E-optimal designs in Fourier regression models on a partial circle

Holger Dette
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum
Germany
email: holger.dette@ruhr-uni-bochum.de
FAX: +49 2 34 32 14 559

Viatcheslav B. Melas
St. Petersburg State University
Department of Mathematics
St. Petersburg
Russia
email: v.melas@pobox.spbu.ru

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Abstract

In the common trigonometric regression model we investigate the E-optimal design problem on the interval $[c, d]$. It is demonstrated that this problem can be reduced to the consideration of the corresponding design problem for the model on the interval $[-a, a]; 0 < a \leq \pi$. In a second step it is shown that the structure of the optimal design for the symmetric design space $[-a, a]$ depends sensitively on the size of the design space and for most cases the E-optimal designs can be found explicitly. Moreover, in the remaining situations a functional approach is proposed and used for the numerical determination of E-optimal designs. The results are illustrated in the linear, quadratic and cubic regression model, for which a complete solution is given.

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1 Introduction

Trigonometric regession models of the form

$$y = \frac{\beta_0}{\sqrt{2}} + \sum_{j=1}^{m} \beta_{2j-1} \sin(jt) + \sum_{j=1}^{m} \beta_{2j} \cos(jt) + \varepsilon, \quad x \in [c, d]; \quad c < d \tag{1.1}$$

($d - c \leq \pi$) are widely used to describe periodic phenomena [see e.g. Mardia (1972), Graybill (1976) or Kitsos, Titterington and Torsney (1988)] and the problem of designing experiments for Fourier regression models has been discussed by several authors [see e.g. Hoel (1965), Karlin and Studden (1966), page 347, Fedorov (1972), page 94, Hill (1978), Lau and Studden (1985), Riccomagno, Schwabe and Wynn (1997), Dette and Haller (1998)]. While most authors concentrate on the design space $[-\pi, \pi]$ much less attention has been paid to the case of a smaller design space [see e.g. Hill (1978)]. This situation is of practical importance because in many applications it is impossible to take observations on the full circle $[-\pi, \pi]$. We refer for example to Kitsos, Titterington and Torsney (1988), who investigated a design problem in rhythmometry involving circadian rhythm exhibited by peak expiratory flow, for which the design region has to be restricted to a partial cycle of the complete 24-hour period. Optimal designs for estimating some of the individual coefficients in the trigonometric regresion model (1.1) have been found explicitly by Dette and Melas (2001). Recently Dette, Melas and Pepelysheff (2001) determined D-optimal designs for the trigonometric regression numerically.
Due to the minimax structure of the E-optimality criterion explicit results for E-optimal designs in regression models are only available in specific situations. Most authors concentrate on the polynomial case or models with a very similar structure [see Melas (1982), Pukelsheim and Studden (1993), Dette (1993), Heiligers (1994, 1998), Chang and Heiligers (1996) or the recent work of Imhof and Studden (2001)]. In the present paper we present a further model, for which the E-optimal designs can be found explicitly in nearly all cases. In Section 2 we demonstrate that the problem of determining E-optimal designs in the trigonometric regression model (1.1) on the the interval $[c, d]$ can be reduced to the corresponding design problem on the symmetric interval $[-a, a]$, where $0 < a \leq \pi$. It is then shown that the structure of the E-optimal design depends sensitivley on the size a of the interval [note that this is similar to the polynomial case, see Melas (2000)]. In Section 2 and 3 we find the E-optimal designs explicitly, whenever $a \in [-a, a] \setminus (\underline{a}, \overline{a})$, where $\underline{a}, \overline{a}$ are given constants depending on the degree of the regression. Moreover, it is demonstrated that the range $\overline{a} - \underline{a}$ not covered by these results is usually rather small. Section 4 contains the discussion of the the linear trigonometric regression model, for which the optimal designs can be found explicitly in all cases. Finally, Section 5 deals with models of degree larger than one in the remaining case $a \in (\underline{a}, \overline{a})$, and the functional approach proposed in Melas (2000) is used to find the E-optimal designs numerically in these cases.

2 Preliminary results and E-optimal designs on large design spaces

Consider the common regression model

$$y = \sum_{j=0}^{k} \theta_j f_j(x) + \varepsilon \ , \qquad x \in \mathcal{X} \ , \tag{2.1}$$

where the explanatory variable varies in the compact design space $\mathcal{X}$, $f_0, \ldots, f_k$ are continuous and linearly independent regression functions and observations at different points are assumed to be independent. An approximate design is a probability measure ξ on $\mathcal{X}$ (or on its Borel field) with finite support [see Kiefer (1974)], where the observations are taken at the support points proportional to the weights of ξ at these points. If $f(x) = (f_0(x), \ldots, f_k(x))^T$ denotes the vector of regression functions, the covariance matrix of the least squares estimator for the parameter $\theta = (\theta_0, \ldots, \theta_k)^T$ based on uncorrelated observations from an approximate design is approximately proportional to the inverse of the information matrix

$$M(\xi) = \int_{\mathcal{X}} f(t) f^T(t) d\xi(t) \ , \tag{2.2}$$

and an optimal design maximizes an appropriate concave function of this matrix [see e.g. Fedorov (1972), Silvey (1980) or Pukelsheim (1993)]. In the present paper we are interested in the E-optimality criterion, which is given by

$$\Phi(\xi) = \lambda_{\min}(M(\xi)) \ , \tag{2.3}$$

where $\lambda_{\min}(A)$ denotes the minimum eigenvalue of a symmetric matrix $A \in \mathbb{R}^{k+1 \times k+1}$. Note that maximizing Φ is equivalent to minimizing the function

$$\frac{1}{\Phi(\xi)} = \lambda_{\max}(M^{-1}(\xi)) = \max_{\|a\|_2 = 1, a \in \mathbb{R}^{k+1}} a^T M^{-1}(\xi) a.$$

The expression $a^T M^{-1}(\xi)a$ is proportional to the variance of the least squares estimate for the linear combination $a^T\theta$ ($a \in \mathbb{R}^{k+1}$) and therefore an E-optimal design minimizes the worst variance over all possible (normalized) linear combinations.
It follows by standard arguments [see e.g. Pukelsheim (1993)] that an E-optimal design exists. For an E-optimal design ξ_E we define $\mathcal{P}_{\xi_E}$ as the eigenspace corresponding to the minimal eigenvalue $\lambda_{\min}(M(\xi_E))$ and

$$\mathcal{P} = \bigcap_{\xi_E \text{ is } E\text{-optimal}} \mathcal{P}_{\xi_E} \tag{2.4}$$

as the intersection of all eigenspaces corresponding to E-optimal designs. It can easily be verified that $\mathcal{P} \neq \emptyset$ and the following Lemma gives a characterization for E-optimal designs [for a proof we refer to Melas (1982) or Pukelsheim (1993)].

Lemma 2.1. *A design ξ^* is E-optimal for the regression model (2.1) if and only if there exists a nonnegative definite matrix $A^* \in \mathbb{R}^{k+1 \times k+1}$ such that* $\operatorname{tr} A^* = 1$ *and*

$$\max_{x \in \mathcal{X}} f^T(x) A^* f(x) \le \lambda_{\min}(M(\xi^*)). \tag{2.5}$$

Moreover, if x^* is a support point of ξ^*, there is equality in (2.5), i.e.

$$f^T(x^*) A^* f(x^*) = \lambda_{\min}(M(\xi^*))$$

and the matrix A^* *can be represented as*

$$A^* = \sum_{i=1}^{s} \alpha_i z_i z_i^T \ , \tag{2.6}$$

where $z_1, \ldots, z_s$ *is an orthonormal basis of the set* $\mathcal{P}$ *defined in (2.4),* $s = \dim \mathcal{P}$ *and* $\alpha_1, \ldots, \alpha_s \geq 0$ *with* $\sum_{i=1}^{s} \alpha_i = 1$.

In the specific situation of the trigonometric regression model (1.1) we have $\mathcal{X} = [c, d]$, $f_0(t) = 1/\sqrt{2}$, $f_{2j}(t) = \cos(jt)$ $(j = 1, \ldots, m)$ and $f_{2j-1}(t) = \sin(jt)$ $(j = 1, \ldots, m)$. Note that we use a slightly different parametrization of the intercept, but most of our results are also valid for the trigonometric regression model with $f_0(t) = 1$. Our first result shows that the E-optimal design in the trigonometric regression model is essentially invariant with respect to transformations of the design space by an additive shift.

Lemma 2.2. *Let*

$$\eta = \begin{pmatrix} t_1 & \ldots & t_n \\ w_1 & \ldots & w_n \end{pmatrix}$$

denote a design on the interval $[c, d]$, $a = (c + d)/2$ *and* ξ_η *be the design obtained by the linear transformation* $t \to t - a$, *i.e.*

$$\xi_\eta = \begin{pmatrix} t_1 - a & \ldots & t_n - a \\ w_1 & \ldots & w_n \end{pmatrix},$$

then the information matrices $M(\eta)$ *and* $M(\xi_\eta)$ *in the trigonometric regression model (1.1) have the same eigenvalues, in particular*

$$\lambda_{\min}(M(\eta)) = \lambda_{\min}(M(\xi_\eta)).$$

Proof: Let $f(t) = (1/\sqrt{2}, \sin t, \cos t, \ldots, \sin(mt), \cos(mt))^T$, then we have for any $\alpha \in \mathbb{R}$

$$f(t + \alpha) = P(\alpha) f(t)$$

where $P(\alpha)$ is a $(2m + 1) \times (2m + 1)$ (block) matrix given by

$$P(\alpha) = \begin{pmatrix} \frac{1}{\sqrt{2}} & & & \\ & Q(\alpha) & & \\ & & \ddots & \\ & & & Q(m\alpha) \end{pmatrix}$$

with

$$Q(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Because $P(\alpha)$ is orthogonal the matrices $M(\eta)$ and

$$M(\xi_\eta) \;=\; \int_{-a}^{a} f(t)f^T(t)d\xi_\eta(t) = \int_{c}^{d} f(t-a)f^T(t-a)d\eta(t) \;=\; P(-a)M(\eta)P^T(-a)$$

have the same eigenvalues and the assertion of the Lemma has been established. $\Box$

From the proof of Lemma 2.2 it follows that for any ϕ_p-criterion in the sense of Pukelsheim (1993) the solution of the ϕ_p-optimal design problem for the trigonometric regression model (1.1) on the interval $[c,d]$ can be obtained from the solution of the corresponding problem on the interval $[-a,a]$ and a linear transformation. For this reason we will restrict our subsequent investigations about E-optimal designs to symmetric intervals of the form $[-a,a]$, where $0 < a \le \pi$. Note that in general an E-optimal design for the trigonometric regression model (1.1) on the interval $[-a,a]$ is not necessarily unique. For example, it follows by Lemma 2.1 that for the full circle $[-a,a] = [-\pi,\pi]$ any design with information matrix

$$M^* \;=\; I_{2m+1} \;=\; \mathrm{diag}(\frac{1}{2},\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}) \in \mathbb{R}^{2m+1\times 2m+1} \tag{2.7}$$

is E-optimal. In particular any design of the form

$$\xi_n^* = \begin{pmatrix} t_1 & \dots & t_n \\ \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix} \tag{2.8}$$

with $n \ge 2m+1$ and

$$t_j = -\pi + \frac{2j-1}{n}\pi\ , \qquad j = 1,\dots,n\ , \tag{2.9}$$

has information matrix M^* [see Pukelsheim (1993)] and is therefore E-optimal for the trigonometric regression model (1.1) on the interval $[-\pi,\pi]$. In the following we will prove that the E-optimal design for the trigonometric regression model is unique, provided that the design space is sufficiently small.
To this end let $\Xi^{(1)}_{(a)}$ denote the set of all designs of the form

$$\xi = \xi(a) = \begin{pmatrix} -t_m & \dots & -t_1 & t_0 & t_1 & \dots & t_m \\ \frac{w_m}{2} & \dots & \frac{w_1}{2} & w_0 & \frac{w_1}{2} & \dots & \frac{w_m}{2} \end{pmatrix}\ ,$$

where $0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = a$ and $w_j > 0\,(j = 0,\dots,m)$ such that $\sum_{j=0}^m w_j = 1$. Furthermore, define

$$\Xi^{(2)}_{(a)} \;=\; \Big\{\xi \mid \mathrm{supp}(\xi) \subset [-a,a], \exists\, A^* \in PD(2m+1):\ \mathrm{tr}\,A^* = 1, \quad f^T(t)A^*f(t) = \lambda_{\min}(M(\xi))\ \ \forall\, t \in [-a,a]\Big\}, \tag{2.10}$$

where $PD(2m+1)$ denotes the set of all positive definite $(2m+1)\times(2m+1)$ matrices. A straightforward calculation shows $\xi^*_{2m+1} \in \Xi^{(2)}_{(a)}$, and with the aid of Lemma 2.1 it is easy to see that the design ξ^*_{2m+1} defined in (2.8) is E-optimal for the trigonometric regression model (1.1) on the interval $[-a, a]$, whenever $a > \bar{a}$, where

$$\bar{a} = \bar{a}(m) = \pi\Big(1 - \frac{1}{2m+1}\Big) \tag{2.11}$$

denotes the largest support point of the design ξ^*_{2m+1}. The following result shows that E-optimal designs for the trigonometric regression model (1.1) on the interval $[-a, a]$ are either in the set $\Xi^{(1)}_{(a)}$ or in $\Xi^{(2)}_{(a)}$ depending on the sign of the quantity $a - \bar{a}$.

