$O(\alpha_s^3)$ Contributions to the Heavy Flavor
Wilson Coefficients of the Structure Function
$F_2(x, Q^2)$ at $Q^2 \gg m^2$

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$D_7$, both mass assignments $m_a = m_2, m_b = m_1$ and $m_a = m_1, m_b = m_2$ yield the same result due to symmetry reasons.

$D_{8a}$ with $m_a = m_2$, $m_b = m_1$ and $D_{8b}$ with $m_a = m_1$, $m_b = m_2$ respectively.

The different operators expressed in terms of the graph polynomial $\Psi_G$ and different Dodgson polynomials of the graph $\tilde{G}$, where the external line of $G$ has been closed.

This ladder graph is two-edge reducible concerning the pairs of edges 1, 8, 2, 5 and 3, 4.

The 3-loop Benz diagram for $I_1(N)$, Eq. (588).

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The 3-loop Benz diagram for $I_3(N)$, Eq. (594).

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The ladder diagram for $I_7(N)$, Eq. (618).

Diagram 8. For this diagram linear reducibility can only be gained through a suitable variable transformation.

Crossed-box topologies with local operator insertions.

Feynman rules of QCD.

Feynman rules for quarkonic composite operators.
This Thesis contains contributions to the following publications:

i) Publications: Journals

1. The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity

2. Massive 3-loop Ladder Diagrams for Quarkonic Local Operator Matrix Elements

3. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$

4. The logarithmic contributions to the $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering

5. Calculating Massive 3-loop Graphs for Operator Matrix Elements by the Method of Hyperlogarithms

6. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity

ii) Publications: Conference Proceedings

1. Heavy Flavor DIS Wilson coefficients in the asymptotic regime
   J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock
   in: Proceedings of the 10th DESY Workshop on Elementary Particle Theory: Loops and Legs in Quantum Field Theory, Wörlitz, Germany, 2010,

2. 3-Loop Heavy Flavor Corrections to DIS with two Massive Fermion Lines
   J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock
3. New Heavy Flavor Contributions to the DIS Structure Function $F_2(x, Q^2)$ at $O(\alpha_s^3)$
   J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock
   in: Proceedings of the 10th International Symposium on Radiative Corrections: Applications of Quantum Field Theory to Phenomenology (RADCOR 2011), Mamallapuram, Tamil Nadu, India, 2011,

4. New Results on the 3-Loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering
   J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock
   in: Proceedings of the 36th International Conference on High Energy (ICHEP 2012), Melbourne, Australia, 2012,

5. Recent Results on the 3-Loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering
   in: Proceedings of the 1st International Workshop on Deep-Inelastic Scattering and Related Subjects (DIS 2013), Marseille, France, 2013,

6. New Results on the 3-Loop Heavy Flavor Corrections in Deep-Inelastic Scattering
   in: Proceedings of the 11th International Symposium on Radiative Corrections ”Application of Quantum Field Theory to Phenomenology” (RADCOR 2013), Durham, United Kingdom, 2013,

7. 3-loop heavy flavor Wilson coefficients in deep-inelastic scattering

8. Recent progress on the calculation of three-loop heavy flavor Wilson coefficients in deep-inelastic scattering
   in: Proceedings of the 12th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory (LL2014), Weimar, Germany, 2014,
9. **3-Loop Heavy Flavor Corrections in Deep-Inelastic Scattering with Two Heavy Quark Lines**


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Three further journal publications are in preparation.
1 Introduction

It has been observed experimentally that the magnetic moment of the proton and the neutron 
derivates significantly from that of the electron. This has been the first hint that nucleons could 
be not point-like elementary particles, cf. [1, 2]. During the 1950s lepton-nucleon scattering 
experiments performed by Hofstadter and collaborators [3–6] showed furthermore that neutrons 
and protons possess extended charge distributions giving further evidence to the existence of a 
nucleonic substructure.

A series of newly discovered hadronic states at various particle accelerators, particularly during 
the late 1950s, led to the development of a deeper understanding of the hadronic composition. 
The various states were classified according to their spin as mesons (integer spin) and baryons 
(half–integer spin). The immense number of newly discovered states was calling for a system-
atic classification based on the properties charge, isospin and mass. Independent attempts 
by M. Gell–Mann and Y. Ne’eman to group the different hadrons into octet and decouplet– 
representations of the flavor group $SU(3)$ with similar properties proved to be very success-
ful [7–9]. The prediction of the mass of the previously unknown $\Omega^–$–baryon with strangeness $-3$ 
and electric charge $-1$ constituted a great success of this model. Based on these discoveries in 
1964 M. Gell–Mann [10], and G. Zweig [11] proposed that baryons were bound states, being 
composed of three spin-$\frac{1}{2}$ particles named quarks and mesons composed of a quark–antiquark pair. 
To describe all hadron states known at that time the three quark flavors $u$, $d$, and $s$ with electric charges $e_u = +2/3$, $e_d = e_s = -1/3$ proved to be sufficient. Despite its 
strong predictive power, the static quark–model also suffered from a fundamental problem. The 
$\Delta^{++}$ and $\Delta^–$–resonances were predicted as spin– and flavor–symmetric wave functions of three 
quarkonic fermions thereby violating the well–established spin–statistic theorem [12–14]. This 
contradiction was finally resolved by introducing a new 3–valued quantum number named color 
[15–18]. Since no colored particles could be detected experimentally it was finally proposed, 
that physical hadrons must be color–singlets [16, 17, 19].

A systematic study of the nucleonic substructure was performed in the late 1960s at the Stanford 
Linear Accelerator SLAC [20, 21], by analyzing the differential deeply analytic cross section of 
unpolarized lepton–nucleon scattering which was parametrized by the non–perturbative structure 
functions $F_2$ and $F_L$ [22, 23]. Kinematically this process is characterized by the energy transfer 
$\nu$ in the rest–frame of the nucleon and the virtual 4–momentum transfer from the lepton to 
the nucleon of $q^2 = -Q^2$. Studying the algebra of currents, Bjorken predicted [24] that in the 
limit $Q^2$, $\nu \to \infty$ with the ratio $Q^2/\nu$ fixed the structure functions do not depend on $Q^2$ and $\nu$ 
individually any longer, but become a function of the Bjorken variable $x = \frac{Q^2}{2M\nu}$ only, where $M$ 
denotes the mass of the hadron and $0 < x < 1$. This behavior became known as Bjorken–scaling 
and has been experimentally confirmed in experiments of the SLAC and MIT groups [25–28], cf. 
also [29–31]. The scaling property constituted further evidence for the point–like substructure of 
hadrons and inspired Feynman to propose the parton–model [32–34]. Furthermore, the prediction 
that the structure Function $F_L$ is much smaller than $F_2$, known as Callan–Gross relation [35], 
could be experimentally confirmed. This could only be explained if the scattering process was 
dominated by spin–1/2 partons.

The group theoretic approach by Gell-Mann and Ne’eman and Feynman’s parton model were 
finally joint, when Bjorken and Paschos suggested, that Feynman’s partons would be identical 
to the quarks proposed by Gell-Mann [36]. The ground for a full theoretical description of the 
force binding the different quarks together was laid in the 1950s when, inspired by the successful 
description of Quantum Electrodynamics as a quantum field theory based on the abelian group
U(1), C.N. Yang and R.L. Mills [37] studied quantum field theories based on non-abelian gauge groups. The renomalizability of massless Yang–Mills theories was proven by G. ’t Hooft [38]. These developments inspired M. Gell-Mann, H. Fritzsch and H. Leutwyler, [39], cf. also [17], to describe quark interactions by a $SU(3)_c$ Yang-Mills theory, thereby giving birth to Quantum Chromodynamics (QCD).

A short time before D. Gross and F. Wilczek, [40], and H. Politzer, [41] discovered asymptotic freedom, calculating the running coupling of QCD to one-loop order which decreases at ever shorter distances or larger momentum transfers. This explains why scaling is compatible with the fact that free quarks could never be observed and makes the use of perturbative computation methods for scattering processes at large enough momentum scales possible.

In 1974 the $J/\psi$ meson has been discovered almost simultaneously at SLAC and at Brookhaven National Laboratory (BNL) [42, 43]. It was described as a meson of a fourth quark, the charm quark, which had been predicted previously [44–47]. A few years later as a first quark of the third family of elementary particles, the bottom quark ($b$) was experimentally found by discovering the $\Upsilon$–resonance [48]. Another quark, the top–quark, had been predicted but due to its large mass it was difficult to be produced in particle accelerators for a longer period and it was observed in 1995 at TEVATRON [49–52], finally.

Together with the electroweak $SU_L(2) \times SU_Y(1)$ sector QCD forms the Standard Model of elementary particle physics. The theoretical description of the electroweak sector of the Standard Model has been developed by S. Glashow [53](1961) and S. Weinberg in 1967 [54], cf. also [55, 56].

G. ’t Hooft and M. Veltman proved that this theory is renormalizable [57], see also [58–61].

One of the cleanest experimental approaches to study the substructure of hadrons are deep-inelastic scattering (DIS) experiments. On the theoretical side, applying QCD to practical calculations was problematic at first, as it is not perturbative in the complete kinematic region. Here the light cone expansions (LCE), the expansion of operator products around light like distances, emerged as an important tool to perform the corresponding computations [62–65]. Applying it to deeply inelastic scattering, one derives a factorization theorem, which separates hadronic bound state effects from the short distance behavior at leading twist [66–68]. The former are described by the parton distribution functions (PDFs) which can be extracted from experimental precision data or computed by using non-perturbative methods at a given scale $Q^2$. The short distance behavior, on the other side, is governed by the process dependent Wilson coefficients, which allow for a perturbative series representation in the strong coupling constant $\alpha_s = g_s^2/(4\pi)$. A first strong success for perturbative QCD computations was the prediction of logarithmic scaling violations in deeply inelastic scattering [69, 70]. These are due to parton–parton interactions and the fact that QCD is not conformally invariant [71]. They could experimentally be first observed in 1975 [72,73].

As it constitutes the most direct way to probe the substructure of hadrons, numerous DIS experiments have been performed since the 1970s [74–79]. Currently the widest kinematic range for DIS cross-section studies has been covered by experiments at HERA at DESY [80–83]. Here the structure function $F_2$ has been measured over a large kinematic range with $6 \cdot 10^{-7} \leq x \leq 0.65$ and $0.045 \leq Q^2 \leq 30000\text{GeV}^2$ [84], while $F_L$ has mainly been measured at fixed targets [85,86]. For a theoretical description the case where only the massless quark flavors ($u, d, s$) contribute has been studied extensively. The massless Wilson coefficients for the $F_2$ and $F_L$ were computed at $O(\alpha_s)$ in [87–89], at second order in [90–100], and at third order in [101–106].

The experimental data from HERA show, however, that especially in the small $x$–region the structure functions receive significant contributions due to charm quarks, cf. [107–110]. Furthermore the DIS-analyses constitute an excellent tool to extract the parton distribution functions.
and QCD–parameters as the strong coupling constant $\alpha_s(M_Z)$ from experimental data. Very recently also the heavy quark masses $m_t$ and $m_c$ have been precisely measured using these processes [111, 112]. Here a missing precise next-to-next-to-leading order (NNLO) heavy flavor description is considered one of the dominant sources for the systematic differences between the PDF extractions of different groups [113–118]. A precise theoretical description of the heavy flavor contributions to DIS would be especially useful to constrain the low–$x$ behavior of the gluon–distribution, which also constitutes an important input for the analysis of Higgs-Boson production at the LHC.

The complete analytic computation of the heavy flavor Wilson coefficients is much harder to obtain, however. The results at leading order were calculated in the late 1970s, [119–123]. At next–to–leading order (NLO) a semi-analytic result for the heavy flavor contributions has been computed in [124–126]. A precise numerical implementation in Mellin space has been given in Ref. [127].

For values of the momentum transfer $Q^2$ much larger than the mass of the heavy quark $m^2$, the heavy flavor Wilson coefficients factorize, cf. Ref. [128], into the process dependent light flavor Wilson coefficients $C_{(g,\bar{g})}(2, L)(x, Q^2/\mu^2)$ and the process independent massive operator matrix elements (OMEs) $A_{ij}(x, \mu^2/m^2)$. Here the OMEs contain all the mass dependence and are given as matrix elements of the different twist-2 local composite operators between the partonic states $| j \rangle$ ($i, j = q, g$). For the structure function $F_2$ this representation holds from $Q^2 \gtrsim 10 m^2$ cf. [128], which covers a very important kinematic region of the deep-inelastic scattering experiments at HERA being widely free of higher twist effects [129–132]. Using this representation the NNLO contributions to the structure function $F_L$ have been obtained in Ref. [133]. However, in this case the factorized representation becomes only valid at values $Q^2 \gtrsim 800 m^2$, [128], and thus does not describe the relevant kinematic region probed at HERA.

The NLO OMEs have been computed in Refs. [128, 134]. They were later re-computed using a different method in Refs. [133, 135–139] and extended up to order $O(\varepsilon)$ in the dimensional regularization parameter $\varepsilon = D − 4$. The latter contribution constitutes an important ingredient to the renormalization of the NNLO OMEs which has been worked out in [139]. Here as a first contribution to the NNLO description a number of fixed Mellin moments $N = 2 \ldots 10$ and in some cases for $N = 2 \ldots 14$ has been calculated. First results for general values of the Mellin-variable $N$ at 3-loop order were obtained in [140, 141], where the $O(C_{A,F}T_F^2 N_F)$ color-contributions to the OMEs contributing to $F_2$ have been computed. The calculation was based on integral representations in terms of (nested) sums over generalized hypergeometric functions [142–149]. These obey a sum representation which allows for a direct extraction of the singularities of the Laurent series expansion in the dimensional regularization parameter $\varepsilon = D − 4$. Afterwards, advanced summation algorithms encoded in the package SIGMA were applied to evaluate the nested sums. The same method has been used later to compute the gluonic OMEs $A_{gg,Q}$ and $A_{gq,Q}$ [150]. These are important to define the variable flavor number scheme (VFNS) for the PDFs. The method of representing integrals by hypergeometric functions is very efficient and allows for a direct computation of QCD Feynman diagrams without relying on integration by parts (IBP)-reductions. It is, however, restricted to a topological subclass of the Feynman integrals contributing to the NNLO OMEs. Generalizations thereof have been thought after and inspired the use of Appell–function representations to solve a large number of ladder topology diagrams [151] and approaches using intermediary Mellin-Barnes representations [152]. The complete $O(C_{A,F}T_F^2)$ contributions to the OME $A_{gg,Q}$ have been computed in Ref. [152]. Furthermore, recently a lot of progress in the calculation of massive operator matrix elements has been made using integration by parts reductions [153] and differential equation methods.
Additionally to the OMEs $A_{qq,Q}^{PS}$ and $A_{gq,Q}$ which have been presented in [140] (see also Section 3 of this Thesis) now also the OMEs $A_{gq,Q}$ [156], $A_{NS,Q}$ [157] and $A_{PS,Q}$ [158] are completely known.

Previous OME-computations have been performed with $N_F$ massless and one heavy quark flavor representing the $c$- or the $b$-quark, depending on the kinematic region one is interested in. There is, however, no strong hierarchy between the quark masses since $m_c^2/m_b^2 \approx 1/10$. The charm quark can thus not be considered massless at scales $\mu^2 \sim m_b^2$ and the sequential decoupling of the individual heavy quark flavors in the usual variable flavor number scheme has to be replaced by a variable flavor scheme which decouples both $c$- and $b$-quarks at the same time. One may, however, compute massive operator matrix elements depending on both heavy quark flavors and define heavy flavor Wilson coefficients containing heavy flavor contributions from both $c$ and $b$-quarks. It is one of the main objectives of this Thesis to set up the framework for these computations.

Additionally to the physical importance it is one of the aims of this Thesis to explore the mathematical methods and structures which are best suited for this class of computations. From a technical viewpoint the computation of the massive OMEs is performed in Mellin space, which renders the factorization relation for the heavy flavor Wilson coefficients a pure product. The OMEs thus become functions of the Mellin variable $N$, which assumes positive integer values. In a final step one desires to find the analytic continuation in order to perform the inverse Mellin transform by a contour integration. One is thus faced by the discrete Mellin–$N$ space and the continuous $x$-space, which are related to each other by the (inverse) Mellin transformation. As some algorithms (or their implementations) are better suited for a computation in a continuous space, e.g. [159], cf. [152, 156–158], and the inverse Mellin transform is hard to obtain for momentum integrals, a further continuous space may be introduced by the construction of appropriate generating functions. In both, the discrete and the continuous space, physical quantities are mostly described within distinctive sets of characteristic functions. For most massless single scale analyses and simple massive topologies at three loops [106,160–165] these are typically nested harmonic sums [166,167]. They obey algebraic, [168], and structural relations, [169–171]. At fixed values of $N$ these results evaluate to multiple zeta values, [172,173].

A first generalization of the harmonic sums is obtained by adding additional weights into the nested sums. This yields the so-called generalized harmonic sums, which have been observed in [174]. Analytic and algorithmic aspects of (inverse) Mellin transforms, asymptotic expansions, etc. have been presented in Ref. [175]. Further extensions to cyclotomic sums [176] and additionally over letters involving binomials of the kind $\binom{2i}{i}$ [177] have been studied as well. The latter have been observed in Feynman diagrams with several massive fermionic lines [152] but also in other diagrams exhibiting a highly nested topology. They appear again in different parts of this Thesis. This includes the case of a diagram with a single massive line and 4-leg operator insertion and more general binomial summation kernels in the case of diagrams with massive lines of unequal mass. Algorithms performing many operations as (inverse) Mellin transforms, (asymptotic) expansions and reductions to a basis for most of these sum structures are available within the computer algebra package HarmonicSums [175,177–179] . On the continuous side, each sum structure is associated to respective iterated integrals structures by an inverse Mellin transform. Here the most commonly observed functions were the harmonic polylogarithms [180] over the alphabet $\{0, \pm 1\}$. But generalizations thereof where soon required. Firstly the alphabets may be enlarged, but lately also more general classes of iterated integrals [176], for example over cyclotomic polynomials or square root-valued integration kernels [177] have been studied.

The outline of this Thesis is as follows. In Section 2 we describe the deeply-inelastic lepton-
nucleon scattering process and its kinematics. We define the structure functions, review the QCD-improved parton model and present the asymptotic structure of the heavy flavor Wilson coefficients. In Section 3 we use representations in terms of special functions and modern summation algorithms to compute the remaining $O(C_{A,F}T_F^2N_F)$ contribution given in terms of the OME $A_{gg,Q}$. Furthermore, a large number of diagrams with two massless loop insertions has been computed in an automatized way. The logarithmic contributions to the heavy flavor Wilson coefficients follow as a direct consequence of the renormalization prescription worked out in Ref. [139]. Their exact form has been determined and is presented together with the respective theoretical background in Chapter 4. The two heavy flavor Wilson coefficients $L_g^S$ and $L_g^{PS}$ are given in complete form. We also present all logarithmic contributions to the $3$-loop Wilson coefficients $H_g^S$ and $H_g^{PS}$, which rely on the anomalous dimensions [69, 70, 100–103, 105, 161, 162, 181] up to three and the OMEs up to two loops [128, 133–139]. Section 5 presents the renormalization of the contributions with two distinct heavy quark flavors. Similar to the case of one heavy quark flavor [139] here the charge is renormalized in the MOM–scheme first using the background field method [182–184] to ensure that external gluons are strictly on–shell and the charge renormalization is covered by that of a propagator. A series of fixed Mellin moments $(N = 2, 4, 6)$ contributing to all the OMEs of the structure function $F_2$ and the variable flavor number scheme (VFNS) has been computed, see Section 6. Here the Feynman diagrams with operator insertions have been mapped to massive tadpoles, which could then be expanded and evaluated using the codes Q2e/Exp [185, 186]. First general-$N$ results have been obtained for the non–singlet OME and all scalar topologies contributing to the OME $A_{gg,Q}$. In the non–singlet case the $N$–dependent contributions decouple completely from the infinite sums stemming from the mass distribution of the diagram. For the diagrams such a decoupling can be achieved in intermediate steps using a split of the integration domain and suitable mappings of integration variables. In Section 7 the application of a direct integration algorithm based on hyperlogarithms [159, 187] for convergent Feynman diagrams with operator insertions is explored. Since it relies on an integration kernel containing only rational or polylogarithmic functions, generating function representations for the Feynman diagrams with operator insertions have been introduced. The algorithm has been implemented in MAPLE [188] and provides a very efficient way to compute diagrams of nested topologies that were previously inaccessible with different methods. A linear reduction algorithm allows to predict which Feynman integrals are suitable for an integration via hyperlogarithms, where in some cases linear reducibility could be re-obtained by a suitable variable mapping. Appendix A presents the conventions used for the computations within this Thesis. The relevant Feynman rules for the computed QCD- and scalar diagrams are given in Appendix B. Appendix C summarizes the $D$-dimensional momentum integration and Appendix D shortly reviews the different special functions including nested sum structures and different forms of iterated integrals. The Mellin moments $N = 2, 4, 6$ for the contributions with both $c$- and $b$-quarks are presented in Appendix F and the contributions $\propto \ln(Q^2/m^2)$ for the heavy flavor Wilson coefficients $H_g^{PS}$ and $H_g^S$ are listed in Appendix E.
Deeply inelastic scattering denotes the scattering of highly energetic leptons off nucleons. This Thesis will focus on DIS via the exchange of a single photon. The electroweak corrections can be found in Ref. [189] to 1-loop and dominant higher order corrections can be found in Refs. [190–192,192–194].

![Figure 1: Schematic diagram of deeply inelastic scattering via single boson exchange](image)

### 2.1 Kinematics

The basic kinematics of the deep-inelastic scattering process is depicted in Figure 1. A lepton with momentum \( l \) is scattered off a nucleon with momentum \( P \) via exchange of a virtual vector boson with 4-momentum \( q \). The nucleon disintegrates and a new hadronic state \( F \) with a 4-momentum \( P_F \) is formed. The 4-momentum of the virtual photon is space-like and the virtuality \( Q^2 \) is defined by

\[
Q^2 = -q^2 = -(l - l')^2 ,
\]

with \( l' \) denoting the 4-momentum of the outgoing lepton. The outgoing hadronic state is measured inclusively, thus only its total 4-momentum \( P_F \) is recorded. In this case the total cross-section depends only on two further independent kinematic variables. These are the total center of momentum energy squared

\[
s \equiv (P + l)^2 ,
\]

and the invariant mass of the final hadron state \( \langle P_F \rangle \),

\[
W^2 \equiv (P + q)^2 = P_F^2 .
\]

Here \( s \) denotes the total center of momentum energy squared and \( W \) is the invariant mass of the final hadron state \( \langle P_F \rangle \). The lepton mass is very small compared to the other mass scales and will be neglected from now on. The process is then described using the following Lorentz-invariant kinematic variables:

\[
\nu \equiv \frac{P.q}{M} = \frac{W^2 + Q^2 - M^2}{2M} ,
\]

\[
x \equiv \frac{-q^2}{2P.q} = \frac{Q^2}{2M\nu} = \frac{Q^2}{W^2 + Q^2 - M^2} ,
\]
\[ y \equiv \frac{P \cdot q}{P \cdot l} = \frac{2M \nu}{s - M^2} = \frac{W^2 + Q^2 - M^2}{s - M^2}. \] 

(7)

Thus \( \nu \) is the total energy in the rest frame of the nucleon, \( x \) is the Bjorken variable and \( y \) is the inelasticity, cf. [195]. Combining the definitions (2, 6) and (7) one obtains

\[ Q^2 = xy s. \]

(8)

Deep-inelastic scattering occurs via the exchange of \( \gamma, Z, W^\pm \)-bosons. In this Thesis we study neutral current unpolarized lepton-nucleon scattering, which is dominated by single photon exchange for not too large virtualities, i.e. \( Q^2 \lesssim 500 \text{GeV}^2 \), cf. [196]. We will thus neglect weak boson contributions from \( Z \)-boson exchange in the following.

The physical region is constrained by several conditions. From Eqs. (4,6) one obtains

\[ W^2 = (P + q)^2 = M^2 + Q^2 \left( \frac{1}{x} - 1 \right) \geq M^2. \]

(9)

As the invariant hadronic mass obey the condition \( W^2 \geq M^2 \) this limits the Bjorken variable \( x \) to the region

\[ 0 \leq x \leq 1. \]

(10)

\( x = 1 \) characterizes elastic scattering processes, while the inelastic region is described by \( x < 1 \) [197].

Further restrictions on the kinematic variables are obtained by demanding a positive energy-transfer in the rest-frame of the nucleon

\[ \nu \geq 0, \quad 0 \leq y \leq 1, \quad s \geq M^2. \]

(11)

The differential cross section reads, cf. [198–201],

\[ l' \frac{d\sigma}{d^3l'} = \frac{1}{32(2\pi)^3(l \cdot P)} \sum_{\eta', \eta, \sigma, F} (2\pi)^4 \delta^4(P_P + l' - P - l) |M_{fi}|^2. \]

(12)

Here the summation is over the the spin components of the leptons, quarks and gluons, \( \eta(\eta') \) and \( \sigma \), respectively. In the unpolarized case the cross section is obtained as an average over leptonic and hadronic spin degrees of freedom. \( M_{fi} \) denotes the transition matrix element for the electromagnetic current, which is given by

\[ M_{fi} = e^2 \bar{u}(l', \eta') \gamma^\mu u(l, \eta) \frac{1}{q^2} \langle P_P | J^{em}_\mu (0) | P, \sigma \rangle, \]

(13)

for \( ep \) scattering at leading order, cf. e.g. [198, 200–202]. \( | P, \sigma \rangle \) and \( \langle P_P | \) denote the initial and final hadron state. \( u(\bar{u}) \) are the bi-spinors of the electron respectively its conjugate, and \( \gamma_\mu \) denote the Dirac-matrices. Furthermore, \( e \) is the electric charge and \( J^{em}_\mu (x) \) the quarkonic contribution to the electromagnetic current operator, which is self-adjoint:

\[ J^1_\mu (x) = J_\mu (x). \]

(14)
For QCD it reads

\[ J_{\mu}^{em}(\xi) = \sum_{f,f'} \bar{\Psi}_f(\xi) \gamma_{\mu} \lambda_{f,f'}^{em} \Psi_{f'}(\xi) , \]  

(15)

where the sum is over the different quark flavors \( f, f' \), \( \Psi_f(\xi) \) denotes the respective quark field of flavor \( f \) and \( \lambda_{f,f'}^{em} \) is a matrix which describes the electromagnetic charges of the different quark flavors. For three quark flavors it reads

\[ \lambda_{f}^{em} = \frac{1}{2} \left( \lambda_{flavor}^3 + \frac{1}{\sqrt{3}} \lambda_{flavor}^8 \right) , \]  

(16)

where the \( \lambda_{flavor}^i \) denote the Gell-Mann matrices, cf. [203, 204]. Using Eqs. (12) and (13) the differential cross section can be rewritten as a contraction of a tensor \( L_{\mu\nu} \) which describes leptonic contributions only and a hadronic tensor \( W_{\mu\nu} \)

\[ \int_0^1 \frac{d\sigma}{dy} = \frac{\alpha^2}{4 P_1 Q^4} L_{\mu\nu} W_{\mu\nu} = \frac{1}{2(s-M^2)} \frac{\alpha^2}{Q^4} \int L_{\mu\nu} W_{\mu\nu} . \]

(17)

Here \( \alpha \) denotes the fine structure constant and in the Born approximation the leptonic and hadronic tensors read

\[ L_{\mu\nu}(l,l') = \sum_{q,q'} \left[ \bar{u}(l',\eta') \gamma^\mu u(l,\eta) \gamma^\nu \right] , \]

(18)

\[ W_{\mu\nu}(q,P) = \frac{1}{4\pi} \sum_{\sigma,F} (2\pi)^4 \delta^4(P_F-q-P) \langle P,\sigma | J_{\mu}^{em}(0) | P_F \rangle \langle P_F | J_{\nu}^{em}(0) | P,\sigma \rangle . \]

(19)

The decomposition into purely leptonic and hadronic tensors relies on general properties of this amplitude and holds for higher orders as well, cf. [205]. The leptonic tensor may be easily evaluated within QED. One obtains

\[ L_{\mu\nu}(l,l') = Tr[l_{\mu} l_{\nu}' + l_{\nu}' l_{\mu} - \frac{Q^2}{2} g_{\mu\nu}] . \]

(20)

The hadronic tensor cannot be computed using purely perturbative methods only due to the non-perturbative effects within the hadronic substructure. One may, however, separate perturbative and non-perturbative contributions and apply perturbative computation methods on the former part.

The hadronic tensor obeys various symmetry conditions, cf. e.g. [206]. These allow to extract the Lorentz structure and to parametrize the hadronic tensor by scalar functions, which contain all the information on the internal structure of the proton. In the general case there are 14 such structure functions, cf. Refs. [205,207], while in the cases of unpolarized DIS via single photon exchange only two structure functions remain. Here the leptonic tensor (18) is symmetric and thus only the symmetric contribution of the hadronic tensor is physically significant. It follows that the hadronic tensor must be expressible as a linear combination of the following tensor structures

\[ g_{\mu\nu} , \ q_{\mu} q_{\nu} , \ P_{\mu} P_{\nu} , \ q_{\mu} P_{\nu} + q_{\nu} P_{\mu} . \]

(21)
From the conservation of the electromagnetic current, Lorentz- and time-reversal invariance one obtains the condition

$$ q_\mu W^{\mu \nu} = 0 . $$

(22)

Imposing gauge invariance on the Ansatz (21) yields a representation of the hadronic tensor in terms of the structure functions $F_{1,2}$

$$ W_{\mu \nu}(q, P) = \frac{1}{2x} \left( -q_{\mu \nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_1(x, Q^2) $$

$$ + \frac{2x}{Q^2} \left( P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} + \frac{q_\mu q_\nu}{4x^2} \right) F_2(x, Q^2) . $$

(23)

Frequently also the longitudinal structure function

$$ F_L(x, Q^2) = F_2(x, Q^2) - 2x F_1(x, Q^2) $$

(24)

is used. The structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ are real functions. Their arguments are Bjorken-$x$ and $Q^2$. In the elastic case $x = 1$ the cross section is determined by the total energy transfer only. Inserting (23) into (17) yields the differential cross section in terms of the structure functions $F_2$ and $F_L$:

$$ \frac{d\sigma}{dx dy} = \frac{2\pi \alpha^2}{xyQ^2} \left\{ 1 + (1 - y)^2 \right\} F_2(x, Q^2) - y^2 F_L(x, Q^2) \right\} . $$

(25)

The structure functions $F_2$ and $F_L$ are extracted from the hadronic tensor (23) by applying the following $D$ dimensional projection operators:

$$ F_L(x, Q^2) = \frac{8x^3}{Q^2} P^\mu W_{\mu \nu}(q, P) , $$

$$ F_2(x, Q^2) = \frac{2x}{D - 2} \left[ (D - 1) \frac{4x^2}{Q^2} P^\mu P_\mu - g^{\mu \nu} \right] W_{\mu \nu}(q, P) . $$

(26)

2.2 The Parton Model

Studying the algebra of electromagnetic currents Bjorken observed [24] that in the limit $Q^2, \nu^2 \to \infty$ keeping the ratio $Q^2/\nu$ constant, the structure functions do not depend on the kinematic variables individually anymore, but on their ratio, i.e. on $x$, Eq. (16),

$$ \lim_{\{Q^2, \nu\} \to \infty, x=\text{const.}} F_{(2, L)}(x, Q^2) = F_{(2, L)}(x) . $$

(27)

This effect became known as Bjorken scaling and has soon after been experimentally confirmed to hold approximately in electron–proton collisions performed at SLAC in 1968, [25–28]. Feynman has shown that this behavior may be explained by considering the proton as a composite object of point-like constituents at large enough scales. During the short interaction time these partons behave as quasi–free particles. The virtual photon scatters of one single parton while the other partons behave as “spectator” partons. The hadronic tensor is thus obtained as an incoherent
sum over the individual partonic contributions weighted by the probability to find a parton of the respective state $i$ and with a momentum fraction $z$ inside the proton. This probability is described by the parton distribution function $f_i(z)$.

In the collinear parton model, the momenta of the interacting parton $p$ is taken to be collinear to the nucleons momentum $P$ and thus

$$p = zP .$$

(28)

In Feynman’s original model protons were always composed of two $u$ and one $d$ quarks. Radiative corrections were neglected and the experimentally observed strict correlation

$$\delta \left( \frac{q \cdot p}{M} - \frac{Q^2}{2M} \right) ,$$

(29)

which implies $z = \frac{q}{P}$, was enforced. The discovery of QCD allowed for a more sophisticated description which included contributions from virtual quark states and gluons. This QCD-improved parton model may be derived by applying the light-cone expansion.

The hadronic tensor is then derived from a partonic tensor $W_{i\mu}^i(\tau, Q^2)$, which describes the photon-parton interaction, $W_{i\mu}$:

$$W_{\mu\nu}(x, Q^2) = \frac{1}{4\pi} \sum_i \int_0^1 dz \int_0^1 d\tau \left[ f_i(z) + f_i(z) \right] W_{i\mu\nu}^i(\tau, Q^2) \delta(x - z\tau) .$$

(30)

Here $f_i(z)$ and $f_i(z)$ denote the parton distribution function of the respective charged parton $i$ and its anti-parton. The partonic tensor $W_{i\mu}$ corresponds to the hadronic tensor (33), where the hadronic states $\langle P |$ are substituted by the corresponding partonic state $\langle p |$ of the interacting parton.

In the Born-approximation the electromagnetic current takes the form

$$\langle i | J^i_{\mu}(\tau) | i \rangle = -ie_i \bar{u} \gamma_\mu u^i ,$$

(31)

where $e_i$ is the charge of the respective parton $i$. This yields the partonic tensor

$$W_{i\mu}(\tau, Q^2) = \frac{2\pi e^2_i}{q \cdot p} \delta(1 - \tau) \left[ 2p^i_\mu p^i_\nu + p^i_\mu q_\nu + p^i_\nu q_\mu - g_{\mu\nu} q \cdot p \right] + O(\alpha_s) .$$

(32)

Using elementary quantum mechanical relations Eq. (19) is rewritten as, cf. [201,208],

$$W_{\mu\nu}(q, P) = \frac{1}{4\pi} \sum_\sigma \int d^4 \xi \exp(iq \xi) \langle P | [J^c_{\mu}(\xi), J^c_{\nu}(0)] | P \rangle = \frac{1}{2\pi} \int d^4 \xi \exp(iq \xi) \langle P | [J^c_{\mu}(\xi), J^c_{\nu}(0)] | P \rangle .$$

(33)

By applying the optical theorem the hadronic tensor is expressed in terms of the forward Compton amplitude for virtual gauge boson nucleon scattering

$$W_{\mu\nu}(q, P) = \frac{1}{\pi} \text{Im} \, T_{\mu\nu}(q, P) ,$$

(34)
with the Compton amplitude

\[
T_{\mu\nu}(q, P) = i \int d^4 \xi \, \exp(iq\xi) \langle P \mid T J_\mu(\xi) J_\nu(0) \mid P \rangle ,
\]

(cf. [203]).

In the Bjorken–limit the light cone expansion of the forward Compton amplitude (35) at twist \(\tau = 2\) yields, cf. [88,209],

\[
T_{\mu\nu}(q, P) \rightarrow \sum_{i,N} \left\{ Q^2 g_{\mu\nu} + g_{\mu\nu} q_i q_j - g_{\mu\nu} q_i q_j \right\} C_{i,2} \left( N, \frac{Q^2}{\mu^2} \right) N_p q_i q_j \left( \frac{2}{Q^2} \right)^N \langle P \mid O_{i,1 \ldots N} (\mu^2) \mid P \rangle .
\]

The light cone expansion of the products of electromagnetic currents in (35) yields different local operators, which in case of single photon exchange and at twist \(\tau = 2\) are given by

\[
O_{q, s, \mu_1, \ldots, \mu_N}^{N} = \sum_{i,N} \left\{ Q^2 g_{\mu\nu} + g_{\mu\nu} q_i q_j - g_{\mu\nu} q_i q_j \right\} C_{i,2} \left( N, \frac{Q^2}{\mu^2} \right) N_p q_i q_j \left( \frac{2}{Q^2} \right)^N \langle P \mid O_{i,1 \ldots N} (\mu^2) \mid P \rangle .
\]

In Eqs. (37-39) \(S\) denotes the symmetrization operator (332) of the Lorentz indices \(\mu_1, \ldots, \mu_N\). \(D_\mu\) is the covariant derivative, \(\psi\) and \(\overline{\psi}\) the quark resp. anti–quark fields and \(F^{a}_{\mu\nu}\) the gluonic field strength tensor with \(a\) the color index in the adjoint representation. Furthermore \(\lambda_r\) is the flavor matrix of \(SU(N_F)\) with \(N_F\) fermion flavors. The labels \(q, g\) in Eqs. (37-39) distinguish quarkonic and gluonic operators.

### 2.3 The light flavor Wilson Coefficients

The structure functions \(F_i\) contains long-distance effects contributions from parton-parton interactions, which make it impossible to describe it using purely perturbative methods. However, for large enough virtualities \(Q^2\) it obeys the factorization relation

\[
F_i(x, Q^2) = \frac{x}{N_F} \sum_q e_q^2 \left[ \Sigma(x, \mu^2) \otimes C_{i,q}^{S} \left( x, \frac{Q^2}{\mu^2} \right) + G(x, \mu^2) \otimes C_{i,g} \left( x, \frac{Q^2}{\mu^2} \right) \right. \\
\left. + N_F \Delta_q(x, \mu^2) \otimes C_{i,q}^{NS} \left( x, \frac{Q^2}{\mu^2} \right) \right] , i = 2, L .
\]

Here \(N_F\) denotes the number of quark flavors, \(e_Q\) are the electric charges and the sum run over light quark flavors which typically are \(u, d, s\). \(G(x, Q^2/\mu^2)\) is the gluonic parton distribution.
function (PDF), $\Sigma(x, \mu^2)$ denotes the singlet combination of the quark and anti-quark PDFs, $f_q$ and $f_{\bar{q}}$ respectively. It reads

$$
\Sigma(N_F, N, \mu^2) = \sum_{i=1}^{N_F} \left[ f_q(N_F, N, \mu^2) + f_{\bar{q}}(N_F, N, \mu^2) \right].
$$

Likewise the non-singlet combination of the quark PDFs is given by

$$
\Delta_q(N_F, N, \mu^2) = f_q(N_F, N, \mu^2) + f_{\bar{q}}(N_F, N, \mu^2) - \frac{1}{N_F} \Sigma(N_F, N, \mu^2).
$$

In Eq. (40) the $C^{S,NS}_{i,q}(2, \mu^2)$ are the light flavor Wilson coefficients, which describe the hard scattering of a photon with a light quark. Here the singlet contribution is usually split into a non-singlet and a pure singlet contribution:

$$
C^{S}_{i,q} = C^{NS}_{i,q} + C^{PS}_{i,q}.
$$

The symbol $\otimes$ denotes the Mellin convolution $f \otimes g$ of two functions $f, g$

$$
[f \otimes g](z) = \int_0^1 dz_1 \int_0^1 dz_2 \delta(z - z_1 z_2) f(z_1) g(z_2).
$$

The Mellin transformation $\mathcal{M}[f](N)$ of a function $f$ is defined by the integral

$$
\mathcal{M}[f](N) = \int_0^1 dz \, z^{N-1} f(z).
$$

The Mellin transformation translates the convolution (44) into a simple product of two functions in Mellin–space,

$$
\mathcal{M}[f \otimes g](N) = \mathcal{M}[f](N) \mathcal{M}[g](N).
$$

It is thus technically advantageous to perform computations of this kind in Mellin–space.

