Gluing copulas

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Abstract

We present a new way of constructing bivariate copulas, by rescaling and gluing two (or more) copulas. Examples illustrate how this construction can be applied to build complicated copulas from simple ones.

1 Introduction

Let $I = [0, 1]$ be the closed unit interval and $I^2 = [0, 1] \times [0, 1]$ the closed unit square. A (two-dimensional) copula is a function $C : I^2 \to I$ satisfying the following conditions:

1. $C(x, 0) = C(0, y) = 0$ for all $x, y \in I$
2. $C(x, 1) = x$ and $C(1, y) = y$ for all $x, y \in I$
3. C is 2-increasing, i.e., $C(x_2, y_2)-C(x_2, y_1)-C(x_1, y_2)+C(x_1, y_1) \geq 0$ for all rectangles $[x_1, x_2] \times [y_1, y_2] \subset I^2$.

These conditions imply further key properties of copulas. In particular, a copula is Lipschitz continuous and increasing in each argument; therefore, its partial derivatives exist a.e. on I^2. There are three distinguished copulas, namely

$$\begin{aligned} C^-(x, y) &= \max(x + y - 1, 0) \\ C^+(x, y) &= \min(x, y) \\ P(x, y) &= xy. \end{aligned}$$

C^+ and C^- are called the Fréchet-Hoeffding upper and lower bound, respectively, since for any copula C and any $(x, y) \in I^2$ we have the estimates

$$C^-(x, y) \leq C(x, y) \leq C^+(x, y). \tag{1}$$

The true importance of copulas to probability theory stems from the well known Sklar theorem [4, 3] which states that, when a joint distribution function has continuous marginal distribution functions, it can be decomposed into the margins and a unique copula.

Theorem 1.1 (Sklar's theorem)**.** *For every two-dimensional distribution function H with marginal distribution functions F, G there exists a copula C such that*

$$H(x, y) = C(F(x), G(y)).$$

Moreover, if F and G are continuous then C is unique.

Conversely, given any copula C and distribution functions F, G the above equation defines a two-dimensional distribution function H with marginal distribution functions F, G.

In view of this theorem, a rich collection of copulas yields an equally rich collection of bivariate distribution functions with arbitrary margins, which proves useful in modeling and simulation.

There are several ways of constructing copulas. Among them are geometric methods (e.g., ordinal sums, shuffles of min, or copulas with prescribed diagonal sections), algebraic methods (e.g., a copula transformation), and methods based on generators, leading to the large class of Archimedean copulas (including the well known Frank and Gumbel families); for details we refer to [3]. For less known constructions see, e.g., [1, 2].

In this paper, we present a new method for constructing copulas, the so-called gluing construction. In its simplest form, it proceeds as follows. Given two copulas C_1, C_2 and a number $\theta \in (0, 1)$, the graphs of C_1 and C_2 are rescaled and pasted into the rectangle $[0, \theta] \times I$ and $[\theta, 1] \times I$, respectively, i.e., they are glued together at the vertical $\{\theta\} \times I$.

This simple gluing construction can be generalized in various directions. First of all, it is possible to glue together not just two but actually countably many copulas at once. Secondly, one can also glue copulas vertically along a horizontal segment $I \times \{\theta\}$. Finally, combining both leads to the gluing method in its most general form.

2 The gluing method for two copulas

Consider two copulas C_1, C_2. For any given parameter $\theta \in (0,1)$, consider the partition $I = [0,\theta] \cup [\theta, 1]$ and set

$$(C_1 \underset{X}{\sqcup} C_2)(x,y) = \begin{cases} \theta C_1(\frac{x}{\theta}, y) & \text{if } 0 \le x \le \theta \\ (1-\theta) C_2\big(\frac{x-\theta}{1-\theta}, y\big) + \theta y & \text{if } \theta \le x \le 1 \end{cases} \tag{2}$$

Thus, $C_1 \underset{X}{\sqcup} C_2$ is the result of gluing C_1 and C_2 horizontally along the x-axis. We claim that it is indeed a copula.

Theorem 2.1. *For any two copulas C_1, C_2 and any $\theta \in (0,1)$, the function $C_1 \underset{X}{\sqcup} C_2$ is again a copula.*

Proof. It follows immediately from $C_i(0,y) = C_i(x,0) = 0$ that

$$(C_1 \underset{X}{\sqcup} C_2)(0,y) = (C_1 \underset{X}{\sqcup} C_2)(x,0) = 0$$

for all $x, y \in I$. Moreover, it is also clear from the construction that

$$(C_1 \underset{X}{\sqcup} C_2)(1,y) = y \qquad \text{and} \qquad (C_1 \underset{X}{\sqcup} C_2)(x,1) = x$$

for every $x, y \in I$, because the corresponding properties hold for C_1 and C_2.

