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Solving 2 and 3-dimensional problems with help of dynamical geometry software

Introduction

The Hungarian tradition - to solve 3-dimensional geometry problem on synthetical way - can be supported by 3D dynamical geometry softwares and suitable worksheet. This method makes possible to balance with time pressure at the secondary mathematics education. In order to show the effectivity of this method we chose a complex and interesting 3-dimensional problem. There is a kind of complexity which is not based on high mathematical knowledge, but on manipulating with 3-dimensional objects. The connection with the perspective drawing in renaissance art makes the problem interesting.

Below we describe the parallel consideration in 3D geometry and its Cabri 3D implementation. The steps of the solution are structured according to the well-known problem solving strategies of Pólya. (The instructions on the worksheet are typing in *italics letters*, the help regarding useful Cabri tools are written in *[brackets]*).

Posing the complex problem

Find for a given quadrangle a pyramid based on it, which can be cut in a square.

Understand the problem

Try to find another connection between the given quadrangle and square.

After an “excursion” on art by web search we find quadrangles demonstrating squares.



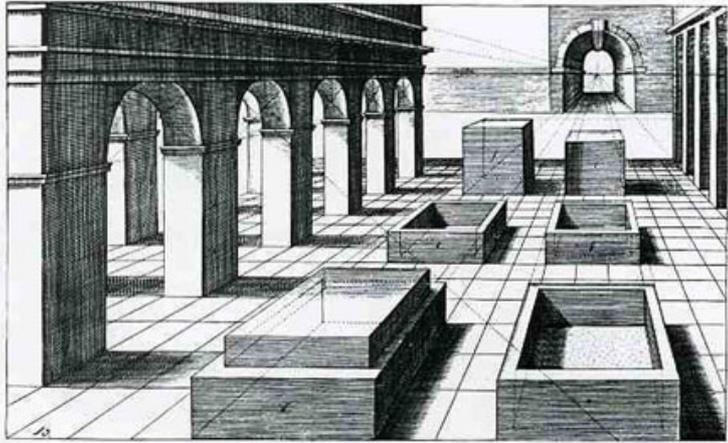
M. C. Escher: Belvedere



Gustave Caillebotte. *Le Pont de l'Europe*.
1876. Oil on canvas, 49 1/8 X 71 1/8".
Musée du Petit Palais, Geneva.

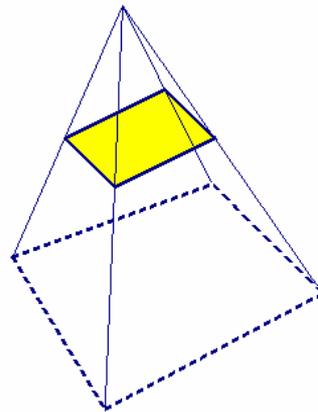
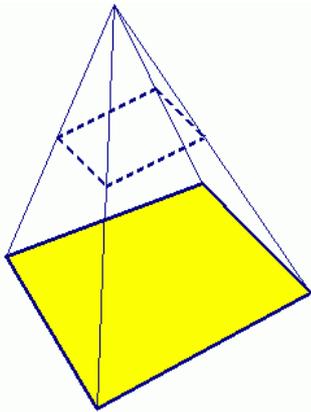


Hieronymus Rodler:
A vault the way you'd like it



3.60 Jan Vredeman de Vries, *Perspective Study*, from *Perspective*,
Leiden, 1604.

These experiences indicate the change of our viewpoint from given quadrangle to square. We are looking for the centre of projection.



Devise a plan and carry out the plan:

Geometry construction

Consider the given quadrangle and suppose that the problem is solved.

We analyze the complete configuration and find out the following facts:

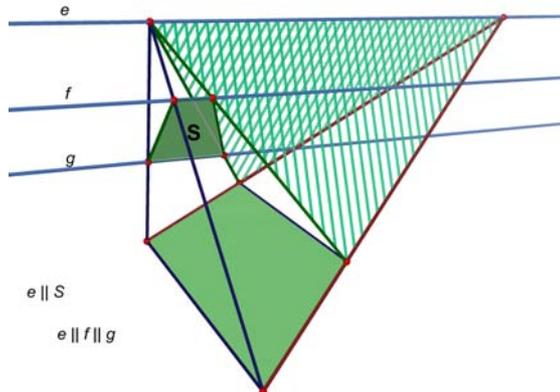
- Iff a **cutting plane is parallel to the intersection line of two other planes**, all the three intersection lines will be parallel.

In order to get a trapezoid as a cutting quadrangle we have to cut the pyramid by plane standing parallel to the intersection of two opposite faces.

Cabri 3D implementation

Quadrangle in Cabri 3D: [polygon tool / three times click on the base plane / double click to finish the quadrangle].

Pyramid: [pyramid tool / click on the quadrangle / SHIFT+click to define 5th vertex "in a space"]



- If we cut the pyramid by plane standing **parallel to intersection of both opposite face-pairs**, the cutting quadrangle will be parallelogram.

