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On Closure Operators in Fuzzy Algebras and Fuzzy Deductive Systems

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The starting point of this paper is the classical well-known theorem due to G. BIRKHOFF, P. HALL, and J. SCHMIDT which establishes a one-to-one correspondence between compact closure operators, inductive closure operators, inductive closure systems, and closure operators generated in universal algebras (and generated in deductive systems, respectively). In the paper presented we make first steps in order to generalize this important theorem to the fuzzy set theory and fuzzy algebra (and fuzzy deductive systems, respectively).

Keywords Compact Closure Operators, Inductive Closure Operators, Inductive Closure Systems, Universal Algebras, Deductive Systems, Fuzzification

1 Introduction

For arbitrary crisp sets A and B by $A \cap B$, $A \cup B$, and $A \setminus B$ we denote the usual intersection, union, and difference of A and B, respectively, furthermore $A \subseteqq B$ means that A is a subset of B. For an arbitrary system $\mathfrak{A}$ of sets by $\bigcap \mathfrak{A}$ and $\bigcup \mathfrak{A}$ we denote the intersection and the union of all sets of $\mathfrak{A}$, respectively. If we have a family $(A_i | i \in I)$ of sets A_i, then we write $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$. The cardinal number of A is denoted by $\operatorname{card} A$, the power set of A by $\mathbb{P}(A)$, the empty set by $\emptyset$, and the empty sequence of elements of a set by e. Hence we define $A^0 =_{\text{def}} \{e\}$. $A^n =_{\text{def}}$ the set of all sequences of elements of A with the length n where n is an integer with $n \geqq 1$. Finally, we define $A^* =_{\text{def}} \bigcup_{n \in \mathbb{N}} A^n$ where $\mathbb{N} = \{0, 1, \cdots\}$.

For compact denotation in the following we shall use sometimes the symbolic of predicate calculus, i. e. $\forall x$ as "for every x", $\exists x$ as "there is an x", $\wedge$ as "and", $\vee$ as "or", $\rightarrow$ as "if - then", $\leftrightarrow$ as "if and only if", $\neg$ as "not".

Remember the definition of a complete lattice

$$\mathfrak{L} = [L, \curlywedge, \curlyvee, 0, u]$$

with the domain L, the intersection operator $\curlywedge$, the union operator $\curlyvee$, the zero element 0, and the unit element u.

Remember that by the definition

$$x \preceqq y =_{\text{def}} x \curlywedge y = x \qquad (x, y \in L)$$

there is introduced a partial order on L. For $K \subseteqq L$ by $\inf K$ and $\sup K$ we denote the infimum and the supremum of K with respect to $\preceqq$, respectively. A set $C \subseteqq L$ is said to be a chain of $\mathfrak{L}$ if and only if

$$\forall x \forall y \big(x, y \in C \rightarrow x \preceqq y \quad \vee \quad y \preceqq x\big) \ .$$

Let $\leqq$ be the natural ordering of real numbers. For an arbitrary set S of real numbers by $\operatorname{Inf} S$ and $\operatorname{Sup} S$ we denote the infimum and the supremum of S with respect to $\leqq$, respectively. By $\langle 0, 1 \rangle$ we denote the set of all real numbers r with $0 \leqq r \leqq 1$.

Let U be an arbitrary non-empty set called universe. A fuzzy set F on U is a mapping $F : U \rightarrow \langle 0, 1 \rangle$, i. e. we do not distinguish between a fuzzy set F and its membership function μ_F. The set of all fuzzy sets on U is denoted by $F\mathbb{P}(U)$.

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We introduce the empty fuzzy set $\emptyset$ on U and the universal fuzzy set $\mathcal{U}$ on U, respectively, for every $x \in U$ defined by

$$\begin{aligned} &\emptyset(x) = 0 \\ \text{and} \quad &\mathcal{U}(x) = 1 . \end{aligned}$$

As usual we define the support supp(F) of a fuzzy set F on U by

$$\operatorname{supp}(F) = \{x | x \in U \wedge F(x) > 0\} .$$

A fuzzy set F is said to be finite and a singleton if and only if supp(F) is finite and card supp$(F) = 1$, respectively.

For $F, G \in F\mathbb{P}(U)$ as usual we put

$$F \sqsubseteq G =_{\text{def}} \forall x \big(x \in U \rightarrow F(x) \leqq G(x)\big)$$

and for $x \in U$ we define

$$\begin{aligned} (F \sqcap G)(x) &=_{\text{def}} \min(F(x), G(x)) \\ (F \sqcup G)(x) &=_{\text{def}} \max(F(x), G(x)) . \end{aligned}$$

Furthermore, for arbitrary $\mathfrak{F} \subseteqq F\mathbb{P}(U), x \in U$ we put

$$\begin{aligned} (\text{INF}\,\mathfrak{F})(x) &=_{\text{def}} \text{Inf}\{F(x) | F \in \mathfrak{F}\} \\ (\text{SUP}\,\mathfrak{F})(x) &=_{\text{def}} \text{Sup}\{F(x) | F \in \mathfrak{F}\} . \end{aligned}$$

Remark

The operations $\sqcap, \sqcup$, INF, SUP are defined in the “standard sense” and used throughout the paper presented. In a forthcoming paper we shall discuss the case if min and max are replaced by an arbitrary t-norm τ and an s-Norm (t-conorm) σ, respectively. Furthermore, INF and SUP should be replaced by the quantifier Q_τ and Q_σ, respectively (see [16–19]).

Finally, remember the following

Theorem 1

$$\mathbb{L} = [\mathbb{P}(U), \cap, \cup, \emptyset, U]$$

and

$$L = [F\mathbb{P}(U), \sqcap, \sqcup, \emptyset, \mathcal{U}]$$

are complete lattices.

In the following chapters 2 and 3 we repeat some facts more or less well-known from lattice theory and universal algebra.

2 A Bijection between Closure Operators and Closure Systems of Complete Lattices

Let $\mathfrak{L} = [L, \wedge, \vee, 0, u]$ be a complete lattice. Assume that $\varphi : L \to L$ and $C \subseteqq L$.

Definition 1

1. *φ is said to be* monotone *on* $\mathfrak{L} =_{\text{def}} \forall x \forall y (x, y \in L \wedge x \leqq y \to \varphi(x) \leqq \varphi(y))$
2. *φ is said to be* embedding *on* $\mathfrak{L} =_{\text{def}} \forall x (x \in L \to x \leqq \varphi(x))$
3. *φ is said to be* closed *on* $\mathfrak{L} =_{\text{def}} \forall x (x \in L \to \varphi(\varphi(x)) \leqq \varphi(x))$
4. *φ is said a* closure operator *of* $\mathfrak{L} =_{\text{def}}$ φ *fulfils the conditions 1, 2, and 3.*

Definition 2

C is said to be a closure system *of* $\mathfrak{L} =_{\text{def}} \forall D (D \subseteqq C \to \inf D \in C)$

Definition 3

1. $\mathrm{SET}(\varphi) =_{\text{def}} \{c \mid c \in L \wedge \varphi(c) \leqq c\}$
2. $\mathrm{FCT}(C)(x) =_{\text{def}} \inf\{y \mid x \leqq y \wedge y \in C\}$

Lemma 2

If φ is monotone on $\mathfrak{L}$, then $\mathrm{SET}(\varphi)$ *is a closure system of $\mathfrak{L}$.*

