1) if and only if the design η_{ξ^*} obtained by the transformation (2.4) is $t(\ell)$-optimal in the ordinary polynomial regression model of degree m on the interval $[\alpha, 1]$.*

Similary, a symmetric design ξ^* on the interval $[-a, a]$ is optimal for estimating the coefficient $\beta_{2\ell-1}$ $(1 \le \ell \le m)$ in the trigonometric regression model (1.1) if and only if the design η_{ξ^*} obtained by the transformation (2.4) is $u(\ell-1)$-optimal in the ordinary weighted *polynomial regression model of degree $m-1$ with efficiency function $\lambda(x) = 1 - x^2$ on the interval $[\alpha, 1]$.*

Proof. We concentrate on the second case. The first assertion follows by exactly the same arguments. By Lemma 2.2 $e_{2\ell-1}$-optimality of ξ^* in the trigonometric regression (1.1) holds if and only if the design η_{ξ^*} obtained by the transformation (2.4) minimizes the criterion

$$\begin{aligned} e_{\ell-1}^T M_s^-(\xi) e_{\ell-1} &= e_{\ell-1}^T (U^{-1})^T \left(\left(\int_\alpha^1 (1-x^2) x^{i+j} d\eta_\xi(x) \right)_{i,j=0}^{m-1} \right)^- U^{-1} e_{\ell-1} \\ &= u^T(\ell-1) \left(\left(\int_\alpha^1 (1-x^2) x^{i+j} d\eta_\xi(x) \right)_{i,j=0}^{m-1} \right)^- u(\ell-1). \end{aligned}$$

The assertion now follows because the matrix in the last expression is the information matrix of a weighted polynomial regression of degree $m-1$ with efficiency function $\lambda(x) = 1 - x^2$ [see Fedorov (1972)].

□

It is obvious that Lemma 2.2 and 2.3 are important tools for the identification of optimal designs for estimating the individual coefficients in the trigonometric regression model (1.1). In the remaining part of this section we will use these results for the derivation of bounds on the number of support points of the optimal designs. To this end we note that the Chebyshev polynomials of the first kind are orthogonal with respect to the arcsine distribution, i.e.

$$\frac{1}{\pi} \int_{-1}^1 T_i(x) T_j(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 1 & \text{if } i = j = 0 \\ \frac{1}{2} & \text{if } i = j \geq 1 \\ 0 & \text{if } i \neq j \end{cases} \tag{2.12}$$

while the corresponding orthogonality relation for the Chebyshev polynomials of the second kind is

$$\frac{2}{\pi} \int_{-1}^1 \sqrt{1-x^2} U_i(x) U_j(x) dx = \begin{cases} 1 & \text{if } i = j \geq 0 \\ 0 & \text{if } i \neq j \end{cases} \tag{2.13}$$

[see e.g. Rivlin (1974)].

Theorem 2.4 *If ξ_k^* denotes a symmetric optimal design for estimating the parameter β_k in the trigonometric regression model (1.1), then*

$$\begin{aligned} \max\{2\ell+1, m-\ell+1\} &\leq \#supp(\xi_{2\ell}^*) \leq 2m+1 \\ \max\{2\ell, m-\ell+1\} &\leq \#supp(\xi_{2\ell-1}^*) \leq 2m \end{aligned}$$

whenever $0 \leq \ell \leq m$.

Proof. We will concentrate on the first case $k = 2\ell$ even, the odd case will follow by similar arguments. Using Lemma 2.2 and the transformation (2.4) the assertion of the theorem in the even case follows if we establish the bounds

$$\max\{\ell + 1, \frac{m - \ell}{2} + 1\} \le \#\mathrm{supp}(\eta_{\xi^*_{2\ell}}) \le m + 1 \tag{2.14}$$

for the support of the e_ℓ-optimal design $\eta_{\xi^*_{2\ell}}$ in the Chebyshev regression model (2.9). This implication is obvious for the lower bound, while the upper bound requires the additional argument that the support of the optimal design $\eta_{\xi^*_{2\ell}}$ must either contain the point 1 or consists of less than $m + 1$ points. Finally $m + 1$ points in the interval $[\alpha, 1]$ including the right boundary correspond to $2m + 1$ points in the interval $[-a, a]$ including the center 0 by the transformation (2.4).
Note that the upper bound in (2.14) follows directly from Pukelsheim (1993) p. 190. Let

$$\eta^*_{\xi_{2\ell}} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ w_1 & w_2 & \dots & w_n \end{pmatrix} \tag{2.15}$$

denote the e_ℓ-optimal design for the Chebyshev regression model (2.9) $(0 \le \ell \le m)$. If $n = m + 1$ there is nothing to prove and we consider now the remaining case $n \le m$. A reformulation of condition (3) of Lemma 2.1 shows that $\eta_{\xi^*_{2\ell}}$ is e_ℓ-optimal in the Chebyshev regression model (2.9) if and only if there exists a constant $h > 0$ and a polynomial

$$\varphi(x) = T_\ell(x) - \sum_{\substack{j=0 \\ j \ne \ell}}^{m} q_j T_j(x)$$

such that

$$\sqrt{h}|\varphi(x)| \le 1 \quad \forall\, x \in [\alpha, 1] \tag{2.16}$$

$$\sqrt{h}|\varphi(x_i)| = 1 \quad i = 1, \dots, n \tag{2.17}$$

$$F D w = 0 \in \mathbb{R}^n \tag{2.18}$$

where $w = (w_1, \dots, w_n)^T$ denotes the vector of the weights of the design $\eta_{\xi^*_{2\ell}}$, F is an $m \times n$ matrix defined by

$$F = \begin{bmatrix} T_0(x_1) & \dots & T_0(x_n) \\ T_1(x_1) & \dots & T_1(x_n) \\ \vdots & \vdots & \vdots \\ T_{\ell-1}(x_1) & \dots & T_{\ell-1}(x_n) \\ T_{\ell+1}(x_1) & \dots & T_{\ell+1}(x_n) \\ \vdots & \vdots & \vdots \\ T_m(x_1) & \dots & T_m(x_n) \end{bmatrix} \in \mathbb{R}^{m \times n} \tag{2.19}$$

and D is a diagonal matrix with entries $\varphi(x_1), \ldots, \varphi(x_n)$ (note that these quantities are all equal in absolute value). Note that for $\ell \geq n$ the upper $n \times n$ block

$$(T_{i-1}(x_j))_{i,j=1}^n$$

of the matrix F in (2.19) is non-singular because of the Chebyshev property of the system $\{T_0(x), \ldots, T_{n-1}(x)\}$ [see Karlin and Studden (1966)]. In this case (2.18) would imply $w = 0$, which is impossible for the optimal design. Consequently we obtain $n > \ell$ which yields one of the lower estimates in (2.14).
In order to establish the second estimate $n \geq (m-\ell)/2 + 1$ we will prove below that the existence of a nontrivial solution $\mu = Dw \in \mathbb{R}^n$ of (2.18) implies that the integral equation

$$\int_{-1}^{1} \ell(x) Q(x) T_\ell(x) \frac{dx}{\sqrt{1-x^2}} = 0 \tag{2.20}$$

holds for all polynomials Q of degree $\leq m-n$, where

$$\ell(x) = \prod_{i=1}^{n} (x - x_i)$$

denotes the supporting polynomial of the design $\eta^*_{\xi_{2\ell}}$. If this equivalence has been established it follows from the assumption $n \leq m$ that the polynomial $\ell(x)T_\ell(x)$ of degree $\ell + n$ is orthogonal to all polynomials $Q(x)$ of degree less or equal than $m-n$. Assume that $n \leq \frac{m-\ell}{2}$, then $\ell(x)T_\ell(x)$ would be of degree $\frac{m+\ell}{2}$ and $m-n$ would be bounded from below by $\frac{m+\ell}{2}$. Consequently we can use $Q(x) = \ell(x)T_\ell(x)$ in (2.20) which is impossible proving that $n \geq \frac{m-\ell}{2} + 1$.
In order to prove the remaining implication

$$(2.18) \quad \Rightarrow \quad (2.20)$$

assume that $\mu \in \mathbb{R}^n$ is a nontrivial solution of $F\mu = 0$, which means that there exist two linearly independent columns of the matrix F (note that $n \leq m$). Let $z, z_1, \ldots, z_{m-n}$ denote complex numbers and define an $(m+1) \times (m+1)$ matrix $B(z)$ by adding an additional row and $m-n+1$ additional columns, i.e.

$$B(z) = \begin{bmatrix} T_0(x_1) & \ldots & T_0(x_n) & T_0(z) & T_0(z_1) & \ldots & T_0(z_{n-m}) \\ T_1(x_1) & \ldots & T_1(x_n) & T_1(z) & T_1(z_1) & \ldots & T_1(z_{n-m}) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ T_m(x_1) & \ldots & T_m(x_n) & T_m(z) & T_m(z_1) & \ldots & T_m(z_{n-m}) \end{bmatrix}. \tag{2.21}$$

If $x_1, \ldots, x_n, z, z_1, \ldots, z_{m-n}$ are all distinct we have

$$\det B(z) \neq 0$$

because this determinant is proportional to a Vandermonde determinant based on these points. Assume that $z_1, \ldots, z_{m-n}$ are distinct and different from the x_i, then we obtain by Laplace's rule the representation

$$\det B(z) = \sum_{i=0}^{m} b_i T_i(z)$$

where $b_\ell = 0$ because the corresponding determinant obtained by deleting the $(n+1)$th column and $(\ell+1)$th row contains two linearly dependent columns [note that this determinant contains the matrix F defined in (2.19)]. Therefore the orthogonality (2.12) implies

$$\int_{-1}^{1} \det B(z) \cdot T_\ell(z) \frac{dz}{\sqrt{1-z^2}} = \sum_{i=0}^{m} b_i \int_{-1}^{1} T_i(z) T_\ell(z) \frac{dz}{\sqrt{1-z^2}} = 0. \tag{2.22}$$

On the other hand the definition (2.21) yields $\det B(z_i) = 0 \quad (i = 1, \ldots, m-n)$ and $\det B(x_i) = 0$ $(i = 1, \ldots, n)$ which shows that

$$\det B(z) = \prod_{i=1}^{n} (z - x_i) \prod_{i=1}^{m-n} (z - z_i) \; = \; \ell(z) Q(z) \; , \tag{2.23}$$

where the last equation defines the polynomial Q. Because $z_1, \ldots, z_{m-n}$ are arbitrary complex numbers the identities (2.22) and (2.23) imply that (2.20) holds for any polynomial of degree $m-n$, which completes the proof of Theorem 2.4.

