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Abstract

We give two proofs of the two-dimensional Borsuk-Ulam theorem. One is completely elementary and does not use homology theory or the mapping degree, while the second one makes use of the recent theory of symplectic quasi-states.

1 Introduction

The two-dimensional Borsuk-Ulam theorem states that a continuous vector field on $\mathbb{S}^2$ takes the same values on at least one pair of antipodal points.

Theorem 1.1 (Borsuk-Ulam for $\mathbb{S}^2$)**.** *Let $\sigma : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map $\sigma(x) = -x$. Then, for every continuous mapping $V : \mathbb{S}^2 \to \mathbb{R}^2$, there exists at least one point $x \in \mathbb{S}^2$ such that $V(x) = V(\sigma(x))$.*

There are several proofs of this result (as well as higher-dimensional generalizations), most of them using some kind of degree argument [Dod, DG, GP] or combinatorial methods [Fan, Had, Tuc].

The aim of this note is to present two proofs in two dimensions which rely on geometric rather than topological arguments. The fundamental idea is the same for both of them: the pair of antipodal points is characterized as the intersection of the zero sets of two real-valued functions on $\mathbb{S}^2$. This intersection result on $\mathbb{S}^2$ can then be established by either elementary arguments (see Section 3) or the recent theory of symplectic quasi-states (see Section 4).

For the historical background and a thorough mathematical discussion of the Borsuk-Ulam theorem in arbitrary dimensions, we refer to [DG, Sect. 5].

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2 Reduction to an intersection result

We represent the given vector field V as $V = (V_1, V_2)$ with two continuous functions $V_i : \mathbb{S}^2 \to \mathbb{R}$. Then the condition $V(x) = V(\sigma(x))$ is equivalent to $f_1(x) = f_2(x) = 0$ for the continuous functions $f_i : \mathbb{S}^2 \to \mathbb{R}$ defined by $f_i(x) = V_i(x) - V_i(\sigma(x))$. In other words, we are looking for intersection points

$$x \in f_1^{-1}(0) \cap f_2^{-1}(0)$$

with each f_i satisfying the relation

$$f \circ \sigma = -f \tag{1}$$

because σ is an involution, i.e., $\sigma \circ \sigma = \mathrm{id}$.

Theorem 2.1 (Intersection result)**.** *Let $f_1, f_2 : \mathbb{S}^2 \to \mathbb{R}$ be smooth functions, both satisfying* (1) *and having 0 as a regular value. Then $f_1^{-1}(0) \cap f_2^{-1}(0)$ contains at least one pair of antipodal points.*

As a corollary, we immediately obtain Theorem 1.1 for smooth vector fields V. Thus, it remains to prove Theorem 1.1 in the continuous setting, taking its smooth version for granted.

For this, we approximate the continuous functions f_1 and f_2 by smooth functions $f_{1,n}$ and $f_{2,n}$, respectively, such that each $f_{i,n}$ possesses 0 as a regular value and satisfies

$$\|f_i - f_{i,n}\|_{C^0} \leq \frac{1}{n} \tag{2}$$

for every $n \in \mathbb{N}$; see [Hir]. Now define

$$\tilde{f}_{i,n}(x) := \frac{f_{i,n}(x) - f_{i,n}(\sigma(x))}{2}.$$

In view of (2) we conclude that

$$\|f_i - \tilde{f}_{i,n}\|_{C^0} = \frac{1}{2}\,\|f_i - f_{i,n} - (f_i \circ \sigma - f_{i,n} \circ \sigma)\|_{C^0} \leq \frac{1}{n}.$$

Applying Theorem 2.1 to $\tilde{f}_{i,n}$, we obtain sequences of antipodal points $x_n, \sigma(x_n)$ such that

$$\tilde{f}_{i,n}(x_n) = \tilde{f}_{i,n}(\sigma(x_n)) = 0$$

for all n. By the compactness of $\mathbb{S}^2$ we may assume that $x_n \to x$ so that the continuity of f_i implies $f_i(x) = f_i(\sigma(x)) = 0$ for $i = 1, 2$. This finishes the proof of Theorem 1.1.

