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in terms of betweenness in  
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## Abstract

We prove that one can define the relation  $\parallel$ , with  $ab \parallel cd$  to be read as ‘ $a = b$  or  $c = d$  or  $ab$  and  $cd$  are parallel lines (or coincide)’ positively existentially in  $L_{\omega_1\omega_1}$  in terms of  $\neq$  and the ternary relation  $B$  of betweenness, with  $B(abc)$  to be read as ‘ $b$  lies between  $a$  and  $c$ ’ in Archimedean ordered affine geometry. We also show that a self-map of an Archimedean ordered translation plane or of a flat affine plane which preserves both  $B$  and  $\neg B$  must be a surjective affine mapping.

**Mathematics Subject Classification:** 51G05, 51F20, 51F05.

## 1 Introduction

There is a large literature on what came to be called *characterizations of geometric transformations under mild hypotheses*, in which classical geometric transformations are characterized as mappings (which may be required to be one-to-one or onto or both) required to preserve only a certain geometric notion, which was thought to be too weak to characterize the geometric transformation in question. The better known among these surprising characterizations are:

- Carathéodory’s characterization of Möbius or conjugate Möbius transformation as one-to-one self-maps of the closed complex plane that map circles (real circles or lines) onto circles (real circles or lines);
- The Mazur-Ulam theorem, stating that surjective isometries of real normed spaces are affine (i. e. map lines onto lines);
- The Beckman-Quarles theorem, stating that self-maps of finite dimensional real Euclidean spaces which map points at unit distance into points at unit distance must be isometries;
- The Alexandrov-Zeeman theorem, characterizing elements of the orthochronous inhomogeneous Lorentz group with dilations as self-transformations of the real Minkowski space that preserve causality.

All such theorems can be rephrased as purely logical statements inside a formalized geometric theory, asserting the definability of a geometric notion in terms of another geometric notion. Rephrasings of a large class of characterizations of geometric transformations under mild hypotheses as definability statements, in which explicit definitions are provided, can be found in [4], [13]-[21], [24] and [25].

The reason why one expects that such rephrasings as definability statements ought to exist for such theorems is encapsulated in the following combination of Beth’s definability, Lyndon’s preservation theorem, and Keisler’s extension thereof (see [8], [5, Th. 6.6.4, Ex. 6.6.2]), which can be transferred from first-order logic to  $\mathcal{L}_{\omega_1\omega}$ , a logic in which one can form countably infinite conjunctions (and disjunctions) of first-order formulas (cf. [9], [12], [1, Chapter VIII]):

**Preservation and Definability Theorem** *Let  $L \subseteq L^+$  be two first order or  $\mathcal{L}_{\omega_1\omega}$  languages containing a sign for an identically false formula,  $\mathcal{T}$  be a theory in  $L^+$ , and  $\varphi(\mathbf{X})$  be an  $L^+$ -formula in the free variables  $\mathbf{X} = (X_1, \dots, X_n)$ . Then the following assertions are equivalent:*

- (i) *there is a positive (positive existential; positive existential with negated equality allowed)  $L$ -formula  $\psi(\mathbf{X})$  such that  $\mathcal{T} \vdash \varphi(\mathbf{X}) \leftrightarrow \psi(\mathbf{X})$ ;*
- (ii) *for any  $\mathfrak{A}, \mathfrak{B} \in \text{Mod}(\mathcal{T})$ , and each  $L$ -epimorphism ( $L$ -homomorphism;  $L$ -monomorphism)  $f : \mathfrak{A} \rightarrow \mathfrak{B}$ , the following condition is satisfied:  
if  $\mathbf{c} \in \mathfrak{A}^n$  and  $\mathfrak{A} \models \varphi(\mathbf{c})$ , then  $\mathfrak{B} \models \varphi(f(\mathbf{c}))$ .*

This theorem tells us that, for every characterization of a geometric transformation under mild hypotheses, i. e. for all statements of type (ii), there must be a definition of the notions preserved by the transformation, i. e. a statement of type (i). One can ask whether there is any legitimate reason for preferring one of the two equivalent variants over the other. Given that the proof of the above theorem is not constructive in the sense that it does not provide a method for finding  $\psi(\mathbf{X})$  in case we know that (ii) holds finding the actual definition is preferable to proving theorems regarding mappings. One may know that (ii) is true, but still not know of any definition  $\psi(\mathbf{X})$ , whereas whenever we have a definition satisfying the required syntactic conditions, the inference to (ii) is immediate.

An interesting theorem of type (ii) was proved recently in [6]. It states that a self-map of an Archimedean ordered affine Pappian plane which preserves both the betweenness and the non-betweenness relation (i. e. both  $B$  and  $\neg B$ ) must be an affine mapping and thus onto.

The aim of this paper is to provide a purely geometric proof of this result, to slightly generalize this result to flat affine planes, as well as to isolate the main reason why the result and the algebraic characterization of such maps given in [6] holds, which is, better than one would expect (for, by our Preservation and Definability Theorem, the equivalent reformulation, as in (i), would ensure only an existential definition in terms of  $B$ ), the positive existential definability of the parallelism relation in terms of betweenness and  $\neq$ .

Main references for the theory of ordered affine planes (in terms of betweenness relations) and the theory of ordered projective planes (in terms of separation relations) are [23, Chapter 9], [27, Chapter V], and [32].

## 2 Positive existential definition of parallelism in terms of betweenness and $\neq$

Ordered affine geometry can be expressed, with only one sort of variables, standing for *points*, in two different languages:

- one in which the only predicate is the ternary predicate  $B$ , with  $B(abc)$  standing for ‘point  $b$  is between  $a$  and  $c$  (or coincides with  $a$  or with  $c$ ’, and
- one with two predicates,  $B$  and  $\parallel$ , with  $ab \parallel cd$  to be read as ‘the lines determined by  $(a, b)$  and  $(c, d)$  are parallel or coincide, or  $a = b$ , or  $c = d$ ’.