$\square$

For a given support the optimal weights of a c-optimal design can be obtained by standard formulas [see e.g. Studden (1968) or Pukelsheim and Torsney (1991)]. The following formulas for the weights of the optimal designs for estimating individual coefficients are obtained from general results on quadrature formulas [see Stroud and Secrest (1966)] and provide an alternative and interesting representation for the weights of the e_k-optimal designs in the case of trigonometric regression. For the sake of simplicity we state these results only for the Chebyshev regression model (2.9), the situation for the model (2.10) is similar (see Theorem 5.5) and the trigonometric case is obtained by the transformation (2.4) (see the following sections).

Lemma 2.5 *Let $n \le m+1$ and*

$$\eta^* = \begin{pmatrix} x_1 & \ldots & x_n \\ w_1 & \ldots & w_n \end{pmatrix}$$

denote an e_ℓ-optimal design in the Chebyshev regression model (2.9) $(0 \le \ell \le m)$, *then the weights can be represented by the formula*

$$w_i = \frac{|A_i|}{\sum_{j=1}^{n} |A_j|} \qquad i = 1, \ldots, n \tag{2.24}$$

where the quantities A_i are given by

$$A_i = \int_{-1}^{1} \ell_i(x) T_\ell(x) \frac{dx}{\sqrt{1-x^2}} \qquad i = 1, \ldots, n \tag{2.25}$$

and

$$\ell_i(x) = \prod_{\substack{j=1 \\ i \neq j}}^{n} \frac{x - x_j}{x_i - x_j}$$

denotes the ith Lagrange interpolation polynomial with nodes $x_1, \ldots, x_n$.

Proof. The proof consists of two steps. At first we will show that for a given (distinct) support the space of solutions of the equation (2.18) is of dimension 1 and as a consequence the vector w is uniquely determined by normalization. Secondly, we will demonstrate that the weights given by (2.24) and (2.25) define such a solution.
For the proof of uniqueness assume that $\mu_1 = (\mu_{11}, \ldots, \mu_{1n})^T$ and $\mu_2 = (\mu_{21}, \ldots, \mu_{2n})^T$ are two linearly independent solutions of the equation $FD\mu = 0$ where the matrix F has been defined in (2.19). Let $\bar{F}$ denote the $(m+1) \times n$ matrix obtained from the $m \times n$ matrix F by adding the row $(T_\ell(x_1), \ldots, T_\ell(x_n))$ between the ℓth and $(\ell+1)$th row, then the identity (2.17) and $FD\mu_i = 0$ $(i = 1, 2)$ imply

$$\bar{F}D\mu_i = \frac{1}{h_i} \sum_{j=1}^{n} \mu_{ij} \cdot e_\ell \quad i = 1, 2.$$

Whenever $n \leq m+1$ the first n rows are linearly independent. If $\sum_{j=1}^{n} \mu_{ij} = 0$ for some $i \in \{1, 2\}$ this would imply $\mu_i = 0$ contradicting to the non triviality of these vectors. Consequently we have $\sum_{j=1}^{n} \mu_{ij} \neq 0$, $i = 1, 2$ which yields (by the same argument) that these vectors are linarly dependent. For this reason the dimension of the space of solutions of the equation (2.18) is one and the component of any nontrivial solution must be all of the same sign (because there exists at least one solution of $FD\mu = 0$ yielding the weights of the e_ℓ-optimal design).
For the second part we distinguish the cases $n = m+1$ and $n \leq m$. For the latter case note that by the proof of Theorem 2.4 the existence of a solution of the system (2.18) implies the identity

$$\int_{-1}^{1} Q(x)\ell(x)T_\ell(x)\frac{dx}{\sqrt{1-x^2}} = 0$$

for all polynomials Q of degreee $\leq m-n$. It then follows from standard results on quadrature formulas [see e.g. Stroud and Secrest (1966), p. 6] that there exists weights $\alpha_1, \ldots, \alpha_n$ such that the identity

$$\int_{-1}^{1} P(x)T_\ell(x)\frac{dx}{\sqrt{1-x^2}} = \sum_{j=1}^{n} \alpha_j P(x_j)T_\ell(x_j) \tag{2.26}$$

holds for all polynomials P of degree m. In other words the quadrature formula (2.26) is exact for all polynomials of degree m. Because $n \leq m$ we can use the Lagrange interpolation polynomials $\ell_1(x), \ldots, \ell_n(x)$ with nodes $x_1, \ldots, x_n$ in (2.26), which yields

$$A_i = \int_{-1}^{1} \ell_i(x)T_\ell(x)\frac{dx}{\sqrt{1-x^2}} = \alpha_i T_\ell(x_i) \quad i = 1, \ldots, n$$

and

$$(2.27) \qquad \int_{-1}^{1} P(x)T_\ell(x)\frac{dx}{\sqrt{1-x^2}} = \sum_{j=1}^{n} A_j P(x_j)$$

for all polynomials P of degree $\leq m$. Note that the A_i do not vanish simultaneously, because otherwise the left hand side of (2.27) would be zero, which is impossible. Moreover, from the orthogonality relation (2.12) for the Chebyshev polynomials of the first kind we have

$$0 = \int_{-1}^{1} T_i(x)T_\ell(x)\frac{dx}{\sqrt{1-x^2}} = \sum_{j=1}^{n} A_j T_i(x_j)$$

whenever $i \in \{0, \ldots, \ell-1, \ell+1, \ldots, m\}$. But this system is equivalent to $FA = 0$, where $A = (A_1, \ldots, A_n)^T$ and F is defined in (2.19), which proves that $D(A_1, \ldots, A_n)^T$ is a solution of (2.18). The assertion of Lemma 2.5 in the case $n \leq m$ now follows from Lemma 2.1 and the first part of this proof.
For the remaining case $n = m+1$ recall the definition of the matrix

$$\bar{F} = (T_i(x_{j+1}))_{i,j=0,\ldots,m} \in \mathbb{R}^{m+1 \times m+1}$$

and define $\bar{F}_i(x)$ as the matrix obtained from $\bar{F}$ by replacing the ith column by the vector $(T_0(x), \ldots, T_m(x))^T$. With these notations the Lagrange interpolation polynomials can be represented as

$$\ell_i(x) = \frac{\det \bar{F}_i(x)}{\det \bar{F}} \quad i = 1, \ldots, m+1$$

[note that $\bar{F}$ is nonsingular by the Chebyshev property of the polynomials $T_0(x), \ldots, T_m(x)$] and we obtain from the orthogonality (2.12)

$$A_i = \int_{-1}^{1} \ell_i(x)T_\ell(x)\frac{dx}{\sqrt{1-x^2}} = c_\ell \frac{\det \bar{F}_{i\ell}}{\det \bar{F}}$$

where $c_0 = \pi$, $c_\ell = \pi/2$ if $\ell \geq 1$ and the matrix $\bar{F}_{i\ell}$ is obtained from $\bar{F}$ deleting the ith column and $(\ell+1)$th row ($i = 1, \ldots, m+1; \ell = 0, \ldots, m$). It now follows from Corollary 8.9 in Pukelsheim (1993) that the quantities defined in (2.24) give the weights of the e_ℓ-optimal design.

□

3 Optimal designs on the full circle and the quadratic trigonometric regression model

In this section we will study the case of the full circle as design space in more detail and indicate that on arbitrary intervals the situation becomes extremely difficult. It turns out that for many but not for all cases optimal designs for estimating the individual coefficients in the trigonometric regression model (1.1) with design space $[-\pi, \pi]$ can be found explicitly.

Theorem 3.1 *Consider the trigonometric regression model (1.1) on the design space* $[-\pi, \pi]$.

(a) For any ℓ *such that* $m/3 < \ell \le m$ *and any* $\beta \in [0, \frac{1}{2\ell}]$ *the design*

$$(3.1) \qquad \xi_{2\ell}^* = \begin{pmatrix} -\pi & -\pi + \frac{\pi}{\ell} & \dots & -\pi + \frac{2\ell-1}{\ell}\pi & \pi \\ \frac{1}{2\ell} - \beta & \frac{1}{2\ell} & \dots & \frac{1}{2\ell} & \beta \end{pmatrix}$$

is optimal for estimating the parameter $\beta_{2\ell}$. *Moreover, in this case* $\Phi_{2\ell}(\xi_{2\ell}^*) = 1$.

(b) For any ℓ *such that* $m/3 < \ell \le m$ *the design* $\xi_{2\ell}^*$ *defined by (3.1) is optimal for the estimation of the intercept* β_0.

(c) For any ℓ *such that* $m/3 < \ell \le m$ *the design*

$$\xi_{2\ell-1}^* = \begin{pmatrix} -\pi + \frac{\pi}{2\ell} & -\pi + \frac{3\pi}{2\ell} & \dots & -\pi + \frac{2\ell-3}{2\ell}\pi & \frac{-\pi}{2\ell} + \pi \\ \frac{1}{2\ell} & \frac{1}{2\ell} & \dots & \frac{1}{2\ell} & \frac{1}{2\ell} \end{pmatrix}$$

is optimal for estimating the coefficient $\beta_{2\ell-1}$. *Moreover, in this case* $\Phi_{2\ell-1}(\xi_{2\ell-1}^*) = 1$.

Proof. We will only consider the first case (a), the remaining parts are treated similary. The proof follows essentially by an application of Lemma 2.1 and discrete orthogonality properties for the Chebyshev polynomials of the first kind. To be precise let $t_i = -\pi + \frac{i}{\ell}\pi$ $(i = 0, \dots, 2\ell)$ and consider the trigonometric polynomial $\varphi(t) = \cos(\ell t)$, which obviously satisfies condition (1) and (2) of Lemma 2.1 with $h = 1$. In order to prove the remaining condition (3) we have to establish the identities

$$(3.2) \qquad s_{2j} = \int_{-\pi}^{\pi} \varphi(t) f_{2j}(t) d\xi_{2\ell}^*(t) = \frac{1}{2\ell} \sum_{i=0}^{2\ell-1} \cos(jt_i)\cos(\ell t_i) = 0$$

for all $j = 0, 1, \dots, \ell-1, \ell+1, \dots, m$, and

$$(3.3) \qquad s_{2j-1} = \int_{-\pi}^{\pi} \varphi(t) f_{2j-1}(t) d\xi_{2\ell}^*(t) = \frac{1}{2\ell} \sum_{i=0}^{2\ell-1} \sin(jt_i)\cos(\ell t_i) = 0$$

for all $j = 1, \dots, m$. Note that the relation (3.3) is obvious by the symmetry of the design $\xi_{2\ell}^*$. For the quantities s_{2j} we obtain with the notation $x_i = \cos(t_i) = \cos(-t_i) = \cos(t_{2\ell-i})$ $i = 0, \dots, \ell$

$$s_{2j} = \frac{1}{2\ell} \{T_j(x_0)T_\ell(x_0) + T_j(x_\ell)T_\ell(x_\ell)\} + \frac{1}{\ell} \sum_{i=1}^{\ell-1} T_j(x_i)T_\ell(x_i).$$

Note that $x_0, \dots, x_\ell$ are the extremal points of the Chebyshev polynomial of the first kind and that orthogonality properties of these polynomials with respect to discrete measures [see Rivlin (1974), Exercise 1.5.28] show that $s_{2j} = 0$ if and only if for all $j \in \{0, \ldots, \ell - 1\} \cup \{\ell + 1, \ldots, m\}$ the quantities

$$\ell + j \quad \text{and} \quad |\ell - j|$$

are not multiples of 2ℓ. A simple calculation shows that this is obviously satisfied if $\ell > m/3$, which completes the proof of the first assertion of Theorem 3.1.