The renormalization prescription, see cf. Section 5, introduces the renormalization scale $\mu$. As physical observables the structure functions are independent of the renormalization scale. This yields the renormalization group equation (RGE) [210–214]

$$
\mathcal{D}(\mu^2) F_{(2,L)}(N, Q^2) = \mu^2 \frac{d}{d\mu^2} F_{(2,L)}(N, Q^2) = 0.
$$

Here the total derivative operator $\mathcal{D}(\mu^2)$ is given by

$$
\mathcal{D}(\mu^2) \equiv \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s(\mu^2)) \frac{\partial}{\partial a_s(\mu^2)} - \sum_i \gamma_m(a_s(\mu^2)) m_i(\mu^2) \frac{\partial}{\partial m_i(\mu^2)},
$$

where the sum runs over all the different masses. The $\beta$–function and the anomalous mass dimensions, $\gamma_m$, are given by

$$
\beta(a_s(\mu^2)) \equiv \mu^2 \frac{\partial a_s(\mu^2)}{\partial \mu^2},
$$
\[ \gamma_{m_i}(a_s(\mu^2)) \equiv -\frac{\mu^2}{m_i(\mu^2)} \frac{\partial m_i(\mu^2)}{\partial \mu^2}. \]  

From Eqs. (47) and (40) one may derive the RGEs for the parton distribution functions [209]

\[
\frac{d}{d\ln \mu^2} \left( \frac{\Sigma(N_F, N, \mu^2)}{G(N_F, N, \mu^2)} \right) = -\frac{1}{2} \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{qg} & \gamma_{gg} \end{pmatrix} \left( \frac{\Sigma(N_F, N, \mu^2)}{G(N_F, N, \mu^2)} \right),
\]

\[
\frac{d}{d\ln \mu^2} \Delta_k(N_F, N, \mu^2) = -\frac{1}{2} \gamma_{qq} \Delta_k(N_F, N, \mu^2),
\]

and for the Wilson coefficients

\[
\frac{d}{d\ln \mu^2} \left( \frac{C^\text{PS}_{q,i}(N_F, N, Q^2/\mu^2)}{C^\text{PS}_{g,i}(N_F, N, Q^2/\mu^2)} \right) = \frac{1}{2} \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{qg} & \gamma_{gg} \end{pmatrix} \left( \frac{C^\text{PS}_{q,i}(N_F, N, Q^2/\mu^2)}{C^\text{PS}_{g,i}(N_F, N, Q^2/\mu^2)} \right),
\]

\[
\frac{d}{d\ln \mu^2} C^\text{NS}_{q,i}(N_F, N, Q^2/\mu^2) = \frac{1}{2} \gamma_{qq} C^\text{NS}_{q,i}(N_F, N, Q^2/\mu^2).
\]

Eqs. (52-54) are the QCD evolution equations of the massless parton densities and Wilson coefficients. The explicit expressions for the anomalous dimensions and massless Wilson coefficients are found in Refs. [69,70,100–103,105,161,162,181] and [87–91,93–103,105,106].

2.4 Heavy flavor corrections in the limit \( Q^2 \gg m^2 \)

In the case of pure photon exchange\(^1\) the heavy quark contribution to the structure functions \( F_{(2,2)}(x, Q^2) \) for one heavy quark of mass \( m \) and \( N_F \) light flavors are given by, cf. [134],

\[
\frac{1}{x} F_{(2,2)}(x, N_F + 1, Q^2, m^2) = 
\sum_{k=1}^{N_F} \epsilon^2_k \left[ L^\text{NS}_{q,(2,2)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes f_k(x, \mu^2, N_F) + f_{\overline{F}}(x, \mu^2, N_F) \right]
\]

\[ + \frac{1}{N_F} L^\text{PS}_{q,(2,2)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) \]

\[ + \frac{1}{N_F} L^\text{S}_{g,(2,2)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, N_F) \]

\[ + \epsilon^2_Q \left[ H^\text{PS}_{q,(2,2)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) \right]
\]

\[ + H^\text{S}_{g,(2,2)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, N_F) \],

Here the argument \((N_F + 1)\) indicates the presence of \( N_F \) massless and one massive quark flavor. One distinguishes the heavy flavor Wilson coefficients \( L^\text{NS}_{i,g}, L^\text{PS}_{i,g}, L^\text{S}_{i,g} \) and \( H^\text{PS}_{i,g}, H^\text{S}_{i,g} \), where the photon couples to a light (respectively heavy) quark line [119].

Generally the presence of one additional massive quark flavors introduces much more complex integrals and functions during the computation of the contributing Feynman diagrams when

\(^1\)For heavy flavor corrections in case of \( W^\pm \)-boson exchange up to \( O(\alpha_s^3) \) see [215–218].
compared to the purely massless case. Due to this we restrict ourselves to the kinematically very interesting region where the the virtuality $Q^2$ is much larger then mass $m^2$, such that power correction in $m^2/Q^2$ maybe be safely neglected, but not so large that logarithms of the kind $\ln(m^2/Q^2)$ become too large to allow for a perturbative series approximation.

This kinematic region the heavy flavor Wilson coefficients factor into the light flavor Wilson coefficients $C_{i,\langle 2, 2 \rangle}^{S, PS, NS} \left( N, N_F + 1, \frac{Q^2}{\mu^2} \right)$ and the process-independent massive operator matrix elements $A_{ij}^{S, PS, NS}$, cf. [139, 219],

$$
C_{j,\langle 2, 2 \rangle}^{S, PS, NS, asymp} \left( N, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = 
\sum_i A_{ij}^{S, PS, NS} \left( N, N_F + 1, \frac{m^2}{\mu^2} \right) C_{i,\langle 2, 2 \rangle}^{S, PS, NS} \left( N, N_F + 1, \frac{Q^2}{\mu^2} \right) + O \left( \frac{m^2}{Q^2} \right). 
$$

The explicit expressions for the massive operator matrix elements are obtained from the Green’s functions

$$
G_{ij}^{Q, \langle 2, 2 \rangle} = J_N h_{ij} Q^{n_\mu} O_{i}^{\mu} Q_{j}^{n_\mu},
$$

$$
G_{ab}^{Q, \langle 2, 2 \rangle} = J_N h_{ab} Q^{n_\mu} O_{i}^{\mu} Q_{j}^{n_\mu}. 
$$

Here $\langle q, k \mid, \langle g, \nu, j \mid$ denote external quark and gluon states, respectively and the indices $i,j$ ($a,b$) are the color indices of the fundamental(adjoint) representation. The operators $O_i$ are defined in (37-39) and the subscript $Q$ indicates that only contributions that contain at least one heavy quark line are taken into account. The local operators (37-39) are traceless and symmetric under the Lorentz group, whereas the Green’s function contains trace terms which do not contribute to the final result. These terms may be projected out from the beginning by contracting with the external source term

$$
J_N \equiv \Delta_{\mu_1} \ldots \Delta_{\mu_N},
$$

where $\Delta_\mu$ represents an arbitrary light-like vector, $\Delta^2 = 0$. The Green’s functions still contains a Lorentz or spinor structure and color indices due to the external partonic states. These structure is mapped to 1 to define the massive operator matrix elements.

For the Green’s functions with external gluonic states one may choose to include unphysical transverse gluon states in the summation over the indices $\mu, \nu$. In this case the respective projection reads [139]

$$
P_g^{(1)} \hat{G}^{ab}_{l,\langle Q \rangle, \mu, \nu} \equiv -\frac{\delta_{ab}}{N_c^2 - 1} \frac{\delta^{\mu\nu}}{D - 2} (\Delta \cdot p)^{-N} \hat{G}^{ab}_{l,\langle Q \rangle, \mu, \nu},
$$

and diagrams with external ghosts have to be included to compensate the unphysical terms. If one uses the physical projector

$$
P_g^{(2)} \hat{G}^{ab}_{l,\langle Q \rangle, \mu, \nu} \equiv \frac{\delta_{ab}}{N_c^2 - 1} \frac{1}{D - 2} (\Delta \cdot p)^{-N} \left( -g^{\mu\nu} + \frac{\phantom{\frac{1}{2}} \Delta^{\mu} \Delta^{\nu}}{\Delta \cdot p} \right) \hat{G}^{ab}_{l,\langle Q \rangle, \mu, \nu},
$$

additional ghost diagrams are not required.

For the Green’s functions with external fermions one applies the projector

$$
P_f \hat{G}^{ij}_{l,\langle Q \rangle} \equiv \frac{\delta^{ij}}{N_c} (\Delta \cdot p)^{-N} \frac{1}{4} \text{Tr} \left[ \not p \hat{G}^{ij}_{l,\langle Q \rangle} \right].
$$
Here and in (60)–(61), $N_c$ denotes the number of colors, cf. Appendix A. The unrenormalized OMEs are then defined by the projections of the Green’s functions $[139]$

$$\tilde{A}_{tq} \left( \frac{\hat{m}_t^2}{\mu^2}, \varepsilon, N \right) = P_\mu(P^{(1,2)}, \mu) \tilde{\rho}_{1\ell(Q), \mu}^a, \quad (63)$$

$$\tilde{A}_{tq} \left( \frac{\hat{m}_t^2}{\mu^2}, \varepsilon, N \right) = P_\mu \tilde{\Gamma}_{1\ell(Q)}, \quad (64)$$

The respective Feynman rules are given in Appendix B. The asymptotic heavy flavor Wilson coefficients are given by $[219]$ \footnote{Ref. [219] contains a few inconsistencies concerning the $\tilde{f}$ notation which have been corrected in Ref. [220].}

$$C_{q,(2,L)}^{NS} \left( N, N_F, \frac{Q^2}{\mu^2} \right) + L_{q,(2,L)}^{NS} \left( N, N_F + 1, \frac{Q^2 - m_q^2}{\mu^2} \right) = A_{qg, L}^{NS} \left( N, N_F + 1, \frac{m_q^2}{\mu^2} \right) C_{q,(2,L)}^{NS} \left( N, N_F + 1, \frac{Q^2}{\mu^2} \right), \quad (65)$$

in the non–singlet case, and for the pure–singlet and singlet contribution

$$C_{g,(2,L)}(N_F) + L_{g,(2,L)}(N_F + 1) = A_{gq, L} \left( N_F + 1, N_F \tilde{\Gamma}_g(2, L) \right) (N_F + 1)$$

$$+ A_{gq, L} \left( N_F + 1 \right) C_{q,(2,L)}^{NS} \left( N_F + 1 \right)$$

$$+ A_{gq, L} \left( N_F + 1 \right) N_F \tilde{\Gamma}_g(2, L) (N_F + 1) \right), \quad (66)$$

The $H_{1,2}$s are decomposed into OMEs and light flavor Wilson coefficients by

$$H_{q,(2,L)}^{PS}(N_F + 1) = A_{qg, L} \left( N_F + 1 \right) \left[ C_{q,(2,L)}^{NS} \left( N_F + 1 \right) + \tilde{C}_{q,(2,L)}^{PS} \left( N_F + 1 \right) \right]$$

$$+ A_{gq, L} \left( N_F + 1 \right) \left[ C_{q,(2,L)}^{NS} \left( N_F + 1 \right) + \tilde{C}_{q,(2,L)}^{PS} \left( N_F + 1 \right) \right]$$

$$+ A_{gq, L} \left( N_F + 1 \right) \left[ C_{q,(2,L)}^{NS} \left( N_F + 1 \right) + \tilde{C}_{q,(2,L)}^{PS} \left( N_F + 1 \right) \right], \quad (68)$$

Here and in the following, the index "asymp" to denote the asymptotic heavy flavor Wilson coefficients is omitted. Expanding the expressions (65–69) up to $O \left( \alpha_s^2 \delta_2 \right)$ yields $[139]$

$$L_{q,(2,L)}^{NS}(N_F + 1) = \alpha_s^2 \left[ A_{qg, L}^{(2), NS} \left( N_F + 1 \right) \delta_2 + \tilde{C}_{q,(2,L)}^{NS} \left( N_F \right) \right]$$
the variable flavor number scheme (VFNS) and parton densities may be defined to describe them. Here we have applied the notation

\[
\tilde{f}(N_F) = \frac{f(N_F)}{N_F}, \quad \hat{f}(N_F) = f(N_F + 1) - f(N_F).
\]

2.5 The variable flavor number scheme (VFNS)

In an exact description only massless particles may be interpreted as partons in hard scattering processes since their lifetime must be large compared to the interaction time \( \sim \frac{1}{Q^2} \) [221]. For large values of \( Q^2 \) the heavy quark flavors may, however, be considered as becoming effectively massless and variable flavor number scheme- parton densities may defined to describe them.

In Mellin–space on obtains the following set of parton densities is obtained, [134] :

\[
f_k(N_F + 1, N, \mu^2, m^2) + f_k(N_F + 1, N, \mu^2, m^2) =
\]
$$A^{NS}_{qq;Q} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot \left[ f_k(N_F, N, \mu^2) + f_{\overline{k}}(N_F, N, \mu^2) \right]$$

$$+ \frac{1}{N_F} A^{PS}_{qq;Q} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot \Sigma(N_F, N, \mu^2)$$

$$+ \frac{1}{N_F} A_{gg;Q} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot G(N_F, N, \mu^2), \quad (76)$$

$$f_Q(N_F + 1, N, \mu^2, m^2) + f_{\overline{Q}}(N_F + 1, N, \mu^2, m^2) =$$

$$A^{PS}_{Qq} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot \Sigma(N_F, N, \mu^2)$$

$$+ A_{Qg} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot G(N_F, N, \mu^2) \cdot (77)$$

The flavor singlet, non–singlet and gluon densities for \((N_F + 1)\) flavors are

$$\Sigma(N_F + 1, N, \mu^2, m^2) = \left[ A^{NS}_{qq;Q} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) + A^{PS}_{qq;Q} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \right]$$

$$+ A^{PS}_{Qq} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot \Sigma(N_F, N, \mu^2)$$

$$+ \left[ A_{gg;Q} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) + A_{Qg} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \right] \cdot G(N_F, N, \mu^2), \quad (78)$$

$$\Delta_k(N_F + 1, N, \mu^2, m^2) = f_k(N_F + 1, N, \mu^2, m^2) + f_{\overline{k}}(N_F + 1, N, \mu^2, m^2)$$

$$- \frac{1}{N_F + 1} \Sigma(N_F + 1, N, \mu^2, m^2), \quad (79)$$

$$G(N_F + 1, N, \mu^2, m^2) = A_{gg;Q} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot \Sigma(N_F, N, \mu^2)$$

$$+ A_{Qg} \left( N, N_F + 1, \frac{\mu^2}{m^2} \right) \cdot G(N_F, N, \mu^2) \cdot (80)$$

These expressions, however, rely on the presence of a strong hierarchy among the masses of the heavy quark flavors \(m_{Q_i}\),

$$m_{Q_1}^2 \ll m_{Q_2}^2 \ll \cdots \ll m_{Q_i} \cdot (81)$$

This condition is not fulfilled when considering charm– and bottom–contributions, with \(\eta = m_c^2/m_b^2 \sim \frac{1}{10}\). Up to two-loop order this does not cause an essential problem since no graphs with both \(c\)- and \(b\)-quarks contribute. This changes at \(O(\alpha_s^3)\) and the respective power corrections have to be taken into account, see Sections 5 and 6.
3 Topologies with massless loops

Among the \( \sim 3000 \) diagrams contributing to the different massive OMEs at 3-loop order which describe the massive contributions to the structure function \( F_2(x, Q^2) \) for \( Q^2 \gg m^2 \), there are many which contain at least one completely massless fermion loop. These massless loops can be integrated out first and yield new propagators with real exponents. Although these real exponents lead to a complication, methods inspired by those at 2-loop order can be applied to treat this class of diagrams. Using this method the complete OME \( A_{qg, Q} \) has been computed [140, 222]. For all contributing diagrams with two massless bubble insertions this method has been completely automatized [223].

3.1 General approach

In [139] a database of fixed Mellin moments for all diagrams contributing to the OMEs \( A_{qg,Q}^{NS}, A_{qg,Q}^{PS}, A_{gq,Q}, A_{Qg}, A_{gg,Q} \) has been given. Codes written in FORM [224] and Maple [188] have been set up to identify all diagrams with at least one massless loop. After applying the Feynman rules of Appendix B the occurring fermionic traces of \( \gamma \)-matrices were calculated using the built-in FORM-functions. The momentum integrals were performed in a loop–by–loop approach: All denominators contributing one loop were combined using the Feynman parametrization in Appendix (C.19). As a next step the momenta are shifted in order to symmetrize the \( D \)-dimensional momentum integral. As a byproduct this yields factors of the form

\[
(\Delta.k_i + \Delta.(P_1(\{x_k\})k_{j_1} + \cdots + P_n(\{x_n\})k_{j_n}))^N ,
\]

with polynomials \( P_i \) in the Feynman parameters. Using the binomial theorem

\[
(\Delta.k_i + \Delta.l)^N = \sum_{j=0}^{N} \binom{N}{j} (\Delta.l)^{N-j}(\Delta.k_i)^j ,
\]

only the first few terms contribute. This is due to the property that symmetric momentum integrals over odd powers of the momentum vanish and the occurrence of the contraction \( \Delta.\Delta = 0 \) for higher powers of \( \Delta.k_i \). A maximum of the first three terms had to be considered in the present computation. The symmetric \( D \)-dimensional momentum integral was finally evaluated by applying the rules in Appendix C and the same steps were repeated for the remaining loop momenta. Here the order in which the momenta are integrated is not arbitrary. More simple special functions representation may be obtained in many cases by specific choices for the integration order of the loop momenta. One–scale one–loop subdiagrams were integrated out first, as in this case the scales factor out of the denominator functions and the Feynman parameter integrals reflect the integral definition of the Euler–B function

\[
B(\alpha, \beta) = \int_{0}^{1} dx_i \ x_i^{\alpha-1} (1 - x_i)^{\beta-1} = \Gamma \left[ \begin{array}{c}
A \\
C
\end{array} \right] A + C ,
\]

where the shorthand notation (D.27) has been applied. To avoid more involved special function representations it is furthermore advantageous to integrate out the loop consisting of all massive propagators as a next step if this does not yield too complicated polynomials \( P_i \) in the operator-insertion (82). After integrating out all the loop momenta using this prescription all Feynman...
parameter integrals obey the general form

\[ I = \int_0^1 \cdots \int_0^1 \left( \prod_{i=1}^n dx_i \ x_i^{\alpha_i} (1 - x_i)^{\beta_i} \right) \delta \left( 1 - \sum_{i} x_j \right) \cdots \delta \left( 1 - \sum_{k} x_k \right) \frac{OP(x_1, \cdots, x_n)}{Q^{\alpha_0}}. \]  

(85)

Here \( \delta (1 - \sum i x_j) \) denotes the Dirac-\( \delta \) distribution whereas the sums within are taken over all Feynman parameters attached to the different propagators within one loop-momentum integration. Furthermore, the \( \alpha_i \) and \( \beta_i \) are either integer numbers or linear polynomials in the dimensional regularization parameter \( \varepsilon \) and \( Q \) is a polynomial in the Feynman parameters describing the mass distribution of the diagram. The precise form of the operator function \( OP(x_1, \cdots, x_n) \) in Eq. (85) depends on the specific operator insertion (see Appendix B):

- operator insertion on a fermionic line
  \[ OP(x_1, \cdots, x_n) = P_1^N \]  
  (86)

- operator insertion on a \( qqg \)-vertex
  \[ OP(x_1, \cdots, x_n) = \sum_{j=0}^N P_1^j P_2^{N-j} = \frac{P_1^{N+1} - P_2^{N+1}}{P_1 - P_2} \]  
  (87)

- operator insertion on a \( qqgg \)-vertex
  \[ OP(x_1, \cdots, x_n) = \sum_{j=0}^N \sum_{l=0}^{N-j} P_2^j P_1^{N-l-j} (P_1 + P_4)^l \]  
  \[ = \sum_{j=0}^N P_2^j \times \frac{(P_1 + P_4)^{N-j+1} - P_1^{N-j+1}}{P_4} \]  
  \[ = \frac{(P_1 + P_4)^{N+2} - (P_1 + P_4) P_2^{N+1}}{P_4 (P_1 + P_4 - P_2)} - \frac{P_1^{N+2} - P_1 P_2^{N+1}}{P_4 (P_1 - P_2)} \]  
  (89)

with the \( P_i \) polynomials in the Feynman parameters \( x_i \). Using the closed forms (87,89) or (90) if the respective denominators factor into Euler-\( B \)-like factors \( x_i^\alpha \) or \( (1 - x_i)^\beta \) in all Feynman parameters. In a next step the \( \delta \)-distributions are integrated out by

\[ \int_0^1 \cdots \int_0^1 dy \ \delta \left( 1 - y \sum_{i=1}^n x_i \right) f(y, x_1, \cdots, x_n) = \theta \left( 1 - \sum_{i=1}^n x_i \right) f \left( 1 - \sum_{i=1}^n x_i, x_1, \cdots, x_n \right), \]  

(91)

where we have introduced the Heaviside function

\[ \theta (x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{otherwise}. \end{cases} \]  

(92)

The Feynman parameter integrals are then remapped to the hypercubus by applying

\[ \int_0^1 \cdots \int_0^1 dx_1 \cdots dx_n \ \theta \left( 1 - \sum_{i=1}^n x_i \right) f(x_1, x_2, \cdots, x_n) \]  

\[ = \int_0^1 \cdots \int_0^1 dx_1 \cdots dx_n \ \theta \left( 1 - \sum_{i=2}^n x_i \right) f \left( x_1 \left( 1 - \sum_{i=2}^n x_i \right), x_2, \cdots, x_n \right) \left( 1 - \sum_{i=2}^n x_i \right). \]  

(93)
The way in which Eqs. (91) and (93) are applied may lead to a higher or lower entanglement of the integrals, specifically with respect to the parts stemming from the operator insertion. We will thus discuss strategies to obtain an integral representation with the least possible complexity. If the remapping (93) of a specific parameter leads to a factorization of polynomials it is applied, e.g.

\[
\int \cdots \int_0^1 dx_1 \cdots dx_n \, \theta \left( 1 - \sum_{i=1}^n x_i \right) \left( x_1 + y_1 \left( 1 - \sum_{i=2}^n x_i \right) \right)^\alpha = \int \cdots \int_0^1 dx_1 \cdots dx_n \, \theta \left( 1 - \sum_{i=2}^n x_i \right) \left( x_1 + y_1 \right)^\alpha \left( 1 - \sum_{i=2}^n x_i \right)^\alpha
\]  

(94)

In a next step we map parameters that do not appear in the operator polynomials or other non-trivial factors. Here we always choose factors such that the corresponding mapping does not lead to additional terms with symbolic or negative exponents. If a Feynman parameter occurs only within one factor with negative or symbolic exponent, it may be integrated out directly. If these steps are not sufficient to obtain a factorization of the polynomials in the operator function into like terms, the binomial theorem is applied in order to impose this factorization.

After applying all these steps in an optimal way all diagrams of this class obey a representation of the form

\[
I \propto \int_0^1 \cdots \int_0^1 dx_1 \cdots dx_n \prod_i x_i^{a_i} (1 - x_i)^{\beta_i} \frac{OP'(x_1, \ldots, x_n)}{(1 - x_k x_l)}
\]

(95)

Here the \( a_i \) and \( \beta_i \) denote linear functions in \( \varepsilon \) and the Mellin variable \( N \), \( x_k, x_l \in \{x_1, \ldots, x_n\} \). The exponent \( \gamma \) may be zero or a linear expression in \( \varepsilon \) and the operator function \( OP' \) may or may not contain finite sums stemming from respective operator Feynman rules. All integrals except for the ones over the Feynman parameters \( x_k \) and \( x_l \) are evaluated using the integral representation of the Euler–B function (84) and its representation in terms of \( \Gamma \)-functions (84). The nested integral over the two remaining Feynman parameters \( x_k \) and \( x_l \) reflects the integral definition of the hypergeometric \( _3F_2 \) function. For the generalized hypergeometric function \( \text{P}_{P+1} F_P \) it is given by, cf. [142,143,225],

\[
\text{P}_{P+1} F_P \left[ \begin{array}{c} a_0, a_1, \ldots, a_P \\ b_1, \ldots, b_P \end{array} ; z \right] = \Gamma \left[ \begin{array}{c} b_1, \ldots, b_P \\ a_1, \ldots, a_P, b_1 - a_1, \ldots, b_P - a_P \end{array} \right] \times \int_0^1 \cdots \int_0^1 \prod_{i=1}^P (dx_i) \, x_i^{a_i - 1} (1 - x_i)^{b_i - a_i - 1} (1 - z x_1 \cdots x_P)^{-a_0}
\]

(96)

It obeys the series representation

\[
\text{P}_F Q \left[ \begin{array}{c} a_1, \ldots, a_P \\ b_1, \ldots, b_Q \end{array} ; z \right] = \sum_{i=0}^{\infty} \frac{(a_1)_{\ldots} (a_P)_i}{(b_1)_{\ldots} (b_Q)_{i}} \frac{z^i}{\Gamma(i+1)}
\]

(97)

with the radius of convergence

\[
|z| < 1, \text{ or } z = 1, \text{ Re} \left( \sum_{i=1}^P b_i - \sum_{i=1}^{P+1} a_i \right) > 0.
\]

(98)
The hypergeometric functions obey a class of contiguous relations \([142, 225–228]\), which may be used in some cases to reduce the occurring \(3F_2\) functions to the Gauss hypergeometric function \(2F_1\). In this case Gauß’ theorem

\[
2F_1 \left[ \begin{array}{c} a, b \\ c \\ 1 \end{array} \right] = \Gamma \left[ \begin{array}{c} c, c - a - b \\ c - a, c - b \end{array} \right], \quad \text{Re}(c - a - b) > 0 ,
\]

is applied to obtain a sum–free representation. All diagrams are thus expressed in terms of nested sums over Γ–functions. This representation now allows to extract the \(\epsilon\)-poles explicitly and to perform the Laurent series expansion using Eqns. (D.23-D.26). In a last step, the remaining sums are evaluated using summation algorithms based on difference fields \([229–235]\) encoded in the packages \texttt{Sigma} \([236, 237]\) and \texttt{EvaluateMultiSums} \([238, 239]\). In order to treat infinite summation bounds and to speed up the manipulations of expressions in terms of indefinite nested sums, the package \texttt{HarmonicSums} \([175, 177–179]\) is utilized in addition.

### 3.2 The OME \(A_qg,Q\)

![Some typical diagrams contributing to the OME \(A_qg,Q\). Thick lines represent heavy quarks, whereas thin ones depict massless fermion flavors.](image)

Most of the diagrams contributing to the operator matrix element \(A_{qg,Q}\) contain at least one single–scale loop and do thereby belong to the class of diagrams discussed in the present section. Furthermore a series of ladder graphs (as the rightmost graph in Figure 2) contributed. This class of graphs may be represented within the same set of special functions and is evaluated with the same methodology. The diagrams contributing to \(A_{qg,Q}\) are very similar to the diagrams describing the \(O(N_F^2)\) contribution to the OME \(A_{Qg}\) with the only difference being the assignment of the masses to the different fermionic lines. All together there are 89 diagrams with external gluons and 8 ghost–diagrams contributing to the OME \(A_{qg,Q}\). They can be grouped into 35 symmetry classes, where all members of a symmetry class yield the same result due to symmetry reasons. In this way we obtain for the constant term of the unrenormalized OME \(\hat{A}_{qg,Q}\) the expression

\[
\epsilon_{qg,Q}^{(3),0} = N_F T_F \left\{ C_F \left[ \frac{N^2 + N + 2}{N(N + 1)(N + 2)} \left( \frac{-56}{9} S_4 + \frac{32}{27} S_4 S_1 + \frac{8}{9} S_2 S_1^2 + \frac{4}{9} S_2 + \frac{4}{27} S_1^4 \right) \right] ight.
\]

\[
+ \frac{256}{9} S_1^3 c_3 \right] - \frac{16(10N^3 + 13N^2 + 29N + 6)}{81N^2(1 + N)(2 + N)} \left[ S_4^2 + 3S_4 S_1 \right]
\]

\[
+ \frac{32(5N^3 - 16N^2 + N - 6)}{81N^2(1 + N)(2 + N)} S_3 + \frac{8(109N^4 + 291N^3 + 478N^2 + 324N + 40)}{27N^2(1 + N)^2(2 + N)} S_2
\]

\[
+ \frac{8(215N^4 + 481N^3 + 930N^2 + 748N + 120)}{81N^2(1 + N)^2(2 + N)} S_1^2 - \frac{R_4}{243N^2(1 + N)^3(2 + N)} S_1
\]
with the polynomials

\[ R_4 = 24368N^5 + 81984N^4 + 179200N^3 + 225232N^2 + 126880N + 21504 , \]
\[ R_5 = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 , \]
\[ R_6 = 13923N^{17} + 180999N^{16} + 1064857N^{15} + 3812487N^{14} + 9348807N^{13} \]
\[ + 16391845N^{12} + 20248499N^{11} + 17070917N^{10} + 11536274N^9 + 11303496N^8 \]
\[ + 13846104N^7 + 16104128N^6 + 22643488N^5 + 29337472N^4 \]
\[ + 26395008N^3 + 15388416N^2 + 5612544N + 995328 , \]
\[ R_7 = 139N^6 + 1093N^5 + 3438N^4 + 5776N^3 + 5724N^2 + 3220N + 752 , \]
\[ R_8 = 1648N^6 + 11104N^5 + 34368N^4 + 63856N^3 + 71904N^2 + 43264N + 10880 , \]
\[ R_9 = 1244N^{10} + 10557N^9 + 40547N^8 + 90323N^7 + 114495N^6 + 49344N^5 \]
\[ - 69902N^4 - 115200N^3 - 64352N^2 - 11264N + 864 , \]
\[ R_{10} = 3315N^{15} + 39780N^{14} + 194011N^{13} + 471164N^{12} + 416251N^{11} - 860568N^{10} \]
\[ - 3525799N^9 - 6015120N^8 - 6333994N^7 - 4373672N^6 - 1907512N^5 \]
\[ - 499824N^4 - 217952N^3 - 264192N^2 - 160128N - 34560 . \]

Here \( C_F, C_A \) and \( T_F \) denote the color factors, cf. (A.20-A.23). \( S_k \equiv S_k(N) \) denotes the single harmonic sum (D.44) and \( \zeta_k = \sum_{k=1}^{\infty} k^{-1} \), \( k \geq 2, k \in \mathbb{N} \) are values of the Riemann \( \zeta \)-function. The heavy flavor Wilson coefficient \( L_{\beta}^F \) is thus completed.

### 3.3 Automatized computation

Additionally to the \( C_AT_F^2N_F \) and the \( C_FT_F^2N_F \) contributions there is a much larger class of diagrams with similar topologies contributing to the relevant operator matrix elements. All

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3In collaboration with Mark Round.
diagrams with at least one massless internal two-point function are in principle computable with the same method. Due to the relatively large number of these diagrams it is advantageous to set up automatized procedures in order to allow for an efficient evaluation.

All contributing diagrams have been generated using QGRAF in Ref. [139]. We used the same diagram database to allow for easier comparison with the fixed Mellin moments computed in this reference. FORM-codes have been set up to implement the Feynman rules including those for the operator insertions, see Appendix B. A Mathematica code then analyzes the specific diagram topologies, selects those diagrams with at least one massless bubble subtopology and chooses an optimal integration order for the different loop-momenta. This integration order is determined by the following criteria

- Integrate all completely massless loops first
- Integrate all completely massive loops
- If in doubt integrate the loop without any operator insertion first
- Otherwise choose next integration momenta randomly.

The momenta are then integrated using a FORM-code and a Feynman-parameter representation is obtained. We apply the strategies outlined in Section 3.1 to map the integrals to the hypercubus in a way that avoids further entanglement among the different Feynman parameter integrals. All Feynman parameter integrals which may be performed without yielding a higher nesting among the remaining integration variables are performed at this step and the binomial theorem is applied until all remaining integrals correspond to the integral definitions of the Euler-B function or a generalized hypergeometric \( pF_Q \)-function. The \( \varepsilon \)-expansion is performed using the sum representation of the hypergeometric functions and the nested sums are evaluated in a final step using the summation packages Sigma [236, 237], EvaluateMultiSums [238, 239] and \( \rho \)-sum [240, 241]. For fixed values of the Mellin variable \( N \), all results were in agreement with the Mellin moments computed in Ref. [139].

All results of the individual diagrams are expressed within the class of nested harmonic sums up to weight \( w = 5 \), and the constants \( \zeta_2 \) and \( \zeta_3 \).

<table>
<thead>
<tr>
<th>OMEs</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{Qg} ) (including ghost diagrams)</td>
<td>100</td>
</tr>
<tr>
<td>( A_{NS} )</td>
<td>29</td>
</tr>
<tr>
<td>( A_{gS\text{,trans}} )</td>
<td>29</td>
</tr>
<tr>
<td>( A_{PS} )</td>
<td>30</td>
</tr>
<tr>
<td>( A_{gg,Q} ) (including ghost diagrams)</td>
<td>89</td>
</tr>
<tr>
<td>( A_{gq,Q} )</td>
<td>27</td>
</tr>
</tbody>
</table>

Table 1: Number of diagrams that could be computed in an automated way via representations in special functions
4 Logarithmic contributions to the heavy flavour Wilson coefficients

The computation of the complete heavy flavour Wilson coefficients at \( O(\alpha_s^3) \) constitutes a very challenging task due to very large expressions and complex mathematical structures. Due to the renormalization procedure [139] we know, however, that all logarithmic contributions in \( (m^2/\mu^2) \) can be expressed in terms of the 3-loop anomalous dimensions and different lower order quantities\(^4\). All 3-loop contributions to the two Wilson coefficients \( L_q^S \) and \( L_q^\text{PS} \) have been computed and they are now presented in a complete form, see also Ref. [220]. The complete Wilson coefficients \( L_q^\text{NS} \) and \( H_q^\text{PS} \) including logarithmic and non-logarithmic contributions can be found in Refs. [157, 158, 223, 244, 245]. Furthermore all logarithmic contributions to the operator matrix elements, which are needed for the variable flavor number scheme have been computed [156].

The single-mass contributions to the operator matrix elements have the general form

\[
\hat{A}^{(3)}_{ij} (\varepsilon) = \left( \frac{m^2}{\mu^2} \right)^{2\varepsilon/3} \left[ \frac{1}{\varepsilon^2} \hat{a}^{(3),3}_{ij} + \frac{1}{\varepsilon} \hat{a}^{(3),2}_{ij} + \frac{1}{\varepsilon} \hat{a}^{(3),1}_{ij} + \hat{a}^{(3),0}_{ij} \right],
\]

where the \( \hat{a}^{(3),k}_{ij} \) are functions of the Mellin variable \( N \) only. After renormalizing the mass, the coupling, the operator and applying mass factorization, the renormalized expressions for the massive operator matrix elements \( A_{ij} \) obey the general structure [139, 246]

\[
A^{(3)}_{ij} \left( \frac{m^2}{\mu_R^2} \right) = \hat{a}^{(3),3}_{ij} \ln \left( \frac{m^2}{\mu_R^2} \right) + \hat{a}^{(3),2}_{ij} \ln^2 \left( \frac{m^2}{\mu_R^2} \right) + \hat{a}^{(3),1}_{ij} \ln \left( \frac{m^2}{\mu_R^2} \right) + \hat{a}^{(3),0}_{ij},
\]

where \( \mu_R \) denotes the renormalization scale. For simplicity we will identify the renormalization scale with the factorization scale from now on, \( \mu_R = \mu_F = \mu \) and use the shorthand notation

\[
L_M = \ln \left( \frac{m^2}{\mu^2} \right).
\]

Due to the form of the mass-dependence in (108) the coefficients \( \hat{a}^{(3),1}, \hat{a}^{(3),2} \) and \( \hat{a}^{(3),3} \) depend on the pole terms of the unrenormalized OME only. The renormalization procedure introduces terms which cancel these pole terms and all logarithmic coefficients of the renormalized OMEs can thus be expressed in terms of anomalous dimensions and lower order quantities. The only contributions that have not been known yet are the constant parts of the 3-loop OMEs \( a_{ij} \). The analytic expression for \( a_{qq}^{\text{PS}} \) has been published in [140] as well as \( a_{qq,Q} \) the computation of which is outlined in Section 3. In the \( \overline{\text{MS}} \)–scheme the renormalized OMEs \( A_{qq}^{\text{PS}} \) and \( A_{gg}^{\text{PS}} \) read [139]

\[
A_{qq,Q}^{(3),\overline{\text{MS}}} = N_F \left\{ \frac{\gamma_{qq}^{(0)} \gamma_{gg}^{(0)} \beta_{0,Q}}{12} L_M^3 + \frac{1}{8} \left( 4 \gamma_{qq}^{(1),\text{PS}} \beta_{0,Q} + \gamma_{qq}^{(0)} \gamma_{gg}^{(1)} \right) L_M^2 \right. \\
+ \frac{1}{4} \left( 2 \gamma_{qq}^{(2),\text{PS}} + \gamma_{qq}^{(0)} \left( 2 a_{gg,Q}^{(2)} - 2 \gamma_{gg}^{(0)} \beta_{0,Q} \right) \right) L_M \right\},
\]

\(^4\)For heavy flavour Wilson coefficients depending on both \( m_b \) and \( m_c \) there are further logarithms in the mass ratio \( \eta = m_c^2/m_b^2 \) which can not be determined by the renormalization procedure only, [242, 243]
$$-\gamma_{gg}^{(0)} \mathcal{A}_{gg,Q}^{(2)} + \frac{\gamma_{gg}^{(0)} \gamma_{gg}^{(0)} \beta_{0,Q} \zeta_{3}}{12} - \frac{\gamma_{gg}^{(1),PS}}{4} \beta_{0,Q} \zeta_{2} + a_{gg,Q}^{(3),PS},$$  \hspace{1cm} (111)$$

and

$$A_{gg,Q}^{(3),MS} = N_F \left[ \frac{\gamma_{gg}^{(0)}}{48} \left\{ \gamma_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \left[ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_{0} \right] \right\} \right] + \frac{1}{8} \left\{ 2\hat{\gamma}_{gg}^{(1)} \beta_{0,Q} - \hat{\gamma}_{gg}^{(1),PS} - \hat{\gamma}_{gg}^{(1),NS} + 2\beta_{1,Q} \right\} \right] \right] L_M + \frac{1}{2} \left\{ \hat{\gamma}_{gg}^{(2)} + \hat{\gamma}_{gg}^{(0)} a_{gg,Q}^{(2)} \right\} L_M$$

$$- a_{gg,Q}^{(2),NS} + \beta_{1,Q}^{(1)} - \frac{\gamma_{gg}^{(0)}}{8} \zeta_{2} \left[ \gamma_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \left[ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_{0} \right] \right] \right\} L_M$$

$$+ \frac{\gamma_{gg}^{(0)}}{16} \left[ \gamma_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \left[ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_{0} \right] \right]$$

$$- \frac{\gamma_{gg}^{(0)}}{16} \left[ \gamma_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \left[ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_{0} \right] \right] \right\} L_M$$

$$+ \frac{\gamma_{gg}^{(0)}}{48} \left[ \gamma_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \left[ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_{0} \right] \right]$$

$$- \frac{\gamma_{gg}^{(0)}}{48} \left[ \gamma_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \left[ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_{0} \right] \right] \right\} L_M$$

$$- \frac{\gamma_{gg}^{(0)}}{16} \left[ \gamma_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \left[ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_{0} \right] \right] \right\} L_M$$

In (111,112) the \(\gamma_{ij}\) denote the anomalous dimensions, and the \(\beta_i\) and \(\beta_{ij,Q}\) coefficients of the QCD-\(\beta\) function, see Eqs. (195-198). The light flavor Wilson coefficients obey the general form

$$C_i \left( N, a_s, Q^2 / \mu^2 \right) = \sum_{l=0}^{\infty} a_s^l \sum_{j=0}^{l} c_{ij}^{l,j} L_Q^j,$$  \hspace{1cm} (113)$$

with the logarithm

$$L_Q = \ln \left( \frac{Q^2}{\mu^2} \right).$$  \hspace{1cm} (114)$$

We use the matrix notation

$$c^{i,k} = (c^{i,k}_q , c^{i,k}_g ),$$  \hspace{1cm} (115)$$

and

$$\gamma^{(k)} = \begin{pmatrix} \gamma^{(k)}_{qq} & \gamma^{(k)}_{qg} \\ \gamma^{(k)}_{gq} & \gamma^{(k)}_{gg} \end{pmatrix}.$$  \hspace{1cm} (116)$$

Here \(c^{0,0} = (1, 0)\) and the first order contributions \(c^{1,0}\) are given in Ref. [89]. The second order contributions have been computed in Refs. [95-97]. The coefficients \(c_{ij}^{l,j}\) with \(j > 1\) follow from the evolution equation obtained by resolving the RGE for the Wilson coefficients. Up to \(O(a_s^2)\) they are given by [247]

$$c^{1,1} = c^{0,0} \gamma^{(0)}$$

$$c^{2,1} = c^{0,0} \gamma^{(1)} + c^{1,0} \left( \gamma^{(0)} - \beta_0 E \right)$$

$$c^{2,2} = \frac{1}{2} c^{1,1} \left( \gamma^{(0)} - \beta_0 E \right)$$

$$c^{3,1} = c^{0,0} \gamma^{(2)} + c^{1,0} \left( \gamma^{(1)} - \beta_1 E \right) + c^{2,0} \left( \gamma^{(0)} - 2\beta_0 E \right)$$

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\[ c^{3.2} = \frac{1}{2} \left( c^{1.1} \left( \gamma^{(1)} - \beta_1 E \right) + c^{2.1} \left( \gamma^{(0)} - 2\beta_0 E \right) \right) \]

\[ c^{3.3} = \frac{1}{3} c^{2.2} \left( \gamma^{(0)} - 2\beta_0 E \right), \]

with \( E \) the unity matrix.

