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Reducing the Number of Inference Steps for Multiple-Stage Fuzzy IF-THEN Rule Bases

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Reducing the Number of Inference Steps for Multiple-Stage Fuzzy IF-THEN Rule Bases*

Stephan Lehmke, Karl-Heinz Temme, and Helmut Thiele

We present a theoretical result concerning the foundations of fuzzy inference systems which use multi-stage fuzzy IF-THEN rule bases, i. e. collections of fuzzy IF-THEN rules in which the conclusion of one rule, say the fuzzy set G in the rule IF F THEN G, may appear as the premise of another rule, say IF G THEN H. This result immediately leads to a method for modifying a fuzzy IF-THEN rule base in order to reduce the number of IF-THEN rules and the number of inference steps needed to obtain the final inference result.

Two of the authors of this paper have investigated under which circumstances the classical rule of *syllogism* may be applied, i. e. under which assumptions concerning the inference mechanism and the structure of the overall rule base the two rules IF F THEN G and IF G THEN H may be replaced by the single rule IF F THEN H.

Several preconditions have to be placed on the inference mechanism and the structure of the rule base to yield the validity of the classical syllogism rule, especially if the premises of other rules in the rule base *overlap* with the fuzzy sets F or G (which will be the case in almost any applicable fuzzy rule base). It turns out that these preconditions are very strong and reduce the usability of this method as a tool for rule base 'compression' considerably.

In the present paper, we discuss a method by which the two rules IF F THEN G and IF G THEN H are replaced by a rule of the form IF F' THEN H, where F' is calculated from F and G.

Keywords. fuzzy IF-THEN rule bases, chaining, multiple-stage inference, inference with fuzzy inputs, syllogism

1 Introduction

The increasing maturity of fuzzy inference systems is bringing about two effects which in some manner enforce one another. On the one hand, the complexity of applications increases, on the other hand, research on fuzzy inference systems is going on and thus such systems are increasingly better understood. Consequently, not only is there demand from the applications to build more and more complex fuzzy inference systems, but also better means to do so are being developed.

Prompted by this development, several new concepts had to be integrated into the theory of fuzzy inference. This paper is concerned with the relationship of two of these

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concepts, *inference with fuzzy inputs* and *multiple-stage fuzzy IF-THEN rule bases*. Both are well established in the domain of *fuzzy expert systems*, but are also beginning to gain importance in the field of *fuzzy control* (see for instance D. DRIANKOV, R. PALM, and H. HELLENDOORN [3]; G. C. MOUZOURIS and J. M. MENDEL [8]).

Inference with fuzzy inputs means that the input to a fuzzy inference system is not a trivial representation of a *crisp* value from the input universe by a *singleton fuzzy set*, but a 'true' fuzzy set on the input universe. In fuzzy expert systems such a fuzzy set may be obtained in the form of a fuzzy term (like 'moderately high temperature'), in fuzzy control it might result from a mathematical representation of sensor accuracy (or rather, inaccuracy).

We speak of *multiple-stage fuzzy IF-THEN rule bases* if a fuzzy term occurring in the conclusion of a fuzzy IF-THEN rule, say the fuzzy set G in the rule IF F THEN G, may appear as the premise of another IF-THEN rule, say IF G THEN H. For a fuzzy inference system, this means that the result of one inference step is used as the input of another inference step. Several such inference steps may have to occur before the final result of the fuzzy inference system is obtained.

If the fuzzy inference system is unable to process fuzzy inputs, multiple-stage inferences are often carried out by inserting a *defuzzification* step between two inference steps. This means that the semantic coupling of the inference steps is 'loosened', which can result in undesirable side effects, and furthermore the inference result becomes dependent on the defuzzification method in an unintuitive way.

If the fuzzy inference system is able to process fuzzy inputs, we can immediately use the fuzzy set resulting from one inference step as the input of the next one, yielding a much tighter semantic coupling of the inference steps, which has many benefits.

In this paper, we investigate one benefit of this tight semantic coupling, namely the oportunity to *reduce* the number of rules in the rule base and the number of inference steps.

Two of the authors of the present paper have investigated under which circumstances the classical rule of *syllogism* may be applied, i. e. under which assumptions concerning the inference mechanism and the structure of the overall rule base the two rules IF F THEN G and IF G THEN H may be replaced by the single rule IF F THEN H (see K.-H. TEMME and H. THIELE [14, 15]; the subject has also been investigated by D. RUAN and others [9–13] and also by S. GOTTWALD [4, 5]).

Several preconditions have to be placed on the inference mechanism and the structure of the rule base to yield the validity of the classical syllogism rule, especially if the premises of other rules in the rule base *overlap* with the fuzzy sets F or G (which will be the case in almost any applicable fuzzy rule base). It turns out that these preconditions are very strong and reduce the usability of this method as a tool for rule base 'compression' considerably.

In the present paper, we discuss a method by which the two rules IF F THEN G and IF G THEN H are replaced by a rule of the form IF F' THEN H, where F' is calculated from F and G. The method can be applied whenever the inference system is of MAMDANI style. As no preconditions have to be placed on the structure of the rule base, this method is of considerable practical relevance.

1.1 Notational Conventions

We denote the **real unit interval**, i. e. the set of all real numbers r with $0 \leqq r \leqq 1$ by $\langle 0, 1\rangle$.

Let U be an arbitrary non-empty set called **universe**. Fuzzy sets on U are mappings $F : U \to \langle 0, 1\rangle$. We denote the (crisp!) set of all fuzzy sets on U by $\mathfrak{F}(U)$.

Given binary operations $\kappa, \alpha : \langle 0, 1\rangle \times \langle 0, 1\rangle \to \langle 0, 1\rangle$, we define a generalized **intersection** and **union** operation $\Cap, \Cup$ on fuzzy sets as follows, given fuzzy sets $F, G \in \mathfrak{F}(U)$ and $x \in U$.

$$(F \Cap G)(x) =_{\text{def}} \kappa(F(x), G(x))$$
$$(F \Cup G)(x) =_{\text{def}} \alpha(F(x), G(x))$$

In the case $\kappa = \min, \alpha = \max$, we get the common fuzzy set intersection $\cap$ and union $\cup$ as defined by L. A. ZADEH [18].

The classical (see [18]) subset relation $\subseteqq$ for fuzzy sets $F, G \in \mathfrak{F}(U)$ is defined as follows:

$$F \subseteqq G =_{\text{def}} \text{ for every } x \in U, F(x) \leqq G(x).$$

We define the **height** $\operatorname{hgt}(F)$ of a fuzzy set $F \in \mathfrak{F}(U)$ by

$$\operatorname{hgt}(F) =_{\text{def}} \operatorname{Sup}\{F(x) | x \in U\}.$$

Functional operators on U are mappings $\Phi : \mathfrak{F}(U) \to \mathfrak{F}(U)$. The **product** $\Phi \circ \Psi$ of two functional operators on U is defined as follows.

$$\Phi \circ \Psi(F) =_{\text{def}} \Psi(\Phi(F)) \qquad (F \in \mathfrak{F}(U))$$

1.2 Fuzzy IF-THEN Rules and Rule Bases

Let a fixed universe U be given. A **fuzzy IF-THEN rule** is a syntactic construct of the form

(1) $$\text{IF } F \text{ THEN } G,$$

where F (called **premise**) and G (called **conclusion**) are fuzzy sets on the universe U.

