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Ben Schweizer

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Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

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BEN SCHWEIZER

TU Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, D-44227 Dortmund, Germany.
ben.schweizer@tu-dortmund.de

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Abstract: We study the evolution of saturation profiles in a porous medium. When there is a more saturated medium on top of a less saturated medium, the effect of gravity is a downward motion of the liquid. While in experiments the effect of fingering can be observed, i.e. an instability of the planar front solution, it has been verified in different settings that the Richards equation with gravity has stable planar fronts. In the present work we analyze the Richards equation coupled to a play-type hysteresis model in the capillary pressure relation. Our result is that, in an appropriate geometry and with adequate initial and boundary conditions, the planar front solution is unstable. In particular, we find that the Richards equation with gravity and hysteresis does not define an L^1-contraction.

1 Introduction

The standard model for the description of saturation distributions in porous media is the Richards equation. Denoting the fluid pressure by p and the volume fraction of pore space that is filled with fluid by s (saturation), the combination of mass conservation and Darcy's law for the velocities yields the Richards equation

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x]) \,. \tag{1.1}$$

In this equation, a normalization of porosity, density, and gravity are performed, the acceleration of gravity is 1 and points in direction $-e_x$. The permeability $k = k(s)$ is a given function $k : \mathbb{R} \to [0, \infty)$. The unknowns are pressure and saturation, two scalar variables which depend on time $t \in [T_-, T_+]$ and position $\tilde{x} \in \Omega$, where $\Omega \subset \mathbb{R}^{n+1}$ is the domain of the porous medium. We decompose the spatial variables according to the direction of gravity, $\tilde{x} = (x, y) \in \Omega \subset \mathbb{R}^{n+1}$ for $x = \tilde{x} \cdot e_x$ and $y \in \mathbb{R}^n$.

Equation (1.1) must be supplemented with a capillary pressure relation

$$p \in \mathcal{F}(s).$$

The simplest possibility (and the standard choice for the Richards equation) is a functional dependence, $p = p_c(s)$, where p_c is a monotone function. In applications, the capillary pressure function can have infinite slope and can even be multivalued, $p_c = \mathcal{F} \subset \mathbb{R} \times \mathbb{R}$. In this degenerate case (even without hysteresis) we must therefore write the inclusion symbol in the above pressure relation. In physical variables, the saturation has only values in $[0, 1]$, k is a function on $[0, 1]$, $p_c = \mathcal{F}$ is a maximal monotone graph $\mathcal{F} \subset [0, 1] \times \mathbb{R}$. Typically, k vanishes for small arguments and p_c is multi-valued in the end-points of its domain. Nevertheless, the instability result of the current work is shown for non-degenerate coefficient functions.

1.1 Gravity wetting fronts

We are interested in the situation that a more saturated medium is above a less saturated medium such that, under the influence of gravity, the saturation increases in the lower part of the medium. A question which receives considerable interest is the following: Does the penetration into the initially dryer medium always happen with a one-dimensional front, i.e. with negligible variations in the horizontal variable y, or can there also appear *fingers,* i.e. smaller structures with a higher saturation along which the fluid moves downwards? We refer to the nice overview in [13] for fingering effects in other physical systems.

In various experimental works, the development of fingers was verified for well-adapted initial and boundary conditions. Early works date into the 1960'ies. One more recent study is [26], where an experimental set-up with finger development is described. It contains the measurement of saturation profiles (non-monotonic in x), and the observation that the time evolution of fingers is approximately given by a translation in x. Fingers are also observed in [3], where also the influence of an increased initial saturation is studied. If the fluid enters a less dry medium, the finger widens and eventually dissapears, and the saturation profile becomes monotonic in x. The importance of a very dry initial condition is also discussed in [16].

To model the experimentally observed fingering effect, the standard Richards equation with a fixed capillary pressure curve seems to be inadequate [11]. This observation coincides with the mathematical analysis of [28] which contains a stability result for one-dimensional front solutions under Richards equation with a *function* $\mathcal{F}$. A stability result for the classical Richards equation can be derived also in degenerate cases, see [18] and [6].

As a consequence, modifications of the Richards equation have been introduced in order to capture gravity fingering. One of the most prominent models was introduced by Hassanizadeh and Gray [12]. Their suggestion is to replace the algebraic relation $p = p_c(s)$ by a kinetic equation such as $\tau\partial_t s = p - p_c(s)$ for some real parameter τ. Such *non-equilibrium Richards equations* (NERE) are studied e.g. in [10, 17]; a *low-frequency instability criterion* is introduced and used to predict an instability in the NERE model. Once more, a low initial saturation is important for a spatial instability. The same model is also analyzed in [20] with the result that non-monotonic one-dimensional profiles can be induced by the NERE model. A two-dimensional numerical simulation shows a non-monotonic finger solution.

Another possible modification of the Richards equation is to introduce a rate independent hysteresis in the form that different capillary pressure curves are used for imbibition and drainage. This most elementary model is actually closely related to the play-type hysteresis studied here. Numerically, gravity fingers for this model were observed in [15]. For a theoretical analysis of different hysteresis models we refer to [30]. We mention that, starting from a quite different model, a higher dimensional instability is also numerically observed in [9].

1.2 A rate independent hysteresis model

It is well known that porous media exhibit hysteresis effects [4]. Furthermore, the importance of hysteresis for the development of gravity fingers seems also to be evident. Much less clear is the choice of an appropriate hysteresis model. Beliaev and Hassanizadeh distinguish in [5] between static capillary pressure hysteresis and a dynamic variant. They give thermodynamic arguments in favor of the (static) play-type hysteresis model and are able to confirm the model to some extend by reported measurements. Furthermore, the model is expanded by the inclusion of dynamic effects in the spirit of [12].

In this work we discuss the play-type hysteresis model in its simplest form and study its possible effects in terms of the gravity fingers instability. We emphasize that the play-type model has many virtues: it gives a reasonable agreement with experimental data, it is rate-independent (as are most of the reported measurements), such that, in particular no additional time-scale (τ) is introduced. Furthermore, the play-type model is thermodynamically consistent, and it can, to some extend, be justified theoretically [21, 22]. We do not doubt the presence of dynamic hysteresis effects in porous media, but we want to analyze here the implications of the purely static hysteresis model. Our main result is a rigorous instability result for the Richards equation with play-type hysteresis.

We next describe this model in more detail. Mathematically, we interpret the operator $\mathcal{F}$ in the relation $p \in \mathcal{F}(s)$ not as an algebraic relation for every time instance, but rather as a map $s|_{[T_-,t]} \mapsto \mathcal{F}(s|_{[T_-,t]}) \in \mathbb{R}$. With a parameter $\gamma > 0$, which is a measure for the difference in pressure between imbibition and drainage, we may specify $\mathcal{F}$ through the differential relation

$$p \in p_c(s) + \gamma \operatorname{sign}(\partial_t s), \tag{1.2}$$

where $\operatorname{sign}(\xi) := [-1, 1]$ for $\xi = 0$ and $\operatorname{sign}(\xi) \in \{\pm 1\}$ for $\xi \neq 0$. Relation (1.2) demands that the pressure p is always in the s-dependent interval $[p_c(s) - \gamma, p_c(s) + \gamma]$. Furthermore, for p strictly between $p_c(s) - \gamma$ and $p_c(s) + \gamma$, the time derivative $\partial_t s$ necessarily vanishes.

The hysteresis relation can be made more general by demanding that, loosely speaking, the effect of different values of γ is averaged. The result is a Prandtl-Ishlinskii hysteresis relation. For a finite number of γ's, the relation can be written as

$$p \in p_j(s_j) + \gamma_j \operatorname{sign}(\partial_t s_j) \quad \forall j = 1, ..., N, \quad s = \sum_{j=1}^{N} c_j s_j. \tag{1.3}$$

Here, c_j and γ_j are given positive numbers for $j = 1, ..., N$. We demand $\sum_j c_j = 1$ such that the saturation is a convex combination of the different s_j, which can be thought of as the saturations in different materials that constitute the porous medium. Regarding physical units, the numbers γ_j are pressure variables. Finally, the functions p_j are monotone graphs. The general Prandtl-Ishlinskii hysteresis can be formulated equivalently, replacing the finite sums by integrals. A homogenization result is derived in [23] for linear laws p_j: a porous medium which consists of different materials that exhibit the play-type hysteresis (1.2) (with different parameters) can be described in its averaged behavior by a Prandtl-Ishlinskii relation.

Our main result is an instability statement for problem (1.1) with an hysteresis operator $\mathcal{F}$ in the capillary pressure relation. We will show the instability for the simple model of play-type hysteresis in (1.2). Since this is contained in the more complex models such as (1.3), the possibility of an instability is clear also for those more complex models.

To conclude the description of our model we finally describe the boundary conditions. We consider a time domain $t \in (T_-, T_+)$ containing $t = 0$ and a spatial domain $(x, y) \in [L_-, L_+] \times Y$, with $Y = [0, L_y)^n$ a rectangle in $\mathbb{R}^n$ with periodically identified boundaries (the relevant cases are $n = 1$ and $n = 2$). Initial values are given by $s_0 : [L_-, L_+] \to \mathbb{R}$ (which we identify with $s_0 : [L_-, L_+] \times Y \to \mathbb{R}$), and boundary conditions $\bar{p}_\pm : (T_-, T_+) \to \mathbb{R}$. The initial and boundary conditions are chosen as

$$s(x, y, t = 0) = s_0(x) \quad \forall y \in Y, x \in (L_-, L_+), \tag{1.4}$$

$$p(x = L_\pm, y, t) = \bar{p}_\pm(t) \quad \forall y \in Y, t \in (T_-, T_+). \tag{1.5}$$

Wetting fronts appear when a more saturated medium is above a less saturated medium. Mathematically, we choose a constant initial saturation s_0, set $\bar{p}_-(t) = p_c(s_0) + \gamma$ for all t, and $\bar{p}_+(t) > p_c(s_0) + \gamma$, at least for $t \in (T_-, 0)$.

1.3 Main result: Instability

Our result is that, for appropriate boundary conditions, the hysteresis system is spatially unstable. By *spatially unstable* we mean that y-independent solutions (planar solutions) are unstable solutions of the higher dimensional system. For the precise definition of stability we refer to Definition 2.2 of the next section.

Theorem 1.1 (Spatial instability) *Let $p_c, k \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $k, \partial_s k, \partial_s p_c > 0$ and let $\gamma > 0$ be given. Then there exist initial and boundary conditions* (1.4)–(1.5) *such that the following holds. If Assumption 3.3 concerning existence and regularity properties of a one-dimensional free boundary problem is satisfied, then the planar front solution of the hysteresis system* (1.1)–(1.2) *is unstable.*

Loosely speaking, instability is defined as follows: for any $\rho > 0$, there is a geometry (given by $L_\pm$ and L_y), a perturbation w (of size less than ρ) of the planar initial values, and a source function f (of size less than ρ), such that the corresponding solution is arbitrarily far from the planar solution for a large observation time T.

Comments on the main theorem.
We impose positivity and monotonicity assumptions on p_c and k that are natural in the context of the Richards equation, but we restrict ourself to the non-degenerate case. In particular, we show that the instability of front solutions is not a consequence of degenerate coefficients.

We only have a weak instability result in the sense that the numbers L_y and T must be chosen in dependence of ρ, i.e. the fingers may be very wide and may develop only at large times. On the one hand, this limitation is consequence of our method of proof. On the other hand, it is not clear if an arbitrarily small perturbation can create fingers of finite size in the non-degenerate setting. Based on the description in [3] concerning the finger widening, we expect that, for a stronger instability result, degenerate coefficient functions must be studied.

Method of proof.
We study a switch in the pressure boundary condition on the upper boundary. Until time $t = 0$, a large pressure on the upper boundary generates an imbibition process, water invades a medium with low saturation from the top. At time $t = 0$, the pressure on the upper boundary jumps to a lower value. This induces a decrease of pressure in a region near the upper boundary, while a (gravity driven) imbibition process continues in the lower part of the domain. This setting is in accordance with experiments.

The switch at time $t = 0$ effectively means that a first evolution process is considered until time 0, while a second evolution process runs after time 0. Both evolution processes are stable — but the combined process is unstable: a small perturbation of initial data at time $t = T_-$ results in a small perturbation at time $t = 0$, but this perturbation changes the second process for all later times. If T_+ is large enough, the perturbation at time $t = T_+$ is large.

The proof of the theorem rests entirely on the analysis of the one-dimensional system, i.e. the system with one spatial variable $x \in (L_-, L_+)$ and a time variable $t \in (T_-, T_+)$. We describe solutions of this system after the switching time with a free boundary problem. The qualitative properties of this free boundary problem can be analyzed, see Figure 1 for an illustration. In particular, there exists a flux parameter $q(t)$ which decreases in time, but does not vanish in the limit of large times. This implies that the front continues to proceed with a finite speed. Since the limiting front speed depends on the saturations at time $t = 0$, this implies that a small perturbation at time $t = 0$ can result in large perturbation at time $t = T_+$. The juxtaposition of slightly perturbed one-dimensional profiles in y-direction provides an unstable solution.

