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Axiomatic Considerations of the Concepts of R-Implication and T-Norm

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Axiomatic Considerations of the Concepts of R-Implication and T-Norm*

Helmut Thiele

Abstract In the paper presented we investigate under which conditions there exists a one-to-one correspondance between a class of generalized R-implications and a class of generalized T-norms based on the mutual definability of these classes. Furthermore, we study which properties of functions of the one class will be translated into properties of functions of the other class by the bijection mentioned above. This paper can be considered as a continuation of author's paper dealing with the same problematics for S-implications on the one hand and S-norms (T-conorms) and negations on the other hand.

Keywords R-implications, T-norms, mutual definability, bijections between classes of generalized R-implications and classes of generalized T-norms, translating properties of R-implications and T-norms by this bijection.

1 Basic Definitions and Fundamental Results

By $\langle 0,1\rangle$ we denote the set of all real numbers r with $0 \leqq r \leqq 1$.

We define

$$\mathrm{FUNCT}(2) =_{\mathrm{def}} \{\Phi | \Phi : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle\} \ .$$

In [26] we have defined the functional operators RIMP and TNOR with

$$\mathrm{RIMP}, \mathrm{TNOR} : \mathrm{FUNCT}(2) \to \mathrm{FUNCT}(2)$$

as follows where $\tau, \pi \in \mathrm{FUNCT}(2)$ and $r, s \in \langle 0,1\rangle$

Definition 1

1. $\mathrm{RIMP}(\tau)(r,s) =_{\mathrm{def}} \sup\{t | t \in \langle 0,1\rangle \wedge \tau(r,t) \leqq s\}$
2. $\mathrm{TNOR}(\pi)(r,s) =_{\mathrm{def}} \inf\{t | t \in \langle 0,1\rangle \wedge \pi(r,t) \geqq s\}$.

We underline that definition 1 generalizes the well-known residuation operation and the generation of a T-norm by a given implication, respectively.

The following six theorems and corollaries one can find already in [26], but without proof.

Theorem 1

For every $r, s \in \langle 0,1\rangle$,

$$\mathrm{TNOR}(\mathrm{RIMP}(\tau))(r,s) \leqq \tau(r,s) \ .$$

Proof

Assume $r, s \in \langle 0,1\rangle$. By definition of the operator TNOR we have to prove

(1) $$\inf\{t | t \in \langle 0,1\rangle \wedge \mathrm{RIMP}(\tau)(r,t) \geqq s\} \leqq \tau(r,s) \ .$$

By definition of inf and RIMP it is sufficient to show

(2) $$\exists t \big(t \in \langle 0,1\rangle \wedge \sup\{t' | t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\} \geqq s \wedge t \leqq \tau(r,s)\big)$$

*Revised version of a paper originally published in *International Conference on Onformation Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU '98)*, Paris, France, July 6–10, 1998 hence by definition of sup it is sufficient to show

(3) $$\exists t \exists t' \big(t, t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t \wedge t' \geqq s \wedge t \leqq \tau(r,s)\big).$$

We put

(4) $$\begin{aligned} t &=_{\text{def}} \tau(r,s) \\ t' &=_{\text{def}} s\ . \end{aligned}$$

Obviously, (3) holds. ■

Theorem 2

If the function τ is monotone and left-hand continuous with respect to $\langle 0,1\rangle$ and its second argument, then for every $r,s \in \langle 0,1\rangle$,

$$\tau(r,s) \leqq \text{TNOR}\big(\text{RIMP}(\tau)\big)(r,s)\ .$$

Proof

Assume $r,s \in \langle 0,1\rangle$. By definition of the operator TNOR we have to prove

(1) $$\tau(r,s) \leqq \inf\{t \mid t \in \langle 0,1\rangle \wedge \text{RIMP}(\tau)(r,t) \geqq s\}\ .$$

