An Algorithm for Heilbronn's Problem*

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Abstract *Heilbronn conjectured that given arbitrary n points from $\mathbf{R}^2$, located in the unit square (or circle), there must be three points which form a triangle of area at most $O(1/n^2)$. This conjecture was proved false by a nonconstructive argument of Komlós, Pintz and Szemerédi [KPS] who showed that there is a configuration of n points in the unit square where all triangles have area at least $\Omega(\log n/n^2)$. In this paper, we provide a polynomial-time algorithm which for every n computes such a configuration of points.*

We then consider a generalization of this problem as introduced by Schmidt [Sc] to convex hulls of $k \geq 4$ points. We obtain the following result: For every k, there is a polynomial-time algorithm which on input n computes n points in the unit square such that the convex hull of any k points has area at least $\Omega(1/n^{(k-1)/(k-2)})$. For $k = 4$, the existence of such a configuration has been proved in [Sc].

1 Introduction

Consider a configuration of n points in the closed unit disc (or square) such that the minimum of the areas of the triangles formed by any three points assumes its maximum possible value. Denote this value by $\Delta^{(3)}(n)$. Heilbronn conjectured that $\Delta^{(3)}(n) = O(1/n^2)$. This conjecture was proved false by Komlós, Pintz and Szemerédi [KPS] who showed (nonconstructively) the existence of a configuration of n points which proves $\Delta^{(3)}(n) = \Omega(\log n/n^2)$. The best known upper bound was given by Roth [Ro] who essentially showed that $\Delta^{(3)}(n) = O(1/n^{1.117})$.

In this paper, we give a polynomial time algorithm which for every n computes a configuration of n points in the unit square which achieves the lower bound of [KPS].

We will discretize Heilbronn's problem by considering only a fixed amount of possible positions for the points. More precisely, we will consider a $T \times T$ grid where T will be of the order of n^α for some $\alpha > 1/2$.

We remark that one of the problems introduced by this discretization is that now "many" point triples lie on one line, hence, yielding a triangle of area zero, whereas in the unit square, the probability of this happening for randomly chosen points is zero.

Our algorithm will find n points on the $T \times T$ grid where every triangle has area at least $\Omega(T^2 \log n/n^2)$. This yields a solution for the problem on the unit square (or disc) by scaling with an appropriate factor.

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The problem of Heilbronn can be transformed into a problem on hypergraphs. For that purpose, we first give some basic definitions.

Definition 1.1 *A hypergraph is a finite set $V \neq \emptyset$ of vertices and a set of (hyper)edges $E_1, \ldots, E_r$ where $\emptyset \neq E_i \subseteq V$ for all $1 \leq i \leq r$. A hypergraph is called k-uniform if every hyperedge has cardinality k. A subset $V' \subseteq V$ is called independent if no hyperedge is a subset of V'.*

2-uniform hypergraphs capture the more usual concept of undirected simple graphs. For graphs without certain cycles (e.g., bipartite graphs), very often better estimations on certain parameters and better algorithms are available than for general graphs. For hypergraphs, the same is often true.

Definition 1.2 *A 2-cycle in a hypergraph is a pair of vertices v, w and a sequence $v \underset{E_1}{\sim} w \underset{E_2}{\sim} v$ such that E_1 and E_2 are different hyperedges, and $\{v,w\} \subseteq E_1$ and $\{v,w\} \subseteq E_2$. (One can reach w from v via E_1 and return to v via E_2.)*

