UNIVERSITY OF DORTMUND

REIHE COMPUTATIONAL INTELLIGENCE

COLLABORATIVE RESEARCH CENTER 531

Design and Management of Complex Technical Processes and Systems by means of Computational Intelligence Methods

Fuzzy Rough Sets versus Rough Fuzzy Sets — An Interpretation and a Comparative Study using Concepts of Modal Logics

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No. CI-30/98

Technical Report ISSN 1433-3325 April 1998

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This work is a product of the Collaborative Research Center 531, "Computational Intelligence", at the University of Dortmund and was printed with financial support of the Deutsche Forschungsgemeinschaft.

Fuzzy Rough Sets versus Rough Fuzzy Sets
An Interpretation and a Comparative Study
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Keywords Rough Sets, Fuzzy Sets, Kripke-Semantics, The S5-System, Crisp and Soft Reachability Relations, Fuzzy Interpretation of Modal Operators

Abstract The starting point of the paper is the (well-known) observation that the "classical" Rough Set Theory as introduced by PAWLAK is equivalent to the S5 Propositional Modal Logic where the reachability relation is an equivalence relation. By replacing this equivalence relation by an arbitrary binary relation (satisfying certain properties, for instance, reflexivity and transitivity) we shall obtain generalized (crisp!!) rough set theories. Our ideas in the paper are:

1. We replace the crisp reachability relation by a binary fuzzy relation whereas the set to be approximated remains crisp. It is very important that the reachability relation is used as a fuzzy relation, i. e. without introducing and using a cut point. Hence, these lower and upper "fuzzy" approximations of the given crisp set are *fuzzy* sets, in general.
2. Vice versa, the given set to be approximated is a fuzzy set, but the reachability relation is crisp. Also in this case the lower and the upper "crisp" approximations of the given fuzzy set are again fuzzy sets, in general.
3. Finally, we define a lower and an upper approximation of a fuzzy set using a binary fuzzy relation. It is interesting that this approach coincides with a concept which we have developed for interpreting the modal operators *Box* and *Diamond* in the framework of Fuzzy Logic.

1 Some Fundamental Notions and Notations

We fix a non-empty set U called universe. For arbitrary subsets $X,Y \subseteqq U$ we denote the union and the intersection of X and Y by $X \cup Y$ and $X \cap Y$, respectively. $\overline{X}$ is the complement of X with respect to U, i. e. the set $U \setminus X$ where $U \setminus X$ denotes the set-theoretical difference of U and X. The empty set is denoted by $\emptyset$.

For crisp binary relations R, i. e. for $R \subseteqq U \times U$, we use the terms "reflexive", "symmetric", "transitive", "tolerance", "semi partial ordering", "equivalence" as usual. In doubt we refer to definition 2 where these notions are generalized to binary fuzzy relations. By U/R we denote the set of equivalence classes from U generated by the equivalence relation R on U.

Assume $R, R' \subseteqq U \times U$. For definiteness we recall

Definition 1

1. $R \cup R' =_{\text{def}} \{[x,y] \mid [x,y] \in R \vee [x,y] \in R'\}$
2. $R \circ R' =_{\text{def}} \{[x,y] \mid \exists z([x,z] \in R \wedge [z,y] \in R')\}$
3. $R^0 =_{\text{def}} \{[x,x] \mid x \in U\}$
4. $R^{n+1} =_{\text{def}} R \circ R^n$ *where* n *is an integer with* $n \geqq 0$

*Revised version of a paper originally published in *5th European Congress on Intelligent Techniques & Soft Computing (EUFIT '97)*, Aachen, Germany, September 8–11, 1997, vol. 1, pages 159–167

5. $R^* =_{\text{def}} \bigcup_{n=0}^{\infty} R^n$.