□

Example 3.2 It is interesting to note that in the case $\ell \leq m/3$ there exists an extremal polynomial for the design $\xi_{2\ell}^*$ such that the inequality (1) of Lemma 2.1 is satisfied (in other words the usual checking condition of the equivalence theorem is fulfilled), but the design $\xi_{2\ell}^*$ does not allow the estimability of the parameter $\beta_{2\ell}$ (because the identity (3) in Lemma 2.1 is not satisfied). For example, consider the case $m = 4$ and $\ell = 1$ for which a possible candidate for e_2-optimality is the design

$$\xi_2^* = \begin{pmatrix} -\pi & 0 & \pi \\ \frac{1}{2} - \beta & \frac{1}{2} & \beta \end{pmatrix}.$$

In this case the polynomial $\varphi(t) = \cos t + \cos(3t)$ satisfies condition (1) and (2) of Lemma 2.1 with $h = 1/4$ but does not satisfy (3). This corresponds to the non-estimability of the parameter β_2 by the design ξ_2^* [i.e. $e_2 \notin M(\xi_2^*)$]. A similar observation can be made in the design problem for the estimation of the parameters corresponding to the sine terms in the model (1.1).
We also mention the two cases not covered by Theorem 3.1 in this example. The optimal design for estimating the coefficient of $\cos(2t)$ in in the trigonometric regression model of degree 4 on the interval $[-\pi, \pi]$ is given by

$$\xi_2^* = \begin{pmatrix} -\frac{2\pi}{3} & -\frac{\pi}{3} & \frac{\pi}{3} & \frac{2\pi}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

with extremal polynomial $\varphi(t) = \sin t + 1/6 \sin(3t)$, while the optimal design for estimating the coefficient of $\sin t$ is given by

$$\xi_1^* = \begin{pmatrix} -\frac{5\pi}{6} & -\frac{\pi}{6} & \frac{\pi}{6} & \frac{5\pi}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

with extremal polynomial $\varphi(t) = \cos t - 1/6 \cos(3t)$. The optimality of these designs can verified by an direct application of Lemma 2.1.

Example 3.3 Consider the case $m = 2$ and the trigonometric regression model (1.1) on the design space $[-\pi, \pi]$. By part (a) of Theorem 3.1 the design

$$\xi_4^* = \begin{pmatrix} -\pi & -\frac{\pi}{2} & 0 & \frac{\pi}{2} & \pi \\ \frac{1}{4} - \beta & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \beta \end{pmatrix} \tag{3.4}$$

is optimal for estimating the coefficient of the term $\cos(2t)$ in (1.1) whenever $\beta \in [0, 1/4]$. Note that formally this design is not symmetric (except if $\beta = \frac{1}{8}$) but this non symmetry is artifical, because the boundary points $-\pi$ and π can be identified due to the 2π-periodicity of the regression functions. Similary, the design

$$\xi_2^* = \begin{pmatrix} -\pi & 0 & \pi \\ \frac{1}{2} - \beta & \frac{1}{2} & \beta \end{pmatrix} \tag{3.5}$$

is optimal for estimating the parameter of the term $\cos t$ whenever $\beta \in [0, 1/2]$ and the designs ξ_2^* and ξ_4^* are optimal for estimating the intercept in the trigonometric regression model (1.1). We also note that there are more optimal designs for estimating the intercept, e.g.

$$\xi_0^* = \begin{pmatrix} -\frac{2\pi}{3} & 0 & \frac{2\pi}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{3.6}$$

The β_0-optimality of the design ξ_0^* can be shown by a direct application of Lemma 2.1 observing that the corresponding blocks in (2.5) are given by $M_s(\xi_0^*) = 0 \in \mathbb{R}^{2\times 2}$ and

$$M_c(\xi_0^*) \;=\; \frac{1}{2} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Finally, by part (c) of Theorem 3.1 we obtain that the designs

$$\xi_3^* = \begin{pmatrix} -\frac{3\pi}{4} & -\frac{\pi}{4} & \frac{\pi}{4} & \frac{3\pi}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{3.7}$$

and

$$\xi_1^* = \begin{pmatrix} -\frac{\pi}{2} & \frac{\pi}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.8}$$

are optimal for estimating the coefficients of the terms $\sin(2t)$ and $\sin(t)$ in the quadratic regression model on the interval $[-\pi, \pi]$, respectively.

Example 3.4 The final example of this section will investigate the optimal designs for the quadratic trigonometric regression model (1.1) on the partial circle $[-a, a]$ indicating the particular difficulties caused by this restriction. We give a complete solution of this problem, where the results are obtained from the following sections. At this point the optimality of the particular designs can be directly verified through Lemma 2.1 (which is left to the reader) and the examples should serve as a motivation for the more technical considerations in the following sections.
It is easy to see that for $a \geq 2/3\pi$ the design given in (3.6) is optimal for estimating the intercept in the trigonometric regression of degree 2 on the interval $[-a, a]$. If $a \leq \frac{2}{3}\pi$ the

situation changes and the design

$$\xi_{0,a}^* = \begin{pmatrix} -a & t^* & 0 & t^* & a \\ w_2^* & w_1^* & w_0^* & w_1^* & w_2^* \end{pmatrix} \tag{3.9}$$

with

$$t^* = t^*(a) = \arccos(\cos(a)/2 + 1/2) \tag{3.10}$$

and

$$w_1^* = w_1^*(a) = \frac{1 + 2\cos a}{5 + 6\cos a + \cos^2 a}, \quad w_2^* = \frac{1 + (\cos a)/2}{5 + 6\cos a + \cos^2 a} \tag{3.11}$$

is e_0-optimal on the interval $[-a, a]$ (see Theorem 4.1 below).
The optimal design for the estimation of the coefficient of $\cos t$ on the interval $[-a, a]$ is obtained as

$$\xi_{2,a}^* = \begin{pmatrix} -a & -\pi + a & \pi - a & a \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{3.12}$$

if $\arccos(-1/3) \leq a \leq \pi$ (see Example 4.5 below) and as

$$\xi_{2,a}^* = \begin{pmatrix} -a & -t^* & 0 & t^* & a \\ w_2^* & \frac{1}{4} & w_0^* & \frac{1}{4} & w_2^* \end{pmatrix} \tag{3.13}$$

in the case $0 \leq a \leq \arccos(-1/3)$, where t^* is defined by (3.10) and the weights w_0^* and w_2^* are given by

$$w_2^* = \frac{1}{16} \frac{\cos a + 3}{\cos a + 1}, w_0^* = \frac{1}{2} - 2w_2^* \tag{3.14}$$

(see Theorem 4.1 below). Finally, the design for estimating the coefficient of $\cos(2t)$ on the interval$[-a, a]$ is given by

$$\xi_{4,a}^* = \begin{pmatrix} -a & -t^* & 0 & t^* & a \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \end{pmatrix} \tag{3.15}$$

where the point t^* is defined by (3.10) (see Example 4.4 below). We also note that the points $1 = \cos 0$, $x^* = \cos t^*$ and $\alpha = \cos a$ are the extremal points of the Chebyshev polynomial of the first kind

$$T_2 \left(\frac{2x - 1 - \alpha}{1 - \alpha} \right)$$

on the interval $[\alpha, 1]$ and demonstrate in the following section that this is a general property of $e_{2\ell}$-optimal designs for sufficiently small design spaces (see Theorem 4.1 below).
The description of the optimal designs for the estimation of the coefficients corresponding to the sine-terms is more complicated. Define e as the unique positive solution of the equation

$$e^4 + 2e^3 \cdot \cos a + e^2 \cdot \sin a - 2e \cdot \cos a - 1 = 0$$

and

$$t^* = t^*(a) = \arccos e. \tag{3.16}$$

If $\frac{\pi}{2} \le a \le \pi$ the optimal design for estimating the coefficient of $\sin t$ on the interval $[-a, a]$ is given by

$$\xi_{1,a}^* = \begin{pmatrix} -\frac{\pi}{2} & \frac{\pi}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.17}$$

(see Corollary 5.1 below) while for $0 < a \le \frac{\pi}{2}$ the design

$$\xi_{1,a}^* = \begin{pmatrix} -a & -t^* & t^* & a \\ \frac{1}{2} - w_1^* & w_1^* & w_1^* & \frac{1}{2} - w_1^* \end{pmatrix} \tag{3.18}$$

with t^* defined by (3.16) and

$$w_1^* = w_1^*(a) = \frac{1}{2} \frac{\cos(a)(\cos(a) - 1)(\cos(a) + 1)(\cos(a)e - 2e^2 + 1)}{(\cos(a) - e)(\cos(a)e^3 + (3 - 2\cos(a)^2)e^2 - 2\cos(a)e + \cos(a)^3 e + \cos(a)^2 - 2)} \tag{3.19}$$

is e_1-optimal on the interval $[-a, a]$ (see Theorem 5.3 below).
Similary, if $\frac{3}{4}\pi \le a \le \pi$ the design

$$\xi_{3,a}^* = \begin{pmatrix} -\frac{3\pi}{4} & -\frac{\pi}{4} & \frac{\pi}{4} & \frac{3\pi}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{3.20}$$

is optimal for estimating the coefficient of $\sin(2t)$ in the quadratic trigonometric regression on the interval $[-a, a]$ (see Corollary 5.1 below), while for $0 < a \le \frac{3}{4}\pi$ the e_3-optimal design is of the form (3.18) with t^* given by (3.16) and weight w_1^* defined by

$$w_1^* = w_1^*(a) = \frac{1}{2} \frac{(\cos(a) - 1)(\cos(a) + 1)(e\cos(a) + 1 - 2e^2)}{(\cos(a) - e)(\cos(a) + e\cos(a)^2 - e^2\cos(a) - e^3)} \tag{3.21}$$

(see Theorem 5.5 below).

4 Optimal designs for estimating individual coefficients of cosine terms on a partial circle

In this section we investigate $e_{2\ell}$-optimal design for $(0 \le \ell \le m)$ for the trigonometric regression model (1.1) with design space $[-a, a]$ in more detail. It is demonstrated that there exists a point, say $a_\ell^* \in (0, \pi]$, such that for all $a \le a_\ell^*$ the optimal design for estimating the parameter $\beta_{2\ell}$ in the trigonometric regression on the interval $[-a, a]$ can be found explicitly. Our second result gives a lower bound for a_ℓ^*, while it is indicated at the end of this section that an explicit solution of the $e_{2\ell}$-optimal design problem for any value of a satisfying $a_\ell^* < a \le \pi$ can only be expected in particular cases.