We proceed by considering the grid points as vertices of a 3-uniform hypergraph $H^{(3)}_{T,L}$ whose hyperedges are determined by the parameters T and L as follows:
We consider an edge $\{p, q, r\}$ to be part of the hypergraph if the triangle on p, q, r in the $T \times T$ grid (which is possibly degenerated) has area at most L, where L will later be chosen such that $L = \Omega(T^2 \log n/n^2)$.
Thus, we have transformed Heilbronn's problem into a problem on hypergraphs: We only need to find a subset V' of the hypergraph vertices such that V' is an independent set and such that $|V'| \geq n$. Then, every triangle determined by points from V' has area bigger than L.
For this problem, we show that there exists a polynomial-time algorithm. It is based on derandomization through potential functions. However, the crucial problem is to determine the number of hyperedges, i.e., the number of triangles of area at most L in the $T \times T$-grid.
Using only that $H^{(3)}_{T,L}$ is a 3-uniform hypergraph, we would only be able to use this approach for a choice of $L = \Omega(T^2/n^2)$. In order to make the approach work for L up to $L = \Omega(T^2 \log n/n^2)$, we have to work harder by counting the number of 2-cycles in the hypergraph. The reason is that for hypergraphs with not too many 2-cycles, the existence of larger independent sets can be shown, and algorithms for finding them also exist.
Furthermore, we are able to demonstrate that our algorithm can be used for another similar problem, first investigated for the special case $k = 4$ by Schmidt [Sc]. Given a configuration of n points in the unit square. What is the minimum area of the convex hull of k chosen points? Choose n points to maximize this minimum area which we denote by $\Delta^{(k)}(n)$.
Schmidt for $k = 4$ proved that there is a configuration which shows $\Delta^{(4)}(n) = \Omega(1/n^{3/2})$. We give an easier proof for the fact that $\Delta^{(4)}(n) = \Omega(1/n^{3/2})$. Moreover, our arguments yield a polynomial time algorithm which for general $k \geq 3$ computes n points where the minimum area is $\Omega(1/n^{(k-1)/(k-2)})$, i.e., $\Delta^{(k)}(n) = \Omega(1/n^{(k-1)/(k-2)})$. We remark, cf. [Sc], that it is not known whether $\Delta^{(k)}(n) = o(1/n)$.

2 Basics from Number Theory

We need a simple result from number theory. For $s \geq 1$, let F_1 be defined by $F_1(s) := \sum_{h=1}^{s} gcd(h,s)$. We are content with the following crude estimate:

Lemma 2.1 *Let $\varepsilon > 0$. Then there is a constant $c_\varepsilon > 0$ such that: $F_1(s) \leq c_\varepsilon \cdot s^{1+\varepsilon}$*

Proof: The constant c_ε can be chosen to make the statement true for $s \leq 2$. Let $s \geq 3$ in the following. For a given divisor d of s, there are at most s/d numbers $h \in \{1, \ldots, s\}$ such that $gcd(h,s) = d$. Hence, we have

$$F_1(s) \leq s \cdot \sum_{d \text{ divisor of } s} 1 = s \cdot \#\text{divisors of s}.$$

From number theory, it is known that the number of divisors of some number $s \geq 3$ is bounded by $s^{c'/(\log\log s)}$ for some constant c'. Thus, $F_1(s) \leq c_\varepsilon \cdot s^{1+\varepsilon}$ for some constant $c_\varepsilon > 0$. $\square$

3 Counting Triangles of Bounded Area

It will be essential for our purposes that the number of triangles with area at most L is not too large. We also need an estimation on the number of 2-cycles in the corresponding hypergraph $H_{T,L}^{(3)}$.

Given a point p, we denote by p_x and p_y its x- and y-coordinate. Define a lexicographic order on the points of the $T \times T$-grid by

$$p < q \quad \Leftrightarrow \quad p_x < q_x \text{ or } (p_x = q_x \text{ and } p_y < q_y).$$

Lemma 3.1 *Choose some $\varepsilon > 0$. There is a constant $c_\varepsilon > 0$ such that in $H_{T,L}^{(3)}$, the number of 3-edges $\{p_1, p_2, p_3\}$ where the triangle (p_1, p_2, p_3) is degenerated, is at most $c_\varepsilon \cdot T^{4+\varepsilon}$.*