Theorem 2.3. *If $a \in [\bar{a}, \pi]$, then any E-optimal design for the trigonometric regression model (1.1) on the interval $[-a, a]$ is contained in the set $\Xi^{(2)}_{(a)}$ defined in (2.10). If $a \in (0, \bar{a})$, then the E-optimal design for the trigonometric regression model on the interval $[-a, a]$ is unique and contained in the set $\Xi^{(1)}_{(a)}$. Moreover, $\xi \in \Xi^{(2)}_{(a)}$ if and only if the information matrix of ξ is of the form (2.7).*

Proof: Let ξ^* denote an E-optimal design for the trigonometric regression model on the interval $[-a, a]$ $(0 < a \leq \pi)$, then it follows by similar arguments as given in the proof of Lemma 2.2 of Dette, Melas and Pepelysheff (2001) that

$$\xi^* \in \Xi^{(1)}_{(a)} \cup \Xi^{(2)}_{(a)}$$

(we only have to replace the equivalence theorem for D-optimality by Lemma 2.1). The same arguments show that if an E-optimal design for the trigonometric regression model (1.1) on the interval $[-a, a]$ belongs to the set $\Xi^{(1)}_{(a)}$, then it is the unique E-optimal design on $[-a, a]$.
We start proving the last assertion of the theorem. If $a > \bar{a}$, then the design ξ^*_{2m+1} defined in (2.8) is E-optimal for the trigonometric regression on the interval $[-a, a]$ and therefore satisfies $M(\xi^*_{2m+1}) = M^*$, where M^* is given in (2.7). Consequently, any design ξ on the interval $[-a, a]$ with $M(\xi) = M^*$ must also be E-optimal and satisfies $\xi \in \Xi^{(2)}_{(a)}$.
Conversely, let

$$\xi = \begin{pmatrix} t_1 & t_2 & \dots & t_n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}$$

denote an arbitrary design on the interval $[-a, a]$, then it is easy to see that the information matrix of ξ in the trigonometric regression model (1.1) satisfies

$$(M(\xi))_{11} = \frac{1}{2}, \quad \operatorname{tr}(M(\xi)) = m + \frac{1}{2}. \tag{2.12}$$

Now assume additionally that ξ is E-optimal and $a > \bar{a}$, then the E-optimality of the design ξ^*_{2m+1} in (2.8) implies

$$\lambda_{\min}(M(\xi)) = \lambda_{\min}(M(\xi^*_{2m+1})) = \lambda_{\min}(M^*) = \frac{1}{2}.$$

On the other hand we have from the well known estimates $(M(\xi))_{ii} \geq \lambda_{\min}(M(\xi)) = \frac{1}{2}$ and the equations in (2.12)

$$m + \frac{1}{2} = \sum_{i=1}^{2m+1} (M(\xi))_{ii} \geq \frac{1}{2}(2m+1) = m + \frac{1}{2}\ , \tag{2.13}$$

which shows

$$(M(\xi))_{ii} = \frac{1}{2}\ , \qquad i = 1, \ldots, 2m+1. \tag{2.14}$$

In the next step let $\alpha = (M(\xi))_{ij} = (M(\xi))_{ji}$ denote the element in the position (i, j) of the information matrix of the design ξ, where $1 \leq i \neq j \leq 2m+1$, and define

$$p = \frac{1}{\sqrt{2}}(e_i - \text{sign}(\alpha)e_j),$$

where $e_i \in \mathbb{R}^{2m+1}$ denotes the ith unit vector. Then $\|p\|_2^2 = 1$ and we obtain

$$\begin{aligned} \frac{1}{2} &= \lambda_{\min}(M(\xi)) \ \leq \ p^T M(\xi) p \ = \ \frac{1}{2}(1, -\text{sign}(\alpha)) \begin{pmatrix} \frac{1}{2} & \alpha \\ \alpha & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -\text{sign}(\alpha) \end{pmatrix} \\ &= \frac{1}{2} - |\alpha| \ \leq \ \frac{1}{2}\ , \end{aligned}$$

which implies $\alpha = (M(\xi))_{ij} = 0$, whenever $1 \leq i \neq j \leq 2m+1$. Consequently the information matrix of any E-optimal design is diagonal, i.e. $M(\xi) = \frac{1}{2}I_{2m+1}$.
Now let $a < \bar{a}$, then it follows from the recent results of Dette, Melas and Pepelysheff (2001) that for any design ξ on the interval $[-a, a]$

$$\det M(\xi) < (\frac{1}{2})^{2m+1}$$

[see the proof of Theorem 3.3 in the same reference]. Because any E-optimal design ξ^* in $\Xi_a^{(2)}$ satisfies $\det M(\xi^*) = \det M^* = 2^{-2m-1}$, there are no E-optimal designs on the interval $[-a, a]$, which belong to the set $\Xi_a^{(2)}$ (if $a < \bar{a}$). Consequently, by the discussion at the beginning of the proof the E-optimal design is unique and an element of the set $\Xi_a^{(1)}$. Finally, if $a > \bar{a}$ we have shown that the information matrix of the E-optimal design for the trigonometric regression model (1.1) is unique and equal to the matrix $M^* = M(\xi^*_{2m+1}) = \frac{1}{2}I_{2m+1}$, where the design ξ^*_{2m+1} is defined by (2.8). Because $\xi^*_{2m+1} \in \Xi_a^{(2)}$ it follows from the definition (2.10) that any E-optimal design belongs to the set $\Xi_a^{(2)}$.

$\square$

Note that Theorem 2.3 provides a solution of the E-optimal design problem in the trigonometric regression model (1.1) on the interval $[-a, a]$ whenever $a > \bar{a} = \pi(1 - 1/(2m+1))$. In this case the solution is not necessarily unique. However, the information matrix corresponding to E-optimal designs is unique although the E-criterion (considered as a mapping on the positive definite matrices) is not strictly concave. If $a < \bar{a}$ the E-optimal design on the interval $[-a, a]$ is unique and will be described explicitly in the following section, if the parameter a is sufficiently small.

3 E-optimal designs on sufficiently small intervals

Throughout this paper let

$$T_k(x) = \cos(k \arccos x) \ , \qquad k \in \mathbb{N}_0 \ ; \tag{3.1}$$

denote the k-th Chebyshev polynomial of the first kind [see Rivlin (1974)] which are orthogonal with respect to the arcsine distribution, i.e.

$$\frac{2}{\pi} \int_{-1}^{1} T_i(x) T_j(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 1 \text{ if } i = j \geq 1 \\ 2 \text{ if } i = j = 0 \\ 0 \text{ if } i \neq j \end{cases} \tag{3.2}$$

It is well know [see Rivlin (1974)] that $T_k(x)$ is the unique solution of the extremal problem

$$\min_{a_0,\dots,a_{k-1} \in \mathbb{R}} \quad \max_{x \in [-1,1]} |2^{k-1} x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0| \ ,$$

and in particular we have equality at the Chebyshev points $s_i = \cos(i\pi/k)$, i.e.

$$T_k(s_i) = (-1)^i \ , \qquad i = 0, \dots, k. \tag{3.3}$$

Throughout this paper let $\alpha = \cos a \in [-1, 1)$, define

$$x_i \ = \ x_i(a) \ = \ \frac{1-\alpha}{2} s_i + \frac{1+\alpha}{2} \ , \qquad i = 0, \dots, m \ , \tag{3.4}$$

as the extremal points of the Chebyshev polynomial of the first kind

$$T_m\left(\frac{2x - 1 - \alpha}{1 - \alpha}\right) = \frac{q_{a0}}{\sqrt{2}} \ + \ \sum_{i=1}^{m} q_{ai} T_i(x) \tag{3.5}$$

on the interval $[\alpha, 1]$ and

$$t_i = t_i(a) = \frac{1}{a} \arccos x_i, \qquad i = 0, \dots, m. \tag{3.6}$$

We will consider designs of the form

$$\hat{\xi}_a = \begin{pmatrix} -at_m & \dots & -at_1 & t_0 & at_1 & \dots & at_m \\ \frac{\hat{w}_m}{2} & \dots & \frac{\hat{w}_1}{2} & \hat{w}_0 & \frac{\hat{w}_1}{2} & \dots & \frac{\hat{w}_m}{2} \end{pmatrix} \tag{3.7}$$

as candidate for the E-optimal design in the trigonometric regression model (1.1) on the interval $[-a, a]$ (note that $\hat{\xi}_a \in \Xi^{(1)}_{(a)}$). The weights in (3.7) are given by

$$\hat{w}_i = \hat{w}_i(a) = \frac{|q_a^T F^{-1} e_i|}{\sum_{j=0}^m |q_a^T F^{-1} e_j|}\ , \qquad i = 0, \ldots, m, \tag{3.8}$$

where $e_i \in \mathbb{R}^{m+1}$ denotes the $(i+1)$th unit vector, the vector $q_a^T = (q_{a0}, \ldots, q_{am}) \in \mathbb{R}^{m+1}$ is defined by the representation (3.5) and the matrix $F \in \mathbb{R}^{m+1 \times m+1}$ is given by

$$F = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ T_1(x_0) & T_1(x_1) & \ldots & T_1(x_m) \\ \vdots & \vdots & \ddots & \vdots \\ T_m(x_0) & T_m(x_1) & \ldots & T_m(x_m) \end{pmatrix}. \tag{3.9}$$

The following result specifies some properties of the design defined in (3.7) and (3.8) and is the main tool for proving its E-optimality for sufficiently small design spaces $[-a, a]$.

Lemma 3.1. *Let $\hat{\xi}_a$ denote the design defined by (3.7) and (3.8), then the following statements are correct.*

(i) If $0 < a \leq \pi/2$, then the weights $\hat{w}_i = \hat{w}_i(a)$ can be represented as

$$\hat{w}_i = \lambda_a (-1)^i \frac{2}{\pi} \int_{-1}^{1} \ell_i(x) T_m\Big(\frac{2x - \alpha - 1}{1 - \alpha}\Big) \frac{dx}{\sqrt{1 - x^2}}\ ,$$

where the constant λ_a is given by

$$\lambda_a = \frac{1}{q_a^T q_a}, \tag{3.10}$$

the vector $q_a^T = (q_{a0}, \ldots, q_{am})$ is defined in the representation (3.5) and

$$\ell_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \tag{3.11}$$

denotes the ith Lagrange interpolation polynomial with knots $x_0, \ldots, x_m$ given by (3.4).

(ii) For all $a \in (0, \pi]$ the quantity λ_a defined in (3.10) is an eigenvalue of the matrix $M(\hat{\xi}_a)$ with corresponding eigenvector $\bar{q}_a = (q_{a0}, 0, q_{a1}, 0, \ldots, 0, q_{am})^T$.

(iii) The support points and weights defined by (3.6) and (3.8), respectively, satisfy

$$\lim_{a \to 0} t_i(a) = \cos\Big(\pi \frac{m - i}{2m}\Big) \qquad i = 0, \ldots, m \tag{3.12}$$

$$\lim_{a \to 0} \hat{w}_i(a) \;=\; \begin{cases} \frac{1}{m} & \text{if } i = 1, \ldots, m-1 \\ \frac{1}{2m} & \text{if } i = 0, m. \end{cases} \tag{3.13}$$

Proof: Let $w = (w_0, \dots, w_m)^T \in \mathbb{R}_+^{m+1}; \sum_{i=0}^m w_i = 1$ and

$$\xi_a(w) = \begin{pmatrix} -at_m & \dots & -at_1 & t_0 & at_1 & \dots & at_m \\ \frac{w_m}{2} & \dots & \frac{w_1}{2} & w_0 & \frac{w_1}{2} & \dots & \frac{w_m}{2} \end{pmatrix}$$

an arbitrary design with positive weights at the points $\pm at_i \, (i = 0, \dots, m)$. It was shown in Theorem 4.1 and 4.3 of Dette and Melas (2001) that for $a \in (0, \pi/2]$ the optimal designs $\xi_{(0)}, \xi_{(2)}, \dots, \xi_{(2m)}$ for estimating the individual coefficients $\beta_0, \beta_2, \dots, \beta_{2m}$, respectively, in the trigonometric regression model (1.1) on the interval $[-a, a]$ are of the form

$$\xi_{(2j)} = \xi_a(w_{(j)}) \ , \qquad j = 0, \dots, m, \tag{3.14}$$

where the weights $w_{(j)} = (w_{(j)0}, \dots, w_{(j)m})^T$ are given by

$$w_{(j)i} = \frac{B_{(j)i}}{\sum_{s=0}^m B_{(j)s}} \ , \qquad i = 0, \dots, m, \tag{3.15}$$

and

$$B_{(j)i} \ = \ (-1)^{m+i-j} \int_{-1}^{1} \ell_i(x) T_j(x) \frac{dx}{\sqrt{1 - x^2}} \ = \ (-1)^{m+i+j} c_j e_i^T F^{-1} e_j \tag{3.16}$$

with $c_0 = \pi/\sqrt{2}$, $c_j = \pi/2 \, (j = 1, \dots, m)$. Note that we use a slightly different notation for the support points t_i and weights $w_{(j)i}$ compared to the cited reference.

