

# $\Gamma$ -limits of convolution functionals

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# Γ-LIMITS OF CONVOLUTION FUNCTIONALS

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ABSTRACT. We compute the  $\Gamma$ -limit of a sequence of non-local integral functionals depending on a regularization of the gradient term by means of a convolution kernel. In particular, as  $\Gamma$ -limit, we obtain free discontinuity functionals with linear growth and with anisotropic surface energy density.

Keywords: Free discontinuities,  $\Gamma$ -convergence, anisotropy.

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## 1. INTRODUCTION

As it is well known, many variational problems which are recently under consideration, arising for instance from image segmentation, signal reconstruction, fracture mechanics and liquid crystals, involve a *free discontinuity set* (according to a terminology introduced in [19]). This means that the variable function  $u$  is required to be smooth outside a surface  $K$ , depending on  $u$ , and both  $u$  and  $K$  enter the structure of the functional, which takes the form given by

$$\mathcal{F}(u, K) = \int_{\Omega \setminus K} \phi(|\nabla u|) dx + \int_{K \cap \Omega} \theta(|u^+ - u^-|, \nu_K) d\mathcal{H}^{n-1},$$

being  $\Omega$  an open subset of  $\mathbb{R}^n$ ,  $K$  is a  $(n - 1)$ -dimensional compact subset of  $\mathbb{R}^n$ ,  $|u^+ - u^-|$  the jump of  $u$  across  $K$ ,  $\nu_K$  the normal direction to  $K$ , while  $\phi$  and  $\theta$  given positive functions, whereas  $\mathcal{H}^{n-1}$  denotes the  $n - 1$ -dimensional Hausdorff measure.

The classical weak formulation for such problems can be obtained considering  $K$  as the set of the discontinuities of  $u$  and thus working in the space of functions with bounded variation. More precisely, the aforementioned weak form of  $\mathcal{F}$  takes on  $BV(\Omega)$  the general form

$$(1.1) \quad \mathcal{F}(u) = \int_{\Omega} \phi(|\nabla u|) dx + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1} + c_0 |D^c u|(\Omega),$$

where  $Du = \nabla u \mathcal{L}^n + (u^+ - u^-) \mathcal{H}^{n-1} + D^c u$  is the decomposition of the measure derivative of  $u$  in its absolutely continuous, jump and Cantor part, respectively, and  $S_u$  denotes the set of discontinuity points of  $u$ .

The main difficulty in the actual minimization of  $\mathcal{F}$  comes from the surface integral

$$\int_{S_u} \theta(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1},$$

which makes it necessary to use suitable approximations guaranteeing the convergence of minimum points and naturally leads to  $\Gamma$ -convergence.

As pointed out in [10], it is not possible to obtain a variational approximation for  $\mathcal{F}$  by the typical integral functionals

$$\mathcal{F}_{\varepsilon}(u) = \int_{\Omega} f_{\varepsilon}(\nabla u) dx$$

defined on some Sobolev spaces. Indeed, when considering the lower semicontinuous envelopes of these functionals, we would be lead to a convex limit, which conflicts with the non-convexity of  $\mathcal{F}$ .

Heuristic arguments suggest that, to get rid of the difficulty, we have to prevent that the effect of *large* gradients is concentrated on *small* regions. Several approximation methods fit this requirements. For instance in [7], [12], [24] the case where the functionals  $\mathcal{F}_{\varepsilon}$  are restricted to finite elements spaces on regular triangulations of size  $\varepsilon$  is considered. In [1], [2], [23] the implicit constraint on the gradient through the addition of a higher order penalization is investigated. Moreover, it is important to mention the AMBROSIO & TORTORELLI approximation (see [4] and [5]) of the Mumford-Shah functional via elliptic functionals.

The study of non-local models, where the effect of a large gradient is spread onto a set of size  $\varepsilon$ , was first introduced by BRAIDES & DAL MASO in order to approximate the Mumford-Shah functional (see [10] and also [11], [13], [14], [15], [16]) by means of the family

$$(1.2) \quad \mathcal{F}_{\varepsilon}(u) = \frac{1}{\varepsilon} \int_{\Omega} f\left(\varepsilon \int_{B_{\varepsilon}(x) \cap \Omega} |\nabla u|^2 dy\right) dx, \quad u \in H^1(\Omega),$$

where, for instance,  $f(t) = t \wedge 1/2$  and  $B_{\varepsilon}(x)$  denotes the ball of centre  $x$  and radius  $\varepsilon$ . A variant of the method proposed in [10] has been used in [22] to deal with the approximation of a functional  $\mathcal{F}$  of the form (1.1), with  $\phi$  having linear growth and  $\theta$  independent on the normal  $\nu_u$  (see also [20] and [21]). More precisely, in [22] the  $\Gamma$ -limit of the family

$$\mathcal{F}_{\varepsilon}(u) = \frac{1}{\varepsilon} \int_{\Omega} f\left(\varepsilon \int_{B_{\varepsilon}(x) \cap \Omega} |\nabla u| dy\right) dx, \quad u \in W^{1,1}(\Omega),$$

for a suitable concave function  $f$ , is computed.

In [25] (see also [13]) the case of an anisotropic variant of (1.2) has been considered. In particular it is proven that the family

$$\mathcal{F}_{\varepsilon}(u) = \frac{1}{\varepsilon} \int_{\Omega} f(\varepsilon |\nabla u|^p * \rho_{\varepsilon}) dx, \quad u \in H^1(\Omega), \quad p > 1,$$

$\Gamma$ -converges to an anisotropic version of the Mumford-Shah functional.

In this paper we investigate the  $\Gamma$ -convergence of the family

$$\mathcal{F}_{\varepsilon}(u) = \frac{1}{\varepsilon} \int_{\Omega} f_{\varepsilon}(\varepsilon |\nabla u| * \rho_{\varepsilon}) dx, \quad u \in W^{1,1}(\Omega).$$

The main difficulty to overcome is the estimate from below for the lower  $\Gamma$ -limit in terms of the surface part, while the contribution arising from the volume and Cantor parts has been treated along the same line of the argument already exploited in [25]. The estimate from above has

been achieved by density and relaxation arguments. We prove that the  $\Gamma$ -limit, in the strong  $L^1$ -topology, is given by

$$\mathcal{F}(u) = \int_{\Omega} \phi(|\nabla u|) dx + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1} + c_0 |D^c u|(\Omega),$$

where  $c_0 = \lim_{t \rightarrow +\infty} \phi(t)/t$  and

$$\theta(s, \nu) = \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} f(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) dx : (u_j) \in W_\nu^{0,s}, \varepsilon_j \rightarrow 0^+ \right\},$$

being  $W_\nu^{a,b}$  the space of all sequences on the cylinder  $Q_\nu$  which converge, shrinking onto the interface, to the function that jumps from  $a$  to  $b$  around the origin (see paragraph 3.1 for details). In section 7 we have been able to show that the method used in [22] to write  $\theta$  in a more explicit form works only if  $n = 1$ . In the case  $n > 1$  such an argument does not work. Let us briefly discuss the reason. Without loss of generality we can suppose  $\nu = \mathbf{e}_1$ . Let  $P_C^\perp$  be the orthogonal projection of  $C$  onto  $\{x_1 = 0\}$ . Denote by  $X$  the space of all functions  $v \in W_{\text{loc}}^{1,1}(\mathbb{R} \times P_C^\perp)$  which are non-decreasing in the first variable and such that there exist  $\xi_0 < \xi_1$  with  $v(x) = 0$  if  $x_1 < \xi_0$  and  $v(x) = s$  if  $x_1 > \xi_1$ . Then, exploiting the same argument as in [22], we have  $\theta(s, \mathbf{e}_1) \geq \inf_X G$ , where

$$G(v) = \int_{-\infty}^{+\infty} f \left( \int_{C(\mathbf{se}_1)} \partial_1 v(z) \rho(z - t\mathbf{e}_1) dz \right) dt.$$

The estimate  $\theta(s, \mathbf{e}_1) \geq \inf_X G$  turns out to be optimal if  $\inf_X G = \inf_Y G$ , where  $Y$  is the space of all functions  $v \in X$  such that  $v$  depends only on the first variable. This is due to the fact that proving the inequality  $\theta(s, \mathbf{e}_1) \geq \inf_X G$  we lose control on all the derivatives  $\partial_i v$  for any  $i = 2, \dots, n$ . In the case  $C = B_1$  and  $\rho = \frac{1}{\omega_n} \chi_{B_1}$ , treated in [22], one is able to prove that  $\inf_X G = \inf_Y G$  computing directly  $\inf_X G$  by a discretization argument (see Prop. 5.7 in [22]). In general,  $\inf_X G = \inf_Y G$  does not hold. Indeed proceeding at first as in the proof of Prop. 5.6 in [22], one is able to show that for any  $C \subset \mathbb{R}^2$  open, bounded, convex and symmetrical set (i.e.  $C = -C$ ) and for  $\rho = \frac{1}{|C|} \chi_C$ , it holds

$$(1.3) \quad \inf_Y G = \int_{-h_1}^{h_1} f \left( \frac{s}{|C|} \mathcal{H}^1(C \cap \{z_1 = t\}) \right) dt.$$

Now if  $C$  is the parallelogram  $C = \{(x, y) \in \mathbb{R}^2 : -2 \leq y \leq 2, x - 1 \leq y \leq x + 1\}$  applying (1.3), we get

$$\inf_Y G' = 2f \left( \frac{2s}{|C|} \right) + 2 \int_0^2 f \left( \frac{sr}{|C|} \right) dr.$$

If we compute  $G$  on the function  $w$  given by

$$w(x, y) = \begin{cases} 0 & \text{if } y > x - 1 \\ s & \text{if } y \leq x - 1 \end{cases},$$

(to do this we notice that the functional  $G$  makes sense also on  $BV_{\text{loc}}(\mathbb{R} \times (-2, 2))$  writing  $D_1 v$  instead of  $\partial_1 v dz$ ) we obtain

$$G(w) = 2f \left( \frac{4s}{|C|} \right).$$

If  $f$  is strictly concave then

$$G(w) < 2f \left( \frac{2s}{|C|} \right) + 2f \left( \frac{2s}{|C|} \right) < 2f \left( \frac{2s}{|C|} \right) + 2 \int_0^2 f \left( \frac{sr}{|C|} \right) dr = \inf_Y G.$$

By a density argument we deduce that  $\inf_X G < \inf_Y G$ .

As a conclusion, it seems that for a generic anisotropic convolution kernel  $\rho_\varepsilon$  the expression for  $\theta$  can not be further simplified when  $n > 1$ .

## 2. NOTATION AND PRELIMINARIES

We will denote by  $L^p(\Omega)$  and by  $W^{k,p}(\Omega)$ , for  $k \in \mathbb{N}$ ,  $k \geq 1$ , and for  $1 \leq p \leq +\infty$ , respectively the classical Lebesgue and Sobolev spaces on  $\Omega$ . The Lebesgue measure of a measurable set  $A \subset \mathbb{R}^n$  will be denoted by  $|A|$ , whereas the Hausdorff measure of  $A$  of dimension  $m < n$  will be denoted by  $\mathcal{H}^m(A)$ . The ball centered in  $x$  with radius  $r$  will be denoted by  $B_r(x)$ , while  $B_r$  stands for  $B_r(0)$ ; moreover, we will use the notation  $\mathbb{S}^{n-1}$  for the boundary of  $B_1$  in  $\mathbb{R}^n$ . The volume of the unit ball in  $\mathbb{R}^n$  will be denoted by  $\omega_n$ , with the convention  $\omega_0 = 1$ . Finally  $\mathcal{A}(\Omega)$  denotes the set of all open subsets of  $\Omega$ .

**2.1. Functions of bounded variation.** For a thorough treatment of  $BV$  functions we refer the reader to [3]. Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . We recall that the space  $BV(\Omega)$  of real *functions of bounded variation* is the space of the functions  $u \in L^1(\Omega)$  whose distributional derivative is representable by a measure in  $\Omega$ , i.e.

$$\int_{\Omega} u \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} \varphi dD_i u, \quad \forall \varphi \in C_c^\infty(\Omega), \forall i = 1, \dots, n,$$

for some  $\mathbb{R}^n$ -valued measure  $Du = (D_1 u, \dots, D_n u)$  on  $\Omega$ . We say that  $u$  has *approximate limit* at  $x \in \Omega$  if there exists  $z \in \mathbb{R}$  such that

$$\lim_{r \rightarrow 0^+} \int_{B_r(x)} |u(y) - z| dy = 0.$$

The set  $S_u$  where this property fails is called *approximate discontinuity set* of  $u$ . The vector  $z$  is uniquely determined for any point  $x \in \Omega \setminus S_u$  and is called the *approximate limit* of  $u$  at  $x$  and denoted by  $\tilde{u}(x)$ . We say that  $x$  is an *approximate jump point* of the function  $u \in BV(\Omega)$  if there exist  $a, b \in \mathbb{R}$  and  $\nu \in \mathbb{S}^{n-1}$  such that  $a \neq b$  and

$$(2.1) \quad \lim_{r \rightarrow 0^+} \int_{B_r^+(x, \nu)} |u(y) - a| dy = 0, \quad \lim_{r \rightarrow 0^+} \int_{B_r^-(x, \nu)} |u(y) - b| dy = 0,$$

where  $B_r^+(x, \nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle > 0\}$  and  $B_r^-(x, \nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle < 0\}$ . The set of approximate jump points of  $u$  is denoted by  $J_u$ . The triplet  $(a, b, \nu)$ , which turns out to be uniquely determined up to a permutation of  $a$  and  $b$  and a change of sign of  $\nu$ , is usually denoted by  $(u^+(x), u^-(x), \nu_u(x))$ . On  $\Omega \setminus S_u$  we set  $u^+ = u^- = \tilde{u}$ . It turns out that for any  $u \in BV(\Omega)$  the set  $S_u$  is countably  $(n-1)$ -rectifiable and  $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ . Moreover,

$$Du \llcorner J_u = (u^+ - u^-) \nu_u \mathcal{H}^{n-1} \llcorner J_u$$

and  $\nu_u(x)$  gives the approximate normal direction to  $S_u$  for  $\mathcal{H}^{n-1}$ -a.e.  $x \in S_u$ .

For a function  $u \in BV(\Omega)$  let  $Du = D^a u + D^s u$  be the Lebesgue decomposition of  $Du$  into absolutely continuous and singular part. We denote by  $\nabla u$  the density of  $D^a u$ ; the measures  $D^j u := D^s u \llcorner J_u$  and  $D^c u := D^s u \llcorner (\Omega \setminus S_u)$  are called the *jump part* and the *Cantor part* of the derivative, respectively. It holds  $Du = \nabla u \mathcal{L}^n + (u^+ - u^-) \nu_u \mathcal{H}^{n-1} \llcorner J_u + D^c u$ . Let us recall the following important compactness Theorem in  $BV$  (see Th. 3.23 and Prop. 3.21 in [3]):

**Theorem 2.1.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  with Lipschitz boundary. Every sequence  $(u_h)$  in  $BV(\Omega)$  which is bounded in  $BV(\Omega)$  admits a subsequence converging in  $L^1(\Omega)$  to a function  $u \in BV(\Omega)$ .*

We say that a function  $u \in BV(\Omega)$  is a *special function of bounded variation*, and we write  $u \in SBV(\Omega)$ , if  $|D^c u|(\Omega) = 0$ . We say that a function  $u \in L^1(\Omega)$  is a *generalized function of bounded variation*, and we write  $u \in GBV(\Omega)$ , if  $u^T := (-T) \vee u \wedge T$  belongs to  $BV(\Omega)$  for every  $T \geq 0$ . If  $u \in GBV(\Omega)$ , the function  $\nabla u$  given by

$$(2.2) \quad \nabla u = \nabla u^T \quad \text{a.e. on } \{|u| \leq T\}$$

turns out to be well-defined. Moreover, the set function  $T \mapsto S_{u^T}$  is monotone increasing; therefore, if we set  $S_u = \bigcup_{T>0} S_{u^T}$ , for  $\mathcal{H}^{n-1}$ -a.e.  $x \in S_u$  we can consider the functions of  $T$  given by  $(u^T)^-(x)$ ,  $(u^T)^+(x)$ ,  $\nu_{u^T}(x)$ . It turns out that

$$(2.3) \quad u^-(x) = \lim_{T \rightarrow +\infty} (u^T)^-(x), \quad u^+(x) = \lim_{T \rightarrow +\infty} (u^T)^+(x), \quad \nu_u(x) = \lim_{T \rightarrow +\infty} \nu_{u^T}(x)$$

are well-defined for  $\mathcal{H}^{n-1}$ -a.e.  $x \in S_u$ . Finally, for a function  $u \in GBV(\Omega)$ , let  $|D^c u|$  be the supremum, in the sense of measures, of  $|D^c u^T|$  for  $T > 0$ . It can be proved that for any Borel subset  $B$  of  $\Omega$

$$(2.4) \quad |D^c u|(B) = \lim_{T \rightarrow +\infty} |D^c u^T|(B).$$

**2.2. Slicing.** In order to obtain the estimate from below of the lower  $\Gamma$ -limit (see next paragraph) we need some basic properties of one-dimensional sections of  $BV$ -functions. We first introduce some notation. Let  $\xi \in \mathbb{S}^{n-1}$ , and let  $\xi^\perp$  be the vector subspace orthogonal to  $\xi$ . If  $y \in \xi^\perp$  and  $E \subseteq \mathbb{R}^n$  we set  $E_{\xi,y} = \{t \in \mathbb{R} : y + t\xi \in E\}$ . Moreover, for any given function  $u : \Omega \rightarrow \mathbb{R}$  we define  $u_{\xi,y} : \Omega_{\xi,y} \rightarrow \mathbb{R}$  by  $u_{\xi,y}(t) = u(y + t\xi)$ . For the results collected in the following Theorem see [3], section 3.11.

