UNIVERSITY OF DORTMUND

REIHE COMPUTATIONAL INTELLIGENCE

COLLABORATIVE RESEARCH CENTER 531

Design and Management of Complex Technical Processes and Systems by means of Computational Intelligence Methods

On Median Quantifiers

Helmut Thiele

No. CI-27/98

Technical Report ISSN 1433-3325 April 1998

Secretary of the SFB 531 · University of Dortmund · Dept. of Computer Science/XI
44221 Dortmund · Germany

This work is a product of the Collaborative Research Center 531, “Computational Intelligence”, at the University of Dortmund and was printed with financial support of the Deutsche Forschungsgemeinschaft.

On Median Quantifiers*

Helmut Thiele

We denote the set of all real numbers r with $0 \leq r \leq 1$ by $\langle 0,1\rangle$. Let μ be a function with

$$\mu : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle .$$

The starting point of the present paper is the following well-known theorem [1, 2] characterizing so-called median functions.

Theorem 1
If 1. $\forall r \forall s (r, s \in \langle 0,1\rangle \to \min(r,s) \leq \mu(r,s) \leq \max(r,s))$

2. *μ is monotone, commutative, associative and continuous*

then there exists exactly one $c \in \langle 0,1\rangle$ such that for every $r, s \in \langle 0,1\rangle$,

1. *if* $c \leq \min(r,s)$ *then* $\mu(r,s) = \min(r,s)$
2. *if* $\max(r,s) \leq c$ *then* $\mu(r,s) = \max(r,s)$
3. *if* $\min(r,s) < c < \max(r,s)$ *then* $\mu(r,s) = c$.

The aim of the present paper is to generalize this theorem to fuzzy quantifiers.

First, we have to introduce the following notations and definitions.

Let U be a non-empty set called universe. A fuzzy set F on U is a mapping

$$F : U \to \langle 0,1\rangle ,$$

i. e. we do not distinguish between a fuzzy set F and its membership function μ_F. By $\boldsymbol{U}$, $\emptyset$, and C_r we denote the universal, the empty, and a constant fuzzy set on U, respectively, for every $x \in U$ defined by

$$\boldsymbol{U}(x) =_{def} 1$$
$$\emptyset(x) =_{def} 0$$
$$C_r(x) =_{def} r$$

where r is a fixed real number from $\langle 0,1\rangle$.

In the following we shall very often use the notation $F\langle x := r\rangle$ where $F \in \mathcal{F}(U)$, $x \in U$, and $r \in \langle 0,1\rangle$. For every $y \in U$ we define this notation as follows

$$(F\langle x := r\rangle)(y) =_{def} \begin{cases} r & \text{if } y = x \\ F(y) & \text{if } y \neq x \end{cases}$$

In the field of two-valued logic A. MOSTOWSKI has introduced the concept of a general quantifier [3]. Following this approach by a general fuzzy quantifier Q on U [4–7] we understand a mapping of the form

$$Q : \mathcal{F}(U) \to \langle 0,1\rangle .$$

For formulating the following lemma 2 and theorem 3 we shall use the special fuzzy quantifiers ALL and EX defined for arbitrary $F \in \mathcal{F}(U)$ by

$$ALL(F) =_{def} \operatorname{Inf}\{F(x) | x \in U\}$$
$$\text{and} \qquad EX(F) =_{def} \operatorname{Sup}\{F(x) | x \in U\} .$$

Furthermore, for arbitrary general fuzzy quantifiers Q on U we define:

*Long version of a paper originally published in *Linz '97 — Enriched Lattice Structures for Many-Valued and Fuzzy Logics*, Bildungszentrum St. Magdalena, Linz, Austria, Feb 25–Mar 1, 1997 (Abstracts), pages 71–74

Definition 1

1. *Q is said to be monotone on* $\mathcal{F}(U)$
   $=_{def} \forall F \forall G (F, G \in \mathcal{F}(U) \wedge F \subseteq G \rightarrow Q(F) \leq Q(G))$

2. *Q is said to be strongly associative on* U *and* $\mathcal{F}(U)$
   $$=_{def} \forall x \forall y \forall F \forall G \begin{pmatrix} x, y \in U \wedge F, G \in \mathcal{F}(U) \\ \rightarrow Q(F\langle x := Q(G)\rangle) = Q(G\langle y := Q(F\langle x := G(y)\rangle)\rangle) \end{pmatrix}$$

3. *Q is said to be weakly continuous with respect to* U
   $=_{def}$ *For every fixed* $x \in U$ *the functions* φ *and* ψ*, for every* $r \in \langle 0, 1\rangle$ *defined by*
   $$\varphi(r) =_{def} Q(\boldsymbol{U}\langle x := r\rangle)$$
   $$\psi(r) =_{def} Q(C_r\langle x := 0\rangle)$$
   are continuous on $\langle 0, 1\rangle$.

