Physics and Philosophy — Issn: 1863-7388 — 2006 — Id: 004

Article

The Phase of a Bose-Einstein Condensate

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Abstract: If two Bose-Einstein condensates are prepared independently and then overlapped, a spatial interference pattern is observed. This prompts the question what determines the phase of the fringe pattern, and whether a condensate has a well-defined value of the phase. This problem has been studied in the literature in detail. The objective of this article is, to present an introduction to the subject and to summarize the discussion for a wider audience.

Keywords: Quantum mechanics, Bose-Einstein condensate, phase

1 Bose-Einstein condensation

Bose-Einstein-condensation is a quantum-statistical effect, which occurs if an ideal gas of indistinguishable bosons is cooled to extremely low temperatures. Under these conditions, the ground state of the system is occupied by a large number of atoms. In the thermodynamic limit, this occupation is macroscopic. To create a Bose-Einstein condensate (BEC), the gas must be cooled to such low temperatures, that the so-called phase-space density $n\lambda_{dB}^3$ exceeds a critical value, which is close to unity. Here, n denotes the spatial density of atoms, $\lambda_{dB} = h/\sqrt{2\pi m k_B T}$ the thermal de-Broglie wavelength, m the mass of an atom, and T the temperature.
Bose-Einstein condensation is closely related to superfluidity in liquid helium and, to some degree, related to superconductivity in solids. Despite this connection, the latter systems are not at all ideal gases. Indeed, the forces between particles are so strong, that a liquid or solid is formed. A quantitative comparison with the ideal-gas theory is therefore difficult.
The first BEC in a dilute gas was created in 1995 (Anderson et al., 1995; Davis et al., 1995; Bradley et al., 1995). Here, the collisions between particles have such a weak effect that a perturbative treatment of the interactions yields a very successful, quantitative description. Many properties known from liquid helium and super-

conductors were reproduced. But now, a quantitative comparison to an ab-initio theory became possible. In addition, many new effects were observed in these systems with methods, which are not applicable in liquid helium or superconductors.

Almost all experiments on BEC of dilute gases use the following two-stage cooling scheme. The first stage uses cooling of atoms with laser light in a so-called magneto-optical trap. This yields a cloud of a cold atomic gas consisting of up to 10^{10} atoms at temperatures around 100 μK and a phase-space density of up to 10^{-6}. For the second stage of the cooling scheme, the cloud is transferred into a magnetic trap and cooled with radio-frequency (rf) induced evaporation. Here, the magnetic trap creates a conservative, harmonic potential of finite height. By lowering the potential height with the rf field, the hotter atoms are allowed to escape from the trap, so that the remaining cloud is colder.

The critical temperature T_C for the phase transition to BEC is typically near 1 μK. At T_C, the ground-state population becomes noticeable. This population grows during further cooling. The atoms in the ground state are referred to as the condensate. The remaining atoms in the excited states form the so-called thermal cloud. Present-day experiments reach a BEC fraction near 100%. Thus, one obtains an almost pure BEC with typically 10^6 atoms and a central density of typically $n = 10^{14}$ cm^{-3}. For further details on the creation of a BEC, the reader is referred to Pethick/Smith (2002); Pitaevskii/Stringari (2003).

2 Condensate wave function

A BEC is a many-body system of indistinguishable particles. An accurate description of the system will therefore typically be formulated in second quantization. For simplicity, the following discussion focuses on the case with a fixed particle number N at $T = 0$, so that all particles occupy the ground state $\chi(\vec{x})$ of the potential. The position representation of the many-body state is the product $\Psi(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N) = \chi(\vec{x}_1)\chi(\vec{x}_2)\ldots\chi(\vec{x}_N)$. In the Fock basis, this corresponds to a single Fock state $|N\rangle$.

A treatment in second quantization is precise but often mathematically cumbersome. Interestingly, many effects can alternatively be described to a good approximation using the condensate wave function $\phi(\vec{x}) = \sqrt{N}\chi(\vec{x})$. This is a wave function in first quantization, but it is normalized to N instead of 1. The condensate wave function describes the long-range order in the system and plays the role of the order parameter in the phase transition to BEC (Pethick/Smith 2002; Pitaevskii/Stringari 2003; Huang 1987, pp. 298–302).

The condensate wave function has a modulus and a phase $\phi(\vec{x}) = \sqrt{n(\vec{x})}\, e^{i\varphi(\vec{x})}$, where the modulus is related to the spatial density $n(\vec{x})$. The phase of the ground state of the harmonic oscillator potential is independent of the position. Hence, the condensate wave function of the BEC in the trap also has a position-independent phase. Interestingly, nothing in the preparation of the BEC prefers a specific value of the phase φ, which prompts the questions what value the phase has and whether a well-defined value of the phase exists at all. These very questions are the subject of the rest of this paper.