**Theorem 2.2.** *Let  $u \in BV(\Omega)$ . Then  $u_{\xi,y} \in BV(\Omega_{\xi,y})$  for every  $\xi \in \mathbb{S}^{n-1}$  and for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \xi^\perp$ . For such values of  $y$  we have  $u'_{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi \rangle$  for a.e.  $t \in \Omega_{\xi,y}$  and  $J_{u_{\xi,y}} = (J_u)_{\xi,y}$ , where  $u'_{\xi,y}$  denotes the absolutely continuous part of the measure derivative of  $u_{\xi,y}$ . Moreover, for every open subset  $A$  of  $\Omega$  we have*

$$\int_{\xi^\perp} |D^c u_{\xi,y}|(A_{\xi,y}) d\mathcal{H}^{n-1}(y) = |\langle D^c u, \xi \rangle|(A).$$

**2.3.  $\Gamma$ -convergence.** For the general theory see [9] and [18]. Let  $(X, d)$  be a metric space. Let  $(\mathcal{F}_j)$  be a sequence of functions  $X \rightarrow \overline{\mathbb{R}}$ . We say that  $(\mathcal{F}_j)$   $\Gamma$ -converges, as  $j \rightarrow +\infty$ , to  $\mathcal{F} : X \rightarrow \overline{\mathbb{R}}$ , if for all  $u \in X$  we have:

a) For every sequence  $(u_j)$  converging to  $u$  it holds

$$\mathcal{F}(u) \leq \liminf_{j \rightarrow +\infty} \mathcal{F}_j(u_j).$$

b) There exists a sequence  $(u_j)$  converging to  $u$  such that

$$\mathcal{F}(u) \geq \limsup_{j \rightarrow +\infty} \mathcal{F}_j(u_j).$$

The *lower* and *upper*  $\Gamma$ -limits of  $(\mathcal{F}_j)$  in  $u \in X$  are defined as

$$\mathcal{F}'(u) = \inf \left\{ \liminf_{j \rightarrow +\infty} \mathcal{F}_j(u_j) : u_j \rightarrow u \right\}, \quad \mathcal{F}''(u) = \inf \left\{ \limsup_{j \rightarrow +\infty} \mathcal{F}_j(u_j) : u_j \rightarrow u \right\}$$

respectively. We extend this definition of convergence to families depending on a real parameter. Given a family  $(\mathcal{F}_\varepsilon)_{\varepsilon > 0}$  of functions  $X \rightarrow \overline{\mathbb{R}}$ , we say that it  $\Gamma$ -converges, as  $\varepsilon \rightarrow 0$ , to  $\mathcal{F} : X \rightarrow \overline{\mathbb{R}}$  if for every positive infinitesimal sequence  $(\varepsilon_j)$  the sequence  $(\mathcal{F}_{\varepsilon_j})$   $\Gamma$ -converges to  $\mathcal{F}$ . If we define the lower and upper  $\Gamma$ -limits of  $(\mathcal{F}_\varepsilon)$  as

$$\mathcal{F}'(u) = \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) : u_\varepsilon \rightarrow u \right\}, \quad \mathcal{F}''(u) = \inf \left\{ \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) : u_\varepsilon \rightarrow u \right\}$$

respectively, then  $(\mathcal{F}_\varepsilon)$   $\Gamma$ -converges to  $\mathcal{F}$  in  $u$  if and only if  $\mathcal{F}'(u) = \mathcal{F}''(u) = \mathcal{F}(u)$ . It turns out that both  $\mathcal{F}'$  and  $\mathcal{F}''$  are lower semicontinuous on  $X$ . In the estimate of  $\mathcal{F}'$  we shall use the following immediate consequence of the definition:

$$\mathcal{F}'(u) = \inf \left\{ \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}(u_j) : \varepsilon_j \rightarrow 0^+, u_j \rightarrow u \right\}.$$

It turns out that the infimum is attained.

An important consequence of the definition of  $\Gamma$ -convergence is the following result about the convergence of minimizers (see, e.g., [18], Cor. 7.20):

**Theorem 2.3.** *Let  $\mathcal{F}_j : X \rightarrow \overline{\mathbb{R}}$  be a sequence of functions which  $\Gamma$ -converges to some  $\mathcal{F} : X \rightarrow \overline{\mathbb{R}}$ ; assume that  $\inf_{v \in X} \mathcal{F}_j(v) > -\infty$  for every  $j$ . Let  $(\sigma_j)$  be a positive infinitesimal sequence, and for every  $j$  let  $u_j \in X$  be a  $\sigma_j$ -minimizer of  $\mathcal{F}_j$ , i.e.*

$$\mathcal{F}_j(u_j) \leq \inf_{v \in X} \mathcal{F}_j(v) + \sigma_j.$$

Assume that  $u_j \rightarrow u$  for some  $u \in X$ . Then  $u$  is a minimum point of  $\mathcal{F}$ , and

$$\mathcal{F}(u) = \lim_{j \rightarrow +\infty} \mathcal{F}_j(u_j).$$

**Remark 2.4.** *The following property is a direct consequence of the definition of  $\Gamma$ -convergence: if  $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$  then  $\mathcal{F}_\varepsilon + \mathcal{G} \xrightarrow{\Gamma} \mathcal{F} + \mathcal{G}$  whenever  $\mathcal{G}: X \rightarrow \overline{\mathbb{R}}$  is continuous.*

**2.4. Supremum of measures.** In order to prove the  $\Gamma$ -liminf inequality we recall the following useful tool, which can be found in [8].

**Lemma 2.5.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and denote by  $\mathcal{A}(\Omega)$  the family of its open subsets. Let  $\lambda$  be a positive Borel measure on  $\Omega$ , and  $\mu: \mathcal{A}(\Omega) \rightarrow [0, +\infty)$  a set function which is superadditive on open sets with disjoint compact closures, i.e. if  $A, B \subset \subset \Omega$  and  $\overline{A} \cap \overline{B} = \emptyset$ , then*

$$\mu(A \cup B) \geq \mu(A) + \mu(B).$$

Let  $(\psi_i)_{i \in I}$  be a family of positive Borel functions. Suppose that

$$\mu(A) \geq \int_A \psi_i \, d\lambda \quad \text{for every } A \in \mathcal{A}(\Omega) \text{ and } i \in I.$$

Then

$$\mu(A) \geq \int_A \sup_i \psi_i \, d\lambda \quad \text{for every } A \in \mathcal{A}(\Omega).$$

**2.5. A density result.** The right bound for the upper  $\Gamma$ -limit from above will be first obtained for a suitable dense subset of  $SBV(\Omega)$ . More precisely, let  $\mathcal{W}(\Omega)$  be the space of all functions  $w \in SBV(\Omega)$  such that

- (a)  $\mathcal{H}^{n-1}(\overline{S}_w \setminus S_w) = 0$ ;
- (b)  $\overline{S}_w$  is the intersection of  $\Omega$  with the union of a finite member of  $(n-1)$ -dimensional simplexes;
- (c)  $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$  for every  $k \in \mathbb{N}$ .

Theorem 3.1 in [17] gives us the density property of  $\mathcal{W}(\Omega)$  we need; here

$$SBV^2(\Omega) = \{u \in SBV(\Omega) : |\nabla u| \in L^2(\Omega), \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

**Theorem 2.6.** *Assume that  $\partial\Omega$  is Lipschitz. Let  $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ . Then there exists a sequence  $(w_h)$  in  $\mathcal{W}(\Omega)$  such that  $w_h \rightarrow u$  strongly in  $L^1(\Omega)$ ,  $\nabla w_h \rightarrow \nabla u$  strongly in  $L^2(\Omega, \mathbb{R}^n)$ , with  $\limsup_{h \rightarrow +\infty} \|w_h\|_\infty \leq \|u\|_\infty$  and such that*

$$\limsup_{h \rightarrow +\infty} \int_{S_{w_h}} \psi(w_h^+, w_h^-, \nu_{w_h}) \, d\mathcal{H}^{n-1} \leq \int_{S_u} \psi(u^+, u^-, \nu_u) \, d\mathcal{H}^{n-1}$$

for every upper semicontinuous function  $\psi$  such that  $\psi(a, b, \nu) = \psi(b, a, -\nu)$  whenever  $a, b \in \mathbb{R}$  and  $\nu \in \mathbb{S}^{n-1}$ .

**2.6. A relaxation result.** To conclude this section we prove a relaxation result which will be used in the sequel. Recall that given  $X$  be a topological space and  $\mathcal{F}: X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , the *relaxed functional* of  $\mathcal{F}$ , denoted by  $\overline{\mathcal{F}}$ , is the largest lower semicontinuous functional which is smaller than  $\mathcal{F}$ .

**Theorem 2.7.** *Let  $\phi: [0, +\infty) \rightarrow [0, +\infty)$  be a convex, non-decreasing and lower semicontinuous function with  $\phi(0) = 0$  and with*

$$\lim_{t \rightarrow +\infty} \frac{\phi(t)}{t} = c \in (0, +\infty).$$

Let  $\theta: [0, +\infty) \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$  be a lower semicontinuous function such that  $\theta(s, \nu) \leq c's$  for any  $(s, \nu) \in [0, +\infty) \times \mathbb{S}^{n-1}$ , for some  $c' > 0$ . For any  $A \in \mathcal{A}(\Omega)$  let

$$\mathcal{F}(u, A) = \begin{cases} \int_A \phi(|\nabla u|) \, dx + \int_{S_u \cap A} \theta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1} & \text{if } u \in SBV^2(\Omega) \cap L^\infty(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Then the relaxed functional of  $\mathcal{F}$  with respect to the strong  $L^1$ -topology satisfies

$$\overline{\mathcal{F}}(u) \leq \int_\Omega \phi(|\nabla u|) \, dx + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1} + c|D^c u|(\Omega)$$



for any  $u \in BV(\Omega)$ .

*Proof.* Combining a standard convolution argument with a well known relaxation result (see, for instance, Th. 5.47 in [3]) we can say that the relaxed functional of

$$\mathcal{G}(u, A) = \begin{cases} \int_A \phi(|\nabla u|) \, dx & \text{if } u \in C^1(\overline{\Omega}) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

is given by

$$\overline{\mathcal{G}}(u, A) = \begin{cases} \int_A \phi(|\nabla u|) \, dx + c|D^s u|(A) & \text{if } u \in BV(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Since  $C^1(\overline{\Omega}) \subseteq SBV^2(\Omega) \cap L^\infty(\Omega)$  then we get  $\mathcal{F}(u, A) \leq \mathcal{G}(u, A)$ . Hence for any  $A \in \mathcal{A}(\Omega)$  and for any  $u \in BV(\Omega)$

$$\overline{\mathcal{F}}(u, A) \leq \int_A \phi(|\nabla u|) \, dx + c|D^s u|(A).$$

We can now conclude using the fact that for every  $u \in BV(\Omega)$  the set function  $\overline{\mathcal{F}}(u, \cdot)$  is the trace on  $\mathcal{A}(\Omega)$  of a regular Borel measure  $\mu$ . This can be proven exactly along the same line of Prop. 3.3 in [6]. Hence

$$\begin{aligned} \overline{\mathcal{F}}(u) &= \mu(\Omega) = \mu(\Omega \setminus S_u) + \mu(\Omega \cap S_u) \\ &\leq \int_\Omega \phi(|\nabla u|) \, dx + c|D^c u|(\Omega) + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1} \end{aligned}$$

which is what we wanted to prove.  $\square$

### 3. STATEMENT OF THE MAIN RESULTS

Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with Lipschitz boundary. Let  $\phi: [0, +\infty) \rightarrow [0, +\infty)$  be a convex and non-decreasing function with  $\phi(0) = 0$  and

$$(3.1) \quad \lim_{t \rightarrow +\infty} \frac{\phi(t)}{t} = c_0 \in (0, +\infty).$$

For any  $\varepsilon > 0$  let  $f_\varepsilon: [0, +\infty) \rightarrow [0, +\infty)$  be such that:

A1)  $f_\varepsilon$  is non-decreasing, continuous, with  $f_\varepsilon(0) = 0$ .

A2) It holds 
$$\lim_{(\varepsilon, t) \rightarrow (0, 0)} \frac{f_\varepsilon(t)}{\varepsilon \phi\left(\frac{t}{\varepsilon}\right)} = 1.$$

A3)  $f_\varepsilon$  converges uniformly on the compact subsets of  $[0, +\infty)$  to a concave function  $f$ .

**Example 3.1.** Given  $f$  and  $\phi$  as above, a possible choice for  $f_\varepsilon$  satisfying A1-A3 is given by

$$f_\varepsilon(t) = \begin{cases} \varepsilon \phi\left(\frac{t}{\varepsilon}\right) & \text{if } 0 \leq t \leq t_\varepsilon \\ f(t - t_\varepsilon) + \varepsilon \phi\left(\frac{t_\varepsilon}{\varepsilon}\right) & \text{if } t > t_\varepsilon \end{cases}$$

where  $t_\varepsilon \rightarrow 0$ , and  $t_\varepsilon/\varepsilon \rightarrow +\infty$ . The only non-trivial assumption to verify is A2. Since  $\varepsilon/t\phi(t/\varepsilon) \rightarrow c_0$  as  $(\varepsilon, t) \rightarrow (0, 0)$ , with  $t \geq t_\varepsilon$ , the check amounts to verify that

$$\lim_{\substack{(\varepsilon, t) \rightarrow (0, 0) \\ t \geq t_\varepsilon}} \frac{f(t - t_\varepsilon) + \varepsilon \phi\left(\frac{t_\varepsilon}{\varepsilon}\right)}{t} = c_0.$$

This follows immediately from  $f(t - t_\varepsilon)/(t - t_\varepsilon) \rightarrow c_0$  and  $\varepsilon/t_\varepsilon\phi(t_\varepsilon/\varepsilon) \rightarrow c_0$  as  $(\varepsilon, t) \rightarrow (0, 0)$ , and  $t \geq t_\varepsilon$ .