Proof

Assume

(1) $$D \subseteqq \mathrm{SET}(\varphi), \text{ i. e.}$$

(2) $$\forall d (d \in D \to \varphi(d) \leqq d)$$

We have to prove

(3) $$\varphi(\inf D) \leqq \inf D.$$

In order to prove (3) it is sufficient to show

(4) $$\forall d (d \in D \to \varphi(\inf D) \leqq d)$$

From $d \in D$ we get

(5) $$\inf D \leqq d$$

hence by monotonicity of φ

(6) $$\varphi(\inf D) \leqq \varphi(d)$$

hence by (2) we obtain (4). ■

Lemma 3

If φ is embedding and closed on $\mathfrak{L}$, then

$$\forall x (x \in L \to \mathrm{FCT}(\mathrm{SET}(\varphi))(x) \leqq \varphi(x))$$

Proof
By definition 3 we have

$$(1)\qquad \begin{aligned}&\mathrm{FCT}(\mathrm{SET}(\varphi))(x)\\ &= \inf\{y \mid x \leqq y \wedge y \in \mathrm{SET}(\varphi)\}\\ &= \inf\{y \mid x \leqq y \wedge \varphi(y) \leqq y \wedge y \in L\}\end{aligned}$$

In order to prove lemma 3 it is sufficient to show

$$(2)\qquad \exists y_0 (y_0 \in L \wedge x \leqq y_0 \wedge \varphi(y_0) \leqq y_0).$$

We put

$$(3)\qquad y_0 =_{\mathrm{def}} \varphi(x).$$

Then we get

$$(4)\qquad \varphi(x) \in L,$$

$$(5)\qquad x \leqq \varphi(x) \text{ by embedding of } \varphi,$$

$$(6)\qquad \varphi(\varphi(x)) \leqq \varphi(x) \text{ by closedness of } \varphi,$$

hence lemma 3 holds.

■

Lemma 4
If φ *is monotone on* $\mathfrak{L}$*, then*

$$\forall x (x \in L \rightarrow \varphi(x) \leqq \mathrm{FCT}(\mathrm{SET}(\varphi))(x))$$

Proof
For every $x \in L$ we have to prove

$$(1)\qquad \varphi(x) \leqq \inf\{y \mid y \in L \wedge x \leqq y \wedge \varphi(y) \leqq y)\} \;.$$

In order to prove (1) it is sufficient to show

$$(2)\qquad \forall y (y \in L \wedge x \leqq y \wedge \varphi(y) \leqq y \rightarrow \varphi(x) \leqq y)$$

From $x \leqq y$ by monotonicity of φ we get $\varphi(x) \leqq \varphi(y)$, hence $\varphi(x) \leqq y$ because of $\varphi(y) \leqq y$.
■

Theorem 5
If φ *is a closure operator of* $\mathfrak{L}$*, then*

$$\mathrm{FCT}(\mathrm{SET}(\varphi)) = \varphi$$

Proof
By lemma 3 and lemma 4 ■

Corollary 6
SET *is an injection from the set of all closure operators of* $\mathfrak{L}$ *into the set of all closure systems of* $\mathfrak{L}$.

Proof
By lemma 2 and theorem 5 ■

Lemma 7
For every subset $C \subseteqq L$,

1. $\mathrm{FCT}(C)$ *is embedding and*
2. $\mathrm{FCT}(C)$ *is monotone.*

Proof
Trivially on the basis of definition 3 ■

Lemma 8
For every subset $C \subseteqq L$,
if C *is a closure system of* $\mathfrak{L}$, *then*

1. $\mathrm{FCT}(C)$ *is a closure operator and*
2. $\mathrm{SET}(\mathrm{FCT}(C)) \subseteqq C$.

Proof
ad 1 Because of lemma 7 it is sufficient to show that FCT(C) is closed, i. e.

(1) $$\forall x(x \in L \rightarrow \mathrm{FCT}(C)(\mathrm{FCT}(C)(x)) \leqq \mathrm{FCT}(C)(x).$$

On the basis of definition 3 it is sufficient to show

(2) $$\inf\{y \mid \mathrm{FCT}(C)(x) \leqq y \wedge y \in C\} \leqq \mathrm{FCT}(C)(x)$$

In order to prove (2) it is sufficient to show

(3) $$\exists y(|FCT(C)(x) \leqq y \wedge y \in C \wedge y \leqq \mathrm{FCT}(C)(x))$$

We put

(4) $$y =_{\mathrm{def}} \mathrm{FCT}(C)(x)\ .$$

In order to prove (3) it is sufficient to show

(5) $$\mathrm{FCT}(C)(x) \in C\ .$$

But, (5) holds because C is a closure system.

ad 2 We have to show

(6) $$\forall x(x \in L \wedge x \in \mathrm{SET}(\mathrm{FCT}(C)) \rightarrow x \in C)$$

For $x \in L$ we assume

(7) $$x \in \mathrm{SET}(\mathrm{FCT}(C)),$$

hence by definition of SET

(8) $$\mathrm{FCT}(C)(x) \leqq x$$

Because of lemma 7, FCT(C) is embedding, i. e.

(9) $$x \leqq \mathrm{FCT}(C)(x),$$

hence from (8) and (9) we get

(10) $$\mathrm{FCT}(C)(x) = x\ .$$

Furthermore, by definition 3 we have

(11) $$\mathrm{FCT}(C)(x) = \inf\{y | x \leqq y \wedge y \in C\}$$

Now, we assumed that C is a closure system, hence

(12) $$\inf\{y | x \leqq y \wedge y \in C\} \in C,$$

hence by (10) we get $x \in C$.

■

Lemma 9
For every subset $C \subseteqq L$,

$$C \subseteqq \mathrm{SET}(\mathrm{FCT}(C)) .$$

Proof
Assume

(1) $$x \in C$$

In order to prove

(2) $$x \in \mathrm{SET}(\mathrm{FCT}(C)),$$

using the definition of SET it is sufficient to show

(3) $$\mathrm{FCT}(C)(x) \leqq x,$$

hence by definition of FCT it is sufficient to show

(4) $$\inf\{y | x \leqq y \wedge y \in C\} \leqq x .$$

But (4) holds because of $x \in C$. ■

Theorem 10
If C *is a closure system of* $\mathfrak{L}$, *then* $\mathrm{SET}(\mathrm{FCT}(C)) = C$.

Proof
By lemma 8 and lemma 9 ■

Corollary 11
1. SET *is a bijection from the set of all closure operators of* $\mathfrak{L}$ *onto the set of all closure systems of* $\mathfrak{L}$.
2. FCT *is the inversion of the mapping* SET.

Proof
By theorem 5 and theorem 10 ■

Now, we are going to add some concepts of topology.

Definition 4
1. φ *is said to be a a* topological mapping *on* $\mathfrak{L}$
$=_{\mathrm{def}} \forall x \forall y (x, y \in L \rightarrow \varphi(x \vee y) \leqq \varphi(x) \vee \varphi(y))$
2. C *is said to be a* topological set *of* $\mathfrak{L}$
$=_{\mathrm{def}} \forall X (X \subseteqq C \wedge X \text{ is finite} \wedge X \text{ is not empty} \rightarrow \mathrm{Sup} X \in C)$

Lemma 12

If φ *is a topological mapping on* $\mathfrak{L}$*, then* $\mathrm{SET}(\varphi)$ *is a topological set of* $\mathfrak{L}$*.*

Proof

Let X be an arbitrary non-empty finite subset of $\mathfrak{L}$. Because φ is a topological mapping, we get

(1) $$\varphi(\sup X) \leqq \sup\{\varphi(x) | x \in X\}$$

Now, we assume additionally that

(2) $$X \subseteq \mathrm{SET}(\varphi), \text{ i. e. } \forall x(x \in X \rightarrow \varphi(x) \leqq x),$$

hence we get

(3) $$\sup\{\varphi(x) | x \in X\} \leqq \sup X,$$

consequently (1) and (3) imply

(4) $$\varphi(\sup X) \leqq \sup X, \text{ i. e. } \sup X \in \mathrm{SET}(\varphi)$$

