

Homogenization of Maxwell's equations with split rings

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Abstract: We analyze the time harmonic Maxwell's equations in a complex geometry. The scatterer $\Omega \subset \mathbb{R}^3$ contains a periodic pattern of small wire structures of high conductivity, the single element has the shape of a split ring. We rigorously derive effective equations for the scatterer and provide formulas for the effective permittivity and permeability. The latter turns out to be frequency dependent and has a negative real part for appropriate parameter values. This magnetic activity is the key feature of a left-handed meta-material.

1 Introduction

In recent years, applied sciences developed a profound interest in meta-materials, with the aim of understanding their astonishing properties, exploring their potential, and optimizing their design. A meta-material can be defined as an artificial periodic structure constituted by assemblies of elementary components of different kinds such as metallic wires or resonators. The microscopic design of such a heterogeneous complex micro-structure leads to an effective behavior of the assembly, which is different to the one of each single homogeneous component.

In the context of diffraction phenomena, the main topic is to construct meta-materials with unusual electromagnetic properties, if possible in a large range of wavelengths. In particular, building a light-transmitting medium with negative refraction index has become a very popular task. Such fascinating materials were discussed in the seminal paper by Veselago in 1967 [25], but there did not exist a substance of that kind. Only in 1997 scientists managed to construct periodic assemblies of wire sub-structures, which act like a medium with negative effective *permittivity* [20] (see also [14]). The break-through regarding negative effective *permeability* was made in 2002 with a construction of O'Brien and Pendry [18] for metallic photonic crystals. For an excellent review and further references we refer to [24].

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With the contribution at hand we present a rigorous derivation of the effective properties of a meta-material containing split rings. We thereby determine the scaling properties of various geometric quantities regarding the size of the ring and the slit. Furthermore, our analysis provides formulas for the effective material parameters in terms of microscopic cell problems. We study the time-harmonic Maxwell equations with frequency ω in a complex geometry. A three-dimensional scatterer $\Omega \subset \mathbb{R}^3$ contains small split rings of typical size $\eta > 0$. The rings are distributed along a grid with grid-size η . Two materials are used, both have the same positive permeability μ_0 . Instead, the conductivity of the two materials is different, it vanishes outside the rings, while we assume that the rings have a large conductivity $\sigma_\eta \sim \eta^{-2}$.

We derive the homogenized equations for this complex geometry. The averaged equations are again Maxwell equations, but the effective parameters are frequency dependent due to local resonance effects. The main issue is the resulting artificial magnetic tensor $\mu(\omega)$, which depends on the microscopic geometry and on the frequency. We show that, for an appropriate choice of the parameters, the effective permeability tensor can have eigenvalues with negative real part, the crucial feature of a left-handed material. We emphasize that this magnetic effect is solely a consequence of an inhomogeneous permittivity ε_η . A mathematical justification of this effect was given in [6] in a very particular case that allowed to reduce the Maxwell system to a Helmholtz type $2D$ scalar equation (polarized magnetic field). On the basis of a more complex two-dimensional model, Kohn and Shipman were able to derive the predicted form of the effective parameters in [16]. To our knowledge, the present article contains the first mathematical justification of the “negative μ ”-effect for general electromagnetic waves and bounded $3D$ diffraction obstacles.

Further Literature. The physics of meta-materials that produce left-handed-media were discussed in [25] and [19]. Effective media with negative permittivity are studied in [29], where an accurate analysis of the plasma frequency is proposed. Magnetic resonators in the form of cylinders and in the form of two concentric split rings were discussed in [20, 15, 8, 7], and explicit formulas for effective quantities are presented. Another method to find effective parameters is the use of the numerical scheme of [21]. We mention [26] for a more physical approach to homogenization and [28] for an analysis of spectra of operators in homogenization problems. A two-step homogenization approach is proposed in [13] for a situation where slabs with negative permittivity or permeability are stacked alternatively in order to obtain a composite with negative index.

Parallel to the physical literature we find mathematical contributions that are aiming at rigorous results on effective quantities in the spirit of [4]. A rigorous homogenization for the Maxwell equations is carried out in [23], [27]. As in our contribution, the method of two-scale convergence of [1] is employed to derive an effective permeability and permittivity. But in their case the coefficients are bounded and non-degenerate and therefore the effective tensors keep all their eigenvalues with positive real part. Degenerate coefficients appear in [12]

where a quasi-static limit approach is performed (cp. [22] for other interesting homogenization effects in degenerate equations). In contrast, there exist very few contributions in the case of high conductivity metallic inclusions for the homogenization of the Maxwell system. In the context of heat equation and linear elasticity, it is well known that non local effective behaviors may appear [2, 3, 5]. Closer to our work are [6] and [14], where unbounded coefficients are studied in a two-dimensional setting. In [9], the microscopically relevant geometry is two-dimensional, but three-dimensional macroscopic effects are determined. While the above mentioned works are related to long wires inside the scatterer, we now extend the methods to work in the split ring geometry.

We note that an important feature of [9] is the three-scale nature of the problem: the periodicity cell contains a substructure which vanishes in the limit. This is also an important feature in the present contribution, with the slit as the thin substructure. In a different context such a substructure was analyzed in [17].

Mathematical problem. Throughout this article we study the system

$$\operatorname{curl} E_\eta = i\omega\mu_0 H_\eta, \quad (1.1)$$

$$\operatorname{curl} H_\eta = -i\omega\varepsilon_\eta\varepsilon_0 E_\eta. \quad (1.2)$$

Here ω is the angular frequency, ε_0, μ_0 are the permittivity and the permeability in vacuum. The wave number is given by $k_0 := \sqrt{\varepsilon_0\mu_0}\omega$. The rings are assumed to have a large conductivity $\sigma_\eta = \eta^{-2}\omega\varepsilon_0\kappa$, where $\eta > 0$ is the (non-dimensional) relative size of the rings and $\kappa > 0$ is a conductivity parameter. The relative permittivity ε_η is related to the conductivity through $\varepsilon_\eta = 1 + i\frac{\sigma_\eta}{\omega\varepsilon_0}$. Denoting the complex domain occupied by the rings as $\Sigma_\eta \subset \Omega$, we therefore have

$$\varepsilon_\eta = \begin{cases} 1 + i\frac{\kappa}{\eta^2} & \text{in } \Sigma_\eta, \\ 1 & \text{in } \mathbb{R}^3 \setminus \Sigma_\eta. \end{cases} \quad (1.3)$$

We consider a domain Σ_η with the split rings arranged regularly along a three dimensional array. In Figure 1 one layer is sketched. Two neighboring ring centers have distance η , the diameter of each ring is of order η , and the circular cross section of each ring has radius $\beta\eta$. On their upper part the rings are not connected but rather have a thin slit of size $\alpha\eta^3$.

Synopsis of the main result. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, the *scatterer*, that contains a family of split rings $\Sigma_\eta \subset \Omega$ of orientation e_3 as described above, and additionally families of rings in the other two orientations e_1 and e_2 without intersections. We study the diffraction of an incoming incident wave of angular frequency ω and study the resulting electromagnetic field (E_η, H_η) , which is determined as the solution of (1.1)–(1.2) in \mathbb{R}^3 with a suitable radiation condition at infinity. Due to the complex geometry of the scatterer, (E_η, H_η) oscillates at scale η . It is therefore a nontrivial task to identify the averaged field (E, H) which is given as a weak limit, $(E_\eta, H_\eta) \rightharpoonup (E, H)$ in L^2_{loc} as $\eta \rightarrow 0$.

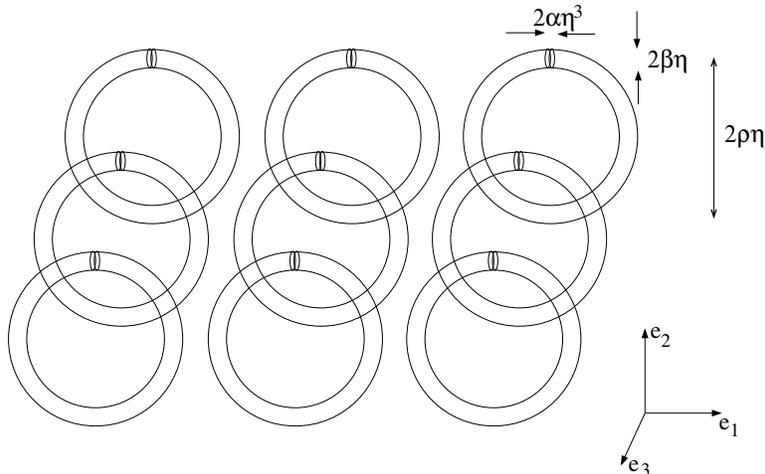


Figure 1: *Sketch of the geometry, showing one layer of rings. The macroscopic domain $\Omega \subset \mathbb{R}^3$ contains $O(\eta^{-3})$ thin split rings of diameter $O(\eta)$. The union of the rings is the complex domain of high conductivity; it is denoted by Σ_η .*

Our results imply that, outside the scatterer Ω , the averaged field (E, H) agrees with the unique solution (\hat{E}, \hat{H}) of a new *effective* diffraction problem on \mathbb{R}^3 of the form

$$\begin{aligned} \text{curl } \hat{E} &= i\omega\mu_0 \hat{\mu}\hat{H}, \\ \text{curl } \hat{H} &= -i\omega\varepsilon_0 \hat{\varepsilon}\hat{E}. \end{aligned}$$

Here, the relative parameters are $\hat{\mu}(x) = \hat{\varepsilon}(x) = 1$ in $\mathbb{R}^3 \setminus \Omega$, whereas for $x \in \Omega$

$$\hat{\varepsilon}(x) = \varepsilon^{\text{eff}}, \quad \hat{\mu}(x) = \mu^{\text{eff}}(\omega).$$

In other words: asymptotically, as $\eta \searrow 0$, the complex structure in Ω looks from the outside like a homogeneous medium, characterized by the effective permittivity and permeability tensors ε^{eff} and $\mu^{\text{eff}}(\omega)$. The two parameters depend on the geometrical characteristics of the rings and on the conductivity parameter κ . In contrast to ε^{eff} which is real, positive, and frequency independent, $\mu^{\text{eff}}(\omega)$ turns out to be *frequency dependent*. The eigenvalues of the effective permeability tensor can have a positive and a negative real part; we obtain a formula for $\mu^{\text{eff}}(\omega)$ which is quite similar to the heuristic one proposed by O'Brien and Pendry in [18]. Moreover, we are able to perform a limit analysis as the conductivity parameter κ increases to infinity. In that case, $\mu^{\text{eff}}(\omega)$ becomes *real* with large negative eigenvalues within some range of frequencies. The effective medium is not dissipative any more and there appears a band gap.

Let us point out that a very simple model of a meta-material with negative $\mu^{\text{eff}}(\omega)$ has been proposed in [6, 7, 8, 15]. It consists of arrays of infinitely long parallel fibers. However, for such a structure, the resonance effect has been evidenced only in a polarized setting (the magnetic field was assumed to be parallel to the fibers), and assuming a *finite conductivity parameter* κ .

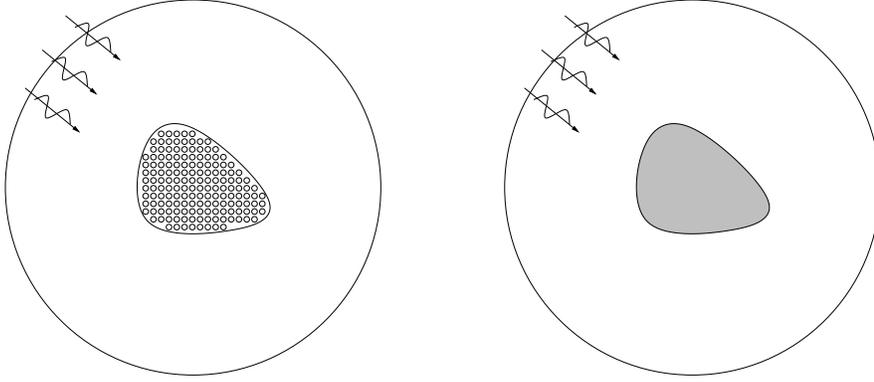


Figure 2: *Illustration of the homogenization process. The multi-ring geometry is replaced by a homogeneous meta-material. The solution (E_η, H_η) in the left geometry is characterized by a highly oscillatory permittivity ε_η . The solution (\hat{E}, \hat{H}) on the right by effective parameters μ^{eff} and ε^{eff} , where μ^{eff} can have a negative real part.*

This paper is organized as follows. Our main results and related discussions are presented in Section 2. The proof of the key convergence result of Theorem 1 is developed in Sections 3–4. In these sections we deal with L^2_{loc} -weakly convergent sequences of vector fields (E_η, H_η) that satisfy (1.1) and (1.2). The energy estimates that allow to deduce the convergence to the effective diffraction problems (Theorems 2 and 3) are established in Section 5.

2 Main homogenization results

We begin with a precise definition of Σ_η , starting from an open domain $\Omega \subset \mathbb{R}^3$ that contains the rings, and the unit cell $Y = (-1/2, 1/2)^3$. The geometry of the rings is determined by the relative first radius $\rho \in (0, 1/2)$ of the ring, a number $\alpha \in (0, 1)$ related to the size of the slit, and a number $\beta \in (0, \rho)$ with $\rho + \beta < 1/2$ for the thickness of the ring. We set $\Sigma_Y^0 = \{(y_1, y_2, 0) \in Y : y_1^2 + y_2^2 = \rho^2\}$ to define the central curve of the ring. The ring is represented by the open subset $\Sigma_Y = B_\beta(\Sigma_Y^0) \subset Y$, the three-dimensional β -ball around Σ_Y^0 . The split ring in the single unit cell is denoted by $\Sigma_Y^\eta \subset Y$,

$$\Sigma_Y^\eta := \Sigma_Y \setminus \left\{ (y_1, y_2, y_3) : |y_1| \leq \alpha\eta^2 \frac{y_2}{\rho}, y_2 > 0 \right\}.$$

We note that, for small η , the two sides of the slit are like two parallel disks of radius β at distance $\alpha\eta^2$. We finally define

$$\Sigma_\eta := \bigcup_{j \in \mathbb{Z}^3 \text{ with } \eta(j+Y) \subset \Omega} \eta(j + \Sigma_Y^\eta). \quad (2.1)$$

The volume fraction of Σ_η is of order 1; in dependence of the geometrical parameters it is approximately $2\rho\pi^2\beta^2$. As a notation for geometrical objects we follow

the convention that a lower index Y marks subsets of the unit cube Y , an upper index η recalls a possible dependence on η . A lower index η denotes subsets of Ω which are created by a periodic repetition of a subset of the unit cube.

Notation. The canonical basis vectors of \mathbb{R}^3 are e_1, e_2, e_3 , normal vectors on surfaces are denoted by n . The characteristic function of a set A is denoted by $\mathbf{1}_A$. For the third order Levi-Civita tensor we write ε_{klm} ; it is totally anti-symmetric with $\varepsilon_{klm} \in \{-1, 0, +1\}$ and the sign convention that $\varepsilon_{1,2,3} = +1$. The wedge-product is $u \wedge v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) = (\sum_{l,m} \varepsilon_{klm} u_l v_m)_k$, and the rotation is defined as $\text{curl } u = \nabla \wedge u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1) = (\sum_{l,m} \varepsilon_{klm} \partial_l u_m)_k$. Geometry dependent constants in \mathbb{R} , \mathbb{C} or \mathbb{C}^3 have lower indices, cell solutions have upper indices. For any complex number $z \in \mathbb{C}$, we denote by $\Re(z)$, $\Im(z)$ the real and imaginary parts.