3 Measuring the relative phase of two BECs

In order to measure the phase of a condensate wave function, one BEC is not sufficient, because the overall phase of a quantum state vector can principally not be measured. Hence, is it necessary to compare the BEC to a phase reference, such as a second BEC. Both BECs must be prepared independently, and subsequently they must be overlapped to create an interference pattern. Results of such an experiment were published in 1997 by the group of W. Ketterle (Andrews et al., 1997). In this experiment, two BECs are created independently in two approximately harmonic traps with a small spatial separation. Next, the harmonic traps are switched off and both clouds fall freely in the gravitational field. In addition, each BEC expands because of its finite kinetic energy. An effectively repulsive atom-atom interaction additionally drives the expansion. After a sufficient expansion time, the clouds overlap almost completely. At this point, the atomic position distribution is measured.

The phase of each BEC becomes position dependent during the expansion. This is true under very general conditions. The basic physics behind this is that, if a particle emitted from a point source traveled a distance $\vec{x}$ in a given time of flight t, then the particle must have had a momentum of $\vec{p} = m\vec{x}/t$. This means that in a small region around this point $\vec{x}$, the wave function must approximately have the form of a plane wave with this momentum $\vec{p}$. This can be illustrated with the textbook example of a freely expanding Gaussian wave packet (Cohen-Tannoudji/Diu/Laloë 1977. p. 61–66). If the initial wave packet has a position-independent phase, then a position dependence of the phase will naturally emerge during the expansion.

In a simplified model, each BEC can be described as a plane wave. Denoting the wave vectors as $\vec{k}_A$ and $\vec{k}_B$, the wave functions of condensates A and B become $\sqrt{n_A}\exp(i\vec{k}_A\vec{x} + i\varphi_A)$ and $\sqrt{n_B}\exp(i\vec{k}_B\vec{x} + i\varphi_B)$. The phase offsets φ_A and φ_B develop from the initial position-independent phases of the trapped BECs. The actual wave functions have a more difficult position dependence, but the plane wave model captures all the physics, which is relevant for the following discussion. The total wave function is the coherent superposition of the two BEC wave functions, so that the total atomic density is

$$n_A + n_B + 2\sqrt{n_A n_B}\cos[(\vec{k}_A - \vec{k}_B)\vec{x} + \varphi_A - \varphi_B] \quad (1)$$

Corresponding interference fringes were clearly observed in the experiment by the Ketterle group. The position of the minima in the interference pattern is sensitive to the relative phase $\varphi_A - \varphi_B$ of the two initially trapped BECs.

If the experiment is repeated many times, one finds a large contrast of the interference fringes in each repetition of the experimental. But, the value of the relative phase $\varphi_A - \varphi_B$ fluctuates randomly from shot to shot. Hence, when averaging over many shots, one does not obtain any interference fringes. This reflects

the fact that nothing in the preparation of the BECs prefers any specific value of the relative phase.
It should be emphasized that the two BECs must be prepared independently, to observe the overall phase of one BEC. There is a large number of alternative experiments that first create a single BEC, then coherently split it into two or more parts, and finally recombine the parts to observe interference. None of these experiments can address the question of the overall phase of the initial condensate, because they observe only the relative phase accumulated between splitting and recombination. These experiments would also show interference if a thermal cloud above T_C were used, as long as the experiment stays within the coherence length of the thermal cloud. The interference of two independently prepared BECs is conceptually different from the split-and-recombine experiments.
It is interesting to note that no interference would be observed, if the experiment were performed with two independently prepared atomic clouds at a temperature above T_C. In this case, almost all atoms are in a singly-occupied quantum state to begin with. Each pair of occupied states could in principle give rise to a well-defined fringe pattern. But the observation of interference fringes requires the recoding of many detector clicks and each pair of states contributes only two detector clicks. Hence, one has to record many clicks from different pairs of states. Thus, one obtains an incoherent mixture of the corresponding interference patterns. This washes out the fringes completely.
A BEC belongs to a special class of systems where many indistinguishable bosons occupy the same quantum state. Other systems with the same property show analogous interference effects. Examples are the interference of light emitted by two independent lasers as well as the Josephson effect in superconductors.