Remark

Choosing premise and conclusion simply as fuzzy sets on a common universe U is a convention to simplify the following mathematical investigations. In applications, IF-THEN rules are usually of a more complex structure. We lose no expressive power by our convention, however.

The use of *logical expressions* involving fuzzy sets on a common universe can be incorporated by application of set operations on these fuzzy sets. Take, for instance, the fuzzy IF-THEN rule

$$\text{IF } F_1 \text{ AND } F_2 \text{ THEN } G_1 \text{ or } G_2,$$

where F_1, F_2, G_1, G_2 are fuzzy sets on a common universe U. We gain a fuzzy IF-THEN rule of the type (1) by defining $F =_{\text{def}} F_1 \cap F_2$ and $G =_{\text{def}} G_1 \cup G_2$, where $\cap, \cup$ are appropriately defined fuzzy set-theoretical operators (see for instance L. A. ZADEH [18]).

The use of fuzzy sets on several different universes in the definition of fuzzy IF-THEN rules is covered by the well-known principle of *cylindrical extension*.

A fuzzy IF-THEN rule base is a finite collection of fuzzy IF-THEN rules. We denote a fuzzy IF-THEN rule base $\mathfrak{R}$ consisting of n fuzzy IF-THEN rules by

$$\mathfrak{R}: \quad \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array}$$

Of course, $F_1, \ldots, F_n, G_1, \ldots, G_n$ are all fuzzy sets on the universe U.

For simplicity, in the following we shall always assume two fuzzy IF-THEN rule bases

$$\mathfrak{R}_1: \quad \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array} \qquad\qquad \mathfrak{R}_2: \quad \begin{array}{llll} \text{IF} & G_1 & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & G_n & \text{THEN} & H_n \end{array}$$

to be given such that inference is done first with rule base $\mathfrak{R}_1$ (given some input fuzzy set F on U) and then with $\mathfrak{R}_2$, taking as input the output of the inference with rule base $\mathfrak{R}_1$. It is easy to see that we lose no expressive power by this simplification. Of course, before carrying out an inference step we have to identify those rules which have to be considered, and we can assume that $\mathfrak{R}_1, \mathfrak{R}_2$ consist exactly of these relevant rules (maybe out of a larger rule base).

That the conclusions $G_1, \ldots, G_n$ occurring in $\mathfrak{R}_1$ have to be identical with the premises of the rules in $\mathfrak{R}_2$ is a constraint which in fact can be relaxed (see section 4).

2 The MAMDANI Case

In this section, we discuss a very common and well-known fuzzy inference mechanism. A slight generalization of this inference mechanism is discussed in the next section.

In the 'classical' fuzzy inference mechanism, we associate with a fuzzy IF-THEN rule

(2) $$\text{IF } F \text{ THEN } G$$

an *interpretation* $R^{F,G}$ in the form of a binary *fuzzy relation* (see L. A. ZADEH [19]) on U, i. e. $R^{F,G} : U \times U \to \langle 0, 1 \rangle$.

$$R^{F,G}(x,y) =_{\text{def}} \min(F(x), G(y)) \qquad (x, y \in U)$$

For an input fuzzy set F' on U, we compute an output fuzzy set G' on U by the following formula, for $y \in U$.

$$G'(y) =_{\text{def}} \operatorname{Sup}\left\{\min\left(F'(x), R^{F,G}(x,y)\right) \middle| \, x \in U\right\} \tag{3}$$

This formula is derived from the *compositional rule of inference* (see L. A. ZADEH [20]). The resulting inference method is also called MAMDANI-inference because it was used in the first documented application of a fuzzy controller (see E. H. MAMDANI and S. ASSILIAN [6]).

Given a fuzzy relation $R : U \times U \to \langle 0,1 \rangle$, formula (3) immediately gives rise to the definition of a *functional operator* Φ^R mapping F' to G', as follows, for $y \in U$.

$$\Phi^R(F')(y) =_{\text{def}} \operatorname{Sup}\left\{\min(F'(x), R(x,y)) \,\middle|\, x \in U\right\} \tag{4}$$

Given a rule base

$$\mathfrak{R} : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array} \tag{5}$$

we have to obtain a *combined* inference result from all the rules. There are two well-known principles to achieve this:

1. First, carry out the *inference* for each rule separately, using (for rule number i) the operator $\Phi^{R^{F_i,G_i}}$ as defined in (4). Then, *aggregate* the inference results into the final output (Principle FITA: **F**irst **i**nference **t**hen **a**ggregation).
2. First, *aggregate* the interpretations $R^{F_1,G_1}, \ldots, R^{F_n,G_n}$ into a combined fuzzy relation $R^{\mathfrak{R}}$. Then, carry out the *inference* for the relation $R^{\mathfrak{R}}$, using the operator $\Phi^{R^{\mathfrak{R}}}$, generating the final output (Principle FATI: **F**irst **a**ggregation **t**hen **i**nference).

In the MAMDANI case, the maximum operator is always used for aggregation.

These principles give rise to the definition of two functional operators on U, FITA and FATI. For the definition of FATI, we need the combined fuzzy relation $R^{\mathfrak{R}}$. This relation is defined as the union of the fuzzy relations interpreting the single rules (we regard a binary fuzzy relation on U as a fuzzy set on $U \times U$).

$$R^{\mathfrak{R}} =_{\text{def}} R^{F_1,G_1} \cup \cdots \cup R^{F_n,G_n}$$

For an input fuzzy set F' on U, we define

$$\begin{aligned} \text{FITA}^{\mathfrak{R}}(F') &=_{\text{def}} \Phi^{R^{F_1,G_1}}(F') \cup \cdots \cup \Phi^{R^{F_n,G_n}}(F') \\ \text{FATI}^{\mathfrak{R}}(F') &=_{\text{def}} \Phi^{R^{\mathfrak{R}}}(F') \end{aligned}$$

In the MAMDANI case discussed in this section, we get the following theorem, well-known from the literature:

Theorem 1
For every rule base $\mathfrak{R}$, $\text{FITA}^{\mathfrak{R}} = \text{FATI}^{\mathfrak{R}}$.

So, for this section it is sufficient to consider only one of the inference principles. We (arbitrarily) choose the principle FITA for the rest of this section.

The matters which were sketched above for the convenience of the reader are discussed in very much detail by H. THIELE in [16].

Given two rule bases

$$(6)\qquad \mathfrak{R}_1 : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array} \qquad\qquad \mathfrak{R}_2 : \begin{array}{llll} \text{IF} & G_1 & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & G_n & \text{THEN} & H_n \end{array}$$

we are now interested in the result of the *two-stage* inference on $\mathfrak{R}_1$ and $\mathfrak{R}_2$, i. e. in the functional operator

$$\text{FITA}^{\mathfrak{R}_1} \circ \text{FITA}^{\mathfrak{R}_2}.$$

Obviously, for an input fuzzy set F' on U, $\left(\text{FITA}^{\mathfrak{R}_1} \circ \text{FITA}^{\mathfrak{R}_2}\right)(F')$ is the result of first executing the inference operator $\text{FITA}^{\mathfrak{R}_1}$ on F' and then executing $\text{FITA}^{\mathfrak{R}_2}$ on the *inference result* of $\text{FITA}^{\mathfrak{R}_1}$.