Let  $C \subset \mathbb{R}^n$  be open, bounded, and connected with  $0 \in C$ . Let  $\rho: C \rightarrow (0, +\infty)$  be a continuous and bounded convolution kernel with

$$\int_C \rho \, dx = 1.$$

For any  $\varepsilon > 0$  and for any  $x \in \mathbb{R}^n$  we will denote by  $C_\varepsilon(x)$  the set  $x + \varepsilon C$ . For any  $x \in \varepsilon C$  let

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right).$$

We consider the family  $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$  of functionals  $L^1(\Omega) \rightarrow [0, +\infty]$  defined by

$$(3.2) \quad \mathcal{F}_\varepsilon(u) = \begin{cases} \frac{1}{\varepsilon} \int_\Omega f_\varepsilon(\varepsilon |\nabla u| * \rho_\varepsilon) \, dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

where, for any  $x \in \Omega$ ,

$$(3.3) \quad |\nabla u| * \rho_\varepsilon(x) = \int_{C_\varepsilon(x) \cap \Omega} |\nabla u(y)| \rho_\varepsilon(y-x) \, dy$$

is a regularization by convolution of  $|\nabla u|$  by means of the kernel  $\rho_\varepsilon$ .

**Remark 3.2.** Notice that with the choice  $C = B_1$  and  $\rho = \frac{1}{\omega_n} \chi_{B_1}$  we get

$$|\nabla u| * \rho_\varepsilon(x) = \int_{B_\varepsilon(x) \cap \Omega} |\nabla u| \, dy$$

and thus the family  $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$  reduces to the case already investigated in [20], [21] and [22].

In order to prove the  $\Gamma$ -convergence of  $\mathcal{F}_\varepsilon$  it is convenient to introduce a localized version of  $\mathcal{F}_\varepsilon$ : more precisely, for each  $A \in \mathcal{A}(\Omega)$  we set

$$(3.4) \quad \mathcal{F}_\varepsilon(u, A) = \begin{cases} \frac{1}{\varepsilon} \int_A f_\varepsilon(\varepsilon |\nabla u| * \rho_\varepsilon) \, dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Clearly,  $\mathcal{F}_\varepsilon(\cdot, \Omega)$  coincides with the functional  $\mathcal{F}_\varepsilon$  defined in (3.2). The lower and upper  $\Gamma$ -limits of  $(\mathcal{F}_\varepsilon(\cdot, A))$  will be denoted by  $\mathcal{F}'(\cdot, A)$  and  $\mathcal{F}''(\cdot, A)$ , respectively.

**3.1. The anisotropy.** In this paragraph we define the surface density

$$\theta: [0, +\infty) \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$$

which will appear in the expression of the  $\Gamma$ -limit of  $\mathcal{F}_\varepsilon$ .

Given  $\nu \in \mathbb{S}^{n-1}$  and  $a, b \in \mathbb{R}$  let us denote by  $u_\nu^{a,b}$  the function  $\mathbb{R}^n \rightarrow \mathbb{R}$  given by

$$u_\nu^{a,b}(x) = \begin{cases} a & \text{if } \langle x, \nu \rangle < 0 \\ b & \text{if } \langle x, \nu \rangle \geq 0. \end{cases}$$

For any  $x \in \mathbb{R}^n$  and any  $\nu \in \mathbb{S}^{n-1}$  let  $P_\nu^\perp(x)$  be the orthogonal projection of  $x$  onto the subspace  $\nu^\perp = \{x \in \mathbb{R}^n : \langle x, \nu \rangle = 0\}$ . We define the cylinder

$$Q_\nu = \{x \in \mathbb{R}^n : |\langle x, \nu \rangle| \leq 1, P_\nu^\perp(x) \in B_1 \cap \nu^\perp\}.$$

Given  $\Omega' \subset \mathbb{R}^n$  with  $Q_\nu \subset \subset \Omega'$  denote by  $W_\nu^{a,b}$  the space of all sequences  $(u_j)$  in  $W_{\text{loc}}^{1,1}(\Omega')$  such that  $u_j \rightarrow u_\nu^{a,b}$  in  $L^1(\Omega')$ , and such that there exist two positive infinitesimal sequences  $(a_j), (b_j)$  with  $u_j(x) = a$  if  $\langle x, \nu \rangle < -a_j$  and  $u_j(x) = b$  if  $\langle x, \nu \rangle > b_j$ . Let

$$(3.5) \quad \theta(s, \nu) = \frac{1}{\omega_{n-1}} \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} f(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) \, dx : (u_j) \in W_\nu^{0,s}, \varepsilon_j \rightarrow 0^+ \right\}.$$

Notice that  $\theta(s, \nu)$  does not depend on the choice of  $\Omega'$ . Let us collect some easy properties of  $\theta$  which immediately descend from the definition.

**Lemma 3.3.** *The following properties hold:*

$$(3.6) \quad \theta \text{ is continuous.}$$

$$(3.7) \quad \theta(s, \nu) = \theta(s, -\nu), \quad \forall s \geq 0, \quad \forall \nu \in \mathbb{S}^{n-1}.$$

$$(3.8) \quad \begin{aligned} & \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} f(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) \, dx : (u_j) \in W_\nu^{0,s}, \varepsilon_j \rightarrow 0^+ \right\} \\ &= \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} f(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) \, dx : (u_j) \in W_\nu^{a,b}, \varepsilon_j \rightarrow 0^+ \right\} \\ & \quad \text{whenever } |a - b| = s. \end{aligned}$$

Moreover, for any  $x_0 \in \mathbb{R}^n$ ,  $\nu \in \mathbb{S}^{n-1}$  and  $s \geq 0$  we have

$$(3.9) \quad \theta(s, \nu) = \frac{1}{\omega_{n-1}} \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{x_0 + Q_\nu} f(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) \, dx : (u_j(\cdot - x_0)) \in W_\nu^{0,s}, \varepsilon_j \rightarrow 0^+ \right\}.$$

**3.2. Main results.** We are now in position to state the main result of the paper.

**Theorem 3.4.** *Let  $\mathcal{F}_\varepsilon$  be as in (3.2), with  $f_\varepsilon$  satisfying conditions A1-A3. Then  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges, with respect to the strong  $L^1$ -topology, as  $\varepsilon \rightarrow 0$ , to  $\mathcal{F} : L^1(\Omega) \rightarrow [0, +\infty]$  given by*

$$\mathcal{F}(u) = \begin{cases} \int_{\Omega} \phi(|\nabla u|) \, dx + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1} + c_0 |D^c u|(\Omega) & \text{if } u \in GBV(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

**Remark 3.5.** *Notice that for any  $u \in GBV(\Omega)$  the expression  $\theta(|u^+ - u^-|, \nu_u)$  turns out to be well defined  $\mathcal{H}^{n-1}$ -a.e.  $x \in S_u$ , since (3.7) holds.*

The proof of Theorem 3.4 will descend combining Proposition 5.10 (the  $\Gamma$ -liminf inequality) with Proposition 6.3 (the  $\Gamma$ -limsup inequality).

As a typical consequence of a  $\Gamma$ -convergence result, we are able to prove a result of convergence of minima by means of the following compactness result for equibounded (in energy) sequences, which will be proved in §4.

**Theorem 3.6.** *Let  $(\varepsilon_j)$  be a positive infinitesimal sequence, and let  $(u_j)$  be a sequence in  $L^1(\Omega)$  such that  $\|u_j\|_\infty \leq M$ , and such that  $\mathcal{F}_{\varepsilon_j}(u_j) \leq M$  for some positive constant  $M$  independent of  $j$ . Then the sequence  $(u_j)$  converges, up to a subsequence, in  $L^1(\Omega)$  to a function  $u \in BV(\Omega)$ .*

**Theorem 3.7.** *Let  $(\varepsilon_j)$  be a positive infinitesimal sequence and let  $g \in L^\infty(\Omega)$ . For every  $u \in L^1(\Omega)$  and  $j \in \mathbb{N}$  let*

$$\mathcal{I}_j(u) = \mathcal{F}_{\varepsilon_j}(u) + \int_{\Omega} |u - g| \, dx, \quad \mathcal{I}(u) = \mathcal{F}(u) + \int_{\Omega} |u - g| \, dx.$$

For every  $j$  let  $u_j \in L^1(\Omega)$  be such that

$$\mathcal{I}_j(u_j) \leq \inf_{L^1(\Omega)} \mathcal{I}_j + \varepsilon_j.$$

Then the sequence  $(u_j)$  converges, up to a subsequence, to a minimizer of  $\mathcal{I}$  in  $L^1(\Omega)$ .

*Proof.* Since  $g \in L^\infty(\Omega)$  and since  $\mathcal{F}_{\varepsilon_j}$  decreases by truncation, we can assume that  $(u_j)$  is equibounded in  $L^\infty(\Omega)$ ; for instance  $\|u_j\|_\infty \leq \|g\|_\infty$ . Applying Theorem 3.6 there exists  $u \in BV(\Omega)$  such that (up to a subsequence)  $u_j \rightarrow u$  in  $L^1(\Omega)$ . By Theorem 2.3, since  $(\mathcal{I}_j)$   $\Gamma$ -converges to  $\mathcal{I}$  (see Th. 3.4 and Remark 2.4),  $u$  is a minimum point of  $\mathcal{I}$  on  $L^1(\Omega)$ .  $\square$

## 4. COMPACTNESS

In this section we prove Theorem 3.6. Let us first recall a useful technical Lemma which can be found in [10], Prop. 4.1. Actually such a Proposition has been proved for  $|\nabla u|^2$ , but, up to simple modifications, the same proof works for  $|\nabla u|$ .

For every  $A \in \mathcal{A}(\Omega)$  and  $\sigma > 0$  we set

$$A_\sigma = \{x \in A : d(x, \partial A) > \sigma\}.$$

**Lemma 4.1.** *Let  $g: [0, +\infty) \rightarrow [0, +\infty)$  be a non-decreasing continuous function such that*

$$\lim_{t \rightarrow 0} \frac{g(t)}{t} = c$$

for some  $c > 0$ . Let  $A \in \mathcal{A}(\Omega)$  with  $A \subset\subset \Omega$ , and let  $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ . For any  $\delta > 0$  and for any  $\varepsilon > 0$  sufficiently small, there exists a function  $v \in SBV(A) \cap L^\infty(A)$  such that

$$\begin{aligned} (1 - \delta) \int_A |\nabla v| \, dx &\leq \frac{1}{\varepsilon} \int_A g\left(\varepsilon \int_{B_\varepsilon(x)} |\nabla u| \, dy\right) \, dx, \\ \mathcal{H}^{n-1}(S_v \cap A_{6\varepsilon}) &\leq \frac{c'}{\varepsilon} \int_A g\left(\varepsilon \int_{B_\varepsilon(x)} |\nabla u| \, dy\right) \, dx, \\ \|v\|_{L^\infty(A)} &\leq \|u\|_{L^\infty(A)} \\ \|v - u\|_{L^1(A_{6\varepsilon})} &\leq c' \|u\|_{L^\infty(A)} \int_A g\left(\varepsilon \int_{B_\varepsilon(x)} |\nabla u| \, dy\right) \, dx, \end{aligned}$$

where  $c'$  is a constant depending only on  $n, \delta$  and  $g$ .

*Proof of Theorem 3.6.* Let  $A \in \mathcal{A}(\Omega)$  with  $A \subset\subset \Omega$  and  $\partial A$  smooth. Let  $r > 0$  such that  $B_r \subset C$ , and let  $m = \inf_{B_r} \rho > 0$ . Then for any  $x \in A$  we have  $B_{r\varepsilon_j}(x) \subset C_{\varepsilon_j}(x)$  and thus for  $j$  sufficiently large,

$$\begin{aligned} |\nabla u_j| * \rho_{\varepsilon_j}(x) &= \int_{C_{\varepsilon_j}(x)} |\nabla u_j(y)| \rho_{\varepsilon_j}(y - x) \, dy \geq \frac{m}{\varepsilon_j^n} \int_{B_{r\varepsilon_j}(x)} |\nabla u_j(y)| \, dy \\ &= mr^n \omega_n \int_{B_{r\varepsilon_j}(x)} |\nabla u_j(y)| \, dy \end{aligned}$$

for any  $x \in A$ . Fix  $\delta > 0$ . By A2 there exist  $t_\delta > 0$  and  $j_\delta$  such that  $f_{\varepsilon_j}(t) \geq (1 - \delta)\varepsilon_j \phi(t/\varepsilon_j)$  for any  $t \in [0, t_\delta]$  and  $j > j_\delta$ . Let  $\alpha, \beta \in \mathbb{R}$ , with  $\alpha > 0$  and  $\beta < 0$ , be such that  $\phi(t) \geq \alpha t + \beta$  everywhere. Then, since  $f_{\varepsilon_j}$  is non-decreasing, we have  $f_{\varepsilon_j}(t) \geq g_{\varepsilon_j}^\delta(t)$  for any  $t \geq 0$ , being

$$g_{\varepsilon_j}^\delta(t) = \begin{cases} (1 - \delta)\alpha t + \varepsilon_j \beta & \text{if } t \in [0, t_\delta] \\ (1 - \delta)\alpha t_\delta + \varepsilon_j \beta & \text{if } t > t_\delta. \end{cases}$$

Therefore, letting  $h_\delta(t) = g_{\varepsilon_j}^\delta(t) - \varepsilon_j \beta$ , we have

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}(u_j, A) &\geq \frac{1}{\varepsilon_j} \int_A h_\delta(|\nabla u_j| * \rho_{\varepsilon_j}) \, dx + \beta |A| \\ (4.1) \quad &\geq \frac{1}{\varepsilon_j} \int_A h_\delta\left(mr^n \omega_n \varepsilon_j \int_{B_{r\varepsilon_j}(x)} |\nabla u_j| \, dy\right) \, dx + \beta |A|. \end{aligned}$$

Let  $\eta_j = r\varepsilon_j$  and  $g_{\delta,m,r}(t) = \frac{1}{r} g_\delta(mr^{n-1} \omega_n t)$ . Notice that, by construction,

$$\lim_{t \rightarrow 0} \frac{g_{\delta,m,r}(t)}{t}$$

exists and is finite. Then inequality (4.1) becomes

$$\mathcal{F}_{\varepsilon_j}(u_j, A) - \beta |A| \geq \frac{1}{\eta_j} \int_\Omega g_{\delta,r,m}\left(\eta_j \int_{B_{\eta_j}(x)} |\nabla u_j| \, dy\right) \, dx.$$

Applying Lemma 4.1 we find a sequence  $(v_j)$  in  $SBV(A)$  and a constant  $C$  independent of  $A$  such that  $\|v_j\|_{BV(A)} \leq C$  and  $\|v_j\|_{L^\infty(A)} \leq C$ . Moreover,

$$(4.2) \quad \|v_j - u_j\|_{L^1(A)} \rightarrow 0.$$

Hence, by Theorem 2.1, the sequence  $(v_j)$  converges, up to a subsequence not relabeled, to some  $u \in BV(A)$ , with  $\|u\|_{BV(A)} \leq C$ . By (4.2) also  $u_j$  converges to  $u$  in  $L^1(A)$ . The arbitrariness of  $A$  and a diagonal argument allow to find a subsequence  $(u_{j_k})$  which converges in  $L^1_{\text{loc}}(\Omega)$  to a function  $u \in BV_{\text{loc}}(\Omega)$ , and the uniform bound of  $\|u_j\|_{L^\infty(\Omega)}$  implies the convergence is strong in  $L^1(\Omega)$ .  $\square$

## 5. THE Γ-LIMINF INEQUALITY

In this section we will prove that for any  $u \in L^1(\Omega)$  the inequality

$$\mathcal{F}(u) \leq \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}(u_j)$$

holds for any  $u_j \rightarrow u$  in  $L^1(\Omega)$ . First we will investigate two particular situations.

**5.1. A preliminary estimate from below in terms of the volume and Cantor parts.** In this paragraph we will take into account a simpler family of functionals. Let  $\alpha, \beta > 0$  and let  $g: [0, +\infty) \rightarrow [0, +\infty)$  given by  $g(t) = \alpha t \wedge \beta$ . Let  $\mathcal{G}_\varepsilon: L^1(\Omega) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$  be defined by

$$\mathcal{G}_\varepsilon(u, A) = \begin{cases} \frac{1}{\varepsilon} \int_A g(\varepsilon |\nabla u| * \rho_\varepsilon) dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

We wish to estimate from below the lower  $\Gamma$ -limit  $\mathcal{G}'(\cdot, A)$  in terms of the volume and the Cantor parts of  $Du$ . To this sake, we apply a slicing procedure, so that at first we will establish a suitable one-dimensional inequality. The idea of the proof is the same as in [25], where the superlinear growth case is treated.