Hence, the only condition to be checked is that $C_1 \underset{X}{\sqcup} C_2$ is 2-increasing. Abbreviating $C = C_1 \underset{X}{\sqcup} C_2$, we have to show that

$$C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \ge 0 \tag{3}$$

for all rectangles $R = [x_1, x_2] \times [y_1, y_2] \subset I^2$. For this, we distinguish two cases.

For the first case, we assume that $R \subset [0,\theta] \times I$ or $R \subset [\theta, 1] \times I$, i.e., $\theta \notin (x_1, x_2)$. Then (3) requires that

$$\theta \cdot \Big[C_1(\frac{x_2}{\theta}, y_2) - C_1(\frac{x_2}{\theta}, y_1) - C_1(\frac{x_1}{\theta}, y_2) + C_1(\frac{x_1}{\theta}, y_1) \Big] \ge 0$$

respectively

$$\begin{aligned} (1-\theta) \cdot \Big[C_2(\frac{x_2-\theta}{1-\theta}, y_2) - C_2(\frac{x_2-\theta}{1-\theta}, y_1) - C_2(\frac{x_1-\theta}{1-\theta}, y_2) \\ + C_2(\frac{x_1-\theta}{1-\theta}, y_1) \Big] + \theta \cdot (y_2 - y_1 - y_2 + y_1) \ge 0 \end{aligned}$$

Since the last term in the second inequality adds up to zero, each inequality follows from the fact that C_1, respectively C_2, is 2-increasing.

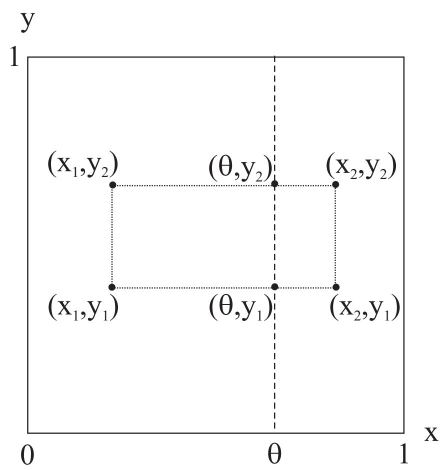


Figure 1: For the proof that $C_1 \underset{X}{\sqcup} C_2$ is 2-increasing

In the second case, where $\theta \in (x_1, x_2)$, we introduce the two auxiliary points (θ, y_1) and (θ, y_2) (see Figure 1) and observe that (3) would follow from the two inequalities

$$\begin{aligned} C(x_2, y_2) - C(x_2, y_1) - C(\theta, y_2) + C(\theta, y_1) &\geq 0 \\ C(\theta, y_2) - C(\theta, y_1) - C(x_1, y_2) + C(x_1, y_1) &\geq 0. \end{aligned}$$

Each single inequality can be treated as in the first case above and, as before, follows from the fact that C_1 and C_2 are 2-increasing. $\square$

Of course, the gluing construction can be also done with respect to the second coordinate which leads to vertical gluing. Namely, given two copulas C_1, C_2 and some $\theta \in (0, 1)$, we define

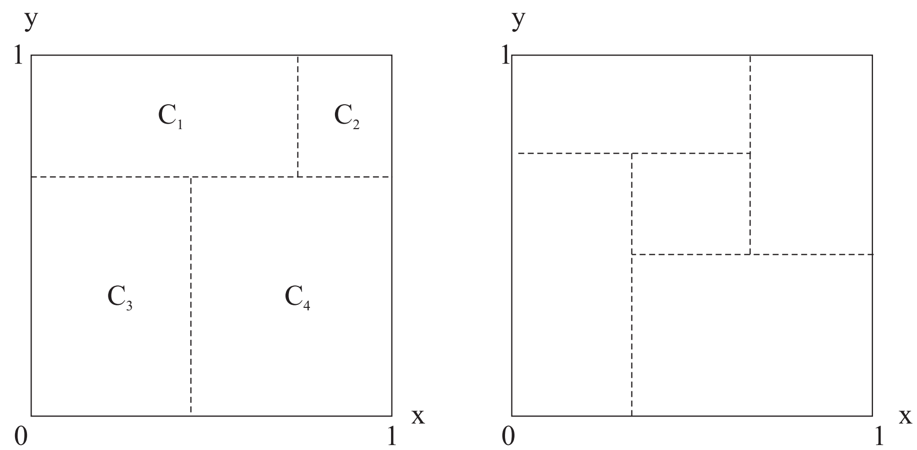
$$(C_1 \underset{Y}{\sqcup} C_2)(x, y) = \begin{cases} \theta C_1(x, \frac{y}{\theta}) & \text{if } 0 \leq y \leq \theta \\ (1-\theta) C_2\big(x, \frac{y-\theta}{1-\theta}\big) + \theta x & \text{if } \theta \leq y \leq 1 \end{cases}$$

The same calculations as above show that $C_1 \underset{Y}{\sqcup} C_2$ is again a copula.

