Relational and Extensional Interpretation of FATI and FITA

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Abstract

A common way to interpret fuzzy IF-THEN rule bases is based on an extensional interpretation, either with a relational approach or without any other constraints. In this paper we investigate both alternatives in combination with the two general paradigms FATI and FITA to process rule bases.

Keywords: fuzzy IF-THEN rule bases, relational interpretation, extensional interpretation, FATI, FITA

1 Introduction

When interpreting Fuzzy IF-THEN rules one can look at one rule defining a fuzzy relation based on the left and right hand side of the rule. Given a fuzzy input set F' the evaluation of this input by a fuzzy rule can be interpreted as the fuzzy image of F' under this relation. When interpreting rule bases, there are two principles to perform the above interpretation on more than one rule. The property, whether the relation solves a particular equation system or not, together with the two principles leads to a combination of four alternatives which are now investigated.

2 The Principles FATI and FITA

The principles FATI (First Aggregation Then Inference) and FITA (First Inference Then Aggregation) have been defined and discussed deeply in [LRTT98]. FATI and FITA are paradigms to interpret Fuzzy IF-THEN rules and rule bases.

When interpreting a rule base according to the FATI principle first one single interpretation I is aggregated based on all rules of the rule base. After that the output set G' is infered from the input set F' by means of I in one inference step.

When interpreting a rule base according to the FITA principle first for each rule an interpretation I_1 to I_n is generated. Then intermediate sets G'_1 to G'_n are infered from the input set F' by means of I_1 to I_n in n parallel inference steps. Finally the output set G' is aggregated from G'_1 to G'_n.

For both paradigms the aggregation resp. generation of the interpretation and/or the aggregation of the output set can be performed as well in an extensional as in a non-extensional way (see below).

The following functional operators are based only on extensional definitions, non-extensional approaches will be discussed in forthcoming papers. Furthermore the interpretations are restricted to relational-based ones.

3 Prerequisites and Definitions

Let

$n \in \mathcal{N}$		
$F, F_i, G, G_i, F', F'_i, G', G'_i : U \to \langle 0,1 \rangle$	$i = 1, \ldots, n$	be fuzzy sets over U,
$\kappa, \pi : \langle 0,1 \rangle^2 \to \langle 0,1 \rangle$		fuzzy connectives,
$S, S_i : U \times U \to \langle 0,1 \rangle$	$i = 1, \ldots, n$	fuzzy relations,
$u, v, x, y \in U$		
IF F_i THEN G_i	$i = 1, \ldots, n$	a fuzzy rule base,
$\alpha, \beta : \langle 0,1 \rangle^n \to \langle 0,1 \rangle$		n-ary functions over the unit intervall.

Definition *$S \circ F$ is the image of F under S, defined as:*

$$(S \circ F)(y) =_{def} \text{Sup}\, \{ \kappa(F(x), S(x,y)) \,|\, x \in U \} \qquad y \in V$$

Various concepts of correctness are discussed in [LRTT98]. A functional operator works locally correct with a given rule base RB and a given interpretation I, if RB is rule-wise correct with respect to I according to definition 3.2.2 on page 27 in [LRTT98].

4 The Functional Operator FATI

According to the principle FATI first one interpretation is generated by aggregating individual interpretations, here based on fuzzy relations and due to the extensional approach in two steps:

1. $S_i(x,y) =_{def} \pi(F_i(x), G_i(y)) \qquad i = 1, \ldots, n$
2. $S(x,y) =_{def} \alpha(S_1(x,y), S_2(x,y), \ldots, S_n(x,y))$

Finally the inference is performed based on the input set F' and the interpretation S as the image of F' under S.