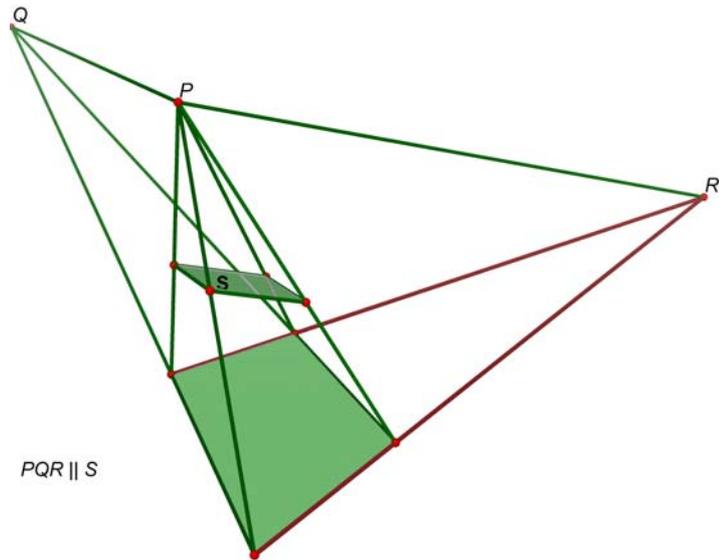
Let PQ is the intersection the one of opposite face-pair, and PR is the intersection of the others. PQ and PR define a plane.

If we cut the pyramid by plane parallel to PQR, the intersection quadrangle will be a parallelogram.

- Finally we have to fix **angles of diagonals and of sides to 90 degrees**. If we choose the wanted (5th) vertex (P) on the **Thales-sphere of segment QR**, the cutting parallelogram will be rectangle, because in this case the QPR angle is right angle, and two edges of parallelogram are parallel to this angle.

- In addition, if we choose wanted vertex on the **Thales-sphere of segment MN**, (where diagonals of given quadrangle intersect line QR,) the cutting rectangle will be square, because its diagonals will be perpendicular to each other.

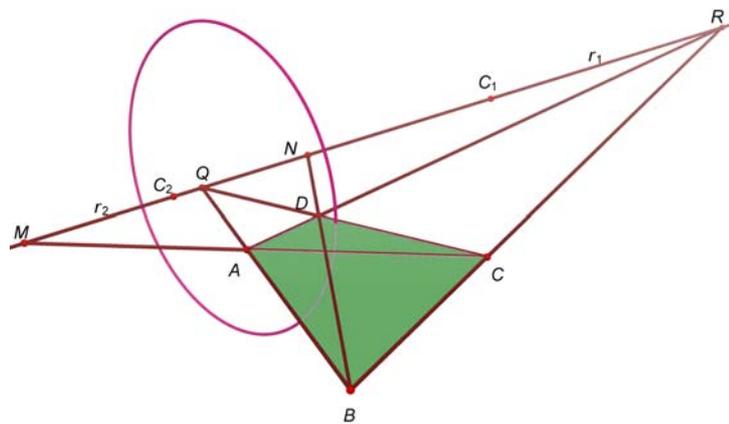
*The order of the last two steps can be changed, first determine a diamond and than a square.



Point Q and R: [Line tool / click on four edges of the pyramid / Intersection point tool / click on lines]

Plane S: [Plane tool / click on P, Q and R. / Parallel tool / click on plane PQR to define parallel to, and somewhere on an edge to define point through]

Parallelogram: [Intersection point tool / click on plane S and four edges of the pyramid. / Polygon tool / click on three intersection points / double click on the 4th point].



Point M and N: [Line tool / click on A & C and B & D and Q & R. / Intersection point tool / click on lines AC, QR and BD, QR]

Thales-spheres: [Midpoint tool / click on Q & R for getting C₁, and M & N for C₂. / Sphere tool / click on C₁ and R, then C₂ and M]

Intersection circle: [Intersection curve tool / click on two spheres. / (Now it is recommended to hide spheres by right click on them and choosing "Hide/Show" command)]

The solution: [Pyramid tool / click on the quadrangle ABCD / click on somewhere on the circle to define 5th vertex] After it the cutting plane can constructed by method above.

Looking back (reflect)

Now we can answer the question if a given quadrangle can represent the central projection of a square.

- If the opposite sides of the given quadrangle are not parallel to each other (arbitrary quadrangle) the two Thales-spheres exist and they always intersect each other because the defining segments (PR and QS) harmonically separate each other.
- When the given quadrangle is a trapezoid, we get instead one of Thales-sphere a plane cutting the other sphere in a circle, too.
- In case of parallelogram our construction does not work. We know that any other construction could not give result because the parallelogram is an affine projection of the square. This means that the centre is an ideal point.

The vertex of the pyramid (this vertex is the centre of the projection) exists if and only if the given quadrangle differs from parallelogram. The parallelogram can not be considered as a perspective drawing of a square.

With help of dynamical geometry software it is possible to demonstrate the steps of the construction and examine the different cases. (The properties of starting quadrangle can be changed directly.)

By this way each user (users with less experiences with 3-dimensional geometry, too) can get an overview about the complex problem by each configuration.

Even beginners in using Cabri 3D could solve the problem during the workshop (in 90 minutes).

References

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