For \( Q^2 = \mu^2 \) the light flavor Wilson coefficients are given in [87–91,93–103,105,106,248]. Renormalizing the OMEs using Eqs. (283,112) and inserting the result into (71,72) yields the complete expression for the respective heavy flavor Wilson coefficients. The leading order contribution to the Wilson coefficient \( L_{q,2}^{PS} \) starts at \( O(a_s^4) \). In Mellin \( N \)-space it reads

\[
L_{q,2}^{PS} = \frac{1}{2} \left[ 1 + (-1)^N \right] \times a_s^4 \left\{ C_F N_F T_F^2 \left[ -\frac{32P_4 L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2} + L_Q \left( \frac{64P_6}{27(N-1)N^4(N+1)^4(N+2)^3} \right. \right. \right.
\]

\[
- \frac{256P_4(-1)^N}{9(N-1)N^2(N+1)^3(N+2)^2} + \frac{2\gamma_0^2 N^2(N+2) L_M^2}{3(N-1)} \right. \right. \]

\[
\left. \left. + \left[ \frac{64(N^2 + N + 2)(8N^3 + 13N^2 + 27N + 16)}{9(N-1)^2N^2(N+1)^3(N+2)} \right] - \frac{64(N^2 + N + 2)^2S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] \right] \left[ L_M \right.
\]

\[
+ \frac{512S_{1,2}}{3(N-1)N(N+1)(N+2)} - \frac{32P_4 L_M^2}{9(N-1)N^3(N+1)^3(N+2)^2}
\]

\[
+ \frac{32P_7}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{32P_3 S_1}{3(N-1)N^3(N+1)^3(N+2)^2}
\]

\[
\left. \left. + \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[ S_1^2 - S_2 \right] \right] \right] \left[ L_M \right.
\]

\[
- \frac{243(N-1)N^5(N+1)^5(N+2)^2}{16P_4 S_1^2} + \frac{32P_3 S_1}{16P_5 S_2}
\]

\[
- \frac{27(N-1)N^3(N+1)^3(N+2)^2}{27(N-1)N^3(N+1)^3(N+2)^2} - \frac{32L_Q^2(N^2 + N + 2)^2}{32N^2 + N + 2)^2 L_M^2}
\]

\[
+ \frac{9(N-1)N^2(N+1)^2(N+2)}{(N-1)N^2(N+1)^2(N+2)} \left[ \frac{64S_1^2}{27} + \frac{32}{9}S_2S_1 + \frac{160S_3}{27} + \frac{256c_3}{9} \right]
\]

\[
+ N_F \delta_{q,4}^{PS}(3)(N_F) \right\}, \]  

with the polynomials

\[ P_1 = 4N^6 + 22N^5 + 48N^4 + 53N^3 + 45N^2 + 36N + 8 \]  

\[ P_2 = N^7 - 15N^6 - 58N^5 - 92N^4 - 76N^2 - 48N - 16 \]  

\[ P_3 = N^7 - 37N^6 - 248N^5 - 799N^4 - 1183N^3 - 970N^2 - 580N - 168 \]  

\[ P_4 = 11N^7 + 37N^6 + 53N^5 + 7N^4 - 68N^3 - 56N^2 - 80N - 48 \]  

\[ P_5 = 49N^7 + 185N^6 + 340N^5 + 287N^4 + 65N^3 + 62N^2 - 196N - 168 \]  

\[ P_6 = 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 \]  

\[ + 1864N^2 + 1248N + 432 \]
\[ P_7 = 143N^{10} + 838N^9 + 1995N^8 + 1833N^7 - 1609N^6 - 5961N^5 - 7503N^4 - 6928N^3 - 4024N^2 - 816N + 144 \]  
\[ P_8 = 176N^{10} + 973N^9 + 1824N^8 - 948N^7 - 10192N^6 - 19173N^5 - 20424N^4 - 16036N^3 - 7816N^2 - 1248N + 288 \]  
\[ P_9 = 1717N^{13} + 16037N^{12} + 66983N^{11} + 161797N^{10} + 241447N^9 + 216696N^8 + 86480N^7 - 67484N^6 - 170003N^5 - 165454N^4 - 81976N^3 - 15792N^2 - 1008N - 864 \]  

(130) \[ (131) \[ (132) \]

For brevity, here and in the following the massless 3-loop Wilson coefficients \( C_{i,j}^k \) have been left symbolically. Their analytic expressions were given in Ref. [106]. Furthermore we have used the shorthand notation

\[ z_{qq}^0 = -\frac{N^2 + N + 2}{N(N + 1)(N + 2)} \]  

(133)

The gluonic Wilson coefficient \( L_{g,2}^S \) reads:

\[
L_{g,2}^S = \frac{1}{2} \left[ 1 + (-1)^N \right] \left\{ a_s T_2^F N_F \left\{ L_M \left\{ \frac{4}{3} z_{qq}^0 S_1 - \frac{16(N^3 - 4N^2 - N - 2)}{3N(N + 1)(N + 2)} \right\} \right. \\
- \frac{4}{3} z_{qq}^0 L_Q L_M \left\} + a_s \left[ N_F T_F^3 \left\{ L_M \left\{ \frac{16}{9} z_{qq}^0 S_1 - \frac{64(N^3 - 4N^2 - N - 2)}{9N(N + 1)(N + 2)} \right\} - \frac{16}{9} z_{qq}^0 L_Q L_M \right\} \\
+ C_A N_F T_F^2 \left\{ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{8}{9} \frac{z_{qq}^0 S_1}{L_M} \right\} \frac{L_Q^3}{L_Q} \\
+ \left\{ \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N - 1)N(N + 1)^2(N + 2)} + \frac{8}{3} \frac{z_{qq}^0 S_1}{L_M} \right\} \frac{L_Q^2}{L_Q} \\
+ \left\{ \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_2}{9(N - 1)N(N + 1)^2(N + 2)} + \frac{8}{3} \frac{z_{qq}^0 S_1}{L_M} \right\} \frac{L_Q}{L_Q} \\
+ \left\{ \frac{32P_{22}S_1}{27(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{64(-1)^NP_{18}}{9(N - 1)N^2(N + 1)^4(N + 2)^4} \right\} \frac{L_M}{L_M} \\
+ \left\{ \frac{16P_{32}}{27(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{L_M^2}{L_M^2} \left\{ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \right\} \frac{L_Q^3}{L_Q} \\
+ \frac{8}{3} \frac{z_{qq}^0 S_1}{L_M} \right\} \frac{L_Q^2}{L_Q} \\
+ \left\{ \frac{32(8N^4 + 13N^3 - 22N^2 - 9N - 26)S_2}{9(N - 1)N(N + 1)^2(N + 2)^2} + \frac{128(N^2 + N + 1)S_3}{9(N + 1)(N + 2)} \right\} \frac{L_Q}{L_Q} \\
+ \frac{64(8N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}}{9(N - 1)N(N + 1)^2(N + 2)^2} + \frac{L_M}{L_M} \left\{ \frac{32P_{26}}{9(N - 1)N^3(N + 1)^3(N + 2)^3} \right\} \frac{L_Q^3}{L_Q} \\
+ \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N + 1)^3(N + 2)^3} - \frac{64(2N - 1)(N^3 + 9N^2 + 7N + 7)S_1}{9(N - 1)N(N + 1)^2(N + 2)^2} \right\} \frac{L_Q}{L_Q} \\
+ \frac{z_{qq}^0 \left[ -\frac{8}{3} S_2^2 + \frac{8S_2}{3} + \frac{16}{3} S_{-2} \right]}{L_M} - \frac{128(N^2 + N + 3)S_{-3}}{3(N + 1)(N + 2)} + \frac{z_{qq}^0 \left[ \frac{8}{9} S_3^2 - 8S_2 S_1 + \frac{32}{3} S_{2,1} \right]}{L_M} \right\} \frac{L_Q}{L_Q} 
\]

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\[
\begin{align*}
+ & \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1)\left[\frac{64}{3}S_{-2}S_1 - 32\zeta_3\right]}{N(N+1)} L_Q + \frac{16P_{12}S_{1}^2}{81N(N+1)(N+2)^3} \\
+ & \frac{8P_{39}}{243(N-1)N^3(N+1)(N+2)^3} + \frac{512}{9}\left(\frac{N^2 + N + 1}{(N-1)N^2(N+1)^2(N+2)^2}\right) \zeta_3 \\
+ & \frac{8P_{39}S_1}{243(N-1)N^4(N+1)(N+2)^4} + L_M^3 \left[ - \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} \right] \\
- & \frac{8\gamma_0 S_1}{9} + \frac{16P_{13}S_2}{81N(N+1)^3(N+2)^3} + \frac{64(5N^4 + 38N^3 + 59N^2 + 31N + 20)S_3}{81N(N+1)^2(N+2)^3} \\
- & \frac{32(121N^3 + 293N^2 + 414N + 224)S_2}{81N(N+1)^2(N+2)} + L_M^2 \left[ - \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right] \\
+ & \frac{8P_{25}}{9(N-1)N^3(N+1)(N+2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N^2(N+1)(N+2)} + \gamma_0^{\frac{4}{3}} S_1^2 \\
+ & \frac{4S_2}{3} + \frac{8}{3} S_2 - 2 \right] + \frac{128(5N^2 + 8N + 10)S_{-3}}{27N(N+1)(N+2)} \\
& \frac{(5N^4 + 20N^3 + 41N^2 + 49N + 20)}{N(N+1)^2(N+2)^2} \\
& + L_M \left[ \frac{32(2N^5 + 21N^4 + 27N^3 + 11N^2 + 25N - 14)S_1}{9(N-1)N(N+1)^2(N+2)^2} + \frac{16P_{27}S_1}{27(N-1)N^3(N+1)^3(N+2)^3} \right] \\
+ & \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3(N+2)^3} - \frac{16}{3} \gamma_0 S_2 S_1 - \frac{64(-1)^N P_{4,1}}{9(N-1)N^2(N+1)(N+2)^4} \\
+ & \frac{16P_{34}}{27(N-1)N^4(N+1)^4(N+2)^4} - \frac{32(2N^5 + 21N^4 + 51N^3 + 23N^2 - 11N - 14)S_2}{9(N-1)N^3(N+1)^3(N+2)^4} \\
+ & \frac{64S_3}{3(N+2)} - \frac{64(2N^5 + 21N^4 + 36N^3 - 7N^2 - 68N - 56)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} + \frac{26S_{-2,1} - 12S_{-3}}{N(N+1)(N+2)} \\
+ & \frac{(N-1)\left[\frac{64}{N+1}S_{-2}S_1 - 32\zeta_3\right]}{N(N+1)} + \gamma_0^{\frac{1}{27}} S_1^4 - \frac{2}{9} S_2 S_1^2 + \left[ \frac{16}{9} S_{2,1} - \frac{40S_3}{27} \right] S_1 \\
+ & \frac{64}{9} \zeta_3 S_1 + \frac{1}{9} S_2^2 + \frac{14S_4}{9} + \frac{32}{9} S_{-4} + \frac{32}{9} S_{3,1} - \frac{16}{9} S_{2,1,1} \right] \\
+ C_F N_F T_F^2 \left[ \frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^4 + 6N^3 - N^2 - 4N + 12)}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{16}{9} \gamma_0 S_1 \right] L_Q^3 \\
+ & \left[ \frac{4P_{31}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{16P_{21}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} + \gamma_0^{\frac{20}{3}} \frac{S_2}{S_1} - 4S_1^2 \right] \\
+ & L_M \left[ \frac{8(N^2 + N + 2)P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{3} \gamma_0 S_1 \right] L_Q^2 + \left[ \frac{16L_{M}^2(N^2 + N + 2)^3}{(N-1)N^3(N+1)^3(N+2)^2} \right] \\
+ & \frac{16P_{22}S_2^2}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^5}{4P_{32}} \\
+ & \frac{45(N-1)^2N^3(N+1)^3(N+2)^4(N+3)^3}{8P_{30}S_1} - \frac{9(N-1)N^4(N+1)^4(N+2)^3}{9(N-1)N^4(N+1)^4(N+2)^3} \\
\right]
\end{align*}
\]
\[ + \frac{16P_{23}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + L_M \left[ -\frac{16P_{28}}{3(N-1)N^4(N+1)^4(N+2)^3} \right] \]
\[ + \frac{16P_7S_1}{3(N-1)N^3(N+1)^3(N+2)^2} + \gamma_{qg}^0 \left( \frac{16S_2}{3} - \frac{16}{3} S_1^2 \right) \]
\[ + \frac{64P_{16}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \gamma_{qg}^0 \left( \frac{8}{3} S_1^3 - \frac{8}{3} S_2 S_1 - \frac{32}{3} S_{2,1} \right) \]
\[ + \frac{\frac{512}{3} S_1 S_{-2} + \frac{256}{3} S_{-3} - \frac{512}{3} S_{-2,1}}{N(N+1)(N+2)} + \frac{64(N-1)\zeta_3}{N(N+1)} \]
\[ + \frac{8(215N^4 + 481N^3 + 930N^2 + 748N + 120)S_2^2}{81N^2(N+1)^2(N+2)} + \frac{P_{40}}{243(N-1)N^6(N+1)^6(N+2)^2} \]
\[ + \frac{4P_{35}S_1}{243(N-1)N^5(N+1)^5(N+2)^2} + L_M \left[ \frac{8(N^2 + N + 2)P_{10} \zeta_3}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9} \gamma_{qg}^0 S_1 \right] \]
\[ + L_M^2 \left[ \frac{4P_{39}}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{16P_{20}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right] \]
\[ + \frac{8(109N^4 + 291N^3 + 478N^2 + 324N + 40)S_2}{27N^2(N+1)^2(N+2)} \]
\[ + \frac{(10N^3 + 13N^2 + 29N + 6) \left[ \frac{16}{9} S_1^3 - \frac{16}{9} S_2 S_1 \right]}{N^2(N+1)(N+2)} + \frac{32(5N^3 - 16N^2 + N - 6)S_3}{81N^2(N+1)(N+2)} \]
\[ + \frac{8P_{19}S_2^2}{27(N-1)N^4(N+1)^4(N+2)^2} - \frac{14S_1^3}{9} \]
\[ P_{17} = 3N^8 + 8N^7 - 2N^6 - 24N^5 + 15N^4 + 88N^3 + 152N^2 + 96N + 48 \] (142)
\[ P_{18} = 5N^8 - 8N^7 - 137N^6 - 436N^5 - 713N^4 - 672N^3 - 407N^2 - 192N - 32 \] (143)
\[ P_{19} = 7N^8 + 4N^7 - 90N^6 - 224N^5 - 21N^4 + 388N^3 + 608N^2 + 336N + 144 \] (144)
\[ P_{20} = 10N^8 + 46N^7 + 105N^6 + 139N^5 + 87N^4 - 17N^3 + 50N^2 + 84N + 72 \] (145)
\[ P_{21} = 19N^8 + 70N^7 + 63N^6 - 41N^5 - 192N^4 - 221N^3 - 142N^2 - 60N - 72 \] (146)
\[ P_{22} = 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 + 373N^3 - 218N^2 - 60N - 72 \] (147)
\[ P_{23} = 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 - 469N^3 - 290N^2 - 60N - 72 \] (148)
\[ P_{24} = 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 + 859N^3 - 778N^2 + 172N + 72 \] (149)
\[ P_{25} = 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 \] (150)
\[ P_{26} = 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 \] (151)
\[ P_{27} = 55N^9 + 336N^8 + 218N^7 - 2180N^6 - 6529N^5 - 9764N^4 - 9368N^3 - 6032N^2 - 2448N - 576 \] (152)
\[ P_{28} = N^{11} - 56N^9 - 236N^8 - 373N^7 + 82N^6 + 1244N^5 + 2330N^4 + 2560N^3 + 1712N^2 + 896N + 288 \] (153)
\[ P_{29} = 33N^{11} + 231N^{10} + 662N^9 + 1254N^8 + 1801N^7 + 2759N^6 + 5440N^5 + 9884N^4 + 12512N^3 + 9200N^2 + 5184N + 1728 \] (154)
\[ P_{30} = 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4 + 12976N^3 + 8016N^2 + 4608N + 1728 \] (155)
\[ P_{31} = 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 + 4645N^6 - 8936N^5 - 11980N^4 - 12352N^3 - 8272N^2 - 4800N - 1728 \] (156)
\[ P_{32} = 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 + 20743N^6 - 24563N^5 + 16702N^4 + 120N^2 + 2432N + 960 \] (157)
\[ P_{33} = 121N^{11} + 988N^{10} + 3554N^9 + 6972N^8 + 7131N^7 - 846N^6 - 14806N^5 + 25354N^4 - 26096N^3 - 16752N^2 - 8352N + 2592 \] (158)
\[ P_{34} = 27N^{12} + 441N^{11} + 2206N^{10} + 5360N^9 + 7445N^8 + 8555N^7 + 18766N^6 + 44852N^5 + 67572N^4 + 63960N^3 + 39632N^2 + 15648N + 2880 \] (159)
\[ P_{35} = 2447N^{12} + 16902N^{11} + 59649N^{10} + 125860N^9 + 128761N^8 - 36530N^7 - 248341N^6 - 304460N^5 + 162188N^4 + 117243N^3 + 29160N^2 + 19440N + 7776 \] (160)
\[ P_{36} = 3361N^{12} + 23769N^{11} + 62338N^{10} + 59992N^9 + 63303N^8 - 317823N^7 + 585520N^6 - 640602N^5 + 430132N^4 - 167536N^3 - 27648N^2 + 9504N + 5184 \] (161)
\[ P_{37} = 76N^{14} + 802N^{13} + 2979N^{12} + 1847N^{11} + 19377N^{10} - 58253N^9 - 26543N^8 + 170601N^7 + 362177N^6 + 225119N^5 - 103240N^4 - 193092N^3 - 137160N^2 - 117072N - 25920 \] (162)
\[ P_{38} = 76N^{14} + 1042N^{13} + 5979N^{12} + 16367N^{11} + 11883N^{10} - 47693N^9 - 125723N^8 - 86079N^7 + 36437N^6 + 22559N^5 - 51700N^4 + 24828N^3 + 132840N^2 + 116208N + 25920 \] (163)
\[ P_{39} = 3180N^{15} + 38835N^{14} + 188728N^{13} + 456665N^{12} + 460954N^{11} - 406761N^{10} - 1972948N^9 - 2827653N^8 - 1857970N^7 + 109786N^6 + 1302824N^5 + 1092456N^4 + 265888N^3 - 227616N^2 - 194688N - 44928 \] (164)
\[ P_{40} = 28503N^{17} + 297639N^{16} + 1232041N^{15} + 2461407N^{14} + 2169615N^{13} + 662941N^{12} \] (165)
The expressions in Eqs. (134),(123) are given in an irreducible basis representation. Apart from those in the 3-loop massless Wilson coefficients the Wilson coefficients \( L_{PS}^{g,2} \) and \( L_{S}^{g,2} \) are given in terms of the harmonic sums (D.44)

\[
S_1, S_{-2}, S_2, S_{-3}, S_3, S_{-4}, S_4, S_{-2,1}, S_{2,1}, S_{3,1}, S_{2,1,1}.
\]

The Mellin transforms of the functions (168) are known \([166, 180]\) and the \(z\)-space expressions for \( L_{PS}^{g,2} \) and \( L_{S}^{g,2} \) have been published in Ref. \([220]\)^{5} together with the logarithmic contributions to the other heavy flavour Wilson coefficients contributing to the structure function \( F_2 \), whereas the non-singlet Wilson coefficient \( L_{NS}^{g,2} \) has been published separately and including the non-logarithmic contributions in Ref. \([157]\).

\^{5}I would like to thank A. Behring for discussions.

\[\begin{align*}
P_{41} & = 75N^{18} + 3330N^{17} + 35497N^{16} + 175010N^{15} + 486862N^{14} + 966996N^{13} \\
& + 2037362 + 3604404N^{11} + 175010N^{10} - 2950602N^9 - 78753403N^8 \\
& - 107977014N^7 - 71548880N^6 + 18344016N^5 + 89016048N^4 + 92657952N^3 \\
& + 58942080N^2 + 25505280 (165) \\
P_{42} & = 325N^{18} + 4280N^{17} + 17759N^{16} - 14880N^{15} - 412326N^{14} - 1696848N^{13} \\
& - 3216546N^{12} - 1169232N^{11} + 8956857N^{10} + 23914216N^9 + 31536899N^8 \\
& + 25361392N^7 + 9982840N^6 - 10154128N^5 - 26098704N^4 - 26761536N^3 \\
& - 17642880N^2 - 8087040N - 1866240 . (166)
\end{align*}\]
Figure 4: The $O(\alpha_s^3)$ contribution by $L_{g,2}^S$ to the structure function $F_2(x, Q^2)$.

Figure 5 displays the contribution of the Wilson coefficient $L_{PS}^S$ to the structure function $F_2(x, Q^2)$ up to $O(\alpha_s^3)$. Here the PDFs of Ref. [249], cf. Eq. (11), have been used. The Figures 3 and 4 illustrate the corresponding two- and three-loop contributions to the Wilson coefficient $L_{g,2}^S$. The $O(\alpha_s^2)$ contribution is smaller than the the $O(\alpha_s^3)$-term. This is due to terms $\propto 1/z$ in the three loop contribution which do not contribute at 2-loop order. Generally the contributions at two- and three-loop order turn out to be minor compared to other contributions to the structure function $F_2(x, Q^2)$ [220].

All logarithmic contributions to the massive operator matrix elements which are needed for the variable flavor number scheme have been published together with the logarithmic contributions to the heavy flavor Wilson coefficients in Ref. [220].
5 Renormalization of massive OMEs with two masses

The Feynman integrals contributing to the various operator matrix elements contain ultraviolet and collinear divergences. We choose to regularize both by applying dimensional regularization in \( D = 4 + \varepsilon \) dimensions. This additionally requires the extension of the Lorentz-metric \( g_{\mu\nu} \) and the Clifford algebra of the Dirac matrices to \( D \) dimensions. The divergences then constitute themselves as poles in a Laurent series expansion around \( \varepsilon = 0 \).

At one and two loop order the two-mass massive operator matrix elements \( \hat{A}_{ij} \) are given in terms of the known single mass contributions since they do not contain more than one internal massive fermion line. The first genuine diagrams with two different masses emerge at \( O(\alpha_s^3) \).

The two-mass OMEs can be decomposed into contributions depending on just a single mass each and contributions stemming from diagrams with both masses

\[
\hat{\chi}^{(l)}_{ij} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) = \left[ \left( \frac{m_1^2}{\mu^2} \right)^{1/2\varepsilon} + \left( \frac{m_2^2}{\mu^2} \right)^{1/2\varepsilon} \right] \hat{A}^{(l)}_{ij} + \hat{\chi}^{(l)}_{ij} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right).
\]  

Here the parts \( \hat{\chi}^{(l)}_{ij} \) correspond the single-mass OMEs the renormalization of which has been derived in Ref. [139] and \( \mu^2 \) is the renormalization scale, which we choose equal to the factorization scale. It would be sufficient to cover the renormalization of the mixed contribution \( \hat{\chi}^{(l)}_{ij} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \) only. However, it is technically advantageous to construct the renormalization prescription for the complete two-mass OMEs. Furthermore, a change in the renormalization scheme as in Eqs. (207,208,241,240) generally introduces a mixing between the different components of Eq. (169). We thus present the renormalization for the complete two-mass OME. In a fixed renormalization scheme. The renormalized mixed-mass contribution is then obtained by subtracting the respective single-mass terms.

To renormalize the operator matrix elements with two masses we follow the renormalization procedure outlined in Ref. [139] before. At various steps adjustments have to be made to account for the second mass which is why we present the different steps here again in complete form. We study the case of \( N_F \) massless and two massive quark flavors as this covers the physically interesting case of contributions due to \( c \)- and \( b \)-quarks and the respective power corrections in

\[
\eta = \frac{m_c^2}{m_b^2},
\]

which are not suppressed by a strong mass hierarchy. Up to \( O(\alpha_s^3) \) no diagrams with more than two massive fermions contribute.

### 5.1 Mass Renormalization

The most frequently used schemes for the mass renormalization schemes are the \( \overline{\text{MS}} \)– and the on–mass shell (OMS) scheme. We renormalize the mass in the OMS scheme first and provide the finite renormalization to switch to the \( \overline{\text{MS}} \)-mass, cf. Eq. (241).

The bare masses \( \hat{m}_i, \ i \in \{1, 2\} \) are expressed by the renormalized on–shell masses \( m_i \) via

\[
\hat{m}_i = Z_{m_i}(m_1, m_2) m_i = m_i \left[ 1 + \hat{a}_s \left( \frac{m_i^2}{\mu^2} \right)^{\varepsilon/2} \delta m_1 + \hat{a}_s^2 \left( \frac{m_i^2}{\mu^2} \right)^\varepsilon \delta m_{2,i} (m_1, m_2) \right] + O(\hat{a}_s^3),
\]
where $\tilde{\alpha}_s$ denotes the bare coupling $g_s^2/(4\pi^2)^2$ and
\[
\delta m^2_2(m_1, m_2) = \delta m^0_2 + \tilde{\delta}_m^i(m_1, m_2).
\] (172)

In Eq. (172) $\delta m^0_2$ is the single mass-contribution, whereas $\tilde{\delta}_m^i$ denotes the additional contribution emerging in the case of two different massive flavors. Note that from order $O(\tilde{\alpha}_s^2)$ onward the $Z$-factor renormalizing $\tilde{m}_1$ depends on $m_2$ and vice versa. For the massive operator matrix elements this is observed at 3-loop order for the first time. The coefficients $\delta m_1$ and $\delta m_2$ have been derived in [251, 252] up to $O(\varepsilon^0)$ and $O(\varepsilon^{-1})$ respectively. The constant part of $\delta m_2$ was given in [253, 254] and the $O(\varepsilon)$-term of $\delta m_1$ in [139],

\[
\begin{align*}
\delta m_0 &= C_F \left[ \frac{6}{\varepsilon} - 4 + \left( 4 + \frac{3}{4} \zeta_2 \right) \varepsilon \right] \\
&= \frac{\delta m_{1^{(-1)}}}{\varepsilon} + \delta m_{1^{(0)}} + \delta m_{1^{(1)}} \varepsilon, \\
\delta m^0_2 &= C_F \left[ \frac{1}{\varepsilon^2} \left( 18 C_F - 22 C_A + 8 T_F(N_F + 1) \right) + \frac{1}{\varepsilon} \left( -\frac{45}{2} C_F + \frac{91}{2} C_A \right) \right. \\
&\quad \left. - 14 T_F(N_F + 1) \right] + C_F \left( \frac{199}{8} - \frac{51}{2} \zeta_2 + 48 \ln(2) \zeta_2 - 12 \zeta_3 \right) + C_A \left( -\frac{605}{8} \right) \\
&\quad + \frac{5}{2} \zeta_2 - 24 \ln(2) \zeta_2 + 6 \zeta_3 \right] + T_F \left[ N_F \left( \frac{45}{2} + 10 \zeta_2 \right) + 1 \left( \frac{69}{2} - 14 \zeta_2 \right) \right] \\
&= \frac{\delta m_{2^{0,(-2)}}}{\varepsilon^2} + \frac{\delta m_{2^{0,(-1)}}}{\varepsilon} + \delta m_{2^{0,(0)}},
\end{align*}
\] (173-176)

\[
\tilde{\delta}_m^i(m_1, m_2) = C_F T_F \left\{ \frac{8}{\varepsilon^2} - \frac{14}{\varepsilon} + 8 r_i^4 H_0^2(r_i) - 8(r_i + 1)^2 (r_i^2 - r_i + 1) H_{-1,0}(r_i) \right. \\
+ 8(r_i - 1)^2 (r_i^2 + r_i + 1) H_{1,0}(r_i) + 8 r_i^2 H_0(r_i) + \frac{3}{2} \left( 8 r_i^2 + 15 \right) \\
+ 2 \left[ 4 r_i^4 - 12 r_i^3 - 12 r_i + 5 \right] \zeta_2 \right\} \\
= \frac{\tilde{\delta} m_{2^{(-2)}}}{\varepsilon^2} + \frac{\tilde{\delta} m_{2^{(-1)}}}{\varepsilon} + \tilde{\delta} m_{2^{i,(0)}},
\] (177-178)

with $i \in \{1, 2\}$,
\[
\left. \begin{align*}
\text{r}_1 &= \sqrt{\eta} \text{ and } \text{r}_2 &= \frac{1}{\sqrt{\eta}}.
\end{align*} \right. \quad (179)
\]

The superscript $i$ for the coefficients $\tilde{\delta} m_{2^{(-2)}}$ and $\tilde{\delta} m_{2^{(-2)}}$ has been dropped as they are independent of the renormalized mass $m_i$. Here $H_d(\zeta)$ denotes harmonic polylogarithms [180] (see Appendix (D.2.1)) with $H_0(\zeta) = \ln(\zeta)$, $H_{-1,0}(\zeta) = \text{Li}_2(-\zeta) + \ln(\zeta) \ln(1 + \zeta)$, $H_{1,0}(\zeta) = \text{Li}_2(1 - \zeta) - \zeta_2$ and $\text{Li}_2(x)$ the dilogarithm (D.33).
Eq. (178) states the complete analytic form of the contribution of the respective other massive flavor to the renormalization of the bare masses. In the present analysis we will focus on \( m_1, m_2 \) being the masses of the charm and bottom quark respectively. Due to the small ratio

\[ \eta \sim 0.1 \]

we may restrict ourselves to the expansions up to \( O(\eta^3 \ln^3(\eta)) \)

\[
\delta m_2^{1,(0)}(m_1, m_2) = C_F T_F \left[ +10\zeta_2 + \frac{45}{2} - 24\zeta_2\eta^{1/2} + 24\eta - 24\zeta_2\eta^{3/2} + 32\left( -\frac{13}{48} \ln(\eta) + \frac{151}{288} + \frac{1}{16} \ln^2(\eta) + \frac{1}{4} \zeta_2 \right) \eta^2 
+ 32\left( -\frac{1}{30} \ln(\eta) - \frac{19}{300} \right) \eta^3 \right] + O(\eta^3 \ln(\eta)^2) \tag{180}
\]

\[
\delta m_2^{2,(0)}(m_1, m_2) = C_F T_F \left[ -2 \ln^2(\eta) + \frac{26}{3} \ln(\eta) + 2\zeta_2 + \frac{103}{18} + 32\left( -\frac{1}{30} \ln(\eta) + \frac{19}{300} \right) \eta + 32\left( -\frac{9}{1120} \ln(\eta) + \frac{1389}{156000} \right) \eta^2 
+ 32\left( -\frac{1}{315} \ln(\eta) + \frac{997}{396900} \right) \eta^3 \right] + O(\eta^3 \ln(\eta)^2) \tag{181}
\]

Applying Eq. (171) we obtain the mass renormalized operator matrix elements by

\[
\hat{A}_{ij}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) = \delta_{ij} + \hat{a}_s \hat{A}_{ij}^{(1)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) + \hat{a}_s^2 \left\{ \hat{A}_{ij}^{(2)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) 
+ \delta m_1 \left[ \left( \frac{m_1^2}{\mu^2} \right)^{\varepsilon/2} \frac{d}{dm_{1}} + \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon/2} \frac{d}{dm_{2}} \right] \hat{A}_{ij}^{(1)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) \right\} 
+ \hat{a}_s^3 \left\{ \hat{A}_{ij}^{(3)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) 
+ \delta m_1 \left[ \left( \frac{m_1^2}{\mu^2} \right)^{\varepsilon/2} \frac{d}{dm_{1}} + \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon/2} \frac{d}{dm_{2}} \right] \hat{A}_{ij}^{(2)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) 
+ \delta m_2(m_1, m_2) \frac{d}{dm_{1}} \frac{m_2 d}{dm_{2}} \hat{A}_{ij}^{(1)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) 
+ \frac{\delta m_1^2}{2} \left[ \left( \frac{m_1^2}{\mu^2} \right)^{\varepsilon/2} \frac{d^2}{dm_{1}^2} + \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon/2} \frac{d^2}{dm_{2}^2} \right] \hat{A}_{ij}^{(1)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) 
+ \delta m_1^2(\frac{m_1^2}{\mu^2})^{\varepsilon/2} \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon/2} \frac{d}{dm_{1}} \frac{d}{dm_{2}} \hat{A}_{ij}^{(1)}(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, N) \right\} \right. \tag{182}
\]

They are symmetric under the interchange of the masses \( m_1 \) and \( m_2 \).
5.2 Renormalization of the Coupling Constant

When renormalizing the coupling constant it is important to note that the factorization relation (56) strictly requires the external massless partonic legs of the operator matrix elements to be on–shell

\[ p^2 = 0 \ , \]  

with \( p \) the external momentum of the OME. This condition would be violated by massive loop corrections to the gluon propagator. We follow [139] and absorb these corrections uniquely into the coupling constant by using the background field method [182–184] to maintain the Slavnov–Taylor-identities of QCD. We thereby adopt a MOM–scheme for the coupling constant. A finite renormalization relates the MOM coupling constants to the \( \overline{\text{MS}} \) scheme and is applied later. For this scheme transformation we assume the decoupling of the two heavy quark flavors. Concerning the light fermionic flavors, the coupling constant is renormalized as in the \( \overline{\text{MS}} \) scheme, with

\[
\hat{a}_s = Z_{g}^{\overline{\text{MS}}} (\varepsilon, N_F) a_s^{\overline{\text{MS}}} (\mu^2) \]  

\[
= a_s^{\overline{\text{MS}}} (\mu^2) \left[ 1 + \delta a_{s,1}^{\overline{\text{MS}}} (N_F) a_s^{\overline{\text{MS}}} (\mu^2) + \delta a_{s,2}^{\overline{\text{MS}}} (N_F) a_s^{\overline{\text{MS}}}^2 (\mu) \right] + O(a_s^{\overline{\text{MS}}^3}) . \]  

(185)

The coefficients \( \delta a_{s,i}^{\overline{\text{MS}}} (N_F) \) are given by

\[
\delta a_{s,1}^{\overline{\text{MS}}} (N_F) = \frac{2}{\varepsilon} \beta_0 (N_F) , \]  

\[
\delta a_{s,2}^{\overline{\text{MS}}} (N_F) = \frac{4}{\varepsilon^2} \beta_0^2 (N_F) + \frac{1}{\varepsilon} \beta_1 (N_F) . \]  

(187)

From the renormalization prescription (185-187) it follows directly that that \( \beta_0 (n_f), \beta_1 (n_f) \) are the coefficients of the QCD \( \beta \)-function for \( N_F \) massless quark flavors

\[
\beta^{\overline{\text{MS}}} (N_F) = -\beta_0 (N_F) a_s^{\overline{\text{MS}}} - \beta_1 (N_F) a_s^{\overline{\text{MS}}}^2 + O \left( a_s^{\overline{\text{MS}}^4} \right) . \]  

(188)

They are given by [40, 41, 255–258]

\[
\beta_0 (N_F) = \frac{11}{3} C_A - \frac{4}{3} T_F N_F , \]  

(189)

\[
\beta_1 (N_F) = \frac{34}{3} C_A^2 - 4 \left( \frac{5}{3} C_A + C_F \right) T_F N_F . \]  

(190)

We split the renormalized gluon self energy \( \Pi \) into purely light and the remaining heavy flavor contributions \( \Pi_L \) and \( \Pi_H \)

\[
\Pi \left( p^2, m_1^2, m_2^2 \right) = \Pi_L \left( p^2 \right) + \Pi_H \left( p^2, m_1^2, m_2^2 \right) . \]  

(191)

The heavy quarks are required to decouple from the running coupling constant and the renormalized OMEs for \( \mu^2 < m_1^2, m_2^2 \) which implies [128]

\[
\Pi_H (0, m_1^2, m_2^2) = 0 . \]  

(192)
We apply the background field method which has the advantage of producing gauge-invariant results also for unphysical quantities as e.g. off-shell Green's functions to compute the heavy flavor contributions to the unrenormalized gluon self-energy \[182, 259\]. Applying the respective Feynman rules \[204\] we obtain the result

\[
\hat{\Pi}_{H,ab,\text{BF}}^{\mu\nu}(p^2, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) = i(-p^2 g^{\mu\nu} + p^\mu p^\nu)\delta_{ab}\hat{\Pi}_{H,\text{BF}}^{\mu\nu}(p^2, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s),
\]

\[
\hat{\Pi}_{H,\text{BF}}(0, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) = \hat{a}_s \frac{2\beta_0}{\varepsilon} \left[ \left( \frac{m_1^2}{\mu^2} \right)^{\varepsilon/2} + \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon/2} \right] \exp\left( \sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left( \frac{\varepsilon}{2} \right)^i \right) + \frac{\hat{a}_s^2}{2} \left[ \left( \frac{m_1^2}{\mu^2} \right)^{\varepsilon} + \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon} \right] \left[ \frac{1}{\varepsilon} \left( -\frac{20}{3} T_F C_A - 4 T_F C_F \right) \right. \\
- \frac{32}{9} T_F C_A + 15 T_F C_F \\
+ \varepsilon \left( -\frac{86}{27} T_F C_A - \frac{31}{4} T_F C_F - \frac{5}{3} \zeta_2 T_F C_A - \zeta_2 T_F C_F \right) \\
+ 2 \left( \frac{2\beta_0}{\varepsilon} \right)^2 \left( \frac{m_1^2}{\mu^2} \right)^{\varepsilon/2} \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon/2} \exp\left( \sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left( \frac{\varepsilon}{2} \right)^i \right) \right] \\
+ O(\hat{a}_s^3), 
\]

(193)

where the masses \( m_1 \) and \( m_2 \) have been renormalized in the on-shell scheme given in Eq. (171). In order to write (193) more compactly we used the notation

\[
f(\varepsilon) \equiv \left( \left( \frac{m_1^2}{\mu^2} \right)^{\varepsilon/2} + \left( \frac{m_2^2}{\mu^2} \right)^{\varepsilon/2} \right) \exp\left( \sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left( \frac{\varepsilon}{2} \right)^i \right),
\]

(194)

and keep this factors unexpanded in the dimensional regularization parameter \( \varepsilon \). Furthermore we denote the contributions to the QCD-\( \beta \) function coefficients by \( \beta_{1,0}^{(2)} \) [40, 41, 128, 139, 255–258]

\[
\beta_{0,Q} = -\frac{4}{3} T_F, 
\]

(195)

\[
\beta_{1,Q} = -4 \left( \frac{5}{3} C_A + C_F \right) T_F, 
\]

(196)

\[
\beta_{1,0}^{(1)} = -\frac{32}{9} T_F C_A + 15 T_F C_F, 
\]

(197)

\[
\beta_{1,0}^{(2)} = -\frac{86}{27} T_F C_A - \frac{31}{4} T_F C_F - \zeta_2 \left( \frac{5}{3} T_F C_A + T_F C_F \right). 
\]

(198)

Eq. (193) differs from the sum of the two individual single-mass contributions [139] by the last term only. This term is due to additional reducible Feynman diagrams in the cases of two heavy quark flavors of different mass.