2.1 The key convergence result

In the following, we make use of two matrices $\mathcal{N} \in \mathbb{R}^{3 \times 3}$ and $\mathcal{M}_\lambda \in \mathbb{C}^{3 \times 3}$. Due to the symmetries in our geometry, both matrices are diagonal,

$$\mathcal{N} := \text{diag}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) \quad , \quad \mathcal{M}_\lambda := \text{diag}(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3 + \lambda \mathbf{m}_0) . \quad (2.2)$$

They are determined through the unit cell problems described in Section 3. The coefficients $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are deduced from (3.5)–(3.6). They are *real and independent of κ and ω* . Instead, the complex coefficients $\mathbf{m}_k(\omega, \kappa)$ defined in (3.24) and (3.25) for $k \in \{0, 1, 2, 3\}$ *do depend on conductivity κ and frequency ω* . However as $\kappa \rightarrow \infty$, the \mathbf{m}_k remain bounded and converge to real coefficients which are still independent of ω . The complex function $\lambda(\omega, \kappa)$ accounts for resonance phenomena: its real part can be very large and negative for suitable values of ω . It takes the form

$$\lambda(\omega, \kappa) := \frac{\varepsilon_0 \mu_0 \omega^2 \mathbf{D}_3(\omega, \kappa)}{-\varepsilon_0 \mu_0 \omega^2 \mathbf{D}_0(\omega, \kappa) - \alpha (\pi \rho)^{-1} + i \kappa^{-1}} , \quad (2.3)$$

where the complex numbers \mathbf{D}_k are flux parameters defined in (3.30). As the \mathbf{m}_k , they become real and independent of ω in the limit $\kappa \rightarrow \infty$ (see Subsection 4.3).

Theorem 1. *Let $\Sigma_\eta \subset \Omega \subset \mathbb{R}^3$ be a split ring scatterer as described above, where the rings are perpendicular to e_3 . Let (E_η, H_η) be a sequence of solutions of (1.1)–(1.2) with the relative permittivity given by*

$$\varepsilon_\eta(x) = \begin{cases} 1 + i \frac{\kappa}{\eta^2} & \text{if } x \in \Sigma_\eta \\ 1 & \text{otherwise.} \end{cases} \quad (2.4)$$

Let Q be a bounded domain containing $\overline{\Omega}$ and assume that (E_η, H_η) satisfies the bound

$$\int_Q |E_\eta|^2 + |H_\eta|^2 \leq C, \quad (2.5)$$

such that, up to a subsequence, $E_\eta \rightharpoonup E$, $H_\eta \rightharpoonup H$ in $L^2(Q, \mathbb{C}^3)$. Then the weak limit (E, H) satisfies in the distributional sense on Q

$$\operatorname{curl} E = i\omega\mu_0 H, \quad (2.6)$$

$$\operatorname{curl} (\hat{M} \cdot H) = -i\omega\varepsilon_0 \hat{N} E. \quad (2.7)$$

Here, the coefficient matrices $\hat{N} : Q \rightarrow \mathbb{R}^{3 \times 3}$ and $\hat{M} : Q \rightarrow \mathbb{C}^{3 \times 3}$ are given by

$$\hat{N}(x) = \begin{cases} \mathcal{N} & x \in \Omega \\ \mathbf{1} & x \notin \Omega \end{cases} \quad \hat{M}(x) = \begin{cases} \mathcal{M}_\lambda & x \in \Omega \\ \mathbf{1} & x \notin \Omega \end{cases}$$

For every frequency ω there exist parameters α and κ such that $\Re(\mathbf{m}_3 + \lambda \mathbf{m}_0) < 0$.

Outline of the proof. In the effective system, equation (2.6) follows immediately by passing to the distributional limit in (1.1). In contrast, the derivation of (2.7) requires a fine analysis of the interactions between the oscillations of the electromagnetic field and the geometry of the split rings. It is performed in two steps in Sections 3 and 4, making use of the notion of two-scale convergence as introduced in [1].

As a preparation to the proof, in Subsection 3.1, we improve (2.5) to

$$\int_Q |\varepsilon_\eta| |E_\eta|^2 + |H_\eta|^2 \leq C. \quad (2.8)$$

Step 1. In Section 3 we derive the equations that are satisfied by the two-scale limits $E_0(x, y)$, $H_0(x, y)$ of a subsequence E_η , H_η . It turns out to be useful to consider a third quantity, the rescaled displacement field $J_\eta = \eta\varepsilon_\eta E_\eta$ with the two scale limit $J_0(x, y)$. This field concentrates in the rings and circulates in the rings with a local flux intensity $j(x)$. We obtain a highly complex system for (H_0, J_0) and determine the solution space. The key result of Section 3 is that the averages $\int_Y E_0(x, y) dy = E(x)$ and $\int_Y H_0(x, y) dy = H(x)$ together with the scalar factor $j(x)$ determine E_0 , H_0 , and J_0 uniquely.

As expected, the microscopic behavior of the electric field is purely electrostatic: $E_0(x, \cdot)$ is determined by 3 periodic shape vector functions $E^k(y)$ with average e_k , defining the positive definite real tensor $\mathcal{N} = \operatorname{diag}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$. The behavior of H_0 is much more intricate. It involves 3 periodic vector fields $H^k(y)$ with average e_k , and, additionally, a field $H^0(y)$ with zero average that describes a circulation through the ring. Starting from these shape functions we introduce circulation vectors $M_k = \mathbf{m}_k e_k$ and $M_0 = \mathbf{m}_0 e_3$ as line integrals and flux intensity constants \mathbf{D}_k as area integrals. All these parameters are complex and frequency dependent (except in the limit $\kappa \rightarrow \infty$).

Step 2. The main result of Section 4 is the relation $j(x) = \lambda(\omega, \kappa) H_3(x)$ of (4.1). It is the result of a microscopic analysis of the electric field in the slit (note that the slit geometry is not exploited in Section 3). The linear relation between the macroscopic strength of the local currents j and the strength of the magnetic field

allows us to close the cell problem: the macroscopic fields $E(x)$, $H(x)$ determine uniquely the two-scale limits $E_0(x, y)$, $H_0(x, y)$, $J_0(x, y)$.

The remaining procedure is straightforward. Using suitable test functions for (1.2), exploiting the tensor \mathcal{N} , the circulation vectors M_k , and the flux coefficients \mathbf{D}_k , we derive the effective equation (2.7). Subsection 4.3 contains a limit analysis for $\kappa \rightarrow \infty$. The results imply that, for appropriate geometries and large κ , $\lambda(\omega, \kappa)$ has a negative real part with large absolute value.

2.2 Effective diffraction problem

We consider the diffraction of a given monochromatic incident electromagnetic field $(E^{\text{in}}, H^{\text{in}})$ with angular frequency ω by the η -periodic split ring structure placed in Ω . In the limit $\eta \rightarrow 0$ we obtain an effective problem that can be interpreted as the diffraction of $(E^{\text{in}}, H^{\text{in}})$ by an homogeneous medium placed in Ω . The permittivity and permeability tensors of the effective medium can be specified in terms of the matrices \mathcal{N} and \mathcal{M}_λ of Theorem 1.

This remarkable and unexpected simplification of the statement of Theorem 1 is obtained by introducing a new effective magnetic field \hat{H} . At first sight, this procedure looks artificial, since \hat{H} differs from the weak limit $H = \lim_\eta H_\eta$ inside the scattering body Ω . However, there holds $\hat{H} = H$ outside Ω , and \hat{H} satisfies the expected continuity conditions across the boundary $\partial\Omega$. As a result, in the limit as $\eta \rightarrow 0$, the obstacle will look *from outside* exactly like an homogeneous material with new effective characteristic tensors (depending on ω).

To make the idea precise, we define

$$\hat{H}(x) := \hat{M}(x)H(x). \quad (2.9)$$

For homogeneity of notations we set $\hat{E} := E$. We can define the effective tensors as

$$\hat{\varepsilon}(x) = \hat{N}(x), \quad \hat{\mu}(x) = (\hat{M}(x))^{-1},$$

so that $\hat{\mu}(x) = \hat{\varepsilon}(x) = 1$ and $\hat{H}(x) = H(x)$ in $\mathbb{R}^3 \setminus \Omega$, whereas, for $x \in \Omega$,

$$\hat{\varepsilon}(x) = \varepsilon^{\text{eff}} := \mathcal{N} \quad , \quad \hat{\mu}(x) = \mu^{\text{eff}}(\omega, \kappa) := (\mathcal{M}_\lambda)^{-1}. \quad (2.10)$$

The effective equations (2.6)–(2.7) can be then rewritten in the form announced in the introduction:

$$\text{curl } \hat{E} = i\omega\mu_0 \hat{\mu} \hat{H}, \quad (2.11)$$

$$\text{curl } \hat{H} = -i\omega\varepsilon_0 \hat{\varepsilon} \hat{E}. \quad (2.12)$$

In the following theorem we consider the diffraction by the ring structure of a *fixed* incident wave $(E^{\text{in}}, H^{\text{in}})$. Such a wave is bounded and satisfies the homogeneous Maxwell equations $\text{curl } E^{\text{in}} = i\omega\mu_0 H^{\text{in}}$ and $\text{curl } H^{\text{in}} = -i\omega\varepsilon_0 E^{\text{in}}$ on \mathbb{R}^3 . For each value of η , the resulting total field (E_η, H_η) is then completely determined by solving (1.1)–(1.2) with the additional requirement that the diffracted fields

$(E_\eta^d, H_\eta^d) = (E_\eta, H_\eta) - (E^{\text{in}}, H^{\text{in}})$ satisfy the Silver-Müller radiation condition at infinity given below.

$$(E_\eta^d, H_\eta^d) = O\left(\frac{1}{|x|}\right), \quad \omega\varepsilon_0\left(\frac{x}{|x|} \wedge E_\eta^d\right) - k_0 H_\eta^d = o\left(\frac{1}{|x|}\right), \quad (2.13)$$

for $|x| \rightarrow \infty$ and with $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$. The proof for existence and uniqueness for this problem is classical and can be found e.g. in [10]. With regard to the uniqueness for limit problem as $\eta \rightarrow 0$, we will always assume for simplicity that the diagonal tensor \mathcal{M}_λ given by (2.2) is invertible and that the imaginary part of its inverse satisfies

$$\Im(\mathcal{M}_\lambda^{-1}) = \Im(\mu^{\text{eff}}) \quad \text{is positive or negative definite.} \quad (2.14)$$

We will show in Section 3 (see Lemma 2) that the first two eigenvalues $\mathbf{m}_1, \mathbf{m}_2$ of \mathcal{M}_λ are equal with strictly negative imaginary part for every ω and κ . Thus (2.14) amounts to check that the imaginary part of the third eigenvalue $\mathbf{m}_3 + \lambda(\omega, \kappa)\mathbf{m}_0$ is negative.

We are now in position to state the homogenization result for diffraction by the split ring structure. It is proved in Section 5 as a consequence of Theorem 1.

Theorem 2 (Effective diffraction problem). *Let the geometry $\Sigma_\eta \subset \Omega \subset \mathbb{R}^3$ and the relative permittivity ε_η be as in Theorem 1. Let $(E^{\text{in}}, H^{\text{in}})$ be an incident wave, and let (E_η, H_η) be the unique solution to (1.1)–(1.2) satisfying the radiation condition (2.13). Then, under condition (2.14), the limit problem (2.6)–(2.7), (2.13) has a unique solution (E, H) and there holds $E_\eta \rightharpoonup E$ and $H_\eta \rightharpoonup H$ in $L_{loc}^2(\mathbb{R}^3, \mathbb{C}^3)$.*

Furthermore we have $(E, H) = (\hat{E}, \hat{H})$ on $\mathbb{R}^3 \setminus \Omega$, where (\hat{E}, \hat{H}) is determined as the solution of the effective diffraction problem (2.11)–(2.12), (2.13).

Interpretation of the result. We may consider \hat{H} as the physically visible magnetic field. It coincides with the average field outside the scatter Ω and enjoys the usual continuity property of the tangential component across the boundary of Ω . This property is encoded in the above theorem by equation (2.12): $\text{curl } \hat{H}$ belongs to $L_{loc}^2(\mathbb{R}^3, \mathbb{C}^3)$ and has therefore no singular part on $\partial\Omega$.

At the same time, \hat{H} has an interpretation as an averaged H_η -field, averaged over lines that do not cross the rings. This follows from the definition of M_k in (3.23). This procedure coincides with that of [16] and we refer to that work for further discussions of this point.

2.3 The case of three ring orientations

Our aim now is to study the case where each cell contains three rings with different orientations. To be precise, we assume that for two constants $0 < q < 1/4$, $0 < \rho < q$ the following central curves for the three rings. $\Sigma_{0,1} := \{(y_1, y_2, y_3) : y_1 = 0, |(y_2, y_3) - (q, q)| = \rho\}$, $\Sigma_{0,2} := \{(y_1, y_2, y_3) : y_2 = 0, |(y_1, y_3) - (q, q)| = \rho\}$,

and $\Sigma_{0,3} := \{(y_1, y_2, y_3) : y_3 = 0, |(y_1, y_2) - (q, q)| = \rho\}$. Starting from these central curves, the three rings and the complex medium in $\Omega \subset \mathbb{R}^3$ are constructed as in the case of a single ring.

The cell problems of Section 3 are changed accordingly, Σ is substituted with the union of three pairwise disjoint connected rings $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. The cell-problem for the electric field can be handled as before and gives a 3 dimensional space of solutions spanned by fields E^1, E^2, E^3 . By symmetry, the associated tensor \mathcal{N} is diagonal with identical positive eigenvalues, hence $\mathcal{N}_{kl} = \mathbf{n} \delta_{kl}$ for some positive real constant \mathbf{n} .

As regards the cell problem for the magnetic field, we find now a 6 dimensional space of solutions spanned by fields H^k , $k = 1, 2, 3$, with average e_k , and the three special fields $H^{0,1}, H^{0,2}, H^{0,3}$ with zero average and passing through $\Sigma_1, \Sigma_2, \Sigma_3$ with a prescribed circulation. As in Subsection 3.4, we may associate 6 circulation vectors M_k, M_k^0 and 18 flux parameters D_{kl}, D_{kl}^0 (representing an averaged flux of H^k or $H^{0,k}$ through the ring Σ_l). By symmetry arguments, it is easy to show that

$$M_k = \mathbf{m} e_k, \quad M_k^0 = \mathbf{m}_0 e_k, \quad D_{kl} = \mathbf{D} \delta_{kl}, \quad D_{kl}^0 = \mathbf{D}_0 \delta_{kl},$$

for suitable scalar functions $\mathbf{m}(\omega, \kappa)$, $\mathbf{m}_0(\omega, \kappa)$, $\mathbf{D}(\omega, \kappa)$, and $\mathbf{D}_0(\omega, \kappa)$.

Eventually, if $H_k(x)$ denotes the local strength of $H^k(y)$ and $j_k(x)$ denotes the local strength of $H^{0,k}(y)$ in the two-scale expansion, we can use the same strategy as in Subsection 4.1 in order to establish linear relations between $H_k(x)$ and $j_l(x)$ for $k, l \in \{1, 2, 3\}$. Still for symmetry reasons, we obtain:

$$j_k(x) = \lambda(\omega, \kappa) H_k(x),$$

where the function $\lambda(\omega, \kappa)$ is given by (2.3), substituting \mathbf{D}_3 with \mathbf{D} .

Eventually, the limit problem associated with this new geometry will involve piecewise constant scalar tensors \hat{n} and \hat{m} with $\hat{n} = \hat{m} = 1$ in $\mathbb{R}^3 \setminus \Omega$ and $\hat{n} = \mathbf{n}$, $\hat{m} = \mathbf{m} + \lambda(\omega, \kappa)\mathbf{m}_0$ in Ω . The constant \mathbf{n} is real, positive, and independent of ω and κ . Instead, \mathbf{m}, \mathbf{m}_0 and $\lambda(\omega, \kappa)$ are complex parameters that do depend on ω and κ . The dissipativity condition (2.14) becomes

$$\Im(\mathbf{m} + \lambda(\omega, \kappa)\mathbf{m}_0) \neq 0. \quad (2.15)$$

Reproducing with minor modifications the arguments in the proof of Theorem 1 and 2, we obtain the following result.