Theorem 4.1 *Consider the trigonometric regression model (1.1) on the interval* $[-a, a]$ *and let*

$$t_i = t_i(a) = \arccos\Big\{\frac{1-\alpha}{2}\cos\frac{i\pi}{m} + \frac{1+\alpha}{2}\Big\} \quad i = 0, \ldots, m, \tag{4.1}$$

$x_i = \cos t_i$ $(i = 0, \ldots, m)$ *denote the extremal points of the* m*th Chebyshev polynomial*

$$T_m\Big(\frac{2x-1-\alpha}{1-\alpha}\Big)$$

of the first kind on the interval $[\alpha, 1] = [\cos a, 1]$, *define weights*

$$w_0 = \frac{A_0}{\sum_{j=1}^m A_j}; \quad w_i = \frac{A_i}{2\sum_{j=1}^m A_j} \quad i = 1, \ldots, m \tag{4.2}$$

where

$$A_i = (-1)^{m-\ell+i} \cdot \int_{-1}^{1} \ell_i(x) T_\ell(x) \frac{dx}{\sqrt{1-x^2}} \quad i = 0, \ldots, m \tag{4.3}$$

and $\ell_i(x)$ *denotes the* i*th Lagrange interpolation polynomial with nodes* $x_i = \cos t_i$ $(i = 0, \ldots, m)$. *For any* $\ell \in \{0, \ldots, m\}$ *the quantity*

$$a^*_\ell = a^*_{\ell,m} = \sup\{a \in (0, \pi] \mid A_i > 0 \quad \textit{for all} \;\; i = 0, \ldots, m\} \tag{4.4}$$

is always positive and the design

$$\xi^*_{2\ell,a} = \begin{pmatrix} -t_m & \ldots & -t_1 & t_0 & t_1 & \ldots & t_m \\ w_m & \ldots & w_1 & w_0 & w_1 & \ldots & w_m \end{pmatrix} \tag{4.5}$$

is optimal for estimating the parameter $\beta_{2\ell}$ *in the trigonometric model (1.1) on the interval* $[-a, a]$, *whenever* $a \leq a^*_\ell$.

Proof. In a first step we will prove that if the parameter a approaches 0 the quantities A_i defined in (4.3) are all positive. To this end let $s_i = \cos(\frac{i\pi}{m})$ denote the extremal points of the Chebyshev polynomial of the first kind $T_m(x)$. Following Sahm (1998) we have

$$\begin{aligned} 2^{m-1}\prod_{\substack{j=0 \\ j\neq i}}^{m}(s_i - s_j) &= \frac{d}{dx}(x^2-1)U_{m-1}(x)\Big|_{x=s_i} \\ &= \frac{d}{dx}(T_{m+1}(x) - xT_m(x))\Big|_{x=s_i} \\ &= (m+1)U_m(s_i) - ms_iU_{m-1}(s_i) - T_m(s_i) = \frac{(-1)^i}{\gamma_i} \end{aligned}$$

where $\gamma_0 = \gamma_m = 1/(2m)$ and $\gamma_i = 1/m$ if $1 \leq i \leq m-1$. For the derivation of this identity we used the well known facts [see Szegö (1959)] $T'_k(x) = kU_{k-1}(x)$, $U_m(s_i) = (-1)^i$

if $1 \le i \le m-1$ and $U_k(1) = (-1)^k U_k(-1) = k+1$. This implies for the weights in (4.3)

$$\begin{aligned} A_i &= (-1)^{m-\ell+i} \cdot \int_{-1}^{1} \prod_{\substack{j=0 \\ j\neq i}}^{m} \frac{x - x_j}{x_i - x_j} T_\ell(x) \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{2^{m-1}(-1)^{m-\ell}}{(1-\alpha)^m} \gamma_i \int_{-1}^{1} \prod_{\substack{j=0 \\ j\neq i}}^{m} (2x - 1 - \alpha - \{1-\alpha\} s_j) \cdot T_\ell(x) \frac{dx}{\sqrt{1-x^2}}. \end{aligned}$$

This means that for $a \to 0$ (which implies $\alpha = \cos a \to 1$) we have

$$\begin{aligned} \lim_{a\to 0} (1-\alpha)^m A_i &= 2^{2m-1}(-1)^{m-\ell} \gamma_i \int_{-1}^{1} (x-1)^m T_\ell(x) \frac{dx}{\sqrt{1-x^2}} \\ &= \gamma_i 2^{2m-1} \int_{-1}^{1} (1+x)^m T_\ell(x) \frac{dx}{\sqrt{1-x^2}} \\ &= \gamma_i 2^{3m-1} \frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} \frac{P_\ell^{(m,-m-1)}(-1)}{P_\ell^{(m,-m-1)}(1)} \\ &= \gamma_i 2^{3m-1} \frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+1-\ell)} \frac{\Gamma(m+1)}{\Gamma(m+1+\ell)} \end{aligned}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the nth Jacobi polynomial [see e.g. Szegö (1959)], the second equality follows by the substitution $x \to -x$ and $T_\ell(-x) = (-1)^\ell T_\ell(x)$, the third equality is obtained from the identity (4.10.11) in Szegö (1959) and the final equality is a consequence of the relations $P_\ell^{(\alpha,\beta)}(1) = \Gamma(\ell+\alpha+1)/\{\Gamma(\alpha+1)\Gamma(\ell+1)\}$ and $P_\ell^{(\alpha,\beta)}(-x) = P_\ell^{(\beta,\alpha)}(x)(-1)^\ell$ [see formula (4.1.1) and (4.1.3) in the same reference]. Consequently, if $a \to 0$ all quantities A_i defined in (4.3) are positive and by continuity the supremum a_ℓ^* defined by (4.4) is also positive.

For the proof of the second assertion of Theorem 4.1 recall that by Lemma 2.2 the $e_{2\ell}$-optimality of the design $\xi_{2\ell,a}^*$ defined by (4.5) in the trigonometric regression model (1.1) is equivalent to e_ℓ-optimality of the design

$$\eta_{\xi_{2\ell,a}^*} = \begin{pmatrix} x_0 & x_1 & \dots & x_m \\ w_0 & 2w_1 & \dots & 2w_m \end{pmatrix}$$

in the Chebyshev regression model (2.9) on the interval $[\cos a, 1]$. We will now use Lemma 2.1 to establish this optimality. To this end assume that $a_\ell^* > a$ and define

$$\varphi(x) = T_m\Big(\frac{2x-1-\alpha}{1-\alpha}\Big) \cdot \varphi_\ell = \sum_{j=0}^{m} b_j T_j(x) \tag{4.6}$$

where the coefficient φ_ℓ is defined by the condition that the coefficient b_ℓ of $T_\ell(x)$ in the above expansion of φ equals one and $\alpha = \cos a$. This polynomial obviously satisfies the condition (1) and (2) of Lemma 2.1 with $h = 1/\varphi_\ell$. As demonstrated in the proof of

Theorem 2.4 the condition (3) of this Lemma is equivalent to the existence of a solution of the equation

$$FDw = 0 \tag{4.7}$$

with positive coefficients, where $D = \operatorname{diag}(1, -1, \ldots, (-1)^m)$ and the matrix F is defined by (2.19). However, it is demonstrated in the second part of the proof of Lemma 2.5, that the vector $(A_0, -A_1, \ldots (-1)^m A_m)$ is always a solution of $F\mu = 0$. Because $A_i > 0\,(i = 0, \ldots, m)$ whenever $a < a_\ell^*$ it follows that the vector $\tilde{w} = (\tilde{w}_0, \ldots, \tilde{w}_m)^T$ with

$$\tilde{w}_i = \frac{A_i}{\sum_{j=0}^m A_j}, \qquad i = 0, \ldots, m$$

is a solution of (4.7) with positive coefficients. Consequently, by Lemma 2.1 the design

$$\eta_{2\ell}^* = \begin{pmatrix} x_0 & x_1 & \ldots & x_m \\ \tilde{w}_0 & \tilde{w}_1 & \ldots & \tilde{w}_m \end{pmatrix}$$

is optimal for estimating the parameter δ_ℓ in the Chebyshev regression model (2.9). The assertion now follows from the above discussion, which shows that the design $\xi_{2\ell,a}^*$ defined in (4.5) is optimal for estimating the coefficient $\beta_{2\ell}$ in the trigonometric regression (1.1) on the interval $[-a, a]$, whenever $a < a_\ell^*$. The remaining assertion for $a = a_\ell^*$ follows by continuity.

□

Corollary 4.2 *If $a \to 0$ the optimal design $\xi_{2\ell,a}^*$ for estimating the parameter $\beta_{2\ell}$ in the trigonometric regression model (1.1) on the interval $[-a, a]$ converges weakly in the following sense*

$$\lim_{a \to 0} \xi_{2\ell,a}^*([-a, at]) = \xi^*([-1, t]) \quad \forall\, t \in [-1, 1]$$

where the design ξ^* is defined by

$$\xi^* = \begin{pmatrix} -y_m & -y_{m-1} & \ldots & -y_1 & y_0 & y_1 & \ldots & y_{m-1} & y_m \\ \frac{1}{4m} & \frac{1}{2m} & \ldots & \frac{1}{2m} & \frac{1}{2m} & \frac{1}{2m} & \ldots & \frac{1}{2m} & \frac{1}{4m} \end{pmatrix}$$

and the points $y_0, \ldots, y_m$ are given by

$$y_i = \cos\Big(\frac{\pi(m-i)}{2m}\Big) \qquad i = 0, \ldots, m.$$

Proof. The assertion for the weights follows from the transformation (2.4) and the first part of the proof of Theorem 4.1, which shows that for $a \to 0$ the weights in the corresponding design problem in the Chebyshev regression model (2.9) satisfy

$$\lim_{a \to 0} \tilde{\omega}_i = \begin{cases} \frac{1}{2m} & \text{if } i = 0, m \\ \frac{1}{m} & \text{if } i = 1, \ldots, m-1\,. \end{cases}$$

	$m=2$	$m=3$	$m=4$	$m=5$
1	$2\pi/3$	0.6881π	0.7411π	0.7666π
$\cos(x)$	0.6082π	0.7323π	0.7311π	0.7765π
$\cos(2x)$	π	$2\pi/3$	0.7576π	0.7598π
$\cos(3x)$		π	0.7048π	0.7709π
$\cos(4x)$			π	0.7323π
$\cos(5x)$				π

Table 4.1. *Critical values a_ℓ^*. defined in (4.4) for various values of ℓ and m. The optimal design for estimating the coeeficient $\beta_{2\ell}$ in the trigonometric regression model (1.1) on the interval $[-a, a]$ is of the form (4.5), whenever $a \le a_\ell^*$.*

The proof is completed by showing that the scaled support points $\tilde{t}_i(a) = t_i(a)/a$ defined in (4.1) satisfy

$$\lim_{a\to 0} \tilde{t}_i(a) \;=\; \lim_{a\to 0} \frac{t_i(a)}{a} \;=\; y_i = \cos\Big(\frac{\pi(m-i)}{2m}\Big)\,, \quad i = 0, \dots, m.$$

To this end we use the expansion $\cos x = 1 - x^2/2 + o(x^2)$ and obtain from (4.1) for $a \to 0$

$$1 - \frac{(\tilde{t}_i(a)a)^2}{2} + o(a^2) = \cos t_i(a) = 1 - \frac{a^2}{4}(1 - \cos\frac{i\pi}{m}) + o(a^2)$$

which gives by the identity $\cos 2x = 2\cos^2 x - 1$

$$\begin{aligned} \tilde{t}_i^2(a) &= \frac{1}{2}(1 - \cos\frac{i\pi}{m}) + o(a^2) = \frac{1}{2}(1 + \cos\frac{\pi(m-i)}{m}) + o(a^2) \\ &= \Big(\cos\frac{\pi(m-i)}{2m}\Big)^2 + o(a^2) \qquad i = 0, \dots, m \end{aligned}$$

and proves the assertion.