A similar argument as given in the proof of Lemma 2.2 of Dette and Melas (2001) shows that the design $\eta_{\xi_{(2j)}}$ obtained by the transformation

$$\eta_\xi(\cos x) = \begin{cases} \xi(x) + \xi(-x) & \text{if } 0 < x \le a \\ \xi(0) & \text{if } x = 0 \end{cases} \tag{3.17}$$

is optimal for estimating the coefficient δ_j in the Chebyshev regression model

$$y = \frac{\delta_0}{\sqrt{2}} \ + \ \sum_{j=1}^m \delta_j T_j(x) + \varepsilon \ , \tag{3.18}$$

and the representation (3.5) in this paper and Lemma 2.1 in the cited reference show

$$q_{aj}^2 = e_j^T M_T^{-1}(\eta_{\xi_{(2j)}}) e_j \ , \quad j = 0, \dots, m, \tag{3.19}$$

where $M_T(\eta)$ denotes the information matrix of the design η in the model (3.18). Moreover, recalling the definition of the matrix F in (3.9) it follows from Lemma 8.9 in Pukelsheim (1993) that

$$q_{aj}^2 = e_j^T M_T^{-1}(\eta_{\xi_{(j)}}) e_j = \Big(\sum_{i=0}^m |e_i^T F^{-1} e_j| \Big)^2 = \Big(\sum_{i=0}^m (-1)^{m+i+j} e_i^T F^{-1} e_j \Big)^2 \tag{3.20}$$

($j=0,\dots,m$), where the last equality is obtained by a careful analysis of the sign pattern in the matrix F^{-1} observing that $a \in (0,\pi/2]$. Because for $a \in (0,\pi/2]$ the sign of q_{aj} is $(-1)^{m-j}$ we obtain from the identity (3.20)

$$q_{aj} = \sum_{i=0}^{m} (-1)^i e_i^T F^{-1} e_j \ , \qquad j = 0,\dots,m \ , \tag{3.21}$$

and the second equality in (3.16) gives for the vector $q_a^T = (q_{a0},\dots,q_{am})^T$

$$\begin{aligned} q_a^T F^{-1} e_i \ &= \sum_{j=0}^{m} q_{aj}(e_j^T F^{-1} e_i) \\ &= \int_{-1}^{1} \ell_i(x) \sum_{j=0}^{m} \frac{q_{aj}}{c_j} T_j(x) \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{2}{\pi} \int_{-1}^{1} \ell_i(x) T_m\Big(\frac{2x-1-\alpha}{1-\alpha}\Big) \frac{dx}{\sqrt{1-x^2}} \end{aligned} \tag{3.22}$$

($i=0,\dots,m$), where we have used the representation (3.5) and the fact that $c_j = \pi/2$ ($j = 1,\dots,m$); $c_0 = \pi/\sqrt{2}$. Moreover, observing that for $a \in (0,\pi/2]$ the sign of q_{aj} and $e_j^T F^{-1} e_i$ is $(-1)^{m-j}$ and $(-1)^{m+i+j}$, respectively, we obtain that the sign of $q_a^T F^{-1} e_i$ is $(-1)^i$. Now the polynomial in (3.5) attains the values $(-1)^i$ at the point x_i ($i=0,\dots,m$) and it follows

$$T_m\Big(\frac{2x-1-\alpha}{1-\alpha}\Big) = \frac{q_{a0}}{\sqrt{2}} \ + \ \sum_{j=1}^{m} q_{aj} T_j(x) = \sum_{j=0}^{m} (-1)^j \ell_j(x). \tag{3.23}$$

Combining these arguments yields

$$\begin{aligned} \sum_{i=0}^{m} |q_a^T F^{-1} e_i| \ &= \sum_{i=0}^{m} (-1)^i q_a^T F^{-1} e_i \\ &= \frac{2}{\pi} \sum_{i=0}^{m} (-1)^i \int_{-1}^{1} \ell_i(x) T_m\Big(\frac{2x-1-\alpha}{1-\alpha}\Big) \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{2}{\pi} \int_{-1}^{1} T_m^2\Big(\frac{2x-1-\alpha}{1-\alpha}\Big) \frac{dx}{\sqrt{1-x^2}} = q_a^T q_a \ , \end{aligned} \tag{3.24}$$

where the last equation is a consequence of the representation (3.5) and the orthogonality relations (3.2). The assertion (i) of Lemma 3.1 now follows from the definition (3.8) and the identity (3.22).

In order to prove the second assertion of Lemma 3.1 let $P \in \mathbb{R}^{2m+1 \times 2m+1}$ denote a permutation matrix such that

$$PM(\hat{\xi}_a)P^T = \bar{M}(\hat{\xi}_a) := \begin{pmatrix} M_c(\hat{\xi}_a) & 0 \\ 0 & M_s(\hat{\xi}_a) \end{pmatrix} \ , \tag{3.25}$$

where the blocks in the matrix $\bar{M}(\hat{\xi}_a)$ are defined by

$$M_c(\xi) = \int_{-a}^{a} f_c(t) f_c^T(t) d\xi(t) \in \mathbb{R}^{m+1\times m+1} , \tag{3.26}$$

$$M_s(\xi) = \int_{-a}^{a} f_s(t) f_s^T(t) d\xi(t) \in \mathbb{R}^{m\times m} , \tag{3.27}$$

and the vectors $f_c(t) \in \mathbb{R}^{m+1}$ and $f_s(t) \in \mathbb{R}^m$ are given by

$$f_c^T(t) = (1/\sqrt{2}, \cos t, \ldots, \cos(mt)), \tag{3.28}$$

$$f_s^T(t) = (\sin t, \ldots, \sin(mt)), \tag{3.29}$$

respectively. Because the matrices $\bar{M}(\hat{\xi}_a)$ and $M(\hat{\xi}_a)$ have the same eigenvalues and its corresponding eigenvectors are related by the transformation $x \to Px$ the assertion (ii) of Lemma 3.1 follows, if we prove that the vector $\tilde{q}_a = P\overline{q}_a = (q_a^T, 0^T)^T \in \mathbb{R}^{2m+1}$ is an eigenvector of the matrix $\bar{M}(\hat{\xi}_a)$ with corresponding eigenvalue

$$\lambda_a = (\tilde{q}_a^T \tilde{q}_a)^{-1} = (q_a^T q_a)^{-1}.$$

But this follows easily observing that the sign of $q_a^T F^{-1} e_i$ is $(-1)^i$ for $a \in (0, \pi]$ and from the representation of the weights $\hat{w}_i$ in (3.8) and (3.24), which gives

$$\begin{aligned} M_c(\hat{\xi}_a) q_a &= \sum_{i=0}^m f_c(at_i) f_c^T(at_i) \hat{w}_i q_a \\ &= \sum_{i=0}^m f_c(at_i) \frac{1}{q_a^T q_a} q_a^T F^{-1} e_i \\ &= \frac{1}{q_a^T q_a} F F^{-1} q_a = \lambda_a q_a \ . \end{aligned}$$

Consequently, we obtain

$$\bar{M}(\hat{\xi}_a) \tilde{q}_a = \lambda_a \tilde{q}_a$$

completing the proof of the second assertion of Lemma 3.1.

For the proof of the remaining third part recall that the sign of q_{aj} and $e_j^T F^{-1} e_i$ is $(-1)^{m-j}$ and $(-1)^{m+i+j}$, respectively. Then (3.21) implies for sufficiently small a

$$|q_{aj}| = \sum_{i=0}^m |e_j^T F^{-1} e_i| \ ,$$

and from the first equation in (3.22) we have

$$\begin{aligned} (-1)^i q_a^T F^{-1} e_i &= (-1)^i \sum_{j=0}^m |q_{aj}| (-1)^{m-j} |e_j F^{-1} e_i| (-1)^{m+i+j} \\ &= \sum_{j=0}^m |q_{aj}| |e_j F^{-1} e_i|. \end{aligned}$$

A summation of these quantities yields for the weights of the design $\hat{\xi}_a$ defined in (3.7)

$$\hat{w}_i = \frac{|q_a^T F^{-1} e_i|}{\sum_{j=0}^m |q_a^T F^{-1} e_i|} = \sum_{j=0}^m w_{(j)i} \cdot \alpha_j(a) \quad , \qquad i = 0, \ldots, m \ ,$$

where

$$\alpha_j(a) = \frac{|q_{aj}|^2}{\sum_{s=0}^m |q_{as}|^2} \ , \qquad j = 0, \ldots, m \ , \tag{3.30}$$

and the weights $w_{(j)i}$ are defined in (3.15), (3.16) and (3.20) corresponding to the optimal design

$$\xi_{(2j)} \ = \ \xi(w_{(j)}) \ = \ \begin{pmatrix} -at_m & \ldots & -at_1 & t_0 & at_1 & \ldots & at_m \\ \frac{w_{(j)m}}{2} & \ldots & \frac{w_{(j)1}}{2} & w_{(j)0} & \frac{w_{(j)1}}{2} & \ldots & \frac{w_{(j)m}}{2} \end{pmatrix}$$

for estimating the individual coefficient β_{2j} in the trigonometric regression model (1.1) on the interval $[-a, a]$, whenever $0 < a < \pi/2$. Note that we use the second representation in (3.16) and the equation (3.20) to find this normalization. In other words: the design $\hat{\xi}_a$ is obtained as a convex combination of the optimal designs for estimating the individual coefficients in the trigonometric regression model (1.1) on the interval $[-a, a]$ (whenever $0 < a < \pi/2$), that is

$$\hat{\xi}_a = \sum_{j=0}^m \alpha_j(a) \xi_{(2j)} \ . \tag{3.31}$$

If $a \to 0$, the representation (3.5) implies that ($\alpha = \cos a$)

$$\lim_{a \to 0} (1 - \alpha)^m q_a \ = \ f \in \mathbb{R}^{m+1} \ , \tag{3.32}$$

where $f = (f_0, \ldots, f_m)^T \neq 0$ denotes the vector in the expansion

$$2^{2m-1}(x-1)^m \ = \ \frac{f_0}{\sqrt{2}} \ + \ \sum_{j=1}^m f_j T_j(x) \ . \tag{3.33}$$

Consequently, we obtain from (3.30) for the weights in the convex combinaion (3.31)

$$\lim_{a \to 0} \alpha_j(a) \ = \ \alpha_j^* \ = \ \frac{|f_j|^2}{\sum_{i=0}^m |f_i|^2} \ , \qquad j = 0, \ldots, m \ . \tag{3.34}$$

Finally, Corollary 4.2 in Dette and Melas (2001) shows that for $j = 0, \ldots, m$ the optimal design $\xi_{(2j)}$ for estimating the individual coefficient β_{2j} in the trigonometric regression model (1.1) on the interval $[-a, a]$ converges weakly in the following sense

$$\lim_{a \to 0} \xi_{(2j)}([-a, at]) = \zeta([-1, t]) \ , \qquad t \in [-1, 1] \ ,$$

where the limiting design ζ is given by

$$\zeta = \begin{pmatrix} -y_m & -y_{m-1} & \ldots & -y_1 & y_0 & y_1 & \ldots & y_{m-1} & y_m \\ \frac{1}{4m} & \frac{1}{2m} & \ldots & \frac{1}{2m} & \frac{1}{2m} & \frac{1}{2m} & \ldots & \frac{1}{2m} & \frac{1}{4m} \end{pmatrix} \tag{3.35}$$

with

$$y_i = \cos\left(\frac{\pi(m-i)}{2m}\right), \quad i = 0, \ldots, m.$$

Consequently, equation (3.31) shows that $\hat{\xi}_\alpha$ has the same weak limit, i. e.

$$\lim_{a\to 0} \hat{\xi}_a([-a, at]) = \zeta[-1, t] \;, \qquad t \in [-1, 1] \;,$$

and assumption (iii) of Lemma 3.1 follows by rewriting this statement in terms of the support points and weights of the designs $\hat{\xi}_a$ and ζ, respectively. □

Theorem 3.2. *For sufficiently small $a > 0$ the design $\hat{\xi}_a$ defined in (3.7) and (3.8) is E-optimal for the trigonometric regression model (1.1) on the interval $[-a, a]$. The minimum eigenvalue is given by $\lambda_{\min}(M(\hat{\xi}_a)) = \lambda_a$ where*

$$\lambda_a^{-1} = q_a^T q_a = \frac{2}{\pi}\int_{-1}^{1} T_m^2\left(\frac{2x-1-\alpha}{1-\alpha}\right)\frac{dx}{\sqrt{1-x^2}}$$

and the vector $q_a = (q_{a0}, \ldots, q_{a0})^T$ is defined by the expansion (3.5).

Proof: Recalling the definition of the design $\hat{\xi}_a$ in (3.7) and (3.8) , we will study the asymptotic behaviour of the matrix

$$a^{4m} M^{-1}(\hat{\xi}_a)$$

as $a \to 0$. To this end let

$$U_k(x) = \frac{\sin((k+1)\mathrm{arccos} x)}{\sin(\arccos x)} \qquad k \geq 0 \tag{3.36}$$

denote the Chebyshev polynomial of the second kind and define

$$u = u(t) = \frac{2(1-\cos t)}{a^2} \;.$$

Obviously $\cos(kt) = T_k(1 - \frac{a^2}{2}u)$, $\sin(kt)/\sin t = U_{k-1}(1 - \frac{a^2}{2}u)$ and consequently there exists an $(m+1) \times (m+1)$ matrix $S_{(1)}$ and an $m \times m$ matrix $S_{(2)}$ such that the vector

$$\bar{f}(t) = \left(f_c^T(t), f_s^T(t)\right)^T \;\in\; \mathbb{R}^{2m+1}$$

can be represented as

$$\bar{f}(t) = SA\tilde{f}(u(t)) \;, \tag{3.37}$$

where

$$\tilde{f}(t) = \left(\frac{1}{\sqrt{2}}, u(t), \ldots, u^m(t), \frac{\sin t}{a}, \frac{\sin t}{a}u(t), \ldots, \frac{\sin t}{a}u^{m-1}(t)\right)^T \tag{3.38}$$

and the matrices A and S are defined by

$$(3.39)\quad A = A(a) = \operatorname{diag}\Big\{1, \frac{a^2}{2}, \ldots, (\frac{a^2}{2})^m, a, a(\frac{a^2}{2}) \ldots, a(\frac{a^2}{2})^{m-1}\Big\} \in \mathbb{R}^{2m+1 \times 2m+1}$$

and

$$(3.40)\qquad S = \begin{pmatrix} S_{(1)} & 0 \\ 0 & S_{(2)} \end{pmatrix} \in \mathbb{R}^{2m+1 \times 2m+1},$$

respectively. It is easy to see that the matrices $S_{(1)}$, $S_{(2)}$ do not depend on the parameter a and are lower triangular with nonvanishing diagonal elements. Consequently, we obtain an alternative representation for the matrix $\bar{M}(\hat{\xi}_a)$ defined in (3.25)

$$(3.41)\qquad \bar{M}(\hat{\xi}_a) = SA\tilde{M}(\hat{\xi}_a)AS^T$$

where

$$(3.42)\qquad \tilde{M}(\xi) = \int \tilde{f}(t)\tilde{f}^T(t)d\xi(t) \in \mathbb{R}^{2m+1 \times 2m+1}.$$