Axiom systems of the first type can be obtained in the 2-dimensional case, by adding the Euclidean parallel axiom, stating the existence and uniqueness of a parallel from a point to a line, to an axiom system for ordered planes, i. e. one with linear order axioms stating that the order is dense and unending, and the Pasch axiom (in the form that implies two-dimensionality, stating that a line which meets one of the sides of a triangle, must meet another side of that triangle as well). In the higher-dimensional or the dimension-free case (in the latter there is only a lower-dimension

axiom, stating that there are three non-collinear points, but no upper-dimension axiom), one obtains an axiom system of the first type by replacing every occurrence of  $L(xyz)$  (which stands for ‘ $x, y,$  and  $z$  are collinear points (not necessarily different points)’)) in an  $L$ -based axiom system for affine geometry (such as that presented in [11]) by  $B(xyz) \vee B(yzx) \vee B(zxy)$ , as well as adding the axioms for dense, unending linear orders and the outer form of the Pasch axiom (see [31, p. 12]).

Axiom systems of the second type were presented, for the plane case, in [32], and, for the dimension-free case, with dimension  $\geq 3$ , in [10] (order axioms, including the invariance of betweenness under parallel projections, in the form of axiom **oPasch** in [32, p. 148] (see A13 in Appendix), need to be added to the axiom system in terms of  $\parallel$  presented in [10] to get an axiom system for ordered affine spaces of dimension  $\geq 3$ ).

It was shown in [32, 4.6] and in [31, Satz 6.62] that axiom systems of the first type cannot consist entirely of  $\forall\exists$ -statements (i. e. of statements all of whose universal quantifiers (if any) precede all existential quantifiers (if any) when written in prenex form), whereas the axiom systems of the second type from both [32] and [10] (together with the order axioms) consist of  $\forall\exists$ -statements. We conclude that an existential definition of  $\parallel$  in terms of  $B$  can not exist, for, if there were such a definition, we could replace it for  $\parallel$  in the  $\forall\exists$  axiom systems for ordered affine geometry of the second type to get a  $\forall\exists$ -axiom system of the first type.

However, there exists a positive definition of  $\parallel$  in first-order logic in terms of  $L$  (and thus, *a fortiori*, in terms of  $B$ , as we can consider all occurrences of  $L(uvw)$  in the definiens to be abbreviations of  $B(uvw) \vee B(vwu) \vee B(wuv)$ ), the definition being valid in all ordered affine planes. It can be formulated as the following  $\forall\exists$ -statement:

$$ab \parallel cd \leftrightarrow (\forall xyz)(\exists uv) (L(abc) \wedge L(abd)) \vee c = d \vee (L(abx) \wedge L(aby)) \quad (1)$$

$$\vee (L(cdx) \wedge L(cdy)) \vee L(xyz) \vee$$

$$(((L(xyu) \wedge L(xyv)) \vee (L(xzu) \wedge L(xzv))) \wedge u \neq v \wedge L(abu) \wedge L(cdv)).$$

That the definiens holds when  $ab \parallel cd$  is seen by noticing that the cases  $a = b$ ,  $c = d$ , or the coincidence of the lines  $ab$  and  $cd$  are part of the definiens, and if  $ab$  and  $cd$  are two different parallel lines, and  $xy$  and  $xz$  are two different lines (i. e. if  $\neg L(xyz)$ ), then at most one of  $xy$  and  $xz$  can be parallel to  $ab$  (and  $cd$ ), so at least one must intersect the lines  $ab$  and  $cd$  in two different points  $u$  and  $v$ .

That the definiens does not hold when  $ab \not\parallel cd$ , i. e. when  $ab$  and  $cd$  are two different lines and there is a point  $p$  with  $L(abp)$  and  $L(cdp)$ , can be seen by choosing  $x$  to be  $p$  and  $y$  and  $z$  to be two points, not on the lines  $ab$  or  $cd$ , such that  $\neg L(xyz)$ . In this case, the lines  $xy$  and  $xz$  intersect both  $ab$  and  $cd$  in the point  $x$ , so there are no different points  $u$  and  $v$  on one of the lines  $xy$  and  $xz$ , such that one lies on  $ab$  and the other one lies on  $cd$ .

That the  $u \neq v$  which appears in (1) can be positively defined in terms of  $L$  (and thus in terms of  $B$ ) has been shown, for all affine spaces of finite dimension  $\geq 2$ , in [13], and thus the definiens in (1) can be rephrased as a positive statement in  $L$ .

The Archimedean nature of an order relation cannot be expressed in first-order logic (if it could, then so would Archimedean ordered fields, which is impossible, as they are isomorphic to subfields of  $\mathbb{R}$ , and, by the Löwenheim-Skolem theorem, any first-order theory with infinite models must have models of any infinite cardinality), but it can be expressed, with  $L := L_B$ , in  $L_{\omega_1\omega}$  (see [1, Chapter VIII] for a definition and for the main properties of this logic) in weak second order logic  $L(II_0)$  (that allows quantification over finite sets, see [30]), in logic with the Ramsey quantifier  $Q^2$  (see [2]), as well as in deterministic transitive closure logic  $L(DTC)$  (see [3, 8.6] for a definition).

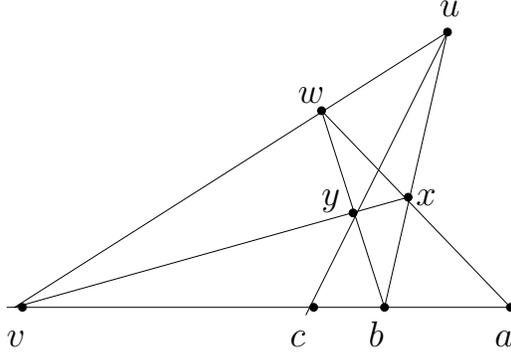


Figure 1: The definition of  $\sigma(abcuvw)$ .