Further literature.
For degenerate Richards equations without hysteresis, existence statements [1, 2, 24] and uniqueness results [1, 6, 18] are available. Concerning the case that hysteresis is included, we are not aware of any result in the degenerate case. In the one-dimensional case, the oil-trapping effect [25] shows that the degeneracy can change qualitative properties of solutions.

Positive results on the stability of planar fronts are available for many systems. In comparison, instability results are rare. As in our approach, long-wave perturbations are considered in [8] to show the (linearized) instability of planar fronts in a reaction diffusion system. Other instability results for planar fronts appear in [27, 7].

The remainder of this text is organized as follows. In Section 2 we recall fundamental facts about the play-type hysteresis model, we introduce one-dimensional front solutions, and give a precise definition of the stability of planar fronts. Section 3 is devoted to the thorough analysis of the one-dimensional hysteresis system to special initial and boundary conditions. The main result is the determination of the limiting flux for large times in Lemmas 3.6 and 3.8. In Section 4, we give the proof of Theorem 1.1.

2 Preliminaries and stability

In this section we collect known properties of the system (1.1)–(1.2). In Subsection 2.1 we recall existence results of [23]. In Subsection 2.2 we define our concept of stability. Subsection 2.3 collects some positive stability results.

2.1 Existence result for a system with hysteresis

Existence properties of the hysteresis system were studied in [23] for s-independent permeability and affine capillary pressure, neglecting gravity. The emphasis in that existence result was to generalize relation (1.2) to a Prandtl-Ishlinskii hysteresis relation in order to treat the system which is obtained after homogenization. In order to specify the results of [23] to our context, we set $\Gamma(x,.) \equiv \delta_\gamma(.)$ and $p_c(\sigma) \equiv a\sigma + b$, and read the results for $s(x,t) := u(x,t)$, $a := a^*$, $b := b^*$, and $k := K^*$, where the letters used in [23] appear on the right hand side of the four settings. Equations (1.8)-(1.10) of [23] with $w(x,t) \equiv w(x,\gamma,t)$ then read

$$w(x,t) = p_c(s(x,t)), \tag{2.1}$$

$$\partial_t s = \nabla \cdot (k \nabla p), \tag{2.2}$$

$$p(x) \in w(x,t) + \gamma\, \mathrm{sign}(\partial_t w), \tag{2.3}$$

and coincide with our system. Theorem 3.2 and case (ii) of Corollary 3.3 in [23] provide the following existence result. The uniqueness is observed in Remark 3.4 of the same article, where the boundary condition is imposed as $p = g$ on $\partial\Omega \times (0,T)$.

Theorem 2.1 (Existence for an hysteresis system, S. 2007, [23]) *Let $\Omega \subset \mathbb{R}^n$ be a rectangle, $T > 0$, $s \mapsto p_c(s)$ strictly monotone affine and $k(x,s) = K^*(x)$ piecewise constant. Let initial and boundary values be given by $s_0 \in L^2(\Omega)$ and $g \in C^1([0,T], H^2(\Omega,\mathbb{R}))$. Then there exists a unique pair (s,p) with*

$$s, \partial_t s \in L^\infty(0,T; L^2(\Omega)), \qquad p \in H^1(0,T; H^1(\Omega)),$$

such that relations (2.1)–(2.3) *are satisfied in the sense of distributions and almost everywhere in $\Omega \times (0,T)$, and the boundary conditions are satisfied in the sense of traces.*

Theorem 2.1 was shown with an approximation procedure. A discretization of Ω with triangles of maximal diameter h replaces the system by an ordinary differential inclusion equation with independent variable t. This equation still contains the inclusion of (2.3). One can treat this degeneracy by replacing the inverse of the sign-function $\mathrm{sign}_\gamma := \gamma\,\mathrm{sign}$ by the Lipschitz function $\psi_\delta^\gamma : \mathbb{R} \to \mathbb{R}$,

$$\psi_\delta^\gamma(r) := \begin{cases} \delta r & \text{for } r \in [-\gamma,\gamma], \\ \gamma\delta + \frac{1}{\delta}(r-\gamma) & \text{for } r > \gamma, \\ -\gamma\delta + \frac{1}{\delta}(r+\gamma) & \text{for } r < -\gamma. \end{cases} \tag{2.4}$$

More specifically, for $\delta > 0$, we solve an ordinary differential equation, which we write as

$$\partial_t s_h = \psi_\delta^\gamma(p_h - p_c(s_h)),$$
$$\nabla^h \left(k(s_h)\nabla^h p_h\right) = \psi_\delta^\gamma(p_h - p_c(s_h)).$$

To make the the method precise, the operator ∇^h is expressed with a finite element method, see (2.3)–(2.6) of [23]. We emphasize that, in the existence result, it is important to send first $\delta \to 0$ with discretization parameter $h > 0$ fixed, and then send $h \to 0$.

With Theorem 2.1 we have an existence and uniqueness result for system (1.1)–(1.2) in the case that Ω is a rectangle, that $k(x,s) = K^*$ is independent of s, that $p_c(.)$ is affine function, and that gravity is neglected. We expect that these assumptions can be relaxed for the existence proof, and that the same existence result can be obtained for smooth strictly positive k, and smooth and strictly monotonically increasing p_c. From now on, we concentrate on stability aspects and skip the further discussion of existence results.

2.2 One-dimensional system and stability property

The one-dimensional system.
Let us now consider one-dimensional solutions to (1.1)–(1.2), i.e. solutions $s(x,y,t) = s(x,t)$ and $p(x,y,t) = p(x,t)$. With x as the only spatial variable and gravity pointing in the negative x-direction, the system for $s(x,t)$ and $p(x,t)$ with $(x,t) \in (L_-,L_+) \times (T_-,T_+)$ reads

$$\partial_t s = \partial_x \left(k(s)[\partial_x p + 1]\right), \tag{2.5}$$

$$p \in p_c(s) + \gamma\,\mathrm{sign}(\partial_t s). \tag{2.6}$$

We recall that we always demand that $\partial_x p$ and $\partial_t s$ are functions in $L^2((L_-,L_+)\times(T_-,T_+))$, that (2.5) holds in the sense of distributions and that (2.6) holds almost everywhere. We note that we must assume for a planar solution that the initial and boundary conditions are y-independent, $s(x,y,t=T_-) = s_0(x)$ and $p(x = L_\pm, y, t) = \bar p_\pm(t)$.

Every solution to this one-dimensional problem is a solution to the higher dimensional problem if we identify s and p with their trivial extensions in the periodic variable $y \in Y = [0,L_y)^n$. We call a solution of problem (2.5)–(2.6) a planar solution of (1.1)–(1.2).

The concept of spatial stability.
Our interest is the stability of a planar solution. More precisely, we are interested in the *spatial stability.* We use this term to indicate that we expect the one-dimensional solution to be stable as a solution of the one-dimensional system, but if we interpret the functions as a solution in higher space dimension, they are unstable. We additionally allow a (small) source term in the conservation law and study, for $f : (L_-,L_+)\times Y\times(T_-,T_+) \to \mathbb{R}$,

$$\partial_t s = \nabla\cdot(k(s)[\nabla p + e_x]) + f, \tag{2.7}$$

$$p \in p_c(s) + \gamma\,\mathrm{sign}(\partial_t s). \tag{2.8}$$

For the definition of stability of system (1.1)–(1.2) we recall that n is the dimension of $Y = [0,L_y)^n$. We include the factor L_y^n in the subsequent L^1-norm estimates. This scaling factor is chosen such that, e.g., $L_y^{-n}\|f\|_{L^1((L_-,L_+)\times Y\times(T_-,T_+))}$ is a measure for the typical height of the function $|f|$, at least for fixed numbers $L_\pm$ and $T_\pm$.

Definition 2.2 (Stability) *Let $s_0 = s_0(x)$ and $\bar p_\pm = \bar p_\pm(t)$ be fixed, and let (s,p) be a corresponding planar solution without sources, i.e. a solution of* (2.5)*–*(2.6)*. We say that (s,p) is* **stable,** *if, for every $\varepsilon > 0$, there exists $\rho > 0$ with the following property: For all $T_- < 0 < T_+$, $L_- < 0 < L_+$, $L_y > 0$, for all perturbations $w \in C^1((L_-,L_+)\times Y)$ of the initial data and all sources $f \in C^1((L_-,L_+)\times Y\times(T_-,T_+))$ with*

$$\|w\|_{L^1((L_-,L_+)\times Y)} + \|f\|_{L^1((L_-,L_+)\times Y\times(T_-,T_+))} < 2\rho L_y^n, \tag{2.9}$$

there exist a solution $(\tilde s,\tilde p)$ to system (2.7)*–*(2.8) *with $\tilde s|_{t=T_-} = s_0 + w$ and*

$$\|\tilde s(.,T_+) - s(.,T_+)\|_{L^1((L_-,L_+)\times Y)} < \varepsilon L_y^n\,. \tag{2.10}$$

Accordingly, we say that (s,p) is **unstable,** *if*

$$\begin{aligned} &\exists\,\varepsilon>0\ \forall\,\rho>0\ \exists T_\pm, L_\pm, L_y \in \mathbb{R},\ w\in C^1((L_-,L_+)\times Y),\ f\in C^1((L_-,L_+)\times Y\times(T_-,T_+))\\ &\qquad \textit{with}\ \ \|w\|_{L^1} + \|f\|_{L^1} \le 2\rho L_y^n \ \ \textit{but}\ \ \|\tilde s(.,T_+) - s(.,T_+)\|_{L^1((L_-,L_+)\times Y)} \ge \varepsilon L_y^n\,. \end{aligned} \tag{2.11}$$

We remark that our stability criterion reflects standard concepts. Nevertheless, many variations of the above definition are possible. We have a rather strong concept of stability since we demand that ρ satisfies the desired inequalities independent of the geometry. Furthermore, in the current context, it seems that the inclusion of the normalizing factors L_y^n makes the stability concept again stronger, since, for large $L_y > 0$, large variations of the initial values are permitted.

An important property of the above definition is observed in the next subsection: every system with an L^1-contraction property is stable in the sense of Definition 2.2.

2.3 Stability results

The system without hysteresis has an L^1-contraction property. We recall here this well-known result (see [1, 31]) and present the proof for the simplest case, namely for strong solutions to the system with non-degenerate coefficient functions. The L^1-contraction property derived in Theorem 2.3 and Remark 2.4 implies the stability of (1.1)–(1.2) in the sense of Definition 2.2 with the choice $\rho = \varepsilon/2$. In particular, the system without hysteresis and the system with constant permeability are both stable.

In the case of strong solutions the result is readily obtained by considering two solutions and by testing the difference of the equations with the sign of the solution difference. The interest in more recent uniqueness studies is to have the same result for degenerate equations, when the distributional derivative $\partial_t s$ is not necessarily an integrable function. In this case, the proof of the contraction property can be performed with the technique of doubling the variables. For this interesting field we refer to [6, 18].

Theorem 2.3 (Stability in absence of hysteresis) *We consider* (2.7)–(2.8) *in the case without hysteresis, i.e. for* $\gamma = 0$,

$$\partial_t s = \nabla \cdot (k(s)[\nabla(p_c(s)) + e_x]) + f. \tag{2.12}$$

Let $k = k(s)$ *and* $p_c = p_c(s)$ *be smooth, independent of* x*, with* k*,* k'*, and* p_c' *strictly positive. Then an* L^1*-contraction property holds. More precisely, for two solutions* s_1 *and* s_2 *with* $\partial_t s_i \in L^2(\Omega \times (T_-, T_+))$ *to the same boundary conditions and with the right hand sides* $f_i \in L^1(\Omega \times (T_-, T_+))$*, there holds*

$$\int_\Omega |s_1 - s_2|(x, t_2)\, dx \le \int_\Omega |s_1 - s_2|(x, t_1)\, dx + \int_{t_1}^{t_2} \int_\Omega |f_1 - f_2|(x, t)\, dx\, dt \tag{2.13}$$

for all $t_2 > t_1$*.*

Proof We note that the non-degenerate problem without hysteresis (2.12) is a standard parabolic problem and existence results are classical. We consider strong solutions s_1 and s_2 to sources f_1 and f_2.

