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Abstract

In this paper we introduce the well-balanced Lévy driven Ornstein-Uhlenbeck process as a moving average process of the form $X_t = \int \exp(-\lambda|t-u|)dL_u$. In contrast to Lévy driven Ornstein-Uhlenbeck processes the well-balanced form possesses continuous sample paths and an autocorrelation function which is decreasing more slowly. Furthermore, depending on the size of λ it allows both for positive and negative correlation of increments. As Ornstein-Uhlenbeck processes X_t is a stationary process starting at $X_0 = \int \exp(-\lambda u)dL_u$. However, by taking a difference kernel we can construct a process with stationary increments starting at zero, which possesses the same correlation structure.

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1 Introduction

Recently moving average processes have attained much attention, both from the theoretical and application side, since they provide a large class of processes, only partly belonging to the class of semimartingales and allowing to model correlation structures including long-range dependence. The theoretical foundations of treating moving average processes with driving Lévy processes have been provided in Rajput and Rosinski (1989) and recently the question under which conditions these type of processes are semimartingales has been considered in Basse and Pedersen (2009). A special case of Lévy driven moving average processes are fractional Lévy motions (cf. Benassi et.al (2004) and Marquardt (2006)), where the kernel function of the fractional Brownian motion is taken, leading to the same correlation structure as fractional Brownian motion. Bender et.al

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(2010) derived conditions on the driving Lévy process and the exponent of the kernel function under which the fractional Lévy motion is a semimartingale. It turns out that this can only be the case in the long memory setting and then the process is of finite variation. Barndorff-Nielsen and Schmiegel (2009) developed the idea of moving average processes further by introducing a stochastic volatility component leading to Brownian semi-stationary processes, which are a very promising class of processes for modelling turbulence. Furthermore, these processes have also been applied to electricity modelling (cf. Barndorff-Nielsen et.al (2010)). However, we can also view the well-known Ornstein-Uhlenbeck process as moving average process, which due to its simple structure is very popular for modelling mean reverting data (e.g. Barndorff-Nielsen and Shephard (2001), Klüppelberg et.al (2009))

Motivated by this we introduce an exponential kernel $\exp(-\lambda|t-\cdot|)$, $\lambda > 0$ on the whole real line leading to the well-balanced Ornstein-Uhlenbeck process. We show that this process is well defined without having to assume further conditions on the driving Lévy process, such as for fractional Lévy motions. The process possess infinitely divisible marginal distributions and is stationary. In contrast to Lévy driven Ornstein-Uhlenbeck processes it possesses continuous sample paths of finite variation and therefore it is a semimartingale with respect to any filtration it is adapted to. Furthermore, the autocorrelation function is decreasing more slowly than the one of the Ornstein-Uhlenbeck process, namely it is of the order $h\exp(-\lambda h)$. In addition the range of the first-order autocorrelation of the increments is $(-0.5, 1)$ in contrast to $(0.5, 0)$ for the Ornstein-Uhlenbeck process. Positive values are often associated to long range dependence, but with the well-balanced Ornstein-Uhlenbeck process we see that this is not true.

Hence the well-balanced Ornstein-Uhlenbeck process might serve as a promising mean process in financial models, e.g. as additive component in stochastic volatility models, since it possesses the following desirable properties:

- the decay of the autocorrelation function is between fast pure exponential decay and long memory,
- the autocorrelation between increments can be positive and negative, depending on λ,
- it is a semimartingale,
- it has an infinitely divisible distribution.

In addition to the well-balanced Ornstein-Uhlenbeck process with the kernel given above we also introduce the process with the corresponding difference kernel $\exp(-\lambda|t-\cdot|) - \exp(-\lambda|\cdot|)$, motivated by the form of the kernel of fractional Brownian motion. This process, in contrast to the previous one, is not stationary, but it possesses stationary increments and starts in zero. Furthermore, the distribution of the squared increments of both processes are obviously equal and the autocorrelation function has the same decay.

Let us give a brief outline on how the paper is organized: in Section 2 we introduce the notation and define the processes, in Section 3 we show that both processes are semimartingales and derive the structure of their characteristics. In Section 4 we provide the moments and correlation structure of the processes. In Section 5 we give a brief empirical example to SAP high frequency data.

2 Definition of the well-balanced Ornstein-Uhlenbeck process

As driving process we consider a Lévy process L given by the characteristic function $E(\exp(iuL_t)) = \exp(t\psi(u))$ with

$$\psi(u) = iu\gamma - \sigma^2\frac{u^2}{2} + \int_{-\infty}^{\infty}\Big(\exp(iux) - 1 - iux1_{|x|\le 1}\Big)\,\nu(dx),$$

where the Lévy measure ν satisfies the integrability condition $\int_{-\infty}^{\infty} 1 \wedge x^2\,\nu(dx) < \infty$.