■

Lemma 13

If C *is a closure system of* $\mathfrak{L}$ *and* C *is a topological set of* $\mathfrak{L}$*, then* $\mathrm{FCT}(C)$ *is a topological mapping on* $\mathfrak{L}$*.*

Proof

We have to prove

(1) $$\forall x \forall y \left(x, y \in L \rightarrow \mathrm{FCT}(C)(x \vee y) \leqq \mathrm{FCT}(C)(x) \vee \mathrm{FCT}(C)(y)\right)$$

In order to prove (1) it is sufficient to show

(2) $$\inf\{z | x \vee y \leqq z \wedge z \in C\} \leqq \inf\{z | x \leqq z \wedge z \in C\} \vee \inf\{z | y \leqq z \wedge z \in C\}$$

Because C is a closure system of $\mathfrak{L}$, we have

(3) $$\inf\{z | x \leqq z \wedge z \in C\} \in C$$

and

(4) $$\inf\{z | y \leqq z \wedge z \in C\} \in C,$$

hence, because C is a topological set of $\mathfrak{L}$, we get

(5) $$\inf\{z | x \leqq z \wedge z \in C\} \vee \inf\{z | y \leqq z \wedge z \in C\} \in C$$

Thus, in order to prove (2), it is sufficient to show

(6) $$x \vee z \leqq \inf\{z | x \leqq z \wedge z \in C\} \vee \inf\{z | y \leqq z \wedge z \in C\}$$

But (6) follows from

(7) $$x \leqq \inf\{z | x \leqq z \wedge z \in C\}$$

and

(8) $$y \leqq \inf\{z | y \leqq z \wedge z \in C\} .$$

■

Theorem 14
If φ *is a closure operator of* $\mathfrak{L}$*, then*
φ *is a topological mapping on* $\mathfrak{L}$ *if and only if* SET(φ) *is a topological set of* $\mathfrak{L}$*.*

Proof
ad 1 (⇓) Trivial by lemma 12

ad 2 (⇑) By theorem 5 we have

(1) $$\varphi = \mathrm{FCT}(\mathrm{SET}(\varphi))\ .$$

Hence by lemma 13 it is sufficient to show that

(2) $$\mathrm{SET}(\varphi) \text{ is a closure system of } \mathfrak{L}$$

and

(3) $$\mathrm{SET}(\varphi) \text{ is a topological set of } \mathfrak{L}\ .$$

But, (2) holds because of lemma 2. Furthermore, (3) is an assumption in case considered.

■

Theorem 15
If C *is a closure system, then*
C *is a topological set of* $\mathfrak{L}$ *if and only if* FCT(C) *is a topological mapping of* $\mathfrak{L}$*.*

Proof
ad 1 (⇓) Trivially by lemma 13

ad 2 (⇑) Because C is a closure system, by theorem 10 we get

(1) $$C = \mathrm{SET}(\mathrm{FCT}(C)),$$

hence by lemma 12, it is sufficient to show that

(2) $$\mathrm{FCT}(C) \text{ is a topological mapping of } \mathfrak{L}\ .$$

But, (2) holds by assumption in case considered.

■

3 On Compactness of Classical Closure Operators. Algebraic Closure Operators. The Theorem of G. BIRKHOFF, P. HALL, and J. SCHMIDT.

We continue with the formulation of the fundamentel theorem due to G. BIRKHOFF, P. HALL, and J. SCHMIDT which is important in many branches of algebra and which gives an algebraic characterization of the (classical) monotonic reasoning (see [9], also [6]). In particular, by this theorem the range of applicability of ZORNs lemma is well defined (see [9]).

In order to formulate this theorem, we fix an arbitrary non-empty set U and consider the complete lattice

$$\mathbb{L} = [\mathbb{P}(U), \cap, \cup, \varnothing, U]$$

where $\mathbb{P}(U)$ is the power set of U, $\cap$ and $\cup$ are the set-theoretical intersection and union of subsets of U, respectively. Furthermore, $\varnothing$ denotes the empty set, and the set U plays the role of the unit element of $\mathbb{L}$.

Because $\mathbb{L}$ is a complete lattice all the considerations, concepts, and results of chapter 2 can be applied to $\mathbb{L}$.

For a mapping $\Phi : \mathbb{P}(U) \to \mathbb{P}(U)$ and a system $\mathfrak{C} \subseteqq \mathbb{P}(U)$ of subsets of U we define the following well-known fundamental concepts:

Definition 5

1. Φ *is said to be* compact *on* $\mathbb{L}$

$$=_{\text{def}} \forall X \forall y \big(X \subseteqq U \wedge y \in \Phi(X) \to \exists X_{fin}(X_{fin} \subseteqq X \wedge X_{fin} \textit{ is finite } \wedge y \in \Phi(X_{fin}))\big)$$

2. Φ *is said to be* strongly compact *on* $\mathbb{L}$

$$=_{\text{def}} \forall X \forall y \begin{pmatrix} X \subseteqq U \wedge y \in X \wedge y \in \Phi(X) \to \\ \exists X_{sing}(X_{sing} \subseteqq X \wedge \operatorname{card} X_{sing} \leqq 1 \wedge y \in \Phi(X_{sing})) \end{pmatrix}$$

3. Φ *is said to be* inductive *on* $\mathbb{L}$

$$=_{\text{def}} \forall \mathfrak{K} \Big(\mathfrak{K} \subseteqq \mathbb{P}(U) \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \textit{ is a chain } \to \Phi(\bigcup \mathfrak{K}) \subseteqq \bigcup \{\Phi(K) | K \in \mathfrak{K}\}\Big)$$

4. $\mathfrak{C}$ *is said to be inductive on* $\mathbb{L}$

$$=_{\text{def}} \forall \mathfrak{K} \Big(\mathfrak{K} \subseteqq \mathfrak{C} \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \textit{ is a chain} \to \bigcup \mathfrak{K} \in \mathfrak{C}\Big)$$

In order to describe the generation of closure operators by universal algebras and deductive systems, respectively, we introduce the following notions. Let n be an integer with $n \geqq 0$ and assume that $X \subseteqq U$.

Definition 6

1. ω *is said to be a (total)* n*-ary* operation *on* U *if and only if* ω *is a mapping*

$$\omega : U^n \to U \qquad (n \geqq 0) .$$

If n *is not specified, we will speak of a* finitary *operation on* U.

2. $A = [U, \Omega]$ *is said to be a (total)* algebra *on* U *if and only if* Ω *is a set of finitary operations on* U.

3. *For given* $X \subseteqq U$ *we put*

$$\overline{\Omega}(X) =_{\text{def}} \left\{ y \,\middle|\, \begin{array}{r} \textit{there exists an integer } n \geqq 0, \textit{ an } n\textit{-ary operation } \omega \in \Omega \\ \textit{and } x_1, \dots, x_n \in X \textit{ such that } y = \omega(x_1, \dots, x_n) \end{array} \right\}$$

4. $X \subseteqq U$ *is said to be* Ω-closed *if and only if* $\overline{\Omega}(X) \subseteqq X$.

Definition 7

1. $\Phi_{\mathfrak{C}}(X) =_{\text{def}} \bigcap \{C | X \subseteqq C \wedge C \in \mathfrak{C}\}$
2. $\Phi_{\Omega}(X) =_{\text{def}} \bigcap \{C | X \subseteqq C \subseteqq U \wedge \overline{\Omega}(C) \subseteqq C\}$
3. $\mathfrak{C}_{\Phi} =_{\text{def}} \{C | C \subseteqq U \wedge \Phi(C) \subseteqq C\}$

4. $\mathfrak{C}_\Omega =_{\text{def}} \left\{C \middle| C \subseteqq U \wedge \overline{\Omega}(C) \subseteqq C\right\}$

For many applications, in particular in logic, it is convenient to generalize the notion of *determininistic* finitary operation introduced by definition 6 to the concept of finitary *non-deterministic, partial* operation (see [9]). With respect to applications in logic, we prefer the term *deduction rule* in this case.