Let  $m \in \mathbb{N}$  odd, let  $A$  be an open interval in  $\mathbb{R}$ , and let  $(\varepsilon_j)$  be a positive infinitesimal sequence. Let  $A_j = \{x \in \varepsilon_j \mathbb{Z} : x \in A\}$ . For any  $j \in \mathbb{N}$  and for any  $x \in A_j$  we define the interval

$$I_j(x) = \left[ x - \frac{m\varepsilon_j}{2}, x + \frac{m\varepsilon_j}{2} \right].$$

**Lemma 5.1.** *Let  $\alpha', \beta' > 0$  and let  $h_j: [0, +\infty) \rightarrow [0, +\infty)$  given by  $h_j(t) = \alpha' t \wedge \frac{\beta'}{\varepsilon_j}$ . Let  $u \in BV(A)$  and let  $u_j \rightarrow u$  in  $L^1(A)$  with  $u_j \in W^{1,1}(A)$  for any  $j \in \mathbb{N}$ . Then*

$$(5.1) \quad \liminf_{j \rightarrow +\infty} \varepsilon_j \sum_{x \in A_j} h_j \left( \int_{I_j(x)} |u'_j| dy \right) \geq \alpha' \int_A |u'| dy + \alpha' |D^c u|(A).$$

*Proof.* For any  $j \in \mathbb{N}$  and  $i = 0, \dots, m-1$  let  $A_j^i = (i\varepsilon_j + m\varepsilon_j \mathbb{Z}) \cap A$ . Obviously  $A_j$  is the disjoint union of  $A_j^i$  for  $i \in \{0, \dots, m-1\}$ . Then

$$\sum_{x \in A_j} h_j \left( \int_{I_j(x)} |u'_j| dy \right) \geq \frac{1}{m} \sum_{i=0}^{m-1} \sum_{x \in A_j^i} m h_j \left( \int_{I_j(x)} |u'_j| dy \right).$$

Now let

$$\overline{A_j^i} = \left\{ x \in A_j^i : \int_{I_j(x)} |u'_j| dx \leq \frac{\beta'}{\alpha' \varepsilon_j} \right\}$$

and let  $v_j \in SBV(A)$  given by

$$v_j(x) = \begin{cases} u_j(x) & \text{if } x \in \bigcup_{y \in \overline{A_j^i}} I_j(y) \\ 0 & \text{otherwise in } A. \end{cases}$$

Hence

$$\begin{aligned} \sum_{x \in A_j^i} m \varepsilon_j h_j \left( \int_{I_j(x)} |u'_j| \, dy \right) &\geq \sum_{x \in \overline{A_j^i}} m \varepsilon_j h_j \left( \int_{I_j(x)} |u'_j| \, dy \right) = \alpha' \sum_{x \in \overline{A_j^i}} \int_{I_j(x)} |u'_j| \, dy \\ &= \alpha' \int_A |v'_j| \, dy. \end{aligned}$$

Observe that since we can suppose, without loss of generality, that

$$\varepsilon_j \sum_{x \in A_j} h_j \left( \int_{I_j(x)} |u'_j| \, dy \right) \leq M$$

for some  $M \geq 0$ , we deduce that

$$M \geq \varepsilon_j \sum_{x \in A_j \setminus \bigcup_{i=0}^{m-1} \overline{A_j^i}} h_j \left( \int_{I_j(x)} |u'_j| \, dy \right) = \varepsilon_j \frac{\beta'}{\varepsilon_j} \# \left( A_j \setminus \bigcup_{i=0}^{m-1} \overline{A_j^i} \right)$$

from which necessarily we have

$$\varepsilon_j \# \left( A_j \setminus \bigcup_{i=0}^{m-1} \overline{A_j^i} \right) \rightarrow 0, \quad \text{as } j \rightarrow +\infty.$$

This implies that  $\|u_j - v_j\|_{L^1(A)} \rightarrow 0$  as  $j \rightarrow +\infty$ . Therefore,  $v_j \rightarrow u$  in  $L^1(A)$ . Finally, by the superadditivity of the  $\liminf$  and by the lower semicontinuity of the total variation, we get

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \varepsilon_j \sum_{x \in A_j} h_j \left( \int_{I_j(x)} |u'_j| \, dy \right) &\geq \frac{1}{m} \sum_{i=0}^{m-1} \liminf_{j \rightarrow +\infty} \sum_{x \in \overline{A_j^i}} m \varepsilon_j h_j \left( \int_{I_j(x)} |u'_j| \, dy \right) \\ &\geq \alpha' \liminf_{j \rightarrow +\infty} \int_A |v'_j| \, dy \geq \alpha' |Du|(A) \\ &\geq \alpha' \int_A |u'| \, dy + \alpha' |D^c u|(A) \end{aligned}$$

which ends the proof.  $\square$

Now, by applying the slicing Theorem 2.2, we will reduce the  $n$ -dimensional inequality to the one-dimensional inequality 5.1. Fix  $\xi \in \mathbb{S}^{n-1}$  and  $\delta \in (0, 1)$ ; consider an orthonormal basis  $\{\mathbf{e}_i\}$  with  $\mathbf{e}_n = \xi$ . Let

$$Q_\delta^\xi = \left\{ x \in \mathbb{R}^n : |\langle x, \mathbf{e}_i \rangle| \leq \frac{\delta}{2}, i = 1, \dots, n \right\}, \quad Q_\delta^\xi(x) = x + Q_\delta^\xi$$

and the lattice  $Z_\delta^\xi = \{x \in \mathbb{R}^n : \langle x, \mathbf{e}_i \rangle \in \delta\mathbb{Z}, i = 1, \dots, n\}$ . In what follows we will denote by  $g_j(t) = \frac{1}{\varepsilon_j} g(\varepsilon_j t)$ ; in particular it holds  $g_j(t) = \alpha t \wedge \frac{\beta}{\varepsilon_j}$  and

$$\mathcal{G}_{\varepsilon_j}(u, A) = \int_A g_j(|\nabla u| * \rho_{\varepsilon_j}) \, dx, \quad u \in W^{1,1}(\Omega).$$

Finally fix  $A \in \mathcal{A}(\Omega)$  and let  $A_\delta^\xi = \{x \in Z_\delta^\xi : Q_\delta^\xi(x) \subset A\}$ . The following Lemma is a standard easy application of the mean value Theorem (see also Lemma 4.2 in [10]).

**Lemma 5.2.** *Let  $u \in W^{1,1}(\Omega)$ . Then there exists  $\tau \in Q_\delta^\xi$  such that*

$$\mathcal{G}_{\varepsilon_j}(u, A) \geq \sum_{x \in A_\delta^\xi} \delta^n g_j(|\nabla u| * \rho_{\varepsilon_j}(x + \tau)).$$

*Proof.* We have

$$\mathcal{G}_{\varepsilon_j}(u, A) \geq \sum_{x \in A_\delta^\xi} \int_{Q_\delta^\xi(x)} g_j(|\nabla u| * \rho_{\varepsilon_j}(y)) \, dy = \int_{Q_\delta^\xi} \sum_{x \in A_\delta^\xi} g_j(|\nabla u| * \rho_{\varepsilon_j}(y + x)) \, dy.$$

Applying the mean value Theorem we get

$$\int_{Q_\delta^\xi} \sum_{x \in A_\delta^\xi} g_j(|\nabla u| * \rho_{\varepsilon_j}(y+x)) \, dy = \sum_{x \in A_\delta^\xi} g_j(|\nabla u| * \rho_{\varepsilon_j}(\tau+x))$$

for some  $\tau \in Q_\delta^\xi$ , which concludes the proof.  $\square$

We are in position to apply the slicing procedure.

**Proposition 5.3.** *Let  $u \in BV(\Omega)$  and  $A \in \mathcal{A}(\Omega)$ . Then*

$$\mathcal{G}'(u, A) \geq \alpha \int_A |\nabla u| \, dx \quad \text{and} \quad \mathcal{G}'(u, A) \geq \alpha |D^c u|(A).$$

*Proof.* Fix  $\xi \in \mathbb{S}^{n-1}$ . For any  $\eta > 0$  let  $P_\eta^\xi$  be the union of the squares  $Q_\eta^\xi(y_i) \subset C$  with  $y_i \in Z_\eta^\xi$  for  $i = 1, \dots, m$ , for some  $m \in \mathbb{N}$  depending on  $\eta$  and  $\xi$ . Let  $\rho_\eta$  be a non-negative constant function on the squares  $Q_\eta^\xi(y_i)$  with  $0 < \rho_\eta \leq \rho$  and such that

$$c_\eta = \int_C \rho_\eta \, dx \rightarrow 1, \quad \text{as } \eta \rightarrow 0.$$

Let  $c_i = \rho_\eta(y_i)$ ; then we can rewrite  $c_\eta$  as  $c_\eta = \sum_{i=1}^m c_i \eta^n$ . Let  $P_{\eta\varepsilon_j}^\xi$  be the union of the squares  $Q_{\eta\varepsilon_j}^\xi(y_i) \subseteq C_{\varepsilon_j}$ , with  $y_i \in Z_{\eta\varepsilon_j}^\xi$ , for  $i = 1, \dots, m$ . Let  $A_j^\xi = A_{\eta\varepsilon_j}^\xi$ ; applying Lemma 5.2, since we can suppose, without loss of generality, that  $u_j \in W^{1,1}(\Omega)$ , there exists  $\tau_j \in Q_{\eta\varepsilon_j}^\xi$  such that

$$\mathcal{G}_{\varepsilon_j}(u_j, A) \geq \sum_{x \in A_j^\xi} (\eta\varepsilon_j)^n g_j(|\nabla u_j| * \rho_{\varepsilon_j}(x + \tau_j)).$$

Let  $B \subset\subset A$ , and, for any  $j$  sufficiently large, let  $v_j(y) = u_j(y + \tau_j)$ . Then we get  $v_j \in W^{1,1}(B)$  and  $v_j \rightarrow u$  in  $L^1(B)$ . Thus

$$\mathcal{G}_{\varepsilon_j}(u_j, A) \geq \sum_{x \in B_j^\xi} (\eta\varepsilon_j)^n g(|\nabla v_j| * \rho_{\varepsilon_j}(x))$$

being  $B_j^\xi = \{x \in Z_{\eta\varepsilon_j}^\xi : Q_{\eta\varepsilon_j}^\xi \subseteq B\}$ . Now, for each  $x \in B_j^\xi$ , we estimate the term  $|\nabla v_j| * \rho_{\varepsilon_j}(x)$ ; we have, for  $j$  large enough,

$$\begin{aligned} |\nabla v_j| * \rho_{\varepsilon_j}(x) &= \int_{C_{\varepsilon_j}} |\nabla v_j(y+x)| \rho_{\varepsilon_j}(y) \, dy \geq \frac{1}{\varepsilon_j^n} \int_{P_{\eta\varepsilon_j}^\xi} |\nabla v_j(y+x)| \rho_\eta\left(\frac{y}{\varepsilon_j}\right) \, dy \\ &\geq \frac{1}{\varepsilon_j^n} \sum_{i=1}^m c_i \int_{Q_{\eta\varepsilon_j}^\xi(y_i)} |\nabla v_j(y+x)| \, dy = \sum_{i=1}^m \frac{c_i \eta^n}{c_\eta} \int_{Q_{\eta\varepsilon_j}^\xi(y_i)} c_\eta |\nabla v_j(y+x)| \, dy. \end{aligned}$$

Since  $\sum_{i=1}^m \frac{c_i \eta^n}{c_\eta} = 1$  and since  $g_j$  is concave we get, for every  $x \in B_j^\xi$ ,

$$g_j(|\nabla v_j| * \rho_{\varepsilon_j}(x)) \geq \sum_{i=1}^m \frac{c_i \eta^n}{c_\eta} g_j\left(c_\eta \int_{Q_{\eta\varepsilon_j}^\xi(y_i)} |\nabla v_j(y+x)| \, dy\right).$$

Thus, reordering the terms, we deduce that

$$\mathcal{G}_{\varepsilon_j}(u_j, A) \geq \sum_{x \in D_j^\xi} (\eta\varepsilon_j)^n g_j\left(c_\eta \int_{Q_{\eta\varepsilon_j}^\xi(x)} |\nabla v_j| \, dz\right)$$

for any  $D \subset\subset B$  and  $j$  sufficiently large, being, as usual,  $D_j^\xi = \{x \in Z_{\eta\varepsilon_j}^\xi : Q_{\eta\varepsilon_j}^\xi \subseteq D\}$ . For convenience we can suppose  $\nabla v_j = 0$  on

$$\mathbb{R}^n \setminus \bigcup_{Q_{\eta\varepsilon_j}^\xi \subseteq D} Q_{\eta\varepsilon_j}^\xi.$$

Let  $\langle \xi \rangle$  be the one-dimensional space generated by  $\xi$ . Let us denote by  $Z_{\eta\varepsilon_j}^{\xi_{\parallel}}$  and by  $Z_{\eta\varepsilon_j}^{\xi_{\perp}}$  the orthogonal projections of  $Z_{\eta\varepsilon_j}^{\xi}$  respectively on  $\langle \xi \rangle$  and  $\xi^{\perp}$ . Then

$$\begin{aligned} \mathcal{G}_{\varepsilon_j}(u_j, A) &\geq \sum_{x \in Z_{\eta\varepsilon_j}^{\xi}} (\eta\varepsilon_j)^n g_j \left( c_{\eta} \int_{Q_{\eta\varepsilon_j}^{\xi}(x)} |\nabla v_j| dz \right) \\ &\geq \sum_{x_{\perp} \in Z_{\eta\varepsilon_j}^{\xi_{\perp}}} \sum_{x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}} (\eta\varepsilon_j)^n g_j \left( c_{\eta} \int_{Q_{\eta\varepsilon_j}^{\xi}(x_{\perp} + x_{\parallel})} |\nabla v_j| dz \right) \end{aligned}$$

where  $x = x_{\parallel} + x_{\perp}$  turns out to be the unique decomposition of any  $x \in Z_{\eta\varepsilon_j}^{\xi}$  with  $x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}$  and  $x_{\perp} \in Z_{\eta\varepsilon_j}^{\xi_{\perp}}$ . Moreover, denoting by  $Q_{\eta\varepsilon_j}^{\xi_{\parallel}}$  and by  $Q_{\eta\varepsilon_j}^{\xi_{\perp}}$  the projections of  $Q_{\eta\varepsilon_j}^{\xi}$  respectively on  $\langle \xi \rangle$  and on  $\xi^{\perp}$ , applying Jensen's inequality we deduce that

$$\begin{aligned} \mathcal{G}_{\varepsilon_j}(u_j, A) &\geq \sum_{x_{\perp} \in Z_{\eta\varepsilon_j}^{\xi_{\perp}}} \sum_{x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}} (\eta\varepsilon_j)^n g_j \left( c_{\eta} \int_{Q_{\eta\varepsilon_j}^{\xi_{\perp}}(x_{\perp})} \int_{Q_{\eta\varepsilon_j}^{\xi_{\parallel}}(x_{\parallel})} |\langle \nabla v_j(z_{\perp} + z_{\parallel}), \xi \rangle| dz_{\parallel} dz_{\perp} \right) \\ &\geq \sum_{x_{\perp} \in Z_{\eta\varepsilon_j}^{\xi_{\perp}}} \sum_{x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}} (\eta\varepsilon_j)^n \int_{Q_{\eta\varepsilon_j}^{\xi_{\perp}}(x_{\perp})} g_j \left( c_{\eta} \int_{Q_{\eta\varepsilon_j}^{\xi_{\parallel}}(x_{\parallel})} |\langle \nabla v_j(z_{\perp} + z_{\parallel}), \xi \rangle| dz_{\parallel} \right) dz_{\perp} \\ &\geq \sum_{x_{\perp} \in Z_{\eta\varepsilon_j}^{\xi_{\perp}}} \int_{Q_{\eta\varepsilon_j}^{\xi_{\perp}}(x_{\perp})} \sum_{x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}} \eta\varepsilon_j g_j \left( c_{\eta} \int_{Q_{\eta\varepsilon_j}^{\xi_{\parallel}}(x_{\parallel})} |\langle \nabla v_j(z_{\perp} + z_{\parallel}), \xi \rangle| dz_{\parallel} \right) dz_{\perp} \\ &\geq \int_{\xi^{\perp}} \sum_{x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}} \eta\varepsilon_j g_j \left( c_{\eta} \int_{Q_{\eta\varepsilon_j}^{\xi_{\parallel}}(x_{\parallel})} |\langle \nabla v_j(z_{\perp} + z_{\parallel}), \xi \rangle| dz_{\parallel} \right) dz_{\perp}. \end{aligned}$$