$\square$

The critical bound a_ℓ^* can be determined numerically from (4.4) and (4.3) by standard numerical integration. Table 4.1 gives some of the critical points a_ℓ^* obtained from Theorem 4.1 for various degrees of the trigonometric regression model. Note that Theorem 4.1 covers a relatively large range of the interval $[0, \pi]$. The following theorem shows that the critical bound of (4.4) is at least $\pi/2$ independent of the parameter which has to be estimated.

Theorem 4.3 *Let x_ℓ^* denote the smallest zero of the polynomial*

$$(4.8) \qquad (\frac{d}{dx})^\ell\{(x+1)U_{m-1}(x)\} \qquad \ell = 1, \dots, m-1$$

and $x_m^* = 0$ *then for any* $\ell \in \{1, \dots, m\}$ *the critical value* a_ℓ^* *defined in (4.4) satisfies*

$$a_\ell^* \geq a_\ell^{**} = \arccos \frac{x_\ell^* + 1}{x_\ell^* - 1}.$$

In particular we have $a_\ell^* \geq a_\ell^{**} > \pi/2$ *for all* $\ell \in \{1, \dots, m\}$ *and for any fixed* ℓ *we have*

$$\lim_{m \to \infty} a_\ell^{**} = \frac{\pi}{2}.$$

Proof. Note that by Lemma 2.3 a design $\xi_{2\ell}^*$ is $e_{2\ell}$-optimal in the trigonometric regression if and only if the measure $\eta_{\xi_{2\ell}^*}$ induced by the transformation (2.4) is $t(\ell)$-optimal in the ordinary polynomial regression on the interval $[\alpha, 1]$, where $t(\ell) = T^{-1}e_\ell$ and T denotes the matrix of coefficients of the Chebyshev polynomial of the first kind defined by (2.12). Let $t_{i,j}$ denote the entries of the matrix T^{-1}, then it follows by Cramer's rule that

$$\begin{aligned} t_{i,j} &= 0 \quad \text{whenever} \quad i + j \text{ is odd} \\ t_{i,j} &= 0 \quad \text{whenever } i < j. \end{aligned}$$

Moreover, the nonvanishing coefficients in the Chebyshev-expansions of the monomials

$$x^k = \sum_{j=0}^{k} \eta_{k,j} T_j(x) \qquad k = 0, \dots, m$$

are all positive [see e.g. Rivlin (1974), Exercise 1.5.32] and as a consequence the vector $t(\ell) = T^{-1}e_\ell$ can be written as

$$t(\ell) = \sum_{j=0}^{\lfloor \frac{m-\ell}{2} \rfloor} \alpha_{\ell,j} e_{\ell+2j} \tag{4.9}$$

with positive coefficient $\alpha_{\ell,j}$ ($j = 0, \dots, \lfloor \frac{m-\ell}{2} \rfloor; \ell = 0, \dots, m$). We will now investigate the $e_{\ell+2j}$-optimal designs in ordinary polynomial regression using recent results of Sahm (2000). Note that the design space, which has to be considered, is the interval $[\alpha, 1]$, where $\alpha \to 1$ as $a \to 0$. Sahm (2000) showed that the structure of the optimal design for estimating the ith coefficient in an ordinary polynomial regression on the interval $[\alpha, 1]$ is determined by the symmetry parameter $s(\alpha) = (\alpha+1)/(\alpha-1)$. In particular he proved that the e_i-optimal design for the ordinary polynomial regression is supported at the transformed Chebyshev points

$$x_j = \cos(t_j) = \frac{1-\alpha}{2} \cos(\frac{j\pi}{m}) + \frac{1+\alpha}{2} \qquad j = 0, \dots, m \tag{4.10}$$

whenever

$$s(\alpha) = \frac{\alpha + 1}{\alpha - 1} < x_i^*,$$

where x_i^* is the smallest zero of the polynomial

$$(\frac{d}{dx})^i\{(x+1)U_{m-1}(x)\}\ ,$$

if $0 \leq i \leq m-1$ and $x_m^* = 0$. Now it is easy to see that the zeros of the polynomials

$$(\frac{d}{dx})^i\{(x+1)U_{m-1}(x)\}$$

and

$$(\frac{d}{dx})^{i+1}\{(x+1)U_{m-1}(x)\}$$

are interlacing and consequently we have for the smallest roots of these polynomials

$$x_i^* < x_j^*$$

whenever $i < j$. This implies that whenever

$$a < a_\ell^{**} = \arccos \frac{x_\ell^* + 1}{x_\ell^* - 1}$$

we have

$$a < \arccos \frac{x_i^* + 1}{x_i^* - 1} \quad i = \ell, \ell+1, \ldots, m$$

and consequently it follows from Sahm (2000) that in this case for all $i = \ell, \ell+1, \ldots, m$ the e_i-optimal designs in the ordinary polynomial regression on the interval $[\alpha, 1]$ are supported at the points in (4.10). We will now prove that the $t(\ell)$-optimal design in the ordinary polynomial regression [which is the e_ℓ-optimal design for the Chebyshev regression (2.9)] is also supported at the full set of Chebyshev points defined in (4.10), whenever $a < a_\ell^{**}$. If this assertion has been proved we obtain from Lemma 2.5 that the weights of the design $\eta_{\xi_{2\ell}^*}$ are given by (2.24) and (2.25), which implies that the quantities

$$A_i = (-1)^{m-\ell+i} \int_{-1}^{1} \ell_i(x) T_\ell(x) \frac{dx}{\sqrt{1-x^2}}$$

defined in (4.3) are positive for all $a \in (0, a_\ell^{**})$. This follows because by the first part of this proof the quantities A_i are positive if $a \to 0$ and they have to be of the same sign, because we will prove below that for all $a \in (0, a_\ell^{**})$ the design $\eta_{\xi_{2\ell}^*}$ is supported at the full set of Chebyshev points.
To this end recall that for $0 < a < a_\ell^{**}$ the $e_{\ell+2j}$-optimal design ($j = 0, \ldots, \lfloor \frac{m-\ell}{2} \rfloor$) is supported at the Chebyshev points defined in (4.10) with extremal polynomial given by (4.6). Lemma 2.1 for the vector $f(x) = (1, x, \ldots, x^m)^T$ shows that the corresponding vector of optimal weights $w^j = (w_0^j, \ldots, w_m^j)^T$ satisfies

$$G_j D w^j = 0\ , \qquad j = 0, \ldots, \lfloor \frac{m-\ell}{2} \rfloor\ ,$$

where the matrix D is given by $D = (-1)^{m-\ell} \cdot \mathrm{diag}(1, -1, \ldots, (-1)^m)$ and the matrix G_j is obtained from the matrix

$$\bar{G} = (x_j^i)_{i,j=0,\ldots,m}$$

by deleting the $(\ell + 2j + 1)$th row. The condition (2) of the same Lemma implies for some $h_j > 0$

$$\bar{G} D w^j = \frac{1}{h_j} e_{\ell+2j} \ , \qquad j = 0, \ldots, \lfloor \frac{m-\ell}{2} \rfloor \ ,$$

which yields that for all $j \in \{0, \ldots, \lfloor \frac{m-j}{2} \rfloor\}$ the $(k+1)$th component of the vector

$$\bar{G}^{-1} e_{\ell+2j}$$

is nonzero and has sign $(-1)^{k+\ell+m}$ (by the pattern of the diagonal elements of the matrix D). Introducing the notation [see Studden (1968)]

$$D_\nu(c) = \begin{vmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 & c_0 \\ x_0 & \ldots & x_{\nu-1} & x_{\nu+1} & \ldots & x_m & c_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_0^m & \ldots & x_{\nu-1}^m & x_{\nu+1}^m & \ldots & x_m^m & c_m \end{vmatrix} \qquad \nu = 0, \ldots, m$$

for a vector $c = (c_0, \ldots, c_m) \in \mathbb{R}^{m+1}$, we obtain for this component the representation

$$(4.11) \quad 0 \neq e_k^T \bar{G}^{-1} e_{\ell+2j} = (-1)^{m-k} \frac{D_k(e_{\ell+2j})}{\det \bar{G}} \qquad j = 0, \ldots, \lfloor \frac{m-\ell}{2} \rfloor, k = 0, \ldots, m$$

and consequently $D_k(e_{\ell+2j})$ has sign $(-1)^\ell$ for all $j = 0, \ldots, \lfloor \frac{m-\ell}{2} \rfloor$. Therefore it follows from the representation (4.9)

$$D_k(t(\ell)) = \sum_{j=0}^{\lfloor \frac{m-\ell}{2} \rfloor} \alpha_{\ell,j} D_k(e_{\ell+2j}) \neq 0$$

for all $k = 0, \ldots, m$ and the results of Studden (1968) show that the $t(\ell)$-optimal design in the ordinary polynomial regression is supported on the full set of Chebyshev points defined by (4.10), whenever $0 < a < a_\ell^{**}$. By the discussion at the beginning of this paragraph the quantities A_i defined in (4.3) are all positive for $a \in (0, a_\ell^{**})$ which implies $a_\ell^{**} \leq a_\ell^*$ and completes the proof of the first part of the theorem.
For the second part we note that all zeros of the polynomial $(x+1)U_{m-1}(x)$ are real and located in the interval $[-1, 1]$ [see e.g. Szegö (1959)] and consequently the roots of the ℓth derivative have the same property, which implies $x_\ell^* > -1$ or equivalently $a_\ell^{**} > \arccos 0 = \pi/2$. Similary the roots of $(x+1)U_{m-1}(x)$ become dense in the interval $[-1, 1]$ as $m \to \infty$ [see Van Assche (1987)] and by of the interlacing property the zeros of the ℓth derivative have the same property, which implies for any fixed ℓ

$$\lim_{m\to\infty} a_\ell^{**} = \lim_{m\to\infty} \arccos \frac{x_\ell^* + 1}{x_\ell^* - 1} = \arccos 0 = \frac{\pi}{2}$$

and completes the proof of the theorem.