4. *Q is said to be weakly commutative with respect to* U
   $=_{def} \forall x \forall y (x, y \in U \rightarrow Q(\boldsymbol{U}\langle x := 0\rangle) = Q(\boldsymbol{U}\langle y := 0\rangle))$

Lemma 2

If 1. $\forall F (F \in \mathcal{F}(U) \rightarrow ALL(F) \leq Q(F) \leq EX(F))$

2. *Q is monotone on* $\mathcal{F}(U)$

3. *Q is strongly associative on* U *and* $\mathcal{F}(U)$

4. *Q is weakly continuous with respect to* U

then

1. $\forall x \forall F (x \in U \wedge F \in \mathcal{F}(U) \wedge Q(\boldsymbol{U}\langle x := 0\rangle) \leq F(x) \rightarrow Q(F) \leq F(x))$

2. $\forall x \forall F (x \in U \wedge F \in \mathcal{F}(U) \wedge F(x) \leq Q(\boldsymbol{U}\langle x := 0\rangle) \rightarrow F(x) \leq Q(F))$

Proof

Case 1 $\text{card}\, U = 1$.

In this case we assume

(1) $$U = \{x\}.$$

Obviously, for arbitrary $F \in \mathcal{F}(U)$ we get

(2) $$ALL(F) = EX(F) = F(x),$$

hence by assumption 1 of lemma 2

(3) $$Q(F) = F(x),$$

i. e. in case 1 lemma 2 holds trivially.

Case 2 $\text{card}\, U \geq 2$.

ad 1 For arbitrary $x \in U$ and $F \in \mathcal{F}(U)$ we have to prove

(4) $$Q(F) \leq F(x).$$

We assume

(5) $$Q(\boldsymbol{U}\langle x := 0\rangle) \leq F(x).$$

Because of $\boldsymbol{U}\langle x := 1\rangle = \boldsymbol{U}$ and $ALL(\boldsymbol{U}) = 1$ by assumption 1 of lemma 2 we have

(6) $$Q(\boldsymbol{U}\langle x := 1\rangle) = 1,$$

hence

(7) $$Q(\boldsymbol{U}\langle x := 0\rangle) \leq F(x) \leq Q(\boldsymbol{U}\langle x := 1\rangle),$$

i. e. for the function φ defined on $\langle 0,1\rangle$ for the fixed $x \in U$ by

$$\varphi(r) =_{def} Q(\boldsymbol{U}\langle x := r\rangle) \qquad (r \in \langle 0,1\rangle)$$

we have

$$\varphi(0) \leq F(x) \leq \varphi(1). \tag{8}$$

Now, by assumption 4 of lemma 2, the function φ is continuous on $\langle 0,1\rangle$, hence by the intermediate value theorem we obtain

$$\exists s\,(s \in \langle 0,1\rangle \wedge \varphi(s) = F(x))\,, \tag{9}$$

i. e.

$$\exists s\,(s \in \langle 0,1\rangle \wedge Q(\boldsymbol{U}\langle x := s\rangle) = F(x))\,. \tag{10}$$

Because of

$$F = F\langle x := F(x)\rangle \tag{11}$$

by (10) we get

$$F = F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle\,, \tag{12}$$

consequently,

$$Q(F) = Q\,(F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle)\,. \tag{13}$$

Obviously, we have

$$\forall r\,(r \in \langle 0,1\rangle \rightarrow F\langle x := r\rangle \subseteq \boldsymbol{U}\langle x := r\rangle)\,, \tag{14}$$

hence for $r =_{def} Q(\boldsymbol{U}\langle x := s\rangle)$

$$F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle \subseteq \boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle\,, \tag{15}$$

consequently by assumption 2 of lemma 2, i. e. by the monotonicity of Q on $\mathcal{F}(U)$,

$$Q\,(F\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle) \leq Q\,(\boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle)\,. \tag{16}$$