The background field is renormalized using the \( Z \)-factor \( Z_A \) which is split into light and heavy quark contributions \( Z_{A,L} \) and \( Z_{A,H} \). It is related to the \( Z \)-factor renormalizing the coupling constant \( g \) via

\[
Z_g = Z_A^{-\frac{1}{2}} = \frac{1}{(Z_{A,L} + Z_{A,H})^{1/2}}. 
\]

(199)
Concerning the light flavors we require the renormalization to correspond to the \( \overline{\text{MS}} \)-scheme with \( N_F \) light flavors

\[
Z_{A,l}(N_F) = Z_{g}^{\overline{\text{MS}}}^{1/2}.
\]  (200)

The heavy flavor contribution is fixed by condition (192) which implies

\[
\Pi_{H,sf}(0, \mu^2, a_s, m_1^2, m_2^2) + Z_{A,H} \equiv 0.
\]  (201)

The \( Z \)-factor in the MOM–scheme is read off by combining (199), (192), (193) and (201)

\[
Z_{g}^{\text{MOM}}(\varepsilon, N_F + 2, \mu, m) \equiv \frac{1}{(Z_{A,l} + Z_{A,H})^{1/2}}.
\]  (202)

Up to \( O(a_s^{\text{MOM}^2}) \) we obtain the renormalization constant

\[
Z_{g}^{\text{MOM}^2}(\varepsilon, m, \mu, N_F + 2) = 1 + a_s^{\text{MOM}^2}(\mu^2) \left[ \frac{2}{\varepsilon} (\beta_0(N_F) + \beta_{0,Q} f(\varepsilon)) \right] \\
+ \frac{1}{\varepsilon} \left( \left( \frac{m_1^2}{\mu^2} \right) \varepsilon + \left( \frac{m_2^2}{\mu^2} \right) \varepsilon \right) \left( \beta_{1,Q} + \varepsilon \beta^{(1)}_{1,Q} + \varepsilon^2 \beta^{(2)}_{1,Q} \right) \\
+ O(\varepsilon^2, a_s^{\text{MOM}^3}).
\]  (203)

We define the coefficients of the MOM–scheme \( Z \)-factor, \( \delta a_{s,1}^{\text{MOM}} \) and \( \delta a_{s,2}^{\text{MOM}} \), analogously to the \( \overline{\text{MS}} \)-coefficients in (185)

\[
\delta a_{s,1}^{\text{MOM}} = \left[ \frac{2 \beta_0(N_F)}{\varepsilon} + \frac{2 \beta_{0,Q} f(\varepsilon)}{\varepsilon} \right],
\]  (204)

\[
\delta a_{s,2}^{\text{MOM}} = \left[ \frac{\beta_1(N_F)}{\varepsilon} + \left\{ \frac{2 \beta_0(N_F)}{\varepsilon} + \frac{2 \beta_{0,Q} f(\varepsilon)}{\varepsilon} \right\}^2 \\
+ \frac{1}{\varepsilon} \left( \left( \frac{m_1^2}{\mu^2} \right) \varepsilon + \left( \frac{m_2^2}{\mu^2} \right) \varepsilon \right) \left( \beta_{1,Q} + \varepsilon \beta^{(1)}_{1,Q} + \varepsilon^2 \beta^{(2)}_{1,Q} \right) \right] + O(\varepsilon^2).
\]  (205)

Finally we express our results in the \( \overline{\text{MS}} \)-scheme. For this transition we assume the decoupling of the heavy quark flavors. The renormalization–scheme transition then follows from the equality of the unrenormalized coupling constant.

\[
Z_{g}^{\overline{\text{MS}}^2}(\varepsilon, N_F + 2) a_s^{\overline{\text{MS}}^2}(\mu^2) = Z_{g}^{\text{MOM}^2}(\varepsilon, m, \mu, N_F + 2) a_s^{\text{MOM}^2}(\mu^2).
\]  (206)

Solving (206) perturbatively we obtain

\[
a_s^{\text{MOM}} = a_s^{\overline{\text{MS}}} - \beta_{0,Q} \left( \ln \left( \frac{m_1^2}{\mu^2} \right) + \ln \left( \frac{m_2^2}{\mu^2} \right) \right) a_s^{\overline{\text{MS}}^2} + \left[ \beta_{0,Q}^2 \left( \ln \left( \frac{m_1^2}{\mu^2} \right) + \ln \left( \frac{m_2^2}{\mu^2} \right) \right)^2 \\
- \beta_{0,Q} \left( \ln \left( \frac{m_1^2}{\mu^2} \right) + \ln \left( \frac{m_2^2}{\mu^2} \right) \right) - 2 \beta_{1,Q} \right] a_s^{\overline{\text{MS}}^3} + O \left( a_s^{\overline{\text{MS}}^4} \right),
\]  (207)
or,
\[
\hat{a}_{ij}^{\text{MOM}} = a_s^{\text{MOM}} + a_s^{\text{MOM}2} \left( \delta a_{s,1}^{\text{MOM}} - \delta a_{s,1}^{\text{MS}} (N_F + 2) \right) + a_s^{\text{MOM}3} \left( \delta a_{s,2}^{\text{MOM}} - \delta a_{s,2}^{\text{MS}} (N_F + 2) \right) - 2\delta a_{s,1}^{\text{MS}} (N_F + 2) \left[ \delta a_{s,1}^{\text{MOM}} - \delta a_{s,1}^{\text{MS}} (N_F + 2) \right] + O(a_s^{\text{MOM}4}),
\]
(208)
as finite renormalization prescriptions describing the scheme–change $\text{MS} \leftrightarrow \text{MOM}$. Note that unlike in Eq. (185) in Eq. (207) and (208) $a_s^{\text{MS}} = a_s^{\text{MOM}} (N_F + 2)$. Applying the coupling renormalization (203) to (183) we obtain as combined formula for the combined mass– and coupling–renormalization up to $O\left(a_s^{\text{MOM}3}\right)$
\[
\hat{A}_{ij} = \delta_{ij} + a_s^{\text{MOM}} \hat{A}_{ij}^{(1)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right)
+ a_s^{\text{MOM}2} \left[ \hat{A}_{ij}^{(2)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + \delta a_{s,1}^{\text{MOM}} \hat{A}_{ij}^{(1)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right]
+ \delta m_1 \left( \frac{m_1^2}{\mu^2} \right)^{\epsilon/2} m_1 \frac{d}{dm_1} + \left( \frac{m_2^2}{\mu^2} \right)^{\epsilon/2} m_2 \frac{d}{dm_2} \right) \hat{A}_{ij}^{(1)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right)
+ a_s^{\text{MOM}3} \left[ \hat{A}_{ij}^{(3)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + \delta a_{s,2}^{\text{MOM}} \hat{A}_{ij}^{(1)} + 2\delta a_{s,1}^{\text{MOM}} \left[ \hat{A}_{ij}^{(2)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \right]
+ \delta m_1 \left( \frac{m_1^2}{\mu^2} \right)^{\epsilon/2} m_1 \frac{d}{dm_1} + \left( \frac{m_2^2}{\mu^2} \right)^{\epsilon/2} m_2 \frac{d}{dm_2} \right) \hat{A}_{ij}^{(2)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right)
+ \left( \delta m_{2,1}(m_1, m_2)m_1 \frac{d}{dm_1} + \delta m_{2,2}(m_1, m_2)m_2 \frac{d}{dm_2} \right) \hat{A}_{ij}^{(1)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right)
+ \frac{\delta m_1^2}{2} \left( \left( \frac{m_1^2}{\mu^2} \right) \right)^{\epsilon/2} m_1 \frac{d^2}{dm_1^2} + \left( \frac{m_2^2}{\mu^2} \right)^{\epsilon/2} m_2 \frac{d^2}{dm_2^2} \right) \hat{A}_{ij}^{(1)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right)
+ \delta m_1^2 \left( \frac{m_1^2}{\mu^2} \right)^{\epsilon/2} \left( \frac{m_2^2}{\mu^2} \right)^{\epsilon/2} m_1 \frac{d}{dm_1} m_2 \frac{d}{dm_2} \hat{A}_{ij}^{(1)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right],
\]
(209)
where the $\epsilon$– and $N$–dependence of the OMEs has been suppressed.

### 5.3 Operator Renormalization

We now remove the UV–divergences by introducing the respective $Z$–factors for the different operators defined in Eqs. (37-39)
\[
\begin{align*}
O_{q,r,\mu_1,\ldots,\mu_N}^{\text{NS}} &= Z_{q,r,\mu_1,\ldots,\mu_N}^{\text{NS}} O_{q,r,\mu_1,\ldots,\mu_N}^{\text{NS}}, \\
O_{i,\mu_1,\ldots,\mu_N}^{\text{S}} &= Z_{i,\mu_1,\ldots,\mu_N}^{\text{S}} O_{i,\mu_1,\ldots,\mu_N}^{\text{S}},
\end{align*}
\]
(210)
(211)
In the singlet case the operator renormalization introduces a mixing between the different operators since they carry the same quantum numbers. Analogously to the OMEs here the Z-factors were split into flavor pure singlet (PS) and flavor non-singlet (NS) contributions

\[ Z_{qq}^{-1} = Z_{qq}^{-1,PS} + Z_{qq}^{-1,NS} . \]  

(212)

Each Z-factor is associated with an anomalous dimension \( \gamma_{ij} \) via

\[ \gamma_{qq}^{NS}(\alpha_s, N_F, N) = \mu \frac{d}{d\mu} \ln \left( Z_{qq}^{NS}(\alpha_s, N_F, \varepsilon, N) \right) , \]

(213)

\[ \gamma_{ij}(\alpha_s, N_F, N) = \mu \frac{d}{d\mu} \ln \left( Z_{ij}(\alpha_s, N_F, \varepsilon, N) \right) . \]

(214)

Here both the anomalous dimensions and the operator Z-factors obey perturbative series expansions in the coupling constant

\[ \gamma^S, PS, NS(\alpha_s, N_F, N) = \sum_{l=1}^{\infty} a_s^{MS} \gamma^{(l-1),S, PS, NS}(N_F, N) \]  

(215)

\[ Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} a_s^k Z_{ij}^{(k)} \]  

(216)

\[ Z_{ij}^{-1} = \delta_{ij} + \sum_{k=1}^{\infty} a_s^k Z_{ij}^{-1,(k)} . \]  

(217)

In order to renormalize the respective operators we consider operator matrix elements with off-shell external legs as a sum of massive and massless contributions:

\[ \hat{A}_{ij}(p^2, m_1^2, m_2^2, \mu^2, a_s^{MOM}, N_F + 2) = \hat{A}_{ij}(\frac{-p^2}{\mu^2}, \alpha_s^{MS}, N_F) \]

\[ + \hat{A}_{ij}^Q(p^2, m_1^2, m_2^2, \mu^2, a_s^{MOM}, N_F + 2) . \]  

(218)

Here the massless contribution depends on \( a_s^{MS} \) since the MOM–scheme in Section 5.2 was constructed in such a way, that it corresponds to the \( \overline{MS} \)–scheme concerning the renormalization of the light quark flavor and gluon contributions. The term \( \delta_{ij} \) does not have any mass-dependence and is considered a part of the light flavor part \( \hat{A}_{ij}(\frac{-p^2}{\mu^2}, \alpha_s^{MS}, N_F) \).

We first consider the renormalization of the purely massless contribution in the \( \overline{MS} \)–scheme [260]

\[ A_{qq}^{NS}(\alpha_s^{MS}, N_F, N) = Z_{qq}^{-1,NS}(\alpha_s^{MS}, N_F, \varepsilon, N) \hat{A}_{qq}^{NS}(\alpha_s^{MS}, N_F, N) \]  

(219)

\[ A_{ij}(\alpha_s^{MS}, N_F, N) = Z_{il}^{-1}(\alpha_s^{MS}, N_F, \varepsilon, N) \hat{A}_{ij}(\alpha_s^{MS}, N_F, \varepsilon, N) \], \( i, j, l = q, g \) .

(220)

Solving (213–214) yields the Z-factors

\[ Z_{ij}(\alpha_s^{MS}, N_F) = \delta_{ij} + a_s^{MS} \frac{\gamma_{ij}^{(0)}}{\varepsilon} + a_s^{MS2} \left\{ \frac{1}{\varepsilon^2} \left( \frac{1}{2} \gamma_{ij}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \right) + \frac{1}{2 \varepsilon} \gamma_{ij}^{(1)} \right\} \]  

46
and by applying the scheme transformation (208). The resulting operator

\[ Z \]

and for the NS- and PS-case

\[ Z_{qq}^{\text{NS}}(a_s^{\text{MOM}}, N_F) = 1 + a_s^{\text{MOM}} \frac{\gamma_{qq}(0)}{\varepsilon} + a_s^{\text{MOM}} \left\{ \frac{1}{\varepsilon} \left( -\frac{1}{2} \gamma_{ij}(1) + \delta a_s^{\text{MOM}} \gamma_{ij}(0) \right) \right\} + a_s^{\text{MOM}} \left\{ \frac{1}{\varepsilon} \left( -\frac{1}{3} \gamma_{kj}(1) + 2 \delta a_s^{\text{MOM}} \gamma_{ij}(0) \right) \right\} \]

(222)

\[ Z_{qq}^{\text{PS}}(a_s^{\text{MOM}}, N_F) = a_s^{\text{MOM}} \left\{ \frac{1}{\varepsilon} \left( -\frac{1}{2} \gamma_{ij}(1) + \delta a_s^{\text{MOM}} \gamma_{ij}(0) \right) \right\} + a_s^{\text{MOM}} \left\{ \frac{1}{\varepsilon} \left( -\frac{1}{3} \gamma_{ij}(1) + 2 \delta a_s^{\text{MOM}} \gamma_{ij}(0) \right) \right\} \]

(223)

The Z-factors describing the UV renormalization of the complete operator matrix elements \( \hat{A}_{ij}(p^2, m_1^2, m_2^2, \mu^2, a_s^{\text{MOM}}, N_F + 2) \) are obtained by inverting (221-223), replacing \( N_F \rightarrow N_F + 2 \) and by applying the scheme transformation (208). The resulting operator Z-factors are

\[ Z_{ij}^{-1}(a_s^{\text{MOM}}, N_F + 2, \mu) = \delta_{ij} - a_s^{\text{MOM}} \frac{\gamma_{ij}(0)}{\varepsilon} + a_s^{\text{MOM}} \left\{ \frac{1}{\varepsilon} \left( -\frac{1}{2} \gamma_{ij}(1) + \delta a_s^{\text{MOM}} \gamma_{ij}(0) \right) \right\} \]

(224)

and

\[ Z_{qq}^{-1,\text{NS}}(a_s^{\text{MOM}}, N_F + 2) = 1 - a_s^{\text{MOM}} \frac{\gamma_{qq}(0)}{\varepsilon} + a_s^{\text{MOM}} \left\{ \frac{1}{\varepsilon} \left( -\frac{1}{2} \gamma_{qq}(1) + \delta a_s^{\text{MOM}} \gamma_{qq}(0) \right) \right\} \]

(225)
Finally the limit $p^2 \to 0$ is performed. Since scale-less diagrams vanish when computing them in dimensional regularization only the Born piece of the massless OME contributes

$$\hat{A}_{ij} \left( 0, \alpha_s^{\overline{MS}}, N_F \right) = \delta_{ij} .$$

One obtains the UV–renormalization prescription

$$\tilde{A}_{ij} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \alpha_s^{\overline{MS}}, N_F + 2 \right) = a_s^{\overline{MS}} \left( \hat{A}_{ij}^{(1),Q} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + Z_{ij}^{-1,1}(N_F + 2, \mu) - Z_{ij}^{1,1}(N_F) \right)
+ a_s^{\overline{MS}} \left( \hat{A}_{ij}^{(2),Q} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + Z_{ij}^{-1,2}(N_F + 2, \mu) \right)
- Z_{ij}^{-1,2}(N_F) + Z_{ik}^{-1,1}(N_F + 2, \mu) \hat{A}_{kj}^{(1),Q} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right)
+ a_s^{\overline{MS}} \left( \hat{A}_{ij}^{(3),Q} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + Z_{ij}^{-1,3}(N_F + 2, \mu) \right)
- Z_{ij}^{-1,3}(N_F) + Z_{ik}^{-1,2}(N_F + 2, \mu) \hat{A}_{kj}^{(2),Q} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right)
+ Z_{ik}^{-1,2}(N_F + 2, \mu) \hat{A}_{kj}^{(1),Q} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) .$$

(229)
Here $Z$–factors at $N_F + 2$ flavors describe the massive case (224-226) while those with argument $N_F$ denote the $Z$–factors for the massless case.

5.4 Mass Factorization

At this stage, only collinear singularities remain. They arise in massless subgraphs only and are therefore independent of the additional heavy quark flavor considered in this analysis. We thus follow [139] directly and remove the collinear singularities via mass factorization

$$A_{ij}(m_1^2/m_2^2, m_2^2/m_2^2, a_{s\text{MOM}}^{\text{MOM}}, N_F + 2) = \tilde{A}_{ij}(m_1^2/m_2^2, m_2^2/m_2^2, a_{s\text{MOM}}^{\text{MOM}}, N_F + 2) \Gamma_{ij}^{-1}. \quad (230)$$

In a fully massless scenario the transition functions $\Gamma_{ij}$ would be related to the light flavor renormalization constant via the identity

$$\Gamma_{ij}(N_F) = Z_{ij}^{-1}(N_F). \quad (231)$$

However in the presence of one or more heavy quark flavors the transition functions apply to massless subgraphs only. Due to this and the subtraction of the $\delta_{ij}$–term in the UV-renormalized OMEs $\tilde{A}_{ij}$ the transition functions contribute up to $O(\alpha^2)$ only and finally the combined renormalization formula

$$A_{ij}(m_1^2/m_2^2, m_2^2/m_2^2, a_{s\text{MOM}}^{\text{MOM}}, N_F + 2) =$$

$$a_{s\text{MOM}}^{\text{MOM}} \left( \tilde{A}_{ij}^{(1)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + Z_{ij}^{-1(1)}(N_F + 2) - Z_{ij}^{-1(1)}(N_F) \right)$$

$$+ a_{s\text{MOM}}^{\text{MOM}^2} \left( \tilde{A}_{ij}^{(2)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + Z_{ij}^{-1(2)}(N_F + 2) - Z_{ij}^{-1(2)}(N_F) \right)$$

$$+ Z_{ik}^{-1(1)}(N_F + 2) \tilde{A}_{kj}^{(1)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + \left[ \tilde{A}_{il}^{(1)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + Z_{il}^{-1(1)}(N_F + 2) \right]$$

$$- Z_{il}^{-1(1)}(N_F) \right)^{-1(1)}(N_F) \right)$$

$$+ a_{s\text{MOM}}^{\text{MOM}^3} \left( \tilde{A}_{ij}^{(3)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + Z_{ij}^{-1(3)}(N_F + 2) - Z_{ij}^{-1(3)}(N_F) \right)$$

$$+ Z_{ik}^{-1(1)}(N_F + 2) \tilde{A}_{kj}^{(2)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + Z_{ik}^{-1(2)}(N_F + 2) \tilde{A}_{kj}^{(1)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right)$$

$$+ \left[ \tilde{A}_{il}^{(1)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + Z_{il}^{-1(1)}(N_F + 2) - Z_{il}^{-1(1)}(N_F) \right] \Gamma_{ij}^{-1(2)}(N_F)$$

$$+ \left[ \tilde{A}_{il}^{(2)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) + Z_{il}^{-1(2)}(N_F + 2) - Z_{il}^{-1(2)}(N_F) \right]$$

$$+ Z_{ik}^{-1(1)}(N_F + 2) \tilde{A}_{kj}^{(1)} Q \left( m_1^2/m_2^2, m_2^2/m_2^2 \right) \right] \Gamma_{ij}^{-1(1)}(N_F) \right) \quad (232)$$
is obtained. Eq. (232) differs from the corresponding renormalization prescription for one heavy quark flavor [139] by the definition of the renormalization constants $Z^{-1,(k)}_{ij}(N_F+2)$ only. Now the term $\delta_{ij}$ is added back to the massive OME. In a final step the coupling constant is transformed to the $\overline{\text{MS}}$-scheme via Eq. (207).

### 5.5 Mass renormalization schemes

For the renormalization of the quark masses the on-shell and the $\overline{\text{MS}}$-scheme are frequently used. The $\overline{\text{MS}}$-renormalization constant is given by

$$\hat{m} = Z_{m}^{\overline{\text{MS}}} m = m \left[ 1 + \hat{a}_s \delta m_1 + \hat{a}_s^2 \delta m_2 \right] + O(\hat{a}_s^3) ,$$

with the coefficients [251],

$$\delta m_1 = \frac{6}{\varepsilon} C_F \equiv \frac{\delta m_1^{(-1)}}{\varepsilon} ,$$

$$\delta m_2 = \frac{C_F}{\varepsilon^2} \left( 18 C_F - 22 C_A + 8 T_F(N_F + 2) \right) + \frac{C_F}{\varepsilon} \left( \frac{3}{2} C_F + \frac{97}{6} C_A - \frac{10}{3} T_F(N_F + 2) \right) \equiv \frac{\delta m_2^{(-2)}}{\varepsilon^2} + \frac{\delta m_2^{(-1)}}{\varepsilon} .$$

The following relations between the coefficients in $\varepsilon$ of the on-shell renormalization constant and the $\overline{\text{MS}}$-renormalization constant are observed:

$$\delta m_1^{(-1)} = \delta m_1^{(-1)} ,$$

$$\delta m_2^{(-2)} = \delta m_2^{(-2)} ,$$

$$\delta m_2^{(-1)} = \delta m_2^{(-1)} - \delta m_1^{(-1)} \delta m_1^{(0)} + 2 \delta m_1^{(0)} (\beta_0 + N_t \beta_{0,Q}) .$$

The scheme transformation are derived from the fact that the bare mass does not depend on the renormalization scheme:

$$\hat{m} = Z_{m}^{\overline{\text{MS}}} m = Z_{m} m$$

After renormalizing the coupling to the $\overline{\text{MS}}$-scheme we obtain the following scheme transformations for the masses $m_i , \ i \in 1,2$:

$$m_i = m_i \left[ 1 + \left\{ \frac{1}{2} \ln \left( \frac{m_i^2}{\mu^2} \right) \delta m_1^{(1)} + \delta m_1^{(0)} \right\} a_s^{\overline{\text{MS}}} + \left\{ \delta m_2^{(0)}(m_1, m_2) - \delta m_1^{(1)} \delta m_1^{(1)} 

+ 2 \delta m_1^{(1)} \beta_0(N_F) + 4 \delta m_1^{(1)} \beta_{0,Q} + \left( -\frac{1}{2} \delta m_1^{(1)} \delta m_1^{(0)} + \delta m_2^{(-1)} + 2 \delta m_1^{(0)} \beta_{0,Q} 

+ \delta m_1^{(0)} \beta_0(N_F) \right) \ln \left( \frac{m_i^2}{\mu^2} \right) + \left( -\frac{1}{4} \delta m_1^{(1)} \beta_0(N_F) + \frac{1}{8} \delta m_1^{(1)} \beta_{0,Q} + \frac{1}{2} \delta m_1^{(1)} \beta_{0,Q} \right) \right\} \right\} \right] .$$

$$\left( \frac{m_i^2}{\mu^2} \right)^2 a_s^{\overline{\text{MS}}} .$$

(240)
Inverting this relation yields

\[ m_i = \frac{1}{m_i} \left( 1 - \delta m_i^{(0)} - \frac{1}{2} \delta m_i^{(-1)} \ln \left( \frac{\mu^2}{m_i^2} \right) \right) a_s^{\overline{MS}} + \left\{ -\delta m_i^{(0)} \ln \left( \frac{\mu^2}{m_i^2} \right) + \left( \frac{1}{2} \delta m_i^{(-1)} \right)^2 \right\} a_s^{\overline{MS}}. \] (241)

We will present all our results in the on-shell scheme. The transformation to the \( \overline{MS} \)–scheme can then easily be performed applying Eq. (241).

### 5.6 One–particle reducible contributions

Starting from \( O(\alpha_s^2) \) the OMEs also contain one-particle reducible contributions. Up to now only the irreducible diagrams were computed. The renormalization acts on the complete OME. Adding the one–particle reducible contributions requires the knowledge of the corresponding quark– and gluon self–energies as well as lower order OMEs.

#### 5.6.1 Self–energy contributions

The scalar self–energies are obtained by projecting out the Lorentz–structure

\[ \tilde{\Pi}_{\mu\nu}^{ab}(p^2, m_1^2, m_2^2, \mu^2, \tilde{a}_s) = i\delta^{ab} \left[ -g_{\mu\nu}p^2 + p_{\mu}p_{\nu} \right] \tilde{\Pi}(p^2, m_1^2, m_2^2, \mu^2, \tilde{a}_s), \] (242)

\[ \tilde{\Pi}(p^2, m_1^2, m_2^2, \mu^2, \tilde{a}_s) = \sum_{k=1}^{\infty} \tilde{\Pi}^{(k)}(p^2, m_1^2, m_2^2, \mu^2), \] (243)

\[ \hat{\Sigma}_{ij}(p^2, m_1^2, m_2^2, \mu^2, \tilde{a}_s) = i \delta_{ij} \hat{\Sigma}(p^2, m_1^2, m_2^2, \mu^2, \tilde{a}_s), \] (244)

\[ \hat{\Sigma}(p^2, m_1^2, m_2^2, \mu^2, \tilde{a}_s) = \sum_{k=2}^{\infty} \tilde{\Sigma}^{(k)}(p^2, m_1^2, m_2^2, \mu^2). \] (245)

We decompose the irreducible two–mass self energies into contributions which depend on one mass only and an additional part stemming from diagrams with both heavy quark flavors

\[ \tilde{\Pi}^{(k)}(p^2, m_1^2, m_2^2, \mu^2) = \tilde{\Pi}^{(k)}(p^2, \frac{m_1^2}{\mu^2}) + \tilde{\Pi}^{(k)}(p^2, \frac{m_2^2}{\mu^2}) + \tilde{\Pi}^{(k)}(p^2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}) \] (246)

and for the quark self–energy

\[ \hat{\Sigma}^{(j)}(p^2, m_1^2, m_2^2, \mu^2) = \hat{\Sigma}^{(j)}(p^2, \frac{m_1^2}{\mu^2}) + \hat{\Sigma}^{(j)}(p^2, \frac{m_2^2}{\mu^2}) + \hat{\Sigma}^{(j)}(p^2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}). \] (247)

Up to two–loop order no diagrams with two heavy flavors contribute

\[ \tilde{\Pi}^{(k)}(p^2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}) = 0 \text{ for } k \in \{1, 2\}, \] (248)
\[ \hat{\Sigma}^{(2)}(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2) = 0 . \] (249)

The single-mass contributions are known from \([139, 261–263]\)

\[ \hat{\Pi}^{(1)}(0, \frac{\hat{m}_2^2}{\mu^2}) = T_F \left( \frac{\hat{m}_2^2}{\mu^2} \right)^{\varepsilon/2} \left\{ -12 C_F + 5 C_A \right\} + \left[ N_F T_F + \frac{8}{27} (35 C_A - 48 C_F) T_F - 12 \frac{C_F}{T_F} \right] \]

\[ \hat{\Pi}^{(2)}(0, \frac{\hat{m}_2^2}{\mu^2}) = T_F \left( \frac{\hat{m}_2^2}{\mu^2} \right)^{\varepsilon} \left\{ -12 C_F + 5 C_A \right\} + \left[ N_F T_F + \frac{8}{27} (35 C_A - 48 C_F) T_F - 12 \frac{C_F}{T_F} \right] \]

\[ + O(\varepsilon^2) , \]

\[ \hat{\Pi}^{(3)}(0, \frac{\hat{m}_2^2}{\mu^2}) = T_F \left( \frac{\hat{m}_2^2}{\mu^2} \right)^{3\varepsilon/2} \left\{ -12 C_F + 5 C_A \right\} + \left[ N_F T_F + \frac{8}{27} (35 C_A - 48 C_F) T_F - 12 \frac{C_F}{T_F} \right] \]

\[ + O(\varepsilon^2) , \]

and for the quark self-energy,

\[ \hat{\Sigma}^{(2)}(0, \frac{\hat{m}_2^2}{\mu^2}) = T_F C_F \left( \frac{\hat{m}_2^2}{\mu^2} \right)^{\varepsilon} \left\{ -12 C_F + 5 C_A \right\} + \left[ N_F T_F + \frac{8}{27} (35 C_A - 48 C_F) T_F - 12 \frac{C_F}{T_F} \right] \]

\[ + O(\varepsilon^2) . \] (253)
\[ + N_F T_F \left\{ \frac{4}{3} \zeta_2 + \frac{674}{81} \right\} + T_F \left\{ \frac{8}{3} \zeta_2 + \frac{604}{81} \right\} + C_A \left\{ \frac{17}{3} \zeta_3 - \frac{5}{3} \zeta_2 + \frac{1879}{162} \right\} \\
+ C_F \left\{ -8 \zeta_3 - \zeta_2 - \frac{335}{18} \right\} + O(\varepsilon). \] (254)

Similarly to other massive processes [264–268], the constant

\[ B_4 = -4 \zeta_2 \ln^2(2) + \frac{2}{3} \ln^4(2) - \frac{13}{2} \zeta_4 + 16 \text{Li}_4 \left( \frac{1}{2} \right) \approx -1.762800093... \] (255)

emerges in Eq (252). At \( O(\alpha_s^3) \) irreducible diagrams with two different masses contribute for the first time. For the gluonic case we computed the respective diagrams up to \( O(\eta^3) \) using the codes Q2e/Exp [185,186]

\[
\hat{\Pi}^{(3)}(0, \hat{m}_1^2, \hat{m}_2^2, \mu^2) = \left\{ \begin{array}{l}
- \frac{1}{\varepsilon^3} \frac{1256}{9} + \frac{1}{\varepsilon^2} \left( -\frac{320}{27} - \frac{64}{3} \left( \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) + \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \right) \right) \\
+ \left( -\frac{128}{315} \right. \\
+ \left( -16/3 + \frac{64}{35} \frac{1}{\eta^2} \frac{256}{315} \frac{\eta^3}{\mu^2} \right) \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) \\
+ \left( -\frac{32}{3} - \frac{64}{3} \right. \\
+ \left( -\frac{10208}{3675} \frac{\eta^2}{\mu^2} + \frac{39616}{99225} \frac{\eta^3}{\mu^2} \right) \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) \\
+ \left( -\frac{64}{9} + \frac{64}{3} \right. \\
+ \left( -\frac{10208}{3675} \frac{\eta^2}{\mu^2} + \frac{39616}{99225} \frac{\eta^3}{\mu^2} \right) \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \\
\left. - \frac{1504}{32} \right. \\
+ \left. \frac{1}{\varepsilon^3} \frac{1256}{9} + \frac{1}{\varepsilon^2} \left[ \frac{560}{27} - \frac{16}{3} \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) - \frac{22}{9} \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \right] \\
+ \frac{1}{\varepsilon} \left[ -8/3 \zeta_2 - 4 \left( \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) + \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \right) \right] - \frac{148}{27} \\
+ \frac{140}{9} \left( \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) + \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \right) \frac{22}{9} \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) \\
+ 4/3 \left( \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) - \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) \right) \\
+ \left( \frac{65}{9} - 2/15 \frac{\eta}{\mu^2} - \frac{16}{21} \frac{\eta^2}{\mu^2} - \frac{50}{189} \frac{\eta^3}{\mu^2} \right) \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) \\
+ \left( \frac{80}{9} + \frac{4}{15} \frac{\eta}{\mu^2} + \frac{32}{21} \frac{\eta^2}{\mu^2} + \frac{100}{189} \frac{\eta^3}{\mu^2} \right) \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \ln \left( \frac{\hat{m}_1^2}{\mu^2} \right) \\
+ \left( -\frac{389}{27} - 2 \zeta_2 - \frac{1924}{225} \frac{\eta}{\mu^2} - \frac{6392}{2205} \frac{\eta^2}{\mu^2} - \frac{20284}{59535} \frac{\eta^3}{\mu^2} \right) \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \\
- \frac{14}{9} \ln^3 \left( \frac{\hat{m}_2^2}{\mu^2} \right) + \left( \frac{65}{9} - 2/15 \frac{\eta}{\mu^2} - \frac{16}{21} \frac{\eta^2}{\mu^2} - \frac{50}{189} \frac{\eta^3}{\mu^2} \right) \ln \left( \frac{\hat{m}_2^2}{\mu^2} \right) \\
\right\} T_F^2 C_F \right. 
\]}
\[- \left( \frac{1139}{243} + \frac{70}{9} \zeta_2 \frac{56}{9} \zeta_3 \frac{34144}{3375} \eta \right) \frac{1292594}{231525} \eta^2 \)
\[- \frac{4231264}{18753525} \eta^3 \right) \right] T_F^2 C_A + O \left( \eta^3 \right) \right) + O \left( \varepsilon \right) . \]

The quarkonic self-energy contributions have been calculated analytically in \( \eta \) using the methods described in Section 6.3

\[ \hat{\Sigma}^{(3)} \left( p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) = C_F T_F^2 \left\{ \frac{128}{9 \varepsilon^2} + \frac{80}{27 \varepsilon} + \frac{2 \ln^2 (\eta)}{3} + \frac{8 \zeta_2}{3} + \frac{604}{81} \right\} + O(\varepsilon) . \]

### 5.6.2 The Operator matrix elements

As in Eqs. (246-247) we define the two-mass OMEs at one-loop order and the irreducible OMEs at \( O(\alpha_s^2) \) by

\[ \hat{A}^{(1)}_{ij} \left( \frac{\hat{m}_1^2}{\mu^2}, \frac{\hat{m}_2^2}{\mu^2} \right) = \hat{\tilde{A}}^{(1)}_{ij} \left( \frac{\hat{m}_1^2}{\mu^2} \right) + \hat{\tilde{A}}^{(1)}_{ij} \left( \frac{\hat{m}_2^2}{\mu^2} \right) , \]

\[ \hat{A}^{(2)}_{ij, \text{irr}} \left( \frac{\hat{m}_1^2}{\mu^2}, \frac{\hat{m}_2^2}{\mu^2} \right) = \hat{\tilde{A}}^{(2)}_{ij, \text{irr}} \left( \frac{\hat{m}_1^2}{\mu^2} \right) + \hat{\tilde{A}}^{(2)}_{ij, \text{irr}} \left( \frac{\hat{m}_2^2}{\mu^2} \right) , \]

where the \( A_{ij} \) functions with one argument denote the usual single-mass OMEs. Using the definitions (246-247) and (258-259) we compose the reducible massive operator matrix elements at \( O(\alpha_s^2) \) by

\[ \hat{\tilde{A}}^{(2), \text{NS}}_{qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) = \hat{\tilde{A}}^{(2), \text{NS,irr}}_{qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) + \hat{\Sigma}^{(2)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) , \]

\[ \hat{\tilde{A}}^{(2)}_{Qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) = \hat{\tilde{A}}^{(2), \text{irr}}_{Qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \]
\[ + \hat{\tilde{A}}^{(1)}_{Qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \hat{\Pi}^{(1)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) , \]

\[ \hat{\tilde{A}}^{(2)}_{gg} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) = \hat{\tilde{A}}^{(2), \text{irr}}_{gg} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) + \hat{\Pi}^{(2)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \]
\[ + \hat{\Pi}^{(1)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) , \]

and at \( O(\alpha_s^3) \) by

\[ \hat{\tilde{A}}^{(3), \text{NS}}_{qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) = \hat{\tilde{A}}^{(3), \text{NS,irr}}_{qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) + \hat{\Sigma}^{(3)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \]

\[ \hat{\tilde{A}}^{(3)}_{Qq} = \hat{\tilde{A}}^{(3), \text{irr}}_{Qq} + \hat{\tilde{A}}^{(2)}_{Qq} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \hat{\Pi}^{(1)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \]
\[ + \hat{\Pi}^{(1)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \hat{\tilde{A}}^{(3)}_{gg} = \hat{\tilde{A}}^{(3), \text{irr}}_{gg} + \hat{\tilde{A}}^{(2)}_{gg} \left( \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \hat{\Pi}^{(1)} \left( 0, \hat{m}_1^2, \hat{m}_2^2, \mu^2 \right) \].
5.7 General Structure of the Massive Operator Matrix Elements

In the case of one heavy quark flavor with mass \( m \), the mass dependence of the unrenormalized massive operator matrix element at order \( \alpha_s^l \) is given by

\[
\hat{A}_{ij}^{(l)} \left( \frac{\hat{m}_1^2}{\mu^2}, \varepsilon, N \right) = \left( \frac{m^2}{\mu^2} \right)^{\frac{l}{2}} \hat{A}_{ij}^{(l)} \left( \varepsilon, N \right).
\] (266)

Here \( \hat{A}_{ij}^{(l)} \left( \varepsilon, N \right) \) does not depend on the mass anymore. It exhibits poles in the dimensional regularization parameter \( \varepsilon \) up to \( \varepsilon^{-l} \)

\[
\hat{A}_{ij}^{(l)} \left( \varepsilon, N \right) = \sum_{k=0}^{\infty} \frac{a(k)}{\varepsilon^{l-k}}.
\] (267)

Furthermore, we adopt the notation of Ref. [139] and denote

\[
a^{(l)} = a^{(l)}, \quad a^{(l+1)} = b^{(l)}. \tag{268}
\]

The unrenormalized operator matrix elements with two massive fermion flavors with masses \( m_1 \neq m_2 \) are split into the respective single-mass terms (266,267) and a part \( \hat{A}_{ij}^{(l)} \left( \frac{\hat{m}_1^2}{\mu^2}, \frac{\hat{m}_2^2}{\mu^2}, \varepsilon, N \right) \) depending on both masses

\[
\hat{A}_{ij}^{(l)} \left( \frac{\hat{m}_1^2}{\mu^2}, \frac{\hat{m}_2^2}{\mu^2}, \varepsilon, N \right) = \left[ \left( \frac{m_1^2}{\mu^2} \right)^{\frac{l}{2}} \right] \hat{A}_{ij}^{(l)} \left( \varepsilon, N \right) + \hat{A}_{ij}^{(l)} \left( \frac{\hat{m}_1^2}{\mu^2}, \frac{\hat{m}_2^2}{\mu^2}, \varepsilon, N \right).
\] (269)

The two flavor contributions \( \hat{A}_{ij}^{(l)} \left( \frac{\hat{m}_1^2}{\mu^2}, \frac{\hat{m}_2^2}{\mu^2}, \varepsilon, N \right) \), \( m_1 \neq m_2 \), to the massive OMEs do not obey a factorization relation as (266) and the mass dependence is pulled into the coefficients of the Laurent expansion

\[
\hat{A}_{ij}^{(l)} \left( \varepsilon, N \right) = \sum_{k=0}^{\infty} a(k) \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right). \tag{270}
\]

Analogously to (268) we define

\[
a^{(l)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \equiv a^{(l)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right). \tag{271}
\]

In the following \( a^{(l,k)} \), \( a^{(l)} \), \( b^{(l)} \) without argument will denote the single mass--quantities corresponding to the definitions in (267,268), while \( a^{(l)} \left( \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \) refers to the two-mass contribution. From Eq. (232) it is obvious that the renormalization of the 3--loop OMEs requires the knowledge of the one--loop OMEs \( A_{ij}^{(1)}(m_1, m_2) \) up to \( O(\varepsilon^2) \) and the two--loop OMEs \( A_{ij}^{(2)}(m_1, m_2) \)

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up to \(O(\varepsilon)\). Up to \(O(\alpha_s^2)\) these two mass quantities can be traced back to the corresponding single-mass quantities by Eqs. (258-259) and (260-262). It is technically advantageous to perform the renormalization on the complete two-flavor OMEs \(\hat{A}_{ij}^{(l)}(\mu^2, \frac{m^2}{\mu^2}, \varepsilon, N)\). For brevity we will present the renormalization formulas for the two-mass contribution \(\hat{A}_{ij}^{(l)}(\mu^2, \frac{m^2}{\mu^2}, \varepsilon, N)\) only, which is obtained after subtracting the respective single-mass contributions, cf. Refs. [139, 246].