By $\{0,1\}$ and $\langle 0,1\rangle$ we denote the set of the numbers $0,1$ and the set of all real numbers r with $0 \leqq r \leqq 1$, respectively. An n-ary BOOLEan function β and ŁUKASIEWICZ' function λ is a mapping $\beta : \{0,1\}^n \to \{0,1\}$ and $\lambda : \langle 0,1\rangle^n \to \langle 0,1\rangle$. λ is said to be ŁUKASIEWICZ' extension of $\beta =_{\text{def}} \forall x_1 \cdots \forall x_n (x_1,\ldots,x_n \in \{0,1\} \to \lambda(x_1,\ldots,x_n) = \beta(x_1,\ldots,x_n))$. By et and seq we denote the binary BOOLEan functions defined by $\mathrm{et}(1,1) = 1$, $\mathrm{et}(1,0) = \mathrm{et}(0,1) = \mathrm{et}(0,0) = 0$ and $\mathrm{seq}(1,1) = \mathrm{seq}(0,1) = \mathrm{seq}(0,0) = 1$, $\mathrm{seq}(1,0) = 0$, respectively. The functions min and $\mathrm{kd}(r,s) =_{\text{def}} \max(1-r,s), (r,s) \in \langle 0,1\rangle)$ are called standard ŁUKASIEWICZ' extensions of et and seq, respectively. Generalized fuzzy conjunctions and fuzzy implications are ŁUKASIEWICZ' extensions of et and seq, respectively. A t-norm τ and an s-norm (t-conorm) σ is a binary monotone, associative and commutative ŁUKASIEWICZ' function satisfying the equation $\tau(r,1) = r$ and $\sigma(r,0) = r$ for every $r \in \langle 0,1\rangle$, respectively. Thus, a t-norm and an s-norm is a generalized fuzzy conjunction and a generalized fuzzy disjunction, respectively.

Fuzzy sets on U are mappings $F : U \to \langle 0,1\rangle$, i. e. we do not distinguish between the fuzzy set F and its membership function μ_F. By $\not{U}$ and $\not{O}$ we denote the universal and the empty fuzzy set on U, i. e. the mapping such that for every $x \in U$ the equations $\not{U}(x) = 1$ and $\not{O}(x) = 0$ hold, respectively. For $F,G : U \to \langle 0,1\rangle$ we define $F \subseteqq G$ by $\forall x(x \in U \to F(x) \leqq G(x))$.

For arbitrary fuzzy sets F and G on U we use the standard operations $\cap$, $\cup$ and $\overline{}$ as usual, i. e. for every $x \in U$ defined by

$$(F \cap G)(x) =_{\text{def}} \min(F(x), G(x))$$
$$(F \cup G)(x) =_{\text{def}} \max(F(x), G(x))$$
$$(\overline{F})(x) =_{\text{def}} 1 - F(x).$$

For binary fuzzy relations S and S' on U, i. e. $S,S' : U \times U \to \langle 0,1\rangle$, the relations $S \cap S'$, $S \cup S'$, and $\overline{S}$ are defined as for fuzzy sets.

In order to formulate some parts of the following definition, we fix a function $\kappa : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$.

Definition 2

1. *S is said to be reflexive on U* $=_{\text{def}} \forall x(x \in U \to S(x,x) = 1)$
2. *S is said to be symmetric on U* $=_{\text{def}} \forall x \forall y(x,y \in U \to S(x,y) = S(y,x))$.
3. *S is said to be* κ*-transitive on U* $=_{\text{def}} \forall x \forall y \forall z(x,y,z \in U \to \kappa(S(x,y), S(y,z)) \leqq S(x,z))$
4. *S is said to be a tolerance relation on U* $=_{\text{def}}$ *S is reflexive and symmetric on U*
5. *S is said to be a semi partial* κ*-ordering relation on U* $=_{\text{def}}$ *S is reflexive and* κ*-transitive on U*
6. *S is said to be a* κ*-equivalence relation on U* $=_{\text{def}}$ *S is reflexive, symmetric, and* κ*-transitive on U*
7. $(S \circ S')(x,y) =_{\text{def}} \mathrm{Sup}\{\min(S(x,z), S'(z,y)) | z \in U\}$
8. $S^0(x,y) =_{\text{def}} \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases} \qquad (x,y \in U)$
9. $S^{n+1} =_{\text{def}} S \circ S^n$
10. $S^*(x,y) =_{\text{def}} \mathrm{Sup}\{S^n(x,y) | n \geqq 0\} \qquad (x,y) \in U)$

If in the points 3, 5, and 6 κ is the function min, we shall call the corresponding relation S standard-transitive on U, semi partial standard-ordering relation on U and standard-equivalence relation on U, respectively.