In the following we give conditions on a kernel function $f(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ such that processes of the form

$$Z_t = \int_{-\infty}^{\infty} f(t,s)\,dL_s, \quad t \ge 0$$

exist. Here L denotes the two-sided version of the Lévy process which is defined in the straight forward way by taking two independent copies $L^{(1)}$ and $L^{(2)}$ and defining

$$L_t := \begin{cases} L_t^{(1)} & \text{if } t \ge 0 \\ -L_{-t-}^{(2)} & \text{if } t < 0. \end{cases}$$

Here and in the following we deal with stochastic integrals on the real line as well as on the positive half line. Integrals on $\mathbb{R}$ are meant in the sense of Rajput and Rosinski (1989), i.e. we associate an independently scattered random measure Λ with the two-sided Lévy process L. For details we refer the reader to Sato (2004) who even treats the more general case of additive processes in law on $[0,\infty)$. The extension to $\mathbb{R}$ is straightforward. Λ is defined on the δ-ring of bounded Borel measurable sets in $\mathbb{R}$ and the integral $\int_{\mathbb{R}} g(s)\,d\Lambda_s$ is introduced in a canonical way for deterministic step functions g. A function f is then called integrable if there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of step functions such that

- $g_n \to f$ a.s with respect to the Lebesgue measure
- $\lim_{n\to\infty}\int_A g_n(s)\,d\Lambda_s$ exists for every $A \in \mathcal{B}(\mathbb{R})$.

If a function f is integrable, we write $\int_{\mathbb{R}} f\,dL_s = \lim_{n\to\infty}\int_A g_n(s)\,d\Lambda_s$. From time to time we will switch between this integral and the classical Itô integral, namely in the case $\int_{\mathbb{R}} 1_{[0,t]}f(s)\,dL_s = \int_0^t f(s)\,dL_s$ for $f \in C_b$. Both integrals coincide for predictable integrands of the type $f(s) = 1_{[0,s)}$. The general case follows by a standard argument using dominated convergence. Before we specify the function $f(t,s)$ let us first briefly look at the setting of a general kernel. Rewriting the criteria of Rajput and Rosinski (1989) for the existence of the integral we obtain: the stochastic integral $\int_{\mathbb{R}} f \, dL_s$ is well defined if for $t \geq 0$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xf(t,s)|^2 \wedge 1 \, \nu(dx) \, ds < \infty$$
$$\int_{-\infty}^{\infty} \sigma^2 f(t,s)^2 \, ds < \infty$$
$$\int_{-\infty}^{\infty} \Big| f(t,s) \Big(\gamma + \int_{-\infty}^{\infty} x \big(1_{|xf(t,s)| \leq 1} - 1_{|x| \leq 1} \, \nu(dx) \big) \Big) \Big| \, ds < \infty$$

(cf. in this context Basse and Pedersen (2009)). Then the characteristic function is given by

$$E(\exp(iuZ_t)) = \exp\left(\int \psi\big(uf(t,s)\big) \, ds \right)$$

and Z_t is infinitely divisible with characteristic triplet $(\gamma_f, \sigma_f^2, \nu_f)$

$$\begin{aligned}
\gamma_f &= \int_{-\infty}^{\infty} f(t,s) \left(\gamma + \int_{-\infty}^{\infty} x \big(1_{|xf(t,s)| \leq 1} - 1_{|x| \leq 1} \, \nu(dx) \big) \right) ds \\
\sigma_f^2 &= \int_{-\infty}^{\infty} \sigma^2 f(t,s)^2 \, ds \\
\nu_f(A) &= (\nu \times \lambda) \Big\{ (x,s) \Big| xf(t,s) \in A \setminus \{0\} \Big\}, \quad A \in \mathcal{B}.
\end{aligned}$$

Furthermore for $u_1, u_2, \cdots, u_m \in \mathbb{R}$ and $-\infty < t_1 < t_2 < \cdots < t_m < \infty$ we obtain

$$E\Big(\exp(\sum_{j=1}^{m} iu_j Z_{t_j}) \Big) = \exp\Big(\int \psi(\sum_{j=1}^{m} u_j f(t_j, s)) \, ds \Big).$$

If we now consider kernels of the form $f(t-s)$ the resulting process Z is stationary since

$$E\big(\exp(iuZ_t) \big) = \exp\Big(\int \psi\big(uf(t-s)\big) \, ds \Big) = \exp\Big(\int \psi\big(uf(x)\big) \, dx \Big).$$

Furthermore it possesses stationary increments, since

$$\begin{aligned}
E\big(\exp(iu(Z_t - Z_s)) \big) &= \exp\Big(\int \psi\big(u(f(t-u) - f(s-u))\big) \, du \Big) \\
&= \exp\Big(\int \psi\big(u(f(t-s+x) - f(x))\big) \, dx \Big).
\end{aligned}$$

If we consider kernels of the form $f(t-s) - f(s)$ we have $Z_0 = 0$ a.s. and stationary increments where the increments have the same distribution as the increments of the process generated by the kernel $f(t-s)$.