We use a Kirchhoff transformation. Choosing a function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi'(s) = k(s) p_c'(s)$, we use the generalized pressure $u = \Phi(s)$ as a new dependent variable. Because of $\nabla u = \Phi'(s)\nabla s = k(s)p_c'(s)\nabla s = k(s)\nabla(p_c(s))$, the equations for s_1 and s_2 transform into

$$\begin{aligned} \partial_t s_1 &= \nabla \cdot (\nabla u_1 + k(s_1) e_x) + f_1, \quad u_1 = \Phi(s_1), \\ \partial_t s_2 &= \nabla \cdot (\nabla u_2 + k(s_2) e_x) + f_2, \quad u_2 = \Phi(s_2). \end{aligned}$$

Let H_η be a family of uniformly bounded smooth functions $H_\eta : \mathbb{R} \to \mathbb{R}$ that are odd and strictly increasing. Using $H_\eta(u_1 - u_2) = -H_\eta(u_2 - u_1)$ as a test-function in the equation for s_1 and $H_\eta(u_2 - u_1)$ as a test-function in the equation for s_2, adding the equations and integrating yields

$$\begin{aligned} &\int_\Omega \partial_t(s_2 - s_1) H_\eta(u_2 - u_1) + \int_\Omega \nabla(u_2 - u_1) \nabla[H_\eta(u_2 - u_1)] \\ &\quad - \int_\Omega \partial_x[k(s_2) - k(s_1)][H_\eta(u_2 - u_1)] = -\int_\Omega (f_1 - f_2) H_\eta(u_2 - u_1)\,. \end{aligned} \tag{2.14}$$

We choose for H_η uniformly bounded and odd approximations of the sign function, $H_\eta(\xi) \to \mathrm{sign}(\xi)$ for every ξ in the limit $\eta \to 0$. In the limit $\eta \to 0$, since the sign of $u_2 - u_1$ is identical to the sign of $s_2 - s_1$, the first integrand converges to $\partial_t |s_1 - s_2|$. At this point we exploit that $\partial_t s_i$ are integrable functions. The second integrand of (2.14) is non-negative for every $\eta > 0$.

It remains to consider the third integral, which we expand by adding and subtracting the same term,

$$
\begin{aligned}
I_3^\eta = &- \int_{\Omega_t} \partial_x [k(s_2) - k(s_1)] \, \{H_\eta(\Phi(s_2) - \Phi(s_1)) - H_\eta(k(s_2) - k(s_1))\} \\
&- \int_{\Omega_t} \partial_x [k(s_2) - k(s_1)] \, H_\eta(k(s_2) - k(s_1)) =: I_{31}^\eta + I_{32}^\eta .
\end{aligned} \tag{2.15}
$$

Let $\hat{H}_\eta$ be a primitive of H_η with $\hat{H}_\eta(0) = 0$. Then the integrand in the last integral is $\partial_x \hat{H}_\eta(k(s_2) - k(s_1))$. The identical boundary conditions for p_1 and p_2 (and thus for s_1 and s_2) imply that

$$
I_{32}^\eta = - \int_{\Omega_t} \partial_x [k(s_2) - k(s_1)] \, H_\eta(k(s_2) - k(s_1)) = - \int_{\Omega_t} \partial_x [\hat{H}_\eta(k(s_2) - k(s_1))] = 0.
$$

Concerning I_{31}^η we note that the factor $H_\eta(\Phi(s_2) - \Phi(s_1)) - H_\eta(k(s_2) - k(s_1))$ is uniformly bounded and converges to 0 pointwise in (x, t). By Lebesgue's convergence theorem, I_{31}^η vanishes in the limit $\eta \to 0$. In the limit $\eta \to 0$ we thus obtain

$$
\int_\Omega \partial_t |s_2 - s_1| \le - \int_\Omega (f_1 - f_2) \operatorname{sign}(u_2 - u_1) \le \int_\Omega |f_1 - f_2| .
$$

An integration from t_1 to t_2 yields the desired estimate (2.13). □

Spatial stability under perturbations in weighted L^2-spaces are analyzed in [28]. The result of that article is another stability result for solutions of the system without hysteresis. The following remark contains the warning that hysteresis does not automatically lead to an instability.

Remark 2.4 (Stability of the hysteresis system for constant k) *We consider problem* (2.7)–(2.8) *with general $\gamma \ge 0$, strictly monotone and smooth p_c, and k independent of s. Then the system has an L^1-contraction property. In particular, in the class of strong solutions, every planar solution is stable.*

Proof We follow the proof of Theorem 2.3. Let (s_1, p_1) and (s_2, p_2) two solutions of (2.7)–(2.8) with $p_i \in p_c(s_i) + \gamma \operatorname{sign}(\partial_t s_i)$ for $i = 1, 2$. Assuming, for notational convenience, $k = 1$, we consider the difference of the two equations $\partial_t s_i = \Delta p_i + f_i$ and multiply with $H_\eta(p_1 - p_2)$. This results in

$$
\begin{aligned}
0 \ge - \int_\Omega \nabla(p_1 - p_2) \nabla(H_\eta(p_1 - p_2)) &= \int_\Omega \partial_t (s_1 - s_2) H_\eta(p_1 - p_2) - \int_\Omega (f_1 - f_2) H_\eta(p_1 - p_2) \\
= \int_\Omega \partial_t (s_1 - s_2) H_\eta(p_c(s_1) - p_c(s_2)) &+ \int_\Omega \partial_t (s_1 - s_2) \, \{H_\eta(p_1 - p_2) - H_\eta(p_c(s_1) - p_c(s_2))\} \\
- \int_\Omega (f_1 - f_2) H_\eta(p_1 - p_2) .&
\end{aligned}
$$

We choose again as sequence of functions H_η odd and bounded approximations of the sign-function. With this choice, by monotonicity of p_c, there holds $H_\eta(p_c(s_1) - p_c(s_2)) \to \operatorname{sign}(p_c(s_1) - p_c(s_2)) = \operatorname{sign}(s_1 - s_2)$. In particular, the first integrand converges to $\partial_t |s_1 - s_2|$ as in the last proof. For the other integral we have a positivity property. Indeed, for $\partial_t s_1 \ge \partial_t s_2$, there holds

$$
p_c(s_1) - p_c(s_2) \in (p_1 - p_2) - \gamma(\operatorname{sign}(\partial_t s_1) - \operatorname{sign}(\partial_t s_2)) \le p_1 - p_2 ,
$$

and therefore, by monotonicity of H_η,

$$
\partial_t (s_1 - s_2) \, \{H_\eta(p_1 - p_2) - H_\eta(p_c(s_1) - p_c(s_2))\} \ge 0.
$$

We find the non-negativity of the integrand. The same argument can be repeated for negative $\partial_t(s_1 - s_2)$. By taking the limit $\eta \to 0$ we find the same contraction result as in Theorem 2.3. □

Our instability result of Theorem 1.1 implies that the hysteresis system with an s-dependent permeability k does not posess the L^1-contraction property.

We mention here that we do expect a stability property in another special situation. We conjecture that (even for an s-dependent positive k) a strictly monotonically increasing (in time) planar solution is stable for the hysteresis system. We note that a strictly increasing planar solution (s,p) satisfies $\partial_t s > 0$ everywhere and hence a system where the hysteresis relation is replaced by $p = p_c(s) + \gamma$. Nevertheless, such a stability result needs a deep analysis since the comparison solution $(\tilde{s}, \tilde{p})$ will, in general, only satisfy the law (1.2).

3 The one-dimensional free boundary problem

In this section we consider once more $x \in (L_-, L_+)$ as the only spatial variable and analyze the hysteresis system (1.1)–(1.2) in its one-dimensional version (2.5)–(2.6). We restrict our analysis to x-independent initial values and piecewise constant boundary conditions. Let the equations be specified by a number $\gamma > 0$ and coefficient functions

$$k, p_c \in C^2(\mathbb{R}, \mathbb{R}) \text{ with } k, k', p_c' > 0 \text{ on } \mathbb{R}. \tag{3.1}$$

In order to specify boundary conditions we use four number $\bar{s}_0$, p_-, p_+^0, and p_+ with

$$\bar{s}_0 \in \mathbb{R}, \quad p_- := p_c(\bar{s}_0) + \gamma, \quad p_+^0 > p_c(\bar{s}_0) + \gamma, \quad p_+ < p_+^0. \tag{3.2}$$

The boundary conditions (1.4)–(1.5) are specified to the following piecewise constant initial saturation and boundary pressures.

$$s(x, t = 0) = \bar{s}_0 \text{ for all } x \in (L_-, L_+), \tag{3.3}$$

$$p(x = L_-, t) = \bar{p}_-(t) := p_- \ \text{ for all } t \in (T_-, T_+), \tag{3.4}$$

$$p(x = L_+, t) = \bar{p}_+(t) := \begin{cases} p_+^0 & \text{for } t \in [T_-, 0), \\ p_+ & \text{for } t \in [0, T_+]. \end{cases} \tag{3.5}$$

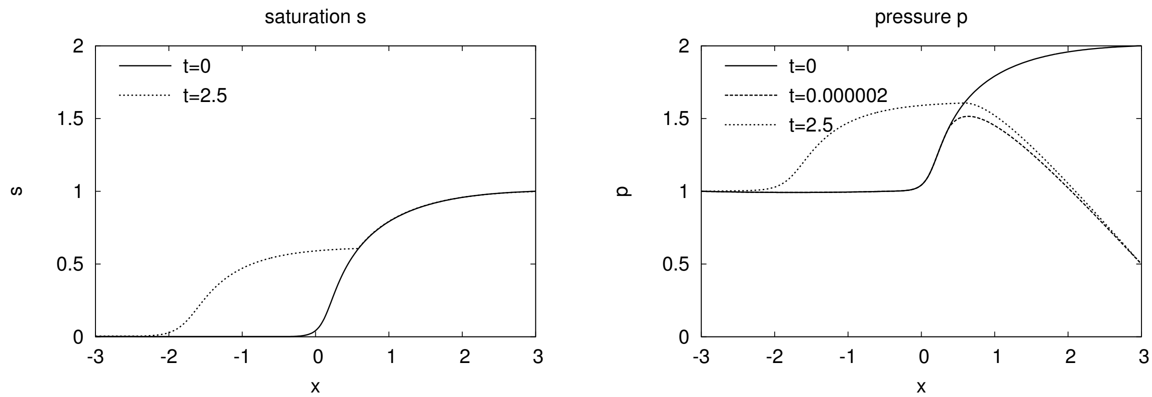

FIGURE 1. *Numerical solutions obtained with the scheme indicated in* (2.4)*. The parameter functions are* $p_c(s) = s$ *and* $k(s) = 0.1 + s^2$*, the parameters are* $\gamma = 1$*,* $s_0 = 0$*,* $p_+^0 = 2$*,* $p_+ = 0.5$*, with the regularization parameter* $\delta = 10^{-3}$*. The saturation at time* 0.000002 *is indistinguishable from the saturation at time* 0*. The large change in the pressure at the switching time* $t = 0$ *is clearly visible. The saturation continues to increase in the left part of the domain despite the low imposed pressure* p_+*. We observe a small, but non-vanishing front speed for large times, as derived in Subsection 3.3. The numerical results are obtained by Andreas Rätz.*

3.1 Behavior of solutions on $(T_-, 0]$.

The boundary conditions are chosen in order to create an imbibition process on the time interval $(T_-, 0]$. The constant function $p \equiv p_-$ together with $s \equiv s_0$ is a solution of (2.5)–(2.6) satisfying the initial condition and the left boundary condition, but the increased pressure $p_+ > p_-$ on the upper boundary initiates the wetting process. During this imbibition process, relation (2.6) reduces to $p = p_c(s) + \gamma$.

Lemma 3.1 (Solution properties on $[T_-, 0)$) *We consider* (2.5)–(2.6) *with conditions* (3.1)–(3.5). *There exists a solution* (s,p) *of this system satisfying*

$$\partial_t s > 0 \quad \text{and} \quad p = p_c(s) + \gamma, \tag{3.6}$$
$$p_- \le p \le p_+^0 \quad \text{and} \quad \partial_x s > 0, \tag{3.7}$$

for all $t \in (T_-, 0)$ *and all* $x \in (L_-, L_+)$.

We note that the uniqueness for system (2.5)–(2.6) is assured by Theorem 2.1 only for constant k, affine p_c, and the case without gravity. For this reason we can not formulate in the above lemma that every solution satisfies the monotonicity properties (3.6)–(3.7).

Proof We analyze (2.5) with (2.6) replaced by $p = p_c(s) + \gamma$, i.e. the equation

$$\partial_t s = \partial_x (k(s)[\partial_x(p_c(s)) + 1]). \tag{3.8}$$

This equation is non-degenerate parabolic and has therefore a classical solution on $(T_-, 0] \times [L_-, L_+]$. Furthermore, solutions of equation (3.8) satisfy a maximum principle. This implies that $p = p_c(s) + \gamma$ satisfies the bounds $p_- \le p \le p_+^0$ of (3.7) and the corresponding lower bound $s \ge \bar{s}_0$.

Differentiating (3.8) with respect to t shows that the time-derivative $v = \partial_t s$ satisfies

$$\partial_t v = \partial_x (k'(s) v [\partial_x (p_c(s)) + 1]) + \partial_x (k(s)[\partial_x (p_c'(s) v)]). \tag{3.9}$$

We consider s as a given function that determines the coefficients of this linear equation for v. Because of the uniform positivity $k > 0$ and $p_c' > 0$, equation (3.9) for v is again a non-degenerate parabolic equation. It is supplemented with the boundary condition $v(L_\pm, t) = 0$. The idea is now to apply the strong maximum principle to v in order to conclude $v \ge 0$. This provides (3.6).