We will choose to express our positive existential definition of  $\parallel$  in terms of  $B$  in  $L'_{\omega_1\omega}$ , where  $L' := L_{B\parallel}$ . *The definition itself is valid in the theory axiomatized by  $(\alpha \vee \beta) \wedge \text{Arch}$ , where by  $\alpha$  we have denoted the conjunction of the  $L'$ -axioms for ordered affine planes (from [32]), by  $\beta$  the conjunction of the  $L'$ -axioms for ordered affine spaces of dimensions  $\geq 3$  (from [10], together with the order axioms), and by  $\text{Arch}$  the Archimedean axiom, which in affine ordered geometry, with no configuration theorems (minor or major Desargues, Pappus) among the axioms, will be stated by using the defined notions  $Z$  and  $\sigma$ , which are defined by*

$$\begin{aligned} Z(abc) &:\Leftrightarrow B(abc) \wedge a \neq b \wedge b \neq c, \\ \sigma(abcuvw) &:\Leftrightarrow (\exists xy) B(bxu) \wedge B(axw) \wedge B(vyx) \wedge B(byw) \wedge B(uyc) \wedge B(bcv). \end{aligned}$$

The meaning of  $Z$  is plain: it stands for strict betweenness. To have some intuition of what  $\sigma$  stands for, let us note that, if we think of the line  $vu$  as the line at infinity, then  $\sigma(abcuvw)$  stands for the fact that the point  $c$  is the reflection of  $a$  in  $b$ , and  $\sigma(abcuvw)$  can thus be thought of as asserting that  $c$  is (a projective geometry view of) the ‘reflection’ of  $a$  into  $b$ , constructed with the help of  $u, v, w, x, y$ .

We are now ready to state the Archimedean axiom, which states that, given two points  $a_1$  and  $a_2$ , a point  $p$  on the ray  $\overrightarrow{a_1a_2}$ , and a line  $uw$  (which we may think of as the ‘line at infinity’) meeting the ray  $\overrightarrow{a_1a_2}$  in a point  $v$  (which we may think of as ‘at infinity’), which is such that  $p$  is strictly between  $a_2$  and  $v$ , the sequence of points  $a_i$ , obtained by iterating the ‘reflection’ operation, first ‘reflecting’  $a_1$  in  $a_2$  to get  $a_3$  (by means of  $\sigma(a_1a_2a_3uvw)$ ), then  $a_2$  in  $a_3$  to get  $a_4$ , and so on, will eventually move past  $p$ , i. e., for some  $n$ , we will find  $p$  lying between  $a_2$  and  $a_{n+2}$ .

$$\begin{aligned} &(\forall a_1a_2vwpx) \neg L(a_1vw) \wedge Z(a_1xw) \wedge Z(a_2xu) \wedge Z(vwu) \wedge Z(a_1a_2v) \wedge Z(a_2pv) \\ &\rightarrow \left[ \bigvee_{n=1}^{\infty} \left( (\exists a_3 \dots a_{n+2}) \bigwedge_{i=1}^n \sigma(a_i a_{i+1} a_{i+2} uvw) \wedge B(a_2 p a_{n+2}) \right) \right], \end{aligned}$$

To improve the readability of the definition of  $\parallel$  in terms of  $B$ , to be stated later, we define further abbreviations:

$$\begin{aligned} \varphi_n(p, q, r, s, u, v) &:\Leftrightarrow (\exists a_3 \dots a_n u_3 \dots u_n) [B(rva_3) \wedge \bigwedge_{i=3}^{n-1} B(ru_i a_{i+1}) \\ &\quad \wedge B(uvu_i) \wedge B(su_i a_i) \wedge B(pq a_i)], \end{aligned}$$

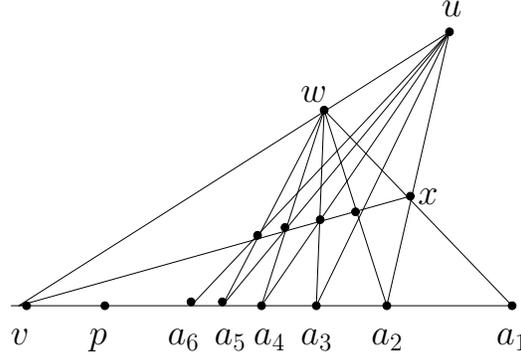


Figure 2: The Archimedean axiom.

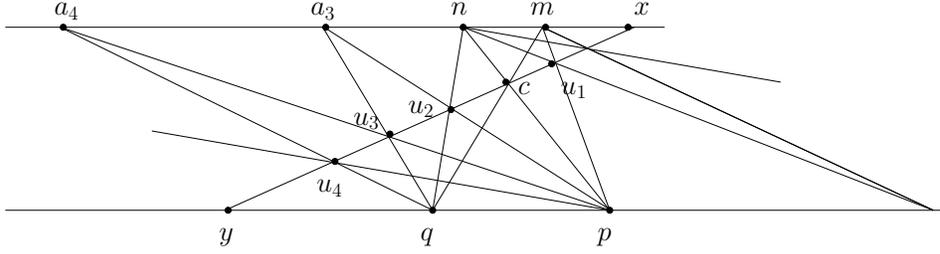


Figure 3:  $\psi(m, n, p, q)$  states that the process described in the picture can be continued indefinitely.

$$\begin{aligned} \psi(m, n, p, q) \quad &:\Leftrightarrow \quad (\exists cu_1u_2xy) [Z(ncp) \wedge Z(mcq) \wedge Z(mu_1p) \wedge Z(nu_2q) \\ &\wedge B(u_2u_1x) \wedge B(nmx) \wedge B(yu_2u_1) \wedge B(yqp) \\ &\wedge \bigwedge_{i=4}^{\infty} \varphi_i(m, n, p, q, u_1, u_2) \wedge \bigwedge_{n=4}^{\infty} \varphi_i(q, p, n, m, u_2, u_1)]. \end{aligned}$$

We are now ready to state our first result:

**Theorem 1** *The relation  $\parallel$  is positively existentially definable in terms of  $B$  and  $\neq$ . The definition, a statement of  $L'_{\omega_1\omega}$ , is*

$$a_1a_2 \parallel b_1b_2 \leftrightarrow a_1 = a_2 \vee b_1 = b_2 \vee \psi(a_1, a_2, b_1, b_2) \vee \psi(a_1, a_2, b_2, b_1). \quad (2)$$

**Proof.** To see that this definition is a theorem of  $(\alpha \vee \beta) \wedge Arch$ , we first notice that the definiens holds whenever  $a_1a_2 \parallel b_1b_2$  holds.