By definition of inf it is sufficient to show

(2) $$\forall t \big(t \in \langle 0,1\rangle \wedge \text{RIMP}(\tau)(r,t) \geqq s \rightarrow \tau(r,s) \leqq t\big),$$

hence by definition of RIMP it is sufficient to show

(3) $$\forall t \big(t \in \langle 0,1\rangle \wedge \sup\{t' \mid t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\} \geqq s \rightarrow \tau(r,s) \leqq t\big).$$

By definition of sup we have

(4) $$\sup\{\tau(r,t' \mid t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\} \leqq t\ .$$

Furthermore, as for every fixed $r \in \langle 0,1\rangle$ the function $\tau(r,s)$ is left-hand continuous with respect to $s \in \langle 0,1\rangle$, we obtain

(5) $$\tau\big(r, \sup\{t' \mid t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\}\big) \leqq \sup\{\tau(r,t') \mid t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\}\ .$$

Hence from (4) and (5) we get

(6) $$\tau\big(r, \sup\{t' \mid t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\}\big) \leqq t\ .$$

In order to prove (3) we assume

(7) $$\sup\{t' \mid t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\} \geqq s\ .$$

Because for every fixed $r \in \langle 0,1\rangle$ the function $\tau(r,s)$ is monotone with respect to $s \in \langle 0,1\rangle$, we obtain

(8) $$\tau(r,s) \leqq \tau\big(r, \sup\{t' \mid t' \in \langle 0,1\rangle \wedge \tau(r,t') \leqq t\}\big),$$

hence from (8) and (6) we get

(9) $$\tau(r,s) \leqq t\ .$$

■

Corollary 3
If the function τ *is monotone and left-hand continuous with respect to* $\langle 0,1\rangle$ *and its second argument, then for every* $r,s \in \langle 0,1\rangle$,

$$\mathrm{TNOR}(\mathrm{RIMP}(\tau))(r,s) = \tau(r,s) \ .$$

Proof
By theorems 1 and 2. ■

Definition 2
1. $\mathrm{FUNCT}(2, M2, LHC2)$
$$=_{\mathrm{def}} \left\{ \varphi \,\middle|\, \begin{array}{l} \varphi : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle \textit{ and } \varphi \textit{ is monotone and left-hand} \\ \textit{continuous with respect to } \langle 0,1\rangle \textit{ and its second argument} \end{array} \right\}$$
2. $\mathrm{FUNCT}(2, M2, RHC2)$
$$=_{\mathrm{def}} \left\{ \varphi \,\middle|\, \begin{array}{l} \varphi : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle \textit{ and } \varphi \textit{ is monotone and right-hand} \\ \textit{continuous with respect to } \langle 0,1\rangle \textit{ and its second argument} \end{array} \right\}$$

Corollary 4
The operator RIMP *is an injection from* $\mathrm{FUNCT}(2, M2, LHC2)$ *into* $\mathrm{FUNCT}(2)$.

Proof
By corollary 3. ■

Now, we are faced with the problem to characterize the image of the class $\mathrm{FUNCT}(2, \mathrm{M2}, \mathrm{LHC2})$ generated by the operator RIMP. The following theorems and corrollaries solve this problem.

Assume $\pi : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$.

Theorem 5
If the function π *is monotone and right-hand continuous with respect to* $\langle 0,1\rangle$ *and its second argument, then for every* $r,s \in \langle 0,1\rangle$,

$$\mathrm{RIMP}(\mathrm{TNOR}(\pi))(r,s) \leqq \pi(r,s) \ .$$

Proof
Like theorem 2. ■

Theorem 6
For every $r,s \in \langle 0,1\rangle$,

$$\pi(r,s) \leqq \mathrm{RIMP}(\mathrm{TNOR}(\pi))(r,s)$$

Proof
Like theorem 1. ■

Corollary 7
If the function π *is monotone and right-hand continuous with respect to* $\langle 0,1\rangle$ *and its second argument, then for every* $r,s \in \langle 0,1\rangle$:

$$\mathrm{RIMP}(\mathrm{TNOR}(\pi))(r,s) = \pi(r,s) \ .$$

Proof
By theorems 5 and 6. ■

Corollary 8

1. TNOR *is a bijection from* $\mathrm{FUNCT}(2, M2, RHC2)$ *onto* $\mathrm{FUNCT}(2, M2, LHC2)$
2. RIMP *is the inversion of* TNOR.