Definition *Functional operator* FATI

$$\text{FATI}(F')(y) =_{def} \text{Sup}\left\{ \kappa(F'(x), S(x,y)) \,\middle|\, x \in U \right\}$$

5 The Functional Operator FITA

According to the principle FITA first n (extensional) interpretations are generated:

$$S_i(x,y) =_{def} \pi(F_i(x), G_i(y)), \qquad i = 1, \ldots, n$$

Then n inferences are performed as images of F' under the corresponding S_i:

$$G'_i(y) = \text{Sup}\left\{ \kappa(F'(x), S_i(x,y)) \,\middle|\, x \in U \right\}, \qquad i = 1, \ldots, n$$

Finally the output set G' is aggregated:

$$G'(y) = \beta(G'_1(y), G'_2(y), \ldots, G'_n(y))$$

Definition *Functional operator* FITA

$$\mathrm{FITA}(F')(y) =_{def}$$
$$\beta(\mathrm{Sup}\,\{\kappa(F'(x), S_1(x,y))\,|\,x \in U\}\,, \ldots\,, \mathrm{Sup}\,\{\kappa(F'(x), S_n(x,y))\,|\,x \in U\})$$

6 Relational and Extensional Interpretations

The functional operators $FATI$ and $FITA$ are defined by means of relations over fuzzy sets. If these relations do not directly depend on x and y but on the membership values $F(x)$ and $G(y)$ we call them "defined in an extensional way" and call functional operators based on them "extensional interpretations".

A relational interpretation is:

1. an extensional interpretation
2. for a given aggregation function α the relation $S(x,y)$ has to be a solution of the relational equation system

If for a given rule base such a solution does not exist, then a relation interpretration is not possible.

7 The Functional Operator FATI with Relational Rule Interpretation

The relational rule interpretation requires, that the relation S is a solution of the equation system:

$$\begin{aligned} S \circ F_1 &= G_1 \\ S \circ F_2 &= G_2 \\ &\vdots \\ S \circ F_n &= G_n \end{aligned}$$

In general this puts certain constraints on the selection of π and α. For particular rule bases there may even exist no solution.

The relational approach guarantees by definition, that the FATI operator works locally correct [Thiel95], e. g. $\mathrm{FATI}(F_i) = G_i$ holds for all $i = 1, \ldots, n$.

8 The Functional Operator FATI with Extensional Rule Interpretation

When using an extensional rule interpretation, π und α can be chosen arbitrarily. This includes the above relational approach but what about the other cases ? One can distinguish two alternatives:

Case 1: All S_i are local solutions $(\forall i : S_i \circ F_i = G_i)$ but $\exists j : S \circ F_j \neq G_j$

Case 2: $S_j \circ F_j \neq G_j$ for at least one j

For both cases at least one point exists, in which a FATI operator based on the F_i and G_i from above does not work locally correctly, e. g. a F_j exists with $\mathrm{FATI}(F_j) \neq G_j$.

9 The Functional Operator FITA with Relational Rule Interpretation

Again the relational rule interpretation requires, that the relation S solves the equation system:

$$\begin{aligned} S \circ F_1 &= G_1 \\ S \circ F_2 &= G_2 \\ &\vdots \\ S \circ F_n &= G_n \end{aligned}$$

As the relation S is not used in the definition of the FITA operator, a relational interpretation for FITA is senseless. One can weaken a relational interpretation for FITA in a way that only local solutions S_i have to exist, but even this weakened approach does not guarantee local correctness as for arbitrary β the local correctness of the S_i is not carried over to the operator FITA.

One can force local correctness if for example the following constraints are put on the S_i and β:

$$\forall i \forall y : \beta(0, ..., 0, G_i(y), 0, ..., 0) = G_i(y) \qquad \text{and}$$
$$\forall i : S_i \circ F_j = \emptyset \text{ for } j \neq i$$

Example: β is a n-ary extension of a S-norm and all F_i are pairwise disjoint.

10 The Functional Operator FITA with Extensional Rule Interpretation

When using an extensional rule interpretation, π und β can be chosen arbitrarily. This includes the above (weakened) relational approach but in addition all cases for which at least one F_j exists with $S_j \circ F_j \neq G_j$.