Theorem 3 (Effective diffraction problem for three ring orientations). *Let $\Sigma_\eta \subset \Omega \subset \mathbb{R}^3$ be given by the three ring geometry in $\Omega \subset \mathbb{R}^3$ defined above and consider the relative permittivity ε_η given by (2.4). Let $(E^{\text{in}}, H^{\text{in}})$ be an incident wave, and let (E_η, H_η) be the unique solution to (1.1)–(1.2) and (2.13).*

Then, under condition (2.15), the limit problem

$$\text{curl } E = i\omega\mu_0 H, \quad (2.16)$$

$$\text{curl } (\hat{m}H) = -i\omega\varepsilon_0 \hat{n}E, \quad (2.17)$$

with the outgoing wave condition (2.13) has a unique solution (E, H) and there holds

$$E_\eta \rightharpoonup E \text{ and } H_\eta \rightharpoonup H \text{ in } L_{loc}^2(\mathbb{R}^3, \mathbb{C}^3).$$

Interpretation with scalar effective coefficients. Following the strategy of Subsection 2.2, we may define the effective fields $\hat{H}(x) := \hat{m}(x)H(x)$ and $\hat{E}(x) := E(x)$ to transform (2.16)–(2.17) into system (2.11)–(2.12), with effective parameters

$$\varepsilon^{\text{eff}} = \mathbf{n} \quad , \quad \mu^{\text{eff}}(\omega, \kappa) = (\mathbf{m} + \lambda \mathbf{m}_0)^{-1}. \quad (2.18)$$

In this case, the limit problem of Theorem 3 can be interpreted as the diffraction system for a homogeneous medium with a *scalar* effective permittivity ε^{eff} and a *scalar* effective permeability μ^{eff} . In other words, the unique solution (\hat{E}, \hat{H}) to (2.11)–(2.12) and (2.13) agrees with the limit pair (E, H) outside of obstacle Ω .

We observe that the real part of $\mu^{\text{eff}}(\omega, \kappa)$ can be negative. This follows in the next paragraph from formula (2.20) for $\mu^{\text{eff}}(\omega, \kappa)$ and its counterpart (2.22) as κ increases to infinity.

Comparison with the O’Brien-Pendry formula. In [18] the authors discuss structured photonic crystals similar to ours. They provide the effective permeability in the form

$$\mu_{\text{eff}}^{BP} = 1 - \frac{f\omega^2}{\omega^2 - \omega_0^2 + i\Gamma\omega} = \frac{(1-f)\omega^2 - \omega_0^2 + i\Gamma\omega}{\omega^2 - \omega_0^2 + i\Gamma\omega}, \quad (2.19)$$

where f is the filling ratio of the structure and Γ a measure of electric resistance that vanishes for high conductivities (see also [20]). To compare our results with the formula of O’Brien-Pendry, we consider the symmetric three ring geometry of Theorem 3 and exploit expressions (2.18) and (2.3). We obtain

$$\mu^{\text{eff}}(\omega, \kappa) = \frac{1}{\mathbf{m}} \frac{(1-f^*)\omega^2 - \omega_0^{*2} + i\Gamma^*\omega}{\omega^2 - \omega_0^{*2} + i\Gamma^*\omega}, \quad (2.20)$$

where we have set, with $\mathbf{D}_0 = -\tau\mathbf{D}$ and $\mathbf{m}_0 = \sigma_0 \mathbf{m}$,

$$f^* := \frac{\sigma_0}{\tau + \sigma_0}, \quad \omega_0^* := \left[\frac{\alpha}{\pi\rho\varepsilon_0\mu_0(\tau + \sigma_0)\mathbf{D}} \right]^{1/2}, \quad \Gamma^* := \frac{1}{\kappa(\tau + \sigma_0)\varepsilon_0\mu_0\omega\mathbf{D}}. \quad (2.21)$$

We emphasize that the coefficients $\mathbf{m}_0, \mathbf{m}, \mathbf{D}_0, \mathbf{D}_3$, characterized in Subsection 3.4 in terms of magnetic shape functions, *are complex and do depend of ω and κ* . In fact, examining the cell problem of Section 3.3, it turns out that they are all functions of the factor $\kappa\omega^2$. The asymptotic analysis as $\kappa \rightarrow \infty$ (or, equivalently, $\kappa\omega^2 \rightarrow \infty$) has been performed in Subsection 4.3 in the case of a single ring orientation, but it can easily be extended to the case of the 3 ring geometry. It provides that, in this limit, all coefficients become *real* with

$$\mathbf{m}_0 > 0, \quad \mathbf{m} \geq \frac{1}{1-f}, \quad \mathbf{D}_3 > 0, \quad \mathbf{D}_0 < 0,$$

where, like in (2.19), f denotes the filling ratio of metallic inclusions. In particular, the coefficients τ and σ_0 in (2.21) are positive and $0 < f^* < 1$. Accordingly we get the following high frequency limit

$$\lim_{\kappa\omega^2 \rightarrow \infty} \mu^{\text{eff}}(\omega, \kappa) = \frac{1}{\mathbf{m}}(1-f^*) < 1-f,$$

which is below the one predicted through (2.19). On the other hand, from (2.20), we deduce the high conductivity limit

$$\lim_{\kappa \rightarrow \infty} \mu^{\text{eff}}(\omega, \kappa) = \frac{1}{\mathbf{m}} \frac{(1 - f^*)\omega^2 - \omega_0^{*2}}{\omega^2 - \omega_0^{*2}}, \quad (2.22)$$

with real positive coefficients f^* and ω_0^* . In conclusion, we confirm asymptotically the existence of a band gap in the range $\omega \in [\omega_0^*, \omega_0^*(1 - f^*)^{-1/2}]$.

3 Two-scale limits and unit cell problems

This section and the next are devoted to the proof of Theorem 1. We start from a sequence (E_η, H_η) of solutions of (1.1)–(1.2) on Q , which satisfies the a priori bound (2.5). For notational convenience we assume that the reference $Q \subset \mathbb{R}^3$ is large enough such that, for an appropriate radius $R > 0$, we have the inclusions $\Omega \subset B_R(0) \subset \bar{B}_{R+1}(0) \subset Q$.

3.1 Two-scale limits

Improved a priori estimate. As announced, we start with the observation that the L^2 -estimate (2.5) can be improved to the energy estimate (2.8). Multiplication of (1.2) with $i\omega^{-1}\bar{E}_\eta$ and integration over a ball $B = B_r$ containing Ω yields

$$\varepsilon_0 \omega \int_{B_r} \varepsilon_\eta |E_\eta|^2 = i \int_{B_r} \text{curl } H_\eta \bar{E}_\eta = i \int_{B_r} H_\eta \text{curl } \bar{E}_\eta + i \int_{\partial B_r} n \wedge H_\eta \cdot \bar{E}_\eta,$$

and therefore by (1.1)

$$\varepsilon_0 \int_{B_r} \varepsilon_\eta |E_\eta|^2 = \mu_0 \int_{B_r} |H_\eta|^2 + i\omega^{-1} \int_{\partial B_r} n \wedge H_\eta \cdot \bar{E}_\eta. \quad (3.1)$$

An integration with respect to $r \in (R, R+1)$ provides a bounded right hand side by the uniform L^2 -bound for (E_η, H_η) . By (1.3), the imaginary part of the left hand member of (3.1) is independent of r , and estimate (2.8) follows.

The two-scale limit triple (E_0, H_0, J_0) . Since E_η and H_η are bounded in $L^2(Q)$ we can, after extraction of a subsequence, consider the two-scale limits for $\eta \rightarrow 0$

$$\begin{aligned} E_\eta(x) &\rightharpoonup E_0(x, y) \text{ weakly in two scales,} \\ H_\eta(x) &\rightharpoonup H_0(x, y) \text{ weakly in two scales,} \end{aligned}$$

for some limit functions $E_0, H_0 \in L^2(Q \times Y, \mathbb{C}^3)$. In the limit $\eta \rightarrow 0$, the slit of the rings vanishes and the geometrically relevant domains are the unit cell Y and the closed ring $\Sigma := \Sigma_Y \subset Y$. For brevity we denote the boundary of the closed ring by $T := \partial\Sigma$, the letter recalls that this is a two-dimensional torus. By n

we denote the normal vector to T , to make a choice, we take n as the outward normal to Σ .

We additionally consider a third quantity, namely the rescaled dielectric field

$$J_\eta := \eta \varepsilon_\eta E_\eta : Q \rightarrow \mathbb{C}^3. \quad (3.2)$$

To leading order, in the rings, this field coincides with $\kappa i \eta^{-1} E_\eta$. The L^2 -norm is finite, since by (2.8)

$$\int_Q |\eta \varepsilon_\eta E_\eta|^2 \leq \sup(\eta^2 |\varepsilon_\eta|) \int_Q |\varepsilon_\eta| |E_\eta|^2 \leq C.$$

We can therefore additionally consider the two-scale limit

$$J_\eta(x) \rightharpoonup J_0(x, y) \text{ weakly in two scales, with } J_0 \in L^2(Q \times Y, \mathbb{C}^3).$$

Sobolev spaces of periodic functions. In the following, $W_{\text{per}}^{1,2}(Y)$ will denote the Hilbert space of *complex* valued Y -periodic functions which are elements of $W_{\text{loc}}^{1,2}(\mathbb{R}^3)$. It is well known (see for instance [10]) that $W_{\text{per}}^{1,2}(Y; \mathbb{C}^3)$ coincides with the set of Y -periodic functions $u : \mathbb{R}^3 \rightarrow \mathbb{C}^3$ such that $\text{div } u$ and $\text{curl } u$ (in the distributional sense) belong to L_{loc}^2 . Furthermore, as an equivalent scalar product, we may consider

$$(u|v) := \int_Y (u\bar{v} + \text{curl } u \cdot \text{curl } \bar{v} + \text{div } u \text{ div } \bar{v}) \, dy.$$

The elements of $W_{\text{per}}^{1,2}(Y; \mathbb{C}^3)$ have well defined traces. For brevity of notation we write $\int_T u$ for integrals over traces looking from the side of Σ . For integrals over traces from $Y \setminus \Sigma$ we write $\int_{T_+} u$. In the appendix we collect some useful integration by parts formulae.

3.2 Cell-problem for E_0 and the tensor \mathcal{N}

The weak limit of E_η in $L^2(Q)$ is recovered by $E(x) = \int_Y E_0(x, y) \, dy$. For $x \notin \Omega$ we have $E_0(x, y) = E(x)$. Indeed, E^η satisfies the Helmholtz equation $\Delta E_\eta + k_0^2 E_\eta = 0$ on $Q \setminus \bar{\Omega}$, and therefore the convergence $E_\eta \rightarrow E$ is uniform on compact subsets of $Q \setminus \bar{\Omega}$ (this fact is well known for general hypo-elliptic operators with constant coefficients).

In contrast, for $x \in \Omega$, $E_0(x, \cdot)$ is not constant. It is determined in terms of its average $E(x)$ by the following equations on the unit cell Y .

$$\begin{aligned} \text{curl}_y E_0 &= 0 \text{ in } Y, \\ \text{div}_y E_0 &= 0 \text{ in } Y \setminus \bar{\Sigma}, \\ E_0 &= 0 \text{ in } \Sigma, \\ E_0 &\text{ is periodic in } Y. \end{aligned} \quad (3.3)$$

Here, the first two equations are derived in a standard way by using equations (1.1)–(1.2) and oscillating test functions of the kind $\eta \psi(x) \theta(x/\eta)$, where ψ is

a smooth scalar function and θ is periodic (and vanishes on Σ when (1.2) is concerned). The third equation is an immediate consequence of the energy estimate (2.8) because of $|\varepsilon_\eta| \rightarrow \infty$ in Σ_η . The first equation implies that $E_0(x, \cdot) = E(x) + \nabla_y \phi(x, \cdot)$, where ϕ is a scalar periodic potential in $W_{\text{per}}^{1,2}(Y, \mathbb{C})$ and $E(x)$ denotes the average of $E_0(x, \cdot)$ on the unit cell, see (A.2). The second equation implies that ϕ is harmonic in $Y \setminus \bar{\Sigma}$, the third equation yields that $\phi(y) + E(x) \cdot y$ constant on the connected subset Σ . Therefore, for a given average electric field $E(x)$, ϕ is determined uniquely (up to constants) by affine boundary values on $\partial\Sigma$. It follows that the microscopic electric field $E_0(x, \cdot)$ can be written as a linear combination

$$E_0(x, y) = \sum_{k=1}^3 E_k(x) E^k(y), \quad (3.4)$$

where the *real valued* shape functions $E^k := e_k + \nabla \phi^k$ are determined in terms of ϕ^k , the unique solution in $W_{\text{per}}^{1,2}(Y)$ of

$$\Delta \phi^k = 0 \quad \text{on } Y \setminus \bar{\Sigma} \quad , \quad \phi^k = -y_k \quad \text{on } \Sigma. \quad (3.5)$$

The symmetric tensor \mathcal{N} . By construction, the fields $\{E^1, E^2, E^3\}$ satisfy $\int_Y E^k \cdot e_l = \delta_{kl}$ and form a basis of the space of solutions for the E_0 -cell problem. However, they are not orthonormal with respect to the usual scalar product in $L^2(Y)$. We define the tensor $\mathcal{N} := \mathcal{N}_{kl}$ through

$$\mathcal{N}_{kl} := \int_Y E^k \cdot \overline{E^l} = \delta_{kl} + \int_{T_+} E^k \cdot n y_l. \quad (3.6)$$

The last equality is obtained with an integration by parts

$$\int_Y E^k \cdot \overline{E^l} = \int_{Y \setminus \Sigma} E^k \cdot (e_l + \nabla \phi^l) = \delta_{kl} - \int_{\partial\Sigma} E^k \cdot n \phi^l = \delta_{kl} + \int_{T_+} E^k \cdot n y_l,$$

where we have used $E^k = 0$ on Σ and $\text{div } E^k = 0$ on $Y \setminus \Sigma$. In particular, the normal trace is well defined, since the divergence is in $L^2(Y \setminus \Sigma)$.

Remark 1. *The previous analysis works in fact for any inclusion Σ such that $\bar{\Sigma} \subset Y$. If we start with a ring configuration which is invariant by rotation along the e_3 -axis, the tensor \mathcal{N} will be diagonal with positive elements $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ such that $\mathbf{n}_1 = \mathbf{n}_2$. If, alternatively, we consider a three rings configuration as depicted in Subsection 2.3, we will end up with a scalar matrix, i.e. $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}_3$. We emphasize that in all cases the real symmetric positive tensor \mathcal{N} does not depend on material parameters or frequency.*

3.3 Cell-problem for the pair (H_0, J_0)

Outside the scatterer, i.e. for $x \in Q \setminus \bar{\Omega}$, we find $H_0(x, y) = H(x)$ with the weak limit H of H_η like for the electric field. For $x \in \Omega$, two of the equations

for the periodic vector field $H_0(x, \cdot)$ are derived just as for $E_0(x, \cdot)$: there holds $\operatorname{curl}_y H_0(x, \cdot) = 0$ in $Y \setminus \bar{\Sigma}$ and $\operatorname{div}_y H_0(x, \cdot) = 0$ in Y . However, the situation is drastically different as $H_0(x, \cdot)$ does not vanish inside Σ . Moreover, the curl-free condition on $Y \setminus \Sigma$ which is not simply connected does not ensure that $H_0(x, \cdot)$ is a gradient of a suitable potential on $Y \setminus \Sigma$; circular fields pointing through the ring are possible.