If $f(t,.) \in L^2(\mathbb{R})$ and the second moment of L exists and the first one vanishes, we denote $E(L_1^2) = V$, then Z_t also exists in the L^2-sense with isometry

$$EZ_t^2 = ||f(t,.)||_{L^2}^2 V.$$

Now we can come back to our special cases and assume $\lambda > 0$. For the stationary Ornstein-Uhlenbeck process the kernel is $\exp(-\lambda(t-s))1_{(-\infty,t]}(s) = \exp(-\lambda \max(t-s,0))$ which obviously leads to a well defined process. For the well-balanced Ornstein-Uhlenbeck process the kernel is

$$\exp(-\lambda|t-s|) = \exp\Big(-\lambda\big(\max(t-s,0)+\max(-(t-s),0)\big)\Big).$$

From this reformulation we can see why we call the process well-balanced Ornstein-Uhlenbeck process, namely

$$X_t = \int_{-\infty}^{\infty} \exp(-\lambda|t-s|)\, dL_s = \int_{-\infty}^{t} \exp(-\lambda(t-s))\, dL_s + \int_{t}^{\infty} \exp(-\lambda(s-t))\, dL_s$$

which is analogous to the well-balanced fractional Lévy motion (cf. Samorodnitsky and Taqqu (1994), Marquardt (2006)). The initial distribution of X is given by

$$X_0 = \int_{-\infty}^{\infty} e^{\lambda|s|}\, dL_s = \int_{-\infty}^{0} e^{\lambda s}\, dL_s + \int_{0}^{\infty} e^{-\lambda s}\, dL_s.$$

As for the fractional kernel we can construct processes with stationary increments starting from zero, which for the Ornstein-Uhlenbeck process leads to

$$\widetilde{U}_t = \int_{-\infty}^{\infty} \exp(-\lambda \max(t-s,0)) - \exp(-\lambda \max(-s,0))\, dL_s$$

and for the well-balanced Ornstein-Uhlenbeck process to

$$Y_t = \int_{-\infty}^{\infty} \exp(-\lambda|t-s|) - \exp(-\lambda|s|)\, dL_s.$$

Now we can provide the characteristic triplet of the process X.

Lemma 2.1. *The well-balanced Ornstein-Uhlenbeck process*

$$X_t = \int_{-\infty}^{\infty} \exp(-\lambda|t-s|)\, dL_s$$

is well-defined and infinitely divisible with characteristic triplet $(\gamma_X, \sigma_X^2, \nu_X)$

$$\begin{aligned}
\gamma_X &= \frac{2}{\lambda}\gamma + \frac{1}{\lambda}\Big(\int_1^{\infty} \nu(dx) - \int_{-\infty}^{-1} \nu(dx)\Big) \\
\sigma_X^2 &= \frac{1}{\lambda}\sigma^2 \\
\nu_X(A) &= (\nu \times \lambda)\Big\{(x,s)\Big| x\exp(-\lambda|t-s|) \in A \setminus \{0\}\Big\}, \quad A \in \mathcal{B}
\end{aligned}$$

if and only if $\lambda > 0$.

Proof. The result follows by straight forward calculations from the general formulae. □

Here we see that in contrast to fractional Lévy motions the well-balanced Ornstein-Uhlenbeck process is well-defined without imposing further conditions on the driving Lévy process.

3 Semimartingale Property and Characteristics

Since the processes X and Y differ only by a random variable which does not depend on $t \geq 0$, in the following we only treat only X. However, the results remain valid for Y.

In a first step we show that $(X_t)_{t\geq 0}$ is a semimartingale with respect to a suitable filtration. In order to do this we introduce the following decomposition

$$\int_{-\infty}^{\infty} e^{-\lambda|t-s|}\, dL_s = e^{-\lambda t}\int_{-\infty}^{0} e^{\lambda s}\, dL_s + e^{-\lambda t}\int_0^t e^{\lambda s}\, dL_s + e^{\lambda t}\int_t^{\infty} e^{-\lambda s}\, dL_s$$

and write the last term as

$$e^{\lambda t}\int_t^{\infty} e^{-\lambda s}\, dL_s = e^{\lambda t}\int_0^{\infty} e^{-\lambda s}\, dL_s - e^{\lambda t}\int_0^{t} e^{-\lambda s}\, dL_s.$$

For reference purposes we write the above representation of $(X_t)_{t\geq 0}$ in a short form

$$X_t = e^{-\lambda t} G + e^{\lambda t} H + e^{-\lambda t} I_t - e^{\lambda t} J_t, \tag{1}$$

using the following notation:

$$I_t := \int_0^t e^{\lambda s}\, dL_s \text{ and } J_t := \int_0^t e^{-\lambda s}\, dL_s$$

and

$$G := \int_{-\infty}^{0} e^{\lambda s}\, dL_s \text{ and } H := \int_0^{\infty} e^{-\lambda s}\, dL_s.$$

The first part $e^{-\lambda t}G + e^{\lambda t}H$ is very simple because G and H are only random variables which are multiplied with a deterministic process of finite variation. On the other hand even this simple part matters if we are concerned with filtrations. Obviously the natural filtration $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t\geq 0}$ of $L^{(1)}$ is not big enough for $(X_t)_{t\geq 0}$ to be adapted to it, since G and H are in general not measurable with respect to any $\mathcal{F}_t^0$. While it is a simple task to attach an independent random variable, which is the case for G (cf. Corollary 1 to Theorem VI.11 in Protter (2005)), it is much more involved by using the common techniques to show that $L^{(1)}$ is still a semimartingale with respect to $\mathcal{G}_t := \sigma(\mathcal{F}_t \cup \int_0^\infty e^{-\lambda s}\, dL_s)$. For further details in this context compare Chapter VI of Protter (2005) and the references given therein.

We will proceed as follows: using the characteristics of the semimartingale $(e^{-\lambda t}I_t + e^{\lambda t}J_t)_{t\geq 0}$ we show that X is a process of finite variation and hence - a posteriori - a semimartingale with respect to any filtration it is adapted to.