In general local correctness is not guaranteed.

11 Conclusions

Among the four alternatives only one approach, the operator FATI with relational rule interpretation, guarantees the correctness of the interpretation. For FITA the relational approach is senseless, maybe the property of local correctness supports the definition of an operator for interpretation a little bit. In all other three cases one has to discuss what it means that a functional operator for the interpretation of a fuzzy rule base does not yield the fuzzy set on the right side of a rule if fed by the fuzzy set of the left side of the rule.

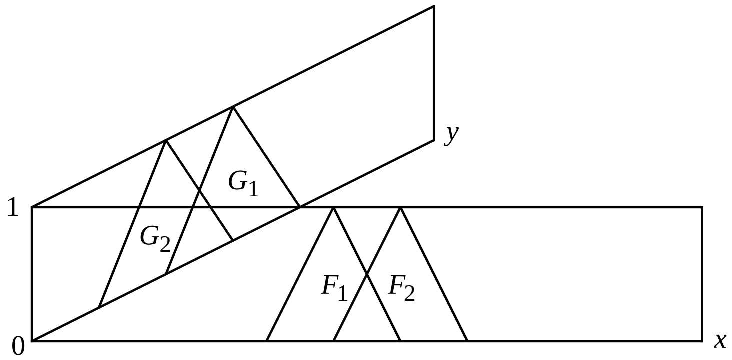
References

[Thiel95] H. THIELE: *On the Calculus of Fuzzy IF-THEN Rules*, in: H.-J. Zimmermann, M. G. Negoita, D. Dascalu (eds): *Real World Applications of Intelligent Technologies*, Editura Academiei Romane, Bucharest, Romania, 1995.

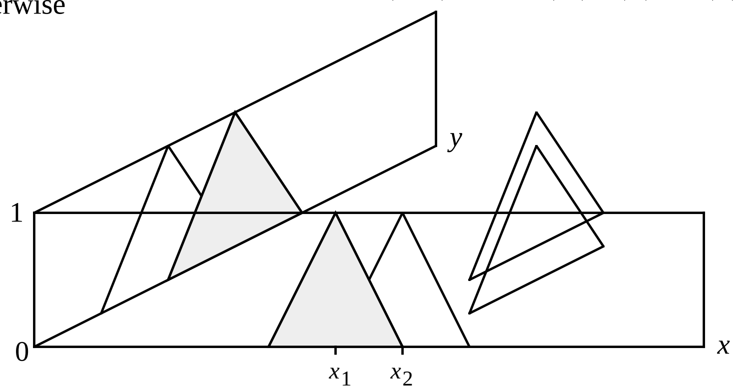
[LRTT98] S. LEHMKE, B. REUSCH, K.-H. TEMME, H. THIELE: *On Interpreting Fuzzy IF-THEN Rule Bases by Concepts of Functional Analysis*, Technical Report CI-19/98, ISSN 1433-3325, Collaborative Research Center 531, University of Dortmund, Germany, February 1998.

Example for FATI with relational rule interpretation

IF F_1 THEN G_1
IF F_2 THEN G_2
$\kappa = \min$
$\alpha = \max$



$$\pi(a,b) = \begin{cases} b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \qquad S(x,y) = \max(\pi(F_1(x), G_1(y)), \pi(F_2(x), G_2(y)))$$



S is a solution of the equation system

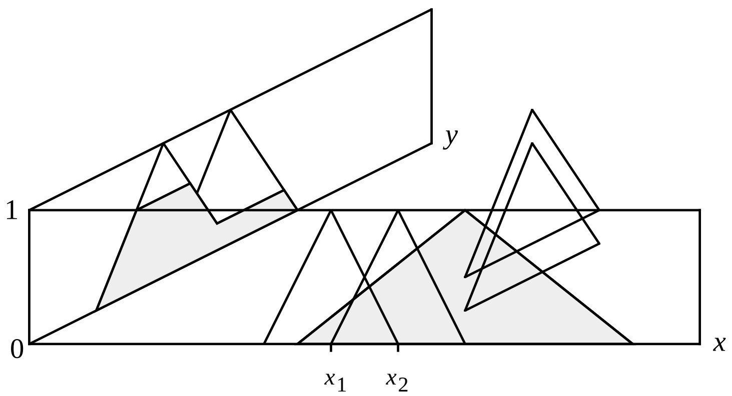
$$S \circ F_1 = G_1$$
$$S \circ F_2 = G_2$$