Our goal is to specify a rule base $\mathfrak{R}_{1,2}$ such that $\text{FITA}^{\mathfrak{R}_1} \circ \text{FITA}^{\mathfrak{R}_2} = \text{FITA}^{\mathfrak{R}_{1,2}}$. This means that instead of executing *two* inference steps, first using the rule base $\mathfrak{R}_1$ and then the rule base $\mathfrak{R}_2$, we wish to execute only *one* inference step on the rule base $\mathfrak{R}_{1,2}$.

In [14, 15], two of the authors of the presented paper have investigated cases in which we can use the following rule base[1].

$$(7)\qquad \mathfrak{R}^{\mathrm{C}}_{1,2} : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & H_n \end{array}$$

This method is of course easy and convenient in several ways. First, it is trivial to calculate the new rule base $\mathfrak{R}^{\mathrm{C}}_{1,2}$ from the given rule bases $\mathfrak{R}_1$ and $\mathfrak{R}_2$. Secondly, not only the number of inference steps, but also the overall size of the rule base is reduced by one half, significantly reducing the computational cost of the inference procedure.

However, in [14, 15], rather tight assumptions had to be made about the inference method, and the MAMDANI inference does **not** meet these assumptions. Indeed, D. DRIANKOV and H. HELLENDOORN [2] give some simple counterexamples to the validity of this straightforward reduction procedure in the MAMDANI case.

In this paper, we consider a more subtle procedure for generating the rule base $\mathfrak{R}_{1,2}$ which still results in a reduction of the number of inference steps and the overall size of the rule base by one half. For every $i \in \{1, \ldots, n\}$, we define a new fuzzy set F_i' by

$$F_i' =_{\text{def}} (F_1 \cap F_{i,1}) \cup \cdots \cup (F_n \cap F_{i,n})$$

where for every $x \in U$ and $i, j \in \{1, \ldots, n\}$,

$$(8)\qquad F_{i,j}(x) =_{\text{def}} \text{hgt}(G_j \cap G_i).$$

[1] Here, “C” stands for “chaining”.

The new rule base $\mathfrak{R}_{1,2}$ is then defined by

(9) $$\mathfrak{R}_{1,2}: \begin{array}{llll} \text{IF} & F_1' & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & F_n' & \text{THEN} & H_n \end{array}$$

Remarks

Definition (9) gives rise to the following remarks:

1. As defined in (8), the fuzzy set $F_{i,j}$ is constant over the whole universe U. It is used to *clip* the fuzzy set F_j at the height of the intersection $G_i \cap G_j$. The clipping is done by the intersection

$$F_j \cap F_{i,j}.$$

2. Alternatively to $\mathfrak{R}_{1,2}$ as defined above, we could consider the following larger rule base $\mathfrak{R}_{1,2}'$, which we give in set notation:

(10) $$\mathfrak{R}_{1,2}' =_{\text{def}} \left\{ \text{IF}\, F_j \cap F_{i,j}\, \text{THEN}\, H_i \,\middle|\, i, j \in \{1, \ldots, n\} \right\}.$$

By the fact that $\cup$ is used for the aggregation of inference results in the MAMDANI case, we can prove that $\mathfrak{R}_{1,2}$ is *equivalent* to $\mathfrak{R}_{1,2}'$ in the sense that both rule bases define the same functional operator.

Proposition 2

For all rule bases $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of the form (6), $\text{FITA}^{\mathfrak{R}_{1,2}} = \text{FITA}^{\mathfrak{R}_{1,2}'}$.

Proof

By expanding definitions, we get for $F' : U \to \langle 0, 1\rangle$ and $y \in U$:

$$\begin{aligned} \text{FITA}^{\mathfrak{R}_{1,2}'}(F')(y) &= \text{FATI}^{\mathfrak{R}_{1,2}'}(F')(y) \\ &= \Phi^{R^{\mathfrak{R}_{1,2}'}}(F')(y) \\ &= \text{Sup}\left\{ \min\left(F'(x), R^{\mathfrak{R}_{1,2}'}(x,y)\right) \,\middle|\, x \in U \right\}, \end{aligned}$$

where $R^{\mathfrak{R}_{1,2}'}(x,y)$ is given by

$$\begin{aligned} R^{\mathfrak{R}_{1,2}'}(x,y) &= \left(R^{F_1 \cap F_{1,1}, H_1} \cup \cdots \cup R^{F_n \cap F_{1,n}, H_1} \cup \cdots \cup R^{F_n \cap F_{n,n}, H_n}\right)(x,y) \\ &= \max\left(R^{F_1 \cap F_{1,1}, H_1}(x,y), \ldots, R^{F_n \cap F_{1,n}, H_1}(x,y), \ldots, R^{F_n \cap F_{n,n}, H_n}(x,y)\right). \end{aligned}$$

For every $i \in \{1, \ldots, n\}$, we get

$$\begin{aligned} &\max\left(R^{F_1 \cap F_{i,1}, H_i}(x,y), \ldots, R^{F_n \cap F_{i,n}, H_i}(x,y)\right) \\ &= \max(\min((F_1 \cap F_{i,1})(x), H_i(y)), \ldots, \min((F_n \cap F_{i,n})(x), H_i(y))) \\ &= \min(\max((F_1 \cap F_{i,1})(x), \ldots, (F_n \cap F_{i,n})(x)), H_i(y)) \\ &= \min(((F_1 \cap F_{i,1}) \cup \cdots \cup (F_n \cap F_{i,n}))(x), H_i(y)) \end{aligned}$$

$$= R^{F_i', H_i}(x,y).$$

It follows that

$$\begin{aligned}
R^{\mathfrak{R}_{1,2}'}(x,y) &= \max\Big(R^{F_1',H_1}(x,y),\ldots,R^{F_n',H_n}(x,y)\Big)\\
&= \Big(R^{F_1',H_1}\cup\cdots\cup R^{F_n',H_n}\Big)(x,y)\\
&= R^{\mathfrak{R}_{1,2}}(x,y),
\end{aligned}$$

thus

$$\begin{aligned}
\mathrm{FITA}^{\mathfrak{R}_{1,2}'}(F')(y) &= \mathrm{Sup}\Big\{\min\Big(F'(x), R^{\mathfrak{R}_{1,2}}(x,y)\Big)\Big| x\in U\Big\}\\
&= \Phi^{R^{\mathfrak{R}_{1,2}}}(F')(y)\\
&= \mathrm{FATI}^{\mathfrak{R}_{1,2}}(F')(y)\\
&= \mathrm{FITA}^{\mathfrak{R}_{1,2}}(F')(y).
\end{aligned}$$

■

3. It is of course interesting to ask whether

$$\mathfrak{R}_{1,2}' = \mathfrak{R}^{\mathrm{C}}_{1,2}\cup\mathfrak{R}^{\mathrm{O}}_{1,2},$$

using the "chained" rule base from (7) and some additional rule base $\mathfrak{R}^{\mathrm{O}}_{1,2}$ considering the "overlap" between the fuzzy sets G_i.