The cases  $a_1 = a_2$  or  $b_1 = b_2$  are taken care of by being part of the definiens. If  $a_1 \neq a_2$  and  $b_1 \neq b_2$ , and the lines determined by  $(a_1, a_2)$  and  $(b_1, b_2)$  coincide, then all the points that have to exist, as required by the definitions of  $\psi$  and  $\varphi$ , can be chosen to be any points lying on the line determined by  $(a_1, a_2)$ , which satisfy the required betweenness relations. Given that the order is dense, such points always exist, and thus the definiens holds.

If the lines determined by  $(a_1, a_2)$  and  $(b_1, b_2)$  are parallel (and thus do not coincide), then  $a_2$  and  $b_2$  either lie on the same side of the line determined by  $(a_1, b_1)$  in the plane  $\pi$  determined by  $(a_1, a_2)$  and  $(b_1, b_2)$ , or they lie on different sides of  $(a_1, b_1)$ . If they lie on the same side of  $(a_1, b_1)$ , then  $\psi(a_1, a_2, b_1, b_2)$ , which can be seen by choosing  $c$  to be the intersection point of segments  $a_2b_1$  and  $a_1b_2$ ,  $u_1$  to be any point with  $Z(a_1u_1b_1)$ ,  $x$  any point with  $Z(a_2a_1x)$ , the intersections of line  $xu_1$  with lines  $a_2b_2$  and  $b_1b_2$  to be  $u_2$  and  $y$ . That  $\varphi_i(a_1, a_2, b_1, b_2, u_1, u_2)$  holds for all  $i \geq 4$



with the  $\forall\exists$ -theory of ordered fields, given that, when written in prenex disjunctive normal form, an  $\forall\exists$ -sentence amounts to the statement that one of a finite set of systems of equations and inequalities has a solution, and the existence of solutions for such systems does not depend on the Archimedean nature of the order). However, as noted earlier, we know from [32, 4.6] that such a  $\delta$  cannot exist for ordered affine planes.

**Corollary** *Let  $\mathfrak{A}$  be an Archimedean ordered affine plane admitting a subplane  $\mathfrak{A}'$  which is ordered with respect to the induced ordering. Then  $\mathfrak{A}'$  carries the parallelism of  $\mathfrak{A}$ , i. e. the relation  $\parallel$  of  $\mathfrak{A}'$  is a restriction of the relation  $\parallel$  of  $\mathfrak{A}$ .*

**Proof.** The relation  $\parallel$  in  $\mathfrak{A}'$  can be defined by the positive existential definition (2) in terms of  $B$ , and we will refer to it only as the relation defined by the definiens in (2).

If  $ab \parallel_{\mathfrak{A}'} cd$ , then there are points in the plane  $\mathfrak{A}'$  for which the definiens of (2) is satisfied. These points belong to  $\mathfrak{A}$  as well, so the definiens holds in  $\mathfrak{A}$ , and thus, by (2),  $ab \parallel_{\mathfrak{A}} cd$ .

Suppose now  $a, b, c, d$  are points belonging to the universe of  $\mathfrak{A}'$ , and that  $ab \parallel_{\mathfrak{A}} cd$ . Were  $ab \parallel_{\mathfrak{A}'} cd$  not to hold, then we would have  $a \neq b, c \neq d$ ,  $a, b, c, d$  are not all on one line, and there would exist a point  $p$  in the universe of  $\mathfrak{A}'$  with  $L(abp)$  and  $L(cdp)$ . Since the point  $p$  belongs to the universe of  $\mathfrak{A}$  as well, we couldn't have  $ab \parallel_{\mathfrak{A}} cd$ .  $\square$

To show that this corollary does make a point, we present examples of Pappian ordered affine planes  $\mathfrak{A}$  admitting an affine subplane  $\mathfrak{A}'$ , which is ordered with respect to the induced ordering and which carries a parallelism distinct from that of  $\mathfrak{A}$ .

**Lemma** *Let  $\mathfrak{P}$  be an ordered projective plane, let  $\mathfrak{A} = \mathfrak{P} \setminus U$  and  $\mathfrak{A}'' = \mathfrak{P} \setminus W$  be affine restrictions of  $\mathfrak{P}$  with respect to two distinct lines  $U$  and  $W$  of  $\mathfrak{P}$ , and let  $\mathfrak{A}'$  be an affine subplane of  $\mathfrak{A}''$  carrying the parallelism of  $\mathfrak{A}''$ . Endow  $\mathfrak{A}$  and  $\mathfrak{A}''$  with the betweenness relations  $B$  and  $B''$  respectively, induced by the ordering of  $\mathfrak{P}$ . If  $U$  does not contain any point lying between (with respect to  $B''$ ) two points of  $\mathfrak{A}'$ , then  $\mathfrak{A}'$  is also a subplane of  $\mathfrak{A}$ , and thus carries the betweenness relation of  $\mathfrak{A}$  (i. e.  $B$  and  $B''$  coincide on  $\mathfrak{A}'$ ), but has a distinct parallelism.*

**Proof.** By the setting above,  $\mathfrak{A}$  and  $\mathfrak{A}''$  are affine planes, which have  $U$  and  $W$  as their lines at infinity, have the points of  $\mathfrak{P} \setminus (U \cup W)$  in common, and in which three points are collinear, if and only if they are collinear in  $\mathfrak{P}$ . In particular, we have  $\mathfrak{A}' \subset \mathfrak{P} \setminus (U \cup W) \subset \mathfrak{A}$ . Since two projective lines are parallel in an affine restriction if and only if they meet on the associated line at infinity, any two lines that are parallel in  $\mathfrak{A}$  are not parallel in  $\mathfrak{A}''$  and vice versa — unless they meet in  $U \cap W$ .