Proof

ad 1.

By thorems 5 and 6 we get

(1) $$\mathrm{FUNCT}(2,\mathrm{M2},\mathrm{RHC2}) \subseteqq \mathrm{RIMP}\,(\mathrm{FUNCT}(2,\mathrm{M2},\mathrm{LHC2}))\,,$$

hence by monotonicity of TNOR

(2) $$\mathrm{TNOR}\,(\mathrm{FUNCT}(2,\mathrm{M2},\mathrm{RHC2})) \subseteqq \mathrm{TNOR}\,(\mathrm{RIMP}\,(\mathrm{FUNCT}(2,\mathrm{M2},\mathrm{LHC2})))\ .$$

By corollary 3 we have

(3) $$\mathrm{TNOR}\,(\mathrm{RIMP}\,(\mathrm{FUNCT}(2,\mathrm{M2},\mathrm{LHC2}))) \subseteqq \mathrm{FUNCT}(2,\mathrm{M2},\mathrm{LHC2}),$$

hence by (2)

(4) $$\mathrm{TNOR}\,(\mathrm{FUNCT}(2,\mathrm{M2},\mathrm{RHC2})) \subseteqq \mathrm{FUNCT}(2,\mathrm{M2},\mathrm{LHC2})\ .$$

ad 2.

In analogy to 1. ■

2 Axioms for Characterizing T-Norms and R-Implications

Definition 3 (Axioms for Characterizing T-Norms)

Assume $\tau : \langle 0,1\rangle \times \langle 0,1\rangle \rightarrow \langle 0,1\rangle$.

TN1. $\forall s\,(s \in \langle 0,1\rangle \rightarrow \tau(0,s) = 0)$

TN2. $\forall r\,(r \in \langle 0,1\rangle \rightarrow \tau(r,0) = 0)$

TN3. $\forall s\,(s \in \langle 0,1\rangle \rightarrow \tau(1,s) = s)$

TN4. $\forall r\,(r \in \langle 0,1\rangle \rightarrow \tau(r,1) = r)$

TN5. $\forall r \forall s \forall t\,(r,s,t \in \langle 0,1\rangle \wedge r \leqq s \rightarrow \tau(r,t) \leqq \tau(s,t))$

TN6. $\forall r \forall s \forall t\,(r,s,t \in \langle 0,1\rangle \wedge s \leqq t \rightarrow \tau(r,s) \leqq \tau(r,t))$

TN7. $\forall r \forall s\,(r,s \in \langle 0,1\rangle \rightarrow \tau(r,s) = \tau(s,r))$

TN8. $\forall r \forall s \forall t\,(r,s,t \in \langle 0,1\rangle \rightarrow \tau(r,\tau(s,t)) = \tau(\tau(r,s),t))$

Definition 4 (Axioms for Characterizing R-Implications)

Assume $\pi : \langle 0,1\rangle \times \langle 0,1\rangle \rightarrow \langle 0,1\rangle$.

RIM1. $\forall s\,(s \in \langle 0,1\rangle \rightarrow \pi(0,s) = 1)$

RIM2. $\forall s\,(s \in \langle 0,1\rangle \rightarrow \pi(1,s) = s)$

RIM3. $\forall r\,(r \in \langle 0,1\rangle \rightarrow \pi(r,1) = 1)$

RIM4. $\forall r \forall s\,(r,s \in \langle 0,1\rangle \wedge r \leqq s \rightarrow \pi(r,s) = 1)$

RIM5. $\forall r \forall s\,(r,s \in \langle 0,1\rangle \wedge \pi(r,s) = 1 \rightarrow r \leqq s)$

RIM6. $\forall r \forall s \forall t \left(r,s,t \in \langle 0,1\rangle \wedge r \leqq s \rightarrow \pi(r,t) \geqq \pi(s,t) \right)$