The argument can be made rigorous with a regularization of the boundary condition in L_+. We use, for $\varepsilon_j \searrow 0$, a sequence of C^∞-functions $p_+^\varepsilon : [T_-, 0] \to \mathbb{R}$ that are monotonically increasing and satisfy $p_+^\varepsilon \equiv p_-$ in the interval $[T_-, T_- + \varepsilon]$. The regularized boundary condition is now $p(x = L_+, t) = p_+^\varepsilon(t)$. The corresponding solutions $(s^\varepsilon, p^\varepsilon)$ and $v^\varepsilon = \partial_t s^\varepsilon$ satisfy (3.8) and (3.9), the boundary conditions $v(L_+, t) = 0$ and $v(L_+, t) = \partial_t p_+^\varepsilon(t) \ge 0$. Furthermore, $v^\varepsilon = 0$ holds in $t = T_-$. The maximum principle for smooth functions yields the non-negativity of v^ε in the whole domain. For a sequence of functions $p_+^\varepsilon(t)$ that approximate p_+^0 for $\varepsilon \to 0$, the solutions of the ε-problem converge to a solution of the original problem. Monotonicity properties of solutions remain valid in the limit and show $\partial_t s \ge 0$. The non-negative function $\partial_t s$ can not vanish identically because of the initial and boundary conditions. Then the strong maximum principle implies the strict inequality (3.6).

In order to conclude the second inequality of (3.7), we repeat the argument with the differentiated equation, this time considering $v = \partial_x s$. The non-negativity of $v = \partial_x s$ on the boundaries $x = L_\pm$ follows from the a priori bounds on s, expressed in $p_- \le p_c(s) + \gamma \le p_+^0$. A regularization argument as above yields $\partial_x s \ge 0$. The strong parabolic maximum principle provides $\partial_x s(L_+, t) > 0$ for all $t \in (T_-, 0]$, since s assumes its maximum at the right boundary (it is not constant by the left boundary condition). This implies also the strict inequality $\partial_x s > 0$ in the interior. □

We emphasize that, during the time-span $(T_-, 0)$, the boundary conditions imply a pure wetting process for equations (2.5)–(2.6). The monotonicity $\partial_t s \ge 0$ together with $p = p_c(s) + \gamma$ is consistent with (2.6). The hysteresis relation has no effect in the time-span $(T_-, 0)$.

Initial values for the time-span $[0, T_+)$.

Our instability result for the hysteresis system is a consequence of properties of the evolution equation (2.5)–(2.6) on the whole interval (T_-, T_+). On the interval $(T_-, 0)$, hysteresis was not relevant. Instead, due to a decreased pressure boundary condition $p_+ < p_+^0$ on the upper boundary (see (3.2)), the hysteresis relation (2.6) will be relevant for $t > 0$.

From now on, our analysis concerns the hysteresis system (2.5)–(2.6) on the time interval $[0, T_+)$. The boundary conditions for the pressure and the initial condition for the saturation are given by

$$p(L_\pm, t) = p_\pm, \qquad s|_{t=0} = s_1. \tag{3.10}$$

Here, the initial saturation s_1 is given as $s_1(x) := s(x, 0)$, where s is the solution of the system on $(T_-, 0]$. From Lemma 3.1 we know that $s_1 \in C^2([L_-, L_+], \mathbb{R})$ is strictly increasing in x.

For the subsequent analysis, a refined study of the situation at $t = 0$ is necessary. For $t > 0$, the pressure value p_+ at the right end point L_+ is below the value $p_c(s_1(L_+)) + \gamma = p_+^0$. Assuming that γ is sufficiently large, the hysteresis relation (2.6) allows that the pressure jumps to a lower value with an unchanged saturation, i.e. $s(x, t) = s_1(x)$ for $t > 0$ sufficiently small and $x < L_+$ sufficiently large. Our next aim is to construct a function $p_1 : [L_-, L_+] \to \mathbb{R}$ which describes initial values for the pressure in the sense that $p(t) \to p_1$ for $t \searrow 0$.

For a given monotone saturation function $s_1 : [L_-, L_+] \to \mathbb{R}$ we consider the following system of equations. The unknowns are the point $x_1 \in (L_-, L_+)$, the flux parameter $q_1 > 0$, and a pressure function $p_1 : [L_-, L_+] \to \mathbb{R}$.

$$p_1 = p_c(s_1) + \gamma \quad \text{on } (L_-, x_1) \tag{3.11}$$

$$k(s_1)[\partial_x p_1 + 1] = q_1 \quad \text{on } (x_1, L_+) \tag{3.12}$$

$$p_1(L_+) = p_+ \tag{3.13}$$

$$p_1(x_1 + 0) = p_c(s_1(x_1)) + \gamma \tag{3.14}$$

$$q_1 = (k(s_1)[\partial_x(p_c(s_1)) + 1])|_{x_1 - 0} \tag{3.15}$$

In these equations, (3.12) is the evolution equation with $\partial_t s$ set to zero, while (3.14) and (3.15) express the continuity of pressure and flux across $x = x_1$.

We note that, on the left interval (L_-, x_1), the pressure p_1 is determined by the algebraic relation (3.11). Once that $q_1 > 0$ is given, the ordinary differential equation (3.12) together with the initial condition (3.13) determines p_1 on the right interval (x_1, L_+). The two free parameters q_1 and x_1 must be determined from the continuity relations (3.14) and (3.15).

For the subsequent construction we introduce the number $q_{\text{ref}} > 0$ as the flux at the outflow boundary for $t = 0$,

$$q_{\text{ref}} := k(s_1)[\partial_x(p_c(s_1)) + 1]|_{x=L_-}\ . \tag{3.16}$$

Due to Lemma 3.1, the reference flux q_{ref} is positive. Furthermore, again by Lemma 3.1, the differential equation $\partial_x(k(s)[\partial_x(p_c(s)) + 1]) = \partial_t s \geq 0$ implies that, at time $t = 0$, the flux is monotonically increasing in x and hence satisfies $k(s_1)[\partial_x(p_c(s_1)) + 1] > q_{\text{ref}}$ on (L_-, L_+). We define additionally a reference pressure function $p_{\text{ref}} : [L_-, L_+] \to \mathbb{R}$ as the solution of the ordinary differential equation

$$k(s_1)[\partial_x p_{\text{ref}} + 1] = q_{\text{ref}} \text{ on } (L_-, L_+) \text{ with } p_{\text{ref}}(x = L_-) = p_-\ . \tag{3.17}$$

The monotonicity of the flux function implies $p_{\text{ref}} \leq p_c(s_1) + \gamma$ and $p_{\text{ref}}(L_+) < p_c(s_1(L_+)) + \gamma = p_+^0$.

Lemma 3.2 (Pressure system for $t = 0$) *Let the initial saturation $s_1 \in C^2([L_-, L_+], \mathbb{R})$ satisfy $\partial_x s_1 > 0$ on $[L_-, L_+]$ and $\partial_x(k(s_1)[\partial_x(p_c(s_1)) + 1]) > 0$ on (L_-, L_+). Let the pressure boundary value p_+ satisfy $p_{\text{ref}}(L_+) < p_+ < p_+^0$. Then problem (3.11)–(3.15) has a unique solution $p_1 \in C^1([L_-, L_+], \mathbb{R})$, $x_1 \in (L_-, L_+)$, $q_1 > 0$. There holds $p_1 \leq p_c(s_1) + \gamma$.*

Proof We consider the map

$$\mathcal{A}_1 : [q_{\text{ref}}, q_{\max}) \ni q \mapsto p \in C^1([L_-, L_+], \mathbb{R}), \tag{3.18}$$

where p is defined as the solution $p = p_1$ of (3.12) and (3.13) to $q_1 = q$. The number $q_{\max} > q_{\text{ref}}$ is defined below. We note already here that, because of $p_{\text{ref}}(L_+) < p_+$, for $q = q_{\text{ref}}$, there holds $p > p_{\text{ref}}$ on $[L_-, L_+]$. We furthermore define the map

$$\mathcal{A}_2 : (q_{\text{ref}}, q_{\max}) \ni q \mapsto \xi_1 \in [L_-, L_+], \tag{3.19}$$

where ξ_1 is the largest intersection point of the two pressure functions as in (3.11), i.e. a point with $p(\xi_1) = p_c(s_1(\xi_1)) + \gamma$ for $p = \mathcal{A}_1(q)$. For $q = q_{\rm ref}$ we have, on the left boundary, $p(L_-) > p_{\rm ref}(L_-) = p_- = p_c(s_1(\xi_1)) + \gamma$, and on the right boundary we have $p(L_+) = p_+ < p_+^0 = p_c(s_1(L_+)) + \gamma$. Therefore, an intersection point $\xi_1 = \mathcal{A}_2(q)$ exists for $q = q_{\rm ref}$. By continuity of the above constructions, there exists a maximal interval $(q_{\rm ref}, q_{\max})$ such that the intersection point $\mathcal{A}_2(q)$ exists for all $q \in (q_{\rm ref}, q_{\max})$.

The value of $q \in (q_{\rm ref}, q_{\max})$ is now chosen in such a way that (3.15) is satisfied for $q_1 = q$, $p = \mathcal{A}_1(q)$ and $x_1 = \mathcal{A}_2(q)$. We verify the existence of such a value q by analyzing relation (3.15) in the limits $q \searrow q_{\rm ref}$ and $q \nearrow q_{\max}$. As a preparation we note the following monotonicity property. Increasing the parameter q increases the values of $\partial_x p$ and hence decreases the values of p for $p = \mathcal{A}_1(q)$ on the whole interval $[L_-, L_+]$. Since $p_c(s_1(.)) + \gamma$ is monotonically increasing in $x \in [L_-, L_+]$, the largest intersection point of the two graphs moves to the left: the map $q \mapsto \xi_1 = \mathcal{A}_2(q)$ is monotonically decreasing.

The limit $q \searrow q_{\rm ref}$ is easily analyzed. The left hand side of (3.15) tends to $q_{\rm ref}$ while the right hand side is strictly above that value (we exploit that the point x_1 moves to the right when q is decreasing). In particular, for q close to $q_{\rm ref}$, the left hand side of (3.15) is smaller than the right hand side.

Regarding the limit $q \nearrow q_{\max}$ we have to distinguish two cases. Let us assume that $q_{\max}$ is finite. Since the intersection point $\mathcal{A}_2(q)$ ceizes to exist at $q = q_{\max}$, by monotonicity of $\mathcal{A}_2$ we can conclude that $\mathcal{A}_2(q) \to L_-$ for $q \nearrow q_{\max}$. In this case, the left hand side of (3.15) tends to a value larger than $q_{\rm ref}$, while the right hand side tends to $q_{\rm ref}$. In particular, relation (3.15) is satisfied for some $q \in (q_{\rm ref}, q_{\max})$. On the other hand, in the case $q_{\max} = \infty$, the left hand side of (3.15) tends to infinity while the right hand side remains bounded. Therefore, also in this case we find $q \in (q_{\rm ref}, q_{\max})$ such that (3.15) holds. In both cases we find the desired solution by setting $q_1 = q$, $x_1 = \mathcal{A}_2(q)$, and, for $p = \mathcal{A}_1(q)$,

$$p_1(x) := \begin{cases} p_c(s_1) + \gamma & \text{for } x < x_1 \\ p(x) & \text{for } x \geq x_1. \end{cases}$$

The inequality $p_1 \leq p_c(s_1) + \gamma$ holds as an equality for $x \leq x_1$. By construction, we have (3.12) and (3.15) satisfied, i.e. $k(s_1)[\partial_x p_1 + 1] = q_1 = (k(s_1)[\partial_x(p_c(s_1)) + 1])(x_1)$. The last expression was assumed to be monotonically increasing in x. This implies that $\partial_x p_1 < \partial_x(p_c(s_1))$ on (x_1, L_+) and thus $p_1 \leq p_c(s_1) + \gamma$ on $[L_-, L_+]$. The regularity of the solution can be read off from the ordinary differential equation. □

3.2 The free boundary problem for $t \in (0, T_+)$

We now study the evolution equations (2.5)–(2.6) for $t > 0$. We expect the following qualitative behavior of solutions. Due to the low pressure boundary condition on the right end point L_+ we expect that, on some interval $(X(t), L_+)$, the hysteresis relation (2.6) is satisfied with $p < p_c(s_1) + \gamma$ and $\partial_t s = 0$. On the left interval $(L_-, X(t))$ we expect further imbibition, i.e. that (2.6) is satisfied with $p = p_c(s) + \gamma$ and $\partial_t s > 0$.

Under these assumptions, by the evolution equation (2.5), the flux $k(s_1)[\partial_x p + 1]$ is constant on the right interval. The equations are

$$k(s_1)[\partial_x p + 1] = q \qquad \text{on } \{(x,t) : X(t) < x < L_+\} \tag{3.20}$$

$$p(X(t) + 0, t) = p_c(s_1(X(t))) + \gamma \tag{3.21}$$

$$p(L_+, t) = p_+ \tag{3.22}$$

We emphasize that the boundary values for p in (3.21) and (3.22) are known (once that $X(t)$ is known). Equation (3.20) can be written as $\partial_x p = -1 + q/k(s_1)$ and integrated for every $q \in \mathbb{R}$. The monotonicity in q shows that, for given $X(t) := \xi$, the equations (3.20)–(3.22) can be solved with some appropriate parameter $q =: Q(\xi)$.