Recall that the ordering of  $\mathfrak{P}$ , defined in terms of a separation relation  $xy|uv$  (which stands for ‘the points  $x$  and  $y$  separate the points  $u$  and  $v$ ’), induces betweenness relations on  $\mathfrak{A}''$  and  $\mathfrak{A}$  by

$$\begin{aligned} B''(bac) &:\Leftrightarrow aw|bc \text{ where } w \text{ is the point of intersection of the lines } ab \text{ and } W \\ B(bac) &:\Leftrightarrow au|bc \text{ where } u \text{ is the point of intersection of the lines } ab \text{ and } U \end{aligned}$$

turning  $\mathfrak{A}''$  and  $\mathfrak{A}$  into ordered affine planes (cf. [23] or [27]). Of course, generally these two betweenness relations differ (given that the corresponding parallelisms differ), but they coincide on  $\mathfrak{A}'$ . For, given any three (distinct) collinear points  $a, b, c$  in  $\mathfrak{A}'$  with  $w$  and  $u$  the points of intersection of the lines  $ab$  with  $W$  and  $U$  respectively, we have that  $uw|bc$  is false (since  $U, W$  do not separate points of  $\mathfrak{A}'$ ), and so that  $aw|bc$  is equivalent to  $au|bc$ .  $\square$

## Examples.

To obtain concrete examples of affine planes as in the above Lemma, we start with an extension  $L/K$  of ordered fields and an element  $t_0 \in L$  such that  $t_0 > k$ , for all  $k \in K$ . Let  $\mathfrak{A}'$  be the ordered affine plane over  $K$ ,  $\mathfrak{A}''$  the ordered affine plane over  $L$ , and  $\mathfrak{P}$  the projective closure of  $\mathfrak{A}''$  where  $W$  denotes the associated line at infinity.

By [23, 9.2], [27, V.1] the ordering of  $\mathfrak{A}''$  uniquely extends to an ordering of  $\mathfrak{P}$  (in terms of a separating relation — in the case of Pappian planes we have  $xy|uv$  if and only if the cross ratio of the four points is negative in the underlying field). We thus get the natural embeddings  $\mathfrak{A}' \subset \mathfrak{A}'' \subset \mathfrak{P}$ , the parallelism of  $\mathfrak{A}'$  is a restriction of that of  $\mathfrak{A}''$ , and the orderings of  $\mathfrak{A}'$  and  $\mathfrak{A}''$  are induced by that of  $\mathfrak{P}$ .

If we choose  $U$  to be the projective line whose affine part in  $\mathfrak{A}''$  is  $\{(x, y) \in L^2 | x = t_0\}$ , the assumptions of the lemma above are obviously fulfilled. Note that there exists a collineation  $\varphi$  of  $\mathfrak{P}$  with  $\varphi(U) = W$  (i. e.  $\varphi \in PGL(3, L)$  and thus order-preserving), which means that  $\mathfrak{A}$  and  $\mathfrak{A}''$  are isomorphic as ordered affine planes.

For concreteness' sake, we present some examples of this construction:

(a) Take  $K := \mathbb{R}$ ,  $L := \mathbb{R}((t))$  to be the field of real Laurent series, fix the unique ordering of  $L$  for which  $t$  is positive, and let  $t_0 := t^{-1}$ . By the above construction, we get a Pappian ordered affine plane  $\mathfrak{A}$  admitting an affine subplane  $\mathfrak{A}'$ , which is even Archimedean ordered with respect to the induced ordering, and which carries a parallelism distinct from that of  $\mathfrak{A}$ .

(b) Let  $\Gamma := \bigoplus_{i=1}^{\infty} \mathbb{Z}$  be the  $\mathbb{N}$ -fold direct sum of  $(\mathbb{Z}, +)$ , ordered lexicographically, i. e.  $(z_1, z_2, \dots) > (y_1, y_2, \dots) :\Leftrightarrow z_i = y_i$  for  $i = 1, \dots, k-1$  and  $z_k > y_k$  for some  $k \in \mathbb{N}$ . Let  $L := \mathbb{Q}((\Gamma))$  be the field of formal power series over  $\mathbb{Q}$  on  $\Gamma$  (the elements of which are formal sums  $f := \sum_{\gamma \in \Gamma} f_{\gamma} t^{\gamma}$  with well-ordered support  $s(f) := \{\gamma \in \Gamma | f_{\gamma} \neq 0\}$ , with  $f_{\gamma} \in \mathbb{Q}$ , the sum and product of which are induced by the rules  $f_{\gamma} t^{\gamma} + g_{\gamma} t^{\gamma} := (f_{\gamma} + g_{\gamma}) t^{\gamma}$  and  $t^{\gamma} \cdot t^{\delta} := t^{\gamma+\delta}$ ), and fix that ordering of  $L$  in which all  $t^{\gamma}$  are positive (i.e.  $f > 0$  if and only if  $f_{\gamma} > 0$  for  $\gamma := \max(s(f))$  and  $f \neq 0$ ), cf. [27, II, §5].

The order-preserving group monomorphism  $\alpha : \Gamma \rightarrow \Gamma$ ,  $(z_1, z_2, \dots) \mapsto (0, z_1, z_2, \dots)$  onto  $\Delta := \bigoplus_{i=2}^{\infty} \mathbb{Z}$  induces an order-preserving field monomorphism  $\bar{\alpha} : L \rightarrow L$ , defined by the rule  $t^{\gamma} \mapsto t^{\alpha(\gamma)}$  onto  $K := \mathbb{Q}((\Delta))$ . Thus  $\bar{\alpha}$  is an order-preserving isomorphism from  $L$  onto a proper subfield  $K$  of  $L$ . Further,  $L$  admits an element  $t_0$  with  $t_0 > k$  for all  $k \in K$ , namely  $t_0 := t^{(1,0,0,\dots)}$ . By the above construction we get a Pappian ordered affine plane  $\mathfrak{A}$  admitting an affine subplane  $\mathfrak{A}'$ , which is ordered with respect to the induced ordering, is even isomorphic to  $\mathfrak{A}$  as an ordered affine plane, but which carries a parallelism distinct from that of  $\mathfrak{A}$ .