$\square$

Example 4.4 Consider the case of estimating the coefficient of the highest cosine term, i.e. $\ell = m$ in Theorem 4.3. In this case the polynomial defined in (4.8) is constant, which implies $a_m^{**} = a_m^* = \pi$, and the e_{2m}-optimal design in the trigonometric regression model on the interval $[-a, a]$ is given by (4.5) for any $a \in (0, \pi]$. Moreover, the weights of the e_{2m}-optimal design can be found explicitly by a careful inspection of the proof of Theorem 4.3, which shows that this design is obtained from the $t(m)$-optimal design in an ordinary polynomial regression on the interval $[\alpha, 1]$ with $\alpha = \cos a$. The representation (4.9) shows that the vectors $t(m)$ and e_m are linearly dependent and consequently this design is the D_1-optimal design in an ordinary polynomial regression on the interval $[\alpha, 1]$. The D_1-optimal design for polynomial regression has been determined by many authors on the interval $[-1, 1]$ [see e.g. Studden (1980, 1982) or Spruill (1990)]. Because this problem is invariant under affine transformations the $t(m)$-optimal design in the ordinary polynomial regression puts masses $\frac{1}{2m}, \frac{1}{m}, \ldots, \frac{1}{m}, \frac{1}{2m}$ at the points $x_0, x_1, \ldots, x_m$ where $x_i = \cos t_i$ $(i = 0, \ldots, m)$ and the nodes t_i are defined by (4.1). Observing the transformation (2.4) it follows that for any $a \in (0, \pi]$ an optimal design for estimating the coefficient β_{2m} in the trigonometric regression model on the interval $[-a, a]$ is given by

$$\xi_{2m}^* = \begin{pmatrix} -t_m & -t_{m-1} & \ldots & -t_1 & t_0 & t_1 & \ldots & t_{m-1} & t_m \\ \frac{1}{4m} & \frac{1}{2m} & \ldots & \frac{1}{2m} & \frac{1}{2m} & \frac{1}{2m} & \ldots & \frac{1}{2m} & \frac{1}{4m} \end{pmatrix},$$

where the support points t_i are defined in (4.1).

Example 4.5 Consider the case of estimating the coefficient β_{2m-2} in the trigonometric regression model (1.1), that is $\ell = m-1$ in Theorem 4.3. In this case we have by induction

$$(\frac{d}{dx})^{m-1}\{(x+1)U_{m-1}(x)\} = (m-1)! \cdot \{1 + mx\}$$

which gives $x_{m-1}^* = -\frac{1}{m}$ and

$$a_{m-1}^{**} = \arccos \frac{1-m}{m+1}.$$

Consequently, the e_{2m-2}-optimal design is supported at the points defined in (4.1) whenever

$$0 < a \leq \arccos(\frac{1-m}{1+m})$$

[see also formula (3.10) in Example 3.4, where the case $m = 2$ is considered].
In this case we are also able to find the optimal designs for $a > a_{m-1}^{**}$ using the recent results of Sahm (2000) and the arguments given in the proof of Theorem 4.3. More precisely, the e_{2m-2}-optimal design for the trigonometric regression on the interval $[-a, a]$ is obtained from the $t(m-1)$-optimal design in the ordinary polynomial regression on $[\alpha, 1]$. Formula

(4.9) shows that this problem is equivalent to the e_{m-1}-optimal design problem in the same model. Theorem 3.2b) of Sahm (2000) shows that in the case

$$\frac{\alpha+1}{\alpha-1} \in [-\frac{1}{m}, -\frac{1}{m} + (1-\frac{1}{m})\frac{1-\cos(\pi/m)}{1+\cos(\pi/m)}]$$

the optimal design is supported at the points

$$x_j = \alpha\frac{1+m\cos(\frac{j\pi}{m})}{1-m} \quad j = 1, \ldots, m \tag{4.12}$$

(the formula for the corresponding weights is omitted for the sake of brevity). The transformation (2.4) shows that for $a \in [\arccos\frac{1-m}{m+1}, \arccos\frac{\alpha^*+1}{\alpha^*-1}]$ the e_{2m-2}-optimal design in the trigonometric regression (1.1) on the interval $[-a, a]$ has only $2m$ support points

$$-t_m - t_{m-1}, \ldots, -t_1, t_1, \ldots, t_m,$$

where $t_j = \arccos x_j$ $(j = 1, \ldots, m)$ and

$$\alpha^* = -\frac{1}{m} + (1-\frac{1}{m})\frac{1-\cos(\pi/m)}{1+\cos(\pi/m)}. \tag{4.13}$$

Consider as a concrete example the case $m = 2$ discussed in Section 3, where $\alpha^* = 0$ and $x_1 = -\alpha$ $x_2 = \alpha$. Here a symmetric e_2-optimal design in the trigonometric regression of degree $m = 2$ on the interval $[-a, a]$ is supported at the four points $-a, -\pi + a, \pi - a, a$, whenever $\arccos(-1/3) \leq a \leq \pi$, as claimed in formula (3.12) of Example 3.4. In the general case $m \geq 3$ it follows that $\alpha^* < 0$ and a third case appears, for which the solution of the optimal design problem in the corresponding polynomial regression cannot be found explicitly [see Sahm (2000)]. In this case the optimal designs for estimating the coefficient β_{2m-2} in the trigonometric regression on the interval $[-a, a]$ is supported on $2m$ points (including the points $-a$ and a) and can be obtained by the methods introduced in Dette, Melas and Pepelysheff (2000) and the transformation (2.4).

5 Optimal designs for estimating individual coefficients of the sine terms on a partial circle

In this section we concentrate on the optimal design problem for the estimation of the individual coefficients corresponding to the sine terms in the trigonometric regression model (1.1). Example 3.4 already indicates that the situation for this case is substantially more difficult. Moreover, it also indicates that the e_1- and e_3-optimal design for the quadratic trigonometric model have the same support points, whenever $a \leq \frac{\pi}{2}$. One of our main results of this section shows that this property is also true for general degree $m \geq 2$. In other words, if a is reasonable small (which will be made precise later) the support points of the e_{2m-1}-optimal design in the trigonometric regression model (1.1) on the interval $[-a, a]$ coincide with the support points of $e_{2\ell-1}$-optimal design for any $\ell \in \{1, \ldots, m\}$.

For this reason we will start our investigations of the sine case with a careful discussion of the optimal design problem for the estimation of the parameter β_{2m-1} in the trigonometric regression model (1.1). In this case we use the implicit function theorem to determine the optimal design, a technique, which was introduced by Melas (1978) in the context of optimal design. Our first result is an immediate consequence of Theorem 3.1 and Lemma 2.1.

Corollary 5.1 *Let $m/3 < \ell \leq m$ and $\pi(1 - 1/2\ell) \leq a \leq \pi$, then the optimal design for estimating the coefficient $\beta_{2\ell-1}$ in the trigonometric regression model (1.1) on the interval $[-a, a]$ is given by the design $\xi^*_{2\ell-1}$ defined in part c) of Theorem 3.1.*

In the following we consider the optimal problem for the estimation of the parameter β_{2m-1} and study the case $0 < a \leq \pi(1 - 1/2m)$ for which the e_{2m-1}-optimal design problem is equivalent to the e_m-optimal design problem in the Chebyshev regression model (2.10) on the interval $[\alpha, 1]$. In this case the function φ in Lemma 2.1 is of the form

$$\varphi_m(x) \;=\; \sqrt{1-x^2}(U_{m-1}(x) + b_{m-2}U_{m-2}(x) + \ldots b_1U_1(x) + b_0) \tag{5.1}$$

and it follows from Threom 2.4 that the e_m-optimal design $\eta_{\xi^*_{2m-1,a}}$ has m support points, including the boundary point α. Morover, the point $x = 1$ cannot be a support point of the design $\eta_{\xi^*_{2m-1,a}}$, because this point is a root of the extremal polynomial φ. Therefore, by the transformation (2.4), the optimal design for estimating the parameter β_{2m-1} in the trigonometric model is of the form

$$\xi_{(\tau,w)} = \begin{pmatrix} -a & at_2 & \ldots & at_m & -at_m & \ldots & -at_2 & a \\ \frac{1}{2}w_1 & \frac{1}{2}w_2 & \ldots & \frac{1}{2}w_m & \frac{1}{2}w_m & \ldots & \frac{1}{2}w_2 & \frac{1}{2}w_1 \end{pmatrix} \tag{5.2}$$

where

$$\begin{aligned} w \in V \;&:=\; \Big\{ w = (w_1, \ldots, w_m)^T \mid w_i > 0; \sum_{j=1}^m w_j = 1 \Big\} \\ \tau \in T \;&:=\; \Big\{ t = (t_2, \ldots, t_m)^T \mid -1 < t_2 < \ldots < t_m < 0 \Big\}. \end{aligned}$$

For $w \in V$ and $\tau \in T$ define $\alpha = x_1 < x_2 < \ldots < x_m < 1$ by $x_i = x_i(\tau) = \cos(at_i)$ $(i = 2, \ldots, m)$ and

$$\eta(\tau, w) = \begin{pmatrix} \alpha & x_2 & \ldots & x_m \\ w_1 & w_2 & \ldots & w_m \end{pmatrix} \tag{5.3}$$

as the design obtained from the measure $\xi(\tau, w)$ by the transformation (2.4), then a straightforward calculation and an application of the Cauchy–Binet formula show

$$\begin{aligned} e_m^T M_s^{-1}(\xi(\tau, w))e_m \;&=\; \frac{\det M_s(\xi(\tau, w))_-}{\det M_s(\xi(\tau, w))} \\ &= \sum_{i=1}^m \frac{1}{(1 - x_i^2)\prod_{j \neq i}(x_j - x_i)^2 w_i} \end{aligned} \tag{5.4}$$

where $x_1 = \alpha$, $A\_ \in \mathbb{R}^{m-1\times m-1}$ denotes the matrix obtained from $A \in \mathbb{R}^{m\times m}$ by deleting the mth row and column. Consequently for given points $x_1, \ldots, x_m$ the optimal weights are obtained as

$$w_i^* = w_i^*(a) = \frac{(-1)^{m-i}\{\sqrt{1-x_i^2}\prod_{j\neq i}(x_i - x_j)\}^{-1}}{\sum_{k=1}^m (-1)^{m-k}\{\sqrt{1-x_k^2}\prod_{j\neq k}(x_k - x_j)\}^{-1}} \quad , \tag{5.5}$$

while the value of the criterion (5.4) for the optimal weights $w^* = (w_1^*, \ldots, w_m^*)$ is given by

$$\psi(x,a) := e_m^T M_s^{-1}(\xi(\tau, w^*))e_m = \left[\sum_{i=1}^m (-1)^{m-i}\{\sqrt{1-x_i^2}\prod_{j\neq i}(x_i - x_j)\}^{-1}\right]^2 . \tag{5.6}$$