Definition 8

1. *d is said to be an n-ary* deduction rule *on U if and only if*

$$d \subseteqq U^n \times U \qquad (n \geqq 0)\ .$$

   If n is not specified, then d is called a finitary *deduction rule on U.*

2. *$\vartheta = [U,D]$ is said to be a* deductive system *on U if and only if D is a set of finitary deduction rules on U, i. e. if $D \subseteqq U^* \times U$.*

3. *For given $X \subseteqq U$ we put*

$$\overline{D}(X) =_{\text{def}} \left\{ y \middle| \begin{array}{c} \text{there exists a natural number } n \geqq 0, n\text{-ary deduction rule} \\ d \in D \text{ and } x_1, \ldots, x_n \in X \text{ such that } [x_1, \ldots, x_n; y] \in d \end{array} \right\}.$$

4. *$X \subseteqq U$ is said to be D-*closed *if and only if $\overline{D}(X) \subseteqq X$.*

5. $\Phi_D(X) =_{\text{def}} \bigcap \left\{C \middle| X \subseteqq C \subseteqq U \wedge \overline{D}(C) \subseteqq C\right\}$

6. $\mathfrak{C}_D(X) =_{\text{def}} \left\{C \middle| C \subseteqq U \wedge \overline{D}(C) \subseteqq C\right\}$

Remark

The definitions of $\Phi_{\mathfrak{C}}$, Φ_Ω, and Φ_D correspond to the definition of FCT(C) (see definition 3). Analogously, the definitions of $\mathfrak{C}_\Phi$, $\mathfrak{C}_\Omega$, and $\mathfrak{C}_D$ correspond to the definition of SET(C)) (see definition 3).

Now, we are able to formulate the theorem due to G. BIRKHOFF, P. HALL, and J. SCHMIDT:

Theorem 16

If Φ is a closure operator on $\mathbb{L}$, then the following propositions 1, 2, 3, 4, and 5 are pairwise equivalent:

1. *Φ is compact on $\mathbb{L}$.*
2. *Φ is inductive on $\mathbb{L}$.*
3. *$\mathfrak{C}_\Phi$ is inductive on $\mathbb{L}$.*
4. *there exists a set Ω of finitary operations on U such that $\Phi = \Phi_\Omega$.*
5. *there exists a set D of finitary deduction rules on U such that $\Phi = \Phi_D$.*

Remarks

1. Concerning history, we want to mention that the role of compactness in studying logical consequence operators was discovered and investigated by A. TARSKI in [11–13] and emphasized by K. SCHRÖTER in [10]. A. MALCEV introduced the model theoretic version of compactness and established its applicability in several branches of algebra (see [8]).

2. The equivalence of 3 and 4 was first proved by J. SCHMIDT in [9], but according to P. COHN in [6], page 81, is an unpublished result of P. HALL, and probably G. BIRKHOFF knew this result (at least part of it) even eearlier (see [1–4]).

3. The equivalence of 1 and 3 was also discovered by J. SCHMIDT and first proved in [9]. He also pointed out the importance of inductive systems of sets for applying ZORN's lemma, where it should be mentioned that the notion of inductiveness can already be found in [5].

4. A proof for the equivalence of 3 and 5 can also be found in [9], with the only difference that the term "finitary deduction rule" is not used there.

5. The equivalence of 2 and 3 is added in this paper. A proof can be carried out easily by using the methods developed in [9].

6. Using the concept of *Clone* (see [6], for instance) we can prove the following modification of theorem 16: In proposition 4 and 5 Ω and D can be taken to be clones of operations and deduction rules, respectively, where a clone of deduction rules is defined in the same way as a clone of operations. If we intend to construct a clone of operations (or deduction rules) by a given compact closure operator Φ, we shall see that the existance of the projection operations follows from the reflexivity of Φ, whereas the monotonicity and the closedness of Φ together imply that the system of operations (deduction rules) is closed with respect to compositions.

Starting with a closure system, we can prove the following

Theorem 17
If $\mathfrak{C}$ is a closure system of $\mathbb{L}$, then the following propositions 1, 2, 3, 4, and 5 are pairwise equivalent:

1. *$\mathfrak{C}$ is inductive on $\mathbb{L}$*
2. *$\Phi_{\mathfrak{C}}$ is inductive on $\mathbb{L}$*
3. *$\Phi_{\mathfrak{C}}$ is compact on $\mathbb{L}$*
4. *there exists a set Ω of finitary operations on U such that $\mathfrak{C} = \mathfrak{C}_{\Omega}$*
5. *there exists a set D of finitary deduction rules on U such that $\mathfrak{C} = \mathfrak{C}_D$.*

Combining the algebraic with the topological concept we obtain the following (possibly not very well-known) theorems:

Theorem 18
If Φ is a closure operator on $\mathbb{L}$, then the following propositions 1, 2, 3, 4, 5, and 6 are pairwise equivalent:

1. *Φ is compact and topological on $\mathbb{L}$;*
2. *Φ is strongly compact on $\mathbb{L}$;*
3. *Φ is inductive and topological on $\mathbb{L}$;*
4. *$\mathfrak{C}_{\Phi}$ is inductive and topological on $\mathbb{L}$;*
5. *there exists a set Ω of 1-ary operations on U such that $\Phi = \Phi_{\Omega}$;*
6. *there exists a set D of 1-ary deduction rules of U such that $\Phi = \Phi_D$.*

Theorem 19
If $\mathfrak{C}$ is a closure system of $\mathbb{L}$, then the following propositions 1, 2, 3, 4, and 5 are pairwise equivalent:

1. *$\mathfrak{C}$ is inductive and topological on $\mathbb{L}$*
2. *$\Phi_{\mathfrak{C}}$ is inductive and topological on $\mathbb{L}$*

3. $\Phi_{\mathfrak{C}}$ *is strongly compact on* $\mathbb{L}$

4. *there exists a set* Ω *of 1-ary operations on* U *such that* $\mathfrak{C} = \mathfrak{C}_\Omega$

5. *there exists a set* D *of 1-ary deduction rules on* U *such that* $\mathfrak{C} = \mathfrak{C}_D$.

4 On Closure Operators generated in Fuzzy Algebras and in Fuzzy Deductive Systems

Now, we intend to "fuzzify" the concepts and results of chapter 3.

We remember that

$$L = [F\mathbb{P}(U), \sqcap, \sqcup, \emptyset, \mathbb{U}]$$

is a complete lattice where $F\mathbb{P}(U)$ is the set of all fuzzy sets on U, furthermore, $\sqcap$ and $\sqcup$ denote the intersection and union of fuzzy sets based on the minimum and maximum function, respectively, and finally, $\emptyset$ and $\mathbb{U}$ are the empty and the universal fuzzy set on U, respectively.

Thus, all considerations of chapter 2 can be applied to the complete lattice L.

For formulating the following definition we assume that $\Psi : F\mathbb{P}(U) \to F\mathbb{P}(U)$, $\vartheta \subseteqq F\mathbb{P}(U)$, and I is a prime minimum-ideal from $\langle 0, 1\rangle$.