Proof: W.l.o.g., let $p_1 < p_2 < p_3$. Let $s = (p_3)_x \Leftrightarrow (p_1)_x$ and $h = (p_3)_y \Leftrightarrow (p_1)_y$. The number of degenerated triangles with $s = 0$ or $h = 0$ is bounded by $c \cdot T^4$. By rotation symmetry (which we account for by an extra constant factor), we may now assume that $s \geq h > 0$. Having chosen p_1 and s and h, there are exactly $gcd(h,s) \Leftrightarrow 1$ many points of the grid on the direct line between p_1 and p_3. As an upper bound for the number of degenerated triangles, we now obtain by Lemma 2.1

$$c'_\varepsilon \cdot T^2 \cdot \sum_{s=1}^{T} \sum_{h=1}^{s} gcd(h,s) \leq c'_\varepsilon \cdot T^2 \cdot \sum_{s=1}^{T} s^{1+\varepsilon} \quad \leq c_\varepsilon \cdot T^{4+\varepsilon}.$$

$\square$

Lemma 3.2 *Given two points $p < q$ on the $T \times T$-grid. Let $s := q_x \Leftrightarrow p_x$ and $h := q_y \Leftrightarrow p_y$.*

a) There are at most $4L$ grid points r such that the following three conditions hold simultaneously:

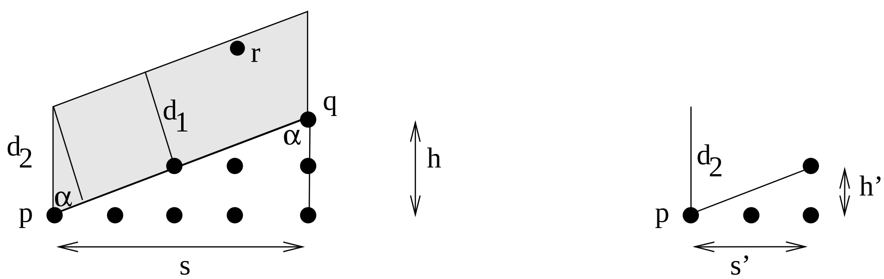
1. $p < r < q$

2. the triangle (p, q, r) is non-degenerated

3. the area of the triangle (p, q, r) is at most L.

b) The number of grid points r which fulfill conditions 2. and 3. is bounded by $12LT/s$ if $s > 0$ and by $4LT$ if $s = 0$.

Proof: We first prove a). By 1.) and 2.), $s > 0$. W.l.o.g., $h \geq 0$. We have the following situation:



Since the area of the triangle is bounded by L, we know that point r must lie in the shaded strip above or below the straight line between p and q, with $d_1 = 2L/\sqrt{s^2 + h^2}$. By a simple geometric argument, $d_1/d_2 = s/\sqrt{s^2 + h^2}$, hence $d_2 = 2L/s$.
If $h = 0$, then the number of points r is trivially bounded by $2 \cdot d_2 \cdot s = 4L$. Otherwise, we have to carefully count the number of grid points falling into the shaded strip. Assume for this purpose w.l.o.g. that the coordinates of p are (0,0). We define $s' = s/gcd(s, h)$ and $h' = h/gcd(s, h)$ and count the number of grid points falling into the shaded strip above the line segment for the x-coordinates $0, \ldots, s' \Leftrightarrow 1$.
For $x = 0$, the points from the y⇔interval $[0, d_2]$ fall into the shaded strip. In general, for $x = i$, the points from the interval $[\frac{ih'}{s'}, \frac{ih'}{s'} + d_2]$ fall into the shaded strip. In order to count the grid points, we only need to count how many points $\frac{c}{s'}$ (c integer) such that $c \equiv 0 \mod s'$ fall into the shaded strip. We have $d_2 = \frac{2L/gcd(s,h)}{s'}$. Choose $D = \lfloor 2L/gcd(s, h) \rfloor$. For $x = 0$, exactly $D + 1$ points of the form $\frac{c}{s'}$ (where c is an integer), fall into the shaded strip. The same for $x = 1, \ldots, s' \Leftrightarrow 1$. Since h' and s' are relatively prime, for $i = 0, \ldots, s' \Leftrightarrow 1$, the value $h' \cdot i$ runs through all elements modulo s' exactly once. Hence, for all $t \in \{0, \ldots, s' \Leftrightarrow 1\}$, the numbers of points of the form t^*/s' such that $t^* \equiv t \bmod s'$, are equal.
In general, we have counted $(D+1)s'$ many points, hence there are $(D+1)s'/s' = (D+1)$ many points of the form cs'/s' in the shaded strip which is the number of grid points in that strip. We are allowed to subtract one since the point p would lead to a degenerate triangle.
Since we are interested in the number of grid points in the shaded area for $x = 0$ to $x = s \Leftrightarrow 1$, we thus count $D \cdot s/s'$ points. Finally, we have to multiply by two (since the strip underneath the line $\overline{pq}$ also needs to be considered). We have counted a certain amount of points too many at the x-coordinate of p, but we additionally have to count the