In order to understand the magnetic activity generated at the scale η , we will couple the equations for H_0 with the two-scale limit J_0 of J_η , which we analyze now. We observe that the field J_η has a vanishing contribution outside Σ_η since

$$\int_{Q \setminus \Sigma_\eta} |J_\eta|^2 = \int_{Q \setminus \Sigma_\eta} |\eta \varepsilon_\eta E_\eta|^2 \leq \sup_{Q \setminus \Sigma_\eta} (\eta^2 |\varepsilon_\eta|) \int_Q |\varepsilon_\eta| |E_\eta|^2 \leq C \eta^2 \rightarrow 0. \quad (3.7)$$

Hence the support of J_0 is contained in $Q \times \bar{\Sigma}$ and (3.14) below follows. Since $J_\eta := \eta \varepsilon_\eta E_\eta$ is a curl, we have $\operatorname{div} J_\eta = 0$ on Q for all η and $\operatorname{div}_y J_0 = 0$ in $Q \times Y$, i.e. (3.13). The relation

$$\eta \operatorname{curl}_x H_\eta = \eta(-i\omega \varepsilon_\eta \varepsilon_0) E_\eta = -i\omega \varepsilon_0 J_\eta$$

yields, in the two-scale limit, (3.9). It remains to verify (3.12), which will follow from

$$\eta \operatorname{curl}_x J_\eta = i\kappa \mathbf{1}_{\Sigma_\eta} \operatorname{curl}_x E_\eta + \eta^2 \operatorname{curl}_x E_\eta - \eta^2 [\varepsilon_\eta] n \wedge E_\eta \mathcal{H}^2[\partial \Sigma_\eta], \quad (3.8)$$

where $[\varepsilon_\eta] = i\kappa \eta^{-2}$ denotes the absolute value of the jump of ε_η . We take the two-scale limit and find

$$\operatorname{curl}_y J_0(x, \cdot) = -\kappa \omega \mu_0 H_0(x, \cdot) \text{ in } \Sigma \setminus S_Y,$$

where $S_Y = \Sigma \cap \{y_1 = 0, y_2 > 0\}$ denotes the limiting position of the slit. But (3.8) implies more: For a test function of the form $\Phi(x) = \psi(x)\varphi(x/\eta)$ with $\psi \in C_0^\infty(Q)$ and $\varphi \in C_0^\infty(\Sigma)$, the two slit contributions of the jump part in (3.8) cancel out in the limit. To make this precise, we calculate for the slit $S_Y^\eta := \{(y_1, y_2, y_3) \in \Sigma : |y_1| < \alpha \eta^2 y_2 / \rho\}$ and the collection of slits S_η

$$\begin{aligned} \lim_{\eta \rightarrow 0} \int_Q \Phi \eta^2 [\varepsilon_\eta] n \wedge E_\eta d\mathcal{H}^2[\partial \Sigma_\eta] &= \lim_{\eta \rightarrow 0} i\kappa \int_{\partial \Sigma_\eta} \Phi \cdot n \wedge E_\eta \\ &= \lim_{\eta \rightarrow 0} i\kappa \int_{S_\eta} \{-\operatorname{curl} \Phi \cdot E_\eta + \Phi \cdot \operatorname{curl} E_\eta\} = 0, \end{aligned}$$

the limits are consequences of the L^2 -bounds for E_η and H_η and the vanishing volume of S_η . This yields the equation on all of Σ , i.e. (3.12).

We summarize the cell problem for (H_0, J_0) . The magnetic field $H_0(x, \cdot)$ satisfies

$$\operatorname{curl}_y H_0 + i\omega \varepsilon_0 J_0 = 0 \text{ in } Y, \quad (3.9)$$

$$\operatorname{div}_y H_0 = 0 \text{ in } Y, \quad (3.10)$$

$$H_0 \text{ is periodic in } Y, \quad (3.11)$$

while the displacement field $J_0(x, \cdot)$ satisfies

$$\operatorname{curl}_y J_0 + \kappa\omega\mu_0 H_0 = 0 \text{ in } \Sigma, \quad (3.12)$$

$$\operatorname{div}_y J_0 = 0 \text{ in } Y, \quad (3.13)$$

$$J_0 = 0 \text{ in } Y \setminus \bar{\Sigma}. \quad (3.14)$$

Special vector fields. In order to evaluate the circulation of the rescaled electric field along the ring, we introduce the following vector fields in $\tau_a, \chi_a : \Sigma \rightarrow \mathbb{R}^3$.

$$\tau_a(y) := \frac{1}{|(y_1, y_2)|} \begin{pmatrix} -y_2 \\ y_1 \\ 0 \end{pmatrix}, \quad \chi_a(y) := \frac{1}{|(y_1, y_2)|} \tau_a.$$

The weight factor in χ_a has been chosen so that $\operatorname{curl} \chi_a = 0$ and $\operatorname{div} \chi_a = 0$ in Σ . The traces of these functions on the torus T are denoted by the same symbols. While τ_a and χ_a point “along the ring”, we additionally need vector fields pointing “through the ring”. We emphasize that we define $\tau_b, \chi_b : T \rightarrow \mathbb{R}^3$ only on T ,

$$\tau_b := n \wedge \tau_a, \quad \chi_b(y) := \frac{1}{|(y_1, y_2)|} \tau_b.$$

We refer to Figure 3 for a sketch regarding the sign convention.

We observe that the extension of χ_a by zero outside Σ (and periodized to all \mathbb{R}^3) is still divergence free. We will use the same symbol to denote this extension. In contrast, due to the tangential jump $-n \wedge \chi_a$ across T , the distribution $\operatorname{curl} \chi_a$ has a singular part $-\chi_b \delta_T$ where $\delta_T = \mathcal{H}^2 \llcorner T$ denotes the surface integral on T . In particular, by (3.9) and (A.1), we have:

$$\int_Y \operatorname{curl} H_0 \cdot \chi_a = \int_T (H_0 \wedge \chi_a) \cdot n = - \int_T H_0 \cdot \chi_b = -i\omega\varepsilon_0 \int_\Sigma J_0 \cdot \chi_a.$$

We are now in position to state the main result of this section

Proposition 1. *The solution space to the cell problem (3.9)–(3.14) is four-dimensional. It is spanned by shape functions $(H^k(y), J^k(y))$, $k = 0, 1, 2, 3$, which are uniquely determined as the solutions of (3.9)–(3.14) with the normalization*

$$\int_Y H^k = e_k, \quad \int_T H^k \cdot \chi_b = \int_\Sigma J^k \cdot \chi_a = 0 \quad \text{for } k \in \{1, 2, 3\}, \quad (3.15)$$

$$\int_Y H^0 = 0, \quad \int_T H^0 \cdot \chi_b = i\omega\varepsilon_0 \int_\Sigma J^0 \cdot \chi_a = 1. \quad (3.16)$$

Our aim is to construct solutions H^k in the function space $W_{\text{per}}^{1,2}(Y, \mathbb{C}^3)$. Because of (3.9) and (3.14), we may as well search for the solution in the closed subspace

$$X = \{u \in W_{\text{per}}^{1,2}(Y, \mathbb{C}^3) : \operatorname{curl} u = 0 \text{ on } Y \setminus \bar{\Sigma}\}.$$

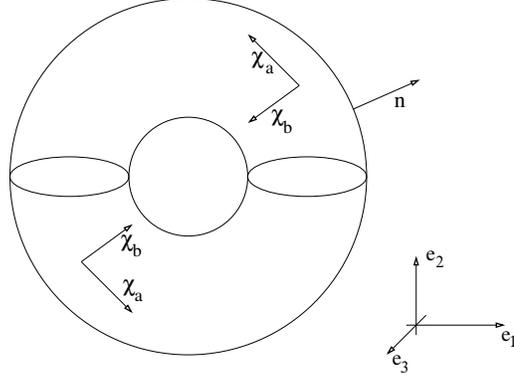


Figure 3: Sketch regarding the signs of different fields on the torus. We show the normal vector n , the tangential field χ_a which is parallel to τ_a , and the tangential field χ_b which is parallel to $\tau_b = n \wedge \tau_a$.

We will employ the Lax-Milgram Theorem to find solutions. To this end we endow X with the sesquilinear form ($k_0^2 = \varepsilon_0 \mu_0 \omega^2$)

$$b(u, v) := \frac{1}{\kappa} \int_{\Sigma} \operatorname{curl} u \cdot \operatorname{curl} \bar{v} + \int_Y \operatorname{div} u \operatorname{div} \bar{v} - ik_0^2 \int_Y u \cdot \bar{v}.$$

The form b is continuous on $X \times X$ and coercive (upon a rotation), since

$$\Re [(1 - i)b(u, u)] = \int_Y \left\{ \frac{1}{\kappa} |\operatorname{curl} u|^2 + |\operatorname{div} u|^2 + k_0^2 |u|^2 \right\}$$

is equivalent to the squared norm in $W_{\text{per}}^{1,2}(Y, \mathbb{C}^3)$. We conclude that the equation $b(u, v) = \langle f, v \rangle \forall v \in X$ has a unique solution u for every element f in X^* , the anti-dual of X , i.e. the space of sesqui-linear continuous forms $X \rightarrow \mathbb{C}$.

Later on, we will provide four special distributions $f = f_k \in W_{\text{per}}^{1,2}(Y, \mathbb{C}^3)^*$ (hence $f \in X^*$), and consider the corresponding solutions $u = U_k$ of the problem $b(u, \cdot) = \langle f, \cdot \rangle$. Up to a normalization, the U_k are the desired functions. More specifically, we will choose all f_k in the subspace

$$\mathcal{F}_{\Sigma} := \{f \in W_{\text{per}}^{1,2}(Y, \mathbb{C}^3)^* : \operatorname{div} f = 0 \text{ on } Y, f = 0 \text{ on } \Sigma\}. \quad (3.17)$$

We have the following lemma, where we write $\int_Y f$ for $\langle f, 1 \rangle$.

Lemma 1. *Let $f \in \mathcal{F}_{\Sigma}$ and let $u \in X$ be the unique solution of the variational equation $b(u, v) = \langle f, v \rangle \forall v \in X$. Then*

- (i) solution property: *The pair $(H_0, J_0) = (u, \frac{i}{\omega \varepsilon_0} \operatorname{curl} u)$ solves equations (3.9)–(3.14) and there holds $-ik_0^2 \int_Y H_0 = \int_Y f$.*
- (ii) uniqueness: *Assume that u satisfies $\int_Y u = 0$ and $\int_T u \cdot \chi_b = 0$. Then u vanishes identically.*

Proof. (i) We first prove (3.10), i.e. $w := \operatorname{div} u = 0$. We choose the curl-free test function $v = \nabla\psi$, where ψ is the unique solution in $W_{\text{per}}^{1,2}(Y)$ of $\Delta\psi + ik_0^2\psi = w$. Then, as v is a gradient and f is divergence free, we have $\langle f, v \rangle = 0$. Therefore

$$0 = b(u, v) = \int_Y w \Delta \bar{\psi} + ik_0^2 \int_Y w \bar{\psi} = \int_Y |w|^2.$$

We now consider a test function v which is smooth and compactly supported in Σ , such that, again, $\langle f, v \rangle = 0$. The variational equation and $\operatorname{div} u = 0$ imply that $\operatorname{curl}(\operatorname{curl} u) - ik_0^2 \kappa u = 0$ holds in the distributional sense on Σ . Therefore $J_0 := \frac{i}{\omega \varepsilon_0} \operatorname{curl} u$ satisfies (3.12). Furthermore, J_0 satisfies (3.13) as a curl, and (3.14) by definition of the space X . The last condition in (i) follows by choosing v constant.

(ii) By the relation $b(u, u) = \langle f, u \rangle$ and the coercivity of $(1-i)b$, it is enough to check that $\langle f, u \rangle = 0$ holds.

Inserting constant functions v , we note that the integral of f vanishes when the integral of u vanishes. Since f has additionally a vanishing divergence, it can be written as $f = \operatorname{curl} \Phi$ for some $\Phi \in L_{\text{per}}^2(Y, \mathbb{C}^3)$. As the vector function $\Phi(y)$ is curl-free on the open set Σ where f vanishes, adding a constant to Φ if necessary, by (see A.4) there exists a scalar function $\rho \in W^{1,2}(\Sigma, \mathbb{C})$ and a complex constant μ such that

$$\Phi(y) = \nabla \rho(y) + \mu \chi_a(y) \text{ for } y \in \Sigma. \quad (3.18)$$

For $u \in X$ we therefore have

$$\langle f, u \rangle = \int_Y \operatorname{curl} \Phi \cdot \bar{u} = \int_Y \Phi \cdot \overline{\operatorname{curl} u} = \int_{\Sigma} \Phi \cdot \overline{\operatorname{curl} u} = \int_{\Sigma} (\nabla \rho + \mu \chi_a) \cdot \overline{\operatorname{curl} u}.$$

It remains to integrate by parts and to exploit $\operatorname{div} \operatorname{curl} u = 0$ and $\operatorname{curl} \chi_a = 0$ on Σ . Regarding boundary integrals, we note that $\operatorname{curl} u$ is divergence free on Y , hence its normal trace on T has no jump. Since $\operatorname{curl} u$ vanishes in $Y \setminus \Sigma$ its normal trace on T vanishes as an element of $W^{-\frac{1}{2},2}(T)$. We therefore obtain

$$\begin{aligned} \int_{\Sigma} \nabla \rho \cdot \overline{\operatorname{curl} u} &= \int_T \rho (\overline{\operatorname{curl} u} \cdot n) = 0, \\ \int_{\Sigma} \chi_a \cdot \overline{\operatorname{curl} u} &= \int_T n \wedge \chi_a \cdot \bar{u} = \int_T \chi_b \cdot \bar{u} = 0. \end{aligned}$$

Thus $\langle f, u \rangle = 0$, which concludes the proof of Lemma 1. \square

Proof of Proposition 1. Let $\mathcal{V} \subset X$ be the subspace of all $u \in X$ such that $(u, \frac{i}{\omega \varepsilon_0} \operatorname{curl} u)$ solves (3.9)–(3.14). The first statement of the proposition can be rephrased by saying that the linear map

$$\mathcal{L} : \mathcal{V} \ni u \mapsto \left(\int_Y u \cdot e_1, \int_Y u \cdot e_2, \int_Y u \cdot e_3, \int_T u \cdot \chi_b \right) \in \mathbb{C}^4$$

is one to one.

Step 1. Injectivity. We prove that \mathcal{L} is injective such that $\dim(\mathcal{V}) \leq 4$. Let $u \in \mathcal{V}$ be a solution with $\int_Y u = 0$ and $\int_T u \cdot \chi_b$. Given u , we *define* the distribution f through $\langle f, v \rangle = b(u, v) \forall v \in X$. The second assertion of Lemma 1 yields $u \equiv 0$ as soon as we show $f \in \mathcal{F}_\Sigma$.

Let $v = \nabla\varphi$ be a gradient. Then $b(u, v) = 0$ because of $\text{curl } \nabla\varphi = 0$ and $\text{div } u = 0$. This shows $\text{div } f = 0$.

Let now v be supported on Σ . Then $b(u, v) = 0$ because of $\text{div } u = 0$ and $\kappa^{-1} \text{curl } \text{curl } u = -ik_0^2 u$. This proves that f vanishes on Σ . We apply Lemma 1 and find the result.

Step 2. Surjectivity. In a second step we prove that \mathcal{V} contains at least four linearly independent solutions which yields $\dim(\mathcal{V}) = 4$ and the surjectivity of \mathcal{L} . To that aim we apply the first statement of Lemma 1 choosing special elements f_0, f_1, f_2, f_3 in \mathcal{F}_Σ . For $k \in \{1, 2, 3\}$, we take $\langle f_k, v \rangle = -ik_0^2 \int_Y g_k \cdot \bar{v}$ where g_k is any divergence-free L^2 function vanishing in Σ such that $\int_Y g_k = e_k$. For instance, for $k \in \{1, 2, 3\}$, we may take g_k to be compactly supported in a small cylinder with principal axis Γ_l and constant in the direction e_k where, with $R = 1/2 - \delta$ close to $1/2$ and $J = (-1/2, 1/2)$,

$$\Gamma_1 = J \times \{R\} \times \{R\}, \quad \Gamma_2 = \{R\} \times J \times \{R\}, \quad \Gamma_3 = \{R\} \times \{R\} \times J. \quad (3.19)$$

being δ so small that $\Gamma_l \cap \bar{\Sigma} = \emptyset$.