Definition 9

1. Ψ *is said to be* compact *on* L

$$=_{\text{def}} \forall F \forall y \left(\begin{array}{l} F \in F\mathbb{P}(U) \wedge y \in U \\ \to \exists F_{fin} \left(\begin{array}{r} F_{fin} \in F\mathbb{P}(U) \wedge F_{fin} \sqsubseteq F \wedge F_{fin} \textit{ is finite} \\ \wedge \Psi(F)(y) \leqq \Psi(F_{fin})(y) \end{array} \right) \end{array} \right)$$

2. Ψ *is said to be* strongly compact *on* L

$$=_{\text{def}} \forall F \forall y \left(\begin{array}{l} F \in F\mathbb{P}(U) \wedge y \in U \\ \to \exists F_{sing} \left(\begin{array}{r} F_{sing} \in F\mathbb{P}(U) \wedge F_{sing} \sqsubseteq F \wedge F_{sing} \textit{ is a singleton} \\ \wedge \Psi(F)(y) \leqq \Psi(F_{sing}(y)) \end{array} \right) \end{array} \right)$$

3. Ψ *is said to be* inductive *on* L

$$=_{\text{def}} \forall \mathfrak{K} \left(\mathfrak{K} \subseteqq F\mathbb{P}(U) \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \textit{ is a chain} \Psi(\text{SUP}\,\mathfrak{K}) \sqsubseteq \text{SUP}\left\{ \Psi(K) \middle| K \in \mathfrak{K} \right\} \right)$$

4. ϑ *is said to be* inductive *on* L

$$=_{\text{def}} \forall \mathfrak{K} \left(\mathfrak{K} \subseteqq \mathfrak{C} \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \textit{ is a } \sqsubseteq \textit{-chain} \to \text{SUP}\,\mathfrak{K} \in \mathfrak{C} \right)$$

Now, we are going to define the concept of an n-ary fuzzy operation on U and of an n-ary fuzzy deduction rule on U.

Definition 10

1. π *is said to be a* (total) n-ary I-fuzzy operation *on* U

$=_{\text{def}}$ *a)* $\pi : U^n \times U \to \langle 0, 1\rangle$ *and*

b) $\forall x_1 \cdots \forall x_n (x_1, \ldots, x_n \in U \to \exists y (y \in U \wedge \pi(x_1, \ldots, x_n, y) \in I)$

c) $\forall x_1 \cdots x_n \forall y \forall y' \left(\begin{array}{l} x_1, \ldots, x_n, y, y' \in U \\ \wedge \pi(x_1, \ldots, x_n, y) \in I \wedge \pi(x_1, \ldots, x_n, y') \in I \rightarrow y = y' \end{array} \right)$

If n is not specified we will speak of a (total) finitary I-fuzzy operation *on* U *with respect to* I.

2. $F = [U, \Pi]$ *is said to be a* (total) I-fuzzy operation *on* U
$=_{\text{def}}$ Π *is a set of (total) finitary* I*-fuzzy operations on* U.

3. *For a given fuzzy set* $F \in F\mathbb{P}(U)$ *we put*

$$\overline{\Pi}(F)(y) =_{\text{def}} \text{Sup}\left\{ \begin{array}{l} \min(F(x_1), \ldots, F(x_n), \pi(x_1, \ldots, x_n; y)) | \\ n \in N \wedge x_1, \ldots, x_n, y \in U \wedge \pi \in \Pi \wedge \pi \text{ is } n\text{-ary} \end{array} \right\}$$

4. F *is said to be* Π-closed
$=_{\text{def}} \overline{\Pi}(F) \sqsubseteq F$

5. $\Psi_\Pi(F) =_{\text{def}} \text{INF}\left\{ G \middle| G \in F\mathbb{P}(U) \wedge F \sqsubseteq G \wedge \overline{\Pi}(G) \sqsubseteq G \right\}$

Now, we are going to fuzzify the concept of deduction rule.

Definition 11

1. δ *is said to be an* n-ary fuzzy deduction rule *on* U

$$=_{\text{def}} \delta : U^n \times U \rightarrow \langle 0, 1 \rangle \; .$$

2. *If n is not specified, we will speak of a finitary fuzzy deduction rule on* U. *If* $x_1, \ldots, x_n, y \in U$, *then we interpret the real number*

$$\delta(x_1, \ldots, x_n, y)$$

as the logical value that the fuzzy deduction rule δ *has the output* y *for the inputs* $x_1, \ldots, x_n$.

3. $\vartheta = [U, \Delta]$ *is said to be a* fuzzy deductive system *on* U
$=_{\text{def}}$ Δ *is a set of finitary fuzzy deduction rules on* U.

4. *For a given fuzzy set* $F \in F\mathbb{P}(U)$ *we put*

$$\overline{\Delta}(F)(y) =_{\text{def}} \text{Sup}\left\{ \begin{array}{l} \min(F(x_1), \ldots, F(x_n), \delta(x_1, \ldots, x_n; y)) | \\ n \in \mathbb{N} \wedge x_1, \ldots, x_n; \in U \wedge \delta \in \Delta \wedge \delta \text{ is } n\text{-ary} \end{array} \right\}$$

5. F *is said to be* Δ-closed
$=_{\text{def}} \overline{\Delta}(F) \sqsubseteq F$

6. $\Psi_\Delta(F) =_{\text{def}} \text{INF}\left\{ G \middle| G \in F\mathbb{P}(U) \wedge F \sqsubseteq G \wedge \overline{\Delta}(G) \sqsubseteq G \right\}$

Because every n-ary I-fuzzy operation can be considered as a special case of an n-ary fuzzy deduction rule, we formulate the following definitions, lemmata, and theorems only for finitary fuzzy deduction rules.

Lemma 20

The mapping

$$\overline{\Delta} : F\mathbb{P}(U) \rightarrow F\mathbb{P}(U)$$

is monotone on L.

Proof

For $F, G \in F\mathbb{P}(U)$ assume

(1) $$F \sqsubseteq G\,.$$

We have to prove

(2) $$\overline{\Delta}(F) \sqsubseteq \overline{\Delta}(G), \text{ i. e.}$$

(3) $$\forall y\big(y \in U \rightarrow \overline{\Delta}(F)(y) \leqq \overline{\Delta}(G)(y)\big)\,.$$

By definition of $\overline{\Delta}$ we have

(4) $$\overline{\Delta}(F)(y) = \operatorname{Sup}\left\{ \begin{matrix} \min(F(x_1), \ldots, F(x_n), \delta(x_1, \ldots, x_n, y)) \big| \\ n \in \mathbb{N} \wedge \delta \in \Delta \wedge \delta \text{ is } n\text{-ary} \wedge x_1, \ldots, x_n \in U \end{matrix} \right\}$$

hence by monotonicity of min and Sup from (1) we get (3). ■

Theorem 21

1. *The set* $\left\{F \middle| F \in F\mathbb{P}(U) \wedge \overline{\Delta}(F) \sqsubseteq F\right\}$ *of all* Δ*-closed fuzzy sets* F *on* U *is a closure system of the lattice* L.
2. Ψ_Δ *is a closure operator of the lattice* L.

Proof

ad 1 By theorem 1, lemma 20, and lemma 2.

ad 2 By theorem 1, theorem 21, and lemma 8.

■

Definition 12

1. $\Delta^{[0]}(F) =_{\text{def}} F$
2. $\Delta^{[k+1]}(F) =_{\text{def}} \Delta^{[k]}(F) \sqcup \overline{\Delta}(\Delta^{[k]}(F))$
3. $\Delta^{[*]}(F) =_{\text{def}} \operatorname{SUP}\left\{\Delta^{[k]}(F) \middle| k \in \mathbb{N}\right\}$

The following concepts of modality of a set Δ of fuzzy deduction rules is sufficient that some of the following theorems can be proved.