3 Generalizations

The gluing method described above can be generalized to infinite partitions. Let $\{J_k\}_{k \in \mathbb{N}}$ be a countable set of closed intervals $J_k = [a_k, b_k] \subset I$ with pairwise disjoint interior such that $\cup_k J_k = I$. There are two possible cases:

1. $a_k < b_k$ for all $k \in \mathbb{N}$, i.e., all intervals J_k are nondegenerate



(a) Gluing four copulas

(b) Not realizable by consecutive gluing

Figure 2: Compositions of horizontal and vertical gluing

2. There are degenerate intervals $J_k = [a_k, b_k]$ with $a_k = b_k$.

Note that the second case can happen, as in the partition

$$I = [0,0] \cup \bigcup_{k \geq 1} \Big[\frac{1}{k+1}, \frac{1}{k}\Big].$$

Let $\{C_k\}$ be a family of copulas $C_k : I^2 \to I$ with the same indexing as $\{J_k\}$. Then we define the function $C : I^2 \to I$ by

$$C(x, y) = (b_k - a_k) C_k\Big(\frac{x - a_k}{b_k - a_k}, y\Big) + a_k y$$

if $x \in [a_k, b_k]$ with $b_k - a_k > 0$, and extend it as a continuous function to degenerate intervals with $a_k = b_k$. Then the same arguments as above show that C is a bivariate copula. Note that the case of a finite partition can also be realized by sequentially applying the $\underset{X}{\sqcup}$-operation.

Finally, we may combine compositions of horizontal and vertical gluing. For instance, given four copulas $C_1, \ldots, C_4$, the copula

$$(C_3 \underset{X}{\sqcup} C_4) \underset{Y}{\sqcup} (C_1 \underset{X}{\sqcup} C_2)$$

might be represented by the partition of I^2 outlined in Figure 2. It is worth mentioning, however, that not every partition of I^2 into rectangles can be realized by consecutive gluing; an example of such a configuration is shown in Figure 2.

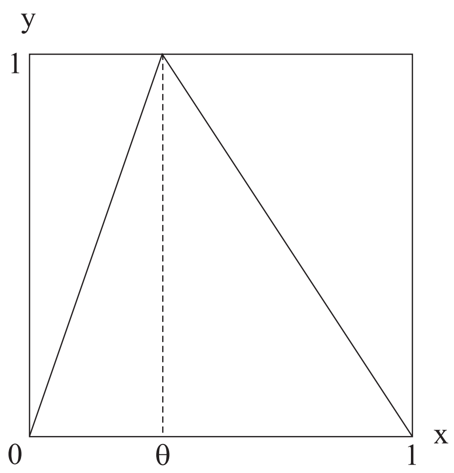


Figure 3: The support of the singular copula C in Example 4.1

4 Examples

We illustrate the gluing construction with some examples. A copula C is called singular if $\frac{\partial^2 C}{\partial x \partial y}$ vanishes almost everywhere in I^2; in this case, the support of C has Lebesgue measure zero in I^2. We refer the reader to [3] for more details.

Example 4.1. Let $\theta \in (0,1)$, and suppose that probability mass θ is uniformly distributed along the line segment joining $(0,0)$ and $(\theta,1)$, and probability mass $1-\theta$ is uniformly distributed along the segment between $(\theta,1)$ and $(1,0)$. Consider the resulting singular copula C whose support consists of these two line segments; see Figure 3. It follows (see [3, Example 3.3]) that

$$C(x,y) = \begin{cases} x & \text{if } x \leq \theta y \\ \theta y & \text{if } \theta y < x < 1-(1-\theta)y \\ x+y-1 & \text{if } 1-(1-\theta)y \leq x. \end{cases}$$

This copula is a standard example of a singular copula. In terms of gluing, C can be written as

$$C = C^{+} \underset{X}{\sqcup} C^{-}$$

where $C^{+}(x,y) = \min(x,y)$ and $C^{-}(x,y) = \max(x+y-1,0)$ is the Fréchet-Hoeffding upper and lower bound, respectively.

Example 4.2. Clearly, using the gluing construction one can combine different dependence relations on different domains. In particular, gluing together an arbitrary copula C and the independence copula $P(x,y) = xy$ yields $P \underset{X}{\sqcup} C$, respectively, $C \underset{X}{\sqcup} P$. Note that $P \underset{X}{\sqcup} C = P$ on $[0,\theta] \times I$, and $C \underset{X}{\sqcup} P = P$ on $[\theta,1] \times I$, respectively; in particular,

$$P \underset{X}{\sqcup} P = P.$$

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