By Lemma 1, the equation $b(u, \cdot) = \langle f_k, \cdot \rangle$ has a solution $u = U_k$ in \mathcal{V} which satisfies the integral condition $\int_Y U_k = ik_0^{-2} \langle f_k, 1 \rangle = \int_Y g_k = e_k$. We observe that the vector fields g_k constructed above does not circulate around the ring.

In contrast, we choose now for f_0 the distribution $\langle f_0, v \rangle := -\int_T \chi_b \cdot \bar{v} = \int_Y \text{curl } \chi_a \cdot \bar{v}$. By definition, f_0 vanishes on Σ , an integration by parts shows that f_0 has vanishing divergence. We can therefore solve $b(u, \cdot) = \langle f_0, \cdot \rangle$ and find a solution $u = U_0$ in \mathcal{V} with $\int_Y U_0 = ik_0^{-2} \int_Y f_0 = 0$. Clearly if U_0 does not vanish identically, by the integral average conditions, $\{U_0, U_1, U_2, U_3\}$ will be a system of four linearly independent solutions in V . To check this last point, we consider a potential vector Φ_a for the divergence-free function χ_a , $\text{curl } \Phi_a = \chi_a$. As χ_a has zero mean value, Φ_a can be chosen in $W_{\text{per}}^{1,2}(Y, \mathbb{C}^3)$ and thus in X (in fact we can explicit $\Phi_a = p_a e_3$ being p_a the weight function introduced in (3.32)). Taking $v = \Phi_a$ as test function in the variational equation, we get

$$b(U_0, \Phi_a) = \langle f_0, \Phi_a \rangle = \int_Y \text{curl } \chi_a \cdot \bar{\Phi}_a = \int_\Sigma |\chi_a|^2 > 0,$$

showing that U_0 cannot vanish identically. The proof of Proposition 1 is achieved. \square

The results of the two previous Subsections can be summarized as follows. With the cell solutions $E^k(y)$, $k = 1, 2, 3$, for the electric field and the cell solutions $(H^k(y), J^k(y))$, $k = 0, 1, 2, 3$, of Proposition 1, the two-scale limits can be

written for $x \in \Omega$ as

$$E_0(x, y) = \sum_{k=1}^3 E_k(x) E^k(y), \quad (3.20)$$

$$H_0(x, y) = j(x) H^0(y) + \sum_{k=1}^3 H_k(x) H^k(y), \quad (3.21)$$

$$J_0(x, y) = j(x) J^0(y) + \sum_{k=1}^3 J_k(x) J^k(y). \quad (3.22)$$

In this expression, the number $j(x) \in \mathbb{C}$ is a measure for *the strength of the electric field in the ring*. Recall that for $x \in Q \setminus \Omega$, one has $E_0(x, y) = E(x)$ and $H_0(x, y) = H(x)$.

3.4 Circulation tensor and flux parameters

In the homogenization process, besides the tensor \mathcal{N} defined in (3.6), several quantities depending on ω, κ and the geometry, will appear to be crucial. They are by-products of the shape functions H^k obtained through the H_0 -cell problem.

The circulation vectors M_k . Recalling definition (3.19) of the reference line segments Γ_l for $l \in \{1, 2, 3\}$, we introduce the vectors $M_k \in \mathbb{C}^3$ for $k \in \{0, 1, 2, 3\}$,

$$M_k \cdot e_l := \int_{\Gamma_l} H^k(y) \cdot e_l d\mathcal{H}^1(y), \quad l \in \{1, 2, 3\}. \quad (3.23)$$

The vector M_k is the average strength of the shape vector field $H^k(y)$ in direction e_l along the curve Γ_l (in agreement with the considerations in [16]).

Let D_Y^0 denote the two-dimensional disk spanning the curve Σ_0 , i.e. $D_Y^0 = \{(y_1, y_2, 0) \in Y : |(y_1, y_2)| < \rho\}$. It is important to notice that $Z := Y \setminus (\bar{\Sigma} \cup D_Y^0)$ is a *simply connected domain* on which the periodic field H^k is curl-free. Therefore in the definition (3.23), the segment Γ_l can be substituted with any oriented curve in Z joining two points a, b on opposite faces of Y such that $b - a = e_l$. In particular, if we consider the e_l -parallel vector flux g_l introduced in the proof of Proposition 1, we obtain that $M_k \cdot e_l = \int_Y H^k \cdot g_l$.

Concerning the vectors M_k , we will exploit the following remarkable characterization of averages of the generalized Poynting vectors $H^k \wedge E^l$.

Lemma 2. *There holds, for every $k \in \{0, 1, 2, 3\}$ and $l \in \{1, 2, 3\}$*

$$\int_Y H^k(y) \wedge E^l(y) dy = M_k \wedge e_l \in \mathbb{C}^3. \quad (3.24)$$

Furthermore, there exist complex coefficients $\mathbf{m}_k = m'_k + i m''_k$ (depending on ω, κ) such that

$$M_k = \mathbf{m}_k(\omega, \kappa) e_k \quad \text{for } k \in \{1, 2, 3\}, \quad M_0 = \mathbf{m}_0(\omega, \kappa) e_3, \quad (3.25)$$

$$m'_k > 0, \quad m''_k < 0 \quad \text{for } k \in \{1, 2, 3\}, \quad \mathbf{m}_1 = \mathbf{m}_2. \quad (3.26)$$

The characterization of (3.25)–(3.26) is a consequence of the symmetries of our particular geometry, whereas (3.24) is a consequence of the fact that Z is simply connected.

Proof. Step 1. We substitute E^l with a periodic vector field p^l that satisfies

$$p^l = E^l \quad \text{on } \partial Y, \quad \text{curl } p^l = 0, \quad \text{supp } p^l \subset Z, \quad \int_Y p^l = e_l. \quad (3.27)$$

To that aim we recall the representation $E^l = e_l + \nabla \phi^l$ with ϕ^l from (3.5), and set $p^l = e_l + \nabla(\theta \phi^l)$, where θ is any periodic and smooth cut-off function such that $\theta = 1$ in a neighborhood of ∂Y and $\theta = 0$ on $\bar{\Sigma} \cup D_Y^0$. Then, since the field $w = E^l - p^l$ is curl-free, vanishes on ∂Y , and agrees with E^l on $\bar{\Sigma}$, integration by parts implies, for any $m \in \{1, 2, 3\}$,

$$\begin{aligned} \int_Y (H^k(y) \wedge w(y)) \cdot e_m \, dy &= - \int_Y y_m \operatorname{div} (H^k(y) \wedge w(y)) \, dy \\ &= - \int_Y y_m \operatorname{curl} H^k \cdot w = - \int_Y y_m \operatorname{curl} H^k \cdot E^l = 0, \end{aligned}$$

where in the last line we used $\operatorname{curl} H^k = 0$ on $Y \setminus \Sigma$ and $w = E^l = 0$ on Σ . Therefore

$$\int_Y (H^k(y) \wedge E^l(y)) \cdot e_m \, dy = \int_Y (H^k(y) \wedge p^l(y)) \cdot e_m \, dy.$$

Now we exploit that the periodic vector field $H^k(y) - M_k$ is curl-free on Z , thus of the form $\nabla \psi^k$ for a suitable scalar potential ψ^k . By construction, averages of $H^k(y) - M_k$ vanish along any curve in Z joining two points a, b on opposite faces of the cube. Therefore ψ^k is periodic. This allows to integrate by parts without boundary integrals and to conclude

$$\begin{aligned} \int_Y (H^k(y) \wedge p^l(y)) \cdot e_m \, dy &= \int_Y (M_k \wedge p^l) \cdot e_m + \int_{Y \setminus \Sigma} \nabla \psi^k \cdot (p^l \wedge e_m) \\ &= (M_k \wedge e_l) \cdot e_m. \end{aligned}$$

This provides (3.24).

Step 2. Consider the reflection $R_3 : (y_1, y_2, y_3) \mapsto (y_1, y_2, -y_3)$. One checks easily that, for all $k \in \{0, 1, 2, 3\}$, $R_3 H^k R_3$ solves the H_0 -cell problem. Additionally, for $k \in \{1, 2\}$, it satisfies the integral conditions $\int_Y R_3 H^k R_3 = R_3 e_k = e_k$ and $\int_T (R_3 H^k R_3) \cdot \chi_b = 0$. By the uniqueness result of Proposition 1, we deduce $R_3 H^k R_3 = H_k$. By the same uniqueness and symmetry arguments using also the reflections $R_1 : (y_1, y_2, y_3) \mapsto (-y_1, y_2, y_3)$ and $R_2 : (y_1, y_2, y_3) \mapsto (y_1, -y_2, y_3)$, we obtain $R_l H^k R_l = H^k$ for $k \in \{1, 2, 3\}$ and $l \neq k$ and, for $k = 0$, $R_l H^0 R_l = H^0$ for $l \neq 3$. Recalling definition (3.23), we derive that

$$M_k \cdot e_l = \int_{\Gamma_l} H^k \cdot e_l = \int_{\Gamma_l} R_l H^k R_l \cdot e_l = - \int_{\Gamma_l} H^k \cdot e_l = 0$$

whenever $l \neq k$ if $k > 0$ or $l \neq 3$ if $k = 0$. This proves (3.25).

By the invariance of all equations with respect to the rotation $y \mapsto e_3 \wedge y$, we similarly obtain that $M_1 \cdot e_1 = M_2 \cdot e_2$. Thus $\mathbf{m}_1 = \mathbf{m}_2$.

Step 3. Let us now prove that all coefficients $\mathbf{m}_k = m'_k + im''_k$ have a positive real part and a negative imaginary part. By Proposition 1 and Lemma 1, H^k is characterized for $k \geq 1$ by the equation

$$b(H^k, v) = -ik_0^2 \frac{1}{|T_k|} \int_{T_k} e_k \cdot \bar{v}, \quad \forall v \in X,$$

where T_k is a small cylinder in Y along the axis Γ_k (see (3.19)). Taking $v = H^k$ we derive

$$\int_Y \frac{1}{\kappa} |\operatorname{curl} H^k|^2 - ik_0^2 \int_Y |H^k|^2 = -ik_0^2 \frac{1}{|T_k|} \int_{T_k} e_k \cdot \overline{H^k} = -ik_0^2 \overline{m_k}.$$

Thus we are led to

$$m'_k = \int_Y |H^k|^2 dy > 0, \quad m''_k = -\frac{1}{\kappa k_0^2} \int_Y |\operatorname{curl} H^k|^2 dy < 0. \quad (3.28)$$

□

The magnetic flux parameters D_k . The remaining relevant quantity is the flux of $H_0(x, \cdot)$ through the ring. Since we should define the quantities as an integral over three-dimensional domains, some care should be employed.

We introduce a parameter z in the disk

$$U = U_{\beta, \rho} := \{z = (r, t) \in \mathbb{R}^2 : (r - \rho)^2 + t^2 \leq \beta^2\}, \quad (3.29)$$

such that the set $\{(r, 0, t) \in \mathbb{R}^3 : (r, t) \in U\}$ represents a cross section of the ring. For every $z = (r, t) \in U$, we denote by Γ_Σ^z the circle $\{(y_1, y_2, y_3) : y_1^2 + y_2^2 = r^2, y_3 = t\}$ passing through the position $(r, 0, t)$. We can now introduce $D_\Sigma^z = \operatorname{conv}(\Gamma_\Sigma^z)$, the two-dimensional disk spanned by Γ_Σ^z . The union of such disks coincides with the convex hull $\operatorname{conv}(\Sigma)$ of Σ . We finally introduce, for $k \in \{0, 1, 2, 3\}$, the weighted magnetic flux as the complex number

$$\mathbf{D}_k(\omega, \kappa) := \int_U \frac{1}{r} \left(\int_{D_\Sigma^{r,t}} H^k(y) \cdot e_3 d\mathcal{H}^2(y) \right) dr dt. \quad (3.30)$$

This number can be rewritten as a bulk integral with respect to a weight function p_a , which is compactly supported in $\operatorname{conv}(\Sigma)$,

$$\mathbf{D}_k(\omega, \kappa) = \int_Y p_a(y) (H^k \cdot e_3) dy, \quad (3.31)$$

if we set

$$p_a(y) := \begin{cases} \log \left(\frac{\rho + \sqrt{\beta^2 - y_3^2}}{\max\{\rho - \sqrt{\beta^2 - y_3^2}, \sqrt{y_1^2 + y_2^2}\}} \right) & \text{if } y \in \operatorname{conv}(\Sigma), \\ 0 & \text{otherwise.} \end{cases} \quad (3.32)$$

Furthermore, noticing that $\chi_a(y) = \frac{1}{r} \tau_a(y)$ holds for every $y \in \Gamma_\Sigma^{r,t}$, and using the Kelvin-Stokes Theorem on each disk D_Σ^z , we may write alternatively $D_k(\omega, \kappa)$ in terms of a potential vector $\psi^k(y)$ of the divergence free field H^k ,

$$\begin{aligned} \mathbf{D}_k(\omega, \kappa) &= \int_U \frac{1}{r} \left(\int_{D_\Sigma^{r,t}} \operatorname{curl} \psi^k(y) \cdot e_3 d\mathcal{H}^2(y) \right) dr dt \\ &= \int_U \left(\int_{\Gamma_\Sigma^{r,t}} \psi^k \cdot \chi_a d\mathcal{H}^1(y) \right) dr dt = \int_\Sigma \psi^k \cdot \chi_a dy. \end{aligned} \quad (3.33)$$

We observe that this formula can be recovered by noticing that $\chi_a = \operatorname{curl}(p_a e_3)$.

Using the symmetry arguments of Lemma 2, it is easy to check that

$$\mathbf{D}_1(\omega, \kappa) = \mathbf{D}_2(\omega, \kappa) = 0. \quad (3.34)$$

4 Macroscopic constitutive laws

4.1 Relation law between $j(x)$ and $H(x)$

In this section we establish a linear relation between the third component of the averaged magnetic field, and the averaged strength of the electric field in the ring, namely

$$j(x) = \lambda(\omega, \kappa) H_3(x). \quad (4.1)$$

The explicit expression of the dimensionless factor $\lambda(\omega, \kappa)$ appeared already in (2.3). The limit of an infinite conductivity $\kappa \rightarrow \infty$ will be studied in Subsection 4.3. We recall that the real part of $\lambda(\omega, \kappa)$ can have both signs.

Showing (4.1) is a delicate task which requires a careful analysis of the electric field in the ring and in the slit. Lemma 3 below makes the following loose statement precise: The field $J_\eta = \eta \varepsilon_\eta E_\eta$ has values of order $O(1)$ in the ring and in the slit. As a consequence, typical values of E_η are of order $O(\eta)$ in the ring, and of order $O(\eta^{-1})$ in the slit. We make use of the special vector field $\chi_a^\eta(x) := \chi_a(x/\eta)$ where χ_a is the periodic function, supported on Σ , which was introduced in Subsection 3.3. The set of all slits S_η is defined by

$$S_\eta := \bigcup_j \{\eta(j + S_Y^\eta)\}, \text{ where } S_Y^\eta := \{y \in \Sigma : |y_1| < \alpha \eta^2 y_2 / \rho\}.$$

Here and in the following, the index j runs over $\{j \in \mathbb{Z}^3 : \eta(j+Y) \subset \Omega\}$. We have therefore a partitioning of the set of closed rings $\bigcup \eta(j + \Sigma)$ into the split-rings Σ_η and the slits S_η . We notice that the volumes are $|\Sigma_\eta| = O(1)$ and $|S_\eta| = O(\eta^2)$.