Proposition 3.1. *The process $(e^{-\lambda t}I_t - e^{\lambda t}J_t)_{t\geq 0}$ is a semimartingale with respect to the filtration $\mathbb{F}^0$.*

Proof. Obviously $e^{-\lambda t}$ and $e^{\lambda t}$ are processes of finite variation on compacts. Furthermore $\int_0^t e^{\lambda s}\, dL_s$ and $\int_0^t e^{-\lambda s}\, dL_s$ are $\mathbb{F}^0$-semimartingales by Jacod and Shiryaev (2003) I.4.34. Since the class of semimartingales forms an algebra (cf. Protter (2005) Theorem IV.67), the statement is proved. □

Proposition 3.2. *The process $(X_t)_{t\geq 0}$ is continuous. In particular the third characteristic ν of the semimartingale $(e^{-\lambda t}I_t + e^{\lambda t}J_t)_{t\geq 0}$ is zero.*

Proof. By the representation (1) above and I.4.36 in Jacod and Shiryaev (2003) we obtain:

$$\Delta X_t = \Delta(e^{-\lambda t}I_t + e^{\lambda t}J_t) = e^{-\lambda t}(e^{\lambda t}\Delta L_t) - e^{\lambda t}(e^{-\lambda t}\Delta L_t) = 0$$

for every $t \geq 0$. Here we denote $\Delta X_t = X_t - X_{t-}$. □

Proposition 3.3. *The second characteristic C of the semimartingale $(e^{-\lambda t}I_t + e^{\lambda t}J_t)_{t\geq 0}$ is zero.*

Proof. We use some well known results on the the square- and the angle-bracket:

$$\begin{aligned}[I, I]_t^c = \left[\int_0^\bullet e^{\lambda s}\, dL_s, \int_0^\bullet e^{\lambda s}\, dL_s\right]_t^c &= \int_0^t e^{2\lambda s} d\, [L, L]_s^c \\ &= \int_0^t e^{2\lambda s}\sigma^2 ds.\end{aligned}$$

We write

$$e^{-\lambda t}I_t = \int_0^t e^{-\lambda s}\, dI_s + \int_0^t I_{s-}\, d(e^{-\lambda s}) + [e^{-\lambda^\bullet}, I]_t$$

and, since $(e^{-\lambda t})_{t\geq 0}$ is a process of finite variation on compacts and continuous, we obtain by Jacod and Shiryaev (2003) Proposition I.4.49

$$\left\langle (e^{-\lambda^\bullet} I_\bullet)^c, (e^{-\lambda^\bullet} I_\bullet)^c \right\rangle_t = \left[e^{-\lambda^\bullet} I_\bullet, e^{-\lambda^\bullet} I_\bullet\right]_t^c = \int_0^t e^{-2\lambda s}\, d\, [I, I]_t^c = \sigma^2 t.$$

Analogously we obtain $\left\langle (-e^{\lambda^\bullet} J_\bullet)^c, (-e^{\lambda^\bullet} J_\bullet)^c \right\rangle_t = \sigma^2 t$ and for the cross terms

$$\left\langle (e^{-\lambda^\bullet} I_\bullet)^c, (-e^{\lambda^\bullet} J_\bullet)^c \right\rangle_t = -\sigma^2 t = \left\langle (-e^{\lambda^\bullet} J_\bullet)^c, (e^{-\lambda^\bullet} I_\bullet)^c \right\rangle_t$$

and therefore $C_t = \left\langle (e^{-\lambda^\bullet} I_\bullet - e^{\lambda^\bullet} J_\bullet)^c, (e^{-\lambda^\bullet} I_\bullet - e^{\lambda^\bullet} J_\bullet)^c \right\rangle_t = 0.$ □

Corollary 3.4. *The process $(X_t)_{t\geq 0}$ is of finite variation on compacts and hence it is a semimartingale with respect to* any *filtration it is adapted to.*

Note that by Proposition 3.2 and Jacod and Shiryaev (2003) Proposition I.4.23 the process $(X_t)_{t\geq 0}$ is even a special semimartingale with respect to every filtration it is adapted to.

For the remainder of the paper we fix the filtration $\mathbb{F}$ which is obtained by defining first $\mathbb{F}^1 = (\mathcal{F}_t^1)_{t\geq 0}$ via $\mathcal{F}_t^1 := \sigma(\mathcal{F}_t^0, G, H)$. Which is completed and made right continuous in the usual way to obtain $\mathbb{F}$.

In Basse and Pedersen (2009) and Bender et.al (2010) the authors treat the case of other kernel functions. However, their conditions on the Lévy process are more restrictive.