RIM7. $\forall r \forall s \forall t \left(r,s,t \in \langle 0,1\rangle \wedge s \leqq t \rightarrow \pi(r,s) \leqq \pi(r,t) \right)$

RIM8. $\forall r \forall s \forall t \left(r,s,t \in \langle 0,1\rangle \rightarrow \left(\pi(r,t) \geqq s \leftrightarrow \pi(s,t) \geqq r \right) \right)$

RIM9. $\forall r \forall s \forall t \left(r,s,t \in \langle 0,1\rangle \rightarrow \pi(r,\pi(s,t)) = \pi(s,\pi(r,t)) \right)$

3 On Translating Properties of Functions by Applying the Functional Operator TNOR

Theorem 9
Assume $\pi : \langle 0,1\rangle \times \langle 0,1\rangle \rightarrow \langle 0,1\rangle$.

1. TNOR(π) *fulfils TN1 if* $\pi(0,0) = 1$
2. TNOR(π) *fulfils TN2 without any assumption for* π
3. TNOR(π) *fulfils TN3 if* π *fulfils RIM2*
4. TNOR(π) *fulfils TN4 if* π *fulfils RIM4 and RIM5*
5. TNOR(π) *fulfils TN5 if* π *fulfils RIM6*
6. TNOR(π) *fulfils TN6 without any assumptions for* π
7. TNOR(π) *fulfils TN7 if* π *fulfils RIM8*
8. TNOR(π) *fulfils TN8 if* π *fulfils RIM5, RIM6, RIM7, RIM9 and* π *is right-hand continuous with respect to* $\langle 0,1\rangle$ *and its second argument.*

Proof
ad 1. TN1
Assume $s \in \langle 0,1\rangle$. We have to prove

(1) $$\mathrm{TNOR}(\pi)(0,s) = 0\ .$$

By definition of TNOR it is sufficient to show

(2) $$\inf\left\{ t \,\middle|\, t \in \langle 0,1\rangle \wedge \pi(0,t) \geqq s \right\} = 0\ .$$

In order to prove (2), by definition of inf it is sufficient to show

(3) $$\exists t \left(t \in \langle 0,1\rangle \wedge \pi(0,t) \geqq s \wedge t = 0 \right)\ .$$

But (3) holds because of $\pi(0,0) = 1$.

ad 2. TN2
Assume $r \in \langle 0,1\rangle$. We have to prove

(1) $$\mathrm{TNOR}(\pi)(r,0) = 0\ .$$

By definition of TNOR it is sufficient to show

(2) $$\inf\left\{ t \,\middle|\, t \in \langle 0,1\rangle \wedge \pi(r,t) \geqq 0 \right\} = 0\ .$$

Because $\pi(r,t) \geqq 0$ holds for every $r,t \in \langle 0,1\rangle$, we get $\pi(r,0) \geqq 0$, hence (2) holds.

ad 3. TN3
Assume $s \in \langle 0,1\rangle$. We have to prove

(1) $$\mathrm{TNOR}(\pi)(1,s) = s\ .$$

By definition of TNOR it is sufficient to show

(2) $$\inf\{t \mid t \in \langle 0,1\rangle \wedge \pi(1,t) \geqq s\} = s\ .$$

Because of RIM2 we have

(3) $$\forall t\,(t \in \langle 0,1\rangle \rightarrow \pi(1,t) = t),$$

hence we have

(4) $$\inf\{t \mid t \in \langle 0,1\rangle \wedge \pi(1,t) \geqq s\} = \inf\{t \mid t \in \langle 0,1\rangle \wedge t \geqq s\}\ .$$

Obviously, we have

(5) $$\inf\{t \mid t \in \langle 0,1\rangle \wedge t \geqq s\} = s,$$

hence from (4) and (5) we get (2).