In the rule base $\mathfrak{R}_{1,2}'$, we 'almost' get the rules from $\mathfrak{R}^{\mathrm{C}}_{1,2}$. If $i = j$, then $G_i\cap G_j = G_i$ and thus, for every $i\in\{1,\ldots,n\}$, we have in $\mathfrak{R}_{1,2}$ a rule of the form

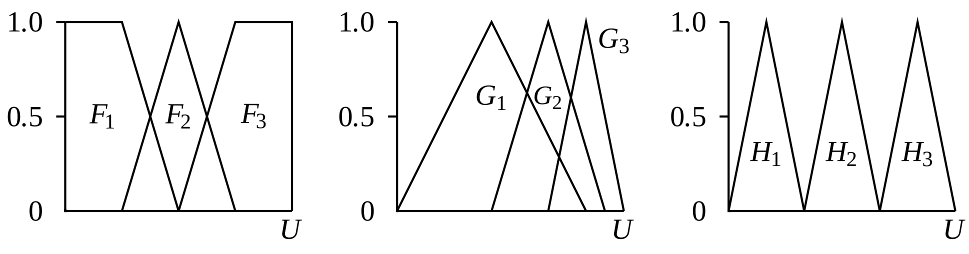
$$\mathrm{IF}\, F_i\cap F_{i,i}\ \mathrm{THEN}\, H_i,$$

where $F_i\cap F_{i,i}$ is F_i clipped at the height of G_i. Thus, $\mathfrak{R}^{\mathrm{C}}_{1,2}$ is contained in $\mathfrak{R}_{1,2}'$ if for every $i\in\{1,\ldots,n\}$, $\mathrm{hgt}(F_i)\leqq\mathrm{hgt}(G_i)$. This is always true if G_i is *normal* in the sense that $\mathrm{hgt}(G_i) = 1$ holds.

Example 1 We consider the following rule bases:

$$\mathfrak{R}_1:\ \begin{array}{llll}\mathrm{IF} & F_1 & \mathrm{THEN} & G_1\\ \mathrm{IF} & F_2 & \mathrm{THEN} & G_2\\ \mathrm{IF} & F_3 & \mathrm{THEN} & G_3\end{array}\qquad\qquad \mathfrak{R}_2:\ \begin{array}{llll}\mathrm{IF} & G_1 & \mathrm{THEN} & H_1\\ \mathrm{IF} & G_2 & \mathrm{THEN} & H_2\\ \mathrm{IF} & G_3 & \mathrm{THEN} & H_3\end{array}$$

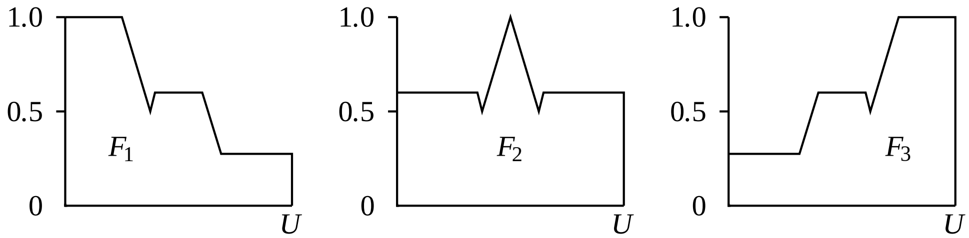
where the fuzzy sets are given as illustrated in the following diagrams, for a suitable universe U:



We obtain the new rule base

$$\text{(11)} \qquad \mathfrak{R}_{1,2}: \quad \begin{array}{llll} \text{IF} & F_1' & \text{THEN} & H_1 \\ \text{IF} & F_2' & \text{THEN} & H_2 \\ \text{IF} & F_3' & \text{THEN} & H_3 \end{array}$$

where the fuzzy sets F_i' are given as illustrated in the following diagrams:



Of course, we have to justify definition (9) by a theorem.

Theorem 3
For all rule bases $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *of the form* (6), $\mathrm{FITA}^{\mathfrak{R}_1} \circ \mathrm{FITA}^{\mathfrak{R}_2} = \mathrm{FITA}^{\mathfrak{R}_{1,2}}$.

Proof
By expanding definitions , we get for $F' : U \to \langle 0, 1\rangle$ and $z \in U$:

$$\begin{aligned}
&\mathrm{FITA}^{\mathfrak{R}_1} \circ \mathrm{FITA}^{\mathfrak{R}_2}(F')(z) \\
&\quad = \left(\Phi^{R^{G_1,H_1}}\left(\mathrm{FITA}^{\mathfrak{R}_1}(F')\right) \cup \cdots \cup \Phi^{R^{G_n,H_n}}\left(\mathrm{FITA}^{\mathfrak{R}_1}(F')\right)\right)(z) \\
&\quad = \max\left(\begin{array}{l} \mathrm{Sup}\left\{\min\left(\mathrm{FITA}^{\mathfrak{R}_1}(F')(y), \min(G_1(y), H_1(z))\right) \middle| y \in U\right\}, \\ \vdots \\ \mathrm{Sup}\left\{\min\left(\mathrm{FITA}^{\mathfrak{R}_1}(F')(y), \min(G_n(y), H_n(z))\right) \middle| y \in U\right\} \end{array}\right).
\end{aligned}$$

We concentrate on the term enclosed in the supremum for some $i \in \{1, \ldots, n\}$.

$$\begin{aligned}
&\min\left(\mathrm{FITA}^{\mathfrak{R}_1}(F')(y), \min(G_i(y), H_i(z))\right) \\
&\quad = \min\left(\max\left(\begin{array}{l} \mathrm{Sup}\left\{\min(F'(x), \min(F_1(x), G_1(y))) | x \in U\right\}, \\ \vdots \\ \mathrm{Sup}\left\{\min(F'(x), \min(F_n(x), G_n(y))) | x \in U\right\} \end{array}\right), \min(G_i(y), H_i(z))\right) \\
&\quad = \min\left(\mathrm{Sup}\left\{\max\left(\begin{array}{l} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{array}\right) \middle| x \in U\right\}, \min(G_i(y), H_i(z))\right)
\end{aligned}$$

$$(12)\quad = \operatorname{Sup}\left\{ \min\left(\max\left(\begin{array}{l} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{array} \right), \min(G_i(y), H_i(z)) \right) \middle| \, x \in U \right\}.$$

For the term enclosed in the supremum, we get

$$\min\left(\max\left(\begin{array}{l} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{array} \right), \min(G_i(y), H_i(z)) \right)$$

$$(13)\quad = \max\left(\begin{array}{l} \min\left(\min(F'(x), \min(F_1(x), G_1(y))), \min(G_i(y), H_i(z))\right), \\ \vdots \\ \min\left(\min(F'(x), \min(F_n(x), G_n(y))), \min(G_i(y), H_i(z))\right) \end{array} \right)$$

$$(14)\quad = \max\left(\begin{array}{l} \min\left(\min(F'(x), \min(F_1(x), \min(G_1(y), G_i(y)))), H_i(z)\right), \\ \vdots \\ \min\left(\min(F'(x), \min(F_n(x), \min(G_n(y), G_i(y)))), H_i(z)\right) \end{array} \right)$$

$$(15)\quad = \min\left(F'(x), \min\left(\max\left(\begin{array}{l} \min(F_1(x), \min(G_1(y), G_i(y))), \\ \vdots \\ \min(F_n(x), \min(G_n(y), G_i(y))) \end{array} \right), H_i(z) \right) \right)$$