same number of points at the x-coordinate of q. Altogether, we have counted (at most) $2D \cdot s/s' = 2gcd(h,s) \cdot \lfloor \frac{2L}{gcd(s,h)} \rfloor \leq 4L$ points r.
For statement b) and $s > 0$, we only need to extend the shaded strip over all x-coordinates. Thus, we get a total number which is at most $4L \cdot (T/s + 2) \leq 12LT/s$. For $s = 0$, the bound $4LT$ is trivially obtained. □

Having counted the number of candidate points for triangles of area at most L very precisely in Lemma 3.2, we can now show the following. Let $CH(p_1, \ldots, p_k)$ denote the convex hull of $p_1, \ldots, p_k$.

Lemma 3.3 *Let $k \geq 3$ be fixed and $L \geq T^{\frac{k-3}{k-2}+\varepsilon}$. There is a constant c_k such that the following holds:*
There are at most $c_k \cdot L^{k-2} \cdot T^4$ many configurations of k points $p_1, \ldots, p_k$ such that the area of the convex hull $CH(p_1, \ldots, p_k)$ is at most L.

Proof: By induction on k. If $k = 3$, then by Lemma 3.1 there are $O(T^{4+\varepsilon})$ many degenerated triangles. There are at most $O(L \cdot T^4)$ many non-degenerated triangles since every such triangle can be constructed by choosing the two lexicographic extremal points (accounting for a factor of T^4), and choosing one of $4L$ possible points. Since $L \geq T^{\varepsilon}$, the total amount is smaller than $c_3 \cdot L \cdot T^4$.
For $k > 3$, assume first that not all points $p_1, \ldots, p_k$ lie on one line. Then there are three points $q_1 < q_2 < q_3$ among them such that the triangle q_1, q_2, q_3 is not degenerated. By Lemma 3.2, there are at most $4L$ possibilities for q_2 when q_1 and q_3 are fixed. The area of the convex hull of the points $\{p_1, \ldots, p_k\} \Leftrightarrow \{q_2\}$ is also bounded by L, hence we can apply the induction hypothesis and we obtain at most $c_{k-1} \cdot L^{k-3} \cdot T^4$ many possibilities to choose these $k \Leftrightarrow 1$ points. Together, this meets the bound in the lemma.
Finally, we have to count the number of configurations of k points such that all of them lie on one line. Such a configuration can be constructed by choosing three points on a line, and then choosing $k \Leftrightarrow 3$ more points on the same line. By Lemma 3.1, there are at most $O(T^{4+\varepsilon})$ many choices for the first three points, for any of the other points, we have a choice of at most T points, which yields a total of $O(T^{4+\varepsilon+k-3}) = O(T^{k+1+\varepsilon})$ which is not larger than $c_k \cdot L^{k-2} \cdot T^4$ since $L \geq T^{\frac{k-3}{k-2}+\varepsilon}$. □

In particular, the previous lemma implies that for $L \geq T^{\varepsilon}$, the number of triples p_1, p_2, p_3 such that the triangle on those points has area bounded from above by L, is at most $c_3 \cdot L \cdot T^4$.
Speaking in terms of the hypergraph $H_{T,L}^{(3)}$, we know that this hypergraph contains at most $c_3 \cdot L \cdot T^4$ hyperedges. Just knowing this number, one could use a derandomization strategy to obtain an independent set of size $\Omega(T^2/n^2)$ for some suitably chosen T. We will later on use such an approach for the case $k \geq 4$, but for $k = 3$, we are able to provide a better algorithm.