### 5.7.1 \(A_{qq,Q}^{NS}\)

The lowest non-trivial flavor non-singlet (NS) contribution is of \(O(\alpha_s^2)\),

\[
A_{qq,Q}^{NS} = 1 + a_s^3 \hat{A}_{qq,Q}^{(2),NS} + a_s^3 \hat{A}_{qq,Q}^{(3),NS} + O(a_s^4)
\]

(272)

Starting from \(a_s^3\) it exhibits non-trivial two-mass contributions

\[
\hat{A}_{qq,Q}^{NS} = 1 + a_s^3 \hat{A}_{qq,Q}^{(3),NS} + O(a_s^4)
\]

(273)

The renormalized two-mass OME in the \textsc{MOM} scheme is obtained from the bare quantities combining Eqs. (209, 232). It is given by

\[
A_{qq,Q}^{(3),NS,MOM}(N_F + 2) = \hat{A}_{qq,Q}^{(3),NS,MOM} + Z_{qq}^{-1,(3),NS}(N_F + N_h) - Z_{qq}^{-1,(3),NS}(N_F) + Z_{qq}^{-1,(1),NS}(N_F + N_h) \hat{A}_{qq,Q}^{(2),NS,MOM} + \hat{A}_{qq,Q}^{(2),NS,MOM} \left[ \hat{A}_{qq,Q}^{(2),NS,MOM} \right]
\]

(274)

After a finite renormalization to the \textsc{MS} scheme and the subtraction of the single-mass contributions one obtains the the pole-structure of the two-flavor piece

\[
\hat{A}_{qq,Q}^{(3),NS} = \frac{16}{3} \frac{1}{\varepsilon^3} \left[ \gamma_{qq}^{(0)} \beta_0^{\gamma_{qq}^{(0)}} + \frac{1}{\varepsilon^2} \left[ \frac{8}{3} \beta_0^{\gamma_{qq}^{(0)}} \gamma_{qq}^{(0)} \beta_0^{\gamma_{qq}^{(0)}} - 4 \gamma_{qq}^{(0)} \beta_0^{\gamma_{qq}^{(0)}} (L_2 + L_1) \right] \right]
\]

(275)

\[
+ \frac{1}{\varepsilon} \left[ -2 \beta_0^{\gamma_{qq}^{(0)}} \gamma_{qq}^{(0)} (L_2 + L_1) - 2 \gamma_{qq}^{(0)} \beta_0^{\gamma_{qq}^{(0)}} (L_2 + L_2L_1 + L_1^2) - 8 \alpha_{qq}^{(0)} \beta_0^{\gamma_{qq}^{(0)}} \right]
\]

with

\[
L_1 = \ln \left( \frac{m_1^2}{\mu^2} \right), \quad L_2 = \ln \left( \frac{m_2^2}{\mu^2} \right).
\]

(276)

The renormalized expression in the \textsc{MS} scheme is given by

\[
A_{qq,Q}^{NS,MS} = \gamma_{qq}^{(0)} \beta_0^{\gamma_{qq}^{(0)}} \left( \frac{2}{3} L_2^3 + \frac{2}{3} L_2^3 + \frac{1}{2} L_2^2 L_1 + \frac{1}{2} L_1^2 L_2 \right) + \beta_0^{\gamma_{qq}^{(0)}} \gamma_{qq}^{(0)} \left( L_2 + L_1^2 \right)
\]

\[
+ \left[ 4 \alpha_{qq}^{NS,2} \beta_0^{\gamma_{qq}^{(0)}} + \frac{1}{2} \beta_0^{\gamma_{qq}^{(0)}} \gamma_{qq}^{(0)} \right] (L_2 + L_1) + 8 \alpha_{qq}^{NS,2} \beta_0^{\gamma_{qq}^{(0)}}
\]

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For the explicit expressions for the anomalous dimensions $\gamma_{ij}$ see Refs. [161, 162] and for the 2–loop constant and $O(\varepsilon)$ part of the OMEs see [135, 137] and references in these papers.

5.7.2 $A_{Qq}^{PS}$

Depending on whether the operator couples to a heavy or a light fermion, there are two pure–
singlet contributions [139]

$$A_{Qq}^{PS} = a_s^2 A_{Qq}^{(2),PS} + a_s^3 A_{Qq}^{(3),PS} + O(a_s^4),$$

(278)

$$A_{qq,Q}^{PS} = a_s^3 A_{qq,Q}^{(3),PS} + O(a_s^4).$$

(279)

Up to $O(a_s^4)$ only the OME $A_{Qq}^{PS}$ contains a generic two–mass contribution, since $A_{qq,Q}^{PS}$ emerges only at $O(a_s^3)$ and contains one internal massless fermion line. One has

$$\hat{A}_{Qq}^{PS} = a_s^3 \tilde{A}_{Qq}^{(3),PS} + O(a_s^4).$$

(280)

The combined renormalization relation at third order is given by

$$A_{Qq}^{(3),MOM} + A_{qq,Q}^{(3),MOM} = \hat{A}_{Qq}^{(3),MOM} + A_{qq,Q}^{(3),MOM} + \hat{A}_{qg,Q}^{(3),MOM} + Z_{qq}^{-1,(3),PS}(N_F + N_h)$$

(281)

$$- Z_{qq}^{-1,(3),PS}(N_F + N_h) - Z_{qq}^{-1,(1)}(N_F + N_h)\hat{A}_{Qq}^{(2),MOM} + Z_{qq}^{-1,(1)}(N_F + N_h)\hat{A}_{gq,Q}^{(2),MOM}$$

$$+ \left[\hat{A}_{Qg}^{(1),MOM} + Z_{qq}^{-1,(1)}(N_F + N_h) - Z_{qq}^{-1,(1)}(N_F)\right]\Gamma_{qq}^{-1,(2)}(N_F) + \left[\hat{A}_{Qq}^{(2),MOM} + Z_{qq}^{-1,(2)}(N_F + N_h)\right]\Gamma_{qq}^{-1,(1)}(N_F).$$

This yields the generic pole structure for the PS two–mass contribution

$$\hat{A}_{Qq}^{(3),PS} = \frac{16}{3} \frac{1}{\varepsilon^3} \hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{0,Q}^{(0)} + \frac{1}{\varepsilon^2} \left[4\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}^{(0)}(L_2 + L_1) + \frac{2}{3} \hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)}(L_2 + L_1) + \frac{1}{2} \hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)} \hat{\gamma}_{qg}^{(0)} + \hat{a}_{Qq}^{(3),PS}(m_1^2, m_2^2, \mu^2)\right].$$

(282)

In the $\overline{MS}$–scheme one obtains the renormalized expression

$$\hat{A}_{Qq}^{\overline{MS},PS} = -\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}^{(0)} \left\{ \frac{1}{2} L_2^2 L_1 + \frac{1}{2} L_1^2 L_2 + \frac{2}{3} L_2^2 + \frac{2}{3} L_1^2 \right\} + \left\{ -\frac{1}{4} \hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)} \beta_{0,Q}^{(0)} \hat{\gamma}_{qg}^{(0)} \hat{\gamma}_{qg}^{(0)} + \hat{a}_{Qq}^{(3),PS}(m_1^2, m_2^2, \mu^2) \right\}$$

$$\times (L_2^2 + L_1^2) + \left\{ 4\hat{a}_{qg}^{(2),PS} \beta_{0,Q}^{(2)} - \hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)} \beta_{0,Q}^{(2)} \hat{\gamma}_{qg}^{(0)} \hat{\gamma}_{qg}^{(0)} + \frac{1}{2} \beta_{0,Q}^{(0)} \hat{\gamma}_{qg}^{(0)} \hat{\gamma}_{qg}^{(0)} + \hat{a}_{Qq}^{(3),PS}(m_1^2, m_2^2, \mu^2) \right\} (L_2 + L_1) + 8\hat{a}_{Qq}^{(2),PS} \beta_{0,Q}^{(2)}$$

$$- 2\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qg}^{(0)} + \hat{a}_{Qq}^{(3),PS}(m_1^2, m_2^2, \mu^2).$$

(283)
5.7.3  $A_{Qg}$

Like in the PS case, there are two different contributions to the OME $A_{Qg}$

$$A_{Qg} = a_s A_{Qg}^{(1)} + a_s^2 A_{Qg}^{(2)} + a_s^3 A_{Qg}^{(3)} + O(a_s^4).$$ \hspace{1cm} (284)

$$A_{qg,Q} = a_s^3 A_{qg,Q}^{(3)} + O(a_s^4).$$ \hspace{1cm} (285)

Of these OMEs only $A_{Qg}$ contains two flavor contributions starting from $O(a_s^2)$

$$\hat{A}_{Qg} = a_s^2 A_{Qg}^{(2)} + a_s^3 A_{Qg}^{(3)} + O(a_s^4).$$ \hspace{1cm} (286)

$$\hat{A}_{qg} = a_s A_{qg}^{(1)} + a_s^2 A_{qg}^{(2)} + a_s^3 A_{qg}^{(3)} + O(a_s^4).$$ \hspace{1cm} (287)

In Eq. (286) the $a_s^2$ contribution consists of one–particle reducible diagrams only, see Eq. (261). As a consequence the flavor dependence factorizes in the $O(a_s^2)$ terms.

The renormalized MOM–scheme two–loop contribution is obtained by

$$A_{Qg}^{(2),\text{MOM}} = \hat{A}_{Qg}^{(2),\text{MOM}} + Z^{-1,2}_{gg}(N_F + N_h) - Z^{-1,2}_{gg}(N_F) + Z^{-1,3}_{gg}(N_F + N_h)\hat{A}_{gg,Q}^{(1),\text{MOM}}$$

$$+ Z^{-1,1}_{gg}(N_F + N_h)\hat{A}_{Qg}^{(1),\text{MOM}} + \left[ \hat{A}_{Qg}^{(1),\text{MOM}} + Z^{-1,1}_{gg}(N_F + N_h) \right] \Gamma_{gg}^{-1,1}(N_F).$$ \hspace{1cm} (288)

The unrenormalized terms are given by

$$\hat{A}_{Qg}^{(2)} = - \frac{1}{\varepsilon^2} 4\beta_0 \gamma^{(0)}_{gg} - \frac{1}{\varepsilon^2} 2\beta_0 \gamma^{(0)}_{gg} \left( \ln \left( \frac{m_b^2}{\mu^2} \right) + \ln \left( \frac{m_c^2}{\mu^2} \right) \right) + \hat{a}_{Qg}^{(2)} \left( \frac{m_b^2}{\mu^2}, \frac{m_c^2}{\mu^2} \right).$$ \hspace{1cm} (289)

The coefficients \( \hat{a}_{Qg}^{(2)} \) and \( \hat{a}_{qg}^{(2)} \left( \frac{m_b^2}{\mu^2}, \frac{m_c^2}{\mu^2} \right) \) are read off Eqn. (261)

$$\hat{a}_{Qg}^{(2)} = - \beta_0 \gamma^{(0)}_{gg} \left\{ \frac{1}{2} (L_1 + L_2)^2 + \tilde{c}_2 \right\}.$$ \hspace{1cm} (290)

$$\hat{a}_{Qg}^{(2)} = \beta_0 \gamma^{(0)}_{gg} \left\{ - \frac{1}{12} (L_2 + L_1)^3 - \frac{1}{2} \tilde{c}_2 (L_2 + L_1) - \frac{1}{3} \tilde{c}_3 \right\}.$$ \hspace{1cm} (291)

The renormalized expression at 2 loops then reads

$$\tilde{A}_{Qg}^{(2),\text{MOM}} = \frac{1}{2} \beta_0 \gamma^{(0)}_{gg} \left( L_1^2 + L_2^2 \right) + \tilde{c}_2 \beta_0 \gamma^{(0)}_{gg} + \hat{a}_{Qg}^{(2)}.$$ \hspace{1cm} (292)

The renormalized 3–loop OMEs in the MOM–scheme are obtained from the charge– and mass–renormalized OMEs by

$$A_{Qg}^{(3),\text{MOM}} + A_{qg,Q}^{(3),\text{MOM}} = \tilde{A}_{Qg}^{(3),\text{MOM}} + \tilde{A}_{qg,Q}^{(3),\text{MOM}} + Z_{gg,Q}^{-1,3}(N_F + N_h) - Z_{gg,Q}^{-1,3}(N_F)$$

$$+ Z_{gg}(N_F + N_h)\tilde{A}_{Qg}^{(1),\text{MOM}} + Z_{gg}(N_F + N_h)\tilde{A}_{gg,Q}^{(2),\text{MOM}} + Z_{gg}(N_F + N_h)\tilde{A}_{Qg}^{(1),\text{MOM}}$$

$$+ Z_{gg}(N_F + N_h)\tilde{A}_{Qg}^{(2),\text{MOM}} + \left[ \tilde{A}_{Qg}^{(1),\text{MOM}} + Z_{gg}(N_F + N_h) \right] \Gamma_{gg}^{-1,1}(N_F + N_h).$$
\[ - Z_{qg}^{-1,1}(N_F) \Gamma_{qg}^{-1,1}(N_F) + \left[ \hat{A}_{qg}^{(2),\text{MOM}} + Z_{qg}^{-1,1}(N_F + N_h) - Z_{qg}^{-1,1}(N_F) \right] \Gamma_{qg}^{-1,1}(N_F) + Z_{qg}^{-1,1}(N_F + N_h) \hat{A}_{qg}^{(1),\text{MOM}} + Z_{qg}^{-1,1}(N_F + N_h) A_{qg}^{(1),\text{MOM}} \right] \Gamma_{qg}^{-1,1}(N_F) + \left[ \hat{A}_{qg}^{(2),\text{PS,MOM}} + Z_{qg}^{-1,1,2}\text{PS}(N_F + N_h) - Z_{qg}^{-1,1,2}\text{PS}(N_F) \right] \Gamma_{qg}^{-1,1}(N_F) \]}

\]

The structure of the unrenormalized OME is more complex than in the NS- or PS case. It is given by

\[ \hat{A}_{qg}^{(3)} = \left[ \frac{28}{3} \beta_0 \beta_0, Q \gamma_{qg}^{(0)} - \frac{8}{3} \gamma_{qg}^{(0)} \beta_0, Q + \frac{14}{3} \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + 24 \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + \frac{1}{3} \gamma_{qg}^{(0)} \beta_0, Q \right] \]

\[ + \frac{1}{2} \left[ \frac{1}{4} \gamma_{qg}^{(0)} \left( \gamma_{qg}^{(0)} \right)^2 + 18 \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + 2 \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + \frac{7}{3} \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q \right] \]

\[ \times \left( L_2 + L_1 \right) + \frac{1}{2} \hat{A}_{qg}^{(2),\text{NS}(1,1)} \left[ \frac{15}{8} \gamma_{qg}^{(0)} \hat{A}_{qg}^{(1)}} + \frac{15}{8} \gamma_{qg}^{(0)} \hat{A}_{qg}^{(1)}} \right] \left( - \frac{1}{8} \hat{A}_{qg}^{(1)}} + \frac{3}{16} \gamma_{qg}^{(0)} \hat{A}_{qg}^{(1)}} \right) \left( L_2 + L_1 \right) \]

\[ + \frac{3}{16} \gamma_{qg}^{(0)} \hat{A}_{qg}^{(1)}} \left( L_2 + L_1 \right) + \frac{1}{2} \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + 8 \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + 4 \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + 12 \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q \]

\[ + 9 \gamma_{qg}^{(0)} \hat{A}_{qg}^{(1)}} + \frac{1}{2} \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q + 8 \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q \]

\[ + \frac{3}{2} \hat{A}_{qg}^{(3)}} \left( m_1^2, m_2^2, \mu^2 \right) . \]

After subtracting the respective one-mass OMEs for both masses \( m_1 \) and \( m_2 \) the two-flavor contribution is obtained. It reads

\[ \hat{A}_{qg}^{(3),\text{MS}} = \left\{ - \frac{9}{4} \beta_0, Q \gamma_{qg}^{(0)} - \frac{7}{96} \gamma_{qg}^{(0)} \left( \gamma_{qg}^{(0)} \right)^2 + \frac{1}{3} \gamma_{gq}^{(0)} \gamma_{qg}^{(0)} \beta_0, Q - \frac{25}{48} \beta_0, Q \gamma_{qg}^{(0)} \gamma_{qg}^{(0)} - \frac{25}{24} \beta_0, Q \right\} \]

\[ \times \left( L_2^2 + L_1^2 \right) + \left\{ \frac{1}{4} \gamma_{qg}^{(0)} \gamma_{qg}^{(0)} \beta_0, Q - \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q - \frac{1}{2} \beta_0, Q \gamma_{qg}^{(0)} \gamma_{qg}^{(0)} - 3 \beta_0, Q \gamma_{qg}^{(0)} \right\} \]

\[ \times \left( L_2^2 L_1 + L_1^2 L_2 \right) + \left\{ \frac{1}{16} \gamma_{qg}^{(0)} \gamma_{qg}^{(0)} \beta_0, Q - \frac{1}{16} \gamma_{qg}^{(0)} \gamma_{qg}^{(0)} \gamma_{qg}^{(0)} - \frac{1}{4} \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q \right\} \]

\[ - \frac{29}{8} \beta_0, Q \delta m_{1,1} \left( \mu^2 \right) + \frac{3}{2} \beta_0, Q \gamma_{qg}^{(0)} \beta_0, Q \delta m_{1,1} \left( \mu^2 \right) . \]

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The matrix element \( A_{gq;Q} \) contributes only from \( \mathcal{O}(a_s^2) \),

\[
A_{gq;Q} = a_s^2 A^{(2)}_{gq;Q} + a_s^3 A^{(3)}_{gq;Q} + \mathcal{O}(a_s^4). 
\]

Diagrams with two different masses however contribute only from \( \mathcal{O}(a_s^3) \)

\[
\tilde{A}_{gq;Q} = a_s^3 \tilde{A}^{(3)}_{gq;Q} + \mathcal{O}(a_s^4). 
\]

The renormalization in the MOM scheme is performed using

\[
A^{(2),\text{MOM}}_{gq;Q} = \tilde{A}^{(2),\text{MOM}}_{gq;Q} + Z_{gq}^{-1,2} (N_F + N_h) - Z_{gq}^{-1,2} (N_F) + \left( \tilde{A}^{(1),\text{MOM}}_{gq;Q} + Z_{gq}^{-1,1} (N_F + N_h) - Z_{gq}^{-1,1} (N_F) \right) \Gamma_{gq}^{-1,1}, 
\]

\[
A^{(3),\text{MOM}}_{gq;Q} = \tilde{A}^{(3),\text{MOM}}_{gq;Q} + Z_{gq}^{-1,3} (N_F + N_h) - Z_{gq}^{-1,3} (N_F) + Z_{gq}^{-1,1} (N_F + N_h) \tilde{A}^{(2),\text{MOM}}_{gq;Q} + \left[ \tilde{A}^{(1),\text{MOM}}_{gq;Q} + Z_{gq}^{-1,1} (N_F + N_h) \right] \Gamma_{gq}^{-1,2} (N_F) + \left[ A^{(2),\text{MOM}}_{gq;Q} + Z_{gq}^{-1,2} (N_F + N_h) \right] \Gamma_{gq}^{-1,1} (N_F) + \tilde{A}^{(2),\text{MOM}}_{gq;Q} \left( Z_{gq}^{-1,1} (N_F + N_h) \right) \Gamma_{gq}^{-1,1} (N_F). 
\]

Applying Eq. (299) yields the unrenormalized expression

\[
\tilde{A}^{(3)}_{gq;Q} = -16 \frac{1}{\varepsilon_3} \gamma_{gq}^{(0)} \beta_0 Q + \left[ \frac{1}{\varepsilon^2} - 12 \gamma_{gq}^{(0)} \beta_0 Q (L_2 + L_1) - 4 \beta_0 Q \tilde{\phi}_{gq}^{(1)} \right] + \frac{1}{\varepsilon} \left[ -6 \gamma_{gq}^{(0)} \beta_0 Q \right] 
\times (L_2 + L_1 L_2 + L_1^2) - 3 \beta_0 Q \tilde{\phi}_{gq}^{(1)} (L_2 + L_1) + \frac{2}{3} \gamma_{gq}^{(2)} - 12 \phi_{gq}^{(2)} \beta_0, 
\]

\[
+ \tilde{a}^{(3)}_{gq;Q} \left( m_1^2, m_2^2, \mu^2 \right), 
\]

and the renormalized operator matrix element

\[
\tilde{A}^{(3),\text{MS}}_{gq;Q} = \gamma_{gq}^{(0)} \beta_0 Q \left( 2L_2^3 + 2L_1^3 + \frac{3}{2} L_2^2 L_1 + \frac{3}{2} L_1^2 L_2 \right) + \frac{3}{2} \beta_0 Q \tilde{\phi}_{gq}^{(1)} (L_2^2 + L_1^2) 
\]
\[ + \left\{ 6a_{gg}^{(2)} \beta_{0,Q} + \frac{3}{2} \gamma_{gg}^{(0)} \beta_{0,Q} \zeta_2 \right\} (L_2 + L_1) + 12 \tilde{a}_{gg,Q}^{(2)} \beta_{0,Q} + \tilde{a}_{gg,Q}^{(3)} \left( m_1^2, m_2^2, \mu^2 \right) \] .

(301)

5.7.5  \( A_{gg,Q} \)

Finally, the matrix element \( A_{gg,Q} \) obeys the expansion

\[ A_{gg,Q} = 1 + a_s A_{gg,Q}^{(1)} + a_s^2 A_{gg,Q}^{(2)} + a_s^3 A_{gg,Q}^{(3)} + O(a_s^4) , \]  

(302)

with two–mass contributions starting at \( O(a_s^2) \),

\[ \tilde{A}_{gg,Q} = + a_s^2 \tilde{A}_{gg,Q}^{(2)} + a_s^3 \tilde{A}_{gg,Q}^{(3)} + O(a_s^4) . \]  

(303)

The renormalization formulas in the \( \text{MOM} \)–scheme read

\[
A_{gg,Q}^{(2),\text{MOM}} = \tilde{A}_{gg,Q}^{(2),\text{MOM}} + Z_{gg}^{-1,(2)} (N_F + N_h) - Z_{gg}^{-1,(2)} (N_F) \\
+ Z_{gg}^{-1,(1)} (N_F + N_h) \tilde{A}_{gg,Q}^{(1),\text{MOM}} + Z_{gg}^{-1,(1)} (N_F + N_h) \tilde{A}_{gg,Q}^{(1),\text{MOM}} \\
+ \left[ \tilde{A}_{gg,Q}^{(1),\text{MOM}} + Z_{gg}^{-1,(1)} (N_F + N_h) - Z_{gg}^{-1,(1)} (N_F) \right] \Gamma_{gg}^{-1,(1)} (N_F) ,
\]  

(304)

\[
A_{gg,Q}^{(3),\text{MOM}} = \tilde{A}_{gg,Q}^{(3),\text{MOM}} + Z_{gg}^{-1,(3)} (N_F + N_h) - Z_{gg}^{-1,(3)} (N_F) \\
+ Z_{gg}^{-1,(2)} (N_F + N_h) \tilde{A}_{gg,Q}^{(1),\text{MOM}} + Z_{gg}^{-1,(2)} (N_F + N_h) \tilde{A}_{gg,Q}^{(1),\text{MOM}} \\
+ \left[ \tilde{A}_{gg,Q}^{(2),\text{MOM}} + Z_{gg}^{-1,(2)} (N_F + N_h) - Z_{gg}^{-1,(2)} (N_F) \right] \Gamma_{gg}^{-1,(1)} (N_F) \\
+ \left[ \tilde{A}_{gg,Q}^{(3),\text{MOM}} + Z_{gg}^{-1,(3)} (N_F + N_h) - Z_{gg}^{-1,(3)} (N_F) \right] \Gamma_{gg}^{-1,(1)} (N_F) .
\]  

(305)

After subtracting all single–mass contributions we obtain the unrenormalized pure two-flavor contribution at 2 loops

\[ \tilde{\tilde{A}}_{gg,Q}^{(2)} = \frac{8 \beta_{0,Q}}{\varepsilon^2} + \frac{4 \beta_{0,Q}^2}{\varepsilon} (L_1 + L_2) + \tilde{a}_{gg,Q} \left( m_1^2, m_2^2, \mu^2 \right) + \varepsilon \tilde{a}_{gg,Q} \left( m_1^2, m_2^2, \mu^2 \right) . \]  

(306)

and the renormalized expression

\[ \tilde{A}_{gg,Q}^{(2),\text{MS}} = - \beta_{0,Q}^{\beta} \left( \ln^2 \left( \frac{m_1^2}{\mu^2} \right) + \ln^2 \left( \frac{m_2^2}{\mu^2} \right) \right) - 2 \beta_{0,Q}^{\beta} \zeta_2 + \tilde{a}_{gg,Q} \left( m_1^2, m_2^2, \mu^2 \right) . \]  

(307)

The \( O(a_s^2) \) contribution consists of one particle reducible contributions only and the coefficients follow from Eq. (262)

\[
a_{gg,Q}^{(2)} = \beta_{0,Q}^{\beta} (L_2 + L_1)^2 + 2 \beta_{0,Q}^{\beta} \zeta_2 ,
\]  

(308)

\[
\tilde{a}_{gg,Q}^{(2)} = \frac{1}{6} \beta_{0,Q}^{\beta} (L_1 + L_2)^3 + \beta_{0,Q}^{\beta} \zeta_2 (L_2 + L_1) + \frac{2}{3} \beta_{0,Q}^{\beta} \zeta_3 .
\]  

(309)
The unrenormalized 3-loop contribution from two masses reads

\[ \hat{A}^{(3)}_{gg;Q} = \frac{1}{\varepsilon^3} \left[ -\frac{10}{3} \hat{g}^{(0)}_{gg} \beta_0 \gamma^{(0)}_{gg} - \frac{56}{3} \beta_0 \beta^2_{0,Q} - \frac{28}{3} \beta^2_{0,Q} \hat{g}^{(0)}_{gg} - 48 \beta^3_{0,Q} \right] + \frac{1}{\varepsilon^2} \left\{ -7 \beta^2_{0,Q} \hat{g}^{(0)}_{gg} \right. \\
\left. -14 \beta_0 \beta^2_{0,Q} - \frac{5}{2} \hat{g}^{(0)}_{gg} \beta_0 \gamma^{(0)}_{gg} - 36 \beta^3_{0,Q} \right\} (L_2 + L_1) + \frac{1}{3} \hat{g}^{(0)}_{gg} \hat{g}^{(1)}_{gg} - \frac{14}{3} \beta_0 \gamma^{(1)}_{gg} + \frac{4}{3} \beta_{1,Q} \beta_{0,Q} - 20 \delta m_1^{-1} \beta^2_{0,Q} \right\] + \frac{1}{\varepsilon} \left\{ \frac{1}{4} \hat{g}^{(0)}_{gg} \hat{g}^{(1)}_{gg} - 15 \delta m_1^{-1} \beta^2_{0,Q} - \frac{7}{2} \beta_0 \gamma^{(1)}_{gg} \right. \\
\left. + \beta_{1,Q} \beta_{0,Q} \right\} (L_2 + L_1) + \left\{ -15 \beta^3_{0,Q} - \frac{11}{8} \hat{g}^{(0)}_{gg} \beta_0 \gamma^{(0)}_{gg} - \frac{13}{2} \beta_0 \beta^2_{0,Q} - \frac{13}{4} \beta^2_{0,Q} \hat{g}^{(0)}_{gg} \right\} \\
\times (L_2^2 + L_1^2) + \left\{ -4 \beta^2_{0,Q} \gamma^{(0)}_{gg} - 24 \beta^3_{0,Q} - 8 \beta_0 \beta^2_{0,Q} - \hat{g}^{(0)}_{gg} \beta_0 \gamma^{(0)}_{gg} \right\} L_2 L_1 - \frac{1}{2} \beta^2_{0,Q} \gamma_{gg} \\\n\left. + \frac{2}{3} \hat{g}^{(2)}_{gg} - 12 \beta_{0,Q} \gamma^{(2)}_{gg} - 18 \beta^3_{0,Q} \zeta_2 + \frac{1}{4} \beta_{0,Q} \gamma_{gg} \gamma^{(0)}_{gg} \hat{g}^{(0)}_{gg} - \beta_0 \beta^2_{0,Q} \zeta_2 - 16 \delta m_1^{(0)} \beta^2_{0,Q} \right. \\
\left. + 4 \beta_{0,Q} \delta m_2^{(-1)} \right\} + \tilde{a}^{(3)}_{gg;Q} (m_1^2, m_2^2, \mu^2) \right). \tag{310} \]

The renormalized result in the \( \overline{\text{MS}} \) scheme is given by

\[ \hat{A}^{(3),\overline{\text{MS}}}_{gg;Q} = + \left\{ \frac{25}{24} \beta^2_{0,Q} \gamma^{(0)}_{gg} + \frac{25}{12} \beta_0 \beta^2_{0,Q} + \frac{9}{2} \beta^3_{0,Q} + \frac{23}{48} \hat{g}^{(0)}_{gg} \beta_0 \gamma^{(0)}_{gg} \right\} (L_2 + L_1) + \left\{ \frac{1}{4} \hat{g}^{(0)}_{gg} \beta_0 \gamma^{(0)}_{gg} \right. \\
\left. + \beta^2_{0,Q} \gamma^{(0)}_{gg} + 2 \beta_0 \beta^2_{0,Q} + 6 \beta^3_{0,Q} \right\} (L_2^2 L_1 + L_1^2 L_2) + \left\{ -\frac{1}{4} \beta_{1,Q} \beta_{0,Q} + \frac{13}{8} \beta_{0,Q} \hat{g}^{(1)}_{gg} \right. \\
\left. + \frac{29}{4} \delta m_1^{(-1)} \beta^2_{0,Q} - \frac{1}{16} \hat{g}^{(0)}_{gg} \hat{g}^{(1)}_{gg} \right\} (L_2^2 L_1 + L_1^2 L_2) + \left\{ -4 \beta_0 \beta^2_{0,Q} \zeta_2 + \frac{9}{4} \beta^2_{0,Q} \zeta_2 \right\} \\
\left. + \frac{27}{2} \beta^3_{0,Q} \zeta_2 - 3 \beta_{0,Q} \delta m_2^{(-1)} + \frac{9}{8} \beta^2_{0,Q} \zeta_2 \gamma^{(0)}_{gg} + 12 \delta m_1^{(0)} \beta^2_{0,Q} + \frac{1}{16} \beta_{0,Q} \gamma_{gg} \gamma^{(0)}_{gg} \gamma^{(0)}_{gg} \right\} \\
\left. + 6 \delta m_2^{(2)} \right\} (L_2 + L_1) - \frac{1}{8} \hat{g}^{(0)}_{gg} \zeta_2 \gamma^{(1)}_{gg} + \frac{1}{4} \beta_{0,Q} \zeta_2 \gamma^{(1)}_{gg} + \frac{1}{3} \beta^2_{0,Q} \zeta_3 + 12 \beta_{0,Q} \tilde{a}^{(2)}_{gg;Q} \right. \\
\left. + 6 \beta^3_{0,Q} \zeta_3 + 16 \delta m_1^{(1)} \beta^2_{0,Q} + \frac{1}{6} \beta^2_{0,Q} \zeta_3 \gamma^{(0)}_{gg} - 2 \beta_{0,Q} \left( \delta^{(1)}_{m_2} + \delta^{(0)}_{m_2} \right) \right. \\
\left. + \frac{9}{2} \delta m_1^{(-1)} \beta^2_{0,Q} \zeta_2 - \frac{1}{12} \beta_{0,Q} \gamma_{gg} \gamma^{(0)}_{gg} \hat{g}^{(0)}_{gg} - \frac{1}{2} \beta_{0,Q} \zeta_2 \beta_{1,Q} \right. \\
\left. + \tilde{a}^{(3)}_{gg;Q} (m_1^2, m_2^2, \mu^2) \right). \tag{311} \]
6 Massive OMEs with two masses

Starting at 3-loop order Feynman diagrams, which carry internal fermion lines of different mass contribute to the OMEs. Contributions of this type are dealt with here for the first time. In Section 6.1 we discuss the decoupling relations for the kinematic region where one has to consider two massive quark flavors. For all OMEs we are calculating a series of Mellin moments \( N = 2, 4, 6 \) \([243,269]\), which are presented in the OMS–scheme in Section 6.2. Results at generals values of the Mellin variable \( N \) are given for the flavor non-singlet case for the vector and axial-vector current and in the case of transversity in Section 6.3 \([243]\). The equal mass case for the non-singlet and pure singlet terms is dealt with in Section 6.4. In Section 6.5 we devise a method to calculate the diagrams contributing to the OME \( A_{gg,Q} \) and compute all scalar topologies both in \( x \)- and in \( N \)-space \([242,270]\).