(c) Examples where the (proper) affine subplane  $\mathfrak{A}'$  is isomorphic to  $\mathfrak{A}$  and carries the same parallelism, may be easier obtained by taking  $L := \mathbb{R}((t))$ , fixing again its ordering with  $t > 0$ , and considering the field monomorphism  $L \rightarrow L$  defined by the rule  $t \mapsto t^2$ . It yields an order-preserving isomorphism from  $L$  onto its subfield  $K := \mathbb{R}((t^2))$ , and thus an order-preserving isomorphism from the affine plane over  $L$  onto its subplane, the affine plane over  $K$ .

Also notice that our Corollary does not simply follow from (1), as there are instances of pairs of Pappian affine planes  $\mathfrak{A}'$  and  $\mathfrak{A}$ , with  $\mathfrak{A}'$  a subplane of  $\mathfrak{A}$ , but such that the coordinatizing field  $K$  of  $\mathfrak{A}'$  is not embeddable in the coordinatizing field  $L$  of  $\mathfrak{A}$ . If we denote by  $\mathfrak{A}'$  the minimal affine plane (i. e. the affine plane over  $GF(2)$ ), which embeds into all affine planes, and thus into all (ordered) affine planes  $\mathfrak{A}$  over ordered fields  $L$ .  $\mathfrak{A}'$  is clearly not orderable, the corresponding parallelisms differ, the projective closure of  $\mathfrak{A}'$ , i. e. the Fano plane (the projective plane over  $GF(2)$ ), does not embed into the projective closure of  $\mathfrak{A}$ , and  $GF(2)$  does not embed

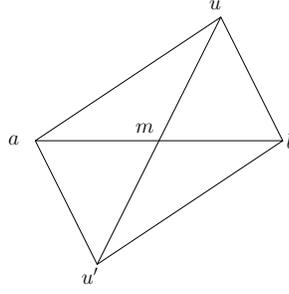


Figure 5: The definition of  $M_u(amb)$ .

into  $L$ . Further examples along this line can be found in [7].

The following result will be used for a short proof of our next theorem, and is of interest in its own right.

**Proposition** *Let  $\mathfrak{P}$  be an Archimedean ordered projective Pappian plane and let  $\mathfrak{P}'$  be a subplane of  $\mathfrak{P}$  endowed with the induced ordering. If  $\mathfrak{P}$  and  $\mathfrak{P}'$  are isomorphic as ordered projective planes, then  $\mathfrak{P} = \mathfrak{P}'$ .*

**Proof.** Choosing a coordinatizing frame of  $\mathfrak{P}'$  we obtain Archimedean ordered fields  $K' \subset K$  where  $K'$  coordinatizes  $\mathfrak{P}'$  and  $K$  coordinatizes  $\mathfrak{P}$ , and where the order-preserving isomorphism between  $\mathfrak{P}$  and  $\mathfrak{P}'$  induces an order-preserving isomorphism between  $K$  and  $K'$ , cf. [23, 4.1 and 9.3] and [27, V.4, Satz 12]. Now the usual proof, showing that any order-preserving automorphism of an Archimedean field is the identity, obviously goes over to order-preserving homomorphisms  $\alpha : K \rightarrow K' \subset K$ : assuming  $\alpha(k) \neq k$ , say  $\alpha(k) < k$ , and choosing an element  $q$  of the prime field  $\mathbb{Q}$  of  $K'$  with  $\alpha(k) < q < k$ , leads to the contradiction  $q < k \Rightarrow q = \alpha(q) < \alpha(k)$ . So we have  $K' = K$ , and therefore  $\mathfrak{P} = \mathfrak{P}'$ .  $\square$

We now turn to our main result, which is Theorem 4.5 of [6], with an alternate wording, for which we provide a purely geometrical proof.

**Theorem 2** *A map  $g : \mathfrak{A} \rightarrow \mathfrak{A}$ , which satisfies  $B(abc) \Leftrightarrow B(g(a)g(b)g(c))$  for all  $a, b, c$  in  $\mathfrak{A}$ , where  $\mathfrak{A}$  is an Archimedean ordered Pappian plane, must be surjective.*

**Proof.** We will present two proofs for this theorem. Our first proof will be purely geometric, while our second proof will use the above Proposition.

1. Let  $f : \mathbb{N} \rightarrow \{0, 1\}$ . We define

$$M_u(amb) := (\exists u') au' \parallel ub \wedge au \parallel a'b \wedge B(amb) \wedge B(umu'),$$

to be read as ‘ $m$  is the midpoint of  $ab$  ( $u$  being an auxiliary point in the construction of  $m$ )’, the formulas  $\delta_n^f(a_1^0, a_2^0, u, x)$ , for all  $n \in \mathbb{N}$ , defined by

$$\begin{aligned} \delta_n^f(a_1^0, a_2^0, u, x) &:= (\exists m_1 \dots m_n a_1^1 \dots a_1^n a_2^1 \dots a_2^n) \bigwedge_{i=0}^n M_u(a_1^i m_i a_2^i) \\ &\quad \wedge B(a_1^i x a_2^i) \wedge \bigwedge_{i=1}^n a_{2-f(i)}^i = m_{i-1} \wedge a_{f(i)+1}^i = a_{f(i)+1}^{i-1}. \end{aligned}$$

In Archimedean ordered Pappian planes, the following statement is valid, stating that, given two segments,  $a_1^0 a_2^0$  and  $b_1^0 b_2^0$ , and a point  $x$ , a lying between  $a_1^0$  and  $a_2^0$ ,