2 “Classical” Crisp Rough Sets and their Crisp Generalizations

Assume that $R \subseteqq U \times U$ is an equivalence relation on U and $X \subseteqq U$. Following PAWLAK [24–26] we define the R-lower approximation $\underline{R}X$ and the R-upper approximation $\overline{R}X$ of X as follows

Definition 3

1. $\underline{R}X =_{\text{def}} \bigcup \{Y | Y \in U/R \wedge Y \subseteqq X\}$
2. $\overline{R}X =_{\text{def}} \bigcup \{Y | Y \in U/R \wedge Y \cap X \neq \emptyset\}$.

Furthermore, following PAWLAK we define

Definition 4

1. *X is said to be R-exact* $=_{\text{def}} \underline{R}X = \overline{R}X$
2. *X is said to be R-rough (R-inexact)* $=_{\text{def}} \underline{R}X \neq \overline{R}X$.

For better understanding some investigations in the next chapters, we quote the following theorem characterizing R-exact sets

Theorem 1
$\underline{R}X = \overline{R}X$ *if and only if there exists* $\mathfrak{C} \subseteqq U/R$ *such that* $X = \bigcup \mathfrak{C}$.

Because R is an equivalence relation on U, we get the following proposition

Proposition 2

1. $\underline{R}X =_{\text{def}} \{x | x \in U \wedge [x]R \subseteqq X\}$
   $= \{x | \forall y([x,y] \in R \rightarrow y \in X)\}$
2. $\overline{R}X =_{\text{def}} \{x | x \in U \wedge [x]R \cap X \neq \emptyset\}$
   $= \{x | \exists y([x,y] \in R \wedge y \in X)\}$.

Now, we incorporate these concepts of rough set theory in the KRIPKE-semantic approach of the (two-valued) modal logic.

For this reason we permit that $R \subseteqq U \times U$ is an arbitrary binary relation on U. We interpret R as a reachability relation of a KRIPKE-frame $\mathfrak{K} = [U, R]$. Then the box and the diamond operator generated by R, respectively, are defined as follows where $X \subseteqq U$.

Definition 5

1. $[R]X =_{\text{def}} \{x | \forall y([x,y] \in R \rightarrow y \in X\}$
2. $\langle R \rangle X =_{\text{def}} \{x | \exists y([x,y] \in R \wedge y \in X)\}$.

Then the announced incorporation is understandable by the following proposition.

Proposition 3
If R is an equivalence relation on U, then

1. $[R]X = \underline{R}X$ and
2. $\langle R \rangle X = \overline{R}X$.

Hence, we can state that the restriction to equivalence relations leads to the well-known S5-System of modal logic or, vice versa, the "classical" crisp rough set theory (as introduced by PAWLAK) can be termed as S5-rough set theory.

Furthermore, if we replace the equivalence relation R by a tolerance relation (i. e. R is reflexive on U and symmetric) or by a semi partial order relation (i. e. R is reflexive on U and transitive) then we will get generalized rough set theories, which could be called B-rough set theory and S4-rough set theory, respectively.

Because of space restriction we cannot discuss this generalized approach. In a forthcoming paper we shall discuss this approach in detail. For more information see, for instance, [23, 28, 34]. A topological oriented generalization can be found in [14].

3 Rough Fuzzy Sets

The starting point of the approach to be presented in this chapter is the following proposition which gives a new interpretation of definition 5. See also [2].