We apply the supremum ranging over y, yielding

$$\operatorname{Sup}\left\{ \min\left(F'(x), \min\left(\max\left(\begin{array}{l} \min(F_1(x), \min(G_1(y), G_i(y))), \\ \vdots \\ \min(F_n(x), \min(G_n(y), G_i(y))) \end{array} \right), H_i(z) \right) \right) \middle| \, y \in U \right\}$$

$$(16)\quad = \min\left(F'(x), \min\left(\operatorname{Sup}\left\{ \max\left(\begin{array}{l} \min(F_1(x), \min(G_1(y), G_i(y))), \\ \vdots \\ \min(F_n(x), \min(G_n(y), G_i(y))) \end{array} \right) \middle| \, y \in U \right\}, H_i(z) \right) \right)$$

$$= \min\left(F'(x), \min\left(\max\begin{pmatrix}\mathrm{Sup}\{\min(F_1(x), \min(G_1(y), G_i(y)))|y \in U\}, \\ \vdots \\ \mathrm{Sup}\{\min(F_n(x), \min(G_n(y), G_i(y)))|y \in U\}\end{pmatrix}, H_i(z)\right)\right)$$

$$= \min\left(F'(x), \min\left(\max\begin{pmatrix}\min(F_1(x), \mathrm{Sup}\{\min(G_1(y), G_i(y))|y \in U\}), \\ \vdots \\ \min(F_n(x), \mathrm{Sup}\{\min(G_n(y), G_i(y))|y \in U\})\end{pmatrix}, H_i(z)\right)\right) \tag{17}$$

$$= \min\left(F'(x), \min\left(\max\begin{pmatrix}\min(F_1(x), \mathrm{hgt}(G_1 \cap G_i)), \\ \vdots \\ \min(F_n(x), \mathrm{hgt}(G_n \cap G_i))\end{pmatrix}, H_i(z)\right)\right)$$

$$= \min\left(F'(x), \min\left(F_i'(x), H_i(z)\right)\right)$$

$$= \min\left(F'(x), R^{F_i', H_i}(x,z)\right).$$

Now, we can assemble the different parts of the proof, yielding

$$\mathrm{FITA}^{\mathfrak{R}_1} \circ \mathrm{FITA}^{\mathfrak{R}_2}(F')(z)$$
$$= \max\begin{pmatrix}\mathrm{Sup}\left\{\min\left(\mathrm{FITA}^{\mathfrak{R}_1}(F')(y), \min(G_1(y), H_1(z))\right)\middle| y \in U\right\}, \\ \vdots \\ \mathrm{Sup}\left\{\min\left(\mathrm{FITA}^{\mathfrak{R}_1}(F')(y), \min(G_n(y), H_n(z))\right)\middle| y \in U\right\}\end{pmatrix}$$

$$
= \max\begin{pmatrix}
\mathrm{Sup}\left\{ \mathrm{Sup}\left\{ \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x),\min(F_1(x),G_1(y))), \\ \vdots \\ \min(F'(x),\min(F_n(x),G_n(y))) \end{pmatrix}, \\ \min(G_1(y),H_1(z)) \end{pmatrix} \middle| \, x\in U \right\} \middle| \, y\in U \right\}, \\
\vdots \\
\mathrm{Sup}\left\{ \mathrm{Sup}\left\{ \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x),\min(F_1(x),G_1(y))), \\ \vdots \\ \min(F'(x),\min(F_n(x),G_n(y))) \end{pmatrix}, \\ \min(G_n(y),H_n(z)) \end{pmatrix} \middle| \, x\in U \right\} \middle| \, y\in U \right\}
\end{pmatrix}
$$

$$
= \max\begin{pmatrix}
\mathrm{Sup}\left\{ \mathrm{Sup}\left\{ \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x),\min(F_1(x),G_1(y))), \\ \vdots \\ \min(F'(x),\min(F_n(x),G_n(y))) \end{pmatrix}, \\ \min(G_1(y),H_1(z)) \end{pmatrix} \middle| \, y\in U \right\} \middle| \, x\in U \right\}, \\
\vdots \\
\mathrm{Sup}\left\{ \mathrm{Sup}\left\{ \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x),\min(F_1(x),G_1(y))), \\ \vdots \\ \min(F'(x),\min(F_n(x),G_n(y))) \end{pmatrix}, \\ \min(G_n(y),H_n(z)) \end{pmatrix} \middle| \, y\in U \right\} \middle| \, x\in U \right\}
\end{pmatrix}
$$

$$
= \max\begin{pmatrix}
\mathrm{Sup}\left\{ \min\left(F'(x), R^{F_1',H_1}(x,z)\right) \middle| \, x\in U \right\}, \\
\vdots \\
\mathrm{Sup}\left\{ \min\left(F'(x), R^{F_n',H_n}(x,z)\right) \middle| \, x\in U \right\}
\end{pmatrix}
$$

$$
= \max\left(\Phi^{R^{F_1',H_1}}(F')(z), \ldots, \Phi^{R^{F_n',H_n}}(F')(z)\right)
$$

$$
= \mathrm{FITA}^{\mathfrak{R}_{1,2}}(F')(z).
$$

Thus, the theorem is proved. ■

From the proof of theorem 3, we can observe the following corollary. For every $i \in \{1, \ldots, n\}$, we define a new fuzzy set H_i' by

$$H_i'(x) =_{\text{def}} (H_1 \cap H_{i,1}) \cup \cdots \cup (H_n \cap H_{i,n})$$

where for every $z \in U$ and $i, j \in \{1, \ldots, n\}$,

$$H_{i,j}(z) =_{\text{def}} \text{hgt}(G_j \cap G_i). \tag{18}$$

Then, we define a new rule base $\mathfrak{R}^*_{1,2}$ by

$$\mathfrak{R}^*_{1,2} : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & H_1' \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & H_n' \end{array} \tag{19}$$

Corollary 4

For all rule bases $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of the form (6)*,* $\text{FITA}^{\mathfrak{R}_1} \circ \text{FITA}^{\mathfrak{R}_2} = \text{FITA}^{\mathfrak{R}^*_{1,2}}$.

Proof

We proceed as in the proof of theorem 3, yielding

$$\begin{aligned}
&\text{FITA}^{\mathfrak{R}_1} \circ \text{FITA}^{\mathfrak{R}_2}(F')(z) \\
&= \max\left(\begin{array}{l} \text{Sup}\left\{\min\left(\text{FITA}^{\mathfrak{R}_1}(F')(y), \min(G_1(y), H_1(z))\right) \middle| y \in U\right\}, \\ \vdots \\ \text{Sup}\left\{\min\left(\text{FITA}^{\mathfrak{R}_1}(F')(y), \min(G_n(y), H_n(z))\right) \middle| y \in U\right\} \end{array} \right) \\
&= \max\left(\begin{array}{l}
\text{Sup}\left\{ \text{Sup}\left\{ \min\left(\begin{array}{l} \max\left(\begin{array}{l} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{array} \right), \\ \min(G_1(y), H_1(z)) \end{array} \right) \middle| x \in U \right\} \middle| y \in U \right\}, \\
\vdots \\
\text{Sup}\left\{ \text{Sup}\left\{ \min\left(\begin{array}{l} \max\left(\begin{array}{l} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{array} \right), \\ \min(G_n(y), H_n(z)) \end{array} \right) \middle| x \in U \right\} \middle| y \in U \right\}
\end{array} \right)
\end{aligned}$$

$$= \operatorname{Sup}\left\{ \operatorname{Sup}\left\{ \max\begin{pmatrix} \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{pmatrix}, \\ \min(G_1(y), H_1(z)) \end{pmatrix}, \\ \vdots \\ \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{pmatrix}, \\ \min(G_n(y), H_n(z)) \end{pmatrix} \end{pmatrix} \middle| x \in U \right\} \middle| y \in U \right\}$$