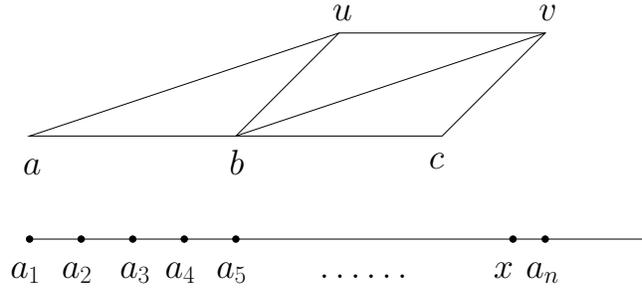


Figure 6: Definition of  $S(abc)$ ; Getting past any given point  $x$  on the ray  $\overrightarrow{a_1 a_2}$ .

and being found in the leftmost or rightmost  $n$ th interval defined by a midpoint of the  $(n - 1)$ st interval of a nested sequence of intervals defined by successive divisions in two of intervals, starting with  $[a_1^0, a_2^0]$ , depending on whether  $f(n)$  is 0 or 1, there is a point  $y$  between  $b_1^0$  and  $b_2^0$  having the same position with respect to the sequence of nested intervals the function  $f$  defines on the interval  $[b_1^0, b_2^0]$ , i. e. for all  $f : \mathbb{N} \rightarrow \{0, 1\}$ , we have:

$$(\forall a_1^0 a_2^0 u b_1^0 b_2^0 v x)(\exists y) \bigwedge_{n=1}^{\infty} \delta_n^f(a_1^0, a_2^0, u, x) \rightarrow \delta_n^f((b_1^0, b_2^0, v, y). \quad (3)$$

The truth of this statement is easily seen by choosing  $y$  such that the ratio in which it divides the segment  $[b_1^0, b_2^0]$  coincides with the ratio in which  $x$  divides the segment  $[a_1^0, a_2^0]$ .

We now prove that  $g$  maps lines onto lines. Let  $a$  and  $b$  be two different points. Let  $y$  be a point between  $g(a)$  and  $g(b)$ . Let  $f : \mathbb{N} \rightarrow \{0, 1\}$  be the map describing  $y$ 's precise location on the segment  $[g(a), g(b)]$  (i. e.  $f(1)$  is 0 if  $y$  belongs to  $[g(a), m]$  and is 1 if  $y$  belongs to  $[m, g(b)]$ , where  $m$  denotes the midpoint of  $[g(a), g(b)]$ , and so on). Let  $x$  be the point on the segment  $[a, b]$  which lies in all the nested intervals defined by  $f$  on  $[a, b]$ . Given that  $g$  preserves, besides  $B$  and  $\neg B$ , midpoints, since it preserves the  $\parallel$  relation,  $g(x)$  belongs to the same nested sequence of intervals  $y$  belongs to. By *Arch*, there can be only one such point, and thus  $y = g(x)$ . Now let (see Figure 6)

$$S(abc) := (\exists uv) L(abc) \wedge \neg L(abu) \wedge au \parallel bv \wedge bu \parallel vc \wedge uv \parallel ab.$$

Given *Arch*,

$$B(a_1 a_2 x) \rightarrow \bigvee_{n=3}^{\infty} (\exists a_3 \dots a_n) \bigwedge_{i=1}^{n-3} S(a_i a_{i+1} a_{i+2}) \wedge B(a_1 x a_n),$$

and, since  $S(abc) \Rightarrow S(g(a)g(b)g(c))$  (given that  $S$  is definable in terms of  $L$  and  $\parallel$ ) every point  $y$  on the line  $g(a)g(b)$  lies between two points  $g(e)$  and  $g(f)$ , with  $e$  and  $f$  on the line  $ab$ , and thus, by (3), there is a point  $x$  between  $e$  and  $f$  (i. e. a point  $x$  on the line  $ab$ ) with  $g(x) = y$ .

Let now  $a, b, c$  be three non-collinear points. Then  $g(a), g(b), g(c)$  are also non-collinear. For any point  $x$  of  $\mathfrak{A}$ , there exist distinct points  $u$  and  $v$  on the lines  $g(a)g(b)$  and  $g(b)g(c)$  respectively, such that  $x$  lies on the line  $uv$ . Given that  $g$  maps lines onto lines, there must be distinct points  $u'$  and  $v'$  on the lines  $ab$  and  $bc$  respectively, such that  $g(u') = u$  and  $g(v') = v$ . Given that  $g$  maps the line  $u'v'$  onto the line  $uv$ , there must be an  $x'$  on  $u'v'$  such that  $g(x') = x$ .

2. Given an order-preserving (and thus injective) mapping  $g : \mathfrak{A} \rightarrow \mathfrak{A}$ , the image  $\mathfrak{A}' = g(\mathfrak{A})$  is an affine subplane of  $\mathfrak{A}$ , isomorphic to  $\mathfrak{A}$  as an ordered affine plane. Furthermore, by the Corollary of Theorem 1 the parallelism of  $\mathfrak{A}'$  is indeed that of  $\mathfrak{A}$ , which means, that the projective closure  $\mathfrak{P}'$  of  $\mathfrak{A}'$  naturally embeds into the projective closure  $\mathfrak{P}$  of  $\mathfrak{A}$ . Since by [23, 9.2] and [27, V.1. Satz 8] (Archimedean) orderings of affine planes uniquely extend to (Archimedean) orderings of their projective closure,  $\mathfrak{P}$  and  $\mathfrak{P}'$  are isomorphic as Archimedean ordered projective planes. By the above Proposition, we obtain  $\mathfrak{P} = \mathfrak{P}'$ , and so  $\mathfrak{A} = \mathfrak{A}'$ .  $\square$

Notice that, by [23, 9.4, Satz 19], Archimedean ordered translation planes (i. e. planes satisfying the minor Desargues axiom, A14 in the Appendix) must be Pappian (see Appendix for the statement of the Pappus axiom), so we could have stated the above theorem (as well as the Proposition preceding it) under the apparently weaker assumption that  $\mathfrak{A}$  is an Archimedean ordered translation plane.