By our above results we know that the second and third characteristic of X (and Y) are zero. In order to write the first characteristic in an neat form we use the integration-by-parts formula and obtain

$$\int_0^t e^{\lambda s}\, dL_s = -\int_0^t L_s \lambda e^{\lambda s}\, ds + L_t e^{\lambda t}$$

respective

$$-\int_0^t e^{-\lambda s}\, dL_s = -\int_0^t L_s \lambda e^{-\lambda s}\, ds - L_t e^{-\lambda t}.$$

Putting these together we have for the first characteristic $B_t^X = X_t - X_0 = Y_t$

$$\begin{aligned}
B_t^X &= (e^{-\lambda t}-1)\int_{-\infty}^0 e^{\lambda s}\, dL_s + (e^{\lambda t}-1)\int_0^\infty e^{-\lambda s}\, dL_s - e^{-\lambda t}\int_0^t L_s\lambda e^{\lambda s}\, ds - e^{\lambda t}\int_0^t L_s\lambda e^{-\lambda s}\, ds \\
&= \int_0^t \Bigg\{ -\left(\int_{-\infty}^0 e^{\lambda s}\, dL_s\right)\lambda e^{-\lambda s} + \left(\int_0^\infty e^{-\lambda s}\, dL_s\right)\lambda e^{\lambda s} \\
&\quad + \lambda e^{-\lambda s}\int_0^s L_r\lambda e^{\lambda r}\, dr - e^{-\lambda s}L_s\lambda e^{\lambda s} - \lambda e^{\lambda s}\int_0^s L_r\lambda e^{-\lambda r}\, dr - e^{\lambda s}L_s\lambda e^{-\lambda s}\Bigg\}\, ds \\
&= \int_0^t \Bigg\{ -G\lambda e^{-\lambda s} + H\lambda e^{\lambda s} - 2L_s\lambda + \lambda e^{-\lambda s}\int_0^s L_r\lambda e^{\lambda r}\, dr - \lambda e^{\lambda s}\int_0^s L_r\lambda e^{-\lambda r}\, dr\Bigg\}\, ds \\
&= \int_0^t \lambda\Bigg\{ -Ge^{-\lambda s} + He^{\lambda s} + \lambda I_s e^{-\lambda s} - \lambda J_s e^{\lambda s} - 2L_s\Bigg\}\, ds.
\end{aligned}$$

If we consider the vector valued process $S = (X, L, G, H)'$ this is even a diffusion with jumps in the sense of Jacod and Shiryaev (2003) Definition III.2.18 since we have a representation of the characteristics which is of the form

$$B_t^S = \int_0^t b(S_s, s)\, ds, \quad C_t^S = \int_0^t c(S_s, s)\, ds \quad \text{and} \quad \nu(\omega; dt, dx) = dt K_t(S_t(\omega), dx)$$

for measurable $b : [0,\infty) \times \mathbb{R}^4 \to \mathbb{R}^4$, $c : [0,\infty) \times \mathbb{R}^4 \to$ {symmetric nonnegative (4×4)-matrices} and K_t is a Borel transition kernel from $[0,\infty) \times \mathbb{R}^4$ into $\mathbb{R}^4$, with $K_t(x, \{0\}) = 0$. Namely for the first component we get

$$\begin{aligned}
(B_t^S)^{(1)} &= \left(\int_0^t \lambda\Big(\lambda X_t - (\lambda+1)Ge^{-\lambda s} - (\lambda-1)He^{\lambda s} - 2L_s\Big)\, ds\right) \\
&= \left(\int_0^t \lambda\Big(\lambda S_s^{(1)} - (\lambda+1)S_s^{(3)}e^{-\lambda s} - (\lambda-1)S_s^{(4)}e^{\lambda s} - 2S_s^{(2)}\Big)\, ds\right).
\end{aligned}$$

By Theorem 2.26 in Jacod and Shiryaev (2003) we can conclude that S is a solution of the following stochastic differential equation:

$$dS_t = \begin{pmatrix} \lambda\Big(\lambda S_t^{(1)} - (\lambda+1)S_t^{(3)}e^{-\lambda t} - (\lambda-1)S_t^{(4)}e^{\lambda t} - 2S_t^{(2)}\Big) \\ \gamma \\ 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dW_t$$

$$+ x^{(2)} \cdot 1_{|x^{(2)}|\le 1} \left(\mu^L(dt,dx) - dt\nu(dx)\right) + x^{(2)} \cdot 1_{|x^{(2)}|>1}\, \mu^L(dt,dx)$$

with initial distribution $S_0 \sim (G+H, 0, G, H)'$.

Summarizing we can see that though integrating with respect to a general Lévy process the special very regular form of the kernel leads to a semimartingale of bounded variation. Hence the well-balanced Ornstein-Uhlenbeck process might serve as mean process in the framework of semimartingale models, e.g. stochastic volatility models in finance.

4 Moments and Correlation Structure

In this section we will analyze the correlation structure of the well-balanced Ornstein-Uhlenbeck process. We will see that though the process is closely related to the stationary version of an Ornstein-Uhlenbeck process the two-sided kernel leads to a different behaviour in the autocorrelation function, namely to a slower decay than the one of the classical Ornstein-Uhlenbeck process and to a bigger range of possible values, including positive ones, in the first order autocorrelation of increments.