We note that Q is decreasing in ξ. For $\xi_2 > \xi_1$, the pressure values satisfy $p_c(s_1(\xi_2)) > p_c(s_1(\xi_1))$. Additionally, the interval (ξ_2, L_+) is shorter than the interval (ξ_1, L_+). This results in $Q(\xi_2) < Q(\xi_1)$.

On the left domain $(L_-, X(t))$ we demand $p = p_c(s) + \gamma$ and the equations

$$\partial_t s = \partial_x \left(k(s)[\partial_x p + 1]\right) \qquad \text{on } \{(x,t) : x < X(t)\} \tag{3.23}$$

$$p(X(t) - 0, t) = p_c(s_1(X(t))) + \gamma, \qquad p(L_-, t) = p_- \tag{3.24}$$

$$\left(k(s)[\partial_x p + 1]\right)(X(t) - 0, t) = Q(X(t)) \tag{3.25}$$

The equations demand with (3.24) and (3.21) the continuity of the pressure, and with (3.20) and (3.25) the continuity of the flux across $X(t)$. One may regard (3.24) as the boundary conditions for p on the left domain, and (3.25) as a transmission condition that determines the free boundary $X(.)$.

The initial conditions are given as

$$X(t = 0) = x_1, \quad s(t = 0) = s_1, \tag{3.26}$$

with x_1 and p_1 given by Lemma 3.2. For the pressure we can expect to recover $p(t = 0) = p_1$.

The analysis of this section is based on the following principal assumption.

Assumption 3.3 Existence and regularity. *We assume that the one-dimensional free boundary problem* (3.20)–(3.25) *has a solution*

$$p \in L^\infty(0, T; H^2((L_-, L_+), \mathbb{R})), \quad X \in C^0([0, T_+], (L_-, L_+)) \cap C^1((0, T_+), \mathbb{R}), \tag{3.27}$$

with s monotonically non-decreasing in t on a cylinder $(L_-, L_+) \times [0, \varepsilon_0)$ for some $\varepsilon_0 > 0$.

Smooth dependence. *Let $(-\delta_0, \delta_0) \ni \delta \mapsto s_1^\delta \in C^2([L_-, L_+])$ be twice differentiable with $s_1^0 = s_1$ and let p^δ be the corresponding solution of the one-dimensional free boundary problem. Then, choosing a smaller $\delta_0 > 0$ if necessary, the map $(x, t, \delta) \mapsto s^\delta(x, t)$ is twice continuously differentiable in δ.*

Remarks on Assumption 3.3.

1.) Regularity. The assumption demands the existence and regularity of a solution to the free boundary problem. We emphasize that smoothness in $t = 0$ is not demanded. Regarding the regularity for $t > 0$ we note that the boundary value problem on the left domain, given by (3.23)–(3.25), can be transformed onto a fixed domain. We may use the independent variable $y \in (-1, 0)$, setting

$$y := \frac{x - X(t)}{X(t) - L_-}, \qquad u(y, t) := p(x, t) = p(y(X(t) - L_-) + X(t), t).$$

For the new unknown $u : (-1, 0) \times (0, T_+) \to \mathbb{R}$ derivatives are calculated as follows.

$$\partial_y u = (X(t) - L_-)\,\partial_x p, \qquad \partial_t u = \partial_t p + R(y, t)\partial_x p \quad \text{ for } \quad R(y, t) = (1 + y)\,\partial_t X(t).$$

These rules allow to transform the parabolic problem on a variable domain (3.23) into a parabolic problem for u on the fixed domain $(-1, 0)$.

We note that the smoothness assumption can be weakened for our result. We only need that on a subset of points (x, t) with arbitrarily large fraction of its measure, the regularity in δ holds. In particular, regularity in δ close to the free boundary and close to time $t = 0$ is not required.

2.) Monotonicity of s. From a physical point of view, the monotonicity of s is very natural: the pressure variations across the domain are given, therefore the saturation cannot decrease, at least if $\gamma > 0$ is chosen sufficiently large. Regarding the mathematical point of view, we note that s is non-decreasing for $x > X(t)$ by (3.28) and that s satisfies a parabolic equation with initial values $\partial_t s > 0$ on $x < X(t)$. Therefore, just like the regularity properties, the monotonicity property can be expected to hold.

An immediate consequence is that the free boundary point $X(.)$ is monotonically non-decreasing in $t \in (0, \varepsilon_0)$. To see this, it suffices to recall that, by (3.24), $X(t)$ is an intersection point of the two graphs $x \mapsto p(x, t)$ and $x \mapsto p_c(s_1(x)) + \gamma$. Since $p = p_c(s) + \gamma$ for $x \leq X(t)$ is non-decreasing in t and $p_c(s_1)$ is increasing in x, the intersection point can not move to the left.

3.) Another interpretation of the assumption. We expect that a stronger version of Theorem 2.1 is valid and that the hysteresis system posesses a solution also in the general case. Then Assumption 3.3 is a statement on the regularity and on structural properties of solutions. It regards the existence of an interface point $X(.)$ that separates a region with $\{\partial_t s > 0\}$ from a region with $\{\partial_t s = 0\}$.

4.) On the verification of the assumption. Concerning the verification of Assumption 3.3 we mention that an existence result for a similar one-dimensional free boundary problem has been derived in [14]. The main idea is to transform (3.23)–(3.25) to a fixed domain, to consider Q as a given smooth function and to modify the initial conditions in order to have a finite speed of X in $t = 0$. Iteration techniques and monotonicity properties of the free boundary (see Lemma 3.5) provide existence and regularity. The proof is lengthy and is therefore not performed here, but intended to be included in a future work.

Consistency.

We recall that the saturation can be recovered from a solution of the free boundary problem by setting

$$s(x,t) := \begin{cases} p_c^{-1}(p(x,t) - \gamma) & \text{if } t \le 0 \text{ or } t > 0, x < X(t) \\ s_1(x) & \text{else.} \end{cases} \tag{3.28}$$

We next want to establish the link between the one-dimensional free boundary problem and the original hysteresis problem. We verify that every solution as in Assumption 3.3 is a solution of the hysteresis problem. The hypothesis $\partial_t s \ge 0$ is checked for solutions in the subsequent Lemma 3.5.

Lemma 3.4 (Consistency) *Let p and X be a solution for the free boundary problem* (3.20)–(3.26) *as described in Assumption 3.3. Let the initial values $s_1 = s|_{t=0}$ be given by a solution s of the hysteresis system on $(T_-, 0)$ as constructed in Lemma 3.1, and by x_1 of Lemma 3.2. Then s of* (3.28) *has, for every $t_0 \in (T_-, 0)$, the regularity*

$$\partial_x s, \partial_t s \in L^\infty(t_0, T_+; L^2((L_-, L_+), \mathbb{R})). \tag{3.29}$$

If $\gamma > 0$ is sufficiently large and if $\partial_t s \ge 0$ is satisfied, the pair (p, s) is a solution to the one-dimensional hysteresis system (2.5)–(2.6) *on (T_-, T_+).*

Proof *Regularity property* (3.29). We first check the regularity on the three subdomain of $(L_-, L_+) \times (0, T_+)$. For $t < 0$, i.e. $t \in (t_0, 0)$, the regularity of the solution is a well-known property of the parabolic evolution equation for $p = p_c(s) + \gamma$. For $t \in (0, T_+)$ and $x > X(t)$, the saturation s is given by $s_1(x)$, such that $\partial_t s = 0$ and $\partial_x s = \partial_x s_1$ hold on that domain. For $t \in (0, T_+)$ and $x < X(t)$, the regularity $p = p_c(s) + \gamma \in L^\infty H^2$ was demanded in Assumption 3.3. Since p_c has a twice differentiable inverse, the regularity carries over to s. The differential equation then implies also a uniform bound for $\partial_t s(., t) \in L^2((L_-, X(t)), \mathbb{R})$.

It remains to consider the interfaces $t = 0$ and $x = X(t)$. The saturation s has the same traces on both sides of $x = X(t)$ by (3.24) and (3.28). In $t = 0$, the saturation has no jump for $x > x_1$ by the construction (3.28). In $t = 0$ and for $x < x_1$, the saturation has no jump because of the initial condition s_1 in the evolution equation (3.23) for $p = p_c(s) + \gamma$. We conclude that both $\partial_x s$ and $\partial_t s$ contain no singular parts and hence that (3.29) holds.

Solution properties. We have already seen that the two distributions $\partial_t s$ and $\partial_x p$ contain no singular parts. Furthermore, the flux function $k(s)[\partial_x p + 1]$ has no jump in $x = X(t)$ for $t > 0$. This follows from the assumption $p \in L^\infty H^2$ which, in turn, is consistent with the flux continuity of equation (3.25). On the basis of the regularity properties, it only remains to verify that (2.5) and (2.6) hold almost everywhere.

For $t < 0$ and for $t > 0$ with $x < X(t)$, the evolution equation is imposed explicitly with $p = p_c(s) + \gamma$. Because of $\partial_t s \ge 0$, both relations (2.5) and (2.6) are satisfied in these regions. In the domain $t > 0$ and $x > X(t)$, the saturation is $s = s_1$ such that the time derivative is $\partial_t s = 0$. Relation (3.20) then implies (2.5) on that domain.

It remains to verify relation (2.6) for $t > 0$ and $x > X(t)$. Because of $\partial_t s = 0$ on that domain, we only have to check that $p_c(s_1) - \gamma \le p \le p_c(s_1) + \gamma$. Since the solutions are bounded, the lower bound $p_c(s_1) - \gamma \le p$ is satisfied for γ sufficiently large. It remains to verify the upper bound $p \le p_c(s_1) + \gamma$. There holds $p(X(t)) = p_c(s_1(X_1(t))) + \gamma$ by (3.21). Using (3.20) and (3.25) we conclude $(k(s_1)[\partial_x p+1])(X(t)+0,t) = q = (k(s)[\partial_x p+1])(X(t)-0,t) \le (k(s_1)[\partial_x(p_c(s_1))+1])(X(t))$, where the last inequality is a consequence of $p \ge p_c(s_1)+\gamma$ for $x < X(t)$, which is due to $\partial_t p = p_c'(s)\partial_t s \ge 0$. The expression $k(s_1)[\partial_x(p_c(s_1))+1])$ is monotonically increasing in $x > X(t)$ such that the $\partial_x p \le \partial_x(p_c(s_1))$ holds in this domain. This implies $p \le p_c(s_1)+\gamma$ for $x > X(t)$, which concludes the proof. □

We note that the continuity of fluxes of (3.25) (together with the continuity of the saturation) implies that $p(.,t) \in H^2((L_-,L_+))$ can be expected. On the other hand, the saturation s will not have this regularity, since, in general, $\partial_x s$ has a jump across $x = X(t)$. This jump is visible in Figure 1.

Qualitative properties of solutions.

We next investigate the qualitative behavior of solutions to the free boundary problem and, in particular, its monotonicity properties. The result will be that the saturation increases everywhere, while the free boundary point X moves to the right. Since Q is a decreasing function, this also yields that the flux parameter $q(t)$ of (3.20) is decreasing: the inflow at the upper boundary is decreasing in t, but, as we will see in the next subsection, it approaches a value $q_\infty > 0$ for $t \to \infty$.

Lemma 3.5 (Monotonicity properties of solutions) *Let p and X be a solution of the free boundary problem* (3.20)–(3.26) *as in Assumption 3.3 with initial values $s_1 = s_{t=0}$ given by a solution s of the hysteresis system on $(T_-,0)$. Then the solution satisfies, for number $s_{\min}, s_{\max}$ that depend on initial and boundary conditions and on p_c the maximum principle $s_{\min} \le s \le s_{\max}$ for all times. Furthermore, for all $t > 0$, we have the monotonicity properties*

$$\partial_t s \ge 0, \quad \partial_x s \ge 0, \quad \partial_t X \ge 0. \tag{3.30}$$

Proof We have derived with (3.29) of Lemma 3.4 the regularity $\partial_x s, \partial_t s \in L^\infty(0,T_+;L^2((L_-,L_+),\mathbb{R}))$, and we have assumed the regularity $X \in C^1((0,T_+))$. Therefore, the non-negativity statements of (3.30) are well-defined in the sense of measurable functions.

Concerning the maximum principle for the saturation: s solves a parabolic equation and hence satisfies a maximum principle on the left domain. It is given by the fixed function s_1 on the right domain, which obeys also a maximum principle by construction on $(T_-,0)$. The continuity condition on the free boundary point implies the maximum principle on the whole space-time domain.