For any  $\sigma > 0$  small let  $D_{\sigma} = \{x \in D : d(x, \partial D) > \sigma\}$  and  $D_{\sigma}^{x_{\perp}} = \{x \in D_{\sigma} : x = x_{\perp} + x_{\parallel}\xi, x_{\parallel} \in \mathbb{R}\}$ , for  $x_{\perp} \in \xi^{\perp}$ . For  $j$  sufficiently large,  $v_j(x_{\perp} + \cdot) \in W^{1,1}(D_{\sigma}^{x_{\perp}})$ . Furthermore,  $v_j \rightarrow u$  in  $L^1(D_{\sigma}^{x_{\perp}})$  for a.e.  $x_{\perp} \in \xi^{\perp}$ . Let  $h_j(t) = g_j(c_{\eta}t)$ ; then, by the very definition of  $g$ , it is easy to see that  $h_j(t) = \alpha c_{\eta}t \wedge \frac{\beta}{\varepsilon_j}$ . We are in position to apply Lemma 5.1 with choice  $\alpha' = \alpha c_{\eta}$  and  $\beta' = \beta$ . Thus

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \sum_{x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}} \eta\varepsilon_j g_j \left( c_{\eta} \int_{Q_{\eta\varepsilon_j}^{\xi_{\parallel}}(x_{\parallel})} |\langle \nabla v_j(z_{\perp} + z_{\parallel}), \xi \rangle| dz_{\parallel} \right) \\ = \liminf_{j \rightarrow +\infty} \sum_{x_{\parallel} \in Z_{\eta\varepsilon_j}^{\xi_{\parallel}}} \eta\varepsilon_j h_j \left( \int_{Q_{\eta\varepsilon_j}^{\xi_{\parallel}}(x_{\parallel})} |\langle \nabla v_j(z_{\perp} + z_{\parallel}), \xi \rangle| dz_{\parallel} \right) \\ \geq \alpha c_{\eta} \int_{D_{\sigma}^{z_{\perp}}} |\langle \nabla u(z_{\perp} + z_{\parallel}), \xi \rangle| dz_{\parallel} + \alpha c_{\eta} |\langle D^c u(z_{\perp} + \cdot), \xi \rangle|(D_{\sigma}^{z_{\perp}}). \end{aligned}$$

Taking into account Theorem 2.2 and Fatou's Lemma we conclude that

$$\liminf_{j \rightarrow +\infty} \mathcal{G}_{\varepsilon_j}(u_j, A) \geq c_{\eta} \alpha \int_{D_{\sigma}} |\langle \nabla u(z), \xi \rangle| dz + c_{\eta} \alpha |\langle D^c u, \xi \rangle|(D_{\sigma}).$$

Since  $c_{\eta} \rightarrow 1$  as  $\eta \rightarrow 0$ , let  $\sigma \rightarrow 0$  and  $D \nearrow A$ . Then

$$(5.2) \quad \mathcal{G}'(u, A) \geq \alpha \int_A |\langle \nabla u(z), \xi \rangle| dz \quad \text{and} \quad \mathcal{G}'(u, A) \geq \alpha |\langle D^c u, \xi \rangle|(A)$$

for any  $\xi \in \mathbb{S}^{n-1}$ . From the first inequality, using the superadditivity of  $\mathcal{G}'$  and Lemma 2.5 we easily deduce that

$$\mathcal{G}'(u, A) \geq \alpha \int_A |\nabla u| dz.$$



Now if  $\psi_\xi = \langle \frac{dD^c u}{|dD^c u|}, \xi \rangle$  the second inequality in (5.2) can be rewritten as

$$\mathcal{G}'(u, A) \geq \alpha \int_A |\psi_\xi| d|D^c u|.$$

Another application of Lemma 2.5 yields

$$\mathcal{G}'(u, A) \geq \alpha \int_A \sup_{\xi \in \mathbb{S}^{n-1}} |\psi_\xi| d|D^c u| \geq \alpha \int_A \sup_{\xi \in \mathbb{S}^{n-1}} \psi_\xi d|D^c u| = \alpha |D^c u|(A).$$

This concludes the proof.  $\square$

**5.2. A preliminary estimate in terms of the surface part.** In this section we will consider the family of functionals  $L^1(\Omega) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$  given by

$$\mathcal{E}_\varepsilon(u, A) = \begin{cases} \frac{1}{\varepsilon} \int_A h(\varepsilon |\nabla u| * \rho_\varepsilon) dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

where  $h: [0, +\infty) \rightarrow [0, +\infty)$  is a non-decreasing concave function with  $h(0) = 0$  and with

$$\lim_{t \rightarrow 0} \frac{h(t)}{t} = c' > 0.$$

The aim of this section is to estimate from below the lower  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon$  in terms of a surface integral; to do this the main idea, as in [22], is to estimate from below the Radon-Nikodym derivative of the lower  $\Gamma$ -limit  $\mathcal{E}'$  with respect to the Hausdorff measure  $\mathcal{H}^{n-1}$  by means of a blow-up argument around a jump point; then the result follows applying Besicovitch's Differentiation Theorem in a standard way.

Given  $x_0 \in \mathbb{R}^n$ ,  $\nu \in \mathbb{S}^{n-1}$  and  $a, b \in \mathbb{R}$ , when considering  $\mathcal{E}'$  for the blow up  $u_{x_0}^{\nu, a, b} = u_\nu^{a, b}(\cdot - x_0)$  (see paragraph 3.1 for the definition of  $u_\nu^{a, b}$ ) on a unit ball  $B_1$  as below (or on a cylinder  $Q_\nu$  as in the sequel), we will assume as  $\Omega$  any set  $\Omega'$  strictly containing  $B_1$  (or  $Q_\nu$ ): the lower  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon(\cdot, A)$  does not change by replacing  $\Omega$  with any  $\Omega' \supset \supset A$ .

For every  $A \in \mathcal{A}(\Omega)$  let  $\mathcal{E}'_-(\cdot, A)$  be the inner regular envelope of  $\mathcal{E}'$ , i.e.

$$\mathcal{E}'_-(\cdot, A) = \sup\{\mathcal{E}'(\cdot, B) : B \in \mathcal{A}(\Omega), B \subset \subset A\}.$$

**Proposition 5.4.** *Let  $u \in BV(\Omega)$  and let  $x_0 \in J_u$ . Then*

$$\liminf_{\varrho \rightarrow 0} \frac{\mathcal{E}'_-(u, B_\varrho(x_0))}{\varrho^{n-1}} \geq \mathcal{E}'(u_{x_0}^{\nu_u(x_0), u^+(x_0), u^-(x_0)}, B_1(x_0)).$$

*Proof.* Let  $\delta \in (0, 1)$ . Then  $\mathcal{E}'_-(u, B_\varrho(x_0)) \geq \mathcal{E}'(u, B_{\delta\varrho}(x_0))$  for every  $\varrho > 0$ . Thus

$$(5.3) \quad \liminf_{\varrho \rightarrow 0} \frac{\mathcal{E}'_-(u, B_\varrho(x_0))}{\varrho^{n-1}} \geq \delta^{n-1} \liminf_{r \rightarrow 0} \frac{\mathcal{E}'(u, B_r(x_0))}{r^{n-1}}.$$

Let us now estimate the lower limit in the right-hand side. Without loss of generality we can assume  $x_0 = 0$ ; moreover, for the sake of simplicity, we will denote by  $u_0$  the function  $u_0^{\nu_u(0), u^+(0), u^-(0)}$ .

Let  $(r_k)$  be a decreasing infinitesimal sequence; for every  $k \in \mathbb{N}$  there exists  $u_j \in W^{1,1}(\Omega)$  such that  $u_j \rightarrow u$  in  $L^1(\Omega)$  and

$$\liminf_{j \rightarrow +\infty} \mathcal{E}_{\varepsilon_j}(u_j, B_{r_k}) \leq \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{2k}.$$

Let  $\bar{j} = j(k)$  be such that  $\varepsilon_{\bar{j}}/r_k \leq 1/k$  and

$$\mathcal{E}_{\varepsilon_{\bar{j}}}(u_{\bar{j}}, B_{r_k}) \leq \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{k},$$

$\|u_{\bar{j}} - u\|_{L^1(\Omega)} \leq \frac{1}{k}$  and such that

$$\int_{B_2} |u_{\bar{j}}(r_k x) - u(r_k x)| dx \leq \frac{1}{k}.$$

Let  $v_k = u_{j(k)}$ . We can suppose that the sequence  $j(k)$  is increasing, and we set  $\sigma_k = \varepsilon_{j(k)}$ . Hence,  $v_k \rightarrow u$  in  $L^1(\Omega)$ ,

$$(5.4) \quad \mathcal{E}_{\sigma_k}(v_k, B_{r_k}) \leq \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{k}$$

and

$$(5.5) \quad \int_{B_2} |v_k(r_k x) - u(r_k x)| \, dx \leq \frac{1}{k}.$$

Inequality (5.4) gives

$$\liminf_{k \rightarrow +\infty} \frac{\mathcal{E}'(u, B_{r_k})}{r_k^{n-1}} \geq \liminf_{k \rightarrow +\infty} \frac{\mathcal{E}_{\sigma_k}(v_k, B_{r_k})}{r_k^{n-1}}$$

while from (5.5) we get

$$\int_{B_2} |v_k(r_k x) - u_0(r_k x)| \, dx \leq \frac{1}{k} + \int_{B_2} |v(r_k x) - u_0(r_k x)| \, dx \rightarrow 0$$

as  $k \rightarrow +\infty$ . Let  $w_k(t) = v_k(r_k t)$ . Then  $w_k \rightarrow u_0$  in  $L^1(B_2)$ ; moreover, for every  $x \in B_{r_k}$  we have, setting  $y = r_k t$  and observing that  $|\nabla w_k(t)| = r_k |\nabla v_k(r_k t)|$ ,

$$\begin{aligned} |\nabla v_k| * \rho_{\sigma_k}(x) &= \int_{C_{\sigma_k}(x)} |\nabla v_k(y)| \rho_{\sigma_k}(y-x) \, dy = \frac{1}{\sigma_k^n} \int_{C_{\sigma_k}(x)} |\nabla v_k(y)| \rho\left(\frac{y-x}{\sigma_k}\right) \, dy \\ &= \frac{r_k^{n-1}}{\sigma_k^n} \int_{C_{\sigma_k/r_k}(x/r_k)} |\nabla w_k(t)| \rho\left(\frac{t}{\sigma_k/r_k} - \frac{x}{\sigma_k}\right) \, dt. \end{aligned}$$

Therefore, setting  $x = r_k z$ , we obtain

$$\begin{aligned} \frac{\mathcal{E}_{\sigma_k}(v_k, B_{r_k})}{r_k^{n-1}} &= \frac{1}{r_k^{n-1} \sigma_k} \int_{B_{r_k}} h(\sigma_k |\nabla v_k| * \rho_{\sigma_k}(x)) \, dx \\ &= \frac{1}{r_k^{n-1} \sigma_k^n} \int_{B_{r_k}} h\left(\frac{r_k^{n-1}}{\sigma_k^{n-1}} \int_{C_{\sigma_k/r_k}(x/r_k)} |\nabla w_k(t)| \rho\left(\frac{t}{\sigma_k/r_k} - \frac{x}{\sigma_k}\right) \, dt\right) \, dx \\ &= \frac{1}{\sigma_k/r_k} \int_{B_1} h\left(\frac{\sigma_k}{r_k} \frac{r_k^n}{\sigma_k^n} \int_{C_{\sigma_k/r_k}(z)} |\nabla w_k(t)| \rho\left(\frac{t-z}{\sigma_k/r_k}\right) \, dt\right) \, dz \\ &= \frac{1}{\sigma_k/r_k} \int_{B_1} h\left(\frac{\sigma_k}{r_k} |\nabla w_k| * \rho_{\sigma_k/r_k}(z)\right) \, dz. \end{aligned}$$

Since  $\sigma_k/r_k \rightarrow 0$ , and  $w_k \rightarrow u_0$  in  $L^1(B_2)$ , by the arbitrariness of  $(r_k)$  and the definition of  $\mathcal{E}'$ , we conclude combining (5.3) with the arbitrariness of  $\delta \in (0, 1)$ .  $\square$

Now we estimate from below  $\mathcal{E}'(u_{x_0}^{\nu, a, b}, B_1(x_0))$ . Without loss of generality, we can assume  $x_0 = 0$  and  $\nu = \mathbf{e}_1$ ; we will denote, for the sake of simplicity, by  $u^{a, b}$  the function  $u_0^{\mathbf{e}_1, a, b}$ . In order to estimate from below  $\mathcal{E}'(u^{a, b}, B_1)$  first we need to consider the problem on a suitable cylinder.

Recall that (see paragraph 3.1)  $Q_{\mathbf{e}_1} = \{x \in \mathbb{R}^n : |x_1| < 1, P_{\mathbf{e}_1}^\perp(x) \in B_1 \cap \mathbf{e}_1^\perp\}$ , being  $P_{\mathbf{e}_1}^\perp(x)$  the orthogonal projection of  $x$  onto the subspace  $\mathbf{e}_1^\perp$ ; for simplicity of notation we will use  $Q$  instead of  $Q_{\mathbf{e}_1}$ .

**Lemma 5.5.** *For any  $A$  open subset of  $Q$  there exist a positive infinitesimal sequence  $(\varepsilon_j)$  and a sequence  $u_j$  in  $W^{1,1}(\Omega')$  converging to  $u^{a, b}$  in  $L^1(\Omega')$  such that*

$$(5.6) \quad \lim_{j \rightarrow +\infty} \mathcal{E}_{\varepsilon_j}(u_j, A) = \mathcal{E}'(u^{a, b}, A)$$

and such that

$$(5.7) \quad u_j(x) = a, \quad \text{if } x_1 \leq -a_j \quad \text{and} \quad u_j(x) = b, \quad \text{if } x_1 \geq b_j$$

for some positive infinitesimal sequences  $(a_j)$  and  $(b_j)$ .

*Proof.* We divide the proof in two steps.