Now let

$$\hat{f}(t) = \Big(\frac{1}{\sqrt{2}}, t^2, \ldots, t^{2m}, t, t^3, \ldots, t^{2m-1}\Big)^T$$

and define for any design ξ

$$(3.43)\qquad \hat{M}(\xi) = \int \hat{f}(t)\hat{f}^T(t)d\xi(t)$$

as the corresponding information matrix. From the expansions

$$1 - \cos(at) = \frac{(at)^2}{2}(1 + o(a)) \quad \text{and} \quad \sin(at) = at(1 + o(a))$$

and (3.38) it is easy to see that

$$\lim_{a \to 0} \tilde{f}(at) = \hat{f}(t).$$

Consequently we obtain from the third part of Lemma 3.1 and the definition (3.42) that

$$(3.44)\qquad \lim_{a \to 0} \tilde{M}(\hat{\xi}_a) = \hat{M}(\zeta)$$

where ζ is the limiting design defined in (3.35). Moreover, from (3.39) we have

$$(3.45)\qquad \lim_{a \to 0}(\frac{a^2}{2})^m A^{-1}(a) = \operatorname{diag}(\underbrace{0, \ldots, 0}_{m}, 1, \underbrace{0, \ldots, 0}_{m}),$$

which gives for the matrix $\bar{M}(\hat{\xi}_a)$ in (3.41)

$$(3.46)\qquad \lim_{a \to 0}(\frac{a^2}{2})^{2m} \bar{M}^{-1}(\hat{\xi}_a) = (S^T)^{-1}DS^{-1}$$

and for the corresponding $(m+1) \times (m+1)$ block

$$(3.47)\qquad \lim_{a \to 0}(\frac{a^2}{2})^{2m} M_c^{-1}(\hat{\xi}_a) = (S_{(1)}^T)^{-1}D_{(1)}S_{(1)}^{-1},$$

where the matrix $D \in \mathbb{R}^{2m+1 \times 2m+1}$ is defined by

$$D = \begin{pmatrix} D_{(1)} & 0 \\ 0 & 0 \end{pmatrix}$$

with

$$D_{(1)} = \Delta e_m e_m^T \in \mathbb{R}^{m+1 \times m+1},$$

$e_m = (0, \ldots, 0, 1)^T \in \mathbb{R}^{m+1}$,

$$\Delta = e_m^T \hat{M}_*^{-1}(\zeta) e_m.$$

and $\hat{M}_*(\zeta)$ denotes the $m+1 \times m+1$ matrix formed by the first $m+1$ rows and columns of the matrix $\hat{M}(\zeta)$ defind in (3.43). Because the matrices D and $D_{(1)}$ have rank one, the matrices on the right hand sides of (3.46) and (3.47) have only one non-vanishing eigenvalue. By the discussion at the end of the proof of Lemma 3.1 we have

$$\lim_{a \to 0} \left(\frac{a^2}{2} \right)^m q_a = f \neq 0 \in \mathbb{R}^{m+1} \ ,$$

where the vector f is defined by the expansion (3.33). Similary, it follows for the eigenvalue λ_a^{-1} of $\bar{M}^{-1}(\hat{\xi}_a)$

$$\lim_{a \to 0} \left(\frac{a^2}{2} \right)^{2m} \lambda_a^{-1} = \lim_{a \to 0} \left(\frac{a^2}{2} \right)^{2m} q_a^T q_a = f^T f \neq 0.$$

Consequently, the continuous dependence of the eigenvalues of a matrix from its elements [see Lancaster (1969)], formula (3.46) and (3.47) imply that for sufficiently small a the matrices

$$(\frac{a^2}{2})^{2m} \bar{M}^{-1}(\hat{\xi}_a), \quad (\frac{a^2}{2})^{2m} M_c^{-1}(\hat{\xi}_a)$$

have a maximal eigenvalue of multiplicity 1, which is given by

$$(\frac{a^2}{2})^{2m} \lambda_a^{-1}.$$

In other words, the minimal eigenvalue λ_a of the matrix $\bar{M}(\hat{\xi}_a)$ has multiplicity 1, provided that a is close to 0.

Now let $0 < a < \bar{a}$ be sufficiently small such that this property is satisfied. By Lemma 3.1 (ii) the vector $\bar{q}_a = (q_{a0}, 0, q_{a1}, 0, \ldots, 0, q_{am})^T$ is the eigenvector corresponding to λ_a and we define $A^* = \lambda_a \bar{q}_a \bar{q}_a^T$. With these notations we have from (3.5)

$$\max_{t \in [-a,a]} f^T(t) A^* f(t) = \max_{t \in [-a,a]} \frac{(q_a^T f_c(t))^2}{q_a^T q_a} = \max_{x \in [\alpha,1]} \frac{T_m(\frac{2x-1-\alpha}{1-\alpha})}{q_a^T q_a} =$$

$$= \frac{1}{q_a^T q_a} = \lambda_a = \lambda_{\min}(\bar{M}(\hat{\xi}_a)) = \lambda_{\min}(M(\hat{\xi}_a))$$

and the optimality of the design $\hat{\xi}_a$ follows from the equivalence theorem given in Lemma 2.1.

Finally, the integral representation of λ_a^{-1} follows from the orthogonality properties (3.2) of the Chebyshev polynomials and the representation (3.5), i. e.

$$\begin{aligned}
\lambda_a^{-1} &= q_a^T q_a = \sum_{i,j=0}^{m} q_{ai} q_{aj} \frac{2}{\pi} \int_{-1}^{1} b_j T_i(x) T_j(x) \frac{dx}{\sqrt{1-x^2}} \\
&= \frac{2}{\pi} \int_{-1}^{1} \Big(\frac{q_{a0}}{\sqrt{2}} + \sum_{i=1}^{m} q_{ai} T_i(x) \Big)^2 \frac{dx}{\sqrt{1-x^2}} \\
&= \frac{2}{\pi} \int_{-1}^{1} T_m^2 \left(\frac{2x-1-a}{1-a} \right) \frac{dx}{\sqrt{1-x^2}},
\end{aligned}$$

where $b_0 = 1/2$, $b_j = 1$, if $j \geq 1$.

□

The following Corollary is an immediate consequence of Theorem 3.2 and its proof.

Corollary 3.3. *Let*

(3.48) $$\underline{a} = \underline{a}(m) = \sup\{a > 0 \mid \lambda_{\min}(M(\hat{\xi}_a)) = \lambda_a\} \ ,$$

where λ_a *is defined in (3.10). Whenever* $0 < a < \underline{a}$*, the E-optimal design for the trigonometric regression model (1.1) on the interval* $[-a, a]$ *is given by the design* $\hat{\xi}_a$ *defined in (3.7) and (3.8). Moreover*

(3.49) $$\underline{a} = \min\{\underline{a}_{(1)}, \underline{a}_{(2)}\},$$

where the quantities $\underline{a}_{(1)}$ *and* $\underline{a}_{(2)}$ *are given by*

(3.50) $$\begin{aligned}
\underline{a}_{(1)} &= \underline{a}_{(1)}(m) = \sup\{a > 0 | \lambda_{\min}(M_c(\hat{\xi}_a)) = \lambda_a\} \\
\underline{a}_{(2)} &= \underline{a}_{(2)}(m) = \sup\{a > 0 | \lambda_{\min}(M_s(\hat{\xi}_a)) > \lambda_a\}
\end{aligned}$$

The quantities $\underline{a}_{(1)}$, $\underline{a}_{(2)}$ have been calculated numerically for lower order trigonometric regression models and are listed in Table 1. Note that these values are rather close to the upper bound $\bar{a} = \pi(1 - 1/(2m+1))$ obtained in Section 2 and consequently Theorem 2.3 and Corollary 3.3 cover a rather large range of the interval $(0, \pi]$ for the parameter a of the design space $[-a, a]$. Moreover, the Table indicates that both values might be equal in general and in Section 5 we will prove that $\underline{a}_{(1)} = \underline{a}_{(2)}$ for all $m \in \mathbb{N}$.

Note that Table 1 does not contain the case $m = 1$, for which a complete analytic solution is presented in the following section. For later purposes we require the following auxiliary result, which is probably of independent interest.

Table 1: *Bounds* $\bar{a}$, $\underline{a} = \min\{\underline{a}_{(1)}, \underline{a}_{(2)}\}$ *obtained in Sections 2 and 3. The E-optimal design for the trigonometric regression model (1.1) on the interval* $[-a, a]$ *can be found analytically, whenever* $a \in (0, \underline{a}] \cup [\bar{a}, \pi]$

m	$\underline{a}_{(1)}$	$\underline{a}_{(2)}$	$\bar{a}$
2	0.741π	0.741π	0.8π
3	0.794π	0.794π	0.857π
4	0.827π	0.827π	0.889π
5	0.851π	0.851π	0.909π

Lemma 3.4. *Let* $0 < a < \bar{a} = \pi(1 - 1/(2m+1))$ *and* ξ^* *denote the E-optimal design for the trigonometric regression model (1.1) on the interval* $[-a, a]$*. If the minimum eigenvalue of the information matrix* $M(\xi^*)$ *has multiplicity 1, then*

$$\xi^* = \hat{\xi}_a,$$

where the design $\hat{\xi}_a$ *is defined in (3.7).*

Proof: From Theorem 2.3 we have that the E-optimal design ξ^* is unique and of the form

$$\xi^* = \begin{pmatrix} -t_m^* & \dots & -t_1^* & t_0^* & t_1^* & \dots & t_m^* \\ \frac{w_m^*}{2} & \dots & \frac{w_1^*}{2} & w_0^* & \frac{w_1^*}{2} & \dots & \frac{w_m^*}{2} \end{pmatrix}.$$

Now let

$$\lambda^* = \lambda_{\min}(M(\xi^*)) = \min\{\lambda_{\min}(M_c(\xi^*)), \lambda_{\min}(M_s(\xi^*))\}$$

denote the minimum eigenvalue of the matrix $M(\xi^*)$ and consider at first the case where $\lambda^* = \lambda_{\min}(M_c(\xi^*))$. Obviously, λ^* is a simple eigenvalue of $M_c(\xi^*)$ and we define $q = (q_0, \dots, q_m)^T$ as the corresponding eigenvector. With the notation $\bar{q} = (q_0, 0, q_1, 0, \dots, 0, q_m)^T$ and $A^* = \bar{q}\bar{q}^T / \bar{q}^T \bar{q}$ it follows from Lemma 2.1 that (note the ξ^* is E-optimal)

$$\lambda^* = \max_{t \in [-a,a]} f^T(t) A^* f(t) = \max_{t \in [-a,a]} \frac{(q^T f_c(t))^2}{q^T q}.$$

Consequently, the polynomial

$$\Psi(x) = q^T f_c(\arccos x)$$

attains its maximum absolute value in the interval $[\alpha, 1]$ at the $m+1$ points $x_i^* = \cos t_i^*$ $(i = 0, \dots, m)$ and must coincide with the polynomial

$$\mp T_m \left(\frac{2x - 1 - \alpha}{1 - \alpha} \right),$$

which implies $\mathrm{supp}(\xi^*) = \mathrm{supp}(\hat{\xi}_a)$, $q = \mp q_a$ and $\lambda^* = \lambda_a = 1/q_a^T q_a$. From the equation

$$M_c(\xi^*) q_a = \lambda_a q_a$$

it is then easy to see that the weights w_i^* must coincide with the weights of the design $\hat{\xi}_a$ given in (3.8) and it follows that $\xi^* = \hat{\xi}_a$.
Secondly, if $\lambda^* = \lambda_{\min}(M_s(\xi^*)) < \lambda_{\min}(M_c(\xi^*))$, then a similar argument shows that ξ^* is concentrated at $2m$ points, which is impossible. □

4 Example: the linear trigonometric regression model on a partial circle

In this section we study the linear trigonometric regression model on the interval $[-a, a]$, which indicates that even this relatively simple case is not trivial. Our next proposition specifies the E-optimal designs in the linear trigonometric regression model. In this case it proves that $\underline{a} = \bar{a}$ and we will show in the following section that this equality only holds in the linear case.

Proposition 4.1. *Consider the linear trigonometric regression model (1.1) on the interval $[-a, a]$.*

(i) If $\bar{a} = 2\pi/3 \leq a \leq \pi$, then an E-optimal design for the model (1.1) is given by

$$\xi_3^* = \begin{pmatrix} \frac{-2\pi}{3} & 0 & \frac{2\pi}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

(ii) If $0 < a \leq \underline{a} = 2\pi/3$, then the E-optimal design for the model (1.1) is unique and given by

$$\xi_a^* = \begin{pmatrix} -a & 0 & a \\ \frac{\mu(a)}{2} & 1 - \mu(a) & \frac{\mu(a)}{2} \end{pmatrix} \tag{4.1}$$

where

$$\mu(a) = \frac{4 + 2\cos a}{4 + 2(1 + \cos a)^2}. \tag{4.2}$$

Proof: The first point and the statement of uniqueness in (ii) follows from Theorem 2.3, which also shows that the E-optimal design is of the form (4.1), whenever $0 < a < \bar{a}$. If a is sufficiently small, we can use Theorem 3.2 and Corollary 3.3 and obtain from the representation of the weights by the first part of Lemma 3.1 [$x_0 = 1$, $x_1 = \alpha = \cos a$,

$q_{a0} = -\sqrt{2}(1+\alpha)/(1-\alpha)$, $q_{a1} = 2/(1-\alpha)]$

$$\begin{aligned} 1-\mu(a) &= \left\{\frac{4+2(1+\alpha)^2}{(1-\alpha)^2}\right\}^{-1} \frac{2}{\pi}\int_{-1}^{1} \frac{x-\alpha}{1-\alpha}\left\{q_{a1}T_1(x)+\frac{q_{a0}}{\sqrt{2}}\right\}\frac{dx}{\sqrt{1-x^2}} = \\ &= 2\frac{1+\alpha+\alpha^2}{4+2(1+\alpha)^2}\ , \end{aligned}$$

where we have used the orthogonality relation for the Chebyshev polynomials of the first kind. The representation (4.2) now follows from a trivial calculation, i.e.

$$\mu(a) = \frac{4+2\alpha}{4+2(1+\alpha)^2}. \tag{4.3}$$

Note that this formula can also be obtained from the representation $w_0 = q_a^T F^{-1} e_0 / q_a^T q_a$, where $e_0 = (1,0)^T$ and

$$F = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & \alpha \end{pmatrix}.$$