Definition 13

1. Δ *is said to be* submodal *with respect to* L

$$=_{\text{def}} \forall y \forall F \left(\begin{matrix} y \in U \wedge F \in F\mathbb{P}(U) \rightarrow \exists n \exists \delta \exists x_1 \cdots x_n \\ \left(\begin{matrix} n \in \mathbb{N} \wedge \delta \in \Delta \wedge \delta \textit{ is } n\textit{-ary} \wedge x_1, \ldots, x_n \in U \wedge \\ \overline{\Delta}(F)(y) = \min(F(x_1), \ldots, F(x_n), \delta(x_1, \ldots, x_n, y)) \end{matrix} \right) \end{matrix} \right)$$

2. Δ *is said to be* strongly submodal *with respect to* L
   $=_{\text{def}}$ *2.1.* Δ *is submodal with respect to* U *and*
   2.2. $\forall y \forall F\left(y \in U \wedge F \in F\mathbb{P}(U) \rightarrow \exists k\left(k \in \mathbb{N} \wedge \Delta^{[*]}(F)(y) = \Delta^{[k]}(F)(y)\right)\right)$

Lemma 22

If Δ *is strongly submodal with respect to* L*, then*

$$\Psi_\Delta(F) \sqsubseteq \Delta^{[*]}(F)\,.$$

Proof

In order to prove the inclusion above, it is sufficient to show

(1) $$F \sqsubseteq \Delta^{[*]}(F)$$

and

(2) $$\overline{\Delta}(\Delta^{[*]}(F)) \sqsubseteq \Delta^{[*]}(F) \ .$$

Proposition (1) holds trivially by definition of $\Delta^{[*]}(F)$ (see definition 12). In order to prove (2) it is sufficient to show

(3) $$\forall y \Big(y \in U \rightarrow \exists k \big(k \in \mathbb{N} \wedge \overline{\Delta}(\Delta^{[*]}(F))(y) \leqq \overline{\Delta}(\Delta^{[k]}(F)(y)) \big) \Big)$$

By definition of $\overline{\Delta}$ and the assumption of submodality for Δ we obtain that there are $n \in \mathbb{N}, x_1, \ldots, x_n \in U, \delta \in \Delta$ such that δ is n-ary and

(4) $$\overline{\Delta}(\Delta^{[*]}(F))(y) = \min\big(\Delta^{[*]}(F)(x_1), \ldots, \Delta^{[*]}(F)(x_n), \delta(x_1, \ldots, x_n, y)\big)$$

Because of the strong submodality of Δ there are $k_1, \ldots, k_n \in \mathbb{N}$ such that

(5) $$\begin{array}{rcl} \Delta^{[*]}(F)(x_1) & = & \Delta^{[k_1]}(F)(x_1) \\ \vdots & & \vdots \\ \Delta^{[*]}(F)(x_n) & = & \Delta^{[k_n]}(F)(x_n) \ , \end{array}$$

hence

(6) $$\overline{\Delta}(\Delta^{[*]}(F))(y) = \min\big(\Delta^{[k_1]}(F)(x_1), \ldots, \Delta^{[k_n]}(F)(x_n), \delta(x_1, \ldots, x_n, y)\big) \ .$$

Because we have

$$\Delta^{[k]}(F) \sqsubseteq \Delta^{[l]}(F)$$

if $k \leqq l$, there exists a natural number $m \in \mathbb{N}$

(7) $$\begin{array}{rcl} \Delta^{[k_1]}(F)(x_1) & \leqq & \Delta^{[m]}(F)(x_1) \\ \vdots & & \vdots \\ \Delta^{[k_1]}(F)(x_n) & \leqq & \Delta^{[m]}(F)(x_n) \ , \end{array}$$

hence by 6

(8) $$\overline{\Delta}(\Delta^{[*]}(F))(y) \leqq \min\big(\Delta^{[m]}(F)(x_1), \ldots, \Delta^{[m]}(F)(x_n), \delta(x_1, \ldots, x_n, y)\big),$$

hence

(9) $$\overline{\Delta}(\Delta^{[*]}(F))(y) \leqq \overline{\Delta}(\Delta^{[m]}(F))(y),$$

i. e. (3) holds. ■

Lemma 23

$$\Delta^{[*]}(F) \sqsubseteq \Psi_{\Delta}(F)$$

Proof
By definition of $\Delta^{[*]}$ it is sufficient to prove

(1) $$\Delta^{[k]}(F) \sqsubseteq \Psi_\Delta(F) \text{ for every } k \in \mathbb{N} .$$

By definition of $\Psi_\Delta(F)$ it is sufficient to prove for $G \in F\mathbb{P}(U)$ and $k \in \mathbb{N}$

(2) $$F \leqq G \wedge \overline{\Delta}(G) \sqsubseteq G \rightarrow \Delta^{[n]}(F) \sqsubseteq G .$$

But (2) will be proved by induction on n. The basic step $n = 0$ is trivial. The induction step n is also trivial if we apply lemma 20, i. e.

(3) $$F \sqsubseteq G \rightarrow \overline{\Delta}(F) \sqsubseteq \overline{\Delta}(G)$$

where $F, G \in F\mathbb{P}(U)$. ∎

Theorem 24
If Δ is strongly submodal with respect to L, then

$$\forall F \left(F \in F\mathbb{P}(U) \rightarrow \Psi_\Delta(F) = \Delta^{[*]}(F) \right) .$$

Proof
By lemma 22 and 23. ∎

Now, we are going to prove the compactness of Ψ_Δ. Therefore we start with the following lemma expressing the compactness of the mapping $\Delta^{[k]}$ where $k \in \mathbb{N}$ and k is fixed.

Lemma 25
If Δ is submodal with respect to L, then

$$\forall k \forall y \forall F \begin{pmatrix} k \in \mathbb{N} \wedge y \in U \wedge F \in F\mathbb{P}(U) \rightarrow \\ \exists F_{fin} \left(F_{fin} \in F\mathbb{P}(U) \wedge F_{fin} \sqsubseteq F \wedge F_{fin} \textit{ is finite} \wedge \Delta^{[k]}(F)(y) \leqq \Delta^{[k]}(F_{fin})(y) \right) \end{pmatrix}$$

Proof
By induction on k.

Basic step. $k = 0$

Then $\Delta^{[0]}(F) = F$, hence we put for $x \in U$,

(1) $$F_{fin}(x) =_{\text{def}} \begin{cases} F(x) & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y . \end{cases}$$

Consequently,

(2) $$\begin{array}{l} F_{fin} \in F\mathbb{P}(U), \\ F_{fin} \sqsubseteq F, \\ F_{fin} \text{ is finite,} \end{array}$$

finally

(3) $$F(y) \leqq F_{fin}(y) .$$

Induction step.

Assume that lemma 25 holds for fixed $k \in \mathbb{N}$, every $F \in F\mathbb{P}(U)$, and every $y \in U$.

By definition of $\Delta^{[k+1]}$ we have

$$\Delta^{[k+1]}(F) = \Delta^{[k]}(F) \sqcup \overline{\Delta}(\Delta^{[k]}(F)), \tag{4}$$

i. e. for every $y \in U$,

$$\Delta^{[k+1]}(F)(y) = \max\left(\Delta^{[k]}(F)(y), \overline{\Delta}\left(\Delta^{[k]}(F)\right)(y)\right) .$$

By induction assumption we get a fuzzy set F_{fin} with

$$\begin{aligned} &F_{fin} \in F\mathbb{P}(U), \\ &F_{fin} \sqsubseteq F, \\ &F_{fin} \text{ is finite,} \end{aligned} \tag{5}$$

$$\Delta^{[k]}(F)(y) \leqq \Delta^{[k]}(F_{fin})(y) . \tag{6}$$

Furthermore, because Δ is submodal with respect to L, we get an $n \in \mathbb{N}$, elements $x_1, \dots, x_n \in U$, and an n-ary fuzzy deduction rule $\delta \in \Delta$ such that

$$\overline{\Delta}\left(\Delta^{[k]}(F)\right)(y) = \min\left(\Delta^{[k]}(F)(x_1), \dots, \Delta^{[k]}(F)(x_n), \delta(x_1, \dots, x_n, y)\right) . \tag{7}$$

Now, by induction assumption for every $i \in \{1, \dots, n\}$ we have an F^i_{fin} such that $F^i_{fin} \in F\mathbb{P}(U)$, $F^i_{fin} \sqsubseteq F$, F^i_{fin} is finite and

$$\Delta^{[k]}(F)(x_i) \leqq \Delta^{[k]}\left(F^i_{fin}\right)(x_i).$$

We define $F^*_{fin} =_{\text{def}} \text{SUP}\left\{F^i_{fin} \middle| i \in \{1, \dots, n\}\right\}$.