6.1 Decoupling two massive quark flavours

In Section 2.2 the decoupling of the heavy flavor Wilson coefficient into light flavor Wilson coefficients and massive operator matrix elements has been discussed for the case of one massive quark flavor. However, the relevant Eqs. (65-69) maybe be generalized to allow for the simultaneous decoupling of more than one heavy quark flavor. This is advised because \( m_c^2/m_b^2 \sim 0.1 \). One obtains

\[
C_{q,(2,L)}^{\text{NS}} \left( N, N_F, \frac{Q^2}{\mu^2} \right) + L_{q,(2,L)}^{\text{NS}} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) = A_{qg,Q}^{\text{NS}} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) C_{q,(2,L)}^{\text{NS}} \left( N, N_F + 2, \frac{Q^2}{\mu^2} \right) ,
\]

in the non–singlet case. For the pure–singlet and singlet contribution the relations read

\[
C_{q,(2,L)}^{\text{PS}} \left( N_F \right) + L_{q,(2,L)}^{\text{PS}} \left( N_F + 2 \right) = \left[ A_{qg,Q}^{\text{PS}} \left( N_F + 2 \right) + A_{qg,Q}^{\text{PS}} \left( N_F + 2 \right) + A_{Qg}^{\text{PS}} \left( N_F + 2 \right) \right]
\times N_F C_{q,(2,L)}^{\text{PS}} \left( N_F + 2 \right)
+ A_{qg,Q}^{\text{PS}} \left( N_F + 2 \right) C_{g,(2,L)}^{\text{NS}} \left( N_F + 2 \right)
+ A_{qg,Q} \left( N_F + 2 \right) N_F C_{g,(2,L)} \left( N_F + 2 \right) ,
\]

\[
C_{g,(2,L)} \left( N_F \right) + L_{g,(2,L)} \left( N_F + 2 \right) = A_{gg,Q} \left( N_F + 2 \right) N_F C_{g,(2,L)} \left( N_F + 2 \right)
+ A_{gQ} \left( N_F + 2 \right) C_{g,(2,L)}^{\text{NS}} \left( N_F + 2 \right)
+ \left[ A_{gg,Q} \left( N_F + 2 \right) + A_{Qg} \left( N_F + 2 \right) \right]
\times N_F C_{q,(2,L)}^{\text{PS}} \left( N_F + 2 \right) .
\]

Here and in the following the mass- and \( Q^2 \)-dependence of the Wilson coefficients and operator matrix elements have been suppressed for brevity. For the \( H_{ij} \) functions the factorization relations into light flavor Wilson coefficients and massive OMEs take the following form in the case of two different masses

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Expanding the expressions (312–316) up to $O(\alpha_s^3)$ then yields

\[ H_{q, (L2, L)}^{PS}(N_F + 2) = A_{qy}^{PS}(N_F + 2) \left[ C_{q, (L2, L)}^{NS}(N_F + 2) + \hat{C}_{q, (L2, L)}^{PS}(N_F + 2) \right] + \left[ A_{qy}^{NS}(N_F + 2) + A_{qy}^{PS}(N_F + 2) \right] \hat{C}_{q, (L2, L)}^{PS}(N_F + 2) + A_{qy}(N_F + 2) \hat{C}_{q, (L2, L)}^{PS}(N_F + 2), \]  

\[ H_{g, (L2, L)}(N_F + 2) = A_{qg}(N_F + 2) \hat{C}_{g, (L2, L)}(N_F + 2) + A_{gq}(N_F + 2) \hat{C}_{g, (L2, L)}^{PS}(N_F + 2) + A_{qg}(N_F + 2) \hat{C}_{g, (L2, L)}^{PS}(N_F + 2) \].

\[ (315) \]

\[ (316) \]

\[
L_{q, (L2, L)}^{NS}(N_F + 2) = \alpha_s^2 \left[ A_{qq}^{(2), NS}(N_F + 2) \delta_2 + \hat{C}_{q, (L2, L)}^{(2), NS}(N_F) \right] + \alpha_s^3 \left[ A_{qq}^{(3), NS}(N_F + 2) \delta_2 + A_{qg}^{(2), NS}(N_F + 2) \hat{C}_{q, (L2, L)}^{(1), NS}(N_F + 2) + \hat{C}_{q, (L2, L)}^{(3), NS}(N_F) \right],
\]

\[ (317) \]

\[
L_{q, (L2, L)}^{PS}(N_F + 2) = \alpha_s^2 \left[ A_{qq}^{(3), PS}(N_F + 2) \delta_2 + A_{qg}^{(2), PS}(N_F + 2) \hat{C}_{q, (L2, L)}^{(1), PS}(N_F + 2) + N_F \hat{C}_{q, (L2, L)}^{(3), PS}(N_F) \right],
\]

\[ (318) \]

\[
L_{g, (L2, L)}^{S}(N_F + 2) = \alpha_s^2 A_{gq}(N_F + 2) \hat{C}_{g, (L2, L)}^{(1), S}(N_F + 1) + \alpha_s^3 \left[ A_{qg}(N_F + 2) \delta_2 + A_{gq}^{(1), S}(N_F + 2) \hat{C}_{g, (L2, L)}^{(2), S}(N_F + 2) + A_{qg}^{(2), S}(N_F + 2) \hat{C}_{g, (L2, L)}^{(1), S}(N_F + 2) + \hat{C}_{q, (L2, L)}^{(3), S}(N_F) \right],
\]

\[ (319) \]

\[
H_{q, (L2, L)}^{PS}(N_F + 2) = \alpha_s^2 \left[ A_{qg}^{(2), PS}(N_F + 2) \delta_2 + \hat{C}_{q, (L2, L)}^{(2), PS}(N_F + 2) \right] + \alpha_s^3 \left[ A_{qg}^{(3), PS}(N_F + 2) \delta_2 + \hat{C}_{q, (L2, L)}^{(3), PS}(N_F + 2) \right] + A_{qg}^{(2)}(N_F + 2) \hat{C}_{g, (L2, L)}^{(1)}(N_F + 2) + A_{qg}^{(2), PS}(N_F + 2) \hat{C}_{q, (L2, L)}^{(1), NS}(N_F + 2),
\]

\[ (320) \]

\[
H_{g, (L2, L)}^{S}(N_F + 2) = \alpha_s^2 \left[ A_{qg}^{(1), S}(N_F + 2) \delta_2 + \hat{C}_{g, (L2, L)}^{(1), S}(N_F + 2) \right] + A_{qg}^{(2), S}(N_F + 2) \hat{C}_{q, (L2, L)}^{(1), S}(N_F + 2) + \alpha_s^3 \left[ A_{qg}^{(3), S}(N_F + 2) \delta_2 + A_{qg}^{(2), S}(N_F + 2) \hat{C}_{q, (L2, L)}^{(1), S}(N_F + 2) \right] + A_{qg}^{(2)}(N_F + 2) \hat{C}_{g, (L2, L)}^{(1)}(N_F + 2) + A_{qg}^{(2), S}(N_F + 2) \left[ C_{g, (L2, L)}^{(2), NS}(N_F + 2) + \hat{C}_{g, (L2, L)}^{(2), PS}(N_F + 2) \right] + A_{qg}^{(1), S}(N_F + 2) \left[ \hat{C}_{q, (L2, L)}^{(2), PS}(N_F + 2) \right],
\]

\[ (321) \]
with \( \delta_2 = 1 \) for the structure functions \( F_2 \) and \( \delta_2 = 0 \) for \( F_L \), respectively. In Eqs (312-321) the following notation has been applied

\[
\hat{f}(x) = \frac{f(x)}{x}, \quad (322)
\]

\[
\hat{f}(x) = f(x + 2) - f(x). \quad (323)
\]

The presence of diagrams with \( c-\) and \( b\)-quarks at 3-loop order yields power corrections in \( \eta = m_c^2/m_b^2 \) to the massive operator matrix elements, which can not be absorbed into the charm–or bottom-densities of the variable flavor number scheme directly. It is therefore advantageous to extend Eqs. (324-328) to allow for a simultaneous decoupling of both heavy quark flavors in the variable flavour number scheme:

\[
f_k(N_F + 2, N, \mu^2, m_1^2, m_2^2) + f_\bar{k}(N_F + 2, N, \mu^2, m_1^2, m_2^2) =
\]

\[
A^{NS}_{qq,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \left[ f_k(N_F, N, \mu^2) + f_\bar{k}(N_F, N, \mu^2) \right]
\]

\[
+ \frac{1}{N_F} A^{PS}_{qq,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F, N, \mu^2)
\]

\[
+ \frac{1}{N_F} A^{PS}_{qg,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F, N, \mu^2), \quad (324)
\]

\[
f_Q(N_F + 2, N, \mu^2, m^2) + f_{\bar{Q}}(N_F + 2, N, \mu^2, m^2) =
\]

\[
A^{PS}_{Qq} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F, N, \mu^2)
\]

\[
+ A_{Qg} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F, N, \mu^2). \quad (325)
\]

The flavor singlet, non–singlet and gluon densities for \( (N_F + 2) \) flavors are given by

\[
\Sigma(N_F + 2, N, \mu^2, m^2) = \left[ A^{NS}_{qq,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A^{PS}_{qq,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right]
\]

\[
+ A^{PS}_{qg,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F, N, \mu^2)
\]

\[
+ \left[ A_{qg,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{Qg} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot G(N_F, N, \mu^2), \quad (326)
\]

\[
\Delta_k(N_F + 2, N, \mu^2, m^2) = f_k(N_F + 2, N, \mu^2, m^2) + f_\bar{k}(N_F + 2, N, \mu^2, m^2)
\]

\[- \frac{1}{N_F + 2} \Sigma(N_F + 2, N, \mu^2, m^2), \quad (327)
\]

\[
G(N_F + 2, N, \mu^2, m^2) = A_{qg,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F, N, \mu^2)
\]

\[
+ A_{qg,Q} \left( N, N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F, N, \mu^2). \quad (328)
\]

Here \( f_k(f_\bar{k}) \) denote the quarkonic parton densities, \( \Sigma \) is the singlet distribution \( \Sigma = \sum_{k=1}^{N_F} (f_k + f_\bar{k}) \) and \( G \) is the gluon density.
6.2 Fixed Mellin Moments

While not yet sufficient for a phenomenological analysis the study of fixed Mellin moments is still interesting for various reasons. They provide a thorough check on the renormalization prescription given in Section 5, and the general $N$ results given later in Sections 6.3 and 6.5.

In theory the knowledge of a sufficiently large number of fixed Mellin moments also allows to construct the general $N$-expression [271]. In many cases it is, however, very difficult to compute such number of fixed Mellin moments with the usual single-moment methods.

In Ref. [139, 246] fixed Mellin moments for the single mass case were computed by projecting the respective integrals onto massive tadpoles and evaluating them using the code MATAD [272].

In the present study we deal with one additional mass. The Feynman integrals are therefore first expanded in the mass ratio $\eta = \frac{m^2}{m^2}$ by an expansion in subgraphs [273–276] using the codes Q2E/Exp [185, 186] which also rely on MATAD to evaluate the single-mass tadpole diagrams.

The Feynman diagrams are generated using QGRAF [277]. In order to take account for the local operator insertions we introduce new additional propagators which either carry an operator insertion or which generate an operator on an attached vertex. In case of operator insertions on a gauge boson, this method leads to a double counting of some vertex diagrams which has to be removed.

After applying the Feynman rules (cf. Appendix B.2) and the projection operators, Eqs. (60-62) the momentum integrals take the form

$$ I^{(l)}(p, m_1, m_2, n_1 \ldots n_j) \equiv \int \frac{d^D k_1}{(2\pi)^D} \ldots \int \frac{d^D k_l}{(2\pi)^D} (\Delta q_1)^{n_1} \ldots (\Delta q_j)^{n_j} f(k_1 \ldots k_l, p, m_1, m_2) . $$

(329)

Here $p$ denotes the external momentum, $p^2 = 0$, $\Delta$ is an arbitrary light–like vector $\Delta^2 = 0$ and $q_i$ are linear combinations of the loop momenta $k_j$ and the external momentum $p$. The exponents $n_i$ are integer-valued and obey $\sum n_i = N$, while the function $f(k_1 \ldots k_l, p, m_1, m_2)$ contains the remaining numerator structure and denominators.

The light-like vector $\Delta$ has been introduced in Eq. (59) to project out trace terms. Removing it again yields the tensor $\tilde{I}_{\mu_1, \ldots, \mu_N}^{(l)}$

$$ I^{(l)}(p, m_1, m_2, n_1 \ldots n_j) = \prod_{j=1}^{N} \Delta^{\mu_j} \tilde{I}_{\mu_1, \ldots, \mu_N}^{(l)}(p, m_1, m_2, n_1 \ldots n_j) . $$

(330)

Since $\prod_{j=1}^{N} \Delta^{\mu_j}$ constitutes a completely symmetric tensor only the purely symmetric part of $\tilde{I}_{\mu_1, \ldots, \mu_N}^{(l)}$ contributes. We thus symmetrize it by shuffling the indices, [166–168,172,174,180], and normalize it by dividing by the number of terms. For the general integral (329) the symmetrized tensor is given by

$$ I_{\mu_1, \ldots, \mu_M}^{(l)}(p, m_1, m_2, n_1 \ldots n_j) = S \tilde{I}_{\mu_1, \ldots, \mu_M}^{(l)}(p, m_1, m_2, n_1 \ldots n_j) . $$

(331)

Here the symmetrization operator $S$ is defined by

$$ Sf_{\mu_1, \ldots, \mu_M} = \frac{1}{M!} \sum_{w} f_{w} , $$

(332)

where the sum is over the words $w$ given by the different orderings of the Lorentz indices $\{\mu_1, \ldots, \mu_M\}$.
For example for $M = 3$ one obtains

$$S f_{\mu_1, \mu_2, \mu_3} = \frac{1}{6} [f_{\mu_1, \mu_2, \mu_3} + f_{\mu_1, \mu_3, \mu_2} + f_{\mu_2, \mu_1, \mu_3} + f_{\mu_2, \mu_3, \mu_1} + f_{\mu_3, \mu_1, \mu_2} + f_{\mu_3, \mu_2, \mu_1}] . \quad (333)$$

The result of the original integral (329) may then be re-obtained by applying the projection operator $[139]$

$$\Pi_{\mu_1 \ldots \mu_N} = F(N) \sum_{i=1}^{[N/2]+1} C(i, N) \left( \prod_{l=1}^{[N/2]-i+1} \frac{g_{\mu_1 \ldots \mu_2}}{p^2} \right) \left( \prod_{k=2[N/2]-2l+3}^{N} \frac{p_{\mu_2}}{b^2} \right) . \quad (334)$$

The prefactors $F(N)$ and the combinatorial factors $C(i, N)$ are given by

$$F^{\text{odd}}(N) = \frac{2^{3/2-N/2} \Gamma(D/2 + 1/2)}{(D - 1) \Gamma(N/2 + D/2 - 1)} , \quad (335)$$

$$C^{\text{odd}}(k, N) = (-1)^{N/2+k+1/2} \frac{2^{2k-N/2-3/2} \Gamma(N + 1) \Gamma(D/2 + N/2 + k - 3/2)}{\Gamma(N/2 - k + 3/2) \Gamma(2k) \Gamma(D/2 + N/2 - 1/2)} , \quad (336)$$

for odd values and

$$C^{\text{even}}(k, N) = (-1)^{N/2+k+1/2} \frac{2^{2k-N/2-2} \Gamma(N + 1) \Gamma(D/2 + N/2 - 2 + k)}{\Gamma(N/2 - k + 2) \Gamma(2k - 1) \Gamma(D/2 + N/2 - 1)} , \quad (337)$$

$$F^{\text{even}}(N) = \frac{2^{1-N/2} \Gamma(D/2 + 1/2)}{(D - 1) \Gamma(N/2 + D/2 - 1/2)} , \quad (338)$$

for even values of the Mellin variable $N$. The prefactors $F^{\text{odd}}(N)$, $F^{\text{even}}(N)$ are chosen such that the projector (334) obeys the normalization condition

$$\Pi_{\mu_1 \ldots \mu_N} p^\mu_1 \ldots p^\mu_N = 1 . \quad (339)$$

The integrals with local operator insertion for fixed values of $N$ are thus represented in terms of tadpole diagrams with a modified numerator structure. The projection operators (334) become very large for large values of $N$, which leads to an exponential increase in the computation time. In the case of two heavy quarks of different mass the computation of the Mellin moment $N = 6$ of the two–mass contributions to the OME $A_{Qg}$ took already about one year of CPU–time.

The pole structure of the unrenormalized OMEs corresponds to the one which was deduced from the renormalization prescription in Section 5. It constitutes an important check on our results. In the following we present the moment $N = 2$ for the two-flavor contributions to the constant parts of the various operator matrix elements as defined in Eq. (169). For $N = 2$ the following non-singlet contribution 2-mass contribution is obtained

$$a_{QgQ}^{\text{NS}(3)}(N = 2) = C_F T_F^2 \left\{ \left( -\frac{1024}{8505} L^2_{\eta} - \frac{190500608}{843908625} - \frac{747008}{2679075} L_{\eta} \right) \eta^3 
+ \left( \frac{7176352}{1157625} - \frac{64}{105} L^2_{\eta} - \frac{33856}{11025} L_{\eta} \right) \eta^2 + \left( \frac{12032}{675} - \frac{512}{45} L_{\eta} \right) \eta 
+ \left( -\frac{1024}{81} - \frac{128}{9} (L_2 + L_1) \right) \zeta_2 - \frac{153856}{2187} + \frac{512}{81} \zeta_3 - \frac{14080}{243} L_1 - \frac{2048}{81} L_2 \right\} .$$

67
Here and in the following we apply the notation

\[ L_1 = \ln \left( \frac{m_1^2}{\mu^2} \right), \quad L_2 = \ln \left( \frac{m_2^2}{\mu^2} \right), \quad L_\eta = \ln (\eta) \equiv \ln \left( \frac{m_2^2}{m_1^2} \right). \]  

(341)

Note that here as well as in all other computed QCD-results within this thesis power corrections \( \propto \eta^4 \) do not appear in the pole-terms in the dimensional regularization parameter \( \varepsilon \).

The constant contribution to the OME \( A_{Qg}^{PS,(3)} \) is given by

\[
\tilde{a}_{Qg}^{PS,(3)}(N = 2) = C_F T_F^2 \left\{ \left( \frac{56086736 - 164464 L_\eta - 2552 L_2^2}{843908625 - 2679075 L_\eta - 8505 L_2^2} \right) \eta^3 + \left( \frac{6008}{4725} L_\eta + \frac{1565036}{496125} \right) \right. \\
- \frac{128}{105} L_2^2 - \frac{67712}{11025} L_\eta \right) \eta^2 + \left( \frac{24064}{675} - \frac{1024}{45} L_\eta \right) \eta + \left( \frac{1472}{81} \right) \\
- \frac{256}{9} \left( L_2 + L_1 \right) \zeta_2 - \frac{256}{27} L_1^2 L_2 + \frac{1024}{81} L_1 \zeta_3 - \frac{26720}{243} L_1 - \frac{3616}{81} L_2 \\
\left. - \frac{1472}{81} L_2 L_1 - \frac{512}{27} L_1 L_2^2 - \frac{266528}{2187} - \frac{1024}{81} L_2^3 - \frac{1472}{81} (L_2^2 + L_1^2) \right\} \\
+ O \left( \eta^4 L_\eta^3 \right).  
\]  

(342)

For \( \tilde{a}_{Qg}^{(3)} \) one obtains

\[
\tilde{a}_{Qg}^{(3)}(N = 2) = C_A T_F^2 \left\{ \left( \frac{828605984 - 5893184 L_\eta + 23872 L_2^2}{843908625 + 2679075 L_\eta + 8505 L_2^2} \right) \eta^3 + \left( \frac{1028192}{99225} L_\eta \right) \right. \\
- \frac{8}{45} L_2^2 \left( \frac{256304}{10125} + \frac{7184}{675} L_\eta - \frac{8}{45} L_2^2 \right) \eta + \left( \frac{74}{81} + \frac{140}{9} (L_2 + L_1) \right) \zeta_2 \\
- \frac{5}{3} L_2 + \frac{772}{81} L_1^3 - \frac{848}{243} L_1 + \frac{280}{27} L_1 L_2 - \frac{35}{81} (L_2^2 + L_1^2) + \frac{104}{27} L_1^2 L_2 \\
- \frac{152}{81} L_2 L_1 + \frac{596}{81} L_2^3 + \frac{78229}{2187} \\
\left. + C_F T_F^2 \left( \frac{320}{27} - \frac{64}{9} (L_2 + L_1) \right) \zeta_2 + \frac{6752}{243} L_2 - \frac{704}{81} L_1 + \frac{1792}{81} \zeta_3 + \frac{128}{81} L_1 - \frac{128}{27} L_1 L_2 \right\}  
\]  

(343)
yields similar integrals as in the case with one massive and one massless fermionic line \[140\].

All non-singlet diagrams at 3-loop order contain two massive fermion loops. One of these may be stripped of its mass by using the Mellin-Barnes representation \[278\]-\[281\], see Figure 6. This yields similar integrals as in the case with one massive and one massless fermionic line \[140\].

Finally the gluonic contributions to the OMEs \(A_{g g, Q}^{(3)}\) and \(A_{g g, Q}^{(3)}\) are given by

\[
\tilde{a}_{g g, Q}^{(3)}(N = 2) = C_F T_F^2 \left\{ \frac{19188592}{120558375} + \frac{153892}{382725} L_\eta + \frac{686}{1215} L_2^2 \right\} \eta^3 + \frac{53824}{33075} L_\eta + \frac{8433658}{3472875} \\
+ \frac{296}{315} L_2^2 \eta^2 + \left( \frac{153872}{10125} - \frac{1412}{675} \right) L_\eta + \frac{14}{45} L_2^3 \eta + \frac{556}{81} \\
- \frac{140}{9} (L_2 + L_1) \left( \frac{214}{27} L_2 - \frac{628}{81} L_2^2 - \frac{272}{81} L_1 - \frac{6682}{243} L_1 - \frac{280}{27} L_1 L_2^2 \\
- \frac{550}{81} (L_2^2 + L_1^2) - \frac{176}{27} L_2 L_2^2 - \frac{568}{81} L_2 L_1 - \frac{524}{81} L_2^3 - \frac{71578}{2187} \right) \\
+ C_F T_F^2 \left\{ \frac{637103552}{843908625} - \frac{4823552}{2679075} L_\eta - \frac{20416}{8505} L_2^2 \right\} \eta^3 + \left( -\frac{752576}{99225} L_\eta \\
- \frac{116240192}{10418625} - \frac{3904}{945} L_2^2 \right) \eta^2 + \left( \frac{702784}{30375} - \frac{14336}{2025} L_\eta - \frac{448}{135} L_2^3 \right) \eta + \left( -\frac{32}{27} \\
+ \frac{64}{9} (L_2 + L_1) \right) L_2 - \frac{5024}{243} L_2^2 + \frac{704}{81} L_2^3 - \frac{1792}{81} L_3 + \frac{736}{81} L_1 + \frac{128}{27} L_1 L_2^2 \\
+ \frac{112}{81} (L_2^2 + L_1^2) - \frac{128}{27} L_2 L_2^2 - \frac{512}{81} L_2 L_1 + \frac{448}{81} L_2^3 + \frac{4448}{243} \right) + O \left( \eta^4 L_3^2 \right), \quad (344)
\]

The results for the constant parts in the Laurent expansion around \(\varepsilon = 0\) for the further Mellin moments \(N = 4\) and \(N = 6\) are listed in Appendix F.

### 6.3 The Non-Singlet Contributions

All non-singlet diagrams at 3-loop order contain two massive fermion loops. One of these may be stripped of its mass by using the Mellin–Barnes representation \[278\]-\[281\], see Figure 6. This yields similar integrals as in the case with one massive and one massless fermionic line \[140\].
Figure 6: One of the massive fermion loop insertion is effectively rendered massless via a Mellin–Barnes representation.

One may now introduce a Feynman parameter representation, integrate the momenta and perform the Feynman parameter integrals in terms of Euler $B$–functions (D.29) analogously to the procedure described in Section 3. The remaining contour integral is then of the general form

$$I = \sum_{j} C_j (\varepsilon, N) \, _4F_3 \left[ \begin{array}{c} a_1(\varepsilon), a_2(\varepsilon), a_3(\varepsilon), a_4(\varepsilon) \\ b_1(\varepsilon), b_2(\varepsilon), b_3(\varepsilon) \end{array} \right],$$

(348)

where the $f_j$ and the $g_j$ are linear functions. Furthermore, the notation

$$\Gamma \left[ \begin{array}{c} a_1, \ldots, a_i \\ b_1, \ldots, b_j \end{array} \right] = \frac{\Gamma(a_1) \cdots \Gamma(a_i)}{\Gamma(b_1) \cdots \Gamma(b_j)}$$

(347)

is applied. After closing the contour in (346) and collecting the residues a linear combination of generalized hypergeometric $4F_3$–functions [142,143] is obtained

For the NS–contributions the arguments of the hypergeometric $pF_q$ are completely independent of the Mellin variable $N$ and each term factors into contributions that describe the operator insertions and the generalized hypergeometric functions covering the mass structure of the diagrams. Due to the fact that the arguments of the hypergeometric functions depend only on the dimensional regularization parameter $\varepsilon$ their respective expansion may be performed with the code HypExp 2 [282]. The results of these expansions are then given in terms of the following (poly)logarithmic functions [283–286], Eq. (D.33),

$$\left\{ \ln(\eta), \ln \left( \frac{1 - \eta_1}{1 + \eta_1} \right), \text{Li}_2(\sqrt{\eta}), \text{Li}_2(\eta), \text{Li}_3(\sqrt{\eta}) \right\},$$

(349)

with $\eta_1 = \sqrt{\eta}$. The prefactor $C_j (\varepsilon, N)$ may contain a sum stemming from the operator insertion on the vertex, see Appendix B. This sum is easily evaluated in terms of single harmonic sums using the summation package Sigma [236,237].

The general pole structure for the unrenormalized two-mass contribution to the OME $A_{qq,Q}^{\text{NS}}$ is given in Eq. (275). The only contribution which is not determined by the renormalization prescription is the constant part, for which we obtain
Here we have used the notation $\eta = \sqrt{\eta}$ and the polynomials $R_i$ read
\begin{align}
R_1 &= 15\eta^2N^4 + 78\eta N^4 + 15N^4 + 30\eta^2N^3 + 156\eta N^3 + 30N^3 + 25\eta^2N^2 + 18\eta N^2 + 25N^2 \\
&\hspace{1cm} + 10\eta^2 N + 4\eta N + 10N + 32\eta \ , \\
R_2 &= 1215\eta^2 N^8 - 1596\eta N^8 + 1215N^8 + 4860\eta^2 N^7 - 6384\eta N^7 + 4860N^7 + 8100\eta^2 N^6 \\
&\hspace{1cm} - 25844\eta N^6 + 8100N^6 + 7290\eta^2 N^5 - 39348\eta N^5 + 7290N^5 + 3645\eta^2 N^4 - 20304\eta N^4 \\
&\hspace{1cm} + 3645N^4 + 810\eta^2 N^3 - 140\eta N^3 + 810N^3 + 432\eta N^2 + 288N + 864 \eta \ .
\end{align}

The pole structure of the unrenormalized transversity OME corresponds to the one in Eq. (275) after substituting the anomalous dimensions $\gamma_{\text{qq}, \text{NS}} \rightarrow \gamma_{\text{qq}, \text{trans}}^{\text{NS}}$. The constant contribution is given by
\begin{align}
\hat{a}^{(3), \text{NS,TR}}_{\text{qq}, Q} = \\
C_F T_F \left\{ \left[ \frac{32}{27} S_1 - \frac{8}{9} \right] \ln^3(\eta) + \left[ \frac{(\eta + 5)(5\eta + 1)}{6\eta} \right] + \frac{(\eta + 1)(5\eta^2 + 22\eta + 5)}{36\eta^{3/2}} \right\} [4S_1 - 3].
\end{align}
\[
\times \ln \left( \frac{1 - \eta}{1 + \eta} \right) + \frac{2(5\eta^2 + 2\eta + 5)}{9\eta} S_1 + \ln(1 - \eta) \left( \frac{16}{3} - \frac{64}{9} S_1 \right) + \frac{32}{9} S_2 \right] \ln^2(\eta) \\
+ \left[ \frac{40(\eta - 1)(\eta + 1)}{9\eta} S_1 - \frac{10(\eta - 1)(\eta + 1)}{3\eta} + \frac{2(\eta + 1)(5\eta^2 + 22\eta + 5)}{9\eta^{3/2}} \right] [4S_1 - 3] \\
\times \text{Li}_2(\eta) + \frac{16\zeta_2}{9} + \frac{2(\eta + 1)(5\eta^2 + 22\eta + 5)}{9\eta^{3/2}} [6 - 8S_1] \text{Li}_3(\eta) \\
+ \frac{(\eta_1 + 1)^2 (5\eta^2 + 22\eta + 5)}{18\eta^{3/2}} [4S_1 - 3] \text{Li}_3(\eta) \\
+ \left( \frac{2}{729\eta} \right) \frac{8(405\eta^2 - 3238\eta + 405)}{27} + \frac{128\zeta_3}{27} - \frac{32\zeta_2}{27} \right] S_1 + \left( \frac{128\zeta_2}{9} + \frac{3712}{81} \right) S_2 \\
- \frac{1280}{81} S_3 + \frac{256}{27} S_4 - \frac{64\zeta_3}{9} - \frac{4R_3}{243N^2(N + 1)^2 \eta} \right) , \quad (353)
\]

with

\[
R_3 = 405\eta^2 N^4 - 532\eta N^4 + 405N^4 + 810\eta^2 N^3 - 1064\eta N^3 + 810N^3 + 405\eta^2 N^2 \\
- 1012\eta N^2 + 405N^2 + 96\eta N + 288\eta , \quad (354)
\]

### 6.4 The contributions with \( m_1 = m_2 \)

Additionally to the case of two fermionic lines of unequal mass all diagrams of this class emerge also with both fermionic lines of the same mass. For the OME \( A_{gg, Q} \) these contributions have been given in Ref. [287].

For the non-singlet contribution one may obtain the equal mass result by taking the limit \( \eta \to 1 \) in Eq. (350) and multiplying by a factor of \( \frac{1}{2} \) to avoid double counting of diagrams. One obtains

\[
\hat{a}_{qq, Q}^{(3), NS} = T^2 C_F \left\{ \frac{128}{27} S_1 - \frac{1024}{27} \zeta_3 S_1 + \frac{64}{9} \zeta_2 S_2 + \frac{256(3N^2 + 3N + 2)}{27N(N + 1)} \zeta_3 - \frac{320}{27} \zeta_2 S_1 \\
- \frac{640}{81} S_3 + \frac{8(3N^4 + 6N^3 + 47N^2 + 20N - 12)}{27N^2(N + 1)^2} \zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 \\
- \frac{4R_4}{729N^4(N + 1)^4} \right\} , \quad (355)
\]

with

\[
R_4 = 417N^8 + 1668N^7 - 4822N^6 - 12384N^5 - 6507N^4 + 740N^3 + 216N^2 + 144N \\
+ 432 . \quad (356)
\]

For the pure-single OME \( A_{qq, Q}^{PS} \) the method described in Section 6.3 yields generalized hypergeometric functions of argument \( \eta \). In the general case the resulting infinite sums were not accessible.
with the present summation technology, but the limit $\eta = 1$ could still be obtained. The single-mass contribution to the constant term of this OME is given by

$$
\tilde{a}_{q_1}^{(3),PS} = \frac{T^2_{\ell} C_F}{(N - 1)N^2(N + 1)^2(2 + N)} \left\{ \left( N^2 + N + 2 \right)^2 \left( \frac{32}{27} S^3 - \frac{512}{27} S_3 \right) + \frac{128}{3} S_{2,1} + \frac{1024}{9} \zeta_3 - \frac{160}{9} S_2 S_1 + \frac{32}{3} \zeta_2 S_1 \right\} - \frac{32 P_1(N)}{9N(N + 2)} \zeta_2 + \frac{32 P_2(N)}{27N(N + 2)(N + 3)(N + 4)(N + 5)} S_2
$$

$$
- \frac{27N(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)}{64P_4(N)} S_1^2 + \frac{81N^2(N + 1)^2(N + 2)(N + 3)(N + 4)(N + 5)}{64P_5(N)} S_1
$$

$$
- \frac{243N^3(N + 1)^2(N + 2)^2(N + 3)(N + 4)(N + 5)}{27}, \quad (357)
$$

where the polynomials $P_i$ are given by

$$
P_1 = 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24, \quad (358)
$$

$$
P_2 = 40N^9 + 625N^8 + 3284N^7 + 5392N^6 - 7014N^5 - 33693N^4 - 47454N^3 - 46100N^2 - 26280N + 7200, \quad (359)
$$

$$
P_3 = 8N^{10} + 133N^9 + 1095N^8 + 5724N^7 + 18410N^6 + 34749N^5 + 40683N^4 + 37370N^3 + 22748N^2 - 3960N - 7200, \quad (360)
$$

$$
P_4 = 52N^{15} + 746N^{14} + 4658N^{13} + 20431N^{10} + 79990N^9 + 251778N^8 + 553796N^7 + 837697N^6 + 886552N^5 + 599060N^4 + 155864N^3 - 82368N^2 - 76896N - 17280, \quad (361)
$$

$$
P_5 = 293N^{15} + 4670N^{14} + 32280N^{13} + 145948N^{12} + 559575N^{11} + 1871440N^{10} + 4877344N^9 + 933994N^8 + 12958212N^7 + 12693884N^6 + 8472792N^5 + 4514336N^4 + 3109248N^3 + 2192832N^2 + 1026432N + 207360. \quad (362)
$$

### 6.5 Scalar $A_{ggQ}$ diagrams with $m_1 \neq m_2$

The factorization into parts depending purely on the Mellin variable $N$ and contributions depending only on the mass ratio $\eta = \frac{m_2}{m_1}$, which has been observed for the non-singlet diagrams, constitutes a very special case. In general a mixing between both variables occurs and more advanced methods are required to perform the calculation. Since the complexity of the mathematical structures contributing to a Feynman diagram depends on the denominator functions mainly and in the present case also on the form of the operator insertion, we first study the scalar topologies contributing to the OME $A_{ggQ}$. Due to the nesting between the Mellin variable and the mass ratio, novel $\eta$–dependent sums and integrals contribute.

#### 6.5.1 Strategy

As we expect new functions to appear in the results and since the construction of the inverse Mellin transforms for these functions is a non-trivial task we opt for an approach were we derive
the $z$-space representation of the respective diagrams directly. The $N$-space representation\(^6\) is then obtained in a final step by using a generating function representation, constructing a difference equation and solving it using \texttt{Sigma} [236,237].

First we introduce Feynman parameters and perform the momentum integration for all closed fermion lines. Therefore, for each massive fermion loop we obtain one effective propagator and each mass is now only contained in one of these.

We detach the mass of one of these effective propagators by using the Mellin-Barnes representation [278–281,288]

\[
\frac{1}{(A + B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\xi \frac{B^\xi}{A^\lambda + \xi} \Gamma(\lambda + \xi) \Gamma(-\xi) \tag{363}
\]

and perform the last momentum integration to obtain expressions like

\[
I_5 = (m_1^2)^{d/2 - \nu_{56}} (m_2^2)^{-\nu_{12347}} \int_{-i\infty}^{+i\infty} d\xi \Gamma \left[ -\xi_0, \nu_5 + \xi_0, \nu_6 + \xi_0, \nu_56 - d/2 + \xi_0, \nu_{12347} - d - \xi_0 \right] \nu_1, \nu_2, \nu_3, \nu_5, \nu_6, \nu_7, \nu_4 - \xi_0, \nu_56 + 2\xi_0 \\
\int_0^1 \cdots \int_0^1 dy_0 \, dy_1 \, dy_2 \, dy_3 \, dz_0 \, dz_1 \, \delta \left( 1 - y_0 - y_1 - y_2 - y_3 \right) \delta \left( 1 - z_0 - z_1 \right) y_1^{\nu_1-1} y_2^{\nu_2-1} y_3^{\nu_3-1} \\
\times (1 - y_1 - y_2 - y_3)^{\nu_2-1} (1 - y_2 - y_3)^{\nu_4-1} \left( y_2 + y_3 \right)^{-N+\nu_4-d/2-\xi_0} \left( y_2 + y_3 \right)^{N+d/2-1-\nu_4+\xi_0} \\
\times (1 - z_1)^{\nu_4-1} \left( (y_1 + y_3)(y_2 + y_3) - y_3 \right)^N \tag{364}
\]

Except for the additional Mellin-Barnes integral the Feynman parameter integrals are now of a similar form as the integrals in Section 3.1 and the appropriate application of the same techniques allows to disentangle all Feynman parameter integrals of this Section into Beta-function integrals of which only one depends on both the Mellin variable $N$ and the Mellin Barnes variable $\xi$. Usually one may rewrite the integrals in terms of $\Gamma$-functions. If this is not the case, we obtain representations as for example

\[
I_{5a} = C_1(m_1, m_2, \varepsilon) \frac{1}{2\pi i} \frac{1}{N + 1} \int_{-i\infty}^{+i\infty} d\xi \left[ \Gamma \left[ 1 + N + \nu_3 \right] \right] \left[ \Gamma \left[ \frac{1 + N + \nu_5}{\nu_3} \right] \right] \\
\times \Gamma \left[ -\frac{\varepsilon}{2} + \nu_4 - \xi, -2 - \varepsilon + \nu_{347} - \xi, -\xi, \nu_5 + \xi, \nu_6 + \xi, -2 - \xi + \nu_{56} + \xi \right] \\
\nu_5 \nu_6, 1 + N + \nu_{37}, -2 - \varepsilon + \nu_4 + 2\nu_4 - \xi \nu_4 - \xi, \nu_{56} + 2\xi \\
\times \int_0^1 dX X^{\xi} X^{1+\varepsilon/2 + N - \nu_4 + \xi} (1 - X)^{\nu_4-1-\xi} \tag{365}
\]

where $C_1$ is a function of the masses $m_1$, $m_2$ and $\varepsilon$. Next, convergent sum representations are derived. Mellin Barnes integrals of the form

\[
\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \Gamma \left[ a_1 + \xi, \cdots, a_i + \xi, b_1 - \xi, \cdots, b_j c_1 + \xi, \cdots c_k + \xi, d_1 - \xi, \cdots, d_l - \xi \right] z^\xi \tag{366}
\]

are usually solved by closing the contour either to the left or to the right and applying Cauchy's theorem

\[
\oint_C f(z) dz = 2\pi i \sum_i \text{res}_z f \tag{367}
\]

\(^6\)The steps to compute these Mellin transforms are included in the computer algebra package \texttt{HarmonicSums} [175,177–179].
If we close the integration contour in (366) to the left (right) the residue sum only converges for 
\( Z > (Z < 1) \), respectively. In (365) we have \( Z = \frac{\eta X}{1 - X} \) which covers both, values below and above 1. We thus follow the method that has been applied in the equal mass case Ref. [289] and split the integration contour and remap the individual parts to the integration domain \([0, 1]\):

\[
\int_{-i\infty}^{+i\infty} d\xi \int_{0}^{1} dX f(\xi, X) \left( \frac{\eta X}{1 - X} \right)^{\xi} = \left( \int_{0}^{1} dX + \int_{1}^{\infty} dX \right) f(\xi, X) \left( \frac{\eta X}{1 - X} \right)^{\xi} = \\
= \int_{-i\infty}^{+i\infty} d\xi \int_{0}^{1} dT \left[ \frac{\eta}{(\eta + T)^{2}} f\left( \xi, \frac{T}{\eta + T} \right) T^{\xi} + \frac{\xi, \eta}{(1 + \eta T)^{2}} f\left( \frac{1}{1 + \eta T} \right) T^{-\xi} \right] = \\
= \int_{-i\infty}^{+i\infty} d\xi \int_{0}^{1} dT T^{\xi} \left[ \frac{\eta}{(\eta + T)^{2}} f\left( \xi, \frac{T}{\eta + T} \right) + \frac{\xi, \eta}{(1 + \eta T)^{2}} f\left( \frac{1}{1 + \eta T} \right) \right].
\tag{368}
\]

A further advantage of this procedure is that the contour integration practically decouples the \( \eta \)-dependence which now only enters indirectly through the \( T \)-integration.

We now follow the well known procedure of deforming the contour integral in order to separate the ascending from the descending poles \(^7\) and apply Cauchy’s theorem to evaluate it. As we are left with only one integration step at this point there are no overlapping singularities anymore. If necessary we map \( T \rightarrow 1 - T \) in order to have singularities which are regulated by \( \varepsilon \) only at \( T = 0 \). These singularities are then written as \( \varepsilon \)-poles explicitly by applying the following integration by parts relation:

\[
\int_{0}^{1} dT T^{-a} f(T) = \left. \frac{1}{1 - a} T^{-a + 1} f(T) \right|_{0}^{1} - \frac{1}{1 - a} \int_{0}^{1} dT T^{-a + 1} f'(T) \tag{369}
\]

Being left with a sum representation and an integration which is regular for \( \varepsilon \to 0 \) we may now perform the Laurent series expansion around \( \varepsilon = 0 \).

**Rewriting the sums.** In order to do this and to perform the infinite sums \(^8\) we apply the package **Sigma** [236, 237]. The sums are then expressed in terms of generalized harmonic sums (D.46) at infinity which have to be rewritten in terms of generalized harmonic polylogarithms at argument \( x = 1 \) using **HarmonicSums** [175,177–179]. These generalized harmonic polylogarithms are iterated integrals over the following alphabet:

\[
\left\{ \frac{d\tau}{\tau}, \frac{d\tau}{\tau + T}, \frac{d\tau}{1 + T\tau^{2}} \right\}. \tag{370}
\]

In order to process them we want the remaining integration variable \( T \) to only appear in the argument of the HPLs. Because of the emergence of letters with non-linear denominators we

---

\(^7\)In some cases an additional regularization parameter was introduced in order to separate overlapping poles.

\(^8\)Due to the integral transformation (368) these infinite sums are independent of the mass ratio \( \eta \) which renders them much easier to solve.
cannot apply the methods given in Section 7.3 directly, although extensions as described in Section 6.5.1 should suffice to transform these HPLs. However, due to the relatively simple structure of the letters in Eq. (370) there is a way based on applying the shuffle relations and rescaling the internal integration variables to rewrite the corresponding iterated integrals in the desired form.