A crisp set $Y \subseteqq U$ can be described by its characteristic function χ_Y defined for every $x \in U$ by

$$\chi_Y(x) =_{\text{def}} \begin{cases} 0 & \text{if } x \notin Y \\ 1 & \text{if } x \in Y \end{cases}$$

Using definition 5 we get the following proposition where $R \subseteqq U \times U$.

Proposition 4
For every $x \in U, X \subseteqq U$,

1. $\chi_{[R]X}(x) = \text{Inf}\{\chi_X(y) | y \in U \wedge [x,y] \in R\}$ and
2. $\chi_{\langle R \rangle X}(x) = \text{Sup}\{\chi_X(y) | y \in U \wedge [x,y] \in R\}$

Now, we replace the "special" fuzzy set $\chi_Y : U \to \{0,1\}$ by a general fuzzy set $F : U \to \langle 0,1 \rangle$ and define the R-lower approximation $[R]F$ and the R-upper approximation $\langle R \rangle F$ of the fuzzy set F as follows where $x \in U$.

Definition 6
1. $([R]F)(x) =_{\text{def}} \text{Inf}\{F(y) | y \in U \wedge [x,y] \in R\}$
2. $(\langle R \rangle F)(x) =_{\text{def}} \text{Sup}\{F(y) | y \in U \wedge [x,y] \in R\}$.

Obviously, $[R]F$ and $\langle R \rangle F$ are again fuzzy sets on U.

Definition 6 is "compatible" with definition 5 in the sense of the following proposition

Proposition 5
1. $\text{KER}([R]F) = [R]\text{KER}(F)$ and
2. $\text{KER}(\langle R \rangle F) = \langle R \rangle \text{KER}(F)$.

The notion of R-exactness can be generalized to fuzzy sets F on U and crisp binary relations R on U as follows

Definition 7

1. *F is said to be R-exact* $=_{\text{def}}$ $[R]F = \langle R\rangle F$
2. *F is said to be R-rough (R-inexact)* $=_{\text{def}}$ $[R]F \neq \langle R\rangle F$.

As we have already stated in [30, 31] this approach is well-known since more than twenty years (see [5, 12, 13, 16, 21, 22, 27, 29]) in order to interpret the modal operators $\Box$ and $\Diamond$ in the frame of fuzzy logic and of possibilistic logic, respectively.

Most of these approaches can be subordinated the following scheme: Given a binary fuzzy relation S on U. Then for a fixed "cutpoint" $c \in \langle 0,1\rangle$ two binary crisp relations R_c and R'_c are defined as follows

$$R_c =_{\text{def}} \{[x,y] \mid x,y \in U \wedge S(x,y) \geqq c\}$$
$$R'_c =_{\text{def}} \{[x,y] \mid x,y \in U \wedge S(x,y) > c\}$$

Then the relations R_c and R'_c are used in order to interpret box and diamond with respect to definition 6.

For definiteness we recall some well-known results.

Theorem 6

For every $R \subseteqq U \times U$ and every fuzzy F and G on U,

1. $[R]F = \overline{\langle R\rangle \overline{F}}$ *and* $\langle R\rangle F = \overline{[R]\overline{F}}$
2. $F \subseteqq G \rightarrow [R]F \subseteqq [R]G$ *and* $\langle R\rangle F \subseteqq \langle R\rangle G$
3. $[R](F \cap G) \supseteqq [R]F \cap [R]G$
4. $\langle R\rangle(F \cup G) \subseteqq \langle R\rangle F \cup \langle R\rangle G$
5. $[R]\mathbb{U} = \mathbb{U}$ *and* $\langle R\rangle(\emptyset) = \emptyset$.

Corollary 7

For every $R \subseteqq U \times U$ and every fuzzy F and G on U,

1. $[R](F \cap G) = [R]F \cap [R]G$
2. $\langle R\rangle(F \cup G) = \langle R\rangle F \cup \langle R\rangle G$
3. $[R](F \cup G) \supseteqq [R]F \cup [R]G$
4. $\langle R\rangle(F \cap G) \subseteqq \langle R\rangle F \cap \langle R\rangle G$.