Notice also that Theorem 2 remains valid if  $\mathfrak{A}$  is an Archimedean ordered  $n$ -dimensional affine space with  $n \geq 3$ . Such spaces are Desarguesian, thus, by Archimedeanity, Pappian, and so Theorem 2 for this  $n$ -dimensional  $\mathfrak{A}$  is implied by its plane case. As the following example shows, Theorem 2 is no longer true in infinite-dimensional Archimedean ordered affine spaces. Let the universe of  $\mathfrak{A}$  consist of all maps  $h : \mathbb{N} \rightarrow \mathbb{Q}$ , and let  $B(abc)$  hold in  $\mathfrak{A}$  if and only if there is a  $t \in \mathbb{Q}$ ,  $0 \leq t \leq 1$ , such that  $b = ta + (1-t)c$ , where  $\lambda h$  is the map defined by  $(\lambda h)(n) = \lambda h(n)$ , for all  $\lambda \in \mathbb{Q}$  and all  $h : \mathbb{N} \rightarrow \mathbb{Q}$ . The map  $\varphi : \mathfrak{A} \rightarrow \mathfrak{A}$ , defined by  $\varphi((h))(1) = 0$  and  $\varphi(h)(n+1) = h(n)$ , for all  $n \in \mathbb{N}$  and all  $h : \mathbb{N} \rightarrow \mathbb{Q}$ , preserves both  $B$  and  $\neg B$ , but is not onto.

By a celebrated result of Prieß-Crampe [26], the completion construction making any Archimedean ordered field a copy of the reals can be carried over to arbitrary (i.e. not necessarily Desarguesian or Pappian) projective planes: Each Archimedean ordered projective plane can be embedded into a topological projective plane the point space of which is a surface. The latter are called *flat* projective planes, and have been thoroughly studied, see [29, Ch. 3], [28]. In particular, any flat projective plane carries a unique ordering, is Archimedean and fulfills, by [28], the following

**Fact** *Let  $\mathfrak{P}$  be a flat projective plane and let  $\varphi : \mathfrak{P} \rightarrow \mathfrak{P}$  be a homomorphism, i. e. a mapping from the point set of  $\mathfrak{P}$  into its point set fulfilling:*

- (i) *collinear points are mapped onto collinear points,*
  - (ii) *the image of  $\varphi$  contains a frame, i. e. four points no three of which are collinear.*
- Then  $\varphi$  is an isomorphism (i. e. it is one-to-one and onto, and preserves both  $L$  and  $\neg L$ ).*

Calling affine planes *flat*, if their projective closures are flat, this result of Salzmann immediately yields the following extension of Theorem 2:

**Corollary** *Let  $\mathfrak{A}$  be a flat affine plane. Then any map  $g : \mathfrak{A} \rightarrow \mathfrak{A}$  fulfilling  $B(abc) \Leftrightarrow B(g(a)g(b)g(c))$  for all points  $a, b, c$  of  $\mathfrak{A}$  is an isomorphism, i. e. must be surjective.*

**Proof.** As above, we have that the image  $\mathfrak{A}'$  of  $g$  is an affine subplane of  $\mathfrak{A}$ , isomorphic to  $\mathfrak{A}$  as an Archimedean ordered affine plane, and that its projective closure  $\mathfrak{P}'$  is a subplane of the projective closure  $\mathfrak{P}$  of  $\mathfrak{A}$ , which is a flat plane. Since any isomorphism of affine planes uniquely extends to an isomorphism of their projective closures, the above Fact gives the assertion.  $\square$

### 3 Appendix

The  $L'$ -axiom system for ordered affine planes from [32] consists of (here  $L(abc)$  is an abbreviation for  $ab \parallel ac \vee a = b \vee a = c$ ):

**A 1**  $ab \parallel cc,$

**A 2**  $ab \parallel ba,$

**A 3**  $a \neq b \wedge ab \parallel pq \wedge ab \parallel rs \rightarrow pq \parallel rs,$

**A 4**  $ab \parallel ac \rightarrow ba \parallel bc,$

**A 5**  $(\exists abc) \neg(ab \parallel ac),$

**A 6**  $(\forall abcp)(\exists q) ab \parallel pq \wedge p \neq q,$

**A 7**  $(\forall abcd)(\exists p) \neg(ab \parallel cd) \rightarrow (pa \parallel pb \wedge pc \parallel pd),$

**A 8**  $B(abc) \rightarrow L(abc),$

**A 9**  $L(abc) \rightarrow (B(abc) \vee B(bca) \vee B(cab)),$

**A 10**  $B(abc) \rightarrow B(cba),$

**A 11**  $B(abc) \wedge B(acd) \rightarrow B(bcd),$

**A 12**  $B(abc) \wedge B(bcd) \wedge b \neq c \rightarrow B(acd),$

**A 13**  $\neg L(abb') \wedge L(abc) \wedge L(ab'c') \wedge bb' \parallel cc' \wedge B(abc) \rightarrow B(ab'c').$

The last axiom, A13, is the outer form of the Pasch axiom, **oPasch** in [32, p. 148], stating the invariance of the betweenness relation under parallel projection. In the presence of the minor Desargues axiom, i. e. of

**A 14**  $\neg L(abp) \wedge \neg L(abr) \wedge ab \parallel pq \wedge ab \parallel rs \wedge ap \parallel bq \wedge ar \parallel bs \rightarrow pr \parallel qs,$

axiom A12 becomes superfluous. If we add *Arch* to A1-A11, A13, A14, then we can prove (see [23, 9.4, Satz 19]) the Pappus axiom, i. e.

**A 15**  $(\bigwedge_{1 \leq i < j \leq 3} \bigwedge_{k=1}^3 \neg L(p_i p_j q_k)) \wedge L(p_1 p_2 p_3) \wedge L(q_1 q_2 q_3) \wedge p_1 q_2 \parallel p_2 q_1$   
 $\wedge p_2 q_3 \parallel p_3 q_2 \rightarrow p_1 q_3 \parallel p_3 q_1.$

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