[see also Pukelsheim (1993), Section 8.9]. A tedious calculation shows that this function is strictly convex as a function of

$$x \in x(T) = \{x = x(\tau) = (\cos at_2, \ldots, \cos at_m)^T \mid \tau = (t_2, \ldots, t_m)^T \in T\}$$

(note that the set $x(T)$ consists of vectors with ordered components). This yields on the one hand that the solution $x^* = x^*(a)$ of the problem

$$\psi(x^*(a), a) = \inf_{\tau\in T} \psi(x(\tau), a) = \inf_{\tau}\inf_{w} e_m^T M_s^{-1}(\xi(\tau, w))e_m$$

is unique and can be obtained from (5.5) with $x = x^*(a)$, where $x^*(a)$ is the unique solution of the equations

$$\frac{\partial}{\partial x_i}\psi(x,a) = 0, \qquad i = 2, \ldots, m\,. \tag{5.7}$$

On the other hand strict convexity also implies that the matrix

$$J(x,a) \ := \left(\frac{\partial^2}{\partial x_i \partial x_j}\psi(x,a)\right)_{i,j=2}^m \tag{5.8}$$

is positive definite for all $x \in x(T)$, $a \in (0, \pi(1-\frac{1}{2m}))$ and by the implicit function theorem [see e.g. Gunning and Rossi (1965)] the mapping

$$x^* \ : \left\{ \begin{array}{ccl} (0, \pi(1-\frac{1}{2m})) & \to & x(T) \\ a & \to & x^*(a) \in \mathbb{R}^{m-1} \end{array} \right. \tag{5.9}$$

is real analytic. In other words: at any point $a_0 \in (0, \pi(1-\frac{1}{2m}))$ there exists a neighbourhood U_0 of a_0 such that $x^*_{|U_0}$ can be expanded in a convergent Taylor series. Note that the mapping x^* in (5.9) describes a complete solution of the e_m-optimal design problem in the Chebyshev regression model (2.10), if the solution for one a_0 can be identified. For this purpose we extend x^* to the region $(-\pi(1-\frac{1}{2m}), \pi(1-\frac{1}{2m}))\backslash\{0\}$ by the definition

$$\tilde{x}^*(a) = x^*(|a|).$$

Observing the obvious relation $\psi(x,a) = \psi(x,-a)$ it follows that the function $\tilde{x}^*$ is also real analytic on the interval $(-\pi(1-\frac{1}{2m}), \pi(1-\frac{1}{2m}))\backslash\{0\}$, and the transformation $x \to$ arccos $(x)/a$ yields that the function

$$\tau^* : \begin{cases} (-\pi(1-\frac{1}{2m}), \pi(1-\frac{1}{2m}))\backslash\{0\} & \to \mathbb{R}^{m-1} \\ a & \to \tau^*(a) \end{cases} \tag{5.10}$$

with

$$\tau^*(a) = \left(\frac{\text{arccos } \tilde{x}_2^*(a)}{a}, \ldots, \frac{\text{arccos } \tilde{x}_m^*(a)}{a} \right) \tag{5.11}$$

is also real analytic. Our next result shows that this property also holds on the complete interval.

Lemma 5.2 *The function τ^* defined in (5.10) and (5.11) can be extended to a real analytic function on the complete interval $(-\pi(1-\frac{1}{2m}), \pi(1-\frac{1}{2m}))$, where the value at the point 0 is defined by*

$$\tau^*(0) := \left(\cos(\pi - \frac{\pi}{2m-1}), \ldots, \cos(\frac{m\pi}{2m-1}) \right). \tag{5.12}$$

Proof. The assertion of Lemma 5.2 is established, if we show that the limit

$$\lim_{a \to 0} \tau^*(a)$$

exists and is equal to the right hand side of (5.12). If $a \to 0$ we obtain from the expansions

$$\begin{aligned} \sin(at) &= at + o(a) \\ \cos(at) &= 1 - \frac{(at)^2}{2} + o(a^2) \end{aligned}$$

the representation ($t_1 = -1$)

$$\left(\frac{1}{2}\right)^{m-1} a^{2(m-1)+1} \psi(x,a) = \sum_{i=1}^{m} \frac{(-1)^{m-i}}{t_i \prod_{j \neq i}(t_i^2 - t_j^2)} + o(a) \tag{5.13}$$

and as $a \to 0$ the minimization of ψ with respect to $x \in x(T)$ is approximately equivalent to the minimization of the right hand side of (5.13) with respect to $\tau = (t_2, \ldots, t_m) \in T$. For this reason we study the function

$$g(\tau) = \sum_{i=1}^{m} \frac{(-1)^{m-i}}{t_i \prod_{j \neq i}(t_i^2 - t_j^2)} \tag{5.14}$$

with $\tau \in T$ and $t_1 = -1$. The same arguments as given for the derivation of (5.6) show that this minimization gives the support points of the unique e_m-optimal design in the polynomial regression

$$E[y|t] = t \sum_{j=0}^{m-1} \delta_j t^{2j}; \quad t \in [-1, 0]. \tag{5.15}$$

It is now easy to see that the measure

$$\eta^* = \begin{pmatrix} \cos\pi & \cos(\pi - \frac{\pi}{2m-1}) & \dots & \cos(\frac{m\pi}{2m-1}) \\ \frac{1}{2m-1} & \frac{2}{2m-1} & \dots & \frac{2}{2(m-1)} \end{pmatrix}$$

is optimal for estimating the coefficient of t^{2m-1} in the model (5.15) and that the corresponding extremal polynomial is given by $\varphi(t) = T_{2m-1}(t)/2^{2m-1}$. Therefore it follows from (5.13) and (5.14) that

$$\lim_{a\to 0} \tau^*(a) = \left(\cos(\pi - \frac{\pi}{2m-1}), \dots, \cos(\frac{m\pi}{2m-1})\right) \tag{5.16}$$

which proves the assertion of the Lemma. We finally note that we also obtain a limit for the corresponding optimal weights in (5.5), i. e.

$$\lim_{a\to 0} w_i^*(a) = \begin{cases} \frac{1}{2m-1} & \text{if } i = 1 \\ \frac{2}{2m-1} & \text{if } i = 2, \dots, m. \end{cases} \tag{5.17}$$

□

Lemma 5.2 shows that in a neighborhood of the point 0 the function

$$\tau^*(a) = (t_2^*(a), \dots, t_m^*(a)),$$

which yields the negative interior support points of the e_{2m-1}-optimal design, can be expanded in a convergent Taylor series, that is

$$\tau^*(a) = \sum_{j=0}^{\infty} \tau_j^* \frac{a^j}{j!} \tag{5.18}$$

where $\tau_0^* = \tau^*(0)$ is defined by the right hand side of (5.12). The coefficients in this expansion can be found recursively as shown in Dette, Melas and Pepelysheff (2000). To be precise, consider the function

$$\Phi(\tau, a) := \psi(x(\tau), a),$$

where $x(\tau) := (\cos(at_2), \dots, \cos(at_m))$ and the function ψ is defined in (5.6) with $x_1 = \cos(-a)$. It then follows that the coefficients in the Taylor expansion (5.18) can be found recursively from $\tau^* = \tau^*(0)$,

$$\tau_{s+1}^* = -J^{-1}(0) \left(\frac{d}{da}\right)^{s+1} g(\tau_{(s)}^*(a), a)\Big|_{a=0} \quad s = 0, 1, \dots \tag{5.19}$$

where

$$\tau_{(s)}^*(a) = \sum_{j=0}^{s} \tau_j^* \frac{a^j}{j!} \tag{5.20}$$

i	0	2	4	6	8	10
t^*_{3i}	-0.8090	0.1839	0.1490	0.0683	-0.0254	-0.0825
t^*_{2i}	-0.3090	0.1839	-0.0412	- 0.0099	0.0148	-0.0049

Table 5.1 *Coefficients in the Taylor expansion (5.23) for the interior negative support points $t^*_2(a)$, $t^*_3(a)$ of the e_5-optimal designs in the cubic trigonometric regression model (1.1) on the interval $[-a, a]$. The e_1- and e_3-optimal designs have the same support points, if $a \le b^*_\ell$, $\ell = 1, 2$, where b^*_ℓ is defined in (5.25).*

is a polynomial of degree s and the functions J and g are defined by

$$J(a) = \left(\frac{\partial^2}{\partial t_i \partial t_j} \Phi(\tau, a) \right)^m_{i,j=2}, \tag{5.21}$$

$$g(\bar{\tau}, a) = \frac{\partial}{\partial \tau} \Phi(\tau, a) \Big|_{\tau=\bar{\tau}}, \tag{5.22}$$

respectively [for a proof see Dette, Melas and Pepelysheff (2000)]. We summarize these results in the following Theorem.

Theorem 5.3 *For any $a \in (0, \pi(1 - \frac{1}{2m})$ the e_{2m-1}-optimal design for the trigonometric regression model (1.1) on the interval $[-a, a]$ is unique and of the form*

$$\xi^*_{2m-1,a} = \begin{pmatrix} -a & at^*_2(a) & \dots & at^*_m(a) & -at^*_m(a) & \dots & -at^*_2(a) & a \\ \frac{1}{2}w^*_1(a) & \frac{1}{2}w^*_2(a) & \dots & \frac{1}{2}w^*_m(a) & \frac{1}{2}w^*_m(a) & \dots & \frac{1}{2}w^*_2(a) & \frac{1}{2}w^*_1(a) \end{pmatrix}$$

where the weights are obtained by formula (5.5) with $x_1 = \cos a$, $x_i = \cos(at^*_i(a))$. The vector of support points $\tau^*(a) = (t^*_2(a), \dots, t^*_m(a))$ is a real analytic function on the interval $(0, \pi(1 - \frac{1}{2m})$ satisfying (5.16) and can be obtained (locally) by the Taylor expansion (5.18) with coefficients τ^*_j, which are calculated recursively by formula (5.19).