Lemma 3. *For every function $\psi \in \mathcal{D}(\Omega)$, there holds*

$$\lim_{\eta \rightarrow 0} \frac{\kappa}{\eta} \int_{\Sigma_\eta} E_\eta(x) \cdot \chi_a^\eta(x) \psi(x) dx = -\frac{1}{\varepsilon_0 \omega} \int_\Omega j(x) \psi(x) dx, \quad (4.2)$$

$$\lim_{\eta \rightarrow 0} \frac{\pi \rho}{\alpha \eta} \int_{S_\eta} E_\eta(x) \cdot \chi_a^\eta(x) \psi(x) dx = -\frac{i}{\varepsilon_0 \omega} \int_\Omega j(x) \psi(x) dx. \quad (4.3)$$

Thus, in the distributional sense in Ω , we have

$$\frac{\omega\varepsilon_0}{\eta} E_\eta \cdot \chi_a^\eta \xrightarrow{*} - \left(\frac{1}{\kappa} + i \frac{\alpha}{\pi\rho} \right) j(x). \quad (4.4)$$

Let us show (4.1), admitting for the moment Lemma 3. The main trick consists in slicing the bulk integral of $\frac{1}{\eta} E_\eta \cdot \chi_a^\eta$ in line integrals along the circles $\Gamma_\eta^z = \bigcup_j \eta(j + \Gamma_Y^z)$, where the circles Γ_Y^z for $z \in U = U_{\beta,\rho} \subset \mathbb{R}^2$ were introduced in Subsection 3.4. In a similar way as for deriving (3.33), we transform the line integrals over $\Gamma_\eta^{r,t}$ into area integrals over the disks $D_\eta^{r,t} = \bigcup_j \eta(j + D_Y^{r,t})$. Recalling that $r \chi_a^\eta$ represents the unit oriented tangent vector to $\Gamma_\eta^{r,t}$, for any smooth function $\psi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} \int_\Omega \frac{1}{\eta} E_\eta \cdot \chi_a^\eta \psi dx &= \int_U \eta \left(\int_{\Gamma_\eta^{r,t}} \psi(x) E_\eta(x) \cdot \chi_a^\eta(x) d\mathcal{H}^1(x) \right) dr dt \\ &= \int_U \frac{1}{r} \left(\int_{D_\eta^{r,t}} \operatorname{curl}_x (\psi E_\eta) \cdot e_3 d\mathcal{H}^2(x) \right) dr dt \\ &= \int_\Omega p_a(x/\eta) [i\omega\mu_0 \psi H_\eta \cdot e_3 + (\nabla\psi \wedge E_\eta) \cdot e_3] d\mathcal{L}^3 \end{aligned}$$

where in the last line p_a stands for the periodic extension of the weight function appearing in (3.32). Note that in the line (or area) integrals above, the parametrization of Γ_η^z or D_η^z with respect to $z = (r, t) \in U_{\beta,\rho}$ induces by change of variables a factor η^2 (or η). We may pass now to the limit in the last integral by using the two-scale convergence of (E_η, H_η) . Recalling that $E_0(x, \cdot)$ vanishes on Σ where p_a is supported, we derive

$$\lim_{\eta \rightarrow 0} \int_\Omega \frac{1}{\eta} E_\eta \cdot \chi_a^\eta \psi dx = i\omega\mu_0 \int_\Omega \left(\int_\Sigma p_a(y) H_0(x, y) \cdot e_3 dy \right) \psi(x) dx.$$

The left hand side limit above can be identified by means of (4.4), while the right hand side can be computed by using (3.21) and (3.31). Since ψ was arbitrary, we can localize in x and conclude, for almost all $x \in \Omega$,

$$-\frac{1}{\omega\varepsilon_0} \left(\frac{1}{\kappa} + i \frac{\alpha}{\pi\rho} \right) j(x) = i\omega\mu_0 \left(D_0(\omega, \kappa)j(x) + \sum_{k=1}^3 D_k(\omega, \kappa)H_k(x) \right).$$

Taking into account (3.34), we are led to (4.1) with

$$\lambda(\omega, \kappa) = \frac{-\varepsilon_0\mu_0\omega^2 D_3(\omega, \kappa)}{\alpha(\pi\rho)^{-1} + \varepsilon_0\mu_0\omega^2 D_0(\omega, \kappa) - i\kappa^{-1}}.$$

At this point, we have a complete description of the microscopic behavior of the magnetic field, using only the averaged magnetic field as an input. Indeed, from (3.21) and (4.1),

$$H_0(x, y) = H_1(x)H^1(y) + H_2(x)H^2(y) + H_3(x) (H^3(y) + \lambda(\omega, \kappa)H^0(y)). \quad (4.5)$$

It remains to prove Lemma 3.

Proof of Lemma 3. Relation (4.4) is a direct consequence of (4.2) and (4.3). In the course of the proof we will use the following fact: for every smooth Y -periodic function $\xi(y)$ and every $\psi \in \mathcal{D}(\Omega)$, we have

$$\lim_{\eta \rightarrow 0} \kappa \int_{\Sigma_\eta} \frac{1}{\eta} E_\eta \cdot \chi_a^\eta(x) \psi(x) \xi(x/\eta) dx = -i \int_{\Omega} \int_Y J_0(x, y) \cdot \chi_a(y) \psi(x) \xi(y) dy dx. \quad (4.6)$$

Indeed, as $-i(\varepsilon_\eta - 1) = \frac{\kappa}{\eta^2} \mathbf{1}_{\Sigma_\eta}$, recalling (3.2),

$$\begin{aligned} \kappa \int_{\Sigma_\eta} \frac{1}{\eta} E_\eta \cdot \chi_a^\eta(x) \psi(x) \xi(x/\eta) dx &= -i\eta \int_{\Omega} (\varepsilon_\eta - 1) E_\eta \cdot \chi_a^\eta(x) \psi(x) \xi(x/\eta) dx \\ &= -i \int_{\Omega} J_\eta \cdot \chi_a^\eta(x) \psi(x) \xi(x/\eta) dx + i\eta \int_{\Omega} E_\eta \cdot \chi_a^\eta(x) \psi(x) \xi(x/\eta) dx. \end{aligned}$$

The L^2 bound for E_η implies that the last integral vanishes for $\eta \rightarrow 0$. The other integral can be evaluated thanks to the two-scale convergence of J_η to J_0 and we are led to (4.6).

Proof of (4.2). It is enough to apply (4.6) with $\xi \equiv 1$, exploiting (3.22) and the normalizations (3.15) and (3.16).

Proof of (4.3). We are going to construct a special η -periodic test-function that coincides with χ_a^η on Σ_η . To that aim, we define, for $\delta > 0$ sufficiently small two subsets of Y ,

$$\Sigma_Y^\delta := B_{\beta+\delta}(\Sigma_0) \text{ (larger ring)}, \quad R_Y^\eta := \{y \in Y : |y_1| \leq \alpha\eta^2 y_2/\rho\} \text{ (wedge)},$$

such that the slit can also be written as $S_Y^\eta = \Sigma \setminus R_Y^\eta$. The parameter $\delta < \min(\beta, 1/2 - \beta)$ will be sent to 0 later on. We use a smooth cut-off function $\xi_\delta : Y \mapsto [0, 1]$, compactly supported in Σ_Y^δ with $\xi_\delta = 1$ on Σ .

Finally, we introduce a piecewise affine function $g_\eta : [0, 2\pi] \mapsto [0, 2\pi]$ as follows. We denote by $\theta_\eta := \alpha\eta^2/\rho$ the number related to the angle of the wedge. The function g_η is defined as the affine interpolation of the four values $g_\eta(0) = \pi$, $g_\eta(\theta_\eta) = \theta_\eta$, $g_\eta(2\pi - \theta_\eta) = 2\pi - \theta_\eta$, and $g_\eta(2\pi) = \pi$. Then, using the polar coordinates $y_1 = -r \sin \theta$, $y_2 = r \cos \theta$ with $\theta \in [0, 2\pi[$ for $y = (y_1, y_2, y_3) \in Y$, we set

$$\varphi^\eta(y) := g_\eta(\theta) \text{ for all } y \in \Sigma_Y^\delta.$$

By construction, φ^η is Lipschitz from Σ to $[0, 2\pi]$ and satisfies

$$\nabla \varphi^\eta = \chi_a \text{ in } \Sigma_Y^\delta \setminus R_Y^\eta, \quad \nabla \varphi^\eta = -\frac{\pi\rho}{\alpha\eta^2} \chi_a \text{ in } \Sigma_Y^\delta \cap R_Y^\eta \quad (4.7)$$

$$|\nabla \varphi^\eta| \leq C \text{ in } \Sigma_Y^\delta \setminus R_Y^\eta, \quad |\nabla \varphi^\eta| \leq C \frac{1}{\eta^2} \text{ in } R_Y^\eta \cap \Sigma_Y^\delta, \quad (4.8)$$

where C is a suitable constant independent of η and δ .

With the above functions we can now perform the limit analysis. Exploiting that $J_\eta = \eta \varepsilon_\eta E_\eta$ is divergence free, we find that for $\psi \in \mathcal{D}(\Omega)$ holds

$$f_\eta := \operatorname{div}(\eta J_\eta \xi_\delta(x/\eta) \psi(x)) = J_\eta \cdot [(\nabla_y \xi_\delta)(x/\eta) \psi(x) + \eta \xi_\delta(x/\eta) \nabla \psi(x)].$$

Observing that $\nabla_y \xi_\delta(x/\eta)$ is bounded and vanishes on Σ_η , we obtain

$$|J_\eta \cdot \nabla_y \xi_\delta(x/\eta) \psi(x)| \leq C\eta |E_\eta|, \quad |J_\eta \cdot \eta \xi_\delta(x/\eta) \nabla \psi| \leq C\eta |J_\eta|.$$

As we know that E_η, J_η are uniformly bounded in $L^2(\Omega)$, we infer that $f_\eta \rightarrow 0$ strongly in $L^2(\Omega)$. We apply f_η to our special test function φ^η and calculate with an integration by parts

$$\begin{aligned} 0 &= \lim_{\eta \rightarrow 0} \int_{\Omega} f_\eta \varphi^\eta \left(\frac{x}{\eta} \right) dx = \lim_{\eta \rightarrow 0} \int_{\Omega} J_\eta \cdot \nabla_y \varphi^\eta(x/\eta) \xi_\delta(x/\eta) \psi(x) dx \\ &= \lim_{\eta \rightarrow 0} (I_\delta^1 + I_\delta^2 + I_\delta^3 + I_\delta^4), \end{aligned} \quad (4.9)$$

where the I_δ^m 's are related to the integration over the four elements of the partition $\Omega = \Sigma_\eta \cup S_\eta \cup [R_\eta \setminus S_\eta] \cup [\Omega \setminus (\Sigma_\eta \cup R_\eta)]$, with the meanwhile standard notation for the union of the wedges $R_\eta := \Omega \cap \cup_j \eta(j + R_Y^\eta)$. Thanks to (4.7) and since $J_\eta = \eta E_\eta$ in $\Omega \setminus \Sigma_\eta$,

$$\begin{aligned} I_\delta^1 &= \int_{\Sigma_\eta} J_\eta \cdot \chi_a^\eta(x) \psi(x) dx \\ I_\delta^2 &= -\frac{\pi\rho}{\alpha} \int_{S_\eta} \frac{E_\eta}{\eta} \cdot \chi_a^\eta(x) \psi(x) dx \\ I_\delta^3 &= \int_{\Omega \setminus (\Sigma_\eta \cup R_\eta)} \eta E_\eta \cdot \nabla_y \varphi^\eta(x/\eta) \xi_\delta(x/\eta) \psi(x) dx \\ I_\delta^4 &= \int_{R_\eta \setminus S_\eta} \eta E_\eta \cdot \nabla_y \varphi^\eta(x/\eta) \xi_\delta(x/\eta) \psi(x) dx \end{aligned}$$

Thanks to (4.8), recalling that ξ_δ and ψ are bounded, we find for I_δ^3 and I_δ^4

$$\begin{aligned} |I_\delta^3| &\leq C\eta \int_{\Omega \setminus (\Sigma_\eta \cup R_\eta)} |E_\eta| \xi_\delta(x/\eta) dx \leq C\eta \sqrt{|\Omega|} \|E_\eta\|_{L^2(\Omega)} \\ |I_\delta^4| &\leq \frac{C}{\eta} \int_{R_\eta \setminus S_\eta} |E_\eta| \xi_\delta(x/\eta) dx \leq \frac{C}{\eta} \|E_\eta\|_{L^2(\Omega)} \left(\int_{R_\eta \setminus S_\eta} |\xi_\delta|^2(x/\eta) dx \right)^{1/2} \\ &\leq C\sqrt{\delta} \|E_\eta\|_{L^2(\Omega)}, \end{aligned}$$

where C is a generic constant, independent of η and δ , and where in the last line we used the fact that the periodic function ξ_δ is compactly supported in Σ_Y^δ so that

$$\int_{R_\eta \setminus S_\eta} |\xi_\delta|^2(x/\eta) dx \leq C \int_{R_Y^\eta \setminus S_Y^\eta} |\xi_\delta|^2(y) dy \leq C |R_Y^\eta \cap (\Sigma_Y^\delta \setminus \Sigma)| \leq C\delta\eta^2.$$

Summarizing, we are led to

$$\limsup_{\eta \rightarrow 0} |I_\delta^3 + I_\delta^4| \leq C\sqrt{\delta}.$$

Sending $\delta \rightarrow 0$ and taking into account (4.9), it follows that

$$\begin{aligned} \lim_{\eta \rightarrow 0} \frac{\pi\rho}{\alpha} \int_{S_\eta} \frac{E_\eta}{\eta} \cdot \chi_a^\eta(x) \psi(x) dx &= \lim_{\eta \rightarrow 0} \int_{\Sigma_\eta} J_\eta \cdot \chi_a^\eta(x) \psi(x) dx \\ &= \int_{\Omega} \int_Y J_0(x, y) \cdot \chi_a \psi(x) dx dy = \frac{1}{i\omega\varepsilon_0} \int_{\Omega} j(x) \psi(x) dx. \end{aligned}$$

This concludes the proof of Lemma 3. \square

4.2 Homogenized equations for $(E(x), H(x))$

We are now in position to establish the homogenized equation (2.7). We recall that (2.6) followed immediately taking weak limits.