Proposition 4.1. *Let $X_t = \int \exp(-\lambda|t-u|)dL_u$ and assume that the driving Lévy process possesses a finite second moment. We denote it by V and the first moment by μ, then we obtain the following characteristic quantities for X*

$$\begin{aligned}
EX_t &= \frac{2\mu}{\lambda} \\
var(X_t) &= \frac{V}{\lambda} \\
cov(X_{t+h}, X_t) &= Vhe^{-\lambda h} + \frac{V}{\lambda}e^{-\lambda h} \\
corr(X_{t+h}, X_t) &= \lambda he^{-\lambda h} + e^{-\lambda h}.
\end{aligned}$$

Proof. From the general form of the characteristic function, we can calculate the second moment of $Z_t = \int f(t,s)dL_s$, provided that L possesses a second moment and both $f(t,.)$ and $f(t,.)^2$ are integrable. We obtain

$$\begin{aligned}
EZ_t &= \int f(t,s)ds \left(\gamma + \int x1_{|x|>1}\nu(dx)\right) \\
EZ_t^2 &= \int f(t,s)^2 ds \left(\sigma^2 + \int x^2\nu(dx)\right) + \left(\int f(t,s)ds\right)^2 \left(\gamma + \int x1_{|x|>1}\nu(dx)\right)^2.
\end{aligned}$$

In the following we denote $\sigma^2 + \int x^2\nu(dx) = V$ and $\gamma + \int x1_{|x|>1}\nu(dx) = \mu$. Using this together with the independent increment property of L, we obtain for $X_t = \int \exp(-\lambda|t-u|)dL_u$ and $s \le t$

$$\begin{aligned}
EX_t &= \frac{2\mu}{\lambda} \\
EX_t^2 &= \frac{V}{\lambda} + \frac{4\mu^2}{\lambda^2}
\end{aligned}$$

$$\begin{aligned} var(X_t) &= \frac{V}{\lambda} \\ E(X_t - X_s)^2 &= \frac{2V}{\lambda}\big(1 - e^{-\lambda(t-s)} - \lambda(t-s)e^{-\lambda(t-s)}\big). \end{aligned}$$

Hence

$$\begin{aligned} Cov(X_t, X_s) &= \frac{1}{2}\big(EX_t^2 - E(X_t - X_s)^2 + EX_s^2\big) - EX_tEX_s \\ &= V(t-s)e^{-\lambda(t-s)} + \frac{V}{\lambda}e^{-\lambda(t-s)} \\ corr(X_t, X_s) &= \lambda(t-s)e^{-\lambda(t-s)} + e^{-\lambda(t-s)}. \end{aligned}$$

□

Comparing this to the well known quantities of a stationary Ornstein-Uhlenbeck process U we see, while the mean and the variance only differ by a multiple of two, the autocovariance and autocorrelation function have an extra term leading to a slower decay. This might be an interesting feature for modelling data, especially coming from finance, where a pure exponential decay often seem too fast to match the empirical autocorrelation properly.

From the form of the second moment we can easily deduce Hölder continuity of the sample paths.

Corollary 4.2. *Assuming a finite second moment of the driving Lévy process, we obtain that the well-balanced Ornstein-Uhlenbeck process is Hölder continuous of the order γ with $\gamma < 0.5$.*

Proof. Using Taylor expansion we obtain

$$E(X_t - X_s)^2 = 2V\Big(\frac{1}{\lambda} - \frac{\exp(-\lambda(t-s))}{\lambda} - (t-s)\exp(-\lambda(t-s))\Big) = 2\lambda V(t-s)^2 + O((t-s)^3).$$

Hence by Kolmogorov-Centsov X_t is Hölder continuous of the order $\gamma < 1/2$. □

This is also different to the classical Ornstein-Uhlenbeck process which inherits the jump property from the driving process.

Also the correlation between increments might be of interest for modelling purposes and follows by direct calculations from the proposition above.

Corollary 4.3. *Assume the same conditions on L as in the previous proposition, then we obtain*

$$\begin{aligned} Corr(X_{k+1} - X_k, X_1 - X_0) &= \exp(-\lambda k)\left(\frac{1}{2} + \frac{1}{2}\frac{1 - \exp(\lambda) + \lambda\exp(\lambda)}{1 - \exp(-\lambda) - \lambda\exp(-\lambda)}\right) \\ &\quad + \lambda k\exp(-\lambda k)\left(\frac{1}{2} + \frac{1}{2}\frac{1 - \exp(\lambda) + \lambda\exp(-\lambda)}{1 - \exp(-\lambda) - \lambda\exp(-\lambda)}\right) \end{aligned}$$

and as a special case the first-order autocorrelation

$$Corr(X_2 - X_1, X_1 - X_0) = \exp(-\lambda)\left(\frac{1+\lambda}{2} + \frac{1}{2}\frac{1+\lambda-\exp(\lambda)+\lambda^2\exp(-\lambda)}{1-\exp(-\lambda)-\lambda\exp(-\lambda)}\right).$$

Note that in contrast to the classical Ornstein-Uhlenbeck process whose autocorrelation function of increments $Corr(U_{k+1} - U_k, U_1 - U_0) = \exp(-\lambda k)(\frac{1}{2} + \frac{1}{2}\frac{1-\exp(\lambda)}{1-\exp(-\lambda)})$ is always negative in the range between -0.5 and 0, we can have positive and negative values, in the range from -0.5 to 1 depending on λ for the well-balanced Ornstein-Uhlenbeck process. Looking for example at the first-order autocorrelation $Corr(X_2 - X_1, X_1 - X_0)$ it is positive for $\lambda < 1.25643$ and negative for bigger values of λ. This provides much more flexibility for modelling, e.g. we can obtain values for the first-order autocorrelation which is often linked to long-range dependence. Assuming that B_t^H denotes a fractional Brownian motion with Hurst parameter $H \in (0,1)$, then by Kettani and Gubner (2006)

$$\frac{\sum_{i=1}^{n-1}(X_i^H - \bar{X}_n^H)(X_{i+1} - \bar{X}_n^H)}{\sum_{i=1}^{n}(X_i^H - \bar{X}_n^H)^2} \to C_H = 2^{2H-1} - 1,$$

where $X_i^H = B_i^H - B_{i-1}^H$ and $\bar{X}_n^H = \frac{1}{n}\sum_{i=1}^n X_i^H$. Hence we can see that as the first-order autocorrelation of the well-balanced Ornstein-Uhlenbeck process $C_H \in (-0.5, 1)$ and $C_H > 0$ for $H > 0.5$.