The FATI operator defined by κ, α, and π works correctly, here shown for the input F_1.

For practical applications the above operator is not very meaningful:

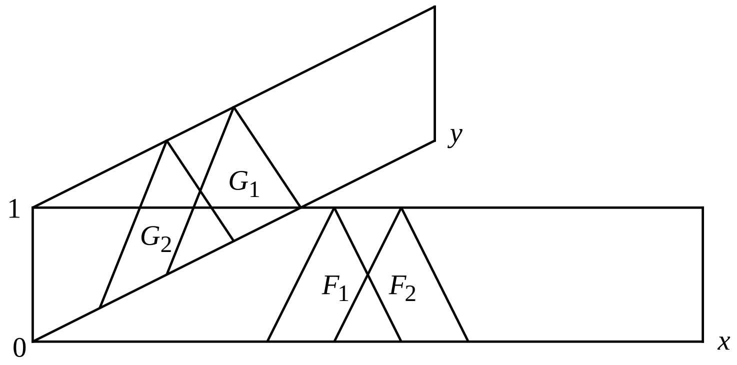
1. The impact of the left sides of the rules depends only on the points x_1 and x_2. For example the operator yields the same result for every other fuzzy set H instead of F_1 as long as $H(x_1) = 1$, $H(x_2) = 0$ and $H(x) < 1$ elsewhere.
2. If for an input set holds either $F(x_1) = 0$ or $F(x_2) = 0$ the operator either generates the empty set or a result which depends only on one of the two rules.

Only if $F(x_1) > 0$ and $F(x_2) > 0$ both right sides of the rules contribute to the result, an example:



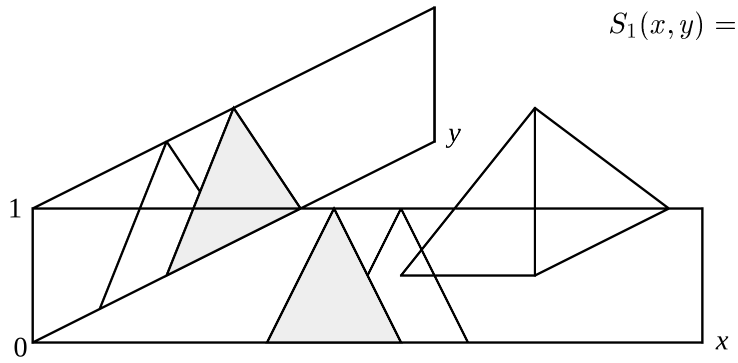
Example for FATI with extensional rule interpretation

IF F_1 THEN G_1
IF F_2 THEN G_2
$\kappa = \min$
$\alpha = \max$



$\pi = \min$

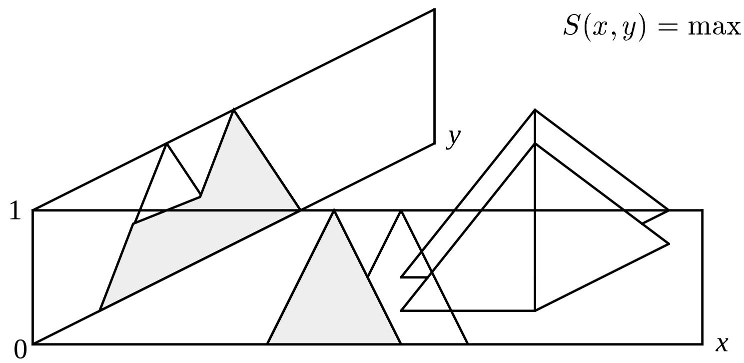
$S_1(x, y) = \min(F_1(x), G_1(y))$



$S_1(x, y) = \min(F_1(x), G_1(y))$ is a solution of $S_1 \circ F_1 = G_1$, hence $\text{FATI}(F_1) = G_1$ (Analog for S_2).
Nevertheless FATI does not work correctly, here shown for F_1:

$\pi = \min$

$S(x, y) = \max(S_1(x, y), S_2(x, y))$



Despite this disadvantage the chosen interpretation is senseful, as for input values out of the interval where the fuzzy sets of the left rule sides overlap, all rules have an impact on the result, which are greater than zero in this range.

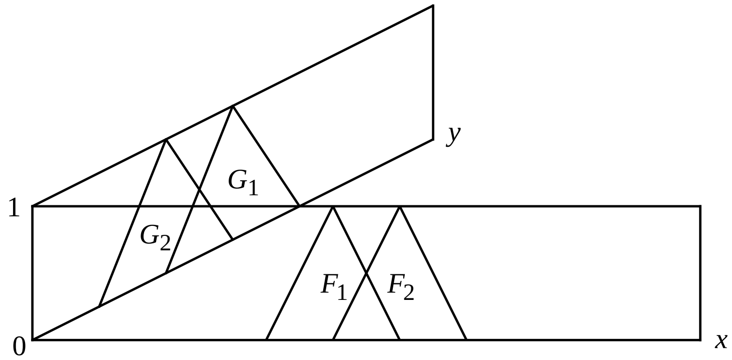
Example for FITA with relational rule interpretation

IF F_1 THEN G_1

IF F_2 THEN G_2

$\kappa = \min$

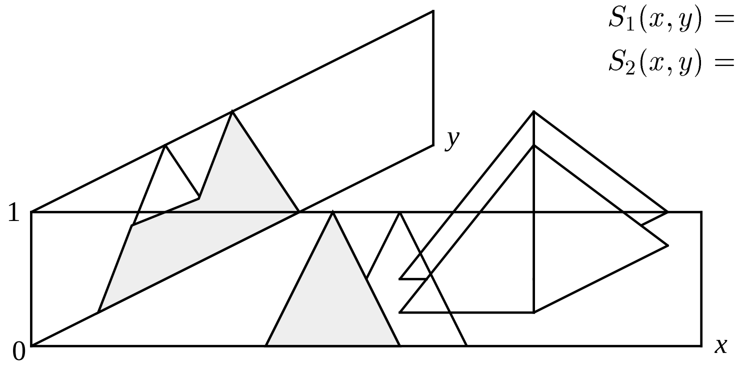
$\beta = \max$



This “standard” interpretation for FITA with $\pi = \min$ works correctly for each single rule, but not for the whole rule base.

$\pi = \min$

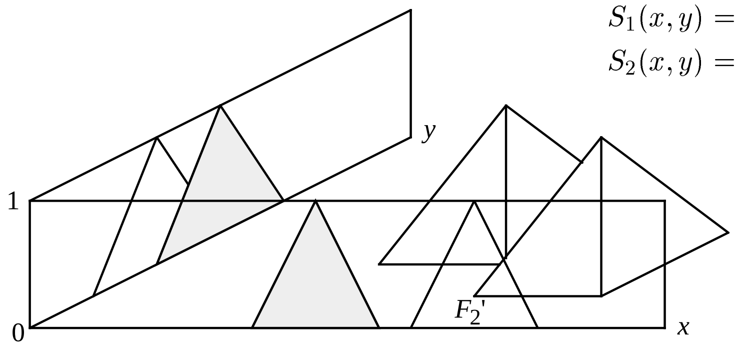
$S_1(x, y) = \min(F_1(x), G_1(y))$
$S_2(x, y) = \min(F_2(x), G_2(y))$