We want to obtain larger independent sets. The key to this is the observation that for hypergraphs with few cycles, an algorithm which computes larger independent sets is known (see [BL]). (Note however that computing maximum independent sets is an NP-hard problem, and it is also hard to approximate in polynomial time, c.f. Håstad [Ha], unless $P = NP$.)
The following lemma estimates the number of 2-cycles:

Lemma 3.4 *The number of 2-cycles in the hypergraph $H^{(3)}_{T,L}$ is at most*

$$C' \cdot (L^2 \cdot T^4 \cdot \log T + L \cdot T^{5+\varepsilon})$$

for some constant $C' > 0$.

Proof: Consider one such 2-cycle. Let us assume first that the triangles building the edges E_1 and E_2 are non-degenerated. Let $\{v, w\} = E_1 \cap E_2$.
Let $s = w_x \Leftrightarrow v_x$ and $h = w_y \Leftrightarrow v_y$. By rotation symmetry (which we will account for by an extra constant factor), we can assume $s > 0$ and $0 \leq h \leq s$.
Using Lemma 3.2, part b), the number of pairs of triangles with area at most L can now be bounded by

$$C \cdot T^2 \cdot \sum_{s=1}^{T} \sum_{h=0}^{s} \left(\frac{LT}{s}\right)^2 \leq C' \cdot L^2 \cdot T^4 \cdot \log T.$$

If both triangles are degenerated, we have 4 points on a line. The proof of Lemma 3.3 shows that there are at most $O(T^{5+\varepsilon})$ such configurations. If one of the triangles is non-degenerated and one is degenerated, then we can construct this situation by choosing one of the $O(LT^4)$ many non-degenerated triangles and choosing one out of at most $O(T)$ possible positions for the fourth point. This also meets the bound of the lemma. $\square$

The next result is taken from [BL] and describes an algorithm for and an extension of a powerful result by Ajtai, Komlós, Pintz, Spencer and Szemerédi [AKPSS], compare also Fundia [Fu].

Theorem 3.5 [BL] *Let $\mathcal{G} = (V, \mathcal{E})$ be a 3-uniform hypergraph on N vertices with average degree $t^2 \geq 1$ where $t \to \infty$ with $N \to \infty$. Let $s_2(\mathcal{G})$ denote the number of 2-cycles in $\mathcal{G}$. If $s_2(\mathcal{G}) \leq N \cdot t^{3-\gamma}$ for some constant $\gamma > 0$, then one can find for any fixed $\delta > 0$ in running time $O\left(|V| + |\mathcal{E}| + s_2(\mathcal{G}) + N^3/t^{3-\delta}\right)$ an independent set of size at least $c(\gamma, \delta) \cdot N/t \cdot (\ln t)^{1/2}$.*

Our earlier considerations have shown that the hypergraph $H^{(3)}_{T,L}$ has average degree $t^2 \leq C \cdot L \cdot T^2$, i.e., $t \leq C' \cdot L^{1/2} \cdot T$. Thus, to apply Theorem 3.5 we need for some constant $\gamma > 0$ that

$$s_2(H^{(3)}_{T,L}) \leq T^2 \cdot (C' \cdot L^{1/2} \cdot T)^{3-\gamma} \ . \qquad (1)$$

By Lemma 3.4, we need to show that $C'' \cdot (L^2 \cdot T^4 \cdot \log T + L \cdot T^{5+\varepsilon}) \leq T^2 \cdot (L^{1/2} \cdot T)^{3-\gamma}$.
Set $A_1 = L^2 \cdot T^4 \cdot \log T$ and $A_2 = L \cdot T^{5+\varepsilon}$ and $B = T^2 \cdot (L^{1/2} \cdot T)^{3-\gamma}$. We will show that $A_1/B = o(1)$ and $A_2/B = o(1)$.