Now, we apply the strong associativity of Q on U and on $\mathcal{F}(U)$, i. e. assumption 3 of lemma 2, to

$$Q\,(\boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle)\,. \tag{17}$$

by putting

$$F =_{def} \boldsymbol{U} \text{ and } G =_{def} \boldsymbol{U}\langle x := s\rangle\,. \tag{18}$$

Hence we obtain

(19)

$$Q\,(\boldsymbol{U}\langle x := Q(\boldsymbol{U}\langle x := s\rangle)\rangle) = Q\,((\boldsymbol{U}\langle x := s\rangle)\langle y := Q\,(\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle)\rangle)\,.$$

Because in case 2 we assumed that $\operatorname{card} U \geq 2$, we can suppose that $y \neq x$, hence we get

$$(\boldsymbol{U}\langle x := s\rangle)(y) = 1, \tag{20}$$

consequently

$$\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle = \boldsymbol{U}\langle x := 1\rangle = \boldsymbol{U}, \tag{21}$$

hence

$$Q\,(\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle) = Q(\boldsymbol{U}) = 1. \tag{22}$$

Furthermore, because of $y \neq x$ we get

$$(\boldsymbol{U}\langle x := s\rangle)\langle y := 1\rangle = \boldsymbol{U}\langle x := s\rangle\,, \tag{23}$$

hence by (10), (22), and (23) we obtain

$$\begin{aligned} Q\,((\boldsymbol{U}\langle x := s\rangle)\langle y := Q\,(\boldsymbol{U}\langle x := (\boldsymbol{U}\langle x := s\rangle)(y)\rangle)\rangle) &= Q\,((\boldsymbol{U}\langle x := s\rangle)\langle y := 1\rangle) \\ &= Q\,(\boldsymbol{U}\langle x := s\rangle) = F(x). \end{aligned} \tag{24}$$

By combining (13), (16), (19), and (24) we obtain (4), i. e.

$$Q(F) \leq F(x).$$

ad 2 For arbitrary $x \in U$ and $F \in \mathcal{F}(U)$ we have to prove

(25) $$F(x) \leq Q(F).$$

We assume

(26) $$F(x) \leq Q(\boldsymbol{U}\langle x := 0\rangle),$$

hence we have

(27) $$0 \leq F(x) \leq Q(\boldsymbol{U}\langle x := 0\rangle).$$

Because of

(28) $$C_0\langle x := 0\rangle = \emptyset$$

and

(29) $$\boldsymbol{U}\langle x := 0\rangle = C_1\langle x := 0\rangle$$

we obtain

(30) $$\begin{aligned} Q(C_0\langle x := 0\rangle) &= Q(\emptyset) \\ &\leq EX(\emptyset) \\ &= 0 \\ &\leq F(x) \\ &\leq Q(\boldsymbol{U}\langle x := 0\rangle) \\ &= Q(C_1\langle x := 0\rangle) \end{aligned}$$

Now, we consider the function ψ defined on $\langle 0,1\rangle$ for the fixed $x \in U$ by

$$\psi(r) =_{def} Q(C_r\langle x := 0\rangle) \qquad (r \in \langle 0,1\rangle).$$

By (30) we have

(31) $$\psi(0) \leq F(x) \leq \psi(1).$$

By assumption 4 of lemma 2, the function ψ is continuous on $\langle 0,1\rangle$, hence by the intermediate value theorem we get

(32) $$\exists s\,(s \in \langle 0,1\rangle \wedge \psi(s) = F(x)),$$

i. e.

(33) $$\exists s\,(s \in \langle 0,1\rangle \wedge Q(C_s\langle x := 0\rangle) = F(x)).$$

We put

(34) $$G =_{def} C_s\langle x := 0\rangle,$$

hence by (11) we obtain

(35) $$Q(F) = Q(F\langle x := Q(G)\rangle).$$

Now, we have

(36) $$F\langle x := Q(G)\rangle \supseteq \emptyset\langle x := Q(G)\rangle,$$

hence by assumption 2 of lemma 2, i. e. by the monotonicity of Q on U and $\mathcal{F}(U)$,

(37) $$Q(F\langle x := Q(G)\rangle) \geq Q(\emptyset\langle x := Q(G)\rangle),$$

hence by (35) and (37)

(38) $$Q(F) \geq Q(\emptyset\langle x := Q(G)\rangle).$$

Now, by the associativity of Q on U and on $\mathcal{F}(U)$ for $F =_{def} \emptyset$ and $G =_{def} C_s\langle x := 0\rangle$ we obtain for $y = x$