Theorem 8

For every $R \subseteqq U \times U$ and every $F : U \rightarrow \langle 0,1\rangle$,

1. *if R reflexive on U, then* $[R]F \subseteqq F \subseteqq \langle R\rangle F$
2. *if R symmetric, then* $F \subseteqq [R]\langle R\rangle F$ *and* $\langle R\rangle[R]F \subseteqq F$
3. *if R transitive, then* $[R]F \subseteqq [R][R]F$ *and* $\langle R\rangle\langle R\rangle F \subseteqq \langle R\rangle F$.

Corollary 9

For every $R \subseteqq U \times U$ and every $F : U \rightarrow \langle 0,1\rangle$,

1. *if R tolerance relation on U,*
   then $[R]F \subseteqq F \subseteqq \langle R\rangle F$ *and* $F \subseteqq [R]\langle R\rangle F$ *and* $\langle R\rangle[R]F \subseteqq F$
2. *if R semi partial order relation on U,*
   then $[R]F \subseteqq F \subseteqq \langle R\rangle F$ *and* $[R]F = [R][R]F$ *and* $\langle R\rangle F = \langle R\rangle\langle R\rangle F$
3. *if R symmetric and transitive,*
   then $\langle R\rangle F \subseteqq [R]\langle R\rangle F$ *and* $\langle R\rangle[R]F \subseteqq [R]F$

4. *if R equivalence relation on U,*
   then $[R]F \subseteqq F \subseteqq \langle R\rangle F$ *and* $[R]F = [R][R]F$ *and* $\langle R\rangle F = [R]\langle R\rangle F$ *and*
   $\langle R\rangle F = \langle R\rangle\langle R\rangle F$ *and* $[R]F = \langle R\rangle[R]F$.

From dynamic propositional fuzzy logic (as introduced in [31]) we quote

Theorem 10
For every $R, R' \subseteqq U \times U$ and every $F : U \to \langle 0,1\rangle$,

1. $[R \cup R']F = [R]F \cap [R']F$
2. $\langle R \cup R'\rangle F = \langle R\rangle F \cup \langle R'\rangle F$
3. $[R \circ R']F = [R][R']F$
4. $\langle R \circ R'\rangle F = \langle R\rangle\langle R'\rangle F$
5. $[R^*]F = F \cap [R][R^*]F$
6. $\langle R^*\rangle F = F \cup \langle R\rangle\langle R^*\rangle F$

The fundamental theorem characterizing R-exact crisp sets (see theorem 1) can be generalized as follows

Theorem 11
For every equivalence relation $R \subseteqq U \times U$ and every fuzzy set $F : U \to \langle 0,1\rangle$, F is R-exact if and only if

1. $\forall K(K \in U/R \to F$ *is constant on* $K)$ *and*
2. $F(x) = \mathrm{Sup}\{\min(\chi_R(x,y), F(y)) | y \in U\}$

4 Fuzzy Rough Sets

In this chapter we ask the question whether a crisp set $X \subseteqq U$ can be approximated by a binary fuzzy reachability relation S on U, i. e. by a mapping $S : U \times U \to \langle 0,1\rangle$. See also [2].

Proposition 4 leads to a correct description of this problem. To this end we replace the crisp relation $R \subseteqq U \times U$ by its characteristic function χ_R. Hence, from proposition 4 we obtain

Proposition 12
1. $\chi_{[R]X}(x) = \mathrm{Inf}\{\mathrm{seq}(\chi_R(x,y), \chi_X(y)) | y \in U\}$
2. $\chi_{\langle R\rangle X}(x) = \mathrm{Sup}\{\mathrm{et}(\chi_R(x,y), \chi_X(y)) | y \in U\}$.

Now, we want to replace the function $\chi_R : U \times U \to \{0,1\}$ by an arbitrary function $S : U \times U \to \langle 0,1\rangle$, i. e. to replace the binary crisp relation R on U by a binary fuzzy relation S on U. Therefore, first of all, we replace the functions seq and et by ŁUKASIEWICZ' extensions of them because seq and et are only defined on $\{0,1\}$. For simplification we choose the standard extensions, i. e. seq and et are replaced by $\mathrm{kd}(r,s) = \max(1-r,s)$ and $\min(r,s)$, respectively.