For some applications it might of course be more realistic not to have a stationary process, but a process with stationary increments like Lévy processes. In the context of well-balanced Ornstein-Uhlenbeck processes we can construct processes with the same correlation structure of increments and hence the same paths regularity by considering the associated difference kernel.

Proposition 4.4. *Let $Y_t = \int_{-\infty}^{\infty} \exp(-\lambda|t-s|) - \exp(-\lambda|s|)dL_s$ and assume that the driving Lévy process possesses a finite second moment. We denote it by V and the first moment by μ, then we obtain the following characteristic quantities for Y*

$$\begin{aligned}
EY_t &= 0\\
var(Y_t) &= Vte^{-\lambda t} + \frac{V}{\lambda}e^{-\lambda t}\\
Corr(Y_{k+1} - Y_k, Y_1 - Y_0) &= \exp(-\lambda k)\left(\frac{1}{2} + \frac{1}{2}\frac{1-\exp(\lambda)+\lambda\exp(\lambda)}{1-\exp(-\lambda)-\lambda\exp(-\lambda)}\right)\\
&\quad +\lambda k\exp(-\lambda k)\left(\frac{1}{2} + \frac{1}{2}\frac{1-\exp(\lambda)+\lambda\exp(-\lambda)}{1-\exp(-\lambda)-\lambda\exp(-\lambda)}\right).
\end{aligned}$$

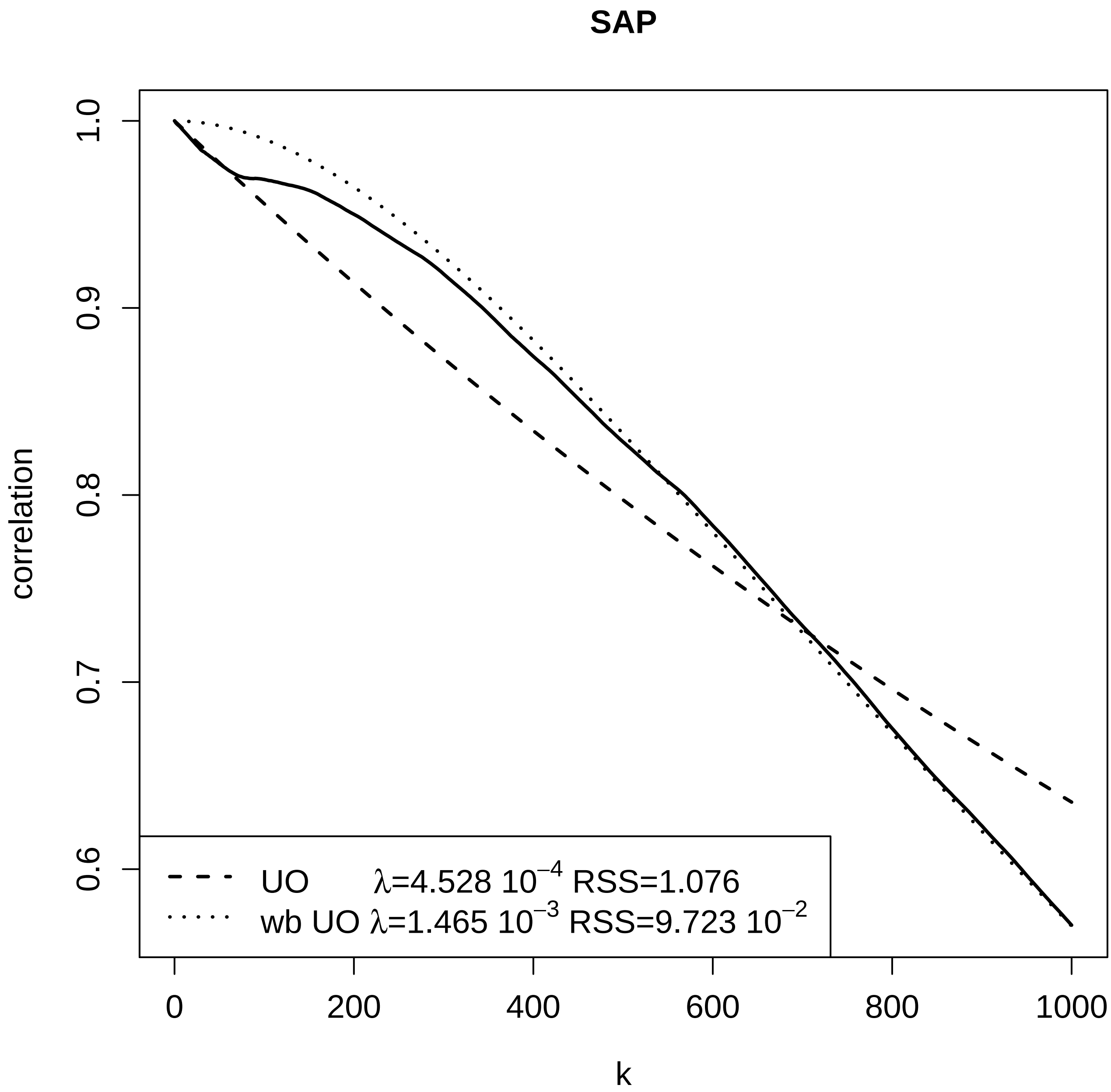
Proof. The proof follows immediately by noting that $Y_t = X_t - X_0$. □