(39) $$Q(\emptyset\langle x := Q(G)\rangle) = Q(G\langle x := Q(\emptyset\langle x := G(x)\rangle)\rangle).$$

By (34), i. e. by definition of G, we have

(40) $$G(x) = (C_s\langle x := 0\rangle)(x) = 0.$$

Furthermore, we get

(41) $$\emptyset\langle x := G(x)\rangle = \emptyset\langle x := 0\rangle = \emptyset,$$

hence

(42) $$Q(\emptyset\langle x := G(x)\rangle) = Q(\emptyset\langle x := 0\rangle) = Q(\emptyset) = 0,$$

thus

(43) $$\begin{aligned} G\langle x := Q(\emptyset\langle x := G(x)\rangle)\rangle &= G\langle x := 0\rangle \\ &= (C_s\langle x := 0\rangle)\langle x := 0\rangle \\ &= C_s\langle x := 0\rangle, \end{aligned}$$

hence by (33) and (43)

(44) $$\begin{aligned} Q(G\langle x := Q(\emptyset\langle x := G(x)\rangle)\rangle) &= Q(C_s\langle x := 0\rangle) \\ &= F(x). \end{aligned}$$

By (38), (39), and (44) we obtain

(45) $$Q(F) \geq F(x),$$

i. e. (25) holds.

■

Theorem 3

If 1. $\forall F(F \in \mathcal{F}(U) \to ALL(F) \leq Q(F) \leq EX(F))$

2. *Q is monotone on $\mathcal{F}(U)$*
3. *Q is strongly associative on U and $\mathcal{F}(U)$*
4. *Q is weakly continuous on U and $\mathcal{F}(U)$*
5. *Q is weakly commutative on U, i. e.*

$$\forall x\forall y(x, y \in U \to Q(\mathbf{U}\langle x := 0\rangle) = Q(\mathbf{U}\langle y := 0\rangle))$$

then there exists exactly one $c \in \langle 0, 1\rangle$ such that for every $F \in \mathcal{F}(U)$,

1. $c \leq ALL(F) \to Q(F) = ALL(F)$
2. $EX(F) \leq c \to Q(F) = EX(F)$
3. $ALL(F) < c < EX(F) \to Q(F) = c.$

Proof

In order to prove the existence of c we fix an $x_0 \in U$ and define $c \in \langle 0, 1\rangle$ by

(1) $$c =_{def} Q(\mathbf{U}\langle x_0 := 0\rangle).$$

Because of assumption 5 we have

(2) $$\forall x\forall y(x, y \in U \to Q(\mathbf{U}\langle x := 0\rangle) = Q(\mathbf{U}\langle y := 0\rangle)),$$

hence c does not depend on the chosen $x_0 \in U$.

ad 1 We assume

(3) $$c \leq ALL(F).$$

By the definition of ALL and (2) we get

(4) $$\forall x(x \in U \to Q(\mathbf{U}\langle x := 0\rangle) \leq F(x)),$$

hence by assertion 1 of lemma 2,

(5) $$\forall x(x \in U \to Q(F) \le F(x)),$$

thus by definition of *ALL*,

(6) $$Q(F) \le ALL(F),$$

hence by assumption 1,

(7) $$Q(F) = ALL(F).$$

ad 2 We assume

(8) $$EX(F) \le c.$$

By (2) and the definition of *EX* we get

(9) $$\forall x(x \in U \to F(x) \le Q(\mathbf{U}\langle x := 0\rangle)),$$

hence by conclusion 2 of lemma 2

(10) $$\forall x(x \in U \to F(x) \le Q(F)),$$

thus by definition of *EX*,

(11) $$EX(F) \le Q(F),$$

hence by assumption 1,

(12) $$Q(F) = EX(F).$$

ad 3 By assumption we have

(13) $$ALL(F) < c < EX(F),$$

hence by definition of *ALL* and of *EX*

(14) there exists an $x_0 \in U$ and a $y_0 \in U$ such that $F(x_0) < c < F(y_0)$.