ad 4. TN4
Assume $r \in \langle 0,1\rangle$. We have to prove

(1) $$\mathrm{TNOR}(\pi)(r,1) = r\ .$$

By definition of TNOR it is sufficient to show

(2) $$\inf\{t \mid t \in \langle 0,1\rangle \wedge \pi(r,t) \geqq 1\} = r\ .$$

In order to show (2), it is sufficient to prove

(3) $$\forall r\,(r \in \langle 0,1\rangle \rightarrow \pi(r,r) = 1)$$

and

(4) $$\forall r \forall t\,(r,t \in \langle 0,1\rangle \wedge \pi(r,t) = 1 \rightarrow t \geqq r)\ .$$

But, RIM4 implies (3) and RIM5 implies (4).

ad 5. TN5
Assume for $r,s,t \in \langle 0,1\rangle$

(1) $$r \leqq s\ .$$

We have to prove

(2) $$\mathrm{TNOR}(\pi)(r,t) \leqq \mathrm{TNOR}(\pi)(s,t)\ .$$

By definition of TNOR it is sufficient to show

(3) $$\inf\{t' \mid t' \in \langle 0,1\rangle \wedge \pi(r,t' \geqq t\} \leqq \inf\{t' \mid t' \in \langle 0,1\rangle \wedge \pi(s,t') \geqq t\}\ .$$

By definition of inf it is sufficient to prove

(4) $$\forall t'\,(t' \in \langle 0,1\rangle \wedge \pi(s,t') \geqq t \rightarrow \pi(r,t') \geqq t)\ .$$

Assume $\pi(s,t') \geqq t$. From $r \leqq s$ and RIM6 we get $\pi(r,t') \geqq \pi(s,t')$, hence (4) holds.

ad 6. TN6
Assume for $r,s,t \in \langle 0,1\rangle$

(1) $$s \leqq t\ .$$

We have to prove

(2) $$\mathrm{TNOR}(\pi)(r,s) \leqq \mathrm{TNOR}(\pi)(r,t)\ .$$

By definition of TNOR it is sufficient to show

(3) $$\inf\{t' | t' \in \langle 0,1\rangle \wedge \pi(r,t') \geqq s\} \leqq \inf\{t' | t' \in \langle 0,1\rangle \wedge \pi(r,t') \geqq t\}\ .$$

By definition of inf it is sufficient to show

(4) $$\forall t' \left(t' \in \langle 0,1\rangle \wedge \pi(r,t') \geqq t \rightarrow \pi(r,t') \geqq s\right)\ .$$

But, (4) holds trivially because of $s \leqq t$.

ad 7. TN7
Assume $r,s \in \langle 0,1\rangle$. We have to prove

(1) $$\mathrm{TNOR}(\pi)(r,s) = \mathrm{TNOR}(\pi)(s,r)\ .$$

By definition of TNOR it is sufficient to show

(2) $$\inf\{t | t \in \langle 0,1\rangle \wedge \pi(r,t) \geqq s\} = \inf\{t | t \in \langle 0,1\rangle \wedge \pi(s,t) \geqq r\}\ .$$

By definition of inf it is sufficient to show

(3) $$\exists t \left(t \in \langle 0,1\rangle \wedge \pi(r,t) \geqq s\right) \quad \text{if and only if} \quad \exists t \left(t \in \langle 0,1\rangle \wedge \pi(s,t) \geqq r\right)\ .$$

But, (3) holds because of assumption RIM8.

ad 8. TN8
Assume $r,s,t \in \langle 0,1\rangle$. We have to prove

(1) $$\mathrm{TNOR}(\pi)(r,\mathrm{TNOR}(\pi)(s,t)) = \mathrm{TNOR}(\pi)(\mathrm{TNOR}(\pi)(r,s),t)\ .$$

By definition of TNOR it is sufficient to show

(2) $$\inf\{u | u \in \langle 0,1\rangle \wedge \pi(r,u) \geqq \mathrm{TNOR}(\pi)(s,t)\} = \inf\{v | v \in \langle 0,1\rangle \wedge \pi(\mathrm{TNOR}(\pi)(r,s),v) \geqq t\}\ .$$