For arbitrary $\psi \in \mathcal{D}(Q, \mathbb{R})$ we use $\psi(x)E^l(x/\eta)$ as a test function in equation (1.2). Since E^l is curl-free and vanishes on Σ_η , integrating by parts in Q we obtain

$$\begin{aligned} -i\omega\varepsilon_0 \int_Q E_\eta(x) \cdot E^l(x/\eta) \psi(x) dx &= \int_Q \operatorname{curl} H_\eta(x) \cdot E^l(x/\eta) \psi(x) dx \\ &= \int_Q H_\eta(x) \cdot (\nabla\psi(x) \wedge E^l(x/\eta)) dx. \end{aligned}$$

We may pass to the limit as $\eta \rightarrow 0$ by using the two-scale convergence of (E_η, H_η) ,

$$i\omega\varepsilon_0 \int_Q \int_Y E_0(x, y) \cdot E^l(y) \psi(x) dy dx = \int_Q \int_Y \nabla\psi(x) \cdot (H_0(x, y) \wedge E^l(y)) dy dx.$$

Since ψ was arbitrary, we deduce the following equation which holds, for every $l \in \{1, 2, 3\}$, in the distributional sense on Q .

$$\operatorname{div}_x \left(\int_Y (H_0(x, y) \wedge E^l(y)) dy \right) = -i\omega\varepsilon_0 \int_Y E_0(x, y) \cdot E^l(y) dy. \quad (4.10)$$

Using the expansions of E_0 in (3.20) and the tensor \mathcal{N} of (3.6) (recalling that E^l is real and \mathcal{N} is symmetric), we find

$$\int_Y E_0(x, y) \cdot E^l(y) dy = \begin{cases} \sum_{k=1}^3 \mathcal{N}_{kl} E_k(x) & \text{if } x \in \Omega \\ E_l(x) & \text{if } x \notin \Omega. \end{cases}$$

For H_0 we use the expansion (4.5) containing $\lambda = \lambda(\omega, \kappa)$, and (3.24) which related generalized Poynting vectors with M_k .

$$\int_Y H_0(x, y) \wedge E^l(y) dy = \begin{cases} \sum_{k=1}^3 H_k(x) M_k \wedge e_l + \lambda H_3(x) M_0 \wedge e_l & \text{if } x \in \Omega \\ H(x) \wedge e_l & \text{if } x \notin \Omega. \end{cases}$$

We recall that \mathcal{M}_λ denotes the 3×3 complex matrix with columns $M_1, M_2, M_3 + \lambda M_0$, see (2.2) and (3.25). Having $\hat{M}(x)$ and $\hat{N}(x)$ defined as in the statement of Theorem 1, in particular as \mathcal{M}_λ and \mathcal{N} inside Ω , we may rewrite the equation (4.10) as

$$\operatorname{div}((\hat{M}(x) \cdot H(x)) \wedge e_l) = -i\omega\varepsilon_0(\hat{N}(x)E(x)) \cdot e_l.$$

Since the left hand side can also be written as $\operatorname{curl}(\hat{M} \cdot H) \cdot e_l$, we arrive at the homogenized equation (2.7).

4.3 The high conductivity limit $\kappa \rightarrow \infty$

In this Subsection we study the case of a large conductivity parameter κ in the rings. Our interest is to find simplified expressions for the coefficients D_k and m_k that enter the effective permeability of the medium. In particular, we will verify sign conditions that guarantee that, indeed, the effective permeability can have a negative real part.

Limit cell problem. As a counterpart of (3.9)-(3.14), the limit sytem of equations characterizing the shape functions $H^k = H \in W_{\text{per}}^{1,2}(Y, \mathbb{R}^3)$ reads

$$\operatorname{curl}_y H = 0 \text{ on } Y \setminus \bar{\Sigma}, \quad (4.11)$$

$$\operatorname{div}_y H = 0 \text{ on } Y, \quad (4.12)$$

$$H = 0 \text{ on } \Sigma. \quad (4.13)$$

Lemma 4 (Large conductivity limit process). *We study $H \in W_{\text{per}}^{1,2}(Y, \mathbb{R}^3)$ solving (4.11)-(4.13).*

(i) *existence and uniqueness: There exist four solutions H^k with the normalization $\int_Y H^k = e_k$ for $k = 1, 2, 3$, $\int_Y H^0 = 0$, and, for the traces from $Y \setminus \Sigma$, with $\int_{\partial\Sigma} H^k \cdot \chi_b = \delta_{k0}$. The normalization determines the four solutions uniquely.*

(ii) *convergence: In the limit process $\kappa \rightarrow \infty$, the fields H_κ^k defined in Proposition 1 satisfy $H_\kappa^k \rightharpoonup H^k$ weakly in $L^2(Y, \mathbb{C}^3)$.*

We emphasize that the H^k are real and independent of the frequency ω .

Proof. (i) *Existence of H^k .* Setting $\delta = \kappa^{-1}$, let us denote by H_δ^k the vector fields of Proposition 1. We recall that these were obtained by solving the variational equation $b_\delta(u, v) = \langle f_k, v \rangle$ for special choices of $f_k \in \mathcal{F}_\Sigma$ of (3.17), where

$$b_\delta(u, v) := \delta \int_\Sigma \operatorname{curl} u \cdot \operatorname{curl} \bar{v} + \int_Y \operatorname{div} u \operatorname{div} \bar{v} - ik_0^2 \int_Y u \cdot \bar{v},$$

on $X = W_{\text{per}}^{1,2}(Y, \mathbb{C}^3) \cap \{\operatorname{curl} u = 0 \text{ on } Y \setminus \bar{\Sigma}\}$. Here, we study the limit $\delta \rightarrow 0$ and therefore define

$$b_0(u, v) := \int_Y \operatorname{div} u \operatorname{div} \bar{v} - ik_0^2 \int_Y u \cdot \bar{v},$$

now on the larger Hilbert space

$$X_0 := \{u \in L^2_{\text{per}}(Y, \mathbb{C}^3) : \operatorname{div} u \in L^2_{\text{per}}, \operatorname{curl} u = 0 \text{ on } Y \setminus \bar{\Sigma}\}$$

which we endow with the scalar product $(u, v)_0 = \int_Y (u \cdot \bar{v} + \operatorname{div} u \operatorname{div} \bar{v})$. It is easy to check that X_0 is a dense subspace. The form b_0 is (up to a rotation) coercive on X_0 . For every distribution $f \in \mathcal{F}_\Sigma$ we find $u \in X_0$ with $b_0(u, \cdot) = \langle f, \cdot \rangle$. As in the proof of Lemma 1 one verifies that u solves (4.11)–(4.13). Inserting the special distributions f_k of Proposition 1 yields the existence of the four fields $H^k(y)$.

For the uniqueness argument we study a solution u of (4.11)–(4.13) and note that u satisfies $b_0(u, \cdot) = \langle f, \cdot \rangle$ for $\langle f, v \rangle := -ik_0^2 \int_Y u \cdot \bar{v}$. It is clear that this distribution vanishes on Σ and has vanishing divergence, hence it is contained in \mathcal{F}_Σ . It remains to show that solutions of $b_0(u, \cdot) = \langle f, \cdot \rangle$ with $f \in \mathcal{F}_\Sigma$ vanish identically if only their normalization averages vanish. This fact follows exactly as in the uniqueness part of Lemma 1.

(ii) *Convergence for $\delta = \kappa^{-1} \rightarrow 0$.* We consider a fixed distribution $f \in \mathcal{F}_\Sigma$ and a sequence $u_\delta \in X$ with $b_\delta(u_\delta, \cdot) = \langle f, \cdot \rangle$ on X . We assume $u_\delta \rightharpoonup u_0$ weakly in $L^2(Y)$ for some function u_0 and note that every normalization of u_δ implies the same normalization of u_0 . Our aim is to show that $u_0 \in X_0$ satisfies $b_0(u, \cdot) = \langle f, \cdot \rangle$ on X_0 . Once this is shown, the uniqueness property of the limit problem implies (ii).

Since u_δ has vanishing divergence on Y and vanishing curl on $Y \setminus \bar{\Sigma}$, these properties remain valid for the weak limit, hence $u_0 \in X_0$. With the test function $v = u_\delta$ we find $b_\delta(u_\delta, u_\delta) = \langle f, u_\delta \rangle$, which implies the upper bound

$$\delta \int_\Sigma |\operatorname{curl} u_\delta|^2 + \int_Y |u_\delta|^2 \leq C$$

for a suitable constant C . In particular, $\delta \operatorname{curl} u_\delta \rightarrow 0$ in L^2 , hence u_0 satisfies

$$\langle f, v \rangle = \lim_{\delta \rightarrow 0} b_\delta(u_\delta, v) = b_0(u, v) \quad \forall v \in X.$$

The space X is dense in X_0 , hence $b_0(u, v) = \langle f, v \rangle$ remains valid for all $v \in X_0$. This shows the claim and concludes the proof. \square

Lemma 4 implies that the complex coefficients $\mathbf{m}_k(\omega, \kappa)$ and $\mathbf{D}_k(\omega, \kappa)$ introduced in Subsection 3.4 converge to *real* and *frequency independent* coefficients \mathbf{m}_k and \mathbf{D}_k as $\kappa \rightarrow \infty$. Indeed, the convergence $H_\kappa^k \rightharpoonup H^k$ allows to pass to the limit in equations (3.23), (3.30), and (3.24). In particular, we find for $k \in \{0, 1, 2, 3\}$ the real limits

$$\mathbf{m}_k := \lim_{\kappa \rightarrow \infty} m_k(\omega, \kappa) = \int_{\Gamma_k} H^k(y) \cdot e_k d\mathcal{H}^1(y) \quad (4.14)$$

$$\mathbf{D}_k := \lim_{\kappa \rightarrow \infty} D_k(\omega, \kappa) = \int_U \frac{1}{r} \left(\int_{D_Y^{r,t}} H^k(y) \cdot e_3 d\mathcal{H}^2(y) \right) dr dt \quad (4.15)$$

$$\mathbf{m}_1 = \mathbf{m}_2, \quad \mathbf{D}_1 = \mathbf{D}_2 = 0.$$

Special functions and sign conditions. In this paragraph, our aim is to give a more precise characterization of the real constants m_k and D_k defined in (4.15) and (4.14). We do so by relating the fields H^k to gradients ∇u^k for four special scalar potentials u^k .

For $k \in \{1, 2, 3\}$, we introduce the solutions $u^k \in W_{\text{per}}^{1,2}(Y \setminus \Sigma)$ of the Neumann problem

$$\Delta u = 0 \quad \text{on } Y \setminus \bar{\Sigma}, \quad \frac{\partial u}{\partial n} = -e_k \cdot n \quad \text{on } \partial \Sigma. \quad (4.16)$$

We set

$$b_{kl} := \int_{Y \setminus \Sigma} (e_k + \nabla u^k) \cdot (e_l + \nabla u^l). \quad (4.17)$$

As is well known (see for instance [11]), the positive definite symmetric matrix $B = (b_{kl})$ is associated to the homogenized equation for the Neumann problem with holes. The related quadratic form can be expressed in term of an infimum problem on the unit cell,

$$Bz \cdot z = \inf_{w \in W_{\text{per}}^{1,2}(Y \setminus \Sigma)} \int_{Y \setminus \Sigma} |z + \nabla w|^2.$$

By taking $w = 0$ as a competitor, we derive in particular the inequality

$$Bz \cdot z \geq (1 - f) |z|^2 \quad \text{with} \quad f := |\Sigma| = 2\pi^2 \rho \beta^2, \quad (4.18)$$

the volume fraction of the rings in Ω . Thus B is invertible with real eigenvalues greater than 1. By the symmetries, it is also easy to check that B is diagonal. Our results are collected as

$$\mathbf{b}_k := b_{kk} \geq 1 - f \quad \text{for } k \in \{1, 2, 3\}, \quad b_{kl} = 0 \quad \text{for } k \neq l. \quad (4.19)$$

For $k = 0$, the special periodic function u^0 must be defined in a slightly different way. We construct a periodic function $u^0 : Z \rightarrow \mathbb{R}$, where $Z = Y \setminus (\bar{\Sigma} \cup D_Y^0)$ is the exterior domain without the central disk. We obtain u^0 by solving the minimization problem

$$\min \left\{ \int_Z |\nabla \varphi|^2 : \varphi(y_1, y_2, 0 \pm 0) = \mp 1 \text{ for } |(y_1, y_2)| \leq \rho - \beta \right\}.$$

This problem has a unique solution $u^0 \in W_{\text{per}}^{1,2}(Z)$ which, by the symmetry with respect to $\{y_3 = 0\}$, satisfies $u^0(y_1, y_2, -y_3) = -u^0(y_1, y_2, y_3)$. Therefore it satisfies the following equations on the half periodic cell,

$$\begin{aligned} \Delta u^0 &= 0 \quad \text{on } Y^- \setminus \bar{\Sigma}, & u^0 &= 1 \quad \text{on } D_Y^0 \setminus \bar{\Sigma}, \\ \partial_n u^0 &= 0 \quad \text{on } \partial \Sigma \cap Y^-, & u^0 &= 0 \quad \text{on } \Gamma_0, \end{aligned}$$

where $Y^- := Y \cap \{y_3 < 0\}$ and $\Gamma_0 = \{y_3 = -1/2\} \cup \{y_3 = 0\} \setminus (\bar{\Sigma} \cup D_Y^0)$.

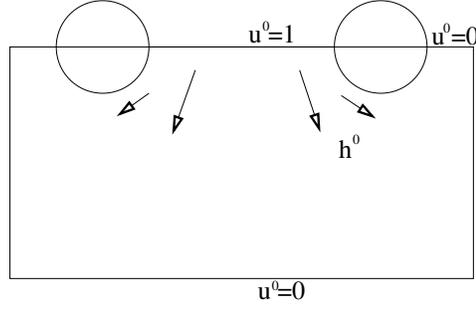


Figure 4: Sketch of the auxiliary function u^0 and the vector field h^0 which has a net flux through the ring.

The jump $[u^0]$ of u^0 across D_Y^0 in the e_3 direction is constant and equal to -2 , whereas on both sides the gradient ∇u^0 has the same trace pointing in the direction of e_3 . We set

$$h^0 := -\frac{1}{4\pi} \nabla u^0 \quad \text{on } Y \setminus \Sigma, \quad h^0 := 0 \quad \text{on } \Sigma. \quad (4.20)$$

By construction, h^0 satisfies (4.11)–(4.13). Since χ_b is divergence free on T , an integration by parts over T yields

$$\int_T h^0 \cdot \chi_b = \frac{1}{4\pi} \int_{\Gamma_\Sigma^{\rho-\beta,0}} (-[u^0]) \chi_b \cdot e_3 = 1.$$

The uniqueness property of Lemma 4 allows us to identify the shape function H^0 ,

$$H^0 = h^0 + \sigma_0 H^3, \quad \text{where } \sigma_0 := -\int_Y h^0 \cdot e_3 dt > 0. \quad (4.21)$$

Here, the positivity of the scalar σ_0 can be checked by writing alternatively

$$\sigma_0 = \frac{1}{2\pi} \int_{Y \setminus \Sigma} \nabla u^0 \cdot e_3 = \frac{1}{2\pi} \left(\int_{\partial \Sigma \cap Y^-} u^0 (-n) \cdot e_3 + \pi(\rho - \beta)^2 \right),$$

where we used symmetry with respect to y_3 and integration by parts. Clearly, by the maximum principle, there holds $0 \leq u^0 \leq 1$ on Y^- , whereas the exterior normal n to Σ satisfies $(-n) \cdot e_3 \geq 0$ on $\partial \Sigma \cap Y^-$. Eventually we need the following constant associated with the set $U_{\rho,\beta}$ (see (3.29)),

$$\gamma := \int_{U_{\rho,\beta}} \frac{1}{r} dr dt.$$

We are now in position to identify the sign of all real constants m_k , D_0 and D_3 which appear in the limit as $\kappa \rightarrow \infty$ in (4.14), (4.15), and in the expression of $\mu^{\text{eff}}(\omega)$ in (2.22).

Lemma 5 (Constants in the limit $\kappa \rightarrow \infty$). *We consider the limiting magnetic fields H^k of (4.11)–(4.13).*

(i) The vector fields H^k are orthogonal in $L^2(Y; \mathbb{R}^3)$. Furthermore,

$$\forall k \geq 1 : \quad \int_Y |H^k|^2 dy = \mathbf{m}_k = \frac{1}{\mathbf{b}_k} \geq \frac{1}{1-f}, \quad \mathbf{m}_0 = \sigma_0 \mathbf{m}_3, \quad (4.22)$$

where f is the volume fraction in (4.18), $\sigma_0 > 0$ is defined by (4.21) and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are the diagonal elements of the Neumann tensor B in (4.19).