We next observe that the three claimed monotonicities hold on a small time-interval $(0,\varepsilon_1)$. Indeed, the monotonicity of s in $t \in [0,\varepsilon_0)$ was demanded in Assumption 3.3. This also implies that X is non-decreasing on $[0,\varepsilon_0)$, see remark 2.) after Assumption 3.3. In $t = 0$, the saturation is strictly monotonic in x by $\partial_x s = \partial_x s_1 > 0$. On $x > X(t)$ holds $s = s_1$ and hence s is monotonic in x for all $t \in (0,\varepsilon_0)$ and $x > X(t)$.

It remains to show the monotonicity $\partial_x s \ge 0$ on the left domain for small times, i.e. for $t \in (0,\varepsilon_1)$ and $x < X(t)$. This can be concluded with a maximum principle argument (similar to Case 1 below). The function $v = \partial_x s$ is initially non-negative, and v is non-negative on the left boundary $x = L_-$ by the maximum principle for s. Furthermore, for $t = 0$, using $\partial_x(p_c(s_1)) > 0$, the flux value satisfies $q = Q(X(0)) > k(s_1(X(0)))$. Since X, Q, s_1, and k are continuous, for small $\varepsilon_1 > 0$, the inequality remains valid and we have $Q(X(t)) > k(s_1(X(t)))$ for all $t \in (0,\varepsilon_1)$. The flux condition (3.25) then implies $\partial_x p = \partial_x(p_c \circ s) > 0$ in $(t, X(t)-0)$ and hence positivity of $v = \partial_x s$ along the right boundary. Since v solves a parabolic equation on a non-decreasing domain, the maximum principle provides $v \ge 0$ for all $t \in (0,\varepsilon_1)$, $x \in (L_-,X(t))$.

Definition of t_*. We define the time instance $t_* > 0$ as the first time instance in which one of the three monotonicity properties fails to hold. More precisely, we set

$$t_* := \sup\left\{t^* \in [0,T_+] : \text{ for all } t \in (0,t^*) \text{ holds } \partial_t X \ge 0 \text{ and } \partial_t s \ge 0, \partial_x s \ge 0 \text{ for a.e. } x\right\}.$$

We note that t_* is well-defined and satisfies $t_* \geq \varepsilon_1$ by the above considerations.

For a contradiction argument, let us assume $t_* < T_+$. In the following, we study properties of solutions in the time instance $t = t_*$, distinguishing three cases. Once we have derived a contradiction, the lemma is shown.

Case 1. $\partial_x s(x, t_*) = 0$ *for some* $L_- \leq x \leq X(t_*)$. We consider the function $v = \partial_x s$ as in Lemma 3.1. The function v is non-negative on $(0, t_*)$ by construction of t_*. Furthermore, the function v satisfies the parabolic equation (3.9) on a strictly increasing interval, and hence satisfies a maximum principle. In particular, the minimum x must lie on the boundary. The case $x = L_-$ is excluded by the fact that the saturation is non-decreasing on $(0, t_*)$ and the left boundary condition is constant in time. We therefore have the minimum in $x = X(t_*)$.

We now read (3.23) as an elliptic equation for p with the non-negative right hand side $\partial_t s \geq 0$. The saturation (and, hence, the pressure $p = p_c(s) + \gamma$) is maximal at the right boundary $x = X(t_*)$ by construction of t_*. The Hopf Lemma for elliptic inequalities then implies $\partial_x p(x, t_*) = p_c'(s(x, t_*))\partial_x s(x, t_*) > 0$, a contradiction.

Case 2. $\partial_t X(t_*) = 0$. We use $v = \partial_t s$, which solves a parabolic equation. Because of $\partial_t X(t_*) = 0$ and (3.24) for $p = p_c(s) + \gamma$ we find $\partial_t p(X(t_*), t_*) = 0$ and hence $v = \partial_t s = 0$ in $X(t_*)$. This yields that $(X(t_*), t_*)$ is a minimum of the non-negative function v, which solves a parabolic equation. In this minimum, there must hold $\partial_x v < 0$ by Hopf's lemma for parabolic equations. This is in contradiction with the time-derivative of equation (3.25), which provides $\partial_t \partial_x p(X(t_*), t_*) = 0$ under our hypothesis $\partial_t X(t_*) = 0$.

Case 3. $\partial_t s(x, t_*) = 0$ *for some* $L_- < x \leq X(t)$. Arguing as above for the non-negative function $v = \partial_t s$, we know that the minimum with value 0 is necessarily attained at $x = X(t_*)$. Then (3.24) yields, in $x = X(t) - 0$,

$$0 = p_c'(s)\partial_t s = \partial_t p = p_c'(s)\,\partial_x s_1\,\partial_t X - p_c'(s)\,\partial_x s\,\partial_t X.$$

Since Case 2 is already excluded, we have $\partial_t X(t_*) > 0$ and conclude $\partial_x s_1 = \partial_x s$ in $X(t) - 0$. This implies for the flux

$$\begin{aligned} Q(X(t)) &\overset{(3.25)}{=} (k(s)[\partial_x(p_c(s)) + 1])\,(X(t) - 0, t) = (k(s_1)[\partial_x(p_c(s_1)) + 1])\,(X(t)) \\ &> (k(s_1)[\partial_x(p_c(s_1)) + 1])\,(X(0)) = Q(X(0)), \end{aligned}$$

the inequality since the flux for $t = 0$ is strictly increasing in x, $\partial_x\,(k(s_1)[\partial_x(p_c(s_1)) + 1]) = \partial_t s|_{t=0} > 0$. On the other hand, as observed after (3.22), Q is non-increasing. In particular, for non-decreasing X, the flux $Q(X(t))$ is non-increasing in t. This provides the desired contradiction in the third and last case. □

3.3 Long-time behavior of solutions

For all times $t > 0$, the saturation on $x < X(t)$ continues to increase by Lemma 3.5. The relevant question is whether the flux at the free boundary point $X(t)$ (which is related to the front-speed of the wetting front) tends to zero for $t \to \infty$, or if it remains finite. The latter case would correspond to a front that continues to proceed with finite speed. Our first aim is now to collect equations that determine the behavior of solutions for large times.

In the free boundary problem, the equations for the right domain suggest a limit problem. If x_∞ denotes the limiting position of the free boundary $X(t)$ (we recall that X is monotonically increasing and bounded by L_+), we can expect that the limiting profile satisfies (3.20) with the right boundary condition (3.22) and the left boundary condition (3.21), which is formulated in (3.31) and (3.32) below.

It remains to formulate a last relation that determines the limiting position x_∞ of the free boundary. We expect that the increasing saturation on the left domain leads to an almost vanishing slope $\partial_x s(X(t)-0, t)$ and hence to an almost vanishing gradient $\partial_x p$ in the limit of large times. By the continuity (3.25) we therefore expect that the flux q coincides with the permeability $k(s_1)$ for large times, which expresses that, in the free boundary point, the flow is purely gravity driven. The limit relation is expressed with (3.33), where we think of a given value of ζ with $|\zeta|$ small. The next lemma is devoted to the limit system which determines $q_\infty > 0$, $x_\infty < L_+$ and $p : (x_\infty, L_+) \to \mathbb{R}$.

$$k(s_1)[\partial_x p + 1] = q_\infty \quad \text{on} \quad x \in (x_\infty, L_+), \tag{3.31}$$

$$p(x = L_+) = p_+, \qquad p(x_\infty) = p_c(s_1(x_\infty)) + \gamma, \tag{3.32}$$

$$k(s_1(x_\infty)) = q_\infty + \zeta. \tag{3.33}$$

Lemma 3.6 (Solvability of system (3.31)–(3.33) for large times) *Let a family of initial values s_1 be given by a twice differentiable map $(-\delta_0, \delta_0) \ni \delta \mapsto s_1^\delta \in C^2([L_-, L_+])$, such that all s_1^δ are as in Lemma 3.2 with $p_+ \in (p_{\text{ref}}^\delta(L_+), p_+^0)$. Then there exists $\delta_1 > 0$ and $\zeta_1 > 0$ such that system* (3.31)–(3.33) *posesses, for all $\zeta \in (-\zeta_1, \zeta_1)$ and all $\delta \in (-\delta_1, \delta_1)$ a unique solution*

$$x_\infty \in (L_-, L_+), \quad q_\infty > 0, \quad p \in H^2((x_\infty, L_+), \mathbb{R}).$$

The solution depends continuously on δ and ζ.

Proof The construction is very similar to the one for Lemma 3.2. Indeed, (3.12)–(3.14) coincide with (3.31)–(3.32). The only difference is the modified flux condition in the free boundary point.

We use $q \in [0, q_1]$ as a parameter, where q_1 is the flux value of (3.12) with corresponding free boundary point x_1. Once more, we denote by $p =: \mathcal{A}_1(q)$ the solution of (3.31) to $q_\infty = q$, with the right boundary condition $p(x = L_+) = p_+$. The rightmost intersection point $\xi \in [L_-, L_+]$ of the graphs of $p = \mathcal{A}_1(q)$ and $p_c(s_1(.)) + \gamma$ is denoted by $\mathcal{A}_2(q) := \xi$. The definitions are just as in Lemma 3.2. In particular, the map $q \mapsto \mathcal{A}_2(q)$ is again monotonically non-increasing.

We claim that the map $\mathcal{A}_2$ is well-defined on $[0, q_1]$. Indeed, for $q \le q_1$, there holds $\partial_x p \le \partial_x p_1$ and therefore, since the same values are assumed in L_+, the comparison result $p \ge p_1$. The function p_1 has an intersection point with $p_c \circ s_1 + \gamma$, namely x_1. Since on the right boundary $p(L_+) = p_+ < p_+^0 = p_c(s_1(L_+)) + \gamma$, also p and $p_c \circ s_1 + \gamma$ have an intersection point $\xi \ge x_1$.

It remains to find $q = q_\infty$ such that also (3.33) is satisfied. We only have to evaluate the two sides of (3.33) in the end-points of the q-interval. For $q = 0$, the number $k(s_1(\mathcal{A}_2(0)))$ is positive, hence greater than q. Instead, for $q = q_1$, there holds $\mathcal{A}_2(q_1) = x_1$ and hence

$$k(s_1(\mathcal{A}_2(q_1))) = k(s_1(x_1)) < (k(s_1)[\partial_x(p_c(s_1)) + 1])\,(x_1) = q_1$$

by (3.15). The continuity of the involved maps and the intermediate value theorem provide the existence of q_∞, such that (3.33) holds as an equality.

Uniqueness of solutions. Let q_∞ and x_∞ define a solution p of the system. We observe that for no other $x > x_\infty$ the function p has an intersection with $p_c \circ s_1 + \gamma$. This follows immediately from

$$k(s_1)[\partial_x p + 1] = q_\infty = k(s_1(x_\infty)) \le k(s_1) \text{ and hence } \partial_x p \le 0,$$

while $\partial_x(p_c \circ s_1) > 0$. Therefore all solutions of (3.31)–(3.33) are detected as $x_\infty = \mathcal{A}_2(q_\infty)$ by the above construction, which used the rightmost intersection point in $\mathcal{A}_2$. But $k(s_1(\mathcal{A}_2(q)))$ in non-increasing in q, while the identity map $q \mapsto q$ is strictly increasing in q. Therefore, (3.33) has at most one solution q_∞.

Continuous dependence. All the constructed maps are continuous in s_1 and ζ. Additionally, the identity $q \mapsto q$ has the derivative 1, which is bounded from below. We conclude that the zero q_∞ depends continuously on δ and ζ. □

Our interest is the limit behavior of solutions to (3.20)–(3.25) for large times. We claim that, in the limit of large t, the flux $q(t) = Q(X(t))$ approaches the limit flux q_∞ of the above lemma (for $\zeta = 0$). Indeed, for increasing t, by Lemma 3.5, the point $X(t)$ is increasing, the flux $q(t) = Q(X(t))$ of (3.20) is decreasing, and, accordingly, the solution p on the right interval is increasing. By boundedness of these quantities, it follows that there exist limits

$$X(t) \nearrow \tilde{x}_\infty, \quad q(t) \searrow \tilde{q}_\infty, \quad p(.,t) \nearrow \tilde{p}(.) \text{ uniformly on } [\tilde{x}_\infty, L_+].$$

for $t \nearrow \infty$. Furthermore, (3.20)–(3.22) imply equations (3.31) and (3.32) for the limit functions. If we can additionally verify (3.33) with $\zeta = 0$, by the uniqueness statement of Lemma 3.6, we have $\tilde{x}_\infty = x_\infty$ and $\tilde{q}_\infty = q_\infty$, and hence the convergence to the limit determined by (3.31)–(3.33).

The true proof is more involved since we have to deal with the dependence on δ, the dependence on the left end-point L_-, and since we need a uniform convergence for $t \to \infty$. Our result will be based on the following claim on ordinary differential equations. We use once more the Kirchhoff transformation and the monotone function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi'(s) = k(s)p_c'(s)$.