4 Spontaneous symmetry breaking

The random shot-to-shot fluctuations of the value of the relative phase have an interesting analogy in a simple mechanical system. Consider a classical particle shaped as a sphere, sitting in the center of a Mexican-hat potential. This location of the particle at the local potential maximum is metastable: if the particle is placed precisely at the center, then it will theoretically remain there indefinitely. But a small mechanical perturbation or a slight imperfection in the preparation of the initial conditions will cause the particle to roll away from the potential maximum towards the ring where the potential energy is minimal. All points in this ring have the same energy, that is the ground state of the system is degenerate. Due to friction, the particle will come to rest at some point near this ring, but it is irreproducible and unpredictable at what point that will be. The symmetry of the original problem is broken in each individual repetition of the experiment. But an average over many shots will reflect the symmetry of the original problem. This is called spontaneous symmetry breaking. This concept is widely used in many fields of physics, most prominently in elementary particle physics. A similar effect exists, e.g., in the thermodynamic description of a ferromagnet (Huang 1987, pp. 298–302).

The analogy to a BEC is obvious: nothing in the preparation of the BEC prefers a specific value of the phase of the condensate wave function. Yet, in each experimental shot, a well-defined value of the phase may be present due to spontaneous symmetry breaking. The value obtained in an individual experimental shot is utterly unpredictable. When creating the BEC with evaporative cooling, one might imagine that the first few atoms, which fall into the ground state, happen to have some value of the phase and that all the following atoms enter the already occupied ground state in an induced process, which leaves the value of the phase of the ground-state population almost unchanged. Note that the use of the condensate wave function relies on spontaneous symmetry breaking.

5 Interference without an initial value of the relative phase

The concept of spontaneous symmetry breaking can explain all observations of the experiment reported in Andrews et al. (1997). Furthermore, spontaneous symmetry breaking assumes that prior to the measurement, the two BECs have a well-defined value of the relative phase for each experimental shot. This prompts the question whether the assumption of such a well-defined value of the phase is necessary to describe the experimental results. The following discussion will show that the answer is, “No”.

Interestingly, one can show that an interference pattern is also obtained if the initial state has no well-defined value of the relative phase (Javanainen/Yoo, 1996; Naraschewski et al., 1996; Cirac et al., 1996; Castin/Dalibard, 1997). An interesting example is the hypothetical case, where each BEC is initially prepared in a Fock state. Here, the concept of spontaneous symmetry breaking is not applicable, because the number-phase uncertainty relation implies that a Fock state cannot have a well-defined value of the phase. More precisely, if a measurement is performed to determine the phase of a Fock state, then the probability distribution for the resulting value of the phase is flat, i.e. each possible value of the phase occurs equally likely. Nevertheless, numerical calculations show that if two BECs in Fock states are overlapped, an interference pattern will be recorded.

An intuitive understanding of this can be gained by considering an atom detector, which produces detection clicks with good spatial and temporal resolution. The absence of a well-defined value of the initial phase δ between the two BECs is equivalent to saying that the probability distribution $p(\delta)$ for having a relative phase δ is equally distributed between 0 and 2π. As a consequence, the probability distribution $p(x_1)$ for the position x_1 of the earliest detector click is also flat. In other words, $p(\delta) = const$ and $p(x_1) = const$.

The central idea is that the first detector click leads to a quantum state reduction. The quantum state before the first click is assumed to be the tensor product of the two Fock states $|N_A\rangle \otimes |N_B\rangle$. Now, the atom detected with the first click might stem from either one of the BECs. Therefore, the state vector for the atoms remaining after the first click is a coherent superposition of the two possibilities that the first detected atom came from BEC A or B

$$c_1 \, |N_A - 1\rangle \otimes |N_B\rangle + c_2 \, |N_A\rangle \otimes |N_B - 1\rangle \tag{2}$$

The relative phase between the coefficients c_1 and c_2 depends on x_1. After the second click, the state is a superposition of $|N_A-2\rangle \otimes |N_B\rangle$ and $|N_A-1\rangle \otimes |N_B-1\rangle$ and $|N_A\rangle \otimes |N_B-2\rangle$. The relative phases in the superposition depend on x_1 and x_2. Therefore, the number uncertainty of each individual BEC grows with the number of detected atoms.

This build-up of a number uncertainty is accompanied by the build-up of a well-defined value of the relative phase, for the following reason: The first click yields some information about the relative phase according to Bayes' theorem

$$p(\delta|x_1) = \frac{p(\delta)}{p(x_1)} \, p(x_1|\delta) \tag{3}$$

Here, $p(x_1|\delta)$ is the conditional probability that the click occurs at position x_1, given that the relative phase has the value δ. One can consider the above example with two interfering plane waves, with $\delta = \varphi_A - \varphi_B$ and $\vec{k} = \vec{k}_A - \vec{k}_B$. One can assume that $n_A = n_B$ and that $\vec{k}$ points along the x axis. From Eq. (1), one obtains $p(x_1|\delta) = const \times [1 + \cos(kx_1 + \delta)]$. As noted earlier, $p(\delta) = const$ and $p(x_1) = const$. Bayes' theorem thus yields the conditional probability that the relative phase of the remaining two BECs takes on the value δ, after the first click is recorded at x_1