A straightforward calculation shows that

$$\lambda_a = \frac{(1-\alpha)^2}{4+2(1+\alpha)^2} \tag{4.4}$$

is the minimum eigenvalue of $M_c(\hat{\xi}_a)$ and has multiplicity 1. Consequently the critical value $\underline{a}$ can be obtained as

$$\begin{aligned} \underline{a} &= \sup\Big\{a \in (0,\bar{a}) | \lambda_{\min}(M_c(\hat{\xi}_a)) < \lambda_{\min}(M_s(\hat{\xi}_a))\Big\} \\ &= \inf\Big\{a \in (0,\bar{a}) | \lambda_{\min}(M_c(\hat{\xi}_a)) = \lambda_{\min}(M_s(\hat{\xi}_a))\Big\}\ , \end{aligned}$$

which gives the equation

$$\lambda_a = \frac{(1-\alpha)^2}{4+2(1+\alpha)^2} = \mu(a)(1-\alpha^2) = \frac{4+2\alpha}{4+2(1+\alpha)^2}(1-\alpha^2) = \frac{(1+\alpha)(4+2\alpha)}{1-\alpha}\lambda_a\ ,$$

where we have used the representation (4.3) and (4.4) for the last equalities. This yields the equation $2\alpha^2 + 7\alpha + 3 = 0$, which gives as unique solution in the interval $[-1,1]$

$$\cos \underline{a} = \underline{\alpha} = -\frac{1}{2}, \quad \underline{a} = \frac{2\pi}{3}.$$

By Theorem 3.2 and Corollary 3.3 the E-optimal design for the trigonometric regression model (1.1) on the interval $[-a,a]$ is given by (4.1) and (4.2), whenever $a \in (0,\underline{a}]$, which proves part (ii) of the proposition. □

5 E-optimal designs on arbitrary intervals

As it was shown in Section 2 we can restrict the discussion of the E-optimal design problem to the case of symmetric intervals $[-a, a]$, $0 < a \leq \pi$. For $0 < a \leq \underline{a} = \underline{a}(m)$ and $\bar{a} = \bar{a}(m) \leq a \leq \pi$ we have already received explicit solutions for the E-optimal design problem in the trigonometric regression model (1.1) on the interval $[-a, a]$. Note that the range $(\underline{a}, \bar{a})$ not covered by these results is rather small (see Table 1) and consequently explicit solutions of the E-optimal design problem are available for most cases. Moreover, in Section 4 we have shown that in the linear trigonometric regression model with $m = 1$ we have $\underline{a} = \bar{a} = 2\pi/3$ and a complete analytic solution is available in this case.
Now we will prove that for $m \geq 2$ it follows that $\underline{a} < \bar{a}$ and elaborate a technique for the case $\underline{a} < a < \bar{a}$, which can be used for the numerical construction of E-optimal designs and is based on a functional approach described in Dette, Melas and Pepelyshev (2000). Roughly speaking this method shows that the support points and weights of the E-optimal design are real analytic functions of the parameter $a \in (\underline{a}, \bar{a})$ and provides a Taylor expansion for these functions, which can be used to find the E-optimal designs numerically. The method will be illustrated for the quadratic and cubic trigonometric regression model at the end of this section.
We begin with a reformulation of Lemma 2.1. To this end let us introduce the function

$$(5.1) \qquad \Psi(x) \;=\; \Psi(x; q, p) \;=\; \frac{(q^T f_{(1)}(x))^2 + (1 - x^2)(p^T f_{(2)}(x))^2}{q^T q + p^T p},$$

where $q = (q_0, \ldots, q_m)^T \in \mathbb{R}^{m+1}$ is an arbitrary vector with $q_m = 1$, $p = (p_0, \ldots, p_{m-1})^T \in \mathbb{R}^m$ is an arbitrary vector and the functions $f_{(1)}(x)$ and $f_{(2)}(x)$ are defined by

$$(5.2) \qquad f_{(1)}^T(x) \;=\; \left(1/\sqrt{2}, T_1(x), \ldots, T_m(x)\right),$$

$$(5.3) \qquad f_{(2)}^T(x) \;=\; (U_0(x), \ldots, U_{m-1}(x)).$$

Due to Theorem 2.3 we can restrict our consideration to the case $a < \bar{a}$ and designs $\xi \in \Xi_a^{(1)}$. The following result is a refinement of Lemma 2.1 for the model at hand.

Lemma 5.1. *For the trigonometric regression model (1.1) on the interval $[-a, a]$ with $0 < a < \bar{a}$ the design*

$$(5.4) \qquad \xi = \begin{pmatrix} -t_m & \ldots & -t_1 & t_0 & t_1 & \ldots & t_m \\ \frac{w_m}{2} & \ldots & \frac{w_1}{2} & w_0 & \frac{w_1}{2} & \ldots & \frac{w_m}{2} \end{pmatrix},$$

with $t_0 = 0$, $t_m = a$ is E-optimal if and only if there exist vectors $q = q(a) = (q_0, \ldots, q_m)^T \in \mathbb{R}^{m+1}$ with $q_m = 1$ and a vector $p = p(a) \in \mathbb{R}^m$, such that the inequality

$$(5.5) \qquad \Psi(x) = \Psi(x; q, p) \leq \lambda_{\min}(M(\xi))$$

holds for all $x \in [\alpha, 1]$, where the function $\Psi(x; q, p)$ is defined in (5.1).

Moreover, if a design ξ of the form (5.4) is E-optimal then these vectors are eigenvectors of the matrices defined by (3.26) and (3.27) corresponding to the minimum eigenvalue $\lambda = \lambda_{\min}(M(\xi))$ of the matrix $M(\xi)$, that is

$$M_c(\xi)q = \lambda q, \;\; M_s(\xi)p = \lambda p, \tag{5.6}$$

and

$$\begin{aligned} \Psi(x_i) &= \Psi(\cos a), \quad , \quad i = 1, \ldots, m-1 \;, \\ \Psi'(x_i) &= 0, \quad , \quad i = 1, \ldots, m-1 \;, \end{aligned} \tag{5.7}$$

where $x_i = \cos t_i$, $i = 0, 1, \ldots, m-1$.

The polynomial $\Psi(x)$ is uniquely determined. The vectors p and q can be chosen such that the polynomials

$$p^T f_{(2)}(x) \quad \textit{and} \quad q^T f_{(2)}(x)$$

have interlacing roots and under this additional condition the vectors p and q are also uniquely determined. If $a \in [0, \underline{a}]$ it follows, that $p = 0$.

Proof. Let us prove that the inequality (5.5) is a necessary condition for E-optimality. To this end assume that a design ξ of the form (5.4) is E-optimal and let A^* be the matrix, defined in Lemma 2.1, such that the inequality (2.5) is satisfied.
Consider the function

$$\Psi(x) = h(\arccos x),$$

where $h(t) = f^T(t)A^*f(t)$, $t = \arccos x$. Note that due to Theorem 2.3 $\Psi(x) \not\equiv \text{const}$ whenever $0 < a < \bar{a}$. Since

$$\sin(k \arccos x) = \sqrt{1-x^2}U_{k-1}(x)$$

and

$$\cos(k \arccos x) = T_k(x),$$

it follows that $\Psi(x)$ is a polynomial of degree $2m$ [note that $\Psi(x)$ is not constant and by Lemma 2.1 has $2m-1$ roots counting multiplicities]. Our polynomial is nonnegative for $-1 \le x \le 1$ due to nonnegative definiteness of the matrix A^*. It is known (see Karlin, Studden, 1966, Ch. 2) that such a polynomial can be represented in the form

$$\Psi(x) = \varphi_1^2(x) + (1-x^2)\varphi_2^2(x), \tag{5.8}$$

where $\varphi_1(x)$ is a polynomial of degree m, $\varphi_2(x)$ is a polynomial of degree $m-1$, i.e.

$$\varphi_1(x) = C_1 \prod_{i=1}^{m} (x - \gamma_i), \;\; \varphi_2 = C_2 \prod_{i=1}^{m-1} (x - \delta_i), \tag{5.9}$$

and that the roots of these polynomials are interlacing, i.e.

$$\gamma_1 \le \delta_1 \le \gamma_2 \le \ldots \le \delta_{m-1} \le \gamma_m,$$

$$\gamma_1 < \gamma_2 < \ldots < \gamma_m, \;\; \delta_1 < \ldots < \delta_{m-1}.$$

Morover, this representation is unique. Since the polynomials $T_0(x), \ldots, T_m(x)$ are linearly independent and the same is true for the polynomials $U_0(x), \ldots, U_{m-1}(x)$, we have

$$\begin{aligned} \varphi_1(x) &= Cq^T f_{(1)}(x), \\ \varphi_2(x) &= Cp^T f_{(2)}(x), \end{aligned}$$

where $C > 0$ is a constant and $q = (q_0, \ldots, q_m)^T \in \mathbb{R}^{m+1}$, $p \in \mathbb{R}^m$ are appropriate vectors with $q_m = 1$. Recalling that the functions $\sqrt{2} f_k(t)$, $k = 0, 1, \ldots, 2m$ are orthonormal with respect to measure $\frac{1}{2\pi} dt$ on the interval $[-\pi, \pi]$ we obtain

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^T(t) A^* f(t) dt = \mathrm{tr} A^* = 1,$$

and, therefore,

$$\begin{aligned} 1 &= \frac{2}{\pi} \int_{-1}^{1} \Psi(x) \frac{dx}{\sqrt{1-x^2}} = \frac{2C}{\pi} \int_{-1}^{1} \Big[(q^T f_{(1)}(x))^2 + (p^T f_{(2)}(x))^2 (1-x^2) \Big] \frac{dx}{\sqrt{1-x^2}} \\ &= C(q^T q + p^T p). \end{aligned}$$

Consequently, $C = 1/(q^T q + p^T p)$ and due to Lemma 2.1 it follows for all $t \in [-a, a]$

$$f^T(t) A^* f(t) \leq \lambda_{\min}(M(\xi)),$$

or equivalently

$$\Psi(x) = \Psi(x; q, p) = \frac{(q^T f_1(x))^2 + (1-x^2)(p^T f_{(2)}(x))^2}{q^T q + p^T p} \leq \lambda \tag{5.10}$$

for all $x \in [\alpha, 1]$, where $\lambda = \lambda_{\min}(M(\xi))$ denotes the minimum eigenvalue of the matrix $M(\xi)$. Therefore condition (5.5) follows from the E-optimality of the design ξ. Due to Lemma 2.1 the left hand side of the inequality (5.10) attains its maximal value λ at the support points $x_i = \cos t_i$, $i = 0, \ldots, m$ (since $\Psi(x) = h(t)$, $t = \arccos x$) and the system of equations in (5.7) provides also a necessary condition for E-optimality.
To prove that (5.6) is also a necessary condition for E-optimality we put $x = \cos t$ and integrate the left hand side of (5.10) with respect to the measure $\xi(dt)$. We receive

$$\frac{q^T M_c(\xi) q + p^T M_s(\xi) p}{p^T p + q^T q} \leq \lambda, \tag{5.11}$$

where the second term should be replaced by zero if $p = 0$. Since

$$\begin{aligned} \min_{\tilde{q}} \frac{\tilde{q}^T M_c(\xi) \tilde{q}}{\tilde{q}^T \tilde{q}} &= \lambda_{\min}(M_c(\xi)) \geq \lambda \,, \\ \min_{\tilde{p}} \frac{\tilde{p}^T M_s(\xi) \tilde{p}}{\tilde{p}^T \tilde{p}} &= \lambda_{\min}(M_s(\xi)) \geq \lambda \,, \end{aligned} \tag{5.12}$$

it follows that q is an eigenvector of the matrix $M_c(\xi)$ corresponding to its minimal eigenvalue λ, that is

$$M_c(\xi)q = \lambda q.$$

Similary, p is either equal to $0 \in \mathbb{R}^m$ or an eigenvector of the matrix $M_s(\xi)$ corresponding to its minimal eigenvalue λ. In both cases we have the equation

$$M_s(\xi)p = \lambda p.$$

Finally, we prove that (5.5) is a sufficient condition for E-optimality of the design ξ. To this end define

$$A = (qq^T + pp^T)/(q^T q + p^T p),$$

then tr $A = 1$ and it follows from (5.5) that for all $t \in [-a, a]$

$$f^T(t)Af(t) \leq \lambda_{\min}(M(\xi)).$$

Due to Lemma 2.1 the design ξ is E-optimal.

Note that the polynomial $\Psi(x)$ is uniquely determined by the conditions (5.7) and (5.5). Moreover, we proved above that the vectors p and q are uniquely determined under the additional condition of interlacing roots.
Let $0 < a < \underline{a}$, then $\lambda_{\min}(M_s(\xi)) > \lambda$ and from (5.11) and (5.12) it follows that $p = 0$. In the case $a = \underline{a}$ the equality $p = 0$ follows from a continuity argument. $\Box$

Lemma 5.1 will be used to obtain a representation for the minimal eigenvalue of the information matrix of the E-optimal design. This representation will be essential for the numerical construction of E-optimal designs.

