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Coding and Mathematical Competitions

1. Introduction

The number of lessons devoted to exact disciplines is decreasing in many countries. Advanced topics are to go, and high – level mathematical education at school meet danger to become occasional and disintegrated.

In this situation mathematical olympiads appear to be a strong stimulating factor. Olympiad "curricula" hasn't changed; it is developed further in an essential way. The standards that were elaborated in olympiad movement during many years in some sense have become the unofficial standards for advanced education in mathematics.

The spectrum of the mathematics covered by olympiads and other contests gradually becomes wider. This is caused by the fact that the traditional areas are more or less exhausted, and fresh ideas are needed to preserve the element of unexpectedness and creativity. Many general topics coming from theoretical computer science have become popular during last years:

- discrete continuity (see, e.g., [1]),
- problems based on the concept of automata (see, e.g., [2]),
- new types of algorithms (see, e.g., [3]),
- new types of impossibility proofs (see, e.g., [4]),

etc.

Here we consider another class of such problems, closely related to the concepts of coding and protocol.

2. Reflection of classical Results in Coding

There are three problem classes of this type. All of them heavily use the concept of the full tree. The first class is based on the following result.

Craft's lemma

Suppose we have a_1 sequences of length 1, a_2 sequences of length 2, …, a_k sequences of length k, all consisting of the letters from the alphabet of size n, such that none of these sequences is an initial fragment of any other. Then the inequality

$$\frac{a_1}{n}+\frac{a_2}{n^2}+\ldots+\frac{a_k}{n^k}\leq 1 \text{ holds.}$$

Example 1 (Latvia, 1995). There are only two sounds in the language COCOCO. No word is the initial fragment of another one. Is it possible that there are 3 words of length 3, 4 words of length 4, 6 words of length 5, 8 words of length 6 and 9 words of length 7 in this language?

Comment. A number of "constructive" problems can be obtained using the fact that Craft's lemma gives also a sufficient condition for the existence of the set of "nonoverlaping" words.

The second class explores also the concept of Hamming distance.

Example 2. In some army each soldier has a code – a 6-digit sequence consisting of the digits 0 and 1 only. Each two codes must differ in at least two positions. What is the largest possible number of soldiers in this army?

Comment. The idea of the "control digit" is used in the solution.

The third class is based on the classical method of obtaining the so – called "information theory lower bound" in the computational complexity theory.

Example 3. There are 6 equally – looking coins, all with different masses. What is the smallest number of weightings on a pan balance that allows us to arrange the coins in the order of their masses?

Comment. For all these classes of problems a tool suitable on the middle and high school level is Dirichlet Principle and its various generalizations. The great educational value of this kind of problems is convincing the students in the value of general approaches and data structures.

3. "I know You know I know…" Type Problems

As far as we know, the first (folklore) problem of this type used in competitions is the following one.

Example 4. Two old and very clever friends meet each other after many years of dissociation. The following conversation takes place (we give, of course, only the essential part of it):

"A. I have three daughters, you know.

B. How old are they?

A. The product of their ages is 36, while the sum of them is equal to the number of the bus just now passing by.

B. I don't know their ages.

A. The eldest daughter has blue eyes.

B. Now I know how old they are."

The reader is also urged to tell how old they are.

The basis of the solution for this and similar problems is to find out what information is encoded in each statement. To obtain the answer (2; 2; 9), the reader should ask himself: **why** B couldn't find the ages knowing both the sum and the product of them?

The problems of this type usually consist of two pieces of mathematics: the **logical part** and the **specific** algebraic/arithmetical/… part to which the logic must be applied. The differences in the difficulty level of the latter one can be significant. If in the example above the factorization of 36 solves the problem, the next example shows a more sophisticated situation.

Example 5 (proposed by Bulgaria for IMO'91). Each of the boys A and B tells the teacher a positive integer but neither of them knows the other's number. The teacher writes two distinct positive integers on the blackboard and announces that one of them is the sum of the numbers they told him. Then he asks A, "Can you guess the sum of the two numbers?" If the answer is "No," the teacher asks B the same question, and so on. Suppose that the boys are intelligent and truthful. Prove that one of their answers eventually will be "Yes."

For a solution, see [5]. Further examples see, e.g., in [6], [7] and [8].

The analysis of the problems of this type provide to the students also deeper insight into the method of mathematical induction (see, e.g., [9]).

The following example shows a link with multi – valued logics.

Example 7 (V.Ufnarovsky). John has guessed a number from the set $\{1;2;3\}$. Ann is allowed to ask him one question with possible answers "yes", "no", "don't know". Can she find out which number was guessed?

4. Problems exploring the idea of a protocol

After the announcement of the idea of public – key cryptosystem by W.Diffie and M.Hellman a lot of applications of this approach have emerged. May be the first contest problem on high – school level where the idea of a protocol is very clearly exploited is the following one.

Example 8 (Russian olympiad, 1997, folklore).

The Judgment of the Council of Sages proceeds as follows: the king arranges the sages in a line and places either a white hat or a black hat on each sage's head. Each sage can see the color of the hats of the sages in front of him, but not of his own hat or of the hats of the sages behind him. Then one by one (in an order of their choosing), each sage guesses a color. Afterward, the king executes those sages who did not correctly guess the color of their own hat.

The day before, the Council meets and decides to minimize the number of executions. What is the smallest number of sages guaranteed to survive in this case?

For the solution see [10], p.94.

Most surprisingly, this idea seems to have passed almost unnoticed, and only few examples are known to the author.

5. Conclusions

It seems that problems of the considered type have a great future in high school students' competitions, not only in problem – solving ones. Not speaking about their significance in the context of growing role of discrete mathematics, they provide rich possibilities for students' research, especially if combining them with the use of nondeterministic, probabilistic and other non – classical types of algorithms. They are closely related also to contests in informatics and linguistics, thus making an additional bridge between these disciplines and mathematics on a high – school level.

Literature

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