Absorbing rational, $N$-dependent factors into the integral. There might be some rational prefactors depending on $N$ left stemming from the integration of the Feynman parameters. These are now pulled into the $T$-integration by performing a partial fraction decomposition and then applying the following partial integration identities:

\[ N^i g(x)^N f(x) = N^{i-1} g(x)^N \frac{f(x)}{g(x)} \left. \frac{1}{dg(x)} \right|_0^1 - \int_0^1 dx \left( g(x) \right)^N \frac{1}{dx} g(x) \frac{1}{dg(x)} \]  
\[ \frac{1}{(N + a)^i} g(x)^N f(x) = \frac{1}{(N + a)^i} g(x)^{N+a} \left( \int dx \frac{f(x)}{g(x)^a} \right)^1_{x=0} - \int_0^1 dx \frac{1}{(N + a)^{i-1}} g(x)^{N+a-1} \frac{dg(x)}{dx} \left( \int dx \frac{f(x)}{g(x)^a} \right). \]  

Especially relation (371) has to be handled with care, as its application may introduce new divergences in each term. This issue is solved by regularizing the remaining integral in (371) by a $\pm$-type distribution which cancels these additional singularities, e.g.

\[ N \int_0^1 dx \ x H_0 (x) \left( \frac{\eta}{\eta + x^2} \right)^N = \eta H_0 (x) \right|_{x=0}^1 + \int_0^1 dx \left( \frac{\eta + x^2}{2x} + x H_0 (x) \right) \left( \frac{\eta}{\eta + x^2} \right)^N \]  
\[ = \frac{\eta}{2} H_0 (x) \right|_{x=0}^1 + \int_0^1 dx \ x \left( \frac{1}{2} + H_0 (x) \right) \left( \frac{\eta}{\eta + x^2} \right)^N \]  
\[ + \frac{\eta}{2} \int_0^1 dx \ \frac{1}{x} \left[ \left( \frac{\eta}{\eta + x^2} \right)^N - 1 \right] + \frac{\eta}{2} H_0 (x) \right|_{x=0}^1 \]  
\[ = \int_0^1 dx \ x \left( \frac{1}{2} + H_0 (x) \right) \left( \frac{\eta}{\eta + x^2} \right)^N \]  
\[ + \frac{\eta}{2} \int_0^1 dx \ \frac{1}{x} \left[ \left( \frac{\eta}{\eta + x^2} \right)^N - 1 \right]. \]  

We now rewrite the remaining integral as a Mellin transform. Considering these expressions one obtains

\[ I = \int_0^1 dx f(x, \eta) (g(x, \eta))^N \]
To do this the class of harmonic polylogarithms is not sufficient and generalizations thereof are required. We thus extend the class of harmonic polylogarithms by introducing addition letters with quadratic polynomials in the respective denominators. These are given by

\[
\{4, i\} \rightarrow \frac{d\tau}{\Phi_4(\tau)},
\]

or more generally

\[
\{\{a, b, c\}, i\} \rightarrow \frac{d\tau \tau^i}{a + b\tau + c\tau^2},
\]

where \(\Phi_4(\tau) = \tau^2 + 1\) denotes the fourth cyclotomic polynomial and \(d\tau\) indicates that the iteration proceeds over \(\tau\). Generalizations of HPLs over the cyclotomic polynomials are also known as cyclotomic HPLs [176]. Thus

\[
H_{0,\{4,1\}}(x) \text{ represents the iterated integral}
\]

\[
H_{0,\{4,1\}}(x) = \int_0^x \frac{d\tau_1}{\tau_1} \int_0^{\tau_1} \frac{d\tau_2}{\tau_2^2 + 1}.
\]

By using this larger functional space, diagram 8a, Figure 14 below, is rewritten as

\[
I = \int_0^1 dx \left( \frac{xH_{\{(1,0,\eta),1,\{(1,0,\eta),0\},0\},0}(x)}{(1 + \eta x^2)^2} \right)^N \left( \frac{\eta x^2}{1 + \eta x^2} \right)^N
\]

\[
= \frac{\eta}{2} \int_0^{\eta/(1+\eta)} dx \left( \frac{\sqrt{x}}{\sqrt{1-x} \sqrt{\eta}} \right)^N
\]

\[
= \frac{1}{32\eta^3} \int_0^{\eta/(1+\eta)} \left\{ \left[ H_0(x) - H_0(\eta) \right] H_1^2(x) + H_1^3(x) - 16 \left( H_{0,\{4,1\},\{4,1\}} \left( \frac{\sqrt{x}}{\sqrt{1-x}} \right) + H_{\{4,1\},0,\{4,1\}} \left( \frac{\sqrt{x}}{\sqrt{1-x}} \right) \right) \right\} x^N,
\]

where in the last step we removed the \(\eta\)-dependence of the argument by again applying a rescaling of the inner integration variables. At this stage it is desirable to remove the square roots in the arguments of the HPLs and to obtain iterated integrals with the argument \(x\) only. In order to obtain this representation we once again exploit the property that taking the derivative reduces the transcendental weight of a hyperlogarithms and use a method similar to the one given in Section 7.3 below, e.g.:

\[
\frac{d}{dx} H_{\{4,1\},0} \left( \frac{\sqrt{x}}{\sqrt{1-x}} \right) = \frac{1}{2} \frac{H_0 \left( \frac{\sqrt{x}}{\sqrt{1-x}} \right)}{1-x}
\]

\[
= \frac{1}{3} \left[ H_0(x) + H_1(x) \right]
\]
\[ H_{(4,1),0} \left( \frac{\sqrt{x}}{\sqrt{1-x}} \right) = \frac{1}{4} (H_{1,0}(x) + H_{1,1}(x)) \] 

However, not all the occurring HPLs can be expressed in terms of generalized HPLs of the previous kind and new, root-valued letters have to be introduced. We thus introduce a new, more general class of iterated integrals which we define recursively by

\[ G(\{f_1(\tau), f_2(\tau), \ldots, f_n(\tau)\}, z) = \int_0^z d\tau_1 \ f_1(\tau_1) G(\{f_2(\tau), \ldots, f_n(\tau)\}, \tau_1) , \] 

with the special cases

\[ G(\{\}, z) = 1 , \] 

and

\[ G(\left\{ \frac{1}{\tau}, \frac{1}{\tau}, \ldots, \frac{1}{\tau} \right\} \text{, } n \text{ times} , z) = \frac{1}{n!} \ln^n(z) . \]

Using these generalized iterated integrals we rewrite the HPLs with root-valued functions in the argument. For example one has

\[ H_{(4,0),\{\eta,0,1\},0} \left( \frac{\sqrt{x}}{\sqrt{1-x}} \right) = \frac{3 - 6x + 3\eta x + 3\eta^2 x + 7x^2 - 2\eta x^2 - 5\eta^2 x^2 - 3x^3 + 3\eta^2 x^3}{3(\eta - 1)\eta} + \frac{2 (1 + \eta) \sqrt{1-x} \sqrt{x}(-1+2x)}{\eta} G(\{\sqrt{1-\tau} \sqrt{\tau}\} , x) - \frac{(-1 + \eta)^2 \sqrt{1-x} \sqrt{x}(-1+2x)}{2\eta} G(\left\{ \frac{\sqrt{1-\tau} \sqrt{\tau}}{-\eta - \tau + \eta \tau} \right\} , x) + \frac{8 (1 + \eta)}{\eta} G(\{\sqrt{1-\tau} \sqrt{\tau}, \sqrt{1-\tau} \sqrt{\tau}\} , x) + \frac{2 (\eta - 1)^2}{\eta} G(\left\{ \sqrt{1-\tau} \sqrt{\tau}, \sqrt{1-\tau} \sqrt{\tau} \right\} , x) + \frac{1 + \eta}{2(\eta - 1)} G(\left\{ \frac{1}{-\eta - \tau + \eta \tau} \right\} , x) . \]

In the present computation similar HPLs up to weight \( w = 3 \) had to be transformed. Due to the size of the expressions and the necessity to cancel spurious terms all relations obeyed by these quantities have to be used. These are:

- shuffle relations
- integration by parts relations, such that only factors with exponents \( \in \{1/2, -1\} \) contribute to the different letters
- shuffling of single square root terms to the end and performing the integrals e.g.:

\[ G(\left\{ \sqrt{\tau}, \frac{1}{\tau + 1} \right\} , x) = \frac{2}{3} \left[ G(\{\sqrt{\tau}\} , x) + x^{3/2} G(\left\{ \frac{1}{1+\tau} \right\} , x) + G(\left\{ \frac{\sqrt{\tau}}{1+\tau} \right\} , x) \right] . \]
These identities have now been implemented in HarmonicSums [175,177–179] and allow a significant simplification of expressions with iterated integrals of this type. Finally the integrals are merged. After the mapping of the integration variables (380) we are left with integrals of the form \( \int_{f(n)}^1 dx \ G(x) \) or \( \int_0^1 f(n) dx \). This is due to the splitting of the \( X \)-integration in Eq. (368) and can be undone by re-merging the integrals

\[
\int_{f(n)}^1 dx \ G(x) = \int_0^1 dx \ G(x) - \int_0^{f(n)} dx \ G(x). \tag{388}
\]

As would have been expected, the integrals \( \int_0^{f(n)} dx \ G(x) \) completely cancel already existing integrals and only trivial integrals of the form

\[
\int_0^{f(n)} dx \ x^\alpha x^N = \frac{1}{\alpha + N + 1} x^{N+\alpha} \bigg|_{x=0} \tag{389}
\]

remain. We now use HarmonicSums [175,177–179] to perform the inverse Mellin transform for terms that do not contain any \( x \)-integration. They usually stem from integration by parts applied in steps (369), (371) or (372). We are left with a \( z \)-space representation for our diagram. This representation usually also includes a part proportional to a \( \delta \)-distribution and a part proportional to a \( \pm \)-distribution.

**Mellin transform and rewriting generalized HPLs.** As a last step we want to generate a \( N \)-space representation for our result, which corresponds to performing the last remaining integration. This is done by using a generating function representation performing the integral into generalized HPLs and then generating a recurrence relation for the \( N \)th coefficient of this result as described in Section 7.4. All this is automatized in the package HarmonicSums [175,177–179] and the resulting recurrences were solved with Sigma [236,237]. The result contains many generalized HPLs at argument 1 stemming from the upper integration limit. In case their letters are free of the mass ratio \( \eta \) they can be evaluated in terms of mathematical constants like \( \pi, \ln 2, \) the Catalan number \( C, \zeta_2, \zeta_3 \) by using standard integration methods or applying the internal integration algorithms of computer algebra packages like Mathematica or Maple. In case these generalized HPLs are not entirely free of \( \eta \) it is desirable to rewrite them as iterated integrals with argument \( \eta \) in order to obtain linear independence and an easier access to series representations. Rewriting these generalized HPLs cannot be done by rescaling integration variables or by just applying (7.3) as due the root valued letters the derivative with respect to an inner variable in general does not lead to a weight reduction in this case. There is, however, an extension to the idea of (7.3): Taking the derivative with respect to inner variables we observe, that only generalized HPLs of a lower weight, GHPLs independent of this variable and the original GHPL itself contribute, e.g.:

\[
\frac{d}{d\eta} G \left( \left\{ \sqrt{\tau} \sqrt{1-\tau}, \frac{\sqrt{\tau} \sqrt{1-\tau}}{-\eta - \tau - \eta\tau} \right\}, 1 \right) = \frac{(1 + \eta) (1 - 8\eta + \eta^2)}{12 (\eta - 1)^4 \eta} \frac{G\left(\left\{\frac{1}{-\eta - \tau + \eta\tau}\right\}, 1\right)}{\eta} - \frac{2}{(\eta - 1) \eta} G \left( \left\{ \sqrt{\tau} \sqrt{1-\tau}, \sqrt{\tau} \sqrt{1-\tau} \right\}, 1 \right)
\]
\[ -\frac{1 + 3\eta}{2(\eta - 1)\eta} G \left( \left\{ \sqrt{\frac{1}{\eta} - \frac{1}{1 - \eta}}, \frac{\sqrt{1 - \tau}}{-\frac{\sqrt{1 - \tau}}{\eta - \tau - \eta\tau}}, 1 \right\} \right). \]  

(390)

Therefore, the linear first order differential operator

\[ \frac{d}{d\eta} + \frac{1 + 3\eta}{2(\eta - 1)\eta} \]

(391)

does lead to a weight reduced expression when applied to the GHPL

\[ G \left( \left\{ \frac{\sqrt{1 - \tau}}{\eta - \tau - \eta\tau}, \frac{\sqrt{1 - \tau}}{\eta - \tau - \eta\tau}, 1 \right\} \right). \]

(392)

The weight reduced expression can be rewritten with the same method and we have to undo the effect of the differential operator by using the general solution for linear first order differential equation

\[ \frac{d}{dx} f(x) + p(x)f(x) = q(x) \]

(393)

\[ \Rightarrow f(x) = \int_a^x (f(a)\delta(x - a) + q(x)) \exp \left( -\int p(x)dx \right) \, d\tau. \]

(394)

Applying this method to the GHPL considered above we obtain

\[ G \left( \left\{ \frac{\sqrt{1 - \tau}}{\eta - \tau - \eta\tau}, \frac{\sqrt{1 - \tau}}{\eta - \tau - \eta\tau}, 1 \right\} \right) = \frac{1 + 4\eta - 2\eta^2}{6(-1 + \eta)^3} - \frac{3(1 - 4\sqrt{\eta} + \eta)}{16(\eta - 1)^2} \zeta_2 \]

\[ + \frac{\sqrt{\eta}}{8(\eta - 1)^2} G \left( \left\{ \frac{1}{\eta - \tau - \eta\tau}, \frac{1}{\eta - \tau - \eta\tau}, 1 \right\} \right) + \frac{(\eta - 3)\eta^2}{4(-1 + \eta)^3} \ln \eta. \]

(395)

For all the GHPLs considered in this section it is always possible to construct a linear first order differential operator \(^9\), which does yield a weight reduced expression when applied to the respective generalized HPL and all the GHPLs could thus be rewritten in terms of GHPLs with argument \(\eta\).

6.5.2 Results

Up to a global prefactor all results are expressed as functions of the mass ratio

\[ \eta = \frac{m_2}{m_1^2} \]

(396)

only. We furthermore define the function \(L_1(\eta)\) which appears frequently

\[ L_1(\eta) = \frac{1}{2} \int_0^\eta \frac{\sqrt{x} \ln^2(x)}{1 - x} \, dx. \]

(397)

\[ = 4H_{-1,0.0}(\sqrt{\eta}) + 4H_{1,0.0}(\sqrt{\eta}) - 8\sqrt{\eta} - \sqrt{\eta} \ln^2(\eta) + 4\sqrt{\eta}\ln(\eta), \]

(398)
and present the results for all scalar two-mass topologies contributing to $A_{gg,Q}$ both in $z$- and in $N$-space. Each term in the $z$-space result of Diagram 1 factors completely into $z$ and $\eta$-dependent contributions. No iterated integrals involving both, $\eta$ and $z$, contribute. The $z$-space result is completely expressible within the class of harmonic polylogarithms

\[
D_1(z) = \left( m_1^2 \right)^{\varepsilon/2} \left( m_2^2 \right)^{\varepsilon-3} \left\{ \begin{array}{l}
\frac{1}{\varepsilon^2} \frac{(1 + \eta^3)}{105} + \frac{1}{\varepsilon} \left[ \frac{74 - 245\eta - 245\eta^2 + 74\eta^3}{44100} - \frac{1}{210} \eta^3 H_0(\eta) \right] \\
- \frac{1}{210} \frac{(1 + \eta^3)}{105} H_1(z) + \frac{5520349 - 792845\eta - 792845\eta^2 + 5520349\eta^3}{115408000} \\
+ \frac{1}{280} \frac{(1 + \eta^3)}{105} \zeta_2 - \frac{(74 - 245\eta - 245\eta^2 + 74\eta^3) H_1(z)}{88200} \\
+ H_0(\eta) \left[ \frac{(-55125 + 37975\eta + 24745\eta^2 + 36181\eta^3)}{22579200} + \frac{1}{420} \eta^3 H_1(z) \right] \\
+ \frac{(525 - 245\eta - 245\eta^2 + 1549\eta^3) H_{0,0}(\eta)}{430080} + \frac{1}{420} \frac{(1 + \eta^3)}{105} H_{1,1}(z) \\
- \frac{(-1 + \eta)^2 (5 + 6\eta + 5\eta^2)}{2048\sqrt{\eta}} \left[ H_{-1,0,0}(\sqrt{\eta}) + H_{1,0,0}(\sqrt{\eta}) \right] \right\}. \tag{399}
\]

Due to the $z$-space structure, in $N$-space only single harmonic sums contribute. The Mellin-space result is given by

\[
D_1(N) = \left( m_1^2 \right)^{\varepsilon/2} \left( m_2^2 \right)^{\varepsilon-3} \left\{ \begin{array}{l}
\frac{1 + (-1)^N}{2} \left[ \frac{\eta^3 + 1}{105\varepsilon^2(N+1)} + \frac{1}{\varepsilon} \left[ \frac{(\eta^3 + 1) S_1(N)}{210(N+1)} \right] \\
- \frac{\eta^3 \ln(\eta)}{210(N+1)} + \frac{(\eta + 1)(2\eta^2(37N - 68) - \eta(319N + 109) + 74N - 136)}{44100(N+1)^2} \right] \right\},
\]

\footnote{Using first order linear differential operators instead of pure differentiation could be used to extend the parametric integration method of Section 7. However, remapping parameters as in Section 7.5 might be a more suitable method to integrate Feynman parameter integrals which are not a priori reducible. Both methods break down when genuine elliptic functions appear.}
\[-\frac{(5\eta^2 + 6\eta + 5)(\eta - 1)^2}{2048\sqrt{\eta}(N + 1)} \left[ H_{-1,0.0}(\sqrt{\eta}) + H_{1,0.0}(\sqrt{\eta}) \right] + \ln(\eta) \left[ \frac{P_1}{22579200(N + 1)^2} \right.\]
\[+ \frac{\eta^3 S_1(N)}{420(N + 1)} + \frac{1549\eta^2 - 245\eta^2 - 245\eta + 525}{860160(N + 1)} \ln^2(\eta) + \frac{P_2(\eta + 1)}{1185408000(N + 1)^3}\]
\[+ \frac{(\eta^3 + 1)(S_1^2(N) + S_2(N))}{840(N + 1)} + \frac{(\eta^3 + 1)\zeta_2}{280(N + 1)}\]
\[+ \frac{(\eta + 1)(2\eta^2(37N - 68) - \eta(319N + 109) + 74N - 136)S_1(N)}{88200(N + 1)^2} \right\}, \tag{400}

with the polynomials \(P_i(\eta, N)\)

\[P_1 = 36181\eta^3N + 89941\eta^3 + 24745\eta^2N + 24745\eta^2 + 37975\eta N + 37975\eta - 55125N - 55125 \tag{401}\]
\[P_2 = 5520349\eta^2N^2 + 10046138\eta^2N + 7348189\eta^2 - 13448794\eta N^2 - 22610228\eta N - 11983834\eta + 5520349N^2 + 10046138N + 7348189. \tag{402}\]

Although topologically very similar to Diagrams \(D_1\), diagrams \(D_{2a}\) and \(D_{2b}\) exhibit much more evolved mathematical structures. As we restrict ourselves to a representation within the class of iterated integrals of argument \(z\), additional root-valued integration kernels had to be introduced. Furthermore, iterated integrals depending on both variables \(\eta\) and \(z\) contribute.

In \(z\)-space diagram \(D_{2a}\) consists of contribution \(D_{2a}^{\text{Reg}}\), which is regular as \(z \to 1\) and a contribution \(D_{2a}^{(+)}\),

\[D_{2a}(z) = D_{2a}^{\text{Reg}}(z) + D_{2a}^{(+)}(z). \tag{403}\]
The latter contains terms $\propto 1/(1 - z)$ or $\propto 1/\sqrt{1 - z^3}$ and requires a regularization via $+\text{-type}$ distribution when performing the Mellin transform. With a function $f^{(+)}(x)$ of the general form

$$f^{(+)}(x) = \sum_{k=0}^{m} (\ln(1-x))^k \frac{p_k(x)}{q_k(x)},$$

we define

$$M[f^{(+)}](N) = \int_{0}^{1} dx \sum_{k=0}^{m} \frac{x^k p_k(x) - p_k(1)}{q_k(x)} (\ln(1-x))^k.$$ 

(405)

This regularization is required for the Mellin transform of the diagrams $D_{2a}$, $D_{2b}$, $D_{8a}$ and $D_{8b}$. For $D_{2a}$ the $+$-part is given by

$$D_{2a}^{(+)} = (m_1^2)^{\varepsilon/2} (m_2^2)^{-3+\varepsilon} \left\{ \begin{array}{l}
- \frac{1}{\varepsilon} \frac{\eta^3}{210(1-z)} - \frac{37\eta^3}{44100(1-z)} + \frac{\eta^3}{420(1-z)} G\left\{ \frac{1}{1-\tau}, \frac{1}{\tau} \right\}, \\
+ \frac{5(-1+\eta)3^4\sqrt{z}}{1536(1-z)^{3/2}} G\left\{ \frac{1}{\sqrt{1-\tau}}, \frac{1}{1-\tau} \right\} + G\left\{ \frac{1}{\tau}, \eta \right\} \frac{\eta^3}{420(1-z)} G\left\{ \frac{1}{1-\tau}, \frac{1}{1-\tau + \eta \tau} \right\}, \\\n- \frac{5\eta^3(1+\eta)3^5\sqrt{z}}{3072(1-z)^{3/2}} G\left\{ \frac{1}{\sqrt{1-\tau}}, \frac{1}{1-\tau} \right\} + \frac{5(-1+\eta)^23^4\sqrt{z}}{12288(1-z)^{3/2}} G\left\{ \frac{1}{\sqrt{1-\tau}}, \frac{1}{1-\tau + \eta \tau} \right\} G\left\{ \frac{1}{\tau}, \eta \right\} G\left\{ \frac{1}{1-\tau}, \frac{1}{1-\tau + \eta \tau} \right\}, \\\n+ \left\{ \frac{1}{\tau}, \eta \right\} G\left\{ \frac{1}{\sqrt{1-\tau}}, \frac{1}{1-\tau + \eta \tau} \right\}\right\},
\end{array} \right.$$

(406)

and the regular contribution to Diagram $2a$ is

$$D_{2a}^{\text{Reg}}(z) = (m_1^2)^{\varepsilon/2} (m_2^2)^{-3+\varepsilon} \left\{ \begin{array}{l}
\delta(1-z) \left[ \frac{1}{\varepsilon^2} \frac{\eta^3}{105} + \frac{1}{\varepsilon} \frac{37\eta^3}{22050} - \frac{1}{210} \frac{\eta^3}{G\left\{ \frac{1}{\tau}, \eta \right\}} - \frac{523\eta^3}{2315250} + \frac{\eta^3 \zeta_2}{280} \\
- \frac{37\eta^3 G\left\{ \frac{1}{\tau}, \eta \right\}}{44100} + \frac{1}{420} \eta^3 G\left\{ \frac{1}{\tau}, \frac{1}{1-\tau} \right\}, \eta \right\}, \\\n- \frac{1}{\varepsilon} \frac{13 - 6z + (3 + 7\eta^2 + 6\eta^3)z^2}{1260z^2} + \frac{Q_4}{11289600\eta^2} + \frac{Q_4}{645120\eta z} G\left\{ \frac{1}{1-\tau}, \frac{1}{\tau} \right\}, \\\n- \frac{Q_4}{645120\eta z} G\left\{ \frac{1}{\tau}, \frac{1}{1-\tau} \right\} - \frac{(\eta - 1)Q_3}{1536\eta \sqrt{1-\tau}} G\left\{ \frac{1}{\sqrt{1-\tau}}, \frac{1}{\tau} \right\} G\left\{ \frac{1}{1-\tau}, \frac{1}{1-\tau + \eta \tau} \right\}, \\\n+ \left\{ \frac{1}{\tau}, \eta \right\} G\left\{ \frac{1}{\sqrt{1-\tau}}, \frac{1}{1-\tau + \eta \tau} \right\} + \frac{Q_3}{3072\eta \sqrt{1-\tau} z^{5/2}} G\left\{ \frac{1}{\sqrt{1-\tau}}, \frac{1}{\tau} \right\} \right\},
\end{array} \right.$$

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with the polynomials

\[ Q_1 = 11025(-1 + z)^3 z + 18375\eta^5 z^4 + \eta^4 z^2 (-9472 + 25725 z - 62475 z^2) + 49\eta^3 z^2 (-1091 - 900 z + 1350 z^2) - \eta(-1 + z)^2 (-8704 - 22050 z + 25725 z^2) - 245\eta^2 z (133 - 253 z + 90 z^2 + 30 z^3) \]

\[ Q_2 = 315(-1 + z)^3 - 525\eta^5 z^3 - 105\eta(-1 + z)^2 (6 + z) + 7\eta^3 z (11 + 180 z + 90 z^2) + 3\eta^2 z (512 - 595 z + 245 z^2) - 105\eta^2 (-1 + 9 z - 18 z^2 + 10 z^3) \]

\[ Q_3 = 3 + (-9 + 4\eta) z + (9 - 8\eta - 6\eta^2) z^2 + (-1 + \eta)^3 (3 + 5\eta) z^3 \]

\[ Q_4 = 105(\eta - 1)^2 (17\eta^2 + 22\eta + 9) z - 3 (35\eta^2 + 302\eta - 105) z + (1715\eta^3 + 945\eta^2 - 387\eta - 945) z^2 + 105(\eta - 1)^3 (\eta + 1)(5\eta + 3) z^3 + 768\eta \]

Performing the Mellin transform by using the regularization \( \epsilon \) yields

\[
D_{2\alpha}(N) = (m_1^{2})^{-\epsilon/2} (m_2)^{-3+\epsilon} \left\{ \frac{1 + (-1)^N}{2} \right\} \frac{\eta^3}{105 \epsilon^2} + \frac{1}{2 \pi i} \left[ -\frac{1}{210} S_1(N) \eta^3 - \frac{1}{210} \ln(\eta) \eta^3 \right.
\]

\[
+ \frac{2N(37N^2 - 105N + 68)\eta^3 - 245(N - 1)N\eta^2 - 210}{44100(N - 1)N(N + 1)} \right] + \left[ S_1^2(N) \right.
\]

\[
+ S_2(N) \right\} \frac{\eta^3}{840} + \frac{2\eta^3}{280} + \left\{ \frac{\eta^3}{840} - \frac{(-1)^N(\eta - 1)^{-N-1} P_4}{(2N - 3)(2N - 1)(2N + 1) 12288} \right.
\]

\[
+ \frac{2^{-2N-13}(2N) P_4}{3(\eta - 1)(N + 1)(2N - 3)(2N - 1)}
\]

\[
+ \left. \frac{2^{-2N-13}(2N) P_4}{3(\eta - 1)(N + 1)(2N - 3)(2N - 1)} \sum_{i_1=1}^{N} \frac{(-1)^i 2^{i_1} (-1 + \eta)^{-i_1}}{(2i_1)(1 + 2i_1)} \right] \ln^2(\eta)
\]

\[
+ \frac{P_1}{1185480000(N - 1)^2 N^2 (N + 1)^2 (2N - 3)(2N - 1)}
\]

\[
- \frac{2^{-2N-11}(2N) P_4}{3\sqrt{\eta}(N + 1)(2N - 3)(2N - 1) \left[ H_{-1,0,0}(\sqrt{\eta}) + H_{1,0,0}(\sqrt{\eta}) \right]}
\]

\[
+ \frac{2^{-2N}(2N) P_4}{12288(\eta - 1)(N + 1)(2N - 3)(2N - 1)}
\]

\[
\times \sum_{i_1=1}^{N} \frac{(-1)^i 2^{i_1} (-1 + \eta)^{-i_1}}{(2i_1)(1 + 2i_1)} \left[ S_{1,1}(1 - \eta, 1, i_1) - S_2(1 - \eta, i_1) \right]
\]

\[
\frac{\eta^{N-1}}{(2i_1)(1 + 2i_1)}
\]

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with the polynomials

\[ P_1 = 14363896\eta^3 N^8 - 4\eta^2 (6247133\eta + 7928445) N^7 \]
\[ -10\eta (1788305\eta^2 - 10831254\eta + 519645) N^6 \]
\[ + (18840889\eta^3 - 108183915\eta^2 + 18290475\eta + 24675735) N^5 \]
\[ + (66146587\eta^3 + 4378395\eta^2 - 17775975\eta + 1881705) N^4 \]
\[ - (78524357\eta^3 - 41113695\eta^2 + 8412075\eta + 86929815) N^3 \]
\[ + 3 (7348189\eta^3 - 4635645\eta^2 + 7657475\eta + 15366365) N^2 \]
\[ - 40320 (245\eta - 424) N - 8467200 \]

\[ P_2 = 5\eta^3 (8N^3 - 12N^2 - 2N + 3) + \eta^2 (-28N^2 + 64N - 9) - 3\eta(2N + 17) + 45 \]

\[ P_3 = \eta^3 (71224N^3 + 217316N^2 - 666110N + 269823) + 24745\eta^2 (4N^2 - 8N + 3) \]
\[ - 3675\eta(14N - 31) - 165375 \]

\[ P_4 = 5\eta^4 (16N^4 - 40N^2 + 9) - 12\eta^3 (8N^3 - 12N^2 - 2N + 3) - 6\eta^2 (4N^2 - 8N + 3) \]
\[ + 12\eta(2N - 3) + 45 \]

Diagram 2b exhibits a very similar structure and is related to diagram 2a by the interchange \( m_1 \leftrightarrow m_2, \eta \to 1/\eta \). Its z-space contributions consists of a part which requires regularization via the + -distribution,

\[ D_{2b}^{(+)}(z) = \left( m_1^2 \right)^{-3+\varepsilon} \left( m_2^2 \right)^{\varepsilon/2} \]

\[ \frac{1}{\varepsilon 210\eta^3(-1+z)} - \frac{37}{44100\eta^3(-1+z)} + \frac{1}{420\eta^3(-1+z)} G \left[ \left\{ \frac{1}{1-\tau} \right\}, z \right] \]

\[ - \frac{5(-1+\eta)^2\sqrt{z}}{1536\eta^4(1-z)^{3/2}} G \left[ \left\{ \sqrt{1-\tau}\sqrt{\tau} \right\}, z \right] + G \left[ \left\{ \frac{1}{\tau} \right\}, \eta \right] - \frac{1}{420\eta^3(-1+z)} \]

\[ + \frac{5(1+\eta)^2\sqrt{z}}{3072\eta^4(1-z)^{3/2}} G \left[ \left\{ \sqrt{1-\tau}\sqrt{\tau} \right\}, z \right] \]

\[ + \frac{5(-1+\eta)^2\sqrt{z}}{12288\eta^4(1-z)^{3/2}} G \left[ \left\{ \sqrt{1-\tau}\sqrt{\tau}, \frac{1}{\eta-\tau+\eta\tau} \right\}, z \right] \]

\[ - \frac{5(1+\eta)^2\sqrt{z}}{3072\eta^4(1-z)^{3/2}} G \left[ \left\{ \sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau} \right\}, z \right] + G \left[ \left\{ \sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau} \right\}, z \right] \]

\[ 85 \]
with 

\[ Q_1 = 315\eta^5(z-1)^3 - 105\eta^4(z-1)^2(z+6) - 105\eta^3(10z^3 - 18z^2 + 9z - 1) \\
+ 7\eta^2z(90z^2 + 180z + 11) + 3\eta z(245z^2 - 595z + 512) - 525z^3, \]  

\[ Q_2 = -315\eta^5(z-1)^3 + 105\eta^4(z-1)^2(z+6) + 105\eta^3(10z^3 - 18z^2 + 9z - 1) \\
- 7\eta^2z(90z^2 + 180z + 11) - 3\eta z(245z^2 - 595z + 512) + 525z^3, \]  

\[ Q_3 = 315\eta^5(z-1)^3z - 3\eta^4(z-1)^2(35z^2 + 210z + 256) - 105\eta^3z(10z^3 - 18z^2 \\
+ 9z - 1) + 35\eta^2z^2(18z^2 + 36z - 49) + 105\eta z^3(7z - 17) - 525z^4, \]  

\[ Q_4 = 11025\eta^5(z-1)^3z - \eta^4(z-1)^2(25725z^2 - 22050z - 8704) - 245\eta^3z(30z^3 \\
+ 90z^2 - 253z + 133) + 49\eta^2z^2(1350z^2 - 900z - 1091) + \eta z^2(-62475z^2 \\
+ 25725z - 9472) + 18375z^4, \]  

\[ Q_5 = 3\eta^4(z-1)^3 - 4\eta^3(z-1)^2z - 6\eta^2z(z-1)^2z^2 - 5z^3 + 12\eta z^3. \]  

In Mellin N-space, this yields

\[ D_{2b}(N) = (m_1^2)^{-3+\varepsilon}(m_2^2)^{\varepsilon/2} \left[ \frac{1 + (-1)^N}{2} \right] \left( \frac{1}{105\varepsilon^2\eta^3} \right). \]
\[
\begin{align*}
\frac{1}{\varepsilon} \left[ -210\eta^3 + 245(N-1)N\eta - 2N(37N^2 - 105N + 68) \right] - \frac{S_1(N)}{210\eta^3} + \ln(\eta) \right]
+ \frac{1}{12288(2N-3)(2N-1)(2N+1)} \times \sum_{i_1=1}^{N} \frac{2^{2i_1}(-1 + \eta)^{-i_1} \eta^{i_1}}{(2i_1)(1 + 2i_1)}
+ \frac{P_1}{118540800\eta^3(N-1)(2N-3)(2N-1)(2N+1)}
+ \frac{2^{2N-12}(2N_N^2)P_3}{2N(2N_N^2)P_3}
+ \frac{1}{3\eta^{-1/2}(N+1)(2N-3)(2N-1)} \left[ H_{-1,0,0}(\sqrt{\eta}) + H_{1,0,0}(\sqrt{\eta}) \right]
+ \frac{1}{12288(\eta-1)\eta^3(N+1)(2N-3)(2N-1)}
\times \sum_{i_1=1}^{N} \frac{2^{2i_1}(-1 + \eta)^{-i_1} \eta^{i_1}}{(2i_1)(1 + 2i_1)}
+ \frac{(\eta-1)^{-N-1}\eta^{-N-2}\eta\eta^2}{6144(2N-3)(2N-1)(2N+1)} \left[ S_{1,1} \left( \frac{-1 + \eta}{\eta}, 1, N \right) - S_2 \left( \frac{-1 + \eta}{\eta}, 1, N \right) \right]
- \frac{1}{22579200\eta^3(N+1)(2N-3)(2N-1)(2N+1)} S_1(N) + \frac{S_3^2(N) + S_2(N)}{840\eta^3}
+ \frac{\zeta_2}{280\eta^3}
\right],
\end{align*}
\]
Diagram 3 displays a particularly simple structure and does only depend on the logarithms $m_a$, $m_b$ and $m_a = m_2$, $m_b = m_1$ yield identical results.

Figure 9: Diagram 3. Due to the symmetry of the diagram both mass assignments $m_a = m_1$, $m_b = m_2$ and $m_a = m_2$, $m_b = m_1$ yield identical results.

$$H_0(\eta) = \ln(\eta) \text{ and } H_0(z) = \ln(z) \text{ in } z\text{-space. } D_3(z) \text{ reads}$$

$$D_3(z) = \left( m_1^2 \right)^{\varepsilon/2} \left( m_2^2 \right)^{\varepsilon-3} \left\{ \frac{(1 + \eta^3)(-1 + z)}{210 \varepsilon z} - \frac{(74 - 245 \eta - 245 \eta^2 + 74 \eta^3)(-1 + z)}{88200 z} + \frac{1}{420} \eta^3 (-1 + z) H_0(\eta) - \frac{1}{420} \eta^3 (1 + \eta^3)(-1 + z) H_0(z) \right\}. \quad (429)$$

In $N$-space this corresponds to an expression in terms of rational functions and $\ln(\eta)$ only. It is given by

$$D_3(N) = \left( m_1^2 \right)^{\varepsilon/2} \left( m_2^2 \right)^{\varepsilon-3} \left[ \frac{1 + (-1)^N}{2} \right] \left\{ \frac{(\eta^3 + 1)}{210 \varepsilon N (N + 1)} + \frac{P_1(\eta + 1)}{88200 N^2 (N + 1)^2} - \frac{\eta^3 \ln(\eta)}{420 N (N + 1)} \right\}, \quad (430)$$

with the polynomial

$$P_1 = \eta^2 (74 N^2 - 346 N + 210) + \eta (-319 N^2 + 101 N + 210) + 74 N^2 - 346 N - 210. \quad (431)$$

The $z$-space expressions for Diagrams $D_{4a}$ and $D_{4b}$ are completely regular as $z \to 1$. For $D_{4a}(z)$ one obtains

$$D_{4a}(z) = \left( m_1^2 \right)^{\varepsilon/2} \left( m_2^2 \right)^{-3+\varepsilon} \left\{ \frac{1}{\varepsilon} \left[ -\frac{(-1 + z)(-1 + 5z + 2z^2)}{60 z^2} \right] + \frac{1}{10} G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] \right\} + \frac{Q_1}{57600 \eta z^2}.$$
Figure 10: Topology 4. $D_{4a}$ is given by assigning $m_a = m_2, m_b = m_1$ and $D_{4b}$ by assigning $m_a = m_1, m_b = m_2$ respectively.
Performing the Mellin transform yields

\[ D_{4a}(N) = \left( m_2 \right)^{\varepsilon/2} \left( m_2 \right)^{-3+\varepsilon} \left[ 1 + \left( -1 \right)^N \right] \left[ -\frac{1}{5\varepsilon(N-1)N(N+1)^2(N+2)} \right] \left[ \eta \left( (4N^2 - 4N - 3)\eta^2 + (4N - 6)\eta - 35 \right) \right] \left[ \frac{(1 - \eta)^{-N}}{1024(N+1)^2(N+2)(2N-3)} \right] \]

with the polynomials

\[
Q_1 = -150\eta^3 z^3 \left( 10 + 41z - 62z^2 + 24z^3 \right) + 75\eta^4 z^2 \left( 12 - 22z + 43z^2 - 38z^3 + 12z^4 \right)
- 20\eta^2 z \left( 82 - 57z + 65z^2 + 135z^3 - 240z^4 + 90z^5 \right)
- 15z \left( 24 - 152z + 662z^2 + 51z^3 - 830z^4 + 420z^5 \right)
+ 2\eta \left( 104 + 696z - 2568z^2 + 7798z^3 + 3195z^4 - 11850z^5 + 5400z^6 \right)
\]

\[
Q_2 = -6 + 67z - 81z^2 - 85z^3 + 105z^4 + 5\eta^4(-1 + z)^2 z(-1 + 3z)
- 10\eta^3(-1 + z)^2 z(1 + 3z) + \eta^2 \left( 2 - 50z + 20z^2 + 160z^3 - 120z^4 \right)
+ 2\eta \left( -6 + 99z + 7z^2 - 45z^3 + 15z^4 \right)
\]

\[
Q_3 = 3 + (-13 + 5\eta) z - 15 \left( -3 + 2\eta + \eta^2 \right) z^2 + 15 \left( 7 - 5\eta - 3\eta^2 + \eta^3 \right) z^3
\]

\[
Q_4 = -6 + 38z - 148z^2 - 4z^3 + 190z^4 - 105z^5 - 5\eta^4 z \left( 3 - 6z + 12z^2 \right)
- 10z^3 + 3z^4 \right) + 10\eta^3 z \left( 3 + 6z^2 - 8z^3 + 3z^4 \right) - 2\eta \left( 6 - 30z \right)
+ 92z^2 + 52z^3 - 60z^4 + 15z^5 \right) + \eta^2 \left( 2 - 17z + 70z^2 + 140z^3
- 280z^4 + 120z^5 \right)
\]

\[
Q_5 = 150\eta^3 z^4 \left( 6 - 8z + 3z^2 \right)
- 75\eta^4 z^3 \left( -6 + 12z - 10z^2 + 3z^3 \right) - 15z \left( 6 - 38z + 148z^2 \right)
+ 4z^3 - 190z^4 + 105z^5 \right) + 10\eta^2 z \left( 3 - 203z + 105z^2
+ 210z^3 - 420z^4 + 180z^5 \right) - 6\eta \left( 40 - 210z - 86z^2 + 460z^3 + 260z^4
- 300z^5 + 75z^6 \right)
\]
with polynomials

\[\begin{align*}
P_1 &= 900\eta^3N^7 - 900\eta^2(2\eta - 1)N^6 - 25\eta \left( 2\eta^2 + 90\eta + 163 \right) N^5 + \left( 2475\eta^3 
&+ 450\eta^2 + 8875\eta + 7296 \right) N^4 + \left( -225\eta^3 + 2250\eta^2 - 725\eta + 6336 \right) N^3 
&- \left( 675\eta^3 + 1350\eta^2 + 8875\eta + 33216 \right) N^2 + 192(25\eta + 27)N + 8640 \\
P_2 &= \eta^3 \left( 8N^3 - 12N^2 - 2N + 3 \right) + 3\eta^2 \left( 4N^2 - 8N + 3 \right) + \eta(45 - 30N) - 105.
\end{align*}\]  