*Step 1.* Fix  $A \in \mathcal{A}(Q)$  with  $A \subset\subset Q$ ,  $\varepsilon, \sigma > 0$  sufficiently small. Let  $\varphi$  given by

$$\varphi(x) = \begin{cases} 0 & x_1 \leq -2\varepsilon - \sigma \\ \text{affine} & -2\varepsilon - \sigma < x_1 < -2\varepsilon \\ 1 & x_1 \geq -2\varepsilon. \end{cases}$$

Obviously we have  $|\nabla\varphi| \leq \frac{1}{\sigma}$ . Let

$$A_\varepsilon = \{x \in \mathbb{R}^n : x_1 < -2\varepsilon - k_1\varepsilon - \sigma\}, \quad B_\varepsilon = \{x \in \mathbb{R}^n : x_1 > -2\varepsilon + \varepsilon k_2\}$$

$$S_\varepsilon = \{x \in \mathbb{R}^n : -2\varepsilon - \varepsilon k_1 - \sigma < x_1 < -2\varepsilon + \varepsilon k_2\}$$

where  $k_1 = \sup_{x \in C} \langle x, \mathbf{e}_1 \rangle$  and  $k_2 = -\inf_{x \in C} \langle x, \mathbf{e}_1 \rangle$ . Let  $u_1, u_2 \in W^{1,1}(\Omega')$  and  $v = \varphi u_1 + (1-\varphi)u_2$ . Then

$$\mathcal{E}_\varepsilon(v, A) = \frac{1}{\varepsilon} \int_{A \cap A_\varepsilon} h(\varepsilon |\nabla u_2| * \rho_\varepsilon) dx + \frac{1}{\varepsilon} \int_{A \cap B_\varepsilon} h(\varepsilon |\nabla u_1| * \rho_\varepsilon) dx + \frac{1}{\varepsilon} \int_{A \cap S_\varepsilon} h(\varepsilon |\nabla v| * \rho_\varepsilon) dx.$$

Taking into account the subadditivity of  $h$  we get

$$\begin{aligned} \frac{1}{\varepsilon} \int_{A \cap S_\varepsilon} h(\varepsilon |\nabla v| * \rho_\varepsilon) dx &\leq \frac{1}{\varepsilon} \int_{A \cap S_\varepsilon} h(\varepsilon (\varphi |\nabla u_1|) * \rho_\varepsilon) dx + \frac{1}{\varepsilon} \int_{A \cap S_\varepsilon} h(\varepsilon ((1-\varphi) |\nabla u_2|) * \rho_\varepsilon) dx \\ &\quad + \frac{1}{\varepsilon} \int_{A \cap S_\varepsilon} h(\varepsilon (|\nabla \varphi| |u_1 - u_2|) * \rho_\varepsilon) dx. \end{aligned}$$

Then

$$\mathcal{E}_\varepsilon(v, A) \leq \mathcal{E}_\varepsilon(u_1, A \cap (B_\varepsilon \cup S_\varepsilon)) + \mathcal{E}_\varepsilon(u_2, A \cap (A_\varepsilon \cup S_\varepsilon)) + \frac{c'}{\sigma} \int_{A \cap S_\varepsilon} |u_1 - u_2| * \rho_\varepsilon dx$$

where we have used  $h(t) \leq c't$  for each  $t \geq 0$ .

*Step 2.* Now let  $(\varepsilon_j)$  be a positive infinitesimal sequence and let  $(v_j)$  be a sequence in  $W^{1,1}(\Omega')$  such that  $v_j \rightarrow u^{a,b}$  in  $L^1(\Omega')$  and

$$\lim_{j \rightarrow +\infty} \mathcal{E}_{\varepsilon_j}(v_j, A) = \mathcal{E}'(u^{a,b}, A).$$

Choosing  $u_1 = v_j$  and  $u_2 = a$  we have, since  $\mathcal{E}_{\varepsilon_j}(u_2, A) = 0$ ,

$$\mathcal{E}_{\varepsilon_j}(\varphi v_j + (1-\varphi)u_2, A) \leq \mathcal{E}_{\varepsilon_j}(v_j, A) + \frac{c'}{\sigma} \int_{\{x_1 < 0\}} |v_j - u_2| * \rho_{\varepsilon_j} dx.$$

By standard properties of the convolution,

$$\int_{\{x_1 < 0\}} |v_j - u_2| * \rho_{\varepsilon_j} dx \leq \|v_j - u_2\|_{L^1(\{x_1 < 0\})} \rightarrow 0$$

as  $j \rightarrow +\infty$ . Therefore, by a diagonal argument, if  $\sigma_h \rightarrow 0$  we can find  $j_h \rightarrow +\infty$  be such that

$$\lim_{h \rightarrow +\infty} \frac{1}{\sigma_h} \int_{\{x_1 < 0\}} |v_{j_h} - u_2| * \rho_{\varepsilon_{j_h}} dx = 0.$$

Thus

$$\limsup_{h \rightarrow +\infty} \mathcal{E}_{\varepsilon_{j_h}}(\varphi v_{j_h} + (1-\varphi)u_2, A) \leq \limsup_{h \rightarrow +\infty} \mathcal{E}_{\varepsilon_{j_h}}(v_{j_h}, A) = \mathcal{E}'(u^{a,b}, A).$$

Setting

$$u_{j_h} = \begin{cases} a & x_1 \leq -2\varepsilon_{j_h} - \sigma_h \\ v_{j_h} & x_1 \geq 0 \end{cases}$$

we easily have  $u_{j_h} \rightarrow u^{a,b}$  in  $L^1(\Omega')$  and  $u_{j_h} = a$  for  $x_1 \leq -a_j$  for a suitable positive infinitesimal sequence  $(a_j)$ . With the same argument one can prove that  $u_{j_h} = b$  for  $x_1 \geq b_j$  for another suitable positive infinitesimal sequence  $(b_j)$ . Thus  $(u_{j_h})$  is optimal and (5.7) hold.  $\square$

**Proposition 5.6.** *We have  $\mathcal{E}'(u^{a,b}, B_1) \geq \mathcal{E}'(u^{a,b}, Q)$ .*

*Proof.* Fix  $\delta \in (0, 1)$ . Let  $(u_j)$  be given by the previous Lemma, applied with  $A = B_1$ . Then  $u_j(x) = a$  if  $x_1 \leq -a_j$ , and  $u_j(x) = b$  if  $x_1 \geq b_j$ , where  $(a_j)$  and  $(b_j)$  are suitable positive infinitesimal sequences. Let  $S_j = (-a_j, b_j) \times \mathbb{R}^{n-1}$ . For  $j$  sufficiently large, we have  $\delta Q \cap S_j \subset \subset B_1$ , from which  $\mathcal{E}_{\varepsilon_j}(u_j, \delta Q \cap B_1) = \mathcal{E}_{\varepsilon_j}(u_j, \delta Q)$ . Then

$$(5.8) \quad \mathcal{E}_{\varepsilon_j}(u_j, B_1) \geq \mathcal{E}_{\varepsilon_j}(u_j, B_1 \cap \delta Q) = \mathcal{E}_{\varepsilon_j}(u_j, \delta Q).$$

Let  $v_j(x) = u_j(\delta x)$ . Then by a simple scaling argument we have  $\mathcal{E}_{\varepsilon_j}(u_j, \delta Q) = \delta^{n-1} \mathcal{E}_{\varepsilon_j/\delta}(v_j, Q)$ . Passing to the limit in (5.8) we get

$$\mathcal{E}'(u^{a,b}, B_1) \geq \delta^{n-1} \liminf_{j \rightarrow +\infty} \mathcal{E}_{\varepsilon_j/\delta}(v_j, Q) \geq \delta^{n-1} \mathcal{E}'(u^{a,b}, Q).$$

We conclude by taking the limit as  $\delta \rightarrow 1^-$ .  $\square$

Now, by an application of the Besicovitch's Differentiation Theorem, we are able to prove the correct estimate from below for the lower  $\Gamma$ -limit of  $\mathcal{E}_{\varepsilon_j}$ . In order to apply such a Theorem, let us consider the set function  $\mathcal{E}'_-(u, \cdot)$ . It is well known that an increasing set function  $\alpha: \mathcal{A}(\Omega) \rightarrow [0, +\infty]$  which satisfies  $\alpha(\emptyset) = 0$ , which is subadditive, superadditive and inner regular, can be extended to a Borel measure on  $\Omega$  (for instance see [18], Th. 14.23). This result can be applied to  $\mathcal{E}'_-(u, \cdot)$ , the subadditivity of  $\mathcal{E}'_-(u, \cdot)$  being the only condition which is not easy to prove, but it can be recovered as in the proof of Prop. 4.3 and Th. 4.6 of [13]; these results are established in the case  $p > 1$ , but the same arguments work if  $p = 1$ .

Denote by  $\mu_u$  the Borel measure on  $\Omega$  which extends  $\mathcal{E}'_-(u, \cdot)$ .

**Lemma 5.7.** *Let  $u \in BV(\Omega)$ . Then  $\mu_u$  is a finite measure.*

*Proof.* Let  $(u_h)$  be a sequence in  $L^1(\Omega)$  converging weakly\* converging to  $u$  in  $BV(\Omega)$ . By definition

$$|Du_h| * \rho_\varepsilon(x) = \int_{C_\varepsilon(x) \cap \Omega} \rho_\varepsilon(x-y) d|Du_h|(y).$$

Since  $Du_h \xrightarrow{*} Du$  as measures, by Fatou's Lemma and taking into account that  $f$  is non-decreasing and continuous, we get

$$(5.9) \quad \liminf_{h \rightarrow +\infty} \frac{1}{\varepsilon} \int_{\Omega} h(\varepsilon |Du_h| * \rho_\varepsilon) dx \geq \frac{1}{\varepsilon} \int_{\Omega} h(\varepsilon \liminf_{h \rightarrow +\infty} |Du_h| * \rho_\varepsilon) dx \geq \frac{1}{\varepsilon} \int_{\Omega} h(\varepsilon |Du| * \rho_\varepsilon) dx.$$

Now let  $u \in BV(\Omega)$  and let  $(u_h)$  be a sequence in  $L^1(\Omega)$  strictly converging to  $u$ . In particular,  $|Du_h| \rightarrow |Du|$  weakly\* as measures (see, for instance, Prop. 3.15 in [3]). Note that that  $D^c u$  vanishes on the sets with finite  $\mathcal{H}^{n-1}$  measure. Moreover, if  $S$  is  $\sigma$ -finite with respect to  $\mathcal{H}^{n-1}$ , then  $\{x \in \Omega : \mathcal{H}^{n-1}(S \cap \partial C_\varepsilon(x)) > 0\}$  is at most countable. Then (see, for instance, Prop. 1.62 in [3]), we have

$$\lim_{h \rightarrow +\infty} |Du_h| * \rho_\varepsilon(x) = |Du| * \rho_\varepsilon(x), \quad \text{a.e. } x \in \Omega.$$

Applying the Dominated Convergence Theorem, we obtain

$$(5.10) \quad \lim_{h \rightarrow +\infty} \frac{1}{\varepsilon} \int_{\Omega} h(\varepsilon |Du_h| * \rho_\varepsilon) dx = \frac{1}{\varepsilon} \int_{\Omega} h(\varepsilon |Du| * \rho_\varepsilon) dx.$$

Combining (5.9) with (5.10) and taking into account that  $\mathcal{E}'_-$  is lower semicontinuous, we have

$$\mathcal{E}'_-(u) \leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} h(\varepsilon |Du| * \rho_\varepsilon) dx.$$

Notice that there exists  $\gamma > 0$  such that  $|C_\varepsilon(x) \cap \Omega| \leq \gamma \varepsilon^n$  for any  $x \in \Omega$ . Denoting by  $M = \sup_C \rho$  and taking into Fubini's Theorem, we get that for sufficiently small  $\varepsilon$ ,

$$\begin{aligned} \int_{\Omega} h(\varepsilon |Du| * \rho_\varepsilon) dx &\leq c' \int_{\Omega} \int_{C_\varepsilon(x) \cap \Omega} \rho_\varepsilon(y-x) d|Du|(y) dx = c' \int_{\Omega} \int_{\Omega} \rho_\varepsilon(y-x) \chi_{C_\varepsilon(x)} dx d|Du|(y) \\ &\leq c' M \int_{\Omega} \int_{\Omega} \frac{|C_\varepsilon(x) \cap \Omega|}{\varepsilon^n} d|Du|(y) \leq c' M \gamma |Du|(\Omega) \end{aligned}$$

and this yields the conclusion.  $\square$

**Proposition 5.8.** *Let  $u \in BV(\Omega)$  and  $A \in \mathcal{A}(\Omega)$ . Then*

$$\mathcal{E}'(u, A) \geq \int_{S_u \cap A} \psi(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1},$$

where

$$\psi(s, \nu) = \frac{1}{\omega_{n-1}} \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} h(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) dx : (u_j) \in W_\nu^{0,s}, \varepsilon_j \rightarrow 0^+ \right\}.$$

*Proof.* For every  $k \in \mathbb{N}$  let  $S_k = \{x \in S_u : |u^+(x) - u^-(x)| > 1/k\}$ . Clearly we have  $\mathcal{H}^{n-1}(S_k) < +\infty$ ; let  $\lambda_k = \mathcal{H}^{n-1} \llcorner S_k$ . Applying the Besicovitch's Differentiation Theorem we deduce that the limit

$$g(x) = \lim_{\varrho \rightarrow 0} \frac{\mu_u(B_\varrho(x))}{\lambda_k(B_\varrho(x))}$$

exists and is finite for  $\lambda_k$ -a.e.  $x \in \Omega$ , and is  $\lambda_k$ -measurable. Moreover, the Radon-Nikodym decomposition of  $\mu_u$  is given by  $\mu_u = g\lambda_k + \mu^s$ , with  $\mu^s \perp \lambda_k$ . By rectifiability for  $\mathcal{H}^{n-1}$ -a.e.  $x \in S_k$  we get

$$\lim_{\varrho \rightarrow 0} \frac{\lambda_k(B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = 1.$$

Thus, for  $\mathcal{H}^{n-1}$ -a.e.  $x_0 \in S_k$  we have, applying Proposition 5.4, Proposition 5.6 and taking into account (5.7),

$$\begin{aligned} g(x_0) &= \lim_{\varrho \rightarrow 0} \frac{\mu_u(B_\varrho(x_0))}{\omega_{n-1}\varrho^{n-1}} = \liminf_{\varrho \rightarrow 0} \frac{\mathcal{E}'_-(u, B_\varrho(x_0))}{\omega_{n-1}\varrho^{n-1}} \\ &\geq \frac{1}{\omega_{n-1}} \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{x_0 + Q_\nu} h(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) dx : (u_j(\cdot - x_0)) \in W_{\nu_u(x_0)}^{u^+(x_0), u^-(x_0)}, \varepsilon_j \rightarrow 0^+ \right\}. \end{aligned}$$

Taking into account (3.8) and (3.9) (which obviously hold for  $h$  instead of  $f$ ) we get

$$\begin{aligned} \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{x_0 + Q_\nu} h(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) dx : (u_j(\cdot - x_0)) \in W_{\nu_u(x_0)}^{u^+(x_0), u^-(x_0)}, \varepsilon_j \rightarrow 0^+ \right\} \\ = \psi(|u^+(x_0) - u^-(x_0)|, \nu_u(x_0)). \end{aligned}$$

Since  $\mu^s$  is non-negative, we deduce that

$$\mathcal{E}'_-(u, A) \geq \int_A \psi(|u^+ - u^-|, \nu_u) d\lambda_k = \int_{S_k \cap A} \psi(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1}.$$

By considering the supremum for  $k \in \mathbb{N}$  we easily obtain

$$\mathcal{E}'_-(u, A) \geq \int_{S_u \cap A} \psi(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1}$$

and the conclusion follows by definition of  $\mathcal{E}'_-$ .  $\square$

**5.3. Proof of the  $\Gamma$ -liminf inequality.** We are ready to prove the  $\Gamma$ -liminf inequality for the family  $(\mathcal{F}_\varepsilon)_{\varepsilon > 0}$ . The main step of the proof consists in combining Proposition 5.3 with Proposition 5.8 and then using a supremum of measures argument.

**Lemma 5.9.** *Let  $\mu$  be as in Lemma 2.5. Let  $\lambda_1, \lambda_2$  be mutually singular Borel measures, and  $\psi_1, \psi_2$  positive Borel functions. Assume that*

$$\mu(A) \geq \int_A \psi_i d\lambda_i$$

for every  $A \in \mathcal{A}(\Omega)$  and  $i = 1, 2$ . Then it holds

$$\mu(A) \geq \int_A \psi_1 d\lambda_1 + \int_A \psi_2 d\lambda_2$$

for every  $A \in \mathcal{A}(\Omega)$ .