Claim 3.7 *Let numbers $s_{\min} < s_{\max}$ and $q_{\min} < q_{\max}$ be given and let $\Phi \in C^1(\mathbb{R})$ be strictly increasing. For arbitrary $\varepsilon_1 > 0$, there exist $L_0, \varepsilon_2 > 0$ such that the following holds. Let $s : (-L_0, 0) \to [s_{\min}, s_{\max}]$ solve*

$$\partial_x(\Phi(s)) + k(s) = q + f \text{ for some } q \in [q_{\min}, q_{\max}], f : (-L_0, 0) \to \mathbb{R} \text{ with } \|f\|_{L^\infty} < \varepsilon_2. \tag{3.34}$$

Then

$$|k(s(0)) - q| < \varepsilon_1. \tag{3.35}$$

Proof Given ε_1, we choose L_0 large enough, such that every solution of the differential equation $\partial_x(\Phi(s)) + k(s) = q$ on $(-L_0, 0)$ solves $|k(s(0)) - q| < \varepsilon_1/2$. For fixed initial value $s(-L_0)$ and fixed q this is possible since the differential equation provides exponential convergence to the solution $s_0 \in \mathbb{R}$ of $k(s_0) = q$. The lower bound L_0 for the interval length can be chosen with continuous dependence on q and on $s(-L_0)$. Since the initial values $s(-L_0)$ and the values of q are chosen in a compact interval, there exists $L_0 > 0$ satisfying the uniform estimate.

Let us now assume that (3.35) does not hold for all f. Then we find a sequence $f_j \to 0$ in $L^\infty((-L_0, 0))$, $q_j \to q$, and solutions s_j to $\partial_x(\Phi(s_j)) + k(s_j) = q_j + f_j$ with $|k(s_j(0)) - q_j| \geq \varepsilon_1$. But then, by the compactness of Arzela-Ascoli, for a subsequence, the solutions s_j converge uniformly to a solution s of the limit problem, which satisfies $|k(s(0)) - q| < \varepsilon_1/2$ by the first step. We find the desired contradiction. □

In the subsequent lemma we consider once more a twice differentiable sequence of initial values $[-\delta_0, \delta_0] \ni \delta \mapsto s_1^\delta \in C^2([L_-, L_+])$ with $s_1^0 = s_1$, all s_1^δ strictly increasing with boundary conditions $(p_c + \gamma)^{-1}(p_+^0)$ and $(p_c + \gamma)^{-1}(p_-)$. The numbers δ_1 and ζ_1 are those of Lemma 3.6. We assume that L_+ is large enough in order to have the initial position of the free boundary positive, $x_1 > 0$.

Lemma 3.8 (Behavior for large times) *We consider solutions of the free boundary problem* (3.20)–(3.25) *on* $(0, T)$*, with position $X^\delta(t)$ and flux constant $q^\delta(t)$ as in Assumption 3.3. The solutions depend on initial values s_1^δ as in Lemma 3.6. Let furthermore x_∞^δ, q_∞^δ, and δ_1 be as in Lemma 3.6. Then, for every $\varepsilon > 0$, there exist $L_0 > 0$ and $m_* > 0$ independent of T such that, for all $L_- \leq -L_0$,*

$$\exists N \subset [-\delta_1, \delta_1] \times [0, T] \text{ with two-dimensional Lebesgue-measure } |N| < m_*, \tag{3.36}$$

such that

$$\left|X^\delta(t) - x_\infty^\delta\right| + \left|q^\delta(t) - q_\infty^\delta\right| < \varepsilon \qquad \forall (\delta, t) \in [-\delta_1, \delta_1] \times [0, T] \setminus N. \tag{3.37}$$

Proof We study solutions of the free boundary problem for various $L_- < 0$ and $\delta \in [-\delta_1, \delta_1]$. We consider (3.20)–(3.22) for a fixed time instance t, and compare these equations with (3.31)–(3.32). The continuous dependence on ζ in Lemma 3.6 provides the existence of $\varepsilon_1 > 0$ (depending on ε, but not on L_- and t) such that, for all $|\delta| \leq \delta_1$,

$$|k(s_1^\delta(X^\delta(t)) - q^\delta(t)| < \varepsilon_1 \quad \Rightarrow \quad |X^\delta(t) - x_\infty^\delta| + |q^\delta(t) - q_\infty^\delta| < \varepsilon. \tag{3.38}$$

We can satisfy the smallness requirement of the left hand side with the help of Claim 3.7. That claim provides $L_0 > 0$ and $\varepsilon_2 > 0$ (both depending on ε_1, but not on δ and t) such that, for

$$\|\partial_x(\Phi(s^\delta)) + k(s^\delta) - q^\delta(t)\|_{L^\infty((-L_0,X^\delta(t)))} < \varepsilon_2, \tag{3.39}$$

the condition $|k(s_1^\delta(X^\delta(t)) - q^\delta(t)| < \varepsilon_1$ is satisfied by (3.35). We therefore define

$$N := \left\{(\delta,t) \in [-\delta_1,\delta_1] \times [0,T] : \|\partial_x \left(\partial_x(\Phi(s^\delta)) + k(s^\delta)\right)\|_{L^1((-L_0,X^\delta(t)))} \ge \varepsilon_2\right\}. \tag{3.40}$$

For all $(\delta,t) \notin N$ holds (3.39), since $q^\delta(t)$ is the value of $\partial_x(\Phi(s^\delta)) + k(s^\delta)$ in the free boundary point $X^\delta(t)$. Therefore (3.37) is satisfied.

It remains to study the measure of N. We calculate, using the evolution equation and $\partial_t s^\delta \ge 0$,

$$\varepsilon_2|N| \le \int_{-\delta_1}^{\delta_1}\int_0^T\int_{-L_0}^{X^\delta(t)} |\partial_t s^\delta| \le \int_{-\delta_1}^{\delta_1} (L_+ + L_0)(s_{\max} - s_{\min}) = 2\delta_1(L_+ + L_0)(s_{\max} - s_{\min}).$$

We find

$$|N| \le \frac{2\delta_1(L_+ + L_0)(s_{\max} - s_{\min})}{\varepsilon_2} =: m_*(\varepsilon).$$

This concludes the proof. □

4 Proof of the instability result

We now turn to the higher dimensional problem and prove our main result, Theorem 1.1. It asserts that the hysteretic Richards equation with time-dependent boundary data shows spatial instability. Our starting point is a solution (s,p) of the one-dimensional problem (2.5)–(2.6) to initial and boundary data given by numbers s_0, p_-, p_+^0, and p_+. Under natural assumptions on these numbers, the qualitative behavior of the solution (s,p) was analyzed in Section 3.

Our aim is the verification of the instability property (2.11) of Definition 2.2 for this quite general one-dimensional solution. We will be able to show the result even for the large deviation $\varepsilon = 1$. We recall that the additional spatial variables are $y \in Y = [0,L_y)^n$. The goal is to construct, given a small $\rho > 0$, numbers $T_\pm, L_\pm, L_y \in \mathbb{R}$, a perturbation $w \in C^1((L_-,L_+) \times Y)$ of the initial data and a source $f \in C^1((L_-,L_+) \times Y \times (T_-,T_+))$, both of size ρ, such that the corresponding solution to the higher dimensional problem deviates by more than ε from the planar solution.

Construction of w and f.
Let (s,p) be a one-dimensional solution as in Section 3. To introduce an x-dependence of the initial perturbation we use an arbitrary nonnegative function

$$w_1 \in C_c^\infty((0,L_+),\mathbb{R}) \text{ with } \|w_1\|_{L^1} \le 1. \tag{4.1}$$

We denote by (s^δ,p^δ) the one-dimensional solution to the perturbed initial values $s_0(x) + \delta w_1(x)$ for $\delta \in \mathbb{R}$. We emphasize at this point that the initial values are not necessarily increasing in x, but the solution $s_1^\delta = s^\delta(t=0)$ is increasing in x for small δ, such that the analysis of Section 3 can be applied.

We next construct a higher dimensional perturbation of the planar initial data $s(x,y) = s(x)$. For notational convenience we assume from now on $n = 1$ and $y \in (0,L_y) \subset \mathbb{R}^1$. To treat the case $n > 1$ it is sufficient to replace y by its first component y_1 in the sequel. As a y-periodic perturbation we use

$$w_2 \in C^\infty((0,L_+) \times [0,L_y),\mathbb{R}), \quad w_2(x,y) := w_1(x)\,\sin(2\pi y/L_y). \tag{4.2}$$

Later on, we will consider the small perturbation $w = \rho w_2$ of the initial values. We construct a comparison solution $(\tilde{s},\tilde{p})$ by juxtaposing solutions of one-dimensional problems, treating $y \in [0,L_y)$ only as a parameter. More precisely, we define $(x,t) \mapsto (\tilde{s}(x,t,y),\tilde{p}(x,t,y))$ as the solution of the one-dimensional system (2.5)–(2.6) with initial condition

$$\tilde{s}(x,t=0,y) = s_0(x) + \rho w_2(x,y). \tag{4.3}$$

In other words, the functions $(x,t) \mapsto (\tilde s, \tilde p)(x,t,y)$ are the one-dimensional solutions with

$$\tilde s(x,t,y) = s^\delta(x,t) \quad\text{to}\quad s_0^\delta(x) = s_0(x) + \delta w_1(x) \quad\text{for}\quad \delta = \rho \sin(2\pi y/L_y). \tag{4.4}$$

In particular, the functions are not solutions of the two-dimensional homogeneous system. But they do satisfy $\tilde p \in p_c(\tilde s) + \gamma\, \mathrm{sign}(\partial_t \tilde s)$ almost everywhere. Furthermore, the conservation law is satisfied up to a small error,

$$\partial_t \tilde s - \nabla \cdot (k(\tilde s)[\nabla \tilde p + e_x]) = \{\partial_t \tilde s - \partial_x (k(\tilde s)[\partial_x \tilde p + 1])\} - \partial_y (k(\tilde s)\partial_y \tilde p). \tag{4.5}$$

We conclude that $(\tilde s, \tilde p)$ solves (2.7)–(2.8) with

$$f = -\partial_y (k(\tilde s)\partial_y \tilde p). \tag{4.6}$$

Lemma 4.1 *For some constant $C_1 = C_1(L_-, T_+) > 0$ the source function f of (4.6) satisfies*

$$\frac{1}{L_y^n} \|f\|_{L^1((L_-,L_+)\times(0,L_y)\times(T_-,T_+))} \le C_1 \rho L_y^{-2}. \tag{4.7}$$

Proof It was part of Assumption 3.3 that the map $\delta \mapsto (s^\delta(x,t), p^\delta(x,t))$ is two times continuously differentiable in a neighborhood of $\delta = 0$, with uniformly bounded first and second derivatives for x and t in compact intervals. In particular, the map $y \mapsto \tilde s(x,t,y)$ is differentiable with all derivatives bounded by some constant C. By (4.4), for $|\rho| \le 1$, it has the derivatives

$$|\partial_y \tilde s(x,y,t)|^2 \le |\partial_\delta s^{\rho \sin(2\pi y/L_y)}(x,t)\, \rho\, \cos(2\pi y/L_y) 2\pi/L_y|^2 \le C\rho L_y^{-2}, \tag{4.8}$$

$$|\partial_y^2 \tilde s(x,y,t)| \le \|\partial_\delta^2 s\|_\infty (2\pi\rho/L_y)^2 + \|\partial_\delta s\|_\infty\, \rho\, (2\pi/L_y)^2 \le C\rho L_y^{-2}, \tag{4.9}$$

and similarly for $\tilde p$. Formula (4.6) for f together with the differentiability of k implies (4.7). □

In the subsequent lemma we assume that (s^δ, p^δ) are solutions of the one-dimensional problem to initial values $s_0(x) + \delta w_1(x)$ with $|\delta| \le 1$. To this family of solutions, Lemma 3.6 provides $0 < \delta_1 \le 1$ with a continuity property of q_∞^δ, and Lemma 3.8 provides a smallness property for $|q^\delta(t) - q_\infty^\delta|$. The result in the next statement is that, for many values of δ, the solution s^δ differs considerably from the solution s^0 to $\delta = 0$.

Lemma 4.2 (Flux variation in the one-dimensional problem) *Let $\delta \in [-\delta_1, \delta_1]$ and let (s^δ, p^δ) be one-dimensional solutions to initial values $s_0(x) + \delta w_1(x)$. Let $q^\delta(t)$ denote the corresponding flux. We set $M_\delta := (\delta_1/2, \delta_1) \subset (-\delta_1, \delta_1)$ with $|M_\delta| = \delta_1/2$.*

Then there exists $\varepsilon_q > 0$ such that, for any fraction $0 < \Theta < 1$, there exist constants $L_0, T_0 > 0$, such that, for all $T > T_0$ and all $L_- < -L_0$, there exists a set $M \subset M_\delta \times [0,T]$ of measure $|M| > \Theta |M_\delta| T$ such that

$$q^\delta(t) - q^0(t) \ge \varepsilon_q \quad \forall (\delta, t) \in M. \tag{4.10}$$

Proof We start with the construction of $\varepsilon_q > 0$. Loosely speaking, we only have to make sure that $q_\infty^\delta > q_\infty + 2\varepsilon_q$ for many values of δ.