Lemma 5.2. *For the trigonometric regression model (1.1) with $m \geq 2$ we have for the quantities $\underline{a}$ and $\bar{a}$ defined in (3.48) and (2.11), respectively,*

$$\underline{a} < \bar{a}.$$

Proof. It is evident that $\underline{a} \leq \bar{a}$. Suppose that $\underline{a} = \bar{a}$, then Theorem 3.2 and Corollary 3.3 show that for $a \leq \underline{a}$ the design $\hat{\xi}_a$ defined by (3.7) and (3.8) is E-optimal. For $a < \bar{a}$ there exists a unique E-optimal design by Theorem 2.3 and a continuity argument shows that there also exists a unique E-optimal design in the case $a = \underline{a} = \bar{a}$, which is of the form

$$\xi^* = \begin{pmatrix} -t_m & \dots & -t_1 & t_0 & t_1 & \dots & t_m \\ \frac{1}{2m+1} & \dots & \frac{1}{2m+1} & \frac{1}{2m+1} & \frac{1}{2m+1} & \dots & \frac{1}{2m+1} \end{pmatrix},$$

where the support points are given by

$$t_i = \frac{\pi i}{m}\Big(1 - \frac{1}{2m+1}\Big) , \qquad i = 0, \ldots, m.$$

Therefore, $\xi^* = \hat{\xi}_a$ and we obtain the equations

$$\cos t_i \;=\; \cos\Big[\frac{\pi i}{m}\Big(1-\frac{1}{2m+1}\Big)\Big] \;=\; \frac{1-\bar{\alpha}}{2}\cos\frac{\pi i}{m} + \frac{1+\bar{\alpha}}{2} \;, \quad i = 0, \ldots, m \;,$$

where $\bar{\alpha} = \cos\bar{a}$, $\bar{a} = \pi(1 - \frac{1}{2m+1})$. In order to prove that this is impossible we note that for $0 < a < \pi$, $1/2 < u < 1$ it follows that

$$\cos au \;>\; \frac{1-\cos a}{2}\cos\pi u + \frac{1+\cos a}{2} \;. \tag{5.13}$$

This inequality can be proved observing that for $a = 0$ and $a = \pi$ we have

$$\cos au - \frac{1-\cos a}{2}\cos\pi u - \frac{1+\cos a}{2} = 0$$

and verifying that the derivative of the left hand side has only one zero in the interval $(0, \pi)$ corresponding to an absolute maximum in this region. Substituting $a = \bar{a}$, $u = i/m$ in (5.13) we obtain a contradiction, which shows that $\underline{a} < \bar{a}$, whenever $m \geq 2$. $\Box$

Throughout the remaining part of this section we assume $m \geq 2$ (the linear case $m = 1$ was discussed in Section 4), $\underline{a} < a < \bar{a}$ and define

$$\begin{aligned}
\tilde{p}(a) \;&=\; p(a) \;\in \mathbb{R}^m \;, \\
\tilde{q}(a) \;&=\; (q_0(a), \ldots, q_{m-1}(a))^T \;\in \mathbb{R}^m \;, \\
x(a) \;&=\; (x_1(a), \ldots, x_{m-1}(a))^T \;\in \mathbb{R}^{m-1} \;, \\
w(a) \;&=\; (w_0(a), \ldots, w_{m-1}(a))^T \;\in \mathbb{R}^m \;,
\end{aligned}$$

where $p(a)$ and $q(a) = (q_1(a), \ldots, q_{m-1}(a), 1)^T$ are the vectors defined by Lemma 5.1, $x_i(a) = \cos t_i(a)$, $i = 1, \ldots, m-1$ and $\{t_i(a)\}_{i=1,\ldots,m-1}$, $\{w_i(a)\}_{i=0,\ldots,m-1}$ correspond to the positive support points and weights of the E-optimal design ξ_a on the interval $[-a, a]$. For arbitrary vectors $\tilde{q} = (q_0, \ldots, q_{m-1})^T$, $\tilde{p} = (p_0, \ldots, p_{m-1})^T$, $x = (x_1, \ldots, x_{m-1})^T$, $w = (w_0, \ldots, w_{m-1})^T$, with $\alpha = \cos a < x_{m-1} < \ldots < x_1 < 1$, $w_i > 0$, $i = 0, \ldots, m-1$, $\sum_{i=0}^{m-1} w_i < 1$ we define the vectors

$$\Theta \;=\; (\theta_0, \ldots, \theta_{4m-2})^T = (\tilde{p}^T, \tilde{q}^T, x^T, w^T)^T \in \mathbb{R}^{4m-1}, \tag{5.14}$$

and similary

$$\Theta(a) \;=\; (\theta_0(a), \ldots, \theta_{4m-2}(a))^T = (\tilde{p}^T(a), \tilde{q}^T(a), x^T(a), w^T(a))^T \in \mathbb{R}^{4m-1}$$

as the vector containing the support points and weights of the E-optimal design and the components of the vectors $q(a)$ and $p(a)$ defined in Lemma 5.1. Let us introduce the function

$$
\begin{aligned}
(5.15) \qquad \lambda(\Theta, a) \;=\; & \sum_{i=0}^{m-1} \frac{(q^T f_{(1)}(x_i))^2 + (1 - x_i^2)(p^T f_{(2)}(x_i))^2}{q^T q + p^T p} w_i + \\
+ \; & \frac{(q^T f_{(1)}(\alpha))^2 + (1 - \alpha^2)(p^T f_{(2)}(\alpha))}{q^T q + p^T p} (1 - w_0 - \ldots - w_{m-1}),
\end{aligned}
$$

where $x_0 = 1$ and the vectors q and p are given by $q = (\tilde{q}^T, 1)^T$, $p = \tilde{p}$. If ξ_a is the E-optimal design on the interval $[-a, a]$, then

$$
\lambda(a) := \lambda(\Theta(a), a) = \lambda_{\min}(M(\xi_a)),
$$

and an immediate differentiation of the function $\lambda(\Theta, a)$ shows that the conditions

$$
(5.16) \qquad \frac{\partial}{\partial \theta_i} \lambda(\Theta, a) \,|_{\Theta = \bar{\Theta}} = 0, \quad i = 0, \ldots, 4m - 2
$$

coincide with conditions (5.6) and (5.7) if $\bar{\Theta} = \Theta(a)$. Therefore, by Lemma 5.1, these conditions are necessary conditions for the vector $\Theta(a)$, which gives the support points and weights of the E-optimal design. We will call the vector equation (5.16) *basic equation.* In order to study the Jacobi matrix of this equation we will present a couple of auxiliary results, which are of independent interest. To this end denote with

$$
(5.17) \qquad \eta = \begin{pmatrix} x_0 & \ldots & x_m \\ w_0 & \ldots & w_m \end{pmatrix}
$$

a design on the interval $[\alpha, 1]$ (with $x_0 = 1$) and let

$$
(5.18) \qquad \xi_\eta = \begin{pmatrix} -t_m & \ldots & -t_1 & t_0 & t_1 & \ldots & t_m \\ \frac{w_m}{2} & \ldots & \frac{w_1}{2} & w_0 & \frac{w_1}{2} & \ldots & \frac{w_m}{2} \end{pmatrix}
$$

be the design corresponding to η by the transformation (3.17), where $t_i = \arccos x_i$, $i = 0, \ldots m$. Similary, for any symmetric design ξ of the form (5.18) on the interval $[-a, a]$ we denote by

$$
\eta_\xi = \begin{pmatrix} x_0 & \ldots & x_m \\ w_0 & \ldots & w_m \end{pmatrix},
$$

with $x_i = \cos t_i$, $i = 0, \ldots, m$, the design on the interval $[\alpha, 1]$ obtained by the transformation (3.17). Finally, $v = v(a)$ denotes the multiplicity of the minimum eigenvalue of the matrix $M_c(\xi_a)$ and $u = u(a)$ is the multiplicity of the minimum eigenvalue of the matrix $M_s(\xi_a)$, where ξ_a denotes the E-optimal design for the trigonometric regression model (1.1) on the interval $[-a, a]$ and the matrices $M_c(\xi_a)$ and $M_s(\xi_a)$ have been defined in (3.26) and (3.27), respectively.

Lemma 5.3. *Let $0 < a \leq \pi$. A design ξ_a of the form (5.18) is an E-optimal design for the trigonometric regression model (1.1) on the interval $[-a, a]$, if and only if*

$$\xi_a = \xi_{\eta_\alpha}, \tag{5.19}$$

where η_α is an E-optimal design for the Chebyshev regression model (3.18) on the interval $[\alpha, 1]$ and $\alpha = \cos a$.
Moreover, the quantities $\underline{a}_{(1)}$ and $\underline{a}_{(2)}$ in (3.50) are equal, i.e.

$$\underline{a}_{(1)} = \underline{a}_{(2)}$$

and the multiplicities $v(a)$ and $u(a)$ of the minimal eigenvalues of the matrices $M_c(\xi_a)$ and $M_s(\xi_a)$ of the E-optimal design ξ_a satisfy

$$v(a) = u(a) + 1,$$

whenever $v(a) > 1$.

Proof. Let us begin with the last assertion, denote with ξ_a the E-optimal design, by $q_{(1)}, \ldots, q_{(v)}$ the eigenvectors of the matrix $M_c(\xi_a)$ corresponding to its minimal eigenvalue $\lambda_{\min}(M_c(\xi_a))$ and define the coordinates of $q_{(j)}$ by $q_{(j)i}$, $i = 0, \ldots, m$, $j = 1, \ldots, v$. Without loss of generality we can choose $q_{(1)}$ such that $q_{(1)m} = 1$, $q_{(2)}$ such that $q_{(2)m} = 0$, $q_{(3)}$ such that $q_{(3)m} = q_{(3)m-1} = 0$ etc. For $v \geq 2$ we introduce the polynomials

$$\begin{aligned}
\varphi_1^2(x) &= \frac{\left(q_{(1)}^T f_{(1)}(x)\right)^2}{q_{(1)}^T q_{(1)}}, \\
\varphi_2^2(x) &= \frac{\left(q_{(i)}^T f_{(1)}(x)\right)^2}{q_{(i)}^T q_{(i)}}, \qquad i \neq 1 \\
g(x) &= \frac{\varphi_1^2(x) + \varphi_2^2(x)}{2},
\end{aligned}$$

where the vectors $f_{(1)}(x)$ and $f_{(2)}(x)$ have been defined in (5.2) and (5.3), respectively. Note that the polynomial g is nonnegative, of degree m and

$$\int g(\cos t)\xi_a(dt) = \lambda_{\min}(M_c(\xi_a)). \tag{5.20}$$

As in the proof of Lemma 5.1 we can find appropriate vectors $q \in \mathbb{R}^{m+1}$ and $p \in \mathbb{R}^m$ such that the polynomial $g(x)$ can be represented in the form

$$g(x) = \bar{\varphi}_1^2(x) + (1 - x^2)\bar{\varphi}_2^2(x), \tag{5.21}$$

where $\bar{\varphi}_1(x) = q^T f_{(1)}(x)$, $\bar{\varphi}_2(x) = p^T f_{(2)}(x)$, Substituting $x = \cos t$, integrating both sides of (5.21) with respect to the measure $\xi_a(dt)$ and taking into account the identitiy (5.20) we receive

$$\begin{aligned}\lambda_{\min}(M_c(\xi_a)) &= q^T M_c(\xi_a) q + p^T M_s(\xi_a) p \\ &\geq \lambda_{\min}(M_c(\xi_a)) q^T q + \lambda_{\min}(M_s(\xi_a)) p^T p.\end{aligned} \tag{5.22}$$

A further integration of the function $g(\cos t)$ with respect to the uniform distribution $dt/2\pi$ on the interval $[-\pi, \pi]$ yields [observing the representation (5.21)]

$$q^T q + p^T p = 1. \tag{5.23}$$

In Section 3 we proved that

$$\lambda_{\min}(M_s(\xi_a)) \geq \lambda_{\min}(M_c(\xi_a)),$$

and consequently (5.22) and (5.23) imply that one of the following conditions holds

(i) $v = 1$, $p = 0$, $\lambda_{\min}(M_c(\xi_a)) = \lambda_{\min}(M(\xi_a)) < \lambda_{\min}(M_s(\xi_a))$,

(ii) $v > 1$, $p \neq 0$ is an eigenvalue of the matrix $M_s(\xi_a)$, $\lambda_{\min}(M_c(\xi_a)) = \lambda_{\min}(M_s(\xi_a))$.

The second part (ii) is an immediate consequence of the previous discussion. For a proof of the first case (i) assume that

$$\lambda = \lambda_{\min}(M_c(\xi_a)) = \lambda_{\min}(M_s(\xi_a))$$

and let p and q be vectors such that $p \neq 0$ and

$$M_c(\xi_a) q = \lambda q, \;\; M_s(\xi_a) p = \lambda p \;\; .$$

We introduce the polynomial

$$g(x) = \bar{\varphi}_1^2(x) + (1 - x^2)\bar{\varphi}_2(x),$$

where $\bar{\varphi}_1(x) = q^T f_{(1)}(x)$, $\bar{\varphi}_2(x) = p^T f_{(2)}(x)$. This polynomial can be represented in the form

$$\left(q_1^T f_{(1)}(x)\right)^2 + \left(q_2^T f_{(1)}(x)\right)^2$$

and a similar calculation as given in previous discussion shows that q_1 and q_2 should be eigenvectors, corresponding to $\lambda_{\min}(M_c(\xi_a))$. Therefore it follows that $v \geq 2$ and this proves that (i) is correct.

In the first case $\lambda_{\min}(M(\xi_a))$ is simple. In the second case $v \geq 2$ and for each eigenvector $q_{(i)}$ there exists an eigenvector $p_{(i)}$ of the matrix $M_s(\xi_a)$. It can be easily checked that the vectors $p_{(i)}$, $i = 2, \ldots, v$ are of the form

$$\begin{array}{l}(p_{(2)0}, \ldots, p_{(2)m-1})^T, \\ (p_{(3)0}, \ldots, p_{(3)m-2}, 0)^T, \\ \qquad\vdots \\ (p_{(v)0}, \ldots, p_{(v)m-v+1}, 0, \ldots, 0)^T.\end{array}$$

Consequently, these vectors are linearly independent, which gives $v(a) \geq u(a) + 1$. In a similar way we can prove that $v(a) \leq u(a) + 1$ and we obtain for the case $v(a) > 1$ that

$$v(a) = u(a) + 1.$$

From (5.22) and (5.23) it also follows that $v(a) > 1$ in the case $\lambda_{\min}(M_c(\xi_a)) = \lambda_{\min}(M_s(\xi_a))$. Recalling the definition of $\underline{a}_{(1)}$ and $\underline{a}_{(2)}$ in (3.50) it thus follows that

$$\begin{aligned} \underline{a}_{(1)} &= \inf\{a \mid v(a) > 1\}, \\ \underline{a}_{(2)} &= \inf\{a \mid \lambda_{\min}(M_c(\xi_a)) = \lambda_{\min}(M_s(\xi_a))\}\ , \end{aligned}$$

and the previous remarks yield

$$\underline{a}_{(1)} \;=\; \underline{a}_{(2)} \;=\; \underline{a}$$

In order to prove the first assertion of Lemma 5.3 let ξ_a be a symmetric E-optimal design of the form (5.18) for the trigonometric regression model (1.1) on the interval $[-a, a]$, then it follows from the previous discussion that

$$\lambda_{\min}(M(\xi_a)) = \lambda_{\min}(M_c(\xi_a)).$$

From the definition of the transformation (3.17) we have

$$M_c(\xi_a) = M_1(\eta_{\xi_a}),$$

where

$$M_1(\eta) = \int f_{(1)}(x) f_{(1)}^T(x) \eta(dx)$$

denotes the information matrix of the design η in the Chebyshev regression model (3.18). Therefore a design ξ_a is an E-optimal design for the regression function $f_c(t)$ on the interval $[-a, a]$ if and only if the design η_{ξ_a} is an E-optimal design in the Chebyshev regression model (3.18) on the interval $[\alpha, 1]$, where $\alpha = \cos a$. Now it is easy to verify that any E-optimal design of the form (5.18) for the regression function $f_c(t)$ on the interval $[-a, a]$ is also an E-optimal design for trigonometric regression model (1.1) on the interval $[-a, a]$ and vice versa. Thus a design ξ_{η_α} of the form (5.18) is an E-optimal design for the trigonometric regression model (1.1) on the interval $[-a, a]$ if and only if the corresponding design η_α is an E-optimal design for the Chebyshev regression model (3.18) on the interval $[\alpha, 1]$ $\Box$

Throughout this paper we denote by $\tau(a)$ the number of common roots of the polynomials $\varphi_1(x) = q^T f_{(1)}(x)$ and $\varphi_2(x) = p^T f_{(2)}(x)$ defined by Lemma 5.1. The following result provides the basis for the implementation of the functional approach.