Note that we can easily also construct a process which only possess this correlation structure for a specific lag and is zero for larger lags. For a kernel on a compact interval $[0, a]$ we obtain the process $X_t = \int_{t-a}^{t} \exp(-\lambda(t-s))dL_s$ which possesses the second moment $EX_t^2 = (1 - \exp(-2\lambda a))/(2\lambda)$. Furthermore for increments $X_t - X_s$ we obtain $E(X_t - X_s)^2 = (1 - \exp(-2\lambda a) - \exp(-\lambda(t-s)) + \exp(-\lambda(2a + s - t)))/\lambda$ if $t - s \leq a$ and if $t - s > a$: $E(X_t - X_s)^2 = EX_t^2 + EX_s^2$. This leads to $Cov(X_t, X_s) = (\exp(-\lambda(t - s)) + \exp(-\lambda(2a + s - t)))/(2\lambda)$ for $t - s \leq a$ and 0 otherwise.

5 Application to SAP high frequency data

Finally we apply the well-balanced Ornstein-Uhlenbeck process to an example of real data and show that the autocorrelation models the empirical autocorrelation quite well. Hence this indeed offers the possibility of adding the well-balanced Ornstein-Uhlenbeck process as an empirically convincing mean process to a classical stochastic volatility model.

We consider one trading day of the SAP share, namely of 1st February 2006 9:00 am to 5:30 pm consisting of 5441 trades.



The picture shows the empirical autocorrelation function as solid line, the dashed line is the fit with a classical Ornstein-Uhlenbeck process and the dotted line with the well-balanced Ornstein-Uhlenbeck process. We can see that the autocorrelation function both visually and by taking the residual sum of squares fits the data much better than the Ornstein-Uhlenbeck process, except for small lags. This might be interpreted as the effects of market microstructure. Namely the two kinks in the empirical curve are at a lag of 75 and 150 respectively. In this setting this correspond to a sampling frequency of 7 minutes and 14 minutes. Values in this range are in the econometrics literature often seen as sampling frequencies from which market microstructure effects start to be negligible. If we start fitting the empirical data only for larger lags that 150, the values of λ and the RSS for the Ornstein-Uhlenbeck process stay the same, whereas the RSS of the well-balanced Ornstein-Uhlenbeck processes decreases to $4.393 10^{-3}$.

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References

[1] Barndorff-Nielsen, O.E. and Schmiegel, J. Brownian Semistationary processes and volatility/intermittency *Thiele Research Reports*, No. 04, March 2009, available at http://www.imf.au.dk/publs?publid=699

[2] Barndorff-Nielsen, O.E. and Shephard, N. Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics (with discussion). *J. Roy. Statist. Soc. Ser. B*, **63** (2001): 167–241.

[3] Barndorff-Nielsen, O.E., Benth, F.E. and Veraart, A.E.D. Modelling electricity forward markets by ambit fields. *CREATES Research Paper*, 2010-41, available at http://ideas.repec.org/s/aah/create.html

[4] Basse, A. and Pedersen, J. Lévy driven moving averages and semimartingales. *Stochastic Process. Appl.*, **119** (2009): 2970–2991.

[5] Benassi, A., Cohen, S. and Istas, J. On roughness indexes for fractional fields. *Bernoulli* **10**-2 (2004): 357–373

[6] Bender, C., Lindner, A. and Schicks, M. Finite variation of fractional Lévy processes. submitted article (2010)

[7] Jacod, J. and Shiryaev, A. *Limit Theorems for Stochastic Processes*. Springer, Berlin 2003 (2nd ed).

[8] Kettani, H. and Gubner, J.A. A Novel Approach to the Estimation of the Long-Range Dependence Parameter. *IEEE Transactions on Circuits and Systems II* **53**-6 (2006): 463–467.

[9] Klüppelberg, C., Meyer-Brandis, T., Schmidt, A. (2009) Electricity spot price modelling with a view towards extreme spike risk. *Quantitative Finance*, to appear.

[10] Marquardt, T. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli* **12**-6 (2006): 1099–1126.

[11] Protter, P.: *Stochastic Integration and Differential Equations.* Springer-Verlag, Berlin 2005 (2nd ed).

[12] Rajput, B.S. and Rosinski, J. Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82** (1989): 451–487.

[13] Samorodnitsky, G. and Taqqu, M. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, New York 1994.

[14] Sato, K. Stochastic integrals in additive processes and application to semi-Lévy processes *Osaka J. Math.* **41** (2004): 211–236.