Now, we define a fuzzy set F' on U as follows:

(15) $$F'(x) =_{def} \begin{cases} F(x) & \text{if } F(x) \le c \\ c & \text{if } F(x) > c \end{cases}.$$

Then we get

(16) $$\forall x(x \in U \to F'(x) \le F(x) \wedge F'(x) \le c).$$

Hence from (16) we get

(17) $$Q(F') \le Q(F)$$

and

(18) $$EX(F') \le c.$$

From (18) by assumption 2 of the theorem we obtain

(19) $$Q(F') = EX(F').$$

Furthermore, by (14) and (15) we have

(20) $$F'(y_0) = c,$$

hence by (18)

(21) $$EX(F') = c,$$

thus by (17), (19), and (21)

(22) $$c \le Q(F).$$

Furthermore, we define a second fuzzy set F'' on U as follows:

(23) $$F''(x) =_{def} \begin{cases} F(x) & \text{if } F(x) \ge c \\ c & \text{if } F(x) < c \end{cases}.$$

Then we get

(24) $$\forall x (x \in U \rightarrow F(x) \leq F''(x) \wedge c \leq F''(x)) .$$

From (24) by monotonicity of Q we obtain

(25) $$Q(F) \leq Q(F''),$$

furthermore, by definition of *ALL*

(26) $$c \leq ALL(F'').$$

From (26) by assumption 1 of the theorem we obtain

(27) $$Q(F'') = ALL(F'').$$

Furthermore, by (14) and (23) we have

(28) $$F''(x_0) = c,$$

hence by (26)

(29) $$ALL(F'') = c,$$

thus by (25), (27), and (29)

(30) $$Q(F) \leq c,$$

hence by (22) and (30)

(31) $$Q(F) = c.$$

In order to prove the uniqueness of c we can assume that card$U \geq 2$ because the case of card$U = 1$ is trivial. Now assume we have $c, c' \in \langle 0, 1\rangle$ which fulfill the conclusion of theorem 3. Without loss of generality we can assume that $c \leq c'$. Then we fix $x_0 \in U$ and define a fuzzy set F for arbitrary $x \in U$ by

(32) $$F(x) =_{def} \begin{cases} c & \text{if } x = x_0 \\ c' & \text{if } x \neq x_0 \end{cases} .$$

Then we get

(33) $$c = ALL(F)$$

and

(34) $$EX(F) = c'.$$

By conclusion 1 for c and conclusion 2 for c' we get

(35) $$Q(F) = ALL(F) = c$$

and

(36) $$Q(F) = EX(F) = c',$$

hence

(37) $$c = c'.$$

■

Remark The strong associativity of a quantifier Q includes a certain version of commutativity of Q, but not its "full" commutativity. This fact will discussed in detail in an extended version of this paper.

Acknowledgement

The author wishes to thank Ulrich Fieseler and Stephan Lehmke for fruitful scientific discussions and their help in preparing the manuscript.

References

[1] DIDIER DUBOIS and HENRI PRADE. *Criteria aggregation and ranking of alternatives in the framework of fuzzy set theory*. In: HANS-JÜRGEN ZIMMERMANN, LOTFI A. ZADEH and B. R. GAINES (editors), *Fuzzy Sets and Decision Processes*, volume 20 of *TIMS Studies in the Management Sciences*, pages 209–240. 1984.

[2] DIDIER DUBOIS and HENRI PRADE. *A Review of Fuzzy Set Aggregation Connectives*. Information Sciences **36**, 85–121, 1985.

[3] ANDRZEJ MOSTOWSKI. *On a generalization of quantifiers*. Fundamenta mathematicae **44**, 12–36, 1957.

[4] HELMUT THIELE. *On Fuzzy Quantifiers*. In: *Proceedings: Fifth International Fuzzy Systems Association World Congress '93*, volume I, pages 395–398, Seoul, Korea, July 4–9, 1993.

[5] HELMUT THIELE. *On T-Quantifiers and S-Quantifiers*. In: *The Twenty-Fourth International Symposium on Multiple-Valued Logic*, pages 264–269, Boston, Massachussets, May 25–27, 1994.

[6] HELMUT THIELE. *On the Concept of Cardinal Number for Fuzzy Sets*. In: *EUFIT '94 — Second European Congress on Intelligent Technologies and Soft Computing*, pages 504–516, Aachen, Germany, September 20–23, 1994. Invited paper.

[7] HELMUT THIELE. *On Fuzzy Quantifiers*. In: Z. BIEN and K. C. MIN (editors), *Fuzzy Logic and its Applications, Information Sciences, and Intelligent Systems*, pages 343–352. Kluwer Academic Publishers, 1995.