For the term inside the supremum, we obtain

$$\max\begin{pmatrix} \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{pmatrix}, \\ \min(G_1(y), H_1(z)) \end{pmatrix}, \\ \vdots \\ \min\begin{pmatrix} \max\begin{pmatrix} \min(F'(x), \min(F_1(x), G_1(y))), \\ \vdots \\ \min(F'(x), \min(F_n(x), G_n(y))) \end{pmatrix}, \\ \min(G_n(y), H_n(z)) \end{pmatrix} \end{pmatrix}$$

$$= \max\begin{pmatrix} \max\begin{pmatrix} \min\left(\min(F'(x), \min(F_1(x), G_1(y))), \min(G_1(y), H_1(z))\right), \\ \vdots \\ \min\left(\min(F'(x), \min(F_n(x), G_n(y))), \min(G_1(y), H_1(z))\right) \end{pmatrix}, \\ \vdots \\ \max\begin{pmatrix} \min\left(\min(F'(x), \min(F_1(x), G_1(y))), \min(G_n(y), H_n(z))\right), \\ \vdots \\ \min\left(\min(F'(x), \min(F_n(x), G_n(y))), \min(G_n(y), H_n(z))\right) \end{pmatrix} \end{pmatrix}$$

$$
= \max\left(\begin{array}{l}
\max\left(\begin{array}{l}
\min(F'(x), \min(F_1(x), \min(\min(G_1(y), G_1(y)), H_1(z)))), \\
\vdots \\
\min(F'(x), \min(F_n(x), \min(\min(G_n(y), G_1(y)), H_1(z))))
\end{array}\right), \\
\vdots \\
\max\left(\begin{array}{l}
\min(F'(x), \min(F_1(x), \min(\min(G_1(y), G_n(y)), H_n(z)))), \\
\vdots \\
\min(F'(x), \min(F_n(x), \min(\min(G_n(y), G_n(y)), H_n(z))))
\end{array}\right)
\end{array}\right)
$$

$$
= \max\left(\begin{array}{l}
\max\left(\begin{array}{l}
\min(F'(x), \min(F_1(x), \min(\min(G_1(y), G_1(y)), H_1(z)))), \\
\vdots \\
\min(F'(x), \min(F_1(x), \min(\min(G_1(y), G_n(y)), H_n(z))))
\end{array}\right), \\
\vdots \\
\max\left(\begin{array}{l}
\min(F'(x), \min(F_n(x), \min(\min(G_n(y), G_1(y)), H_1(z)))), \\
\vdots \\
\min(F'(x), \min(F_n(x), \min(\min(G_n(y), G_n(y)), H_n(z))))
\end{array}\right)
\end{array}\right)
$$

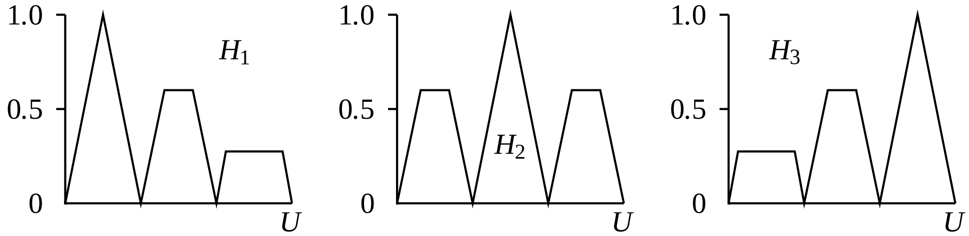
$$
= \max\left(\begin{array}{l}
\min\left(F'(x), \min\left(F_1(x), \max\left(\begin{array}{l}
\min(\min(G_1(y), G_1(y)), H_1(z)), \\
\vdots \\
\min(\min(G_1(y), G_n(y)), H_n(z))
\end{array}\right)\right)\right), \\
\vdots \\
\min\left(F'(x), \min\left(F_n(x), \max\left(\begin{array}{l}
\min(\min(G_n(y), G_1(y)), H_1(z)), \\
\vdots \\
\min(\min(G_n(y), G_n(y)), H_n(z))
\end{array}\right)\right)\right)
\end{array}\right).
$$

From this point, we proceed exactly as in the proof of theorem 3, step (15). ■

Example 2 For the rule base from example 1, we obtain the new rule base

$$
\mathfrak{R}^*_{1,2}: \quad \begin{array}{llll}
\text{IF} & F_1 & \text{THEN} & H_1' \\
\text{IF} & F_2 & \text{THEN} & H_2' \\
\text{IF} & F_3 & \text{THEN} & H_3'
\end{array} \tag{20}
$$

where the fuzzy sets H_i' are given as illustrated in the following diagrams:



3 A Generalization of the MAMDANI Case

We discuss a generalization of the MAMDANI inference, where the minimum is replaced by a binary operator $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$.

We associate with a fuzzy IF-THEN rule

$$\text{IF}\, F \text{ THEN}\, G$$

an *interpretation* $R_\kappa^{F,G}$ in the form of a binary *fuzzy relation* on U, i. e. $R_\kappa^{F,G} : U \times U \to \langle 0,1\rangle$.

(21) $$R_\kappa^{F,G}(x,y) =_{\text{def}} \kappa(F(x), G(y)) \qquad (x, y \in U)$$

Given a fuzzy relation $R : U \times U \to \langle 0,1\rangle$ and a binary operator $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$, we define a *functional operator* Φ_κ^R taking a fuzzy set $F' : U \to \langle 0,1\rangle$ to $G' : U \to \langle 0,1\rangle$, as follows, for $y \in U$.

(22) $$\Phi_\kappa^R(F')(y) =_{\text{def}} \text{Sup}\left\{\kappa(F'(x), R(x,y)) \,\middle|\, x \in U\right\}$$

Given a rule base

$$\mathfrak{R} : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array}$$

the operators FATI and FITA are defined as in section 2, using the functional operator Φ_κ^R. We still use the maximum operator for aggregation, that is, the combined fuzzy relation $R_\kappa^{\mathfrak{R}}$ is defined exactly as in section 2, by

$$R_\kappa^{\mathfrak{R}} =_{\text{def}} R_\kappa^{F_1,G_1} \cup \cdots \cup R_\kappa^{F_n,G_n}$$

For an input fuzzy set F' on U, we define

$$\begin{aligned} \text{FITA}_\kappa^{\mathfrak{R}}(F') &=_{\text{def}} \Phi_\kappa^{R_\kappa^{F_1,G_1}}(F') \cup \cdots \cup \Phi_\kappa^{R_\kappa^{F_n,G_n}}(F') \\ \text{FATI}_\kappa^{\mathfrak{R}}(F') &=_{\text{def}} \Phi_\kappa^{R_\kappa^{\mathfrak{R}}}(F') \end{aligned}$$

Again, we can cite the following theorem from the literature:

Theorem 5

If κ *is non-decreasing wrt. the second argument, then for every rule base* $\mathfrak{R}$*,* $\mathrm{FITA}_\kappa^{\mathfrak{R}} = \mathrm{FATI}_\kappa^{\mathfrak{R}}$.

Again, we (arbitrarily) choose the principle FITA for the rest of this section.

Given two rule bases

$$
\mathfrak{R}_1 : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array} \qquad \mathfrak{R}_2 : \begin{array}{llll} \text{IF} & G_1 & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & G_n & \text{THEN} & H_n \end{array} \tag{23}
$$

we define a new rule base $\mathfrak{R}_{1,2}$ as follows.