Theorem 5.4. *Consider the trigonometric regression model (1.1) on the interval $[-a, a]$, where $0 < a \leq \pi$ and $m \geq 2$. Then $\underline{a} < \bar{a}$ and there exists a number $\nu \in \mathbb{N}$ and real quantities*

$$\underline{a} = a_1 < a_2 < a_3 < \ldots < a_\nu = \bar{a}$$

such that the vector-function

$$\Theta^* : \begin{cases} (\underline{a}, \bar{a}) & \to \mathbb{R}^{4m-1} \\ a & \to \Theta(a) \end{cases} \tag{5.24}$$

is uniquely determined, real analytic on the set

$$\bigcup_{j=1}^{\nu-1} (a_j, a_{j+1}) \tag{5.25}$$

and satisfies the system of equations

$$\frac{\partial}{\partial \theta_i} \lambda(\Theta, a)\Big|_{\Theta=\Theta(a)} \;=\; 0, \qquad i = 0, \ldots, 4m-2, \tag{5.26}$$

where the function $\lambda(\Theta, a)$ is defined in (5.15).

Proof. We have already proved above that the vector-function Θ^* is uniquely determined and satisfies (5.26). It is also obviously continuous. In order to study its analytic properties we define

$$G(\Theta, a) = \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \lambda(\Theta, a) \right)_{i,j=0}^{4m-2},$$

as the Jacobi matrix of the system (5.26) and denote by

$$J = J(a) = G(\Theta(a), a), \tag{5.27}$$

the corresponding value at the point $\Theta = \Theta(a)$. A straightforward but tedious differentiation shows that this matrix is of the form

$$J = h \begin{pmatrix} S & B_{(1)}^T & B_{(2)}^T \\ B_{(1)} & D & 0 \\ B_{(2)} & 0 & 0 \end{pmatrix}, \tag{5.28}$$

where $h = 1/(q^T q + p^T p)$, $q = q(a)$, $p = p(a)$. The matrices in the block matrix (5.28) are given by

$$S = \begin{pmatrix} M_{(1)} & 0 \\ 0 & M_{(2)} \end{pmatrix},$$

where

$$M_{(1)} = M_s(\xi_a) - \lambda I_m \ .$$

Similary, if A_- denotes the matrix A with deleted last row and last column, $M_{(2)}$ is defined by

$$M_{(2)} \;=\; (M_c(\xi_a) - \lambda I_{m+1})_-,$$

$$D \;=\; \mathrm{diag}\,\{d_{11}, \ldots, d_{m-1,m-1}\},$$

where the elements of the matrix D are given by

$$d_{ii} = \left((q^T f_{(1)}(x))^2 + (p^T f_{(2)}(x))^2(1-x^2)\right)''\Big|_{x=x_i(a)}, \quad i = 1,\ldots,m-1 \ ,$$

and

$$\begin{aligned}
B_{(1)}^T &= \left(B_{(1)1}^T \vdots B_{(1)2}^T\right), \\
B_{(1)1}^T &= \left((f_{(1)}(x)_{-} q^T f_{(1)}(x))' w_i\Big|_{x=x_i(a)}\right)_{i=1,\ldots,m-1}, \\
B_{(1)2} &= \left((f_{(1)}(x)(1-x^2))' w_i\Big|_{x=x_i(a)}\right)_{i=1,\ldots,m-1}, \\
B_{(2)}^T &= \left(B_{(2)1}^T \vdots B_{(2)2}^T\right), \\
B_{(2)1}^T &= \left(f_{(1)}(x_i)_{-} q^T f_{(1)}(x_i)\right)_{i=0,\ldots,m-1}, \\
B_{(2)2}^T &= \left(f_{(2)}(x_i) p^T f_{(2)}(x_i)(1-x^2)\right)_{i=0,\ldots,m-1},
\end{aligned}$$

where b_{-} denotes the vector b with deleted last element. Let $\tilde{a} \in (\underline{a}, \bar{a})$ such that the following condition is satisfied:

(A) *there exists a neighbourhood $\tilde{U}$ of the point $\tilde{a}$ such that for all $a \in \tilde{U}$ we have*

$$\tau = \tau(\tilde{a}) = \tau(a).$$

Denote by $\delta_1, \ldots, \delta_\tau$ the common roots of the polynomials

$$\begin{aligned}
\varphi_1(x) &= q^T(a) f_{(1)}(x), \\
\varphi_2(x) &= p^T(a) f_{(2)}(x),
\end{aligned}$$

by $\gamma_1, \ldots, \gamma_{m-\tau}$ the remaining roots of the polynomial $\varphi_1(x)$ and by $\kappa_1, \ldots, \kappa_{m-1-\tau}$ the remaining roots of the polynomial $\varphi_2(x)$, that is

$$\begin{aligned}
\varphi_1(x) &= \prod_{i=1}^{\tau}(x-\delta_i) \prod_{i=1}^{m-\tau}(x-\gamma_i), \\
\varphi_2(x) &= \kappa_{m-\tau} \prod_{i=1}^{\tau}(x-\delta_i) \prod_{i=1}^{m-\tau-1}(x-\kappa_i)
\end{aligned}$$

(recall, that it was shown in the proof of Lemma 5.1 that φ_1 and φ_2 have simple roots, which are interlacing and note that $\kappa_{m-\tau}$ denotes <u>not</u> a root of the polynomial φ_2 but its leading coefficient). Define the vector

$$\begin{aligned}\hat{\Theta}(a) &= \hat{\Theta} = (\gamma_1, \ldots, \gamma_{m-\tau}, \kappa_1, \ldots, \kappa_{m-\tau}, \delta_1, \ldots, \delta_\tau, \tilde{x}(a), \tilde{w}(a))^T \\ &= (\hat{\theta}_0, \ldots, \hat{\theta}_{4m-1-\tau})^T.\end{aligned}$$

and note that in a neighbourhood of the point $\tilde{a}$ there exists essentially a one to one correspondence between the points $\hat{\Theta}(a)$ and $\Theta(a)$. Consider the matrix

$$\tilde{J} = H^T J H, \qquad (5.29)$$

with

$$H = \Big(\partial\hat{\theta}_i/\partial\theta_j\Big)_{i=0,j=0}^{4m-1-\tau,4m-2}.$$

We will prove below that the matrix $\tilde{J}$ is nonsingular for any point $\tilde{a}$ satisfying the condition (A). Because $\tau(a) \in \{1, 2, \ldots, m\}$ it therefore follows that all points $a \in (\underline{a}, \bar{a})$ except for a finite set denoted with $\{a_1, \ldots, a_\nu\}$ satisfy condition (A). Therefore the vector-function

$$\Theta^+ : a \to \hat{\Theta}(a)$$

is a real analytic vector-function on the set (5.25) due to the well known Implicit Function Theorem (Gunning, Rossi, 1965). Because the coefficients of a polynomial are analytic functions of its zeros it follows that the vector-function Θ^* is also real analytic on the same set.

The proof of the nonsingularity of the matrix $\tilde{J}$ is tedious and we indicate the main steps. Denote by $\mathcal{P}$ the eigenspace of the matrix $M_c(\xi_a)$ corresponding to its minimal eigenvalue $\lambda_{\min}(M_c(\xi_a))$ and by $\mathcal{P}_\tau$ the subspace of all vectors $r = (r_0, r_1, \ldots, r_m)^T$ such that the polynomial $\sum_{j=0}^m r_j x^j$ has the form

$$\prod_{i=1}^{\tau}(x - \delta_i) \sum_{j=0}^{m-\tau} \tilde{r}_j x^j$$

for some vector $\tilde{r} = (\tilde{r}_0, \ldots, \tilde{r}_{m-\tau})^T$ of size $m - \tau + 1$. For the sake of tranparency we introduce the notation $F(x) = (1, x, \ldots, x^m)^T$ and define for vectors r, $s \in \mathbb{R}^{m+1}$:

$$< r, s > = \int_{-1}^{1} \big(r^T F(x)\big)\big(s^T F(x)\big)\,\eta_{\xi_a}(dx) \cdot \left[\frac{2}{\pi}\int_{-1}^{1} \big(r^T F(x)\big)\big(s^T F(x)\big)\frac{dx}{\sqrt{1-x^2}}\right]^{-1},$$

A straightforward calculation shows that the condition

$$\frac{\partial\lambda(\hat{\Theta}, a)}{\partial\gamma_i} = 0$$

is equivalent to the condition

$$< q_{\gamma_i}, q >= \lambda(\hat{\Theta}, a),$$

where the vector $q_{\gamma_i} \in \mathcal{P}_\tau$ is defined by

$$q_{\gamma_i}^T F(x) = \frac{1}{x-\gamma_i} q^T F(x) = \frac{d}{dx} q^T F(x)\Big|_{x=\gamma_i}$$

for any $i = 1, \ldots, m-\tau$. This means that

$$q_{\gamma_i} \in \mathcal{P} \ , \quad i = 1, \ldots, m-\tau$$

[note that the vectors $q_{\gamma_1}, \ldots, q_{\gamma_{m-\tau}}$ are linearly independent]. Note that a direct calculation gives

$$\frac{\partial^2}{\partial\gamma_j\partial\gamma_i}\lambda(\hat{\Theta}, a) \ = \ \Big(\frac{q_{\gamma_i}^T M(\xi_a) q_{\gamma_i}}{q_{\gamma_j}^T q_{\gamma_i}} - \lambda(\hat{\Theta}, a)\Big)\frac{1}{q_{\gamma_j}^T q_{\gamma_i}}, \qquad i,j = 1, \ldots, m-\tau.$$

Since $q_{\gamma_j} \in \mathcal{P}$ we obtain

$$\frac{\partial^2}{\partial\gamma_i\partial\gamma_i}\lambda(\hat{\Theta}, a) \ = \ 0 \ , \quad i,j = 1, \ldots, m-\tau,$$

In a similar way it follows that

$$p_{\kappa_i} \in \mathcal{P}_{(2)}, \quad i = 1, \ldots, m-\tau,$$

where $\mathcal{P}_{(2)}$ is the eigenspace, corresponding to $\lambda_{\min}(M_{(s)}(\xi_a))$, and we obtain by the same arguments

$$\frac{\partial^2}{\partial\kappa_i\partial\kappa_j}\lambda(\hat{\Theta}, a) \ = \ 0 \ , \quad i,j = 1, \ldots, m-\tau.$$

It is easy to check that for $a \in (\underline{a}, \bar{a})$ it follows that $\tau \geq 1$. Moreover, using the above formulas we receive that the matrix $\tilde{J}$ has the structure indicated in Table 2, where A is a nonnegative definite matrix and D is the negative definite matrix, defined above.

If $b \neq 0$ and the matrices $C = (C_1 \vdots C_2)$, B_2 and B_1 have full rank it follows by similar arguments as given in Dette, Melas and Pepelysheff (2001) with the help of the Frobenius formula that $\det \tilde{J} \neq 0$. The verification of the listed conditions is equivalent to the verification that certain polynomials are not identically zero. This can be done by the standard technique of counting zeros and is left to the reader. Thus $\det \tilde{J} \neq 0$ for any point a satisfying condition (A). $\Box$

Table 2: *Structure of the matrix $\tilde{J}$ defined in (5.29)*

	1	$m-\tau$	$m-1$	$m-1$	m
1	0	b^T			
$m-\tau$	b	0	V^T	B_1^T	C_1^T
$m-1$		V	A	B_2^T	C_2^T
$m-1$		B_1	B_2	D	0
m		C_1	C_2	0	0

Since the vector function $\Theta(a) = \Theta(\arccos \alpha)$ is real analytic on the set defined by (5.25) it can be expanded into Taylor series in a neighbourhood of any point $\tilde{a} \neq a_j$, $j = 1, \ldots, \nu$, $\underline{a} < \tilde{a} < \bar{a}$ and we obtain for its components an expansion of the form

$$\theta_i(a) = \sum_{k=0}^{\infty} \theta_{i,k}(\alpha - \tilde{\alpha})^k \ , \quad i = 0, \ldots, 4m-2,$$

where $\tilde{\alpha} = \cos \tilde{a}$, $\alpha = \cos a$. For the determination of the coefficents $\{\theta_{i,k}\}$ the general recurrent formulas introduced in Dette, Melas, Pepelyshev (2000) can be applied provided that initial conditions $\theta_{i,0}$, $i = 0, \ldots, 4m-2$ are known. To find such initial coefficients $\Theta^{(0)} = (\theta_{0,0}, \ldots, \theta_{4m-2,0})^T$ we solve the equation

$$Q(\Theta^{(0)}) \ := \sum_{i=0}^{4m-2} \left(\frac{\partial}{\partial \theta_i} \lambda(\Theta, \, \tilde{a}) \Big|_{\Theta = \Theta^{(0)}} \right)^2 = 0$$

for some $\tilde{a}$, which can be done by standard numerical algorithms. To obtain an approximation of the function $\Theta(a)$ with a given precision we have to find one or several points $\tilde{a}_1, \ldots, \tilde{a}_k$, construct the corresponding Taylor series and verify that the calculated design is E-optimal with sufficient precision (note that $\Theta(a)$ contains also the vectors $p(a)$ and $q(a)$ for the equivalence theorem in Lemma 5.1). In the following examples we will illustrate this approach for the quadratic and cubic trigonometric regression model on the interval $[-a, a]$.