*Proof.* Let  $E \subseteq \Omega$  be such that  $\lambda_1(\Omega \setminus E) = 0$  and  $\lambda_2(E) = 0$ . Then we can suppose that  $\psi_1 = 0$  on  $\Omega \setminus E$  and  $\psi_2 = 0$  on  $E$ . Then  $\max\{\psi_1, \psi_2\} = \psi_1 + \psi_2$ . We conclude by applying the Lemma 2.5 with the choice  $\lambda = \lambda_1 + \lambda_2$ .  $\square$

**Proposition 5.10.** *Let  $u \in L^1(\Omega)$  and  $A \in \mathcal{A}(\Omega)$ . Then*

$$\mathcal{F}'(u, A) \geq \int_A \phi(|\nabla u|) \, dx + \int_{S_u \cap A} \theta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1} + c_0 |D^c u|(A).$$

*Proof.* First notice that we can suppose  $u \in GBV(\Omega)$ . Indeed, if  $(\mathcal{F}_{\varepsilon_j}(u_j))$  is bounded and  $u_j \rightarrow u$  in  $L^1(\Omega)$  then  $u \in GBV(\Omega)$ : it suffices to apply Theorem 3.6 to  $u_j^T = -T \vee u_j \wedge T$ , hence we get  $u^T \in BV(\Omega)$  which means  $u \in GBV(\Omega)$ .

Now the key point of the proof is the construction of a suitable family of functions below  $f_{\varepsilon_j}$ .

*Step 1.* Let  $\delta \in (0, 1)$ . We claim that there exists  $t_\delta > 0$  and for any  $h \in \mathbb{N}$  and for any  $\varepsilon > 0$  there exist  $c_h^\delta > 0$ ,  $d_h^\delta < 0$  and  $g_h^\delta: [t_\delta, +\infty) \rightarrow \mathbb{R}$  such that if we let

$$f_\varepsilon^{h,\delta}(t) = \begin{cases} c_h^\delta t + \varepsilon d_h^\delta & \text{if } t \in [0, t_\delta] \\ c_h^\delta t_\delta + \varepsilon d_h^\delta + g_h^\delta(t) & \text{if } t > t_\delta \end{cases}$$

we have:

$$(5.11) \quad \sup_h (c_h^\delta t + d_h^\delta) = (1 - \delta)\phi(t), \quad \forall t \geq 0.$$

$$(5.12) \quad f_\varepsilon(t) \geq f_\varepsilon^{h,\delta}(t), \quad \forall t \geq 0, \forall h \in \mathbb{N}, \text{ for } \varepsilon \text{ sufficiently small.}$$

$$(5.13) \quad f_\varepsilon^{h,\delta} \text{ is continuous, non-decreasing and concave for any } \varepsilon > 0 \text{ and any } h \in \mathbb{N}.$$

$$(5.14) \quad f_\varepsilon^{h,\delta} - \varepsilon d_h^\delta \text{ converges to } (1 - \delta)f \text{ uniformly on compact sets of } [0, +\infty) \text{ as } h \rightarrow +\infty.$$

First of all we point out that

$$(5.15) \quad \lim_{t \rightarrow 0} \frac{f(t)}{t} = c_0.$$

Indeed, by A2 for any  $\sigma \in (0, 1)$  there exist  $t_\sigma, \varepsilon_\sigma > 0$  such that  $f_\varepsilon(t) \leq (1 + \sigma)\varepsilon\phi(t/\varepsilon)$  for each  $t \in [0, t_\sigma]$  and for each  $\varepsilon \in (0, \varepsilon_\sigma]$ . Since  $\phi(s) \leq c_0 s$  for any  $s \geq 0$ , we have  $f_\varepsilon(t)/t \leq (1 + \sigma)c_0$ . By A3 the previous inequality reduces to  $f(t)/t \leq (1 + \sigma)c_0$ . On the other hand there exist  $t'_\sigma, \varepsilon'_\sigma > 0$  such that  $f_\varepsilon(t) \geq (1 - \sigma)\varepsilon\phi(t/\varepsilon)$  for each  $t \in [0, t'_\sigma]$  and for each  $\varepsilon \in (0, \varepsilon'_\sigma]$ . Since  $\phi(s) \geq c_0 s - q$ , for a suitable  $q > 0$ , we have  $f_\varepsilon(t)/t \geq (1 - \sigma)(c_0 t - \varepsilon q)$ . We thus get  $f(t)/t \geq (1 - \sigma)c_0$ . By the arbitrariness of  $\sigma > 0$  we have (5.15).

Formula (5.15) is useful in order to construct the family  $(f_\varepsilon^{h,\delta})$  as follows. By A2 there exists  $t_\delta > 0$  such that  $f_\varepsilon(t) \geq (1 - \delta)\varepsilon\phi(t/\varepsilon)$  for each  $t \in [0, t_\delta]$  and for each  $\varepsilon$  sufficiently small. Fix  $h \in \mathbb{N}$  with  $h > 0$  and let  $(\ell_h)_{h \in \mathbb{N}}$  be a family of affine functions such that  $\sup_h \ell_h(t) = \phi(t)$  for any  $t \geq 0$  (recall that  $\phi$  is convex); we let  $\ell_h(t) = c_h t + d_h$ . Let  $c_h^\delta = (1 - \delta)c_h$  and  $d_h^\delta = (1 - \delta)d_h$ . Then (5.11) holds and we obtain  $f_\varepsilon(t) \geq c_h^\delta t + \varepsilon d_h^\delta$  for all  $t \in [0, t_\delta]$ . Now it is easy to conclude the construction of  $f_\varepsilon^{h,\delta}$  in such a way (5.12), (5.13) and (5.14) hold: for instance connecting the graphic of the affine piece with a suitable rotation and truncation of the graph of  $f$  (see also (5.15)).

*Step 2.* Let  $\delta \in (0, 1)$  and let  $(f_{\varepsilon_j}^{h,\delta})$  be the family constructed in step 1. Let  $\psi_h^\delta = f_{\varepsilon_j}^{h,\delta} - \varepsilon_j d_h^\delta$ . Then we get

$$(5.16) \quad \mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_A \psi_h^\delta(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)) \, dx + d_h^\delta |A|$$

for any  $u \in W^{1,1}(\Omega)$  and  $A \in \mathcal{A}(\Omega)$ . Let  $A', A''$  be open disjoint subsets of  $A$  such that  $|A''| < \delta$ ,  $S_u \subset A''$ . Therefore,

$$(5.17) \quad \mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_{A'} \psi_h^\delta(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)) \, dx + \frac{1}{\varepsilon_j} \int_{A''} \psi_h^\delta(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)) \, dx + d_h^\delta |A'| + \delta d_h^\delta.$$

In particular we get

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_{A'} \psi_h^\delta(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)) \, dx + d_h^\delta |A'|$$

Notice that  $\psi_h^\delta$  is linear in  $[0, t_\delta]$ . Applying Proposition 5.3 with the choice  $g = \psi_h^\delta \wedge \psi_h^\delta(t_\delta)$  we obtain

$$\mathcal{F}'(u, A) \geq c_h^\delta \int_{A'} |\nabla u| \, dx + c_h^\delta |D^c u|(A) + d_h^\delta |A'| = (1 - \delta) \int_{A'} \ell_h(|\nabla u|) \, dx + (1 - \delta) c_h |D^c u|(A').$$

Since  $\mathcal{F}'(u, \cdot)$  is a superadditive function on open sets of  $\Omega$  with disjoint compact closures, by applying Lemma 2.5 and (5.11) we get, by the arbitrariness of  $A'$  and  $\delta$ ,

$$(5.18) \quad \mathcal{F}'(u, A) \geq \int_A \phi(|\nabla u|) \, dx + c_0 |D^c u|(A).$$

Now (5.17) implies also

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_{A''} \psi_h^\delta(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)) \, dx.$$

Applying now Proposition 5.8 with the choice  $h = \psi_h^\delta$  we deduce that

$$\mathcal{F}'(u, A) \geq \int_{S_u \cap A''} \theta_h^\delta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1},$$

being

$$\theta_h^\delta(s, \nu) = \frac{1}{\omega_{n-1}} \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} \psi_h^\delta(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) \, dx : (u_j) \in W_\nu^{0,s}, \varepsilon_j \rightarrow 0^+ \right\}.$$

Using (5.14) and the arbitrariness of  $\delta$ , it follows that  $\theta_h^\delta \rightarrow \theta$  as  $h \rightarrow +\infty$  and  $\delta \rightarrow 0$ . Applying once more Lemma 2.5, by the arbitrariness of  $A''$ , we have

$$(5.19) \quad \mathcal{F}'(u, A) \geq \int_{S_u \cap A} \theta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1}.$$

Applying Lemma 5.9 choosing  $\lambda_1 = \mathcal{L}^n$ ,  $\lambda_2 = \mathcal{H}^{n-1} \llcorner J_u$ ,  $\lambda_3 = |D^c u|$  and taking into account (5.18) and (5.19), we immediately obtain  $\mathcal{F}'(u) \geq \mathcal{F}(u)$  for any  $u \in BV(\Omega)$ .

Let us now consider the case  $u \in GBV(\Omega)$ . Let  $(u_j)$  be a sequence in  $W^{1,1}(\Omega)$  converging to  $u$  in  $L^1(\Omega)$  and such that

$$\lim_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}(u_j) = \mathcal{F}'(u).$$

Define  $u_j^T = (-T) \vee u_j \wedge T$ , and  $u^T = (-T) \vee u \wedge T$ . Since  $u_j^T \rightarrow u^T$  in  $L^1(\Omega)$ , and  $u^T \in BV(\Omega)$ , we have

$$\mathcal{F}'(u) = \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}(u_j) \geq \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}(u_j^T) \geq \mathcal{F}(u^T).$$

Applying (2.2), (2.3) and (2.4) and taking into account the continuity of  $\theta$  we obtain

$$\lim_{T \rightarrow +\infty} \left( \int_{\Omega} \phi(|\nabla u^T|) \, dx + \int_{S_{u^T}} \theta(|(u^T)^+ - (u^T)^-|, \nu_{u^T}) \, d\mathcal{H}^{n-1} + c_0 |D^c u^T|(\Omega) \right) = \mathcal{F}(u)$$

so we are done.  $\square$

## 6. THE Γ-LIMSUP INEQUALITY

In this section we will prove that  $\mathcal{F}''(u) \leq \mathcal{F}(u)$  for any  $u \in L^1(\Omega)$ ; since, by definition,  $\mathcal{F}(u) = +\infty$  for any  $u \in L^1(\Omega) \setminus GBV(\Omega)$ , it is sufficient to consider the case  $u \in GBV(\Omega)$ .

**Lemma 6.1.** *Let  $(\varepsilon_j)$  be a positive infinitesimal sequence,  $\nu \in \mathbb{S}^{n-1}$  and  $s \geq 0$ . Let  $(u_j) \in W_\nu^{0,s}$  be such that*

$$\omega_{n-1} \theta(s, \nu) = \lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} f(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) \, dx.$$

Then for any  $r > 0$  there exists a positive infinitesimal sequence  $\sigma_j$  and  $(v_j) \in W_\nu^{0,s}$  such that for any  $\sigma > 0$  it holds

$$\omega_{n-1} r^{n-1} \theta(s, \nu) = \lim_{j \rightarrow +\infty} \frac{1}{\sigma_j} \int_{rQ_\nu^\sigma} f(\sigma_j |\nabla v_j| * \rho_{\sigma_j}) dx,$$

where  $Q_\nu^\sigma = \{x \in Q_\nu : |\langle x, \nu \rangle| < \sigma\}$ .

*Proof.* Let  $\sigma_j = r\varepsilon_j$  and  $v_j(x) = u_j(rx)$ . Then by the change of variables  $x = rz$  and  $y = rt$  we get

$$\begin{aligned} \frac{1}{\sigma_j} \int_{rQ_\nu^\sigma} f(\sigma_j |\nabla v_j| * \rho_{\sigma_j}) dx &= \frac{r^n}{\sigma_j} \int_{Q_\nu} f\left(\frac{\sigma_j}{r} \int_{C_{\sigma_j/r}} |\nabla v_j(rz - rt)| \rho_{\sigma_j/r}(t) dt\right) dz \\ &= \frac{r^{n-1}}{\varepsilon_j} \int_{Q_\nu} f\left(\varepsilon_j \int_{C_{\varepsilon_j}} |\nabla u_j(z - t)| \rho_{\varepsilon_j}(t) dt\right) dz. \end{aligned}$$

Passing to the limit as  $j \rightarrow +\infty$  we get

$$\lim_{j \rightarrow +\infty} \frac{1}{\sigma_j} \int_{rQ_\nu^\sigma} f(\sigma_j |\nabla v_j| * \rho_{\sigma_j}) dx = r^{n-1} \theta(s, \nu).$$

Since the transition set of the optimal sequence  $(u_j)$  shrinks onto the interface (see (5.7) or the definition of  $W_\nu^{0,s}$ ) we deduce that

$$\lim_{j \rightarrow +\infty} \frac{1}{\sigma_j} \int_{rQ_\nu^\sigma} f(\sigma_j |\nabla v_j| * \rho_{\sigma_j}) dx = \lim_{j \rightarrow +\infty} \frac{1}{\sigma_j} \int_{rQ_\nu^\sigma} f(\sigma_j |\nabla v_j| * \rho_{\sigma_j}) dx$$

for any  $\sigma > 0$ , hence we conclude.  $\square$

**Proposition 6.2.** *For any  $u \in \mathcal{W}(\Omega)$  it holds  $\mathcal{F}''(u) \leq \mathcal{F}(u)$ .*

*Proof.* By the very definition of  $\mathcal{W}(\Omega)$  (see paragraph 2.5) the set  $S_u$  is contained in the union of a finite collection  $K_1, \dots, K_m$  of  $(n-1)$ -dimensional simplexes; it will not be restrictive to assume  $m = 1$  and  $K = K_1 \subseteq \{x \in \mathbb{R}^n : x_1 = 0\}$ . Fix  $h \in \mathbb{N}$ ,  $h \geq 1$ . Let  $\Omega_h = \{x \in \Omega \setminus K : d(x, K) > 1/h\}$ . Let  $S$  be the relative boundary of  $K$ ; obviously it holds  $\mathcal{H}^{n-1}(S) = 0$ . Let  $K_h = \{x \in K : d(x, S) > 1/h\}$ . Let  $k \in \mathbb{N}$ ,  $k \geq 1$ ,  $x_1, \dots, x_k \in K_h$  and  $r \geq 0$  be such that  $B_r(x_i)$  are pairwise disjoint,  $B_r(x_i) \cap \{x_1 = 0\} \subseteq K_h$  for any  $i = 1, \dots, k$  and

$$(6.1) \quad \mathcal{H}^{n-1}\left(K_h \setminus \left(\bigcup_{i=1}^k B_r(x_i) \cap \{x_1 = 0\}\right)\right) < \frac{1}{h}.$$

Let  $Q_h = \{x \in rQ_{\mathbf{e}_1} : |x_1| < 1/h\}$  and  $Q_h(x) = x + Q_h$  for any  $x \in \mathbb{R}^n$ . Moreover, let  $Q_h^+ = Q_h \cap \{x_1 > 0\}$  and  $Q_h^- = Q_h \cap \{x_1 < 0\}$ . At this point we divide the proof in two steps.