Then, for $L = c \cdot T^2 \log n / n^2$, we have:

$$
\begin{aligned}
\frac{A_1}{B} &= \frac{L^2 \cdot T^4 \cdot \log T}{T^2 \cdot (L^{1/2} \cdot T)^{3-\gamma}} = \frac{L^{1/2+\gamma/2}}{T^{1-\gamma}} \cdot \log T \\
&= C \cdot \frac{(\log n)^{1/2+\gamma/2}}{n^{1+\gamma}} \cdot T^{2\gamma} \cdot \log T \\
&= o(1) \qquad (2)
\end{aligned}
$$

which holds for $0 < \gamma \leq 1/10$ and, say, $T \leq n^5$. Moreover,

$$
\begin{aligned}
\frac{A_2}{B} &= \frac{L \cdot T^{5+\varepsilon}}{T^2 \cdot (L^{1/2} \cdot T)^{3-\gamma}} = \frac{T^{\gamma+\varepsilon}}{L^{1/2-\gamma/2}} \\
&= C' \cdot \frac{1}{(\log n)^{1/2-\gamma/2}} \cdot \frac{n^{1-\gamma}}{T^{1-2\gamma-\varepsilon}} \\
&= o(1) \qquad (3)
\end{aligned}
$$

which holds for $0 < \gamma < 1/2$ and $T \geq n^{\frac{1-\gamma}{1-2\gamma-\epsilon}}$.
Hence, by (2) and (3), we infer for $n^{\frac{1-\gamma}{1-2\gamma-\epsilon}} \leq T \leq n^5$ and $0 < \gamma \leq 1/10$ that

$$s_2(H_{T,L}^{(3)}) \leq T^2 \cdot (L^{1/2} \cdot T)^{3-\gamma} .$$

The assumptions of Theorem 3.5 are fulfilled for $T = n^{1+\beta}$ where $\beta > 0$ is a small constant, and we obtain an independent set in $H_{T,L}^{(3)}$ of size

$$
\begin{aligned}
\Omega\left(\frac{T^2}{t} \cdot (\log t)^{1/2}\right) &= \Omega\left(\frac{T^2}{L^{1/2} \cdot T} \cdot \log^{1/2}(L^{1/2} T)\right) \\
&= \Omega\left(\frac{n}{\log^{1/2} n} \cdot \log^{1/2}(\frac{\log^{1/2} n}{n} \cdot T^2)\right) \\
&= \Omega(n) .
\end{aligned}
$$

By Theorem 3.5 the running time for the algorithm is given by

$$O\left(T^2 + L \cdot T^4 + L^2 \cdot T^4 \cdot \log T + L \cdot T^{5+\varepsilon} + T^6/(L^{1/2} \cdot T)^{3-\delta}\right) = O(n^{5+\delta'})$$

for $T \leq n^2$ and for some small constant $\delta' = \delta'(\beta, \varepsilon)$. Thus, using rescaling, we have obtained the following result.

Theorem 3.6 *One can find in polynomial time n points in the unit square such that the minimum area of the arising triangles is $\Omega(\log n / n^2)$.*

4 Polygons with k sides

For general k, one may consider the following problem which for $k = 4$ has been treated in [Sc] with the same bound given below:

Find n points in the unit square such that the convex hull of every choice of k points has an area at least $c \cdot n^{-(k-1)/(k-2)}$.
We show that the following algorithm can be used to find such a configuration of points. The average degree of a k-uniform hypergraph with N points and e edges is defined as $d_{avg} = ek/N$.
Turán's Theorem for Hypergraphs $\mathcal{G}$ gives a lower bound for the independence number $\alpha(\mathcal{G})$, cf. Spencer [Sp], see also [BL]:

Theorem 4.1 [Sp] *Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform hypergraph, $k \geq 2$, with average degree $d_{avg} \geq 1$ and $|V| = N$. Let $t := (d_{avg})^{\frac{1}{k-1}}$ Then,*

$$\alpha(\mathcal{G}) \geq c_k \cdot \frac{N}{t} \ . \tag{4}$$

Moreover, an independent set of size at least $c_k N/t$ can be found in time $O(|V| + |\mathcal{E}|)$.