For every $i \in \{1,\ldots,n\}$, we define a new fuzzy set F_i' by

$$
F_i' =_{\text{def}} (F_1 \cap_\kappa F_{i,1}) \cup \cdots \cup (F_n \cap_\kappa F_{i,n}) \tag{24}
$$

where for every $x \in U$ and $i, j \in \{1,\ldots,n\}$,

$$
F_{i,j}(x) =_{\text{def}} \mathrm{hgt}(G_j \cap_\kappa G_i).
$$

The new rule base $\mathfrak{R}_{1,2}$ is then defined by

$$
\mathfrak{R}_{1,2} : \begin{array}{llll} \text{IF} & F_1' & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & F_n' & \text{THEN} & H_n \end{array} \tag{25}
$$

We can then formulate the following theorem:

Theorem 6

For all rule bases $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *of the form* (23) *and every* $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$*, if* κ *is non-decreasing, associative and continuous, then* $\mathrm{FITA}_\kappa^{\mathfrak{R}_1} \circ \mathrm{FITA}_\kappa^{\mathfrak{R}_2} = \mathrm{FITA}_\kappa^{\mathfrak{R}_{1,2}}$.

Proof

The proof is analogous to the proof of theorem 3. Here, we only note the points where we need the respective properties of κ.

By expanding definitions, we get for $F' : U \to \langle 0,1\rangle$ and $z \in U$:

$$
\begin{aligned}
&\mathrm{FITA}_\kappa^{\mathfrak{R}_1} \circ \mathrm{FITA}_\kappa^{\mathfrak{R}_2}(F')(z) \\
&\qquad = \max\left(\begin{array}{l} \mathrm{Sup}\left\{ \kappa\left(\mathrm{FITA}_\kappa^{\mathfrak{R}_1}(F')(y), \kappa(G_1(y), H_1(z)) \right) \middle| \, y \in U \right\}, \\ \vdots \\ \mathrm{Sup}\left\{ \kappa\left(\mathrm{FITA}_\kappa^{\mathfrak{R}_1}(F')(y), \kappa(G_n(y), H_n(z)) \right) \middle| \, y \in U \right\} \end{array} \right).
\end{aligned}
$$

For the term enclosed in the supremum, we get the following equation for every $i \in \{1, \dots, n\}$. We only have to employ the continuity and non-decreasingness of κ in step (12).

$$\kappa\Big(\mathrm{FITA}^{\mathfrak{R}_1}_{\kappa}(F')(y), \kappa(G_i(y), H_i(z))\Big)$$

$$= \mathrm{Sup}\left\{ \kappa\left(\max\begin{pmatrix} \kappa(F'(x), \kappa(F_1(x), G_1(y))), \\ \vdots \\ \kappa(F'(x), \kappa(F_n(x), G_n(y))) \end{pmatrix}, \kappa(G_i(y), H_i(z)) \right) \middle| \, x \in U \right\}.$$

For the term enclosed in the supremum, we get, by employing the non-decreasingness of κ in steps (13) and (15) and the associativity of κ in steps (14) and (15),

$$\kappa\left(\max\begin{pmatrix} \kappa(F'(x), \kappa(F_1(x), G_1(y))), \\ \vdots \\ \kappa(F'(x), \kappa(F_n(x), G_n(y))) \end{pmatrix}, \kappa(G_i(y), H_i(z)) \right)$$

$$= \kappa\left(F'(x), \kappa\left(\max\begin{pmatrix} \kappa(F_1(x), \kappa(G_1(y), G_i(y))), \\ \vdots \\ \kappa(F_n(x), \kappa(G_n(y), G_i(y))) \end{pmatrix}, H_i(z) \right) \right).$$

By employing the non-decreasingness and continuity of κ in steps (16) and (17), we obtain

$$\mathrm{Sup}\left\{ \kappa\left(F'(x), \kappa\left(\max\begin{pmatrix} \kappa(F_1(x), \kappa(G_1(y), G_i(y))), \\ \vdots \\ \kappa(F_n(x), \kappa(G_n(y), G_i(y))) \end{pmatrix}, H_i(z) \right) \right) \middle| \, y \in U \right\}$$

$$= \kappa\left(F'(x), \kappa\left(\max\begin{pmatrix} \kappa(F_1(x), \mathrm{hgt}(G_1 \cap_{\kappa} G_i)), \\ \vdots \\ \kappa(F_n(x), \mathrm{hgt}(G_n \cap_{\kappa} G_i)) \end{pmatrix}, H_i(z) \right) \right)$$

$$= \kappa\big(F'(x), \kappa\big(F_i'(x), H_i(z)\big)\big)$$

$$= \kappa\Big(F'(x), R^{F_i', H_i}_{\kappa}(x, z)\Big).$$

The rest of the proof is identical to the proof of theorem 3. ■

Remark
The assumptions made in the theorem above wrt. κ include the case that κ is a t-norm.

Again, we obtain the following corollary. For every $i \in \{1, \dots, n\}$, we define a new fuzzy set $H_i{}'$ by

$$H_i{}'(x) =_{\text{def}} (H_1 \cap_\kappa H_{i,1}) \cup \cdots \cup (H_n \cap_\kappa H_{i,n})$$

where for every $z \in U$ and $i, j \in \{1, \dots, n\}$,

$$H_{i,j}(z) =_{\text{def}} \text{hgt}(G_j \cap_\kappa G_i). \tag{26}$$

Then, we define a new rule base $\mathfrak{R}^*_{1,2}$ by

$$\mathfrak{R}^*_{1,2} : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & H_1{}' \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & H_n{}' \end{array} \tag{27}$$

Corollary 7
For all rule bases $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *of the form* (23) *and every* $\kappa : \langle 0,1 \rangle \times \langle 0,1 \rangle \to \langle 0,1 \rangle$*, if* κ *is non-decreasing, associative and continuous, then* $\text{FITA}_\kappa^{\mathfrak{R}_1} \circ \text{FITA}_\kappa^{\mathfrak{R}_2} = \text{FITA}_\kappa^{\mathfrak{R}^*_{1,2}}$.

Proof
Identical to the proof of corollary 4. ∎

4 Conclusions

It should be obvious that theorem 6 offers a powerful tool for reducing the size and the complexity of multiple-stage fuzzy IF-THEN rule bases for some common inference systems. Still, it is interesting to discuss some further results and directions for future research.

4.1 Additional results

Additionally to the two main theorems 3 and 6, we can obtain the following interesting results. As theorem 3 is a special case of theorem 6, we shall refer only to theorem 6 in the following.

4.1.1 Combination of Arbitrary Rule Bases

To maintain the connection with the *chaining* results from [14, 15], we have so far assumed the given rule bases to be of the form (23), i. e. we have assumed that the *conclusions* of the first rule base are *exactly* the *premises* of the second.

Inspection of the proof of theorem 6 yields that in fact, the theorem is *independent* of this assumption. That is, we can assume two rule bases

$$(28) \qquad \mathfrak{R}_1 : \begin{array}{llll} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array} \qquad \mathfrak{R}_2 : \begin{array}{llll} \text{IF} & G_1' & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & G_m' & \text{THEN} & H_m \end{array}$$

to be given where n, m are positive integers and $G_1, \ldots, G_n, G_1', \ldots, G_m'$ are *arbitrary* fuzzy sets on U, and define a new rule base $\mathfrak{R}_{1,2}$ as follows.