Example 5.4 In the case $m = 3$ the optimal design for the estimation of the coefficient $\sin(3x)$ in the cubic trigonometric regression model (1.1) on the interval $[-a, a]$ for $0 < a < 5\pi/6$ is given by

$$\xi^*_{5,a} = \begin{pmatrix} -a & at^*_2(a) & at^*_3(a) & -at^*_3(a) & -at^*_2(a) & a \\ \frac{1}{2}w^*_1(a) & \frac{1}{2}w^*_2(a) & \frac{1}{2}w^*_3(a) & \frac{1}{2}w^*_3(a) & \frac{1}{2}w^*_2(a) & \frac{1}{2}w^*_1(a) \end{pmatrix}$$

where the weights are obtained from formula (5.5). It is easy to see that $t^*_i(a)$ defines an even function and Table 5.1 shows the first six nonvanishing coefficients of the expansion

$$t^*_i(a) = \sum_{j=0}^{\infty} \frac{t^*_{i,2j}}{(2j)!} \left(\frac{a}{\pi}\right)^{2j}. \tag{5.23}$$

i	0	2	4	6	8
t^*_{4i}	-0.9010	0.0996	0.1011	0.0873	0.0576
t^*_{3i}	-0.6235	0.2239	0.0710	-0.0284	-0.0356
t^*_{2i}	-0.2225	0.1243	-0.0301	0.0016	0.0034

Table 5.2 *Coefficients in the Taylor expansion (5.23) for the interior negative support points $t^*_2(a)$, $t^*_3(a)$, $t^*_4(a)$ of the e_7-optimal designs in the trigonometric regression model (1.1) of degree $m = 4$ on the interval $[-a, a]$. The e_1-, e_3- and e_5-optimal designs have the same support points, if $a \leq b^*_\ell$, $\ell = 1, 2, 3$, where b^*_ℓ is defined in (5.25).*

The corresponding functions and weights are depicted in Figure 5.1 for $a \in (0, 5\pi/6)$. Similarly, the e_7-optimal design for the model of degree 4 is of the form

$$\xi^*_{7,a} = \begin{pmatrix} -a & at^*_2(a) & at^*_3(a) & at^*_4(a) & -at^*_4 & -at^*_3 & -at^*_2 & a \\ \frac{1}{2}w^*_1(a) & \frac{1}{2}w^*_2(a) & \frac{1}{2}w^*_3(a) & \frac{1}{2}w^*_4(a) & \frac{1}{2}w^*_4(a) & \frac{1}{2}w^*_3(a) & \frac{1}{2}w^*_2(a) & \frac{1}{2}w^*_1(a) \end{pmatrix}$$

where the weights are obtained from (5.5) and the coefficients in the Taylor expansion (5.23) are listed in Table 5.2 (the odd coefficients are 0 because the supporting points are even functions of a). These functions are illustrated on Figure 5.2.

Theorem 5.5 *Let $-a = at^*_1(a) < at^*_2(a) < \ldots < at^*_m(a) < 0$ denote the negative interior support points of the e_{2m-1}-optimal design in the trigonometric regression model (1.1) on the interval $[-a, a]$, where $a \in (0, \pi(1 - \frac{1}{2m}))$, define $x_i = \cos(at^*_i(a))$ $(i = 2, \ldots, m)$, $x_1 = \cos a$,*

$$B_i = B_i(a) = \int_{-1}^{1} \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \sqrt{\frac{1 - x^2}{1 - x_i^2}} U_{\ell-1}(x) dx, \quad i = 1, \ldots, m,$$

and

$$w^*_i = w^*_i(a) = \frac{|B_i|}{\sum_{j=1}^m |B_j|}, \quad i = 1, \ldots, m. \tag{5.24}$$

If $a \to 0$ we have

$$\lim_{a \to 0} w^*_i(a) = \begin{cases} \frac{2}{2m-1} & \text{if } j = 2, \ldots, m \\ \frac{1}{2m-1} & \text{if } j = 1 \end{cases}.$$

Moreover, if $1 \leq \ell \leq m - 1$ and

$$a < b^*_\ell := b^*_{\ell,m} := \sup\Big\{ a \in (0, \pi(1 - \frac{1}{2\ell})) \,|\, w_i > 0 \ \text{ for all } i = 1, \ldots, m \Big\}, \tag{5.25}$$

then the $e_{2\ell-1}$-optimal design is given by

$$\xi^*_{2\ell-1,a} = \begin{pmatrix} at_1^*(a) & \dots & at_m^*(a) & -at_m^*(a) & \dots & -at_1^*(a) \\ \frac{1}{2}w_1^* & \dots & \frac{1}{2}w_m^* & \frac{1}{2}w_m^* & \dots & \frac{1}{2}w_1^* \end{pmatrix}.$$

In other words, if $a < b^*_\ell$, then the $e_{2\ell-1}$- and the e_{2m-1}-optimal designs in the trigonometric regression model (1.1) on the interval $[-a, a]$ have the same support. Moreover for any $\ell \in \{1, \dots, m-1\}$ we have $b^*_\ell \geq \pi/2$.

Proof. The Proof of Theorem 5.5 is similar to the proof of the corresponding statements for the cosine case and for the sake of brevity we only sketch the main differences. By Lemma 2.2 the design $\xi^*_{2\ell-1,a}$ is $e_{2\ell-1}$-optimal if and only if the design

$$\eta_{\xi^*_{2\ell-1,a}} = \begin{pmatrix} x_1^*(a) & \dots & x_m^*(a) \\ w_1^* & \dots & w_m^* \end{pmatrix}.$$

is e_ℓ-optimal in the Chebyshev regression model (2.10), where $x_i^* = x_i^*(a) = \cos at_i^*$ $(i = 1, \dots, m)$. Consider the Chebyshev expansion of the polynomial in (5.1) and define $\varphi_\ell(x) = \varphi_m(x)/b_\ell$. We will now use this polynomial as extremal polynomial in Lemma 2.1 to establish e_ℓ-optimality of the design $\eta_{\xi^*_{2\ell-1,a}}$. Note that this design has the same support points as the e_m-optimal design and for this reason the e_m-optimality of the design $\eta_{\xi^*_{2m-1,a}}$ implies that the polynomial φ_ℓ satisfies the conditions (1) and (2) of Lemma 2.1. Let

$$F \;=\; \Big((-1)^{j+1}\sqrt{1 - x_j^{*2}}\,U_{i-1}(x_j^*)\Big)_{i,j=1,\dots,m}$$

then condition (3) is satisfied, if we can find a vector $\tilde{w} = (\tilde{w}_1, \dots \tilde{w}_m)^T$, which has nonvanishing coefficients of the same sign and satisfies

$$F\tilde{w} = c_\ell e_\ell \;\in \mathbb{R}^m \tag{5.26}$$

for some constant $c_\ell \in \mathbb{R}$ (the corresponding weigths are then obtained by normalization). A similar analysis as given in the proof of Lemma 2.5 shows the solution of (5.26) is of the form $c(B_1, \dots, B_m)^T$, where

$$B_i = B_i(a) = \int_{-1}^{1} \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \sqrt{\frac{1 - x^2}{1 - x_i^2}}\, U_{\ell-1}(x)dx, \quad i = 1, \dots, m.$$

Moreover, if $a \to 0$ it follows by similar arguments as given in the proof of Lemma 5.2 that the weights defined by (5.24) satisfy

$$\lim_{a \to 0} w_i^*(a) = \begin{cases} \frac{1}{2m-1} & \text{if } i = 1 \\ \frac{2}{2m-1} & \text{if } i = 2, \dots, m. \end{cases} \tag{5.27}$$

Consequently, the quantity b_ℓ^* defined by (5.25) is positive and Lemma 2.1 shows that the design $\eta_{\xi^*_{2\ell-1,a}}$ is e_ℓ- optimal, whenever $a < b_\ell^*$, which proves the first part of the assertion. For the remaining part assume that $a \le \pi/2$ and note that the quantities B_i are proportional to

$$\tilde{B}_i = \int_{-\pi}^{\pi} \sin t \sin(\ell t) \prod_{j\neq i} (\cos t_j - \cos t) dt, \qquad i = 1, \ldots, m \ .$$

Observing that for $t, t_1, \ldots, t_m \in [-\pi/2, \pi/2]$ the inequalities $\cos t \ge 0$,

$$\prod_{j\neq i}(\cos t_j - \cos t) + \prod_{j\neq i}(\cos t_j + \cos t) > 2\prod_{j\neq i}\cos t_j$$

$$\prod_{j\neq i}(\cos t_j - \cos t) - \prod_{j\neq i}(\cos t_j + \cos t) < 0$$

hold, it follows for odd ℓ

$$\begin{aligned}
\tilde{B}_i &= \int_{-\pi}^{\pi} \sin t \sin(\ell t) \prod_{j\neq i}(\cos t_j - \cos t) dt \\
&= \int_0^{\pi} \sin t \sin(\ell t) \Big[\prod_{j\neq i}(\cos t_j - \cos t) + \prod_{j\neq i}(\cos t_j + \cos t)\Big] dt \\
&= 2\int_0^{\pi/2} \sin t \sin(\ell t) \Big[\prod_{j\neq i}(\cos t_j - \cos t) + \prod_{j\neq i}(\cos t_j + \cos t)\Big] dt \\
&> 4\prod_{j\neq i}\cos t_j \cdot \int_0^{\pi/2} \frac{\cos(\ell-1)t - \cos(\ell+1)t}{2} dt \quad \ge \quad 0 \ ,
\end{aligned}$$

which implies that $B_i = c_i\tilde{B}_i \neq 0$ for all $i = 1, \ldots, m$, whenever $a \le \pi/2$. Conseqently the definition (5.25) implies $b_\ell^* \ge \pi/2$, which proves the second part of the assertion in the case of an odd index ℓ. The even case is similar and therefore omitted.

□

For lower degree trigonometric regression the critical bounds are listed in Tabel 5.3. If a is smaller than the correesponding bound, the $e_{2\ell-1}$-optimal design in the trigonometric regression model (1.1) has the same support points as the optimal design for estimating the coefficient of $\sin(mx)$, which can be obtained by a Taylor expansion using the coefficients in Table 5.1 (for $m = 3$) and Table 5.2 (for $m = 4$). The corresponding weights are obtained by numerical integration using formula (5.24). We have illustrated these calculations in the cubic trigonometric regression model in Figure 5.3 and 5.4, which show the support points and corresponding weigths of the optimal designs for estimating the coefficient of $\sin x$ and $\sin(2x)$, respectively (the corresponding designs for the highest sine term can be found in Figure 5.1).

	$m = 2$	$m = 3$	$m = 4$	$m = 5$
$\sin(x)$	$\pi/2$	0.59π	0.66π	0.66π
$\sin(2x)$	$\cdot$	0.53π	0.65π	0.69π
$\sin(3x)$	$\cdot$	$\cdot$	0.63π	0.71π
$\sin(4x)$	$\cdot$	$\cdot$	$\cdot$	0.67π

Table 5.3 *Critical values b_ℓ^* defined in (5.25) for various values of ℓ and m. The optimal design for estimating the coeeficient $\beta_{2\ell-1}$ in the trigonometric regression model (1.1) on the interval $[-a, a]$ has the same support points as the $e_{2m-1}-$ optimal design, whenever $a \leq b_\ell^*$.*

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