Thus, using $S : U \times U \to \langle 0,1\rangle$ as approximating fuzzy relation we define

Definition 8
1. $([S]X)(x) =_{\mathrm{def}} \mathrm{Inf}\{\max(1-S(x,y), \chi_X(y)) | y \in U\}$
2. $(\langle S\rangle X)(x) =_{\mathrm{def}} \mathrm{Sup}\{\min(S(x,y), \chi_X(y)) | y \in U\}$.

In the following chapter we shall replace the crisp set X and its characteristic function χ_X by an arbitrary fuzzy set F on U as in definition 9 described. When studying the approaches determined by definition 8 and 9 we recognized that the properties of $[S]X$ and $[S]F$ essentially depend on S, whereas of fixed S these widely coincide for X and F. The same holds for $\langle S\rangle X$ and $\langle S\rangle F$.

Hence. we shall not study $[S]X$ and $\langle S\rangle F$ separately, but refer to the next chapter.

5 Fuzzy Box and Fuzzy Diamond

Continuing the discussion after definition 8 for arbitrary $S : U \times U \to \langle 0,1\rangle$, $F : U \to \langle 0,1\rangle$, and $x \in U$ we define

Definition 9

1. $([S]F)(x) =_{\text{def}} \text{Inf}\{\max(1-S(x,y),F(y))|y \in U\}$
2. $(\langle S\rangle F)(x) =_{\text{def}} \text{Sup}\{\min(S(x,y),F(y))|y \in U\}$.

Notice that we already introduced these definitions in 1993 (see [30], furthermore see [31]) in order to construct "soft" (i. e. fuzzy) interpretations of the modal operators $\Box$ and $\Diamond$.

Analogous to theorem 6 we get

Theorem 13
For every $S : U \times U \to \langle 0,1\rangle$ and every $F,G : U \to \langle 0,1\rangle$,

1. $[S]F = \overline{\langle S\rangle\overline{F}}$
2. $F \subseteqq G \to [S]F \subseteqq [S]G \wedge \langle S\rangle F \subseteqq \langle S\rangle G$
3. $[S](F \cap G) \supseteqq [S]F \cap [S]G$
4. $\langle S\rangle(F \cup G) \subseteqq \langle S\rangle F \cup \langle S\rangle G$
5. $[S]\mathcal{U} = \mathcal{U} \wedge \langle S\rangle\emptyset = \emptyset$.

Corollary 14
For every $S : U \times U \to \langle 0,1\rangle$ and every $F,G : U \to \langle 0,1\rangle$,

1. $[S](F \cap G) = [S]F \cap [S]G$
2. $\langle S\rangle(F \cup G) = \langle S\rangle F \cup \langle S\rangle G$
3. $[S](F \cup G) \supseteqq [S]F \cup [S]G$
4. $\langle S\rangle(F \cap G) \subseteqq \langle S\rangle F \cap \langle S\rangle G$.

Theorem 15
For every $S : U \times U \to \langle 0,1\rangle$ and every $F : U \to \langle 0,1\rangle$,

1. *if S is reflexive on U, then $[S]F \subseteqq F \subseteqq \langle S\rangle F$*
2. a) *if S is symmetric on U and $\forall x\forall y(x,y \in U \to \min(S(x,y),1-S(x,y)) \leqq F(x))$, then $F \subseteqq [S]\langle S\rangle F$*
   b) *if S is symmetric on U and $\forall x\forall y(x,y \in U \to \max(S(x,y),1-S(x,y)) \geqq F(x))$, then $\langle S\rangle[S]F \subseteqq F$*
3. *if S is standard-transitive on U, then $[S]F \subseteqq [S][S]F$ and $\langle S\rangle\langle S\rangle F \subseteqq \langle S\rangle F$*

Remark In contrast to theorem 8 where for a symmetric $R \subseteqq U \times U$ we could state

$$F \subseteqq [R]\langle R\rangle F \qquad \text{and} \qquad \langle R\rangle[R]F \subseteqq F,$$

we have to accept that for arbitrary $S : U \times U \to \langle 0,1\rangle$ and $F : U \to \langle 0,1\rangle$ the inclusions

$$F \subseteqq [S]\langle S\rangle F \qquad \text{and} \qquad \langle S\rangle[S]F \subseteqq F$$

do not hold, in general.