By definition of inf it is sufficient to prove

(3) $$\exists u \left(u \in \langle 0,1\rangle \wedge \pi(r,u) \geqq \mathrm{TNOR}(\pi)(s,t)\right)$$

if and only if

(4) $$\exists v \left(v \in \langle 0,1\rangle \wedge \pi(\mathrm{TNOR}(\pi)(r,s),v) \geqq t\right)\ .$$

I (↓).
Assume for $u \in \langle 0,1\rangle$

(5) $$\pi(r,u) \geqq \mathrm{TNOR}(\pi)(s,t), \text{ i. e.}$$
(6) $$\pi(r,u) \geqq \inf\{w | w \in \langle 0,1\rangle \wedge \pi(s,w) \geqq t\}\ .$$

Because π is right-hand continuous with respect to $\langle 0,1\rangle$ and its second argument, there exists a $w \in \langle 0,1\rangle$ such that

(7) $$\pi(r,u) \geqq w$$

and

(8) $$\pi(s,w) \geqq t\ .$$

We have to prove (4), i. e.

(9) $$\exists v\big(v \in \langle 0,1\rangle \wedge \pi\big(\inf\{x|x \in \langle 0,1\rangle \wedge \pi(r,x) \geqq s\}, v\big) \geqq t\big)\ .$$

Because of RIM6 we get

(10) $$\pi\big(\inf\{x|x \in \langle 0,1\rangle \wedge \pi(r,x) \geqq s\}, v\big) \geqq \sup\{\pi(x,v)|x \in \langle 0,1\rangle \wedge \pi(r,x) \geqq s\},$$

hence it is sufficient to show

(11) $$\exists v\big(v \in \langle 0,1\rangle \wedge \sup\{\pi(x,v)|x \in \langle 0,1\rangle \wedge \pi(r,x) \geqq s\} \geqq t\big)\ .$$

By definition of sup it is sufficient to show

(12) $$\exists v \exists x\big(v,x \in \langle 0,1\rangle \wedge \pi(x,v) \geqq t \wedge \pi(r,x) \geqq s\big)\ .$$

Put

(13) $$v = \pi(s,u)$$

and

(14) $$x =_{\text{def}} r\ .$$

Then it is sufficient to prove

(15) $$\pi(r,\pi(s,u)) \geqq t$$

and

(16) $$\pi(r,r) \geqq s\ .$$

We prove (15).
By (7) and RIM7 we get

(17) $$\pi(s,\pi(r,u)) \geqq \pi(s,w),$$

hence by (8)

(18) $$\pi(s,\pi(r,u)) \geqq t,$$

hence by RIM9, (15) holds.
Finally, (16) holds because of RIM5.

II (↑).
In analogy to I. ■

4 On Translating Properties of Functions by Applying the Functional Operator RIMP

Theorem 10
Assume $\tau : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$.

1. RIMP(τ) *fulfils RIM1 if* $\tau(0,1) = 0$

2. $\mathrm{RIMP}(\tau)$ *fulfils RIM2 if* τ *fulfils TN3*
3. $\mathrm{RIMP}(\tau)$ *fulfils RIM3 without any assumption*
4. $\mathrm{RIMP}(\tau)$ *fulfils RIM4 if* τ *fulfils TN4*
5. $\mathrm{RIMP}(\tau)$ *fulfils RIM5 if* τ *fulfils TN4 and* τ *is left-hand continuous with respect to* $\langle 0,1\rangle$ *and its second argument*
6. $\mathrm{RIMP}(\tau)$ *fulfils RIM6 if* τ *fulfils TN5*
7. $\mathrm{RIMP}(\tau)$ *fulfils RIM7 without any assumption*
8. $\mathrm{RIMP}(\tau)$ *fulfils RIM8 if* τ *fulfils TN6 and TN7*
9. $\mathrm{RIMP}(\tau)$ *fulfils RIM9 if* τ *fulfils TN3, TN6, TN8, and* τ *is left-hand continuous with respect to* $\langle 0,1\rangle$ *and its second argument.*

Proof
Like theorem 9. ■

5 Conclusions

The theorems and corollaries above give the possibility to derive numerous "translating" and "bijection" theorems for classes of functions. In the following we discuss only one example.