(ii) The flux constants satisfy $\mathbf{D}_1 = \mathbf{D}_2 = 0$ whereas

$$\mathbf{D}_0 = -2\pi\gamma \int_Y |H^0|^2 dy, \quad \mathbf{D}_3 = 2\pi\gamma \sigma_0 \int_Y |H^3|^2 dy = 2\pi\gamma \sigma_0 m_3. \quad (4.23)$$

Proof. (i). The real valued shape functions H^k obtained in Lemma 4 can be related to the solutions u^k of the Neumann problem (4.16). We claim that, for $k \geq 1$,

$$H^k(y) = \begin{cases} \frac{1}{b_k}(e_k + \nabla u^k(y)) & \text{if } y \in Y \setminus \Sigma \\ 0 & \text{if } y \in \Sigma \end{cases} \quad (4.24)$$

Indeed, $e_k + \nabla u^k$ is periodic, divergence free in $Y \setminus \Sigma$ and has a vanishing normal trace on T . Thus its extension by zero over Σ is divergence free. It is also curl-free in $Y \setminus \bar{\Sigma}$ and, by (4.17) and (4.19), it satisfies $\int_{Y \setminus \Sigma} (e_k + \nabla u^k) \cdot e_l = b_k \delta_{kl}$. By the uniqueness of the solution of (4.11)–(4.13) with a given integral average on Y , we deduce (4.24). Clearly the associated circulation vector M_k (see (3.23) and (4.14)) satisfies $M_k = \frac{1}{b_k} e_k$, thus $m_k = \frac{1}{b_k}$ for $k \geq 1$. Let now $l \in \{0, 1, 2, 3\}$. As H^l is divergence free and vanishes in Σ , an integration by parts provides

$$\int_Y H^k \cdot H^l = \frac{1}{b_k} \int_{Y \setminus \Sigma} (e_k + \nabla u^k) \cdot H^l = \begin{cases} \frac{1}{b_k} & \text{if } k = l \\ 0 & \text{else.} \end{cases}$$

which implies the orthogonality conditions. Subsequently we obtain (4.22) by taking $l = k$ with the help of (4.19). The last relation for m_0 is a consequence of (4.21) and of the fact that the circulation tensor of h^0 vanishes by the periodicity of u^0 .

(ii). We compute D_k from (4.15). We observe that, since H^k is divergence-free, the flux of $H^k(y)$ across the disks $D_Y^{r,t}$ is independent of $(r, t) \in U = U_{\eta, \rho}$. Recalling that H^k vanishes in Σ , for every $k \geq 0$ we find

$$D_k = \int_U \frac{1}{r} \left(\int_{D_Y^{r,t}} H^k(y) \cdot e_3 d\mathcal{H}^2(y) \right) = \gamma \int_{D_Y^0} H^k(y) \cdot e_3 d\mathcal{H}^2(y).$$

This averaged flux can be generated also from a different expression. Exploiting (4.20), as the jump of u^0 across D_Y^0 is 2, an integration by parts leads to

$$\int_Y H^k \cdot h^0 = -\frac{1}{4\pi} \int_{Y \setminus D_Y^0} H^k \cdot \nabla u^0 = -\frac{1}{2\pi} \int_{D_Y^0} H^k(y) \cdot e_3 d\mathcal{H}^2(y) = -\frac{1}{2\pi\gamma} D_k.$$

The relations in (4.23) are then deduced by taking $k \in \{0, 3\}$ and by observing that due to the orthogonality conditions and (4.21) we have

$$\int_Y H^0 \cdot h^0 = \int_Y |H^0|^2, \quad \int_Y H^3 \cdot h^0 = -\sigma_0 \int_Y |H^3|^2.$$

This concludes the proof. \square

5 Proof of Theorem 2

Uniqueness. By linearity, we are reduced to show that (E, H) vanishes whenever it solves (2.6)–(2.7) and (2.13) with a vanishing incoming field. As $\hat{M} = \hat{N} = 1$ outside Ω , the real part of the outgoing flux of the Poynting vector through the boundary of a ball B_R such that $\Omega \subset B_R$ is independent of R . Thus, exploiting (2.13) with $(E^i, H^i) = (0, 0)$, we deduce that

$$\Re \left(\int_{\partial Q} (E \wedge \bar{H}) \cdot n(x) \right) = \lim_{R \rightarrow \infty} \Re \left(\int_{\partial B_R} (E \wedge \bar{H}) \cdot n(x) \right) = 0. \quad (5.1)$$

On the other hand, (\hat{E}, \hat{H}) agrees with (E, H) on ∂Q and satisfies (2.11)–(2.12). Integrating by parts over Q we obtain

$$\begin{aligned} \Re \left(\int_{\partial Q} (E \wedge \bar{H}) \cdot n(x) \right) &= \Re \left(\int_Q \operatorname{curl} \hat{E} \cdot \bar{\hat{H}} - \operatorname{curl} \hat{E} \cdot \bar{\hat{H}} \right) \\ &= -\omega \Im \left(\int_Q \left\{ \mu_0 \hat{\mu}(x) \hat{H} \cdot \bar{\hat{H}} + \varepsilon_0 \hat{\varepsilon}(x) \hat{E} \cdot \bar{\hat{E}} \right\} \right) \\ &= -\omega \mu_0 \int_Q \Im(\mu^{\text{eff}}) H \cdot \bar{H}, \end{aligned}$$

where in the last line we used the fact that the function $\hat{\varepsilon}(x)$ is real whereas $\Im(\mu^{\text{eff}})$ vanishes outside Ω . Thus from (5.1) and condition (2.14), we get $H = \hat{H} = 0$ on Ω . It is then standard to deduce that (E, H) vanishes everywhere on \mathbb{R}^3 .

Assuming the energy bound (2.5). In this case Theorem 1 is applicable and Theorem 2 follows provided we show that the convergence of (E_η, H_η) can be extended to all \mathbb{R}^3 and that in addition the limit (E, H) still satisfies the radiation condition (2.13). To that aim we apply Lemma 2.1 of [9], which is based the fact that all components of (E_η, H_η) solve the homogeneous Helmholtz equation $\Delta u + k_0^2 u = 0$ in $\mathbb{R}^3 \setminus \Omega$. This yields, by hypoellipticity, the uniform convergence for (E_η, H_η) as well as of all derivatives on compact subsets of $\mathbb{R}^3 \setminus \Omega$. The stability of the condition (2.13) is then deduced by passing to the limit in the so called Stratton-Chu integral identities.

Proving the energy bound. We now prove that if (2.5) is not true then we are led to a contradiction. Possibly after extracting a subsequence, we may assume that

$$t_\eta := \left(\int_Q |E_\eta|^2 + |H_\eta|^2 \right)^{1/2} \rightarrow \infty.$$

We normalize the fields and define

$$\tilde{E}_\eta := \frac{1}{t_\eta} E_\eta, \quad \tilde{H}_\eta := \frac{1}{t_\eta} H_\eta, \quad \text{with } \int_Q |\tilde{E}_\eta|^2 + |\tilde{H}_\eta|^2 = 1. \quad (5.2)$$

Choosing a subsequence $\eta \rightarrow 0$ we may assume that $\tilde{E}_\eta \rightharpoonup \tilde{E}$ and $\tilde{H}_\eta \rightharpoonup \tilde{H}$ in $L^2(Q)$. Applying the previous step, substituting (E^i, H^i) with $(\frac{E^i}{t_\eta}, \frac{H^i}{t_\eta})$, we obtain that (\tilde{E}, \tilde{H}) solves the effective diffraction problem (2.11)–(2.12) with a vanishing incident wave in relation (2.13). The uniqueness argument proved before shows then that $\tilde{E} = \tilde{H} = 0$. Furthermore, the convergence of $(\tilde{E}_\eta, \tilde{H}_\eta)$ to zero holds uniformly on compact subsets of $\mathbb{R}^3 \setminus \Omega$, in particular on ∂Q . Thus by (3.1) and the normalization (5.2)

$$\int_{\Sigma_\eta} \frac{1}{\eta^2} |\tilde{E}_\eta|^2 \rightarrow 0, \quad \varepsilon_0 \lim_\eta \int_Q |\tilde{E}_\eta|^2 = \mu_0 \lim_\eta \int_Q |\tilde{H}_\eta|^2 = \frac{\mu_0 \varepsilon_0}{\mu_0 + \varepsilon_0} \quad (5.3)$$

The desired contradiction arises when applying compensated compactness and the following

Claim: *There holds*

$$\frac{1}{\eta^2} \tilde{E}_\eta \mathbf{1}_{\Sigma_\eta} \rightarrow 0 \text{ strongly in } H^{-1}(Q). \quad (5.4)$$

Indeed, assuming the claim we have

$$\text{curl } \tilde{H}_\eta = -i\omega\varepsilon_0 \left(1 + \frac{i\kappa}{\eta^2} \mathbf{1}_{\Sigma_\eta} \right) \tilde{E}_\eta \rightarrow 0 \text{ strongly in } H^{-1}(Q), \quad \text{div } \tilde{H}_\eta = 0,$$

and therefore $\int_Q |\tilde{H}_\eta|^2 \rightarrow 0$ which is incompatible with (5.3).

Proof of Claim (5.4). We omit the tilde symbol in this proof. We define the index set $I = I_\eta \subset \mathbb{Z}^3$ by $j \in I_\eta \iff \eta(j + Y) \subset \Omega$, and denote the connected component of Σ_η contained in cell $Y_\eta^j = \eta(j + Y)$ as Σ_η^j . For an arbitrary function $\varphi \in H^1(Q)$ we write

$$a_\eta(\varphi) := \int_{\Sigma_\eta} \frac{1}{\eta^2} E_\eta \varphi = \sum_{j \in I} \int_{\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta (\varphi - \varphi_j) + \sum_{j \in I} \varphi_j \int_{\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta, \quad (5.5)$$

where $\varphi_j = \frac{1}{\eta^3} \int_{Y_\eta^j} \varphi$ is the average of φ on cell j . For the first sum we use the Poincaré inequality for functions with vanishing average to find

$$\begin{aligned} \sum_{j \in I} \int_{\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta (\varphi - \varphi_j) &\leq \sum_{j \in I} \left(\int_{\Sigma_\eta^j} \left| \frac{1}{\eta} E_\eta \right|^2 \right)^{1/2} \left(\int_{\Sigma_\eta^j} \left| \frac{1}{\eta} (\varphi - \varphi_j) \right|^2 \right)^{1/2} \\ &\leq \|\eta^{-1} E_\eta\|_{L^2(\Sigma_\eta)} \|\nabla \varphi\|_{L^2(Q)} = o(1). \end{aligned}$$

In order to find a uniform bound for the second sum in (5.5), we must control, for $k = 1, 2, 3$, the averages

$$\int_{\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta \cdot e_k = \int_{\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta \cdot (\nabla x_k) d\mathcal{L}^3(x) = \int_{\partial\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta^- \cdot n x_k d\mathcal{H}^2(x),$$

where we wrote E_η^- for the trace of E_η on the boundary $\partial\Sigma_\eta^j$ if we take the trace from the interior of Σ_η^j . Instead, we write E_η^+ for the trace from the opposite side. We can exploit the fact that $\varepsilon_\eta E_\eta$ has vanishing divergence, hence

$$\frac{1}{\eta^2} E_\eta^- \cdot n = \frac{1}{i\kappa} (E_\eta^+ - E_\eta^-) \cdot n.$$

We choose a function $\theta_k \in C_0^1(Y, \mathbb{R})$ with $\theta_k(y) = y_k$ on $\partial\Sigma$ and set $\theta_k^\eta(\eta(j+y)) = \eta\theta_k(y)$. With this function we calculate

$$\int_{\partial\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta^- \cdot n x_k d\mathcal{H}^2(x) = \int_{\partial\Sigma_\eta^j} \frac{1}{i\kappa} (E_\eta^+ - E_\eta^-) \cdot n \theta_k^\eta = \int_{Y_\eta^j} \frac{1}{i\kappa} E_\eta \cdot \nabla \theta_k^\eta.$$

Since $\nabla \theta_k^\eta$ is bounded, we can put together the pieces to find

$$\begin{aligned} a_\eta(\varphi) &= o(1) + \sum_{j \in I} \sum_{k=1}^3 (\varphi_j \cdot e_k) \int_{\Sigma_\eta^j} \frac{1}{\eta^2} E_\eta \cdot e_k \\ &= o(1) + \sum_{j \in I} \sum_{k=1}^3 (\varphi_j \cdot e_k) \int_{Y_\eta^j} \frac{1}{i\kappa} E_\eta \cdot \nabla \theta_k^\eta. \end{aligned}$$

If we introduce $\bar{\varphi}_\eta, \bar{E}_\eta^k$ on Ω as the cell-wise constant functions with

$$\bar{\varphi}_\eta(x) = \varphi_j, \quad \bar{E}_\eta^k(x) = \int_{Y_\eta^j} E_\eta \cdot \nabla \theta_k^\eta \quad \text{for all } x \in Y_\eta^j,$$

the expression simplifies to

$$a_\eta(\varphi) = o(1) + \frac{1}{i\kappa} \sum_{k=1}^3 \int_{\Omega} (\bar{\varphi}_\eta \cdot e_k) \bar{E}_\eta^k.$$

We can now analyze the expression $a_\eta(\varphi_\eta)$ for an arbitrary sequence of functions $\varphi_\eta \rightharpoonup \varphi$ in $H^1(Q)$. We construct $\bar{\varphi}_\eta, \bar{E}_\eta^k$ as above and find the following. The convergence $\varphi_\eta \rightharpoonup \varphi$ in $H^1(Q)$ implies $\varphi_\eta \rightarrow \varphi$ in $L^2(Q)$ and therefore also $\bar{\varphi}_\eta \rightarrow \varphi$ in $L^2(Q)$ by Jensen's inequality. The two-scale convergence $E_\eta \rightharpoonup 0$ implies, for $\psi \in C_0^1(Q)$,

$$\int_Q \bar{E}_\eta^k(x) \psi(x) = \int_Q E_\eta(x) \cdot \nabla \theta_k^\eta(x) \psi(x) + o(1) \rightarrow 0,$$

and therefore $\bar{E}_\eta^k \rightharpoonup 0$. We conclude $a_\eta(\varphi_\eta) \rightarrow 0$ and hence the result.

A Existence of potentials

Integration by parts. Let $G \subset \mathbb{R}^3$ be an open subset with Lipschitz boundary and outer unit normal n . Then the relation $\operatorname{div}(U \wedge V) = \operatorname{curl} U \cdot V - \operatorname{curl} V \cdot U$ implies for $U, V \in W^{1,p}(G)$ the formula

$$\int_G (\operatorname{curl} U \cdot V - \operatorname{curl} V \cdot U) = \int_{\partial G} (U \wedge V) \cdot n = \int_{\partial G} (n \wedge U) \cdot V. \quad (\text{A.1})$$

Potentials in Y . Every field $u \in L^2(Y; \mathbb{C}^3)$ with $\operatorname{curl} u = 0$ is a gradient of a periodic function (up to constants). There exists $\varphi \in W_{\text{per}}^{1,2}(Y)$ such that

$$u = z + \nabla \varphi, \quad z = \int_Y u. \quad (\text{A.2})$$

Furthermore, every field $u \in L^2(Y; \mathbb{C}^3)$ with $\operatorname{div} u = 0$ is a curl of a periodic function (up to constants). There exists $\psi \in W_{\text{per}}^{1,2}(Y)$ such that

$$u = z + \operatorname{curl} \psi, \quad z = \int_Y u. \quad (\text{A.3})$$

In addition, ψ can be chosen so that $\operatorname{div} \psi = 0$. The potentials φ and ψ can be found with the help of appropriate Poisson problems.

Potentials in Σ . With the full torus Σ , the nontrivial topology changes the above situation. The special field $\chi_a : \Sigma \rightarrow \mathbb{R}^3$ has a vanishing curl, but it is not a gradient. Since the torus has genus 1, there is only one such function (up to factors). In this work we exploited the following fact:

For every field $u \in L^2(\Sigma, \mathbb{C}^3)$ with $\operatorname{curl} u = 0$, there exists $\mu \in \mathbb{C}$, $\rho \in W^{1,2}(\Sigma, \mathbb{C})$, and $z \in \mathbb{C}^3$ such that

$$u = \nabla \rho + \mu \chi_a + z. \quad (\text{A.4})$$

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