Table 3: *Coefficients in the Taylor expansion (5.30) for the quadratic trigonometric regression model ($m = 2$), where* $0.741 < a/\pi < 4/5 = 0.8$

	0	1	2	3	4	5	6	7
p_0	0.4771	-0.0781	-1.3312	-1.9692	1.2116	3.8592	8.7587	-2.8454
p_1	-0.4928	2.0781	-0.5175	0.0124	1.9268	-1.7913	0.9055	-7.3152
q_0	-0.3532	-2.7276	11.4353	-92.0212	896.9923	-9.90e+03	1.18e+05	-1.46e+06
q_1	-0.3794	-2.5761	15.1122	-109.6045	1.05e+03	-1.15e+04	1.36e+05	-1.69e+06
$1 - x_1$	0.7588	-1.2582	1.5801	3.9027	0.3966	-11.1756	-20.1314	6.0409
w_1	0.1862	0.1994	0.5826	0.2185	-2.1883	-5.1277	0.2418	30.3188
w_2	0.2289	-0.4732	-0.3163	0.5386	-0.2008	0.5346	3.5705	-0.3601

Example 5.5. Consider the quadratic trigonometric regression model (1.1) on the interval $[-a, a]$

$$\beta^T f(t) = \beta_0/\sqrt{2} + \beta_1 \cos t + \beta_2 \sin t + \beta_3 \cos 2t + \beta_4 \sin 2t.$$

By the discussion of Section 2 it follows that for $\bar{a} = 0.8\pi \le a \le \pi$ an E-optimal design is given by

$$\begin{pmatrix} -\frac{4\pi}{5} & -\frac{2\pi}{5} & 0 & \frac{2\pi}{5} & \frac{4\pi}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$

Similary, Corollary 3.3 and Theorem 3.2 show that for $0 < a \le \underline{a} \approx 0.741\pi$ the unique E-optimal design is given by

$$\begin{pmatrix} -a & -t(a) & 0 & t(a) & a \\ \frac{w_2}{2} & \frac{w_1}{2} & w_0 & \frac{w_1}{2} & \frac{w_2}{2} \end{pmatrix}$$

where

$$t(a) = \arccos\Big(\frac{1 + \cos a}{2}\Big)$$

and the weights w_0, w_1 and w_2 can be found by fomula (3.8). In the intermediate case

$$\underline{a} = 0.741\pi < a < 0.8\pi = \bar{a}$$

we will construct the E-optimal design by the functional approach. Note that due to Theorem 2.3 an E-optimal design is of the form

$$\begin{pmatrix} -t_2 & -t_1 & t_0 & t_1 & t_2 \\ \frac{w_2}{2} & \frac{w_1}{2} & w_0 & \frac{w_1}{2} & \frac{w_2}{2} \end{pmatrix},$$

where $t_0 = 0,\ t_2 = a$. Since $w_0 + w_1 + w_2 = 1$ it is enough to consider the weights w_1 and w_2 and the point $x_1 = \arccos t_1$. We take $\tilde{a} = 0.77\pi \approx (\bar{a} + \underline{a})/2$. The first Taylor coefficients for the parameters

$$\begin{aligned} q_0 &= q_0(\mathrm{arccos}\alpha), \\ q_1 &= q_1(\mathrm{arccos}\alpha), \\ p_0 &= p_0(\arccos\alpha), \\ p_1 &= p_1(\mathrm{arccos}\alpha), \\ 1 - x_1 &= 1 - x_i(\arccos\alpha), \\ w_1 &= w_1(\mathrm{arccos}\alpha), \\ w_2 &= w_2(\arccos\alpha), \end{aligned}$$

in the expansion

$$\Theta(\arccos\alpha) = \sum_{n=0}^{\infty} \Theta^{(n)}(\alpha - \cos\tilde{a})^n \tag{5.30}$$

are listed in Table 3. The dependence of the support points and weights of the E-optimal design in the trigonometric regression model from the parameter $a \in (\underline{a}, \bar{a})$ is illustrated in Figure 1. In the present case it follows that $a_1 = \underline{a} < a_2 = \bar{a}$ and for $a_1 < a < a_2$ we have

$$\tau(a) = 1,\ u(a) = 1,\ v(a) = 2.$$

It is also interesting to note that for $0 < a < a_1 = \underline{a}$ we have

$$u(a) = 0,\ v(a) = 1,$$

while for the case $\bar{a} = a_2 < a < \pi$ it follows that

$$u(a) = 2,\ v(a) = 3.$$

In other words, if the parameter a is increased from 0 to π the multiplicity of the minimum eigenvalue of the information matrix of the E-optimal design changes from 1 to 5 by steps of size 2.

Example 5.6. Consider the cubic trigonometric regression model on the interval $[-a, a]$, i.e. $m = 3$. Then, similar to the preceding example an E-optimal design can be found in an explicit form whenever $0 < a \le \underline{a} \approx 0.794\pi$ and $\bar{a} \le a \le \pi$, $\bar{a} = 6/7\pi \approx 0.857\pi$. In the case $a > \bar{a}$ the design

$$\begin{pmatrix} -\frac{6\pi}{7} - & \frac{4\pi}{7} & -\frac{2\pi}{7} & 0 & \frac{2\pi}{7} & \frac{4\pi}{7} & \frac{6\pi}{7} \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \end{pmatrix}.$$

is E-optimal (but not necessarily unique), while in the case $a < \underline{a}$ the support points of the unique E-optimal design are given by

$$\pm a \ , \quad \pm \arccos\Big(\frac{3+\cos a}{4}\Big) \ , \quad \pm \arccos\Big(\frac{1+3\cos a}{4}\Big)$$

and the weights are obtained from formula (3.8). It was found numerically that

$$a_1 = \underline{a} \ < \ a_2 \ \approx 0.8113\pi \ < \ a_3 \ = \ \bar{a} \ = \ 6/7\pi \approx 0.857\pi .$$

and for $a \in (a_1, a_2)$ the first coefficients for the Taylor expansion at the point $\bar{a}_1 = 0.81\pi$ are presented in Table 4, while Table 5 contains the corresponding coefficients for the case $a \in (a_2, a_3)$ (for the expansion at the point $\bar{a}_2 = 0.83\pi$). Note that the multiplicities of the minimal eigenvalues of the matrices $M_s(\xi_a)$ and $M_c(\xi_a)$ are given by

$$\begin{array}{llll} u(a) = 0 \ , & v(a) = 1 & \text{if} & a \in (0, a_1) \\ u(a) = 1 \ , & v(a) = 2 & \text{if} & a \in (a_1, a_2) \\ u(a) = 2 \ , & v(a) = 3 & \text{if} & a \in (a_2, a_3) \\ u(a) = 3 \ , & v(a) = 4 & \text{if} & a \in (a_3, \pi) \ , \end{array}$$

where $a_1 = \underline{a}$ and $a_3 = \bar{a}$.
The behaviour of the optimal design points and weights is presented in Figure 2. It can be verified numerically that the points and weights can be determined with high precision, which is illustrated in Figure 3. This figure shows the extremal polynomial

$$\frac{(p^T f_{(1)}(x))^2 + (q^T f_{(2)}(x))^2}{p^T p + q^T q}$$

in the equivalence theorem for various values of a (note that by Lemma 5.1 this function has be less or equal than the minimum eigenvalue of the information matrix corresponding to the E-optimal design with equality at the support points).

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Table 4: *Coefficients in the Taylor expansion (5.30) for the cubic trigonometric regression model* ($m = 3$)*, where* $0.794 < a/\pi < 0.8113$

	0	1	2	3	4	5	6	7
p_0	-0.3965	4.0928	-0.3055	-12.3495	-11.4981	761.5907	2.99e+05	1.21e+08
p_1	0.6477	-1.8835	-3.2233	-5.1845	1.6191	811.6268	3.06e+05	1.24e+08
p_2	-0.5608	1.6899	3.0650	3.2670	-2.9128	-928.5566	-3.66e+05	-1.48e+08
q_0	0.1501	2.0088	-28.2704	430.0770	-8.39e+03	1.96e+05	-3.60e+06	6.06e+08
q_1	0.2599	4.2543	-37.1810	612.9954	-1.27e+04	2.96e+05	-7.06e+06	3.19e+08
q_2	0.2219	3.4819	-34.7987	534.1822	-1.11e+04	2.56e+05	-7.21e+06	-1.42e+08
$1 - x_1$	0.4047	-1.4247	2.2884	9.9250	15.0767	-48.0838	-405.3233	-1.19e+03
$1 - x_2$	1.3565	0.3121	0.7705	2.0655	2.2545	-12.2375	-83.8545	-224.4858
w_1	0.0966	0.4030	1.2655	1.5036	-9.9210	-81.2046	-281.6821	-1.1661
w_2	0.1397	-0.3048	1.2041	5.8567	-1.1242	-68.7337	-146.2008	467.5664
w_3	0.2164	-0.4311	-2.8260	-3.6782	21.5715	101.4223	64.3300	-810.3346

Table 5: *Coefficients in the Taylor expansion (5.30) for the cubic trigonometric regression model* ($m = 3$)*, where* $0.8113 < a/\pi < 6/7 = 0.857$

	0	1	2	3	4	5	6	7
p_0	-0.0674	9.6717	12.2522	43.8534	-2.67e+03	6.11e+04	-1.65e+06	4.47e+07
p_1	0.8655	5.9768	-7.3377	1.8819	-2.75e+03	6.44e+04	-1.71e+06	4.64e+07
p_2	-0.7126	-3.5536	9.0726	-65.9985	3.11e+03	-7.76e+04	2.07e+06	-5.59e+07
q_0	0.6670	9.0433	-68.2559	962.3302	-1.87e+04	4.10e+05	-9.68e+06	2.41e+08
q_1	0.5298	5.6016	-35.9075	387.2603	-6.62e+03	1.30e+05	-2.86e+06	6.81e+07
q_2	0.0868	-0.7658	22.7645	-454.2934	9.99e+03	-2.39e+05	5.93e+06	-1.52e+08
$1 - x_1$	0.3917	-0.3409	-1.0703	2.8779	38.4908	158.1277	27.2050	-3.51e+03
$1 - x_2$	1.2958	-1.9386	1.7869	25.6580	62.2862	-124.9227	-1.40e+03	-3.53e+03
w_1	0.1197	0.6556	-1.8890	-4.5152	46.8271	18.8599	-223.6196	-876.7235
w_2	0.1383	0.0358	1.8823	1.6989	-29.6814	-78.2035	-40.4011	1.61e+03
w_3	0.1826	-1.0283	0.8572	5.4231	-35.5716	57.4784	271.3875	-1.15e+03

References

F-C. Chang, B. Heiligers (1996). E-optimal designs for polynomial regression without intercept. J. Statist. Plann. Inf. 55, 371-387.

H. Dette (1993). A note on E-optimal designs for weighted polynomial regression. Ann. Statist. 21, 767-771.

H. Dette, G. Haller (1998). Optimal designs for the identification of the order of a Fourier regression. Ann. Statist. 26, 1496-1521.

H. Dette, V.B. Melas (2001). Optimal designs for estimating individual coefficients in Fourier regression models. Preprint, Ruhr-Universität Bochum. http://www.ruhr-uni-bochum.de/mathematik3/preprint.htm.

H. Dette, V.B. Melas, A. Pepelysheff (2000). Optimal designs for estimating individual coefficients in polynomial regression — a functional approach. Preprint, Ruhr-Universität Bochum. http://www.ruhr-uni-bochum.de/mathematik3/preprint.htm.

H. Dette, V.B. Melas, A. Pepelysheff (2001). D-optimal designs for trigonometric regression models on a partial circle. Preprint, Ruhr-Universität Bochum. http://www.ruhr-uni-bochum.de/mathematik3/preprint.htm.

V.V. Fedorov (1972). Theory of optimal experiments. Academic Press, New York.

F.A. Graybill (1976). Theory and application of the linear model. Wadsworth, Belmont CA.

R.C. Gunning, H. Rossi (1965). Analytic Functions of Several Complex Variables. Prentice Hall, New York.

B. Heiligers (1994). E-optimal designs in weighted polynomial regression. Ann. Statist. 22, 917-929.

B. Heiligers (1998). E-optimal designs in spline regression. J. Statist. Plann. Inf. 75, 159-172.

P.D.H. Hill (1978). A note on the equivalence of D-optimal design measures for three rival linear models. Biometrika 65, 666-667.

P. Hoel (1965). Minimax design in two-dimensional regression. Ann. Math. Statist. 36, 1097-1106.

L. Imhof, W.J. Studden (2001). E-optimal designs for rational models. Ann. Statist. 29, Vol. 3.

S. Karlin, W.J. Studden (1966). Tchebycheff Systems: with Applications in Analysis and Statistics. Interscience, New York.

J.C. Kiefer (1974). General equivalence theory for optimum designs (approximate theory). Ann. Statist. 2, 849-879.

C.P. Kitsos, D.M. Titterington, B. Torsney (1988). An optimal design problem in rhythmometry. Biometrics 44, 657-671.

H.O. Lancaster (1969). The Chi-Squared Distribution. Wiley, New York.

T.S. Lau, W.J. Studden (1985). Optimal designs for trigonometric and polynomial regression. Ann. Statist. 13, 383-394.

K. Mardia (1972). The statistics of directional data. Academic Press, New York.

V.B. Melas (1982). A duality theorem and E-optimality. Industrial Laboratory 48, 295-296 (translated from the Russian).

V.B. Melas (2000). Analytic theory of E-optimal designs for polynomial regression. In: Advances in Stochastic Simulation Methods (eds. N. Balakrishnan, V.B. Melas, S. Ermakov), Birkhäuser, Boston, 85-115.

F. Pukelsheim (1993). Optimal Design of Experiments. Wiley, New York.

F. Pukelsheim, W.J. Studden (1993). E-optimal design for polynomial regression. Ann. Statist. 21, 401-415.

E. Riccomagno, R. Schwabe, H.P. Wynn (1997). Lattice-based D-optimum design for Fourier regression. Ann. Statist. 25, 2313-2327.

T.J. Rivlin (1974). Chebyshev Polynomials. Wiley, New York.

S.D. Silvey (1980). Optimal Design. Chapman and Hall, London.