To this end we define

Definition 5
1. $\mathrm{FUNCT}(2,C2) =_{\mathrm{def}} \left\{\varphi \,\middle|\, \begin{array}{l}\varphi:\langle 0,1\rangle\times\langle 0,1\rangle\to\langle 0,1\rangle \textit{ and } \varphi \textit{ is continuous}\\ \textit{with respect to } \langle 0,1\rangle \textit{ and its second argument}\end{array}\right\}$
2. $\mathrm{FUNCT}(2,M2,C2) =_{\mathrm{def}} \left\{\varphi \,\middle|\, \begin{array}{l}\varphi\in \mathrm{FUNCT}(2,C2) \textit{ and } \varphi \textit{ is monotone with}\\ \textit{respect to } \langle 0,1\rangle \textit{ and its second argument}\end{array}\right\}$

Lemma 11
For every φ*, if* $\varphi \in \mathrm{FUNCT}(2,M2,C2)$*, then* $\mathrm{TNOR}(\varphi)$ *and* $\mathrm{RIMP}(\varphi)$ *belong to* $\mathrm{FUNCT}(2,M2,C2)$.

Proof
Like corollary 8. ■

Definition 6
Assume $\pi,\tau:\langle 0,1\rangle\times\langle 0,1\rangle\to\langle 0,1\rangle$.
π *is said to be an R-implication*
$=_{\mathrm{def}}$ π *fulfils*

RIM2. $\forall s\left(s\in\langle 0,1\rangle\to\pi(1,s)=s\right)$

RIM4. $\forall r\forall s\left(r,s\in\langle 0,1\rangle\wedge r\leqq s\to\pi(r,s)=1\right)$

RIM5. $\forall r\forall s\left(r,s\in\langle 0,1\rangle\wedge\pi(r,s)=1\to r\leqq s\right)$

RIM6. $\forall r\forall s\forall t\left(r,s,t\in\langle 0,1\rangle\wedge r\leqq s\to\pi(r,t)\geqq\pi(s,t)\right)$

RIM8. $\forall r\forall s\forall t\left(r,s,t\in\langle 0,1\rangle\to\left(\pi(r,t)\geqq s\leftrightarrow\pi(s,t)\geqq r\right)\right)$

RIM9. $\forall r\forall s\forall t\left(r,s,t\in\langle 0,1\rangle\to\pi(r,\pi(s,t))=\pi(s,\pi(r,t))\right)$

Remember the concept of T-norm which can be defined as follows:
τ is said to be a T-norm
$=_{\text{def}}$ τ fulfils the axioms TN4, TN5, TN7, and TN8.

Then we get the following theorem:

Theorem 12

1. *If* τ *is a T-norm with* $\tau \in \mathrm{FUNCT}(2, C2)$*,*
   then $\mathrm{RIMP}(\tau)$ *is an R-implication with* $\mathrm{RIMP}(\tau) \in \mathrm{FUNCT}(2, C2)$
2. *If* π *is an R-implication with* $\pi \in \mathrm{FUNCT}(2, C2)$*,*
   then $\mathrm{TNOR}(\pi)$ *is a T-norm with* $\mathrm{TNOR}(\pi) \in \mathrm{FUNCT}(2, C2)$
3. RIMP *and* TNOR *are bijections between the class of T-norms and the class of R-implications (restricted in both cases to the class* $\mathrm{FUNCT}(2, C2)$*).*

Proof
By theorems 9, 10, corollaries 3, 7, 8, and lemma 11. ■

Remark
Because T-norms and R-implications are monotone with respect to $\langle 0,1\rangle$ and their second argument, in the theorem above FUNCT(2,C2) can be replaced by FUNCT(2,M2,C2).

In a forthcoming paper we will publish a proof of theorem 10 and discuss further examples.

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