Then we have $F^*_{fin} \in F\mathbb{P}(U)$,

$$\begin{aligned} &F^i_{fin} \sqsubseteq F^*_{fin} \sqsubseteq F \\ &F^*_{fin} \text{ is finite} \end{aligned} \tag{8}$$

and

$$\Delta^{[k]}(F)(x_i) \leqq \Delta^{[k]}\left(F^i_{fin}\right)(x_i) \leqq \Delta^{[k]}\left(F^*_{fin}\right)(x_i) \tag{9}$$

for every $i \in \{1, \dots, n\}$.

From (7) and (9) we obtain

$$\overline{\Delta}\left(\Delta^{[k]}(F)\right)(y) \leqq \min\left(\Delta^{[k]}\left(F^*_{fin}\right)(x_1), \dots, \Delta^{[k]}\left(F^*_{fin}\right)(x_n), \delta(x_1, \dots, x_n, y)\right) . \tag{10}$$

We put

$$F^{**}_{fin} =_{\text{def}} F_{fin} \sqcup F^*_{fin} .$$

hence by (5) and (8) we get

$$\begin{aligned} &F^{**}_{fin} \in F\mathbb{P}(U) \\ &F_{fin} \sqsubseteq F^{**}_{fin} \sqsubseteq F \\ &F^*_{fin} \sqsubseteq F^{**}_{fin} \sqsubseteq F \\ &F^{**}_{fin} \text{ is finite.} \end{aligned} \tag{11}$$

By (6) we obtain

$$\Delta^{[k]}(F)(y) \leqq \Delta^{[k]}\left(F_{fin}\right)(y) \leqq \Delta^{[k]}\left(F_{fin}^{**}\right)(y). \tag{12}$$

Furthermore by (10) and (11)

$$\begin{aligned} &\overline{\Delta}\left(\Delta^{[k]}(F)\right)(y) \\ &\leqq \min\left(\Delta^{[k]}\left(F_{fin}^{*}\right)(x_1), \dots, \Delta^{[k]}\left(F_{fin}^{*}\right)(x_n), \delta(x_1, \dots, x_n, y)\right) \\ &\leqq \min\left(\Delta^{[k]}\left(F_{fin}^{**}\right)(x_1), \dots, \Delta^{[k]}\left(F_{fin}^{**}\right)(x_n), \delta(x_1, \dots, x_n, y)\right) \\ &\leqq \overline{\Delta}\left(\Delta^{[k]}\left(F_{fin}^{**}\right)\right)(y) \end{aligned} \tag{13}$$

Finally, by (4), (12), and (13) we get

$$\Delta^{[k+1]}(F)(y) \leqq \Delta^{[k+1]}\left(F_{fin}^{**}\right)(y) \ . \tag{14}$$

Thus, the induction step is finished. ■

Theorem 26
If Δ is strongly submodal, then Ψ_Δ is compact, i. e.

$$\forall F \forall y \left(F \in F\mathbb{P}(U) \wedge y \in U \rightarrow \exists F_{fin} \left(\begin{array}{r} F_{fin} \in F\mathbb{P}(U) \wedge F_{fin} \sqsubseteq F \wedge F_{fin} \text{ is finite} \\ \wedge \Psi_\Delta(F)(y) \leqq \Psi_\Delta\left(F_{fin}\right)(y) \end{array} \right) \right)$$

Proof
Because Δ is strongly submodal by theorem 24 we get

$$\Psi_\Delta(F)(y) = \mathrm{SUP}\left\{\Delta^{[k]}(F)(y) \middle| k \in \mathbb{N}\right\}, \tag{1}$$

because Δ is strongly submodal, by definition 13 we have

$$\exists k \left(k \in \mathbb{N} \wedge \Psi_\Delta(F)(y) = \Delta^{[k]}(F)(y)\right), \tag{2}$$

thus, by lemma 25 we finish the proof of theorem 26. ■

Consider a mapping

$$\Psi : F\mathbb{P}(U) \rightarrow F\mathbb{P}(U) \ .$$

We are going to investigate the problem under which conditions for Ψ there exists a set Δ of fuzzy deduction rules on U such that

$$\Psi = \Psi_\Delta \ .$$

We start with the following lemma

Lemma 27

$$\forall y \forall F \left(y \in U \wedge F \in F\mathbb{P}(U) \rightarrow \overline{\Delta}(F)(y) \leqq \Psi_\Delta(F)(y)\right)$$

Proof

By definition of Ψ_Δ we have

(1) $$\Psi_\Delta(F)(y) =_{\text{def}} \operatorname{Inf}\left\{G(y) \middle| F \sqsubseteq G \wedge \overline{\Delta}(G) \sqsubseteq G \wedge G \in F\mathbb{P}(U)\right\}$$

hence by definition of Inf it is sufficient to prove

(2) $$\forall G\left(G \in F\mathbb{P}(U) \wedge F \sqsubseteq G \wedge \overline{\Delta}(G) \sqsubseteq G \rightarrow \overline{\Delta}(F)(y) \leqq G(y)\right)$$

From $F \sqsubseteq G$ by monotonicity of $\overline{\Delta}$ (lemma 20) we get $\overline{\Delta}(F) \sqsubseteq \overline{\Delta}(G)$, hence by $\overline{\Delta}(G) \sqsubseteq G$ we obtain $\overline{\Delta}(F) \sqsubseteq G$, i. e. $\overline{\Delta}(F)(y) \leqq G(y)$ for every $y \in U$.

Now, by using Ψ we construct a set Δ_Ψ of fuzzy deduction rules as follows. Therefore we fix an $n \in \mathbb{N}$. and an $\mathfrak{x} \in U^n$. Let $\text{SET}_\mathfrak{x}$ be the set of elements of U belonging to $\mathfrak{x}$. For and arbitrary $y \in U$ and $F \in F\mathbb{P}(U)$ we define

$$F_\mathfrak{x}(y) =_{\text{def}} \begin{cases} F(y) & \text{if } y \in \text{SET}_\mathfrak{x} \\ 0 & \text{if } y \notin \text{SET}_\mathfrak{x} \end{cases} .$$

Hence $F_\mathfrak{x}$ is finite and $F_\mathfrak{x} \sqsubseteq F$. Furthermore $\operatorname{supp}(F_\mathfrak{x}) = \text{SET}_\mathfrak{x}$. ■

Definition 14

1. $\delta_\Psi^{n,F}(\mathfrak{x}, y) =_{\text{def}} \Psi(F_\mathfrak{x})(y)$ *where* $\mathfrak{x} \in U^n$ *and* $y \in U$
2. $\Delta_\Psi =_{\text{def}} \left\{\delta_\Psi^{n,F} \middle| n \in \mathbb{N} \wedge F \in F\mathbb{P}(U)\right\}$

Lemma 28

If Ψ *is compact then*

$$\forall F\left(F \in F\mathbb{P}(U) \rightarrow \Psi(F) \sqsubseteq \Psi_{\Delta_\Psi}(F)\right) .$$