To force correctness for example the above rule base could be changed as follows:

$\pi = \min$

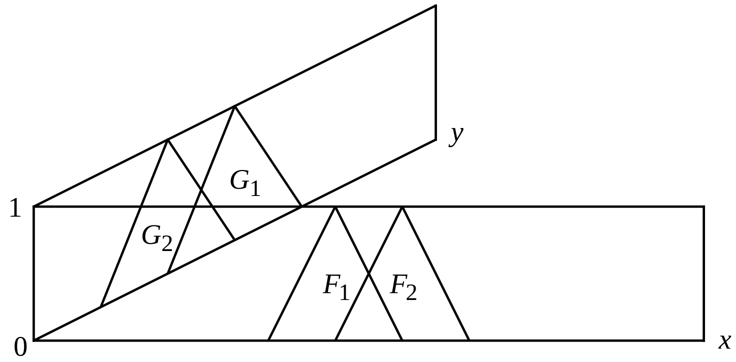
$S_1(x, y) = \min(F_1(x), G_1(y))$
$S_2(x, y) = \min(F_2'(x), G_2(y))$



A relational rule interpretation for FITA makes no sense.

Example for FITA with extensional rule interpretation

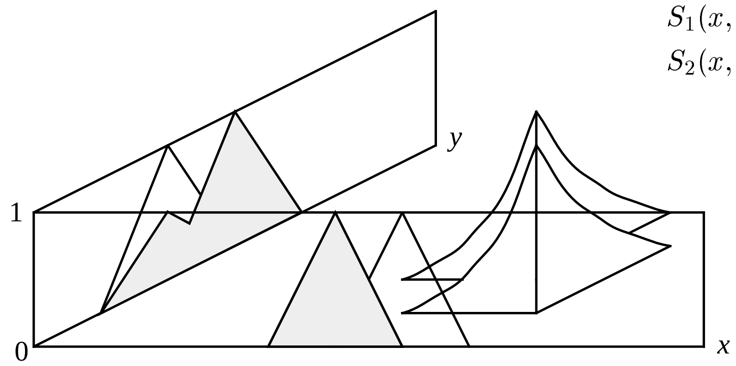
IF F_1 THEN G_1
IF F_2 THEN G_2
$\kappa = \min$
$\beta = \max$



In another frequently used interpretation the product is used for the definition of the relation:

$\pi(a, b) = a * b$

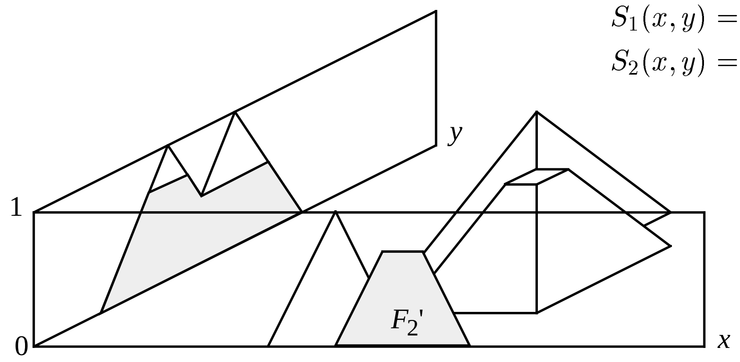
$S_1(x, y) = F_1(x) * G_1(y)$
$S_2(x, y) = F_2(x) * G_2(y)$



Even the local correctness can be invalid for a single rule; in the following example the "height" rule is not fulfilled:

$\pi = \min$

$S_1(x, y) = \min(F_1(x), G_1(y))$
$S_2(x, y) = \min(F_2'(x), G_2(y))$



Although (local) correctness is not given in most of the cases, the extensional rule interpretation for FATI is senseful, as it combines the rules in a way of which the designer had thought of intuitively.