Step 1. Limiting fluxes. We analyze the system (3.31)–(3.33), which determines the limiting speed q_∞ for arbitrary values of the one-dimensional initial saturation s_0. For the perturbed initial values $s_0 + \delta w_1$ we denote the corresponding limiting speed by q_∞^δ. The comparison principle for the parabolic system on the time interval $(T_-, 0)$ implies that the values of the saturation in $t = 0$ are ordered. For $\delta > 0$,

$$s_1^\delta := s^\delta(t=0) \ge s_1^0 = s^0(t=0),$$

with strict inequality $s_1^\delta > s_1^0$ on $(0, L_+)$. We claim that this implies

$$q_\infty^\delta > q_\infty. \tag{4.11}$$

In order to show (4.11) we argue by contradiction and assume that $q_\infty^\delta \le q_\infty$. We first exploit (3.33) with $\zeta = 0$, $s_1^\delta \ge s_1$, and monotonicity of k to find

$$k(s_1(x_\infty^\delta)) \le k(s_1^\delta(x_\infty^\delta)) = q_\infty^\delta \le q_\infty.$$

The monotonicity of $k \circ s_1$ in x then yields $x_\infty^\delta \le x_\infty$. The differential equation (3.31) now implies for the limiting pressure functions p and p^δ on (x_∞, L_+)

$$\partial_x p^\delta = \frac{q_\infty^\delta}{k(s_1^\delta)} - 1 \le \frac{q_\infty}{k(s_1^\delta)} - 1 < \frac{q_\infty}{k(s_1)} - 1 = \partial_x p.$$

The identical boundary conditions $p(x = L_+) = p_+ = p^\delta(x = L_+)$ imply the strict inequality $p^\delta > p$ on (x_∞, L_+). The point x_∞^δ is defined as the maximum of p^δ, and p^δ lies above p, hence we conclude $p^\delta(x_\infty^\delta) > p(x_\infty)$. In particular, exploiting (3.33) and (3.32),

$$q_\infty^\delta = k(s_1^\delta(x_\infty^\delta)) = (k \circ (p_c + \gamma)^{-1})(p^\delta(x_\infty^\delta)) > (k \circ (p_c + \gamma)^{-1})(p(x_\infty)) = k(s_1(x_\infty)) = q_\infty.$$

This is in contradiction with our assumption $q_\infty^\delta \le q_\infty$, hence (4.11) is verified.

Step 2. Choice of ε_q, T_0 and L_0. By inequality (4.11), we have $q_\infty^\delta - q_\infty > 0$ for all $\delta \in (0, \delta_1)$. By the continuity result of Lemma 3.6, for $M_\delta = (\delta_1/2, \delta_1)$, we find $\varepsilon_q > 0$ such that

$$q_\infty^\delta - q_\infty \ge 2\varepsilon_q \forall \delta \in M_\delta.$$

We are now given $\Theta < 1$, and our aim is to find numbers $T_0, L_0 > 0$ with the desired property. Lemma 3.8 with $\varepsilon := \varepsilon_q/2$ implies the existence of $L_0 > 0$ and $m_* > 0$. We use this number L_0 and it remains to choose T_0 based on the given values of m_* and Θ. For all $(\delta, t) \in [-\delta_1, \delta_1] \times [0, T] \setminus N$ we find, by (3.37) and the triangle inequality,

$$|q^\delta(t) - q^0(t)| \ge |q_\infty^\delta - q_\infty| - |q^0(t) - q_\infty| - |q^\delta(t) - q_\infty^\delta| \ge 2\varepsilon_q - \varepsilon_q/2 - \varepsilon_q/2 = \varepsilon_q.$$

Choosing T_0 large enough we achieve that the portion of N (which has measure at most m_*, independent of T) in the set $M_\delta \times [0, T]$ is smaller than $1 - \Theta$. Setting $M = M_\delta \times [0, T] \setminus N$, this concludes the proof. □

Proof of Theorem 1.1. We actually prove a stronger result, namely (2.11) for any $\varepsilon > 0$. Let therefore $\varepsilon > 0$ be arbitrary, the choice $\varepsilon = 1$ provides a proof of the theorem. Let now ρ be an arbitrary (small) positive number, describing the size of the initial perturbation w and of the source f. Our aim is to find $T_\pm, L_\pm, L_y > 0, w$, and f as in (2.11).

We choose $\delta_1 \le \rho$ with the properties as in Lemma 3.6. Upon lowering $\rho > 0$, we may assume $\rho = \delta_1$. We now consider, as in (4.1)– (4.3), $\tilde{s}(x, t, y) = s^\delta(x, t)$ with $\delta = \rho \sin(2\pi y/L_y)$, and f as in (4.6). We set $T_- = -1$ and choose L_+ large enough to satisfy $x_1 > 0$.

Step 1. Smallness of w and f. The choice of w implies

$$\frac{1}{L_y^n} \|w\|_{L^1((L_-, L_+) \times Y)} = \frac{1}{L_y^n} \int_Y \int_{L_-}^{L_+} \rho w_1(x) |\sin(2\pi y/L_y)| \, dx \, dy \le \rho. \tag{4.12}$$

Regarding f we have to satisfy

$$\frac{1}{L_y^n} \|f\|_{L^1((L_-, L_+) \times Y \times (T_-, T_+))} \le \rho. \tag{4.13}$$

Relation (4.7) for the norm of f implies that (4.13) can be satisfied in the end of our other constructions by choosing L_y large enough.

For later use we note that, by continuity in δ, we find bounds $s_{\min} \le s^\delta \le s_{\max}$ and $q_{\min} \le q^\delta \le q_{\max}$, where the bounds are independent of δ, T, L_-, and L_y.

Step 2. Choice of T and L_-. Our aim is to select T and L_- in order to satisfy

$$\frac{1}{L_y^n} \|\tilde{s}(., T) - s(., T)\|_{L^1((L_-, L_+) \times Y)} \ge \varepsilon. \tag{4.14}$$

Lemma 4.2 provides a positive number ε_q, which measures variations of the flux. We now choose the number $\Theta < 1$ large enough to have

$$\Theta\varepsilon_q > 2(1-\Theta)(q_{\max} - q_{\min}). \tag{4.15}$$

With this choice of Θ, we select $M_\delta = (\delta_1/2, \delta_1)$, T_0, and L_0 by Lemma 4.2.

We consider the total mass on the left domain (for notational convenience and without loss of generality we assume the initial saturation to be $s_0 = 0$, such that also $s_{\min} = 0$),

$$m^\delta(t) := \int_{L_-}^{X^\delta(t)} s^\delta(x,t)\,dx.$$

The time increment is calculated with the conservation law (3.23),

$$\begin{aligned} \frac{d}{dt} m^\delta(t) &= s^\delta(X^\delta(t),t)\,\partial_t X^\delta(t) + \int_{L_-}^{X^\delta(t)} \partial_t s^\delta(x,t)\,dx \\ &= s^\delta(X^\delta(t),t)\,\partial_t X^\delta(t) + \left(k(s^\delta)[\partial_x p^\delta + 1]\right)\Big|_{L_-}^{X^\delta(t)}. \end{aligned} \tag{4.16}$$

We abbreviate the outflow at $x = L_-$ by $q_-^\delta(t) := \left(k(s^\delta)[\partial_x p^\delta + 1]\right)(L_-)$. Integrating (4.16) over $[0,T]$ yields, for $\delta = 0$,

$$m^0(T) - m^0(0) \le s_{\max} L_+ + \int_0^T q^0(t)\,dt - \int_0^T q_-^0(t)\,dt. \tag{4.17}$$

On the other hand, we can derive a lower bound for $\delta \in M_\delta$. We use the estimate $q^\delta(t) - q^0(t) \ge \varepsilon_q$ of (4.10) on the large set $M \subset M_\delta \times [0,T]$ (depending on T, but with a bounded volume fraction), and the estimate $q^\delta(t) - q^0(t) \ge -(q_{\max} - q_{\min})$ on the remainder. Integrating (4.16) over $[0,T]$ and M_δ and exploiting (4.17) yields

$$\begin{aligned} \int_{M_\delta} m^\delta(T) - m^\delta(0)\;d\delta &\ge \int_{M_\delta}\int_0^T q^\delta(t)\,dt\,d\delta - \int_{M_\delta}\int_0^T q_-^\delta(t)\,dt\,d\delta \\ &\ge \int_{M_\delta}\int_0^T q^0(t)\,dt\,d\delta + |M_\delta| T\Theta\varepsilon_q - |M_\delta| T(1-\Theta)(q_{\max} - q_{\min}) - \int_{M_\delta}\int_0^T q_-^\delta(t)\,dt\,d\delta \\ &\ge \int_{M_\delta} m^0(T) - m^0(0)\;d\delta - s_{\max} L_+ |M_\delta| + \int_{M_\delta}\int_0^T (q_-^0 - q_-^\delta)(t)\,dt\,d\delta + \frac{1}{2}|M_\delta| T\Theta\varepsilon_q. \end{aligned}$$

We can therefore compare the total mass at time T for $\delta \in M_\delta$ and for $\delta = 0$. We use that the initial mass $m^\delta(0)$ is bounded, independent of δ, and that the total outflow can be bounded by an arbitrary positive number by enlarging L_- (e.g. by 1, see below). We find, with C independent of T,

$$\int_{M_\delta} (m^\delta(T) - m^0(T))\;d\delta \ge -C + T\Theta\varepsilon_q\delta_1/4.$$

We now transform this lower bound for a mass difference into a lower bound for the L^1-norm of $(\tilde{s} - s)(T)$ as required for (4.14). With the set $M_y := \{y \in Y : \rho\sin(2\pi y/L_y) \in M_\delta\}$ we find

$$\begin{aligned} \frac{1}{L_y^n}\|\tilde{s}(.,T) - s(.,T)\|_{L^1((L_-,L_+)\times Y)} &= \frac{1}{L_y^n}\int_Y\int_{L_-}^{L_+} |s^{\rho\sin(2\pi y/L_y)} - s^0|(T)\,dx\,dy \\ &\ge \frac{1}{L_y^n}\int_{M_y}\int_{L_-}^{L_+} (s^{\rho\sin(2\pi y/L_y)} - s^0)(T)\,dx\,dy \ge 2\frac{1}{L_y^n}\int_{M_\delta}\int_{L_-}^{L_+} (s^\delta - s^0)(T)\,dx\,\frac{L_y^n}{2\pi\rho}d\delta \\ &\ge \frac{1}{\pi\rho}\int_{M_\delta}\int_{L_-}^{L_+} (s^\delta - s^0)(T) \ge -C + T\Theta\varepsilon_q\delta_1/(4\pi\rho), \end{aligned}$$

where we used in the last line once more the bounds for the saturation and $L_+ = 1$. We can choose T large in order to have the right hand side large. In particular, we can achieve that (4.14) holds.

Boundedness of $q_-^\delta(t)$. We claim that, for fixed T and constant $C = 1$, we can achieve the bound

$$\int_0^T |q_-^\delta|(t)\, dt \le C \tag{4.18}$$

by imposing a large lower bound on $|L_-|$. This can be seen as follows. With $p = p_c(s) + \gamma$, the saturation s solves the parabolic problem (3.23) on $(L_-, 0)$, with constant Dirichlet data on the left boundary and constant (and matching) initial values. The Dirichlet data on the right boundary are bounded, $s_{\min} \le s \le s_{\max}$.

Transformed solutions $\hat{s}(x) = s(|L_-|x)$ solve the same quasilinear parabolic problem on a fixed spatial domain $(-1, 0)$ with a new time variable. The transformed solution satisfies an upper bound for the Neumann data on the left, $|\partial_x s^\delta(x = -1)| \le C_N$ for all times. This implies $|\partial_x s(x = L_-)| \le C_N/|L_-|$. Choosing $|L_-|$ large (in dependence of $s_{\min}$, $s_{\max}$, and T) provides (4.18) and concludes the proof. □

5 Conclusions

We studied the Richards equation with gravity. It is well known that the classical Richards equation (without hysteresis) defines an L^1-contraction and hence a stable evolution. Even in the "unstable" situation that a more saturated medium is above a less saturated medium, the classical Richards equation will therefore not show an instability; the model predicts a stable planar wetting front, contradicting experiments.

We have therefore included a play-type hysteresis relation between pressure and saturation as suggested and discussed in [5]. Our rigorous analysis shows that this modified Richards equation does not define an L^1-contraction. Instead, for appropriate boundary data, we have shown that an arbitrarily small perturbation of the initial values can lead to the development of fingers.

Our results are obtained for non-degenerate coefficient functions. In this setting, the instability can be shown only on large domains, i.e. with wide fingers. In order to observe fingers in an arbitrary finite domain, we believe that degenerate coefficients must be considered. This is in accordance with experiments, where authors report that very dry sand must be used in order to obtain fingering effects.

The proof of our instability result was based on the analysis of a one-dimensional free boundary problem. The obtained saturation profile in the fingers is monotone in our setting. In order to obtain the experimentally observed non-monotone profiles, it might be necessary to include additionally dynamic hysteresis.

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