Proof

We have to prove that for every $y \in U$ and $F \in F\mathbb{P}(U)$,

(1) $$\Psi(F)(y) \leqq \Psi_{\Delta_\Psi}(F)(y) .$$

By compactness of Ψ we have

(2) $$\exists F_{fin}\left(F_{fin} \in F\mathbb{P}(U) \wedge F_{fin} \text{ is finite } \wedge F_{fin} \sqsubseteq F \wedge \Psi(F)(y) \leqq \Psi\left(F_{fin}\right)(y)\right)$$

Assume $\operatorname{card}\operatorname{supp}\left(F_{fin}\right) = n$. We define

$$\delta_\Psi^{n,F_{fin}}(\mathfrak{x}, y) =_{\text{def}} \Psi\left(\left(F_{fin}\right)_\mathfrak{x}\right)(y) .$$

Then we get for $\mathfrak{x}_0 = \{x_1, \ldots, x_n\} = \operatorname{supp}\left(F_{fin}\right)$

(3) $$\begin{aligned} &\Psi\left(F_{fin}\right)(y) \\ &= \Psi\left(\left(F_{fin}\right)_{\mathfrak{x}_0}\right)(y) \\ &= \delta_\Psi^{n,F_{fin}}(\mathfrak{x}_0)(y) \\ &\leqq \operatorname{Sup}\left\{\delta_\Psi^{n,F_{fin}}(\mathfrak{x})(y) \middle| \mathfrak{x} \in U^n\right\} \\ &= \overline{\delta_\Psi^{n,F_{fin}}}\left(F_{fin}\right)(y) \\ &\leqq \overline{\Delta_\Psi}\left(F_{fin}\right)(y) \\ &\leqq \overline{\Delta_\Psi}(F)(y) \quad \text{by lemma 20} \\ &\leqq \Psi_{\Delta_{Psi}}(F)(y) \quad \text{by lemma 27} \end{aligned}$$

Thus, lemma 28 holds. ■

For proving the following lemma 29 we need the concept of strong submodality of an operator Ψ.

Definition 15

Ψ *is said to be* strongly submodal *on* U

$=_{\text{def}}$

For every $y \in U, F \in F\mathbb{P}(U)$ *there exist* $n^0 \in \mathbb{N}, x_1^0, \dots, x_{n^0}^0 \in U$ *such that*

$$\operatorname{Sup}\left\{ \min\left(\Psi(F)(x_1), \dots, \Psi(F)(x_n), \Psi\left(H_{[x_1, \dots, x_n]}\right)(y)\right) \middle| \, n \in \mathbb{N} \wedge H \in F\mathbb{P}(U) \wedge x_1, \dots, x_n \in U \right\}$$
$$= \min\left(\Psi(F)(x_1^0), \dots, \Psi(F)(x_{n^0}^0), \Psi\left(\Psi(F)_{[x_1^0, \dots, x_{n^0}^0]}\right)(y)\right)$$

Lemma 29

If Ψ *is a closure operator on* L *and* Ψ *is strongly submodal, then*

$$\forall F\left(F \in F\mathbb{P}(U) \rightarrow \Psi_{\Delta_\Psi} \sqsubseteq \Psi(F)\right) .$$

Proof

For every $y \in U$ and every $F \in F\mathbb{P}(U)$ we have to prove

(1) $$\Psi_{\Delta_\Psi}(F)(y) \leqq \Psi(F)(y) .$$

By definition of Ψ_{Δ_Ψ} we have

(2) $$\Psi_{\Delta_\Psi}(F)(y) = \operatorname{Inf}\left\{G(y) \middle| F \sqsubseteq G \wedge \overline{\Delta_\Psi}(G) \sqsubseteq G\right\},$$

hence

(3) $$\forall G\left(F \sqsubseteq G \wedge \overline{\Delta_\Psi}(G) \sqsubseteq G \rightarrow \Psi_{\Delta_\Psi}(F)(y) \leqq G(y)\right) .$$

Put

(4) $$G =_{\text{def}} \Psi(F)$$

Hence it is sufficient to prove

(5) $$F \sqsubseteq \Psi(F)$$

and

(6) $$\overline{\Delta_\Psi}(\Psi(F)) \sqsubseteq \Psi(F) .$$

The condition (5) holds trivially because Ψ is embedding.

Now, we are going to show (6), i. e. we have to prove

(7) $$\forall y\left(y \in U \rightarrow \overline{\Delta_\Psi}(\Psi(F))(y) \leqq \Psi(F)(y)\right)$$

By definition of $\overline{\Delta_\Psi}$ we have

(8) $$\overline{\Delta_\Psi}(\Psi(F))(y) = \operatorname{Sup}\left\{ \min\left(\Psi(F)(x_1), \dots, \Psi(F)(x_n), \delta_\Psi^{n,H}(x_1, \dots, x_n, y)\right) \middle| \, n \in \mathbb{N} \wedge H \in F\mathbb{P}(U) \wedge x_1, \dots, x_n \in U \right\}$$

Now, by definition of $\delta_{\Psi}^{n,H}$ we have

$$(9) \qquad \delta_{\Psi}^{n,H}(\mathfrak{x},y) =_{\text{def}} \Psi\left(H_{\mathfrak{x}}\right)(y) \quad \text{where } \mathfrak{x} \in U^n \text{ and } y \in U$$

hence from (8),

$$(10) \qquad \overline{\Delta_{\Psi}}\left(\Psi(F)\right)(y) = \operatorname{Sup}\left\{ \begin{array}{c} \min\left(\Psi(F)(x_1), \ldots, \Psi(F)(x_n), \Psi\left(H_{\mathfrak{x}}\right)(y)\right) \Big| \\ n \in \mathbb{N} \wedge H \in F\mathbb{P}(U) \wedge \mathfrak{x} \in U^n \wedge \mathfrak{x} = [x_1, \ldots, x_n] \end{array} \right\}$$

Hence, because Ψ is strongly submodal, there are $n^0, x_0^1, \ldots, x_{n^0}^0 \in U$ such that

$$(11) \qquad \overline{\Delta_{\Psi}}\left(\Psi(F)\right)(y) = \min\left(\Psi(F)(x_1^0), \ldots, \Psi(F)(x_n^0), \Psi\left(\Psi(F)_{[x_1^0, \ldots, x_{n^0}^0]}\right)(y)\right).$$

Now, we have

$$(12) \qquad \Psi(F)_{[x_1^0, \ldots, x_{n^0}^0]} \sqsubseteq \Psi(F)$$

hence by monotonicity of Ψ

$$(13) \qquad \Psi\left(\Psi(F)_{[x_1^0, \ldots, x_{n^0}^0]}\right) \sqsubseteq \Psi\left(\Psi(F)\right),$$

hence by closedness of Ψ

$$(14) \qquad \Psi\left(\Psi(F)_{[x_1^0, \ldots, x_{n^0}^0]}\right) \sqsubseteq \Psi(F),$$

hence by 11 we get

$$(15) \qquad \overline{\Delta_{\Psi}}\left(\Psi(F)\right)(y) \leqq \Psi(F)(y)$$

for every $y \in U$.

Thus, (6) holds, i. e. lemma 29 is proved. ■

Theorem 30
If Ψ is a strongly submodal compact closure operator, then

$$\forall F\left(F \in F\mathbb{P}(U) \rightarrow \Psi(F) = \Psi_{\Delta_{\Psi}}(F)\right)$$

Proof
By lemma 28 and lemma 29. ■

5 Concluding Remarks

Because of restricted space in chapter 4 we could not develop a fuzzification of the whole theorems 16, 17, 18, and 19. In a forthcoming paper we shall continue the investigations started in the paper presented.

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