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The notion of regulated function in the calculus teaching by Professor Igor Kluvánek

Everybody familiar with basic calculus remembers properties of continuous functions defined on a bounded closed interval I . Some of the properties can be extended to suitable discontinuous function, namely to functions having right and left limits in each point of I , such functions are called regulated. We shall deal with a special class of regulated functions consisting of piece-wise functions.

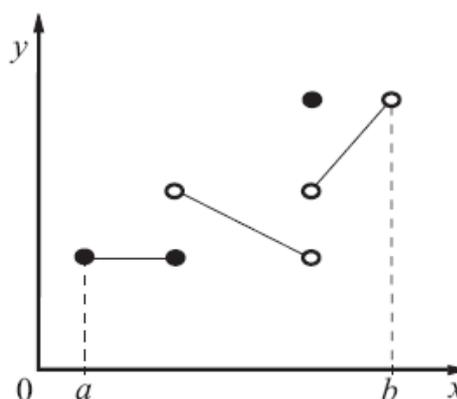
Introduction

Definition 1: A function f is said to be regulated on an interval I if

- the function f has the left limit at every point of the interval I except the left end point and
- the function f has the right limit at every point of the interval I except the right end point.

In this definition, we do not require that the right limit and the left limit of the function at a point are the same. The function is not required to be defined at every point of the interval I .

A graph of a regulated function on $I = \langle a; b \rangle$.



Properties of the regulated functions

Lemma 1: Each function continuous on an interval I is regulated on I .

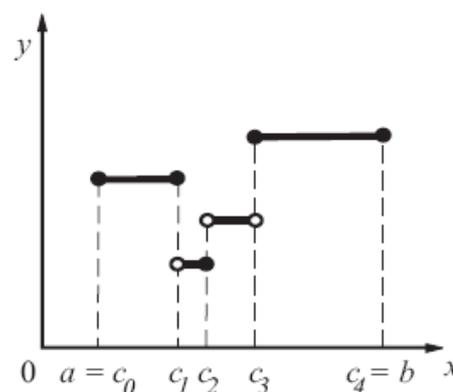
Lemma 2: Each monotonic and bounded function on an interval I is regulated on I .

Theorem 1: Let f be a regulated function on a bounded and closed interval I . Then f is bounded on it.

An important class of a regulated function consists of a piece-wise constant ones.

Definition 2: A function f is said to be piece-wise constant on an interval I with end points a and b , $a < b$, (we admit $a = -\infty$ and $b = \infty$), if there is natural number n and interior points $c_j, j = 1, 2, \dots, n - 1$ of the interval I such that $c_{j-1} < c_j$, for $j = 2, 3, \dots, n - 1$, and if $c_0 = a$ and $c_n = b$, then the function f is constant on each of the interval (c_{j-1}, c_j) , for $j = 1, 2, \dots, n$.

Because of its shape piece-wise constant functions are also called step functions. It follows from the definition that if f is regulated on a interval I , then it is also regulated in each interval J ($J \subseteq I$).



The following theorem states that a function regulated on a bounded closed interval can be replaced with arbitrary accuracy by a step function.

Theorem 2 : For a and b be number such that $a < b$ let f be a function regulated on the interval $\langle a; b \rangle$ and let ε be a positive number. Then there exists a step function g such that

$$|f(x) - g(x)| < \varepsilon$$

for each point x of the interval $\langle a; b \rangle$ belonging to the domain of f . If the function f is continuous on the interval $\langle a; b \rangle$, then we can choose the function g to be right-continuous at each point of the interval $\langle a; b \rangle$, or left-continuous at each point of the interval $(a; b)$.

This theorem has numerous applications. It says, among other things, that for each regulated function on a compact interval (bounded and closed interval) an arbitrarily accurate table exists.

x	\sqrt{x}
20	4,4721
21	4,5826
22	4,6904
23	4,7958
24	4,8990
25	5,0000
26	5,0990
27	5,1962
28	5,2915
29	5,3852
30	5,4772

Example : In the Table, the values of the function \sqrt{x} at the integers in the interval $\langle 20; 30 \rangle$ are given. So, in this case $f(x) = \sqrt{x}$, $a = 20$, $b = 30$, $n=10$, $x_0 = a = 20$, $x_1 = 21$, $x_2 = 22$, and so on, $x_9 = 29$, $x_{10} = 30$. $y_0 = 4,4721$, $y_1 = 4,5826$, and so on, $y_9 = 5,3852$, $y_{10} = 5,4772$. For example, the approximate value of \sqrt{x} at 27 is 5,1962, etc.

To determine approximately $\sqrt{26,4}$, we use the described procedure and find out from the Table that $\sqrt{26,4}$ is approximately 5,0990. Similarly, the approximate value of \sqrt{x} at 27,8 is 5,2915, the approximate value of \sqrt{x} at 21,5 is 4,5826.

It is now apparent that a table of a function f in an interval $\langle a; b \rangle$ together with the described procedure for determining approximate values of f represents a step function g defined on the interval $\langle a; b \rangle$. The number which this procedure produces as an approximate value of the function f at a point $x \in \langle a, b \rangle$ is equal to the value of the step function g at x . If the rule of taking y_{j-1} as an approximate value of f at $\frac{x_{j-1} + x_j}{2}$, for every $j = 1, 2, \dots$, n is adopted, then the corresponding step function g is left-continuous at each point of the interval $\langle a; b \rangle$.

Conclusion

Generalization of the properties any continuous function on the interval $\langle a; b \rangle$ leads to the notion of the regulated function. First advance of this type of functions are application in different approximation regulated function by piece-wise function. Second advance is that the regulated function is integrable in the interval $\langle a; b \rangle$. These properties we can use the effective methods for actual calculation of integral.

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