$$p(\delta|x_1) = const \times [1 + \cos(kx_1 + \delta)] \tag{4}$$

Hence, the probability for the position of the second click x_2 is no longer equally distributed. Indeed, x_2 is most likely near x_1 (modulo $2\pi/k$). The second click again yields information about the value of the relative phase, thus changing the probability for the following detection, etc. During the first ten, or so, detection events, the initially flat probability distribution $p(\delta)$ evolves into a distribution with almost all probability in one narrow peak. In other words, a rather well-defined value of the relative phase builds up. The subsequent clicks do not change this value much any more. This is illustrated in Fig. 2 of Cirac et al. (1996).

Here, the particular value of the relative phase, which is found in one experimental shot, has a clear origin. It is determined by the quantum state reduction resulting from the first few detection events. This has nothing to do with imperfections in the preparation of the initial state.

In conclusion, the presence of interference fringes in the experiment cannot answer the question about the existence of a well-defined value of the relative phase between the two BECs prior to the measurement.

6 Atom-number shot noise

The previous section discussed the case of Fock states, where it is guaranteed that initially there is no well-defined value of the relative phase. Present-day experiments do not deal with BECs in a Fock state. Instead, the repeated production of BECs typically produces at least shot noise in the atom number (and in most cases even broader atom-number distributions). That is, the atom number in BEC A can be described with a Poisson distribution

$$p_A(N_A) = \frac{\mu^{N_A}}{N_A!} e^{-\mu} \quad \text{with} \quad \mu = \langle N_A \rangle \tag{5}$$

There is an analogous distribution for BEC B. The density matrix of the two BECs prior to the measurement is then

$$\rho = \sum_{N_A} p_A(N_A) |N_A\rangle\langle N_A| \otimes \sum_{N_B} p_B(N_B) |N_B\rangle\langle N_B| \tag{6}$$

This Fock-state picture with shot noise has no well-defined value of the relative phase.
Let us now compare this density matrix to the one for the model with spontaneous symmetry breaking. In second quantization, a state with spontaneously broken symmetry is represented by a Glauber state $|\alpha\rangle$. This is also called a coherent state and can be expanded in the Fock-state basis as

$$|\alpha_A\rangle = e^{-|\alpha_A|^2/2} \sum_{N_A} \frac{\alpha_A^{N_A}}{\sqrt{N_A!}} |N_A\rangle \tag{7}$$

where $\alpha_A = \sqrt{\langle N_A \rangle}\, e^{i\varphi_A}$. There is an analogous state for BEC B. With spontaneous symmetry breaking, φ_A and φ_B are assumed to be fixed for each individual shot, but they fluctuate randomly from shot to shot. The density matrix must therefore be averaged over a flat distribution of φ_A and φ_B

$$\rho = \int_0^{2\pi} \frac{d\varphi_A}{2\pi} |\alpha_A\rangle\langle\alpha_A| \otimes \int_0^{2\pi} \frac{d\varphi_B}{2\pi} |\alpha_B\rangle\langle\alpha_B| \tag{8}$$

One can show that this density matrix is mathematically identical to the above density matrix, Eq. (6), of the Fock-state picture with shot noise (Cirac et al., 1996; Castin/Dalibard, 1997). Therefore, all predictions for any observables are identical with the two models. In particular, both models predict the same interference pattern. Consequently, no present-day experiment can distinguish between the two models.
If future experiments should manage to reproducibly produce BECs in a single Fock states, then a description with an well-defined value of the initial relative phase would be impossible because of the number-phase uncertainty relation. While the realization of such experiments is not principally forbidden by fundamental laws of physics, a substantial increase in experimental control would be required. Such a development would certainly take a while.
To summarize, the observed interference between two BECs can be described easily in first quantization using the condensate wave function, which is based on spontaneous symmetry breaking. With this approach, a well-defined value of the phase between the two BECs is assumed to exist prior to the measurement. This value of the relative phase fluctuates randomly from shot to shot. But

the assumption of this well-defined value of the relative phase is not necessary to explain the existence of interference fringes. A model in second quantization requires considerably more computational effort, but it also produces interference fringes, even if there is no well-defined value of the relative phase before the measurement. Such a value gradually builds up during the first few detection events. According to this view, the initial value of the phase of the BEC is only "a convenient fiction" (Mølmer, 1997). In the end, both models are equally successful in describing present-day experiments.

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