For every $i \in \{1, \ldots, m\}$, we define a new fuzzy set F_i' by

$$F_i' =_{\text{def}} (F_1 \mathbin{\text{⋒}} F_{i,1}) \cup \cdots \cup (F_n \mathbin{\text{⋒}} F_{i,n})$$

where for every $x \in U$ and $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$,

$$(29) \qquad F_{i,j}(x) =_{\text{def}} \text{hgt}(G_j \mathbin{\text{⋒}} G_i').$$

The new rule base $\mathfrak{R}_{1,2}$ is then defined by

$$(30) \qquad \mathfrak{R}_{1,2} : \begin{array}{llll} \text{IF} & F_1' & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & F_m' & \text{THEN} & H_m \end{array}$$

We can then formulate the following theorem:

Theorem 8
For all rule bases $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of the form (28) *and every $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$, if κ is non-decreasing, associative and continuous, then* $\text{FITA}_\kappa^{\mathfrak{R}_1} \circ \text{FITA}_\kappa^{\mathfrak{R}_2} = \text{FITA}_\kappa^{\mathfrak{R}_{1,2}}$.

Proof
Identical to the proof of theorem 6. ■

With theorem 8, we have obtained the most general result, enabling us to reduce practically *every* multiple-stage inference of MAMDANI style.

Of course, an analogous result to corollary 7 can also be proved.

4.1.2 Reduction of More than Two Steps

Obviously, if we are given rule bases with more than two stages, then the reduction process may be iterated. By the result of theorem 8, we are completely free respecting the order of reduction steps.

Thus, we can conclude that whenever inference is carried out in the (generalized) MAMDANI style presented here, ***every* multi-stage rule base may be reduced to a single-stage rule base**.

4.1.3 Criteria for Chainability

In [14, 15], *chainability* of rule bases interpreted by FITA has been defined as follows:

Definition 1
Given two rule bases

$$\mathfrak{R}_1 : \begin{array}{cccc} \text{IF} & F_1 & \text{THEN} & G_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & G_n \end{array} \qquad \mathfrak{R}_2 : \begin{array}{cccc} \text{IF} & G_1 & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & G_n & \text{THEN} & H_n \end{array}$$

we define the following rule base:

$$\mathfrak{R}^C_{1,2} : \begin{array}{cccc} \text{IF} & F_1 & \text{THEN} & H_1 \\ & & \vdots & \\ \text{IF} & F_n & \text{THEN} & H_n \end{array}$$

We say that $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *are* ***chainable*** *wrt.* $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$
$=_{\text{def}} \text{FITA}_\kappa^{\mathfrak{R}_1} \circ \text{FITA}_\kappa^{\mathfrak{R}_2} = \text{FITA}_\kappa^{\mathfrak{R}^C_{1,2}}$.

Different criteria for chainability have been derived in [14, 15], but so far the (generalized) MAMDANI case has not been characterized. Using the result of theorem 6, we can now formulate criteria for the chainability in the (generalized) MAMDANI case. First of all, we obtain

Lemma 9
For all rule bases $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *of the form* (28) *and every* $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$*, if* κ *is non-decreasing, associative and continuous, then* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *are* ***chainable*** *wrt.* κ *if and only if* $\text{FITA}_\kappa^{\mathfrak{R}^C_{1,2}} = \text{FITA}_\kappa^{\mathfrak{R}_{1,2}}$ *(where* $\mathfrak{R}_{1,2}$ *is defined by* (25)*).*

By inspection of the rule bases $\mathfrak{R}^C_{1,2}$ and $\mathfrak{R}_{1,2}$, we can derive more convenient criteria. We concentrate on the following criterium:

Lemma 10
For all rule bases $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *of the form* (28) *and every* $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$*, if* κ *is non-decreasing, associative and continuous, then* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *are* ***chainable*** *wrt.* κ *if for every* $i \in \{1,\ldots,n\}$*,* $F_i' = F_i$ *(where* F_i' *is defined by* (24)*).*

Proof
Trivially, $\mathfrak{R}^C_{1,2} = \mathfrak{R}_{1,2}$ in this case. ■

Of course $\text{FITA}_\kappa^{\mathfrak{R}^C_{1,2}} = \text{FITA}_\kappa^{\mathfrak{R}_{1,2}}$ is a weaker condition than $\mathfrak{R}^C_{1,2} = \mathfrak{R}_{1,2}$, that is, there may be cases when $\mathfrak{R}^C_{1,2} \neq \mathfrak{R}_{1,2}$, but still $\text{FITA}_\kappa^{\mathfrak{R}^C_{1,2}} = \text{FITA}_\kappa^{\mathfrak{R}_{1,2}}$. This is the reason for replacing the "if and only if" condition in Lemma 9 by "if" in Lemma 10.

If $\mathfrak{R}^{C}_{1,2} \neq \mathfrak{R}_{1,2}$, but $\mathrm{FITA}_{\kappa}^{\mathfrak{R}^{C}_{1,2}} = \mathrm{FITA}_{\kappa}^{\mathfrak{R}_{1,2}}$ holds, this is a sign that there is some 'redundancy' in the rule bases $\mathfrak{R}_1, \mathfrak{R}_2$. Thus by accepting the stronger condition $\mathfrak{R}^{C}_{1,2} = \mathfrak{R}_{1,2}$, we are saying that all information present in $\mathfrak{R}_1, \mathfrak{R}_2$ is actually *needed*.

The criterion $F_i' = F_i$ for every $i \in \{1, \ldots, n\}$ is already easily testable: we simply calculate F_i' and compare.

For a faster testing procedure, we can derive a simpler criterion, under certain additional assumptions.

Lemma 11

1. *For $i \in \{1, \ldots, n\}$, if G_i is normal, then $F_i \subseteqq F_i'$.*
2. *For $i \in \{1, \ldots, n\}$, $F_i' \subseteqq F_i$ holds if and only if for every $j \in \{1, \ldots, n\}$, $F_j \cap_{\kappa} F_{i,j} \subseteqq F_i$.*

Proof

Trivial from the definitions. ■

Thus, a criterion for $F_i' = F_i$ is derived by the combination of items 1 and 2 of the lemma above. Especially criterion 2 is very strong, however. Indeed, inspection of the criteria derived so far leads to the result that apart from trivial borderline cases, chainability for MAMDANI style inference systems will only be achieved if all the G_i are normal and for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, G_i and G_j do not overlap (compare example 1), a condition which can hardly be met.

This situation is different for the generalized MAMDANI case, however, because if κ is chosen to be, for instance, a t-norm other than the minimum, criterion 2 of Lemma 11 is less strong because every t-norm lies below the minimum and thus $F_j \cap_{\kappa} F_{i,j}$ is likely to be smaller than $F_j \cap F_{i,j}$, making it more easy to fulfill criterion 2.

If chainability is desired, it is thus advisable to consider *generalized* MAMDANI inference systems, for instance *bold* inference, using the *bold* t-norm for κ.

4.2 Further Research

The "generalized" MAMDANI case presented here is actually not very general. In applications, often different t-norms are employed for the interpretation of a rule as a fuzzy relation (equation (21)) and for the definition of the inference operator (equation (22)). Inspection of the proof of theorem 6 yields that our reduction method is likely not to be applicable to this case in full generality (see especially step (14) in the proof of theorem 3). We have to develop additional criteria under which the reduction method is applicable in more general cases of inference procedures.

Furthermore, in section 4.1.3 we did not derive an "if and only if" criterion for chainability which can be checked by inspection of a rule base. However, starting from lemma 9, it may be possible to obtain such a result.

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