*Step 1.* Take a function  $v \in \mathcal{W}(\Omega)$  with  $S_v \subseteq K$  and such that  $v$  is constant in any  $x_i + Q_h^+$  and in any  $x_i + Q_h^-$ . Denote by  $v_i^+$  the value of  $v$  in  $x_i + Q_h^+$  and by  $v_i^-$  the value of  $v$  in  $x_i + Q_h^-$ . We claim that

$$(6.2) \quad \mathcal{F}''(v) \leq \int_{\Omega} \phi(|\nabla v|) dx + \sum_{i=1}^k \int_{K \cap B_r(x_i)} \theta(|v_i^+ - v_i^-|, \mathbf{e}_1) d\mathcal{H}^{n-1} + c|Dv|(\Omega'_h),$$

for some  $c > 0$ , where

$$\Omega'_h = \Omega \setminus \left(\Omega_h \cup \bigcup_{i=1}^k (x_i + Q_h)\right).$$

Let  $(\varepsilon_j)$  be a positive infinitesimal sequence and let  $\delta \in (0, 1)$ . Accordingly to Lemma 6.1, let us define  $v_j \in \mathcal{W}(\Omega)$  be such that we have

$$(6.3) \quad \lim_{j \rightarrow +\infty} \mathcal{F}_{\sigma_j}(v_j, x_i + \delta Q_h) = (\delta r)^{n-1} \theta(|v_i^+ - v_i^-|, \mathbf{e}_1),$$



where  $\sigma_j = r\varepsilon_j$ . Otherwise in  $\Omega$  we set  $v_j = v$ . Then, using the same argument as in the proof of Lemma 5.7, we deduce that

$$(6.4) \quad \frac{1}{\sigma_j} \int_{\Omega} f_{\sigma_j}(\sigma_j |\nabla v_j| * \rho_{\sigma_j}) \, dx \leq \mathcal{F}_{\sigma_j}(v, \Omega_h) + \sum_{i=1}^k \mathcal{F}_{\sigma_j}(v_j, x_i + \delta Q_h) + c|Dv|(\Omega'_{h,\delta}),$$

being

$$\Omega'_{h,\delta} = \Omega \setminus \left( \Omega_h \cup \bigcup_{i=1}^k (x_i + \delta Q_h) \right).$$

The first term on the right-hand side of (6.4) is given by

$$\frac{1}{\sigma_j} \int_{\Omega_h} f_{\sigma_j}(\sigma_j |\nabla v| * \rho_{\sigma_j}) \, dx.$$

By standard properties of the convolution we have  $|\nabla v| * \rho_{\sigma_j} \rightarrow |\nabla v|$  in  $L^1(\Omega)$  and a.e. in  $\Omega$ . From A2 we deduce that

$$(6.5) \quad \lim_{\varepsilon \rightarrow 0} \frac{f_{\varepsilon}(\varepsilon t_{\varepsilon})}{\varepsilon} = \phi(t)$$

whenever  $t_{\varepsilon} \rightarrow t$ , for each  $t \geq 0$ . By the Dominated Convergence Theorem we get

$$\lim_{j \rightarrow +\infty} \frac{1}{\sigma_j} \int_{\Omega_h} f_{\sigma_j}(\sigma_j |\nabla v| * \rho_{\sigma_j}) \, dx = \int_{\Omega_h} \phi(|\nabla v|) \, dx \leq \int_{\Omega} \phi(|\nabla v|) \, dx.$$

Passing to the limsup in (6.4), using (6.3) and using the arbitrariness of  $\delta \in (0, 1)$  we get (6.2).

*Step 2.* For any  $i = 1, \dots, k$  let

$$u_i^+ = \int_{B_r(x_i) \cap K} u^+ \, d\mathcal{H}^{n-1}, \quad u_i^- = \int_{B_r(x_i) \cap K} u^- \, d\mathcal{H}^{n-1}$$

and

$$u_i(x) = \begin{cases} u_i^+ & \text{if } (x_i)_1 - x_1 > 0 \\ u_i^- & \text{if } (x_i)_1 - x_1 \leq 0 \end{cases}, \quad x \in B_r(x_i).$$

For any  $h \in \mathbb{N}$ ,  $h \geq 1$ , let  $u_h = u_i$  on  $Q_h(x_i)$  and  $u_h = u$  otherwise in  $\Omega$ . Applying step 1 with the choice  $v = u_h$  we get

$$\mathcal{F}''(u_h) \leq \int_{\Omega} \phi(|\nabla u|) \, dx + \sum_{i=1}^k \int_{K \cap B_r(x_i)} \theta(|u_i^+ - u_i^-|, \mathbf{e}_1) \, d\mathcal{H}^{n-1} + c|Du|(\Omega'_h).$$

Now  $|\Omega'_h| \rightarrow 0$ . Furthermore, taking into account (6.1) we deduce that  $\mathcal{H}^{n-1}(S_u \cap \Omega'_h) \rightarrow 0$  as  $h, k \rightarrow +\infty$ . Hence  $|Du|(\Omega'_h) \rightarrow 0$  as  $h, k \rightarrow +\infty$ . Exploiting the uniform continuity of the traces of  $u$  and the continuity of  $\theta$ , we also get

$$\sum_{i=1}^k \int_{K \cap B_r(x_i)} \theta(|u_i^+ - u_i^-|, \mathbf{e}_1) \, d\mathcal{H}^{n-1} \xrightarrow{h, k \rightarrow +\infty} \int_{S_u} \theta(|u^+ - u^-|, \mathbf{e}_1) \, d\mathcal{H}^{n-1}$$

and the lower semicontinuity of  $\mathcal{F}''$  yields the conclusion.  $\square$

**Proposition 6.3.** *Let  $u \in GBV(\Omega)$ . Then it holds  $\mathcal{F}''(u) \leq \mathcal{F}(u)$ .*

*Proof.* First let  $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ . We can apply Theorem 2.6, choosing

$$\psi(a, b, \nu) = \theta(|a - b|, \nu)$$

(see (3.6) and (3.7)). Then there exists a sequence  $w_j \rightarrow u$  in  $L^1(\Omega)$ , with  $w_j \in \mathcal{W}(\Omega)$ , such that  $\nabla w_j \rightarrow \nabla u$  strongly in  $L^2(\Omega, \mathbb{R}^n)$  and

$$(6.6) \quad \limsup_{j \rightarrow +\infty} \int_{S_{w_j}} \theta(|w_j^+ - w_j^-|, \nu_{w_j}) \, d\mathcal{H}^{n-1} \leq \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \, d\mathcal{H}^{n-1}.$$

By the lower semicontinuity of  $\mathcal{F}''$  and by Proposition 6.2 we deduce that, applying the Dominated Convergence Theorem and (6.6),

$$\mathcal{F}''(u) \leq \liminf_{j \rightarrow +\infty} \mathcal{F}''(w_j) \leq \int_{\Omega} \phi(|\nabla u|) dx + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1}.$$

Using relaxation Theorem 2.7 we get

$$\mathcal{F}''(u) \leq \int_{\Omega} \phi(|\nabla u|) dx + \int_{J_u} \theta(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1} + c_0 |D^c u|(\Omega)$$

for each  $u \in BV(\Omega)$ . Finally, let  $u \in GBV(\Omega)$  and, for any  $T > 0$ ,  $u^T = -T \vee u \wedge T$ . Then  $u^T \in BV(\Omega)$  for each  $T > 0$  and  $u^T \rightarrow u$  in  $L^1(\Omega)$  as  $T \rightarrow +\infty$ . Taking into account (2.2), (2.3) and (2.4) we obtain, exploiting again the lower semicontinuity of  $\mathcal{F}''$  and the continuity of  $\theta$ ,

$$\begin{aligned} \mathcal{F}''(u) &\leq \limsup_{T \rightarrow +\infty} \left( \int_{\Omega} \phi(|\nabla u^T|) dx + \int_{S_{u^T}} \theta(|(u^T)^+ - (u^T)^-|, \nu_{u^T}) d\mathcal{H}^{n-1} + c_0 |D^c u^T|(\Omega) \right) \\ &= \int_{\Omega} \phi(|\nabla u|) dx + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1} + c_0 |D^c u|(\Omega) \end{aligned}$$

which is what we wanted to prove.  $\square$

## 7. COMPUTATION OF $\theta$ IN THE ONE-DIMENSIONAL CASE

In this section we are able to give an explicit formula for  $\theta$  if  $n = 1$  along the same line of the discretization argument used in [22].

Let  $n = 1$ , then we can set  $\Omega = (a, b)$ ,  $C = I$  to be an open interval around 0,  $\rho: I \rightarrow (0, +\infty)$  continuous and bounded with

$$\int_I \rho dt = 1.$$

For any  $\varepsilon > 0$  let  $\rho_\varepsilon(t) = 1/\varepsilon \rho(t/\varepsilon)$  and  $I_\varepsilon(x) = x + \varepsilon I$ .

**Theorem 7.1.** *It holds*

$$\theta(s) = \int_{-\infty}^{+\infty} f(s\rho(t)) dt.$$

*Proof.* In the one-dimensional setting the expression for  $\theta$  given by (3.5) reads

$$\theta(s) = \inf \left\{ \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{-1}^1 f(\varepsilon_j |u'_j| * \rho_{\varepsilon_j}) dt : (u_j) \in W^{0,s}, \varepsilon_j \rightarrow 0^+ \right\},$$

being  $W^{0,s}$  the space of all sequences  $(u_j)$  in  $W_{\text{loc}}^{1,1}(\Omega')$ ,  $(-1, 1) \subset \Omega'$ , such that  $u_j \rightarrow s\chi_{(0,+\infty)}$  in  $L^1(\Omega')$ , and such that there exist two positive infinitesimal sequences  $(a_j), (b_j)$  with  $u_j(t) = 0$  if  $t < -a_j$  and  $u_j = s$  if  $t > b_j$ . Let  $(u_j) \in W^{0,s}$  and

$$v_j(t) = \int_{-1}^t (u'_j(r))^+ dr.$$

Moreover, let  $w_j = 0 \vee v_j \wedge s$ . Then  $(w_j) \in W^{0,s}$  and by the change of variables  $y = \varepsilon_j z$  and  $t = \varepsilon_j r$  we get

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{-1}^1 f(\varepsilon_j |u'_j| * \rho_{\varepsilon_j}) dt &\geq \frac{1}{\varepsilon_j} \int_{-1}^1 f\left(\int_{I_{\varepsilon_j}} w'_j(t+y) \rho\left(\frac{y}{\varepsilon_j}\right) dy\right) dt \\ &= \frac{1}{\varepsilon_j} \int_{-1}^1 f\left(\varepsilon_j \int_I w'_j(t + \varepsilon_j z) \rho(z) dz\right) dt = \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\left(\varepsilon_j \int_I w'_j(\varepsilon_j r + \varepsilon_j z) \rho(z) dz\right) dr \\ &= \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\left(\int_I \tilde{w}'_j(r+z) \rho(z) dz\right) dr, \end{aligned}$$

where  $\tilde{w}_j(t) = w_j(\varepsilon_j t)$ . Since  $(w_j) \in W^{0,s}$  then the previous inequality becomes

$$\frac{1}{\varepsilon_j} \int_{-1}^1 f(\varepsilon_j |u'_j| * \rho_{\varepsilon_j}) dt \geq \int_{-\infty}^{+\infty} f\left(\int_I \tilde{w}'_j(t+z)\rho(z) dz\right) dt.$$

Denoting by  $X$  the space of all functions  $v \in W_{\text{loc}}^{1,1}(\mathbb{R})$  which are non-decreasing and such that there exist  $\xi_0 < \xi_1$  with  $v(t) = 0$  if  $t < \xi_0$  and  $v = s$  if  $t > \xi_1$ , we are led to solve the minimization problem  $\inf_X G$ , being

$$G(v) = \int_{-\infty}^{+\infty} f\left(\int_{I(t)} v'(x)\rho(x-t) dx\right) dt, \quad v \in X.$$

By a simple regularization argument it is not restrictive to assume  $f \in C^2(0, +\infty)$  and  $f$  strictly concave. For each  $k \in \mathbb{N}$ , with  $k \geq 1$ , we now consider a discrete version  $G_k$  of  $G$  defined on the space of the functions on  $\mathbb{R}$  which are constant on each interval of the form

$$J_i^k = \left[\frac{i}{k}, \frac{i+1}{k}\right), \quad i \in \mathbb{Z}.$$

We define  $X_k$  as the set of the functions  $v: \mathbb{R} \rightarrow [0, s]$ , such that:

- a)  $v$  is constant on any  $J_i^k$ ; denote by  $v^i$  the value of  $v$  on  $J_i^k$ .
- b)  $v^i \leq v^{i+1}$  for any  $i \in \mathbb{Z}$ .
- c)  $v^i = 0$  if  $i < i_0$  and  $v^i = s$  if  $i > i_1$  for some  $i_0 < i_1$ .

Let  $I^k = \{i \in \mathbb{Z} : J_i^k \subset I\}$ . Finally, let  $G_k: X_k \rightarrow \mathbb{R}$  be defined by

$$G_k(v) = \frac{1}{k} \sum_{i \in \mathbb{Z}} f\left(\sum_{h \in I^k} (v^{i+h+1} - v^{i+h})\rho_h^k\right), \quad \rho_h^k = \int_{J_h^k} \rho(z) dz.$$

Obviously  $G_k$  admit minimizers on  $X_k$ . We claim that each minimizer of  $G_k$  on  $X_k$  takes only the values 0 and  $s$ .

Let  $v$  be a minimizer of  $G_k$  on  $X_k$ . Suppose, by contradiction, that there exists  $i_0 \in \mathbb{Z}$  with  $v^{i_0} = c \in (0, s)$ . We can assume that for a suitable  $r \in \mathbb{N}$  it holds

$$v^{i_0-1} < c, \quad c = v^{i_0} = v^{i_0+1} = \dots = v^{i_0+r}, \quad v^{i_0+r+1} > c.$$

Given  $t \in \mathbb{R}$  sufficiently small, we define  $v_t \in X_k$  letting  $v_t^{i_0+l} = c + t$ , if  $0 \leq l \leq r$ , and  $v_t = v$  otherwise. It is easy to see that for some  $\alpha_i^k, \beta_i^k \neq 0$  which do not depend on  $t$ , we have

$$G_k(v_t) = \frac{1}{k} \sum_{i \in J} f(\alpha_i^k + t\beta_i^k)$$

for some finite set  $J \subset \mathbb{Z}$ . The function  $t \mapsto G_k(v_t)$  is twice continuously differentiable in  $t = 0$ , due to the smoothness of  $f$  and we have

$$\frac{d^2}{dt^2} G_k(v_t) \Big|_{t=0} = \frac{1}{k} \sum_{i \in J} f''(\alpha_i^k) (\beta_i^k)^2 < 0$$

by the strict concavity of  $f$ . This contradicts the fact that  $v$  is a minimizer for  $G_k$  on  $X_k$ .

Since  $G_k$  is invariant under translation, we have already shown that

$$\min_{X_k} G_k = G_k(\hat{v})$$

where  $v = s\chi_{(0, +\infty)}$ . Since

$$G_k(\hat{v}) = \frac{1}{k} \sum_{i \in \mathbb{Z}} f(s\rho_{-i}^k).$$

by the definition of the Riemann integral as the limit of the Riemann sums, we deduce that

$$\liminf_{k \rightarrow +\infty} \min_{X_k} G_k \geq \int_{-\infty}^{+\infty} f(s\rho(t)) dt.$$

Given  $\sigma > 0$  let  $v_\sigma \in X$  be such that  $\inf_X G \geq G(v_\sigma) - \sigma$ . Let  $w_\sigma: \mathbb{R} \rightarrow [0, s]$  given by

$$w_\sigma(t) = w_\sigma^i = \int_{J_i^k} v_\sigma(r) dr, \quad t \in J_i^k.$$

Notice that  $w_\sigma \in X_k$ . Let  $k$  be sufficiently large such that  $G(v_\sigma) \geq G_k(w_\sigma) - \sigma$ . Hence

$$G(v_\sigma) \geq \liminf_{k \rightarrow +\infty} \min_{X_k} G_k - \sigma \geq \int_{-\infty}^{+\infty} f(s\rho(t)) dt - \sigma.$$

By the arbitrariness of  $\sigma > 0$  we obtain

$$\theta(s) \geq \inf_X G \geq \int_{-\infty}^{+\infty} f(s\rho(t)) dt.$$

If we let

$$u_j(t) = \begin{cases} 0 & \text{if } t \leq -\varepsilon_j \\ \frac{s}{\varepsilon_j}t + s & \text{if } t \in (-\varepsilon_j, 0) \\ s & \text{if } t \geq 0 \end{cases}$$

for  $\varepsilon_j \rightarrow 0^+$ , we have  $(u_j) \in W^{0,s}$  and a straightforward computation shows that

$$\lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{-1}^1 f(\varepsilon_j |u_j'| * \rho_{\varepsilon_j}) dt = \int_{-\infty}^{+\infty} f(s\rho(t)) dt$$

and this yields the conclusion.  $\square$

**Remark 7.2.** Observe that when  $I = (-1, 1)$  and  $\rho = \frac{1}{2}\chi_{(-1,1)}$  we get

$$\theta(s) = 2f\left(\frac{s}{2}\right).$$

Hence we recover the case investigated in [21].

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