Corollary 16
For every $S : U \times U \to \langle 0,1\rangle$ and every $F : U \to \langle 0,1\rangle$,

1. *if S is a tolerance relation on U, then*
   $[S]F \subseteqq F \subseteqq \langle S\rangle F$ *and* $F \subseteqq [S]\langle S\rangle F$ *and* $\langle S\rangle[S]F \subseteqq F$
2. *if S is a semi partial standard-order relation on U, then*
   $[S]F \subseteqq F \subseteqq \langle S\rangle F$ *and* $[S]F = [S][S]F$ *and* $\langle S\rangle F = \langle S\rangle\langle S\rangle F$
3. *if S is symmetric on U and standard-transitive on U, then*
   a) *if* $\forall x \forall y(x,y \in U \to \min(S(x,y), 1-S(x,y)) \leqq F(x))$*, then* $\langle S\rangle F \subseteqq [S]\langle S\rangle F$
   b) *if* $\forall x \forall y(x,y \in U \to \max(S(x,y), 1-S(x,y)) \geqq F(x))$*, then* $\langle S\rangle[S]F \subseteqq [S]F$
4. *if S is a standard-equivalence relation on U, then*
   a) $[S]F \subseteqq F \subseteqq \langle S\rangle F$ *and*
   b) $[S]F = [S][S]F$ *and*
   c) $\langle S\rangle F = \langle S\rangle\langle S\rangle F$ *and*
   d) *if* $\forall x \forall y(x,y \in U \to \min(S(x,y), 1-S(x,y)) \leqq F(x))$*, then* $\langle S\rangle F = [S]\langle S\rangle F$
   e) *if* $\forall x \forall y(x,y \in U \to \max(S(x,y), 1-S(x,y)) \geqq F(x))$*, then* $[S]F = \langle S\rangle[S]F$

Analogous to “soft” dynamic logic we have

Theorem 17
For every $S, S' : U \times U \to \langle 0,1\rangle$ and every $F : U \to \langle 0,1\rangle$,

1.	$[S \cup S']F = [S]F \cap [S']F$	4.	$\langle S \circ S'\rangle F = \langle S\rangle\langle S'\rangle F$
2.	$\langle S \cup S'\rangle F = \langle S\rangle F \cup \langle S'\rangle F$	5.	$[S^*]F = F \cap [S][S^*]F$
3.	$[S \circ S']F = [S][S']F$	6.	$\langle S^*\rangle F = F \cup \langle S\rangle\langle S^*\rangle F$

Comparing theorems (corollaries) 6, 7, 8, 9, and 10 with 13, 14, 15, 16, and 17, respectively, we can state that for the box operator (and also for the diamond operator) almost the same regularities hold independent of the fact whether a crisp relation R or a fuzzy relation S occurs “inside” the operator.

Finally, we state that there are surprising analogies between Fuzzy Diamond as we have defined above and the “classical” Generalized Modus Ponens as it was introduced by L. A. ZADEH in [36].

We recall: Assume F, G, F', G' are fuzzy sets on U. The Generalized Modus Ponens is an inference rule of the form

$$\frac{\begin{array}{l} F \\ IF\ F'\ \text{THEN}\ G' \end{array}}{G},$$

with the following semantics. Given a function imp : $\langle 0,1\rangle^2 \to \langle 0,1\rangle$ called "implication". The function imp is used to interpret the IF-THEN rule IF F' THEN G' by defining the binary relation

$$S(x,y) =_{\text{def}} \text{imp}(F'(x), G'(y)) \quad (x,y \in U)$$

on U. Then the fuzzy set G derived from F by IF F' THEN g' is defined as

$$G(x) =_{\text{def}} \text{Sup}\{\min(F(y), S(y,x) | y \in U\}$$

for $x \in U$, i. e. by the Compositional Rule of Inference.