It remains to prove the intersection result given in Theorem 2.1. For this, let ω be the standard area form on $\mathbb{S}^2$, normalized such that

$$\int_{\mathbb{S}^2} \omega = 1.$$

According to Jordan's Curve Theorem, every embedded circle on $\mathbb{S}^2$ divides $\mathbb{S}^2$ into two domains. We call an embedded circle an *equator* if these two domains have equal area 1/2.

The following theorem states that the zero set of a function satisfying (1) must contain an equator. Since two equators on $\mathbb{S}^2$ intersect in at least two points, Theorem 2.1 is an immediate consequence of Theorem 2.2. Note, however, that the two intersection points can be antipodal points themselves, so the intersection of the two equators may consist of just one pair of antipodal points (e.g., when the two equators are great circles).

Theorem 2.2 (Existence of an equator)**.** *Let $f : \mathbb{S}^2 \to \mathbb{R}$ be a smooth function, satisfying* (1) *and having 0 as a regular value. Then $f^{-1}(0)$ contains an equator.*

In the remaining part of the paper, we provide two proofs of Theorem 2.2, one elementary and one based on symplectic quasi-states.

3 An elementary proof of Theorem 2.2

Since 0 is a regular value for f and $\mathbb{S}^2$ is compact, the zero set $f^{-1}(0)$ is the union of finitely many, pairwise disjoint, embedded circles $C_j \subset \mathbb{S}^2$; see, e.g., [GP]. Each of the circles C_j divides $\mathbb{S}^2$ into two domains $D_j^\pm$. If some C_j is an equator, the theorem is proven.

Therefore, we assume that none of the C_j is an equator. Then, for every j, the two domains $D_j^\pm$ have different areas, and we write D_j^- for the smaller one. Let us denote by C that circle C_j whose domain D_j^- has maximal area, and call $D^\pm$ the corresponding domains on $\mathbb{S}^2$. Note that

$$\int_{D^-} \omega < \frac{1}{2} < \int_{D^+} \omega. \tag{3}$$

Because of (1), $f^{-1}(0)$ is invariant under σ, so $\sigma(C)$ is also one of the circles C_j, and there are only two cases: either $\sigma(C) = C$ or $\sigma(C) \cap C = \emptyset$.

If $\sigma(C) = C$ then, since σ preserves the non-oriented area, (3) implies that $\sigma(D^\pm) = D^\pm$, so σ preserves orientation which is a contradiction. Therefore,

$$\sigma(C) \cap C = \emptyset.$$

Then C and $\sigma(C)$ bound some annulus-like region $A \subset \mathbb{S}^2$; see Figure 1. We claim that

$$\sigma(A) = A.$$

Consistently neglecting boundaries, we see that the domain D^+ consists of two parts, the annulus A and another domain D. Since $C = \partial D^-$ and $\sigma(C) = \partial D$, we have either $\sigma(D^-) = D$ or $\sigma(D^-) = D^- \cup A$, the latter being impossible since σ preserves the non-oriented area. Hence $\sigma(D) = \sigma(\sigma(D^-)) = D^-$ which implies $\sigma(A) = A$.

Finally, we will see that

$$f|_A = 0, \tag{4}$$

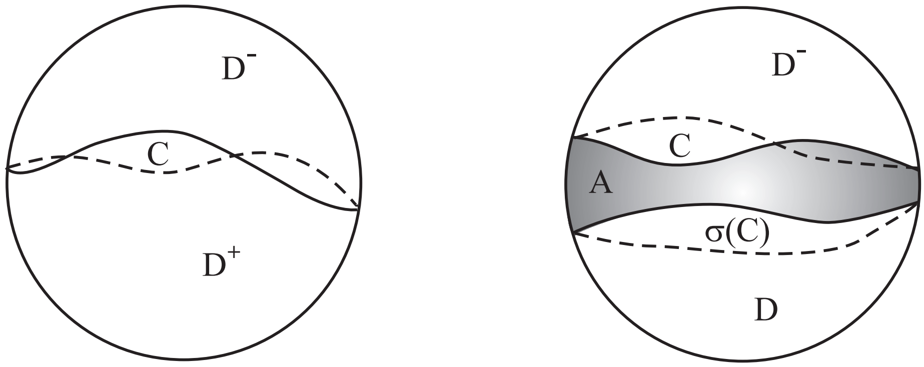

Figure 1: $\sigma(C) = C$ [left] and $\sigma(C) \cap C = \emptyset$ [right]

which, of course, is impossible because 0 is a regular value of f.