For the sake of completeness we sketch the algorithm of Theorem 4.1 which uses the method of conditional probabilities, [Ra], [AS].
Proof. Let $V = \{v_1, \ldots, v_n\}$. Assign a weight $p_i \in [0, 1]$ to every vertex v_i. Define a potential function V by $V(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i \Leftrightarrow \sum_{\{v_{i_1}, \ldots, v_{i_k}\} \in \mathcal{E}} \prod_{j=1}^{k} p_{i_j}$.

In the beginning, $p_1 = \ldots = p_n = p$, hence, $V(p, \ldots, p) = p \cdot n \Leftrightarrow p^k \cdot \dfrac{n \cdot t^{k-1}}{k}$ which is maximal for $p = 1/t$, namely $\frac{k-1}{k} \cdot \frac{n}{t}$.
Subsequently, in each step i, $i = 1, \ldots, n$, we choose either $p_i = 0$ or $p_i = 1$ in order to maximize the value of the current potential $V(p_1, \ldots, p_i, p, \ldots, p)$. As $V(p_1, p_2, \ldots, p_n)$ is linear in each p_i, we have either $V(p_1, \ldots, p_{i-1}, p, \ldots, p) < V(p_1, \ldots, p_{i-1}, 1, p, \ldots, p)$ or $V(p_1, \ldots, p_{i-1}, p, \ldots, p) \leq V(p_1, \ldots, p_{i-1}, 0, p, \ldots, p)$. In the first case, we take vertex v_i, else we discard it. Finally, we have $V(p_1, \ldots, p_n) \geq V(p, \ldots, p)$.
Examining this process more closely shows that the set $V' = \{v_i \in V \mid p_i = 1\}$ is an independent set and has size at least $\frac{k-1}{k} \cdot \frac{n}{t}$. The running time is $O(|V| + |\mathcal{E}|)$. $\square$

By Lemma 3.3, the average degree of the hypergraph $H_{T,L}^{(k)}$ is $d_{avg} \leq c'_k \cdot L^{k-2} \cdot T^2$, if we choose L such that $L \geq T^{\frac{k-3}{k-2}+\varepsilon}$. We can apply Theorem 4.1 with $t \leq L^{\frac{k-2}{k-1}} \cdot T^{\frac{2}{k-1}}$ which yields an independent set of size at least $d_k \cdot T^{2-\frac{2}{k-1}} / L^{\frac{k-2}{k-1}} = d_k \cdot (T^2)^{\frac{k-2}{k-1}} / L^{\frac{k-2}{k-1}}$. We choose some $L = c_2 \cdot T^2 / n^{(k-1)/(k-2)}$ (for some appropriate $c_2 > 0$), then this bound is at least n. In order to fulfill the restriction $L \geq T^{\frac{k-3}{k-2}+\varepsilon}$, it is sufficient to choose the $T \times T$-grid fine enough, namely $T \geq c \cdot n^{\frac{k-1}{k-1-\varepsilon(k-2)}}$, i.e., $T = n^{1+\beta}$ for some small constant $\beta > 0$ is enough. These considerations together with Theorem 4.1 yield the following result:

Theorem 4.2 *For every $k \geq 3$, there is a polynomial time algorithm which on input n computes a configuration of n points in the unit square such that the convex hull of any k points is at least $\Omega(1/n^{(k-1)/(k-2)})$.*

5 Open Problems

An open question which we have not yet been able to resolve is whether the case $k = 4$ or in general $k \geq 4$ can also be improved by a logarithmic factor, that is, by the factor $(\log n)^{1/(k-2)}$. The problem is to count the 2-cycles accurately. The question is also whether for these improvements, the construction can also be made explicit.

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