Thus, we can state that $G = \langle S^{-1}\rangle F$, where $S^{-1}(x,y) =_{\text{def}} S(y,x) \quad (x,y \in U)$, i. e. the fuzzy set G obtained by the Compositional Rule of Inference from F and S can also be constructed by applying the fuzzy diamond operator to F, but this operator "attached" with the inverse relation S^{-1}.

6 Non-Standard Fuzzy Box and Fuzzy Diamond

In many applications of fuzzy logic the function min is replaced by a t-norm τ and the function $\max(1-r,s)$ by another "implication" whereas it is to notice that the concept "implication" is not so uniquely defined and used as the concept of t-norm and of s-norm (for instance, S-implication, R-implication, etc.).

To this end we fix a function $\pi : \langle 0,1\rangle^2 \to \langle 0,1\rangle$ called "implication" and a function $\kappa : \langle 0,1\rangle^2 \to \langle 0,1\rangle$ called conjunction. Using these functions we define for $S : U\times U \to \langle 0,1\rangle$ and $F : U \to \langle 0,1\rangle$,

Definition 10

1. $([S,\pi]F)(x) =_{\text{def}} \text{Inf}\{\pi(S(x,y), F(y)) | y \in U\}$
2. $(\langle S,\kappa\rangle F)(x) =_{\text{def}} \text{Sup}\{\kappa(S(x,y), F(y)) | y \in U\}$.

Because of space restrictions we cannot carry out the "whole" program analogous to chapter 5. As it were an "example" we formulate only the following theorem

Theorem 18

For every $S : U\times U \to \langle 0,1\rangle$ and every $F : U \to \langle 0,1\rangle$,

1. *if $\pi(r,s) = 1-\kappa(r,1-s)$ for every $r,s \in \langle 0,1\rangle$, then $[S,\pi]F = \langle S,\kappa\rangle F$*
2. *if κ is monotone with respect to the second argument and $F \subseteqq G$, then $\langle S,\kappa\rangle F \subseteqq \langle S,\kappa\rangle G$.*
3. *if S is reflexive on U and $\kappa(1,s) = s$ for every $s \in \langle 0,1\rangle$, then $F \subseteqq \langle S,\kappa\rangle F$*
4. *if S is κ-transitive and κ is monotone with respect to the first argument and κ is continuous with respect to the second argument and κ is associative, then $\langle S,\kappa\rangle\langle S,\kappa\rangle F \subseteqq \langle S,\kappa\rangle F$.*

It is well-known that by such replacements many theorems fail or they must be reformulated. The same holds if we define and study the so-called non-standard fuzzy box and non-standard fuzzy diamond. While theorem 18 expresses some known properties of box and diamond (see the corresponding items of theorems 6, 8, 13, 15), other properties depend essentially on π and κ.

In a forthcoming paper we will investigate the following two areas of problems:

1. to study the question which assumptions for π and κ are sufficient to ensure $[S,\pi]$ and $\langle S,\kappa\rangle$ fulfil certain desired properties
2. to study $[S,\pi]$ and $\langle S,\kappa\rangle$ for special choices of π and κ, for instance

$$\begin{array}{lcl} \pi(r,s) = \min(1, 1-r+s) & & \pi(r,s) = 1-r+rs \\ \kappa(r,s) = \max(0, r+s-1) & \text{or} & \kappa(r,s) = r \cdot s \end{array}$$

where $r,s \in \langle 0,1\rangle$.

Acknowledgement. The author wishes to thank Stephan Lehmke and Mario Thalheim for fruitful scientific discussions on the subject and for their help in preparing the manuscript.

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