In order to show (4) we observe that, since D^- has maximal area, A cannot contain any C_j in its interior that is non-contractible in A. Now, if $A = \sigma(A)$ contains a circle C_j which is contractible in A, and hence bounds an open disk $B \subset A$, then it also contains $\sigma(C_j)$ which bounds the open disk $\sigma(B) \subset A$. In view of (1), the function f has different signs on B and $\sigma(B)$. Moreover, there exists a path $\gamma : [0, 1] \to A$ with $\gamma(0) \in B$ and $\gamma(1) \in \sigma(B)$ such that there are $0 < a < b < 1$ with $f(\gamma(t)) = 0$ if, and only if, $t \in \{a, b\}$. Considering the sign of f along γ one sees that this contradicts the fact that 0 is a regular value of f. Consequently, A does not contain any C_j in its interior, so f has a fixed sign on $A = \sigma(A)$ which, in view of (1), implies (4).

Therefore, our starting assumption that none of the C_j is an equator leads to a contradiction, proving Theorem 2.2.

4 A symplectic proof of Theorem 2.2

Since an area form in two dimensions is the same as a symplectic form, the above proof of Theorem 2.2 suggests a symplectic approach in which Theorem 2.1 would become a Lagrangian intersection result. Surprisingly, such an approach exists. However, it is not elementary anymore—it is based on the recent notion of symplectic quasi-states which was introduced by Entov and Polterovich in [EP1]. We refer the interested reader to [EP1, EP2, EPZ] for a detailed description of quasi-states in symplectic topology.

A *symplectic quasi-state* on a closed connected symplectic manifold (M, ω) is a functional

$$\zeta : C^0(M, \mathbb{R}) \to \mathbb{R}$$

satisfying the following properties:

1. $\zeta(1) = 1$
2. $\zeta(F) \leq \zeta(G)$ if $F \leq G$

3. $\zeta(aF+bG) = a\zeta(F)+b\zeta(G)$ for all $a, b \in \mathbb{R}$ and all smooth functions F, G with vanishing Poisson bracket.

It is shown in [EP2, EP1] that for $(M, \omega) = (\mathbb{S}^2, \omega)$ with $\int_{\mathbb{S}^2} \omega = 1$, a symplectic quasi-state ζ can be constructed as follows. For any given Morse function $F : \mathbb{S}^2 \to \mathbb{R}$ with distinct critical values, there is a unique connected component C_F of the level lines of F such that the ω-area of every connected component of its complement is at most $1/2$. Then set

$$\zeta(F) = F(C_F)$$

and extend it to all $F \in C^0(\mathbb{S}^2, \mathbb{R})$ by continuity with respect to the C^0-norm. Note that ζ is invariant under any diffeomorphism of $\mathbb{S}^2$ which preserves the non-oriented area, in particular,

$$\zeta(F) = \zeta(F \circ \sigma). \tag{5}$$

Now assume that f is a smooth function on $\mathbb{S}^2$, satisfying (1) and having 0 as a regular value. Then (5), together with the linearity of ζ, implies that

$$\zeta(f) = \zeta(f \circ \sigma) = \zeta(-f) = -\zeta(f)$$

so that $\zeta(f) = 0$. But, by definition of ζ, this means that $f^{-1}(0)$ contains an equator C_f, and Theorem 2.2 is proven.

5 Concluding remarks

If $i \neq \mathrm{id}$ is any involution on $\mathbb{S}^2$ then a classical result—apparently due to Brouwer [Bro], see also [Ker]—states that i is topologically conjugate to a Euclidean rotation about an angle of π, the reflection along an equatorial plane, or the antipodal map σ.

Theorem 5.1. *Let $i : \mathbb{S}^2 \to \mathbb{S}^2$ be any involution. Then, for every continuous mapping $V : \mathbb{S}^2 \to \mathbb{R}^2$, there exists at least one point $x \in \mathbb{S}^2$ such that $V(x) = V(i(x))$.*

Proof. If i has fixed points, the theorem is trivially true. If i is fixed point free it is topologically conjugate to σ, and applying Theorem 1.1 yields the result. □

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