Interchanging the masses \(m_1 \leftrightarrow m_2\) yields

\[\begin{align*}
D_{4b}(z) &= (m_1^2)^{-3+\varepsilon} (m_2^2)^{\varepsilon/2} \left\{ \frac{1}{\varepsilon} \left[ -\frac{(z - 1)(2z^2 + 5z - 1)}{60z^2} + \frac{1}{10} G(\{1\over \tau\}, z) \right] 
&+ \frac{G(\{1\over \tau\}, z) Q_3}{28800\eta^3z^2} + \frac{Q_4}{57600\eta^3z^2} - \frac{1}{\eta^3z} \frac{(z - 1)Q_1}{1920} G(\{1\over 1 - \tau\}, z) \right\}.
\end{align*}\]  

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\[\begin{align*}
&\quad + \frac{(1 - z)^{3/2} (\eta + 1)}{480 \eta^3 z^{5/2}} Q_5 \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) + G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{\tau} \right\}, z \right) \right] \\
&\quad + \frac{(1 - z)^{3/2} (\eta - 1)^2 Q_5}{1920 \eta^3 z^{5/2}} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) + G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{\tau} \right\}, z \right) \right] \\
&\quad + \frac{(\eta - 1)(\eta + 1)(7\eta^2 + 2\eta - 1)}{16 \eta^3} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right. \\
&\quad \left. + \frac{1}{4 \eta^3} (\eta - 1)^3 (7\eta^2 + 2\eta - 1) \right] + \frac{1}{60 \eta^2} G \left( \left\{ \frac{1}{\tau} \right\}, \eta \right) \\
&\quad + \frac{(1 - z)^{3/2} (\eta - 1)^2 Q_5}{480 \eta^3 z^{5/2}} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) + \frac{(35\eta^2 + 3\eta - 8)}{1920 \eta^2} G \left( \left\{ \frac{1}{-\eta - \tau + \eta \tau} \right\}, z \right) \right] \\
&\quad - \frac{(\eta - 1)(\eta + 1)(7\eta^2 + 2\eta - 1)}{64 \eta^3} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right. \\
&\quad \left. - \frac{1}{\eta^3} (\eta - 1)(\eta + 1)(7\eta^2 + 2\eta - 1) \right] + \frac{1}{3840} G \left( \left\{ \frac{1}{\tau}, \frac{1}{\tau} \right\}, \eta \right) \\
&\quad + \frac{1}{\eta^2} (105\eta^3 - 180\eta^2 - 5\eta + 64) \left[ G \left( \left\{ \frac{1}{\tau}, \frac{1}{\tau} \right\}, \eta \right) + \frac{1}{\eta^3} (\eta - 1)^2 (7\eta^2 + 2\eta - 1) \right] \\
&\quad \times G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) + \frac{1}{\eta^2} (7\eta^3 - 5\eta^2 - 3\eta + 1) \\
&\quad \times G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, \eta \right) + \frac{1}{60} (3 + \frac{1}{\eta^2}) G \left( \left\{ \frac{1}{\tau}, \frac{1}{\tau} \right\}, z \right) - \frac{\zeta_2}{60 \eta^2} \right].
\end{align*}\]
In Mellin space $D_{4b}$ takes the form

$$D_{4b}(N) = (m_1^2)^{-3+\varepsilon} (m_2^2)^{-\varepsilon/2} \left[ 1 + (-1)^N \frac{1}{2} \right] \left\{ \frac{1}{1 + 5(N - 1)N(N + 1)^2(N + 2)} \right. $$

$$+ \frac{-35\eta^2 + (6 - 4N)\eta - 4N^2 + 4N + 3)(\eta - 1)^{-N}\eta^{N-3}}{1024(N + 1)^2(N + 2)(2N - 3)} $$

$$+ \frac{P_1 2^{-2N-9}(2N-2)N-1}}{\eta^3(N+1)^2(N+2)(2N-3)} $$

$$+ \frac{2^{-2N-11}(2N-2)N-1}}{\eta^3(N+1)^2(N+2)(2N-3)} \sum_{i_1=1}^{N} \frac{2^{2i_1}(-1 + \eta)^{-i_1}\eta^{i_1}}{(-2^{2i_1})} \ln^2(\eta) $$

$$+ \frac{35\eta^2 + (6 - 24N)\eta - 12N^2 + 12N + 9}{768\eta^3(N+1)^2(N+2)(2N-3)} $$

$$+ \frac{P_2 2^{-2N-7}(2N-2)N-1}}{\eta^3(N+1)^2(N+2)(2N-3)} $$

$$+ \frac{2^{-2N}(2N-2)N-1}}{1024\eta^4(N+1)^2(N+2)(2N-3)} \sum_{i_1=1}^{N} \frac{2^{2i_1}}{(-1+2i_1)} \ln(\eta) $$

$$+ \frac{(35\eta^2 + (6 - 4N)\eta - 4N^2 + 4N + 3)(\eta - 1)^{-N}\eta^{N-3}}{512(N + 1)^2(N + 2)(2N - 3)} S_1 \left( \frac{\eta - 1}{\eta}, N \right)$$

$$+ \frac{28800\eta^3(N - 1)^2N^2(N + 1)^3(N + 2)(2N - 3)}{P_1 2^{-2N-6}(2N-2)N-1}} - \frac{2^{-2N-7}(2N-2)N-1}}{\eta^5/2N(N+1)^2(N+2)(2N-3)} $$

$$\times \left[ H_{-1.0,0}(\sqrt{\eta}) + H_{1,0,0}(\sqrt{\eta}) \right] P_1 + \frac{2^{-2N}(2N-2)N-1}}{1024\eta^4N(N+1)^2(N+2)(2N-3)} $$

$$\times \sum_{i_1=1}^{N} \frac{2^{2i_1}(-1 + \eta)^{-i_1}\eta^{i_1}}{(-1+2i_1)} \left[ S_{1,1} \left( \frac{-1+\eta}{\eta}, i_1 \right) - S_2 \left( \frac{-1+\eta}{\eta}, i_1 \right) \right]$$

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\[\begin{align*}
&+ \sum_{i_1=1}^{N} \frac{2^{n_1} \left[ \frac{1}{n_1} - \frac{S_1(n_1)}{n_1} \right]}{(-2+2i_1)^{n_1}} + \frac{(35n^2 + (36 - 24N)\eta - 12N^2 + 12N + 9)}{768n^2(N+1)^2(N+2)(2N-3)} \cdot S_1(N) \\
&+ \frac{(35n^2 + (6 - 4N)\eta - 4N^2 + 4N + 3)(\eta - 1)^{-N}n^{-3}}{512(N+1)^2(N+2)(2N-3)} \cdot \left[ S_2 \left( \frac{\eta - 1}{\eta}, N \right) - S_{1,1} \left( \frac{\eta - 1}{\eta^2}, 1, N \right) \right],
\end{align*}\]

with polynomials
\[
P_1 = 105\eta^3 + 15\eta^2(2N - 3) - 3\eta \left( 4N^2 - 8N + 3 \right) - 8N^3 + 12N^2 + 2N - 3
\]
\[
P_2 = 192\eta^3 \left( 38N^4 + 33N^3 - 173N^2 + 27N + 45 \right) - 25\eta^2(163N^2 - 29N - 192) + 450\eta(N - 1)^2N^2 \left( 2N^2 - N - 3 \right) + 225(N - 1)^2N^2
\times \left( 4N^3 - 7N - 3 \right)
\]

In \( z \)-space diagram \( D_{5a} \) is given by

![Diagram](attachment:diagram.png)

Figure 11: This diagram depicts \( D_{5a} \) with \( m_a = m_2 \), \( m_b = m_1 \) and \( D_{5b} \) with \( m_a = m_1 \), \( m_b = m_2 \) respectively.

\[
D_{5a}(z) = \left( m_1^{2v/2}(m_2^2) \right)^{-3+\epsilon} \left\{ \frac{1}{\tau} - \frac{(-1 + z)(-1 + 5z + 2z^2)}{45\tau^2} + \frac{2}{15} G \left( \left\{ \frac{1}{\tau} \right\}, z \right) \right.
- \frac{Q_1}{201600n^2} \cdot \frac{1}{105} \eta^3 z^2 G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] + \frac{(-1 + z)Q_2}{6720n^2} G \left[ \left\{ 1 \right\}, z \right] \\
+ \frac{Q_3}{100800n^2} G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] + \frac{(-1 + z)Q_4}{840n^2/2} G \left[ \left\{ 1 \right\}, z \right] \\
+ \frac{1}{105} \eta^3 G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] + \frac{-1 + z + 9\eta^2 + \eta^4}{4n^2/2} G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] \\
- \frac{(1 + \eta)(1 - z)^{3/2}}{1680n^{2/2}} G \left[ \left\{ \sqrt{1 - \tau^2}, \frac{1}{1 - \tau} \right\}, z \right] \\
+ G \left[ \left\{ \sqrt{1 - \tau^2}, \frac{1}{1 - \tau} \right\}, z \right] + G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] + \frac{Q_5}{6720n^2} + \frac{1}{105} \eta^3 G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] \\
- \frac{(1 + \eta)(1 - z)^{3/2}}{1680n^{2/2}} G \left[ \left\{ \sqrt{1 - \tau^2}, \frac{1}{1 - \tau} \right\}, z \right] - \frac{\eta(-175 - 35n - 16n^2 - 16\eta^2)}{1680}
\]
with polynomials

\[ Q_1 = 30\eta^3 z^2 (216 - 146z - 105z^2 + 70z^3) + 525\eta^4 z^2 (12 - 22z + 43z^2 - 38z^3 + 12z^4) + 75z (24 - 152z + 662z^2 + 51z^3 - 830z^4 + 420z^5) + 20\eta^2 z (290 - 237z + 535z^2 + 756z^3 - 1554z^4 + 630z^5) - 2\eta (672 + 2248z - 11984z^2 + 37234z^3 + 15345z^4 - 55590z^5 + 25200z^6) \]  
\[ Q_2 = 35\eta^4 (-1 + z)z^2 (-1 + 3z) + 2\eta^3 z (3 + 35z - 175z^2 + 105z^3) + 5(-67z + 81z^2 + 85z^3 - 105z^4) - 2\eta (-30 + 509z + 17z^2 - 251z^3 + 105z^4) + 2\eta^2 (-5 + 47z - 28z^2 - 294z^3 + 210z^4) \]  
\[ Q_3 = 525\eta^4 z^3 (-6 + 12z - 10z^2 + 3z^3) + 30\eta^3 z^2 (-108 - 32z + 210z^2 - 280z^3 + 105z^4) - 75z (6 - 38z + 148z^2 + 4z^3 - 190z^4 + 105z^5) + 10\eta^2 z (15 - 751z + 225z^2 + 798z^3 - 1512z^4 + 630z^5) - 2\eta (560 - 2910z - 854z^2 + 7380z^3 + 4020z^4 - 5340z^5 + 1575z^6) \]  
\[ Q_4 = -15 + (65 - 21\eta)z + (-225 + 126\eta + 35\eta^2) z^2 + 105(-5 + 3\eta + \eta^2) z^3 \]  
\[ Q_5 = 35\eta^4 z^3 (3 - 6z + 12z^2 - 10z^3 + 3z^4) + 2\eta^3 z (-3 - 32z + 210z^2 - 280z^3 + 105z^4) - 8\eta (30 - 154z + 492z^2 + 268z^3 - 356z^4 + 105z^5) - 5(6 - 38z + 148z^2 + 4z^3 - 190z^4 + 105z^5) + \eta^2 (10 - 69z + 150z^2 + 532z^3 - 1008z^4 + 420z^5) \] .
In Mellin-space one obtains

\[
D_{\alpha a}(N) = \left( m_1^2 \right)^{\varepsilon/2} \left( m_2^2 \right)^{-3+\varepsilon} \left\{ \frac{1 + (-1)^{N}}{2} \right\} \left( \varepsilon 15(N - 1)N(N + 1)^2(N + 2) \right.
\]

\[
+ \left[ \eta((4N^2 - 8N + 3)\eta^2 - 4(N + 1)\eta + 25)(1 - \eta)^{-N} \right. \\
\frac{256}{256(\eta - 1)(N + 1)(N + 2)(2N - 3)(2N - 1)}
\]

\[
- \left. \frac{2^{-2N}(2N)_N P_2}{512(\eta - 1)(N + 1)^2(N + 2)(2N - 3)(2N - 1)} \right\} \eta \sum_{i_1 = 1}^{N} \frac{2^{2i_1}(1 - \eta)^{-i_1}}{(2i_1)}
\]

\[
+ \left[ \eta((4N^2 - 4N - 3)\eta^2 + 25) \right. \\
\frac{384(1 + 1)^2(N + 2)(2N - 3)}{384(N + 1)^2(N + 2)(2N - 3) - (\eta - 1)(N + 1)^2(N + 2)(2N - 3)(2N - 1)}
\]

\[
\left. \times \sum_{i_1 = 1}^{N} \frac{2^{2i_1}(1 - \eta)^{-i_1} S_1(1 - \eta, i_1) P_2}{(2i_1)} \right] \ln(\eta)
\]

\[
- \frac{2^{-2N-6}(2N)_N P_2}{(N + 1)^2(N + 2)(2N - 3)(2N - 1)}
\]

\[
+ \left( \frac{2^{-2N-7}(2N)_N P_2}{(N + 1)^2(N + 2)(2N - 3)(2N - 1)} \right) H_{1,0,0} (\sqrt{\eta}) + H_{1,0,0} (\sqrt{\eta})
\]

\[
+ \frac{2^{-2N}(2N)_N \eta P_2}{256(\eta - 1)(N + 1)^2(N + 2)(2N - 3)(2N - 1)}
\]

\[
\times \left. \sum_{i_1 = 1}^{N} \frac{2^{2i_1}(1 - \eta)^{-i_1} S_2(1 - \eta, i_1) - S_1,1 (1 - \eta, 1, i_1)}{(2i_1)} \right]
\]

\[
+ \left[ \eta((4N^2 - 8N + 3)\eta^2 - 4(N + 1)\eta + 25)(1 - \eta)^{-N} \right. \\
\frac{128}{128(\eta - 1)(N + 1)(N + 2)(2N - 3)(2N - 1)}
\]

\[
\left. \times \sum_{i_1 = 1}^{N} \frac{2^{2i_1}(1 - \eta)^{-i_1}}{(2i_1)} S_1(1 - \eta, i_1) \right]
\]

\[
+ \left[ \eta(3(4N^2 - 4N - 3)\eta^2 + 25) \right. \\
\frac{384(1 + 1)^2(N + 2)(2N - 3)}{384(N + 1)^2(N + 2)(2N - 3)}
\]

\[
\left. \times \sum_{i_1 = 1}^{N} \frac{2^{2i_1}(1 - \eta)^{-i_1}}{(2i_1)} S_1(1 - \eta, i_1) \right]
\]

\[
+ \frac{\eta((4N^2 - 8N + 3)\eta^2 - 4(N + 1)\eta + 25)(1 - \eta)^{-N} S_1(1 - \eta, N)}{128(\eta - 1)(N + 1)(N + 2)(2N - 3)(2N - 1)} + \frac{\eta(3(4N^2 - 4N - 3)\eta^2 + 25) S_1(N)}{384(N + 1)^2(N + 2)(2N - 3)} \right\},
\]

(457)

with the polynomials

\[
P_1 = 900\eta^3 N^7 - 900\eta^2 (2\eta + 1) N^6 - 25\eta (27\eta^2 - 90\eta - 89) N^5 + (2475\eta^3 \\
- 450\eta^2 - 4625\eta - 5504) N^4 - (225\eta^3 + 2250\eta^2 - 175\eta + 3264) N^3 \\
+ (-675\eta^3 + 1350\eta^2 + 4625\eta + 22784) N^2 - 96(25\eta + 46) N - 5760
\]

(458)
\[ P_2 = \eta^3 (8N^3 - 12N^2 - 2N + 3) + \eta^2 (-4N^2 + 8N - 3) + 9\eta(2N - 3) + 75. \]  

The mass reversed diagram \( D_{56} \) obeys the z-space representation

\[
D_{56}(z) = \left( m_2^2 \right)^{-3 + \varepsilon} \left( m_2^2 \right)^{\varepsilon/2} \left\{ \frac{1}{\varepsilon} \left[ \frac{(z - 1)(2z^2 + 5z - 1)}{45z^2} + \frac{2}{15} G \left( \left\{ \frac{1}{\tau} \right\}, z \right) \right] + \frac{G \left( \left\{ \frac{1}{\tau} \right\}, z \right) Q_3}{100800\eta^3 z^2} - \frac{Q_4}{201600\eta^4 z^2} + \frac{(\eta - 1)(\eta + 1)(5\eta^2 + 2\eta + 1)}{8\eta^3} \right\} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) + G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right] + \frac{(\eta - 1)^3(5\eta^2 + 2\eta + 1)}{32\eta^3} \left[ G \left( \left\{ \frac{1}{\eta - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z \right) + G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right] + \frac{(1 - z)^{3/2}(\eta + 1)Q_5}{1680\eta^3 z^{5/2}} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right] + \frac{(1 - z)^{3/2}(\eta - 1)^2Q_5}{6720\eta^3 z^{5/2}} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right] + \frac{Q_2}{6720\eta^3 z^2} - \frac{(1 - z)^{3/2}(\eta + 1)Q_5}{1680\eta^3 z^{5/2}} \times G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, z \right\} + \frac{1}{\eta^3} \left( 35\eta^2 + 175\eta^3 - 16z + 16\eta z \right) G \left( \left\{ \frac{1}{\eta - \tau + \eta \tau} \right\}, z \right) \right] - \frac{(1 - z)^{3/2}(\eta - 1)^2Q_5}{6720} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right] - \frac{(\eta - 1)^3(5\eta^2 + 2\eta + 1)}{32\eta^3} \left[ G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{\eta - \tau + \eta \tau} \right\}, z \right) - \frac{zG \left( \left\{ \frac{1}{\tau} \right\}, \eta \right)}{105\eta^3} \right] \left[ G \left( \left\{ \frac{1}{\tau} \right\}, \eta \right) \right] + \frac{(1 - z)^{3/2}}{\eta^3 z^{5/2}} \left( \frac{1}{840}(\eta - 1)Q_5 \right) G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, z \right\} + \frac{1}{\eta^3} \frac{1}{105} zG \left( \left\{ \frac{1}{\tau} \right\}, \frac{1}{1 - \tau} \right) \right) + \frac{1}{16} + \frac{1}{384\eta} + \frac{5\eta}{128} + \frac{z}{105\eta^3} G \left( \left\{ \frac{1}{\tau} \right\}, \eta \right) + \frac{1}{\eta^4} (\eta - 1)^2(5\eta^2 + 2\eta + 1) \times G \left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, z \right\} + \frac{(5\eta^3 - 3\eta^2 - \eta - 1)}{256\eta^{5/2}} G \left( \left\{ \frac{\sqrt{\tau}}{1 - \tau}, \frac{1}{\tau} \right\}, \eta \right) \right] + \frac{1}{105} (7 + \frac{z}{\eta^3}) G \left( \left\{ \frac{1}{\tau}, \frac{1}{\tau} \right\}, z \right) - \frac{zG_3}{105\eta^3} \right\} \] 

(460)
\[
Q_1 = 5\eta^4(105z^4 - 85z^3 - 81z^2 + 67z - 6) + 2\eta^3(105z^4 - 251z^3 + 17z^2 \\
+ 509z - 30) + \eta( - 420z^4 + 588z^3 + 56z^2 - 94z + 10) - 2\eta z(105z^3 \\
- 175z^2 + 35z + 3) - 35(z - 1)^2z(3z - 1),
\]
\[
Q_2 = 5\eta^4(105z^5 - 190z^4 + 4z^3 + 148z^2 - 143z + 6) + \eta^3(210z^5 - 712z^4 \\
+ 536z^3 + 984z^2 - 518z + 60) - \eta^2(420z^5 - 1008z^4 + 532z^3 + 150z^2 \\
- 489z + 10) + 2\eta z(-105z^4 + 280z^3 - 210z^2 + 32z + 108) \\
- 35z^2(3z^3 - 10z^2 + 12z - 6),
\]
\[
Q_3 = -75\eta^4z(105z^5 - 190z^4 + 4z^3 + 148z^2 - 38z + 6) - 2\eta^3(1575z^6 \\
- 5340z^5 + 4020z^4 + 7380z^3 - 854z^2 - 2910z + 560) + 10\eta^2z(630z^5 \\
- 1512z^4 + 798z^3 + 225z^2 - 751z + 15) + 30\eta z^2(105z^4 - 280z^3 \\
+ 210z^2 - 32z - 108) + 525z^3(3z^3 - 10z^2 + 12z - 6),
\]
\[
Q_4 = 75\eta^4z(420z^5 - 830z^4 + 51z^3 + 662z^2 - 572z + 24) - 2\eta^3(25200z^6 \\
- 55590z^5 + 15345z^4 + 37234z^3 - 37184z^2 + 2248z + 672) \\
+ 20\eta^2z(630z^5 - 1554z^4 + 756z^3 + 535z^2 - 867z + 290) \\
+ 30\eta z^2(70z^3 - 105z^2 - 146z + 216) \\
+ 525z^3(12z^3 - 38z^2 + 43z - 22).
\]
\[
Q_5 = -105z^3 - 35\eta z^2(3z + 1) - 21\eta^2z(15z^2 + 6z - 1) \\
+ 5\eta^3(105z^5 + 45z^3 - 3z^3 + 13z + 3).
\]

In Mellin space one obtains

\[
D_{56}(N) = (m_1^2)^{-3+\epsilon} (m_2^2)^{\epsilon/2} \left\{ \frac{1}{2} \left[ 1 + \frac{(-1)^N}{2} \right] \right\} \left\{ -\frac{4}{\epsilon 15(N - 1)N(N + 1)^2(N + 2)} \\
+ \frac{25\eta^2 - 4(N + 1)\eta + 4N^2 - 8N + 3}{256(N + 1)(N + 2)(2N - 3)(2N - 1)} \eta^{N - 1} \eta^{N - 2} \\
+ \frac{2}{2\eta^8 - 9(2N)^2} \sum_{i = 1}^{N} \frac{2^{2i}(1 + \eta^{-1})^{-i} \eta^{i}}{(2i)} \right\} \ln^2(\eta) \\
+ \frac{25\eta^2 + 12N^2 - 12N - 9}{384\eta^3(N + 1)^2(N + 2)(2N - 3)} + \frac{\eta^3(N + 1)^2(N + 2)(2N - 3)(2N - 1)}{2\eta^8 - 8(2N)^2} \sum_{i = 1}^{N} \frac{2^{2i}(1 + \eta^{-1})^{-i} \eta^{i}}{(2i)} S_1 \left( \frac{1 + \eta}{\eta}, i \right) \\
\frac{25\eta^2 - 4(N + 1)\eta + 4N^2 - 8N + 3}{128(N + 1)(N + 2)(2N - 3)(2N - 1)} \eta^{N - 1} \eta^{N - 2} \sum_{i = 1}^{N} \frac{2^{2i}(1 + \eta^{-1})^{-i} \eta^{i}}{(2i)} S_1 \left( \frac{\eta}{\eta}, i \right) \\
+ \frac{14400\eta^3(N - 1)^2N^2(N + 1)^3(N + 2)(2N - 3)}{P_2 \eta^2 - 6(2N)^2 \eta} \frac{2}{\eta^8 - 7(2N)^2} P_1 \\
+ \frac{P_2}{\eta^8} \frac{1}{\eta^8 - 7(2N)^2} P_1 \\
\frac{25\eta^2 - 4(N + 1)\eta + 4N^2 - 8N + 3}{128(N + 1)(N + 2)(2N - 3)(2N - 1)} \eta^{N - 1} \eta^{N - 2} \\
- \frac{2\eta^8 - 7(2N)^2 P_1}{\eta^8} P_2 \\
\frac{14400\eta^3(N - 1)^2N^2(N + 1)^3(N + 2)(2N - 3)}{P_2 \eta^2 - 6(2N)^2 \eta} \frac{2}{\eta^8 - 7(2N)^2} P_1 \\
+ \frac{P_2}{\eta^8} \frac{1}{\eta^8 - 7(2N)^2} P_1.
\]
\[ \times \left[ H_{-1,0,0}(\sqrt{\eta}) + H_{1,0,0}(\sqrt{\eta}) \right] + \frac{2 - 2N(2N)P_1}{(2N)N(2N-3)(2N-1)} \]
\[ \times \sum_{i_1=1}^N 2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1} \left[ S_{1,1} \left( \frac{-1+\eta}{\eta}, 1, i_1 \right) - S_2 \left( \frac{-1+\eta}{\eta}, i_1 \right) \right] \]
\[ + \frac{(\eta - 1)^{-N-1}\eta^{-2}(25\eta^2 - 4(N+1)\eta + 4\eta^2 - 8N + 3)}{128(N+1)(N+2)(2N-3)(2N-1)} \left[ S_2 \left( \frac{\eta - 1}{\eta}, N \right) \right. \]
\[ \left. - S_{1,1} \left( \frac{\eta - 1}{\eta}, 1, N \right) \right] + \frac{(25\eta^2 + 12\eta^2 - 12N - 9)}{384\eta^3(N+1)(N+2)(2N-3)} S_1(N) \right \}, \tag{466} \]

where we abbreviated the polynomials

\[ P_1 = 75\eta^3 + 9\eta^2(2N-3) + \eta(-4\eta^2 + 8N - 3) + 8N^2 - 12N^2 - 2N + 3 \tag{467} \]
\[ P_2 = 64\eta^3(86N^4 + 51N^3 - 356\eta^2 + 69N + 90) - 25\eta^2(N - 1)^2N \tag{468} \]
\[-7N - 96) + 450\eta(N - 1)^2N^2(2N^2 - N - 3) - 225(N - 1)^2N^2 \times (4N^3 - 7N - 3) \]

The diagrams \( D_{6a,b} \) and \( D_{8a,b} \), see Figures 12 and 14 respectively, are of a topology with one fermionic triangle and one fermion-bubble. For \( D_{6a}(z) \) one obtains

\[ D_{6a}(z) = (m_1^2)^{-3+\varepsilon} \left( m_2^2 \right)^{\varepsilon/2} \left\{ \right. \]
\[ - \frac{1}{45z^2} + \frac{1}{\varepsilon} \left[ \frac{22 + 26z}{180} + \frac{25}{27} \right] + \frac{1}{90}G \left( \{ \frac{1}{1 - \tau} \}, z \right) - \frac{1}{90}G \left( \{ \frac{1}{\tau} \}, z \right) \]
\[ + \frac{G \left( \{ \frac{1}{1 - \tau} \}, z \right) Q_3}{201600(\eta - 1)^2\eta^3z^2} + \frac{Q_4}{1814400\eta^3z^2} + \frac{1}{(\eta - 1)\eta^3z} \frac{G \left( \{ \frac{1}{1 - \tau} \}, z \right) Q_2}{201600} \]
\[ + \frac{1}{180} \left[ G \left( \{ \frac{1}{1 - \tau} \}, z \right) - G \left( \{ \frac{1}{\tau} \}, z \right) \right] + \frac{\sqrt{1 - z} (\eta + 1)Q_5}{\eta^3\tau^{5/2}} \frac{6720}{180} \]
\[ \times \left[ G \left( \{ \sqrt{1 - \tau}\sqrt{1 - \tau} \}, z \right) + G \left( \{ \sqrt{1 - \tau}\sqrt{1 - \tau} \}, \frac{1}{\tau} \right) \right] \]

Figure 12: \( D_{6a} \) with \( m_a = m_2, m_b = m_1 \) and \( D_{6b} \) with \( m_a = m_1, m_b = m_2 \) respectively.

\[ 99 \]
\[
\begin{aligned}
&\left( -\frac{25\eta^3 + 26\eta^2 + 23\eta + 8}{3360(\eta - 1)\eta^2} \right) G\left( \left\{ \frac{1}{-\eta - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z \right) \\
&+ G\left( \left\{ \frac{1}{-\eta - \tau + \eta \tau}, \frac{1}{\tau} \right\}, z \right) \Bigg[ \frac{\sqrt{1 - z} (\eta - 1)^2 Q_5}{26880} \right] G\left( \left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{-\eta - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z \right) \\
&+ G\left( \left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{-\eta - \tau + \eta \tau}, \frac{1}{\tau} \right\}, z \right) - \frac{1}{\eta^3} \left. \frac{1}{420} \right] G\left( \left\{ \left\{ \frac{1}{1 - \tau}; \frac{1}{1 - \tau} \right\}, z \right) \\
&+ G\left( \left\{ \frac{1}{1 - \tau}, \frac{1}{1 - \tau} \right\}, z \right) + \frac{1}{\eta^3} \left. \frac{1}{420} (\eta - 1) \right] G\left( \left\{ \left\{ \frac{1}{1 - \tau}; -\eta - \tau + \eta \tau, \frac{1}{1 - \tau} \right\}, z \right) \\
&+ G\left( \left\{ \frac{1}{1 - \tau}; -\eta - \tau + \eta \tau, \frac{1}{\tau} \right\}, z \right) \right] - \frac{1}{\eta^3} \left. \frac{(\eta + 1)(75\eta^3 - 63\eta^2 - 35\eta - 105)}{1680} \right] \\
&\times G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) + G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \\
&- \frac{(\eta - 1)^2 (75\eta^3 - 63\eta^2 - 35\eta - 105)}{6720\eta^3} G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \\
&+ \frac{1}{\eta^3} \left. \frac{1}{420} \right] G\left( \left\{ \frac{1}{1 - \tau}, \frac{1}{1 - \tau} \right\}, z \right) \right] + \frac{1}{\eta^3} \left. \frac{(\eta + 1)(75\eta^3 - 63\eta^2 - 35\eta - 105)}{1680} \right] \\
&\times G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) + \frac{1}{\eta^3} \left. \frac{(\eta - 1)^2 (75\eta^3 - 63\eta^2 - 35\eta - 105)}{6720} \right] \\
&G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \right] G\left( \left\{ \frac{1}{\tau} \right\}, \eta \right) \\
&+ \frac{\sqrt{1 - z} (\eta - 1)^2 Q_5}{3360} - \frac{1}{\eta^3} \left. \frac{1}{420} \right] G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) - \frac{1}{\eta^3} \left. \frac{1}{420} \right] G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, \eta \right) \\
&- \frac{1}{\eta^3} \left. \frac{1}{840} (\eta - 1)(75\eta^3 - 63\eta^2 - 35\eta - 105) \right] G\left( \left\{ \sqrt{1 - \tau} \sqrt{\tau}, \sqrt{1 - \tau} \sqrt{\tau}, \frac{1}{1 - \tau} \right\}, z \right) \\
&+ \left[ - \frac{1}{120} + \frac{G\left( \left\{ \frac{1}{1 - \tau} \right\}, z \right)}{420\eta^3} \right] \zeta_2 \right). \tag{469}
\end{aligned}
\]

\(Q_1 = -15\eta^5 \left( 15\eta^5 - 25\eta^4 - 8\eta^3 - 34\eta^2 - 14\eta + 3 \right) + 3\eta^4 (63\eta^5 - 55\eta^4 - 96\eta^3 \\
- 10\eta^2 + 34\eta - 15) + \eta^3 (330\eta^5 - 826\eta^4 + 294\eta^3 + 987\eta^2 - 478\eta + 105) \\
+ \eta^2 (126\eta^5 - 154\eta^4 + 294\eta^3 - 273\eta^2 + 178\eta - 15) + \eta z (-\eta 105\eta^4 \\
+ 35\eta^3 + 210\eta^2 - 105\eta - 12) - 105\eta^2 (3\eta^3 - 7\eta^2 + 6\eta - 3) \right), \tag{470}
Performing the Mellin transformation using HarmonicSums [175, 177–179] one obtains

\[
Q_2 = 75\eta^5(z - 1)^2(15z^3 + 5z^2 - 13z + 3) + \eta^4 \left( -945z^5 + 825z^4 + 1440z^3 + 150z^2 - 1247z + 225 \right) - \eta^3 \left( 1650z^5 - 4130z^4 + 1470z^3 + 4935z^2 - 2602z + 525 \right) - 5\eta^2 \left( 216z^5 - 154z^4 + 294z^3 - 273z^2 + 78z - 15 \right) + 35\eta z (15z^4 - 5z^3 - 30z^2 + 15z + 53) + 525(z - 1)^2z(3z^2 - z + 1),
\]

\[
Q_3 = 75\eta^5z(15z^5 - 25z^4 + 8z^3 + 34z^2 - 14z + 3) + \eta^4 \left( -945z^6 + 825z^5 + 1440z^4 + 722z^2 - 2015z + 560 \right) + \eta^3 \left( -1650z^6 + 4130z^5 - 1470z^4 - 4935z^3 + 2558z^2 + 1715z - 560 \right) - 5\eta^2 \left( 216z^5 - 154z^4 + 294z^3 - 273z^2 + 122z - 15 \right) + 5\eta z^2 \left( 105z^4 - 35z^3 - 210z^2 + 105z - 324 \right) + 525z^3(3z^3 - 7z^2 + 6z - 3),
\]

\[
Q_4 = 3375\eta^4z(60z^5 - 110z^4 - 27z^3 + 146z^2 - 56z + 12) - 8\eta^3 \left( 46575z^6 - 93825z^5 - 4050z^4 + 114075z^3 - 51319z^2 + 7605z + 3780 \right) + 450\eta^2z \left( 168z^5 - 434z^4 + 77z^3 + 574z^2 - 232z + 290 \right) - 360\eta z^2 \left( 525z^4 - 875z^3 + 700z^2 - 175z + 853 \right) + 23625z^3 \left( 12z^3 - 26z^2 + 19z - 10 \right),
\]

\[
Q_5 = 105(1 - 2z)z^3 + 35\eta z^2 \left( -2z^2 + z + 1 \right) - 21\eta z \left( 6z^3 - 3z^2 - 4z + 1 \right) + 5\eta^3 \left( 30z^4 - 15z^3 - 23z^2 + 11z - 3 \right).
\]
Here we abbreviated the polynomials

\[ P_1 = 91582N^7 - 147453N^6 - 132764N^5 + 236946N^4 + 235222N^3 - 442293N^2 + 32760N + 75600, \]
\[ P_2 = 375\eta^2(N + 1) - 6\eta(4N^2 + 23N + 19) + 420N^3 - 676N^2 - 13N + 123, \]
\[ P_3 = -450\eta^3(2N^2 - N - 3) + \eta^2(656N^2 - 328N - 609) + 18\eta(26N^2 - 11N - 42) + 315(4N^2 - 4N - 3), \]
\[ P_4 = 448\eta^3(3 - 2N)^2 + \eta^2(-5600N^2 + 2800N + 6525) - 30\eta(224N^2 - 106N - 345) - 1575(4N^2 - 4N - 3), \]
\[ P_5 = 4\eta^2(2N^3 - 5N^2 + 3N - 10) - 25\eta N(N^2 - 1) - 30N(N^2 - 1), \]
\[ P_6 = 1125\eta^3 + 189\eta^2(2N - 3) - 35\eta(4N^2 - 8N + 3) + 105(8N^3 - 12N^2 - 2N + 3) \]
\[ P_7 = 2\eta^3(2N - 3)(49057N^6 - 10080N^5 - 62834N^4 + 15120N^3 + 215377N^2 - 55440N - 50400) - 225\eta^2(N - 1)^2N(N + 1)(1376N^3 - 1248N^2 - 539N - 840) - 90\eta(N - 1)^2N^2(N + 1)(2N - 3)(2231N + 581) - 70875(N - 1)^2N^2(N + 1) \times (2N - 3)(2N + 1). \]

Assigning the masses for this diagram the other way yields diagram \(D_{6b} \). Its \(z\)-space representation reads

\[
D_{6b}(z) = \left( m_1^{2\epsilon/2} m_2^{-3+\epsilon} \right).
\]
\[ \frac{(1 + \eta)(-75 + 63\eta + 35\eta^2 + 105\eta^3)}{1680\eta} \left[ G \left( \left\{ \sqrt{1 - \frac{\tau}{\sqrt{\tau}}, \sqrt{1 - \frac{\tau}{\sqrt{\tau}}}, \frac{1}{\tau} \right\}, z \right) \right. \\
+ G \left( \left\{ \sqrt{1 - \frac{\tau}{\sqrt{\tau}}, \sqrt{1 - \frac{\tau}{\sqrt{\tau}}}, \frac{1}{\tau} \right\}, z \right) \right] - \frac{(-1 + \eta)^2(-75 + 63\eta + 35\eta^2 + 105\eta^3)}{6720\eta} \]
\]
\[ \times \left[ G \left( \left\{ \sqrt{1 - \frac{\tau}{\sqrt{\tau}}, \sqrt{1 - \frac{\tau}{\sqrt{\tau}}}, \frac{1}{\tau} \right\}, z \right) \right. \\
+ G \left( \left\{ \sqrt{1 - \frac{\tau}{\sqrt{\tau}}, \sqrt{1 - \frac{\tau}{\sqrt{\tau}}}, \frac{1}{\tau} \right\}, z \right) \right] \right\}, \quad (483) \]

with

\[ Q_1 = 23625\eta^4 z^3 (10 + 19z - 26z^2 + 12z^3) - 360\eta^3 z^2 (-853 - 175z) + 700z^2 - 875z^2 + 525z^4 + 3375z (12 - 56z + 146z^2 - 27z^3 - 110z^4 + 60z^5) + 450\eta z^2 (290 - 232z + 574z^2 + 77z^3 - 434z^4 + 168z^5) - 8\eta (3780 + 7605z - 51319z^2 + 114075z^3 - 4050z^4 - 93825z^5 + 465z^6) \]
\[ (484) \]

\[ Q_2 = -525\eta^5 z^3 (3 - 6z - 7z^2 + 3z^3) - 5\eta^4 z^2 (-324 + 105z - 210z^2 - 35z^3 + 105z^4) - 75z (3 - 14z + 34z^2 - 8z^3 - 25z^4 + 15z^5) + 5\eta^3 z (-15 + 122z - 273z^2 + 294z^3 - 154z^4 + 126z^5) + \eta^2 (-560 + 2015z - 722z^2 - 150z^3 - 1440z^4 - 825z^5 + 945z^6) + \eta^2 (560 - 1715z - 2558z^2 + 4935z^3 + 1470z^4 - 4130z^5 + 1650z^6) \]
\[ (485) \]

\[ Q_3 = 15 + (-55 + 21\eta)z + (115 - 84\eta - 35\eta^2) z^2 - (75 + 63\eta + 35\eta^2 + 105\eta^3) z^3 + 2 (-75 + 63\eta + 35\eta^2 + 105\eta^3) z^4 \]
\[ (486) \]

\[ Q_4 = -525\eta^5 (-1 + z)^2 z (1 - z + 3z^2) - 75(-1 + z)^2 (3 - 13z + 5z^2 + 15z^3) - 35\eta z (53 + 15z - 30z^2 - 5z^3 + 15z^4) + 5\eta^3 (-15 + 78z - 273z^2 + 294z^3 - 154z^4 + 126z^5) + \eta^2 (525 - 260z + 4935z^2 + 1470z^3 - 4130z^4 + 1650z^5) \]
\[ (487) \]

\[ Q_6 = 105\eta^3 z^2 (3z^3 - 7z^2 + 6z - 3) + \eta^4 z (105z^4 - 35z^3 - 210z^2 + 105z + 12) + \eta^3 (-126z^5 + 154z^4 - 294z^3 + 273z^2 - 178z + 15) - \eta^2 (330z^5 - 826z^4 + 294z^3 + 987z^2 - 478z + 105) - 3\eta (63z^5 - 55z^4 - 96z^3 - 10z^2 + 34z - 15) + 15 (15z^5 - 25z^4 - 8z^3 + 34z^2 - 14z + 3) \]
\[ (488) \]

In Mellin space this corresponds to

\[ D_{ob}(N) = \left( m_1^{\nu/2} m_2^{\nu} \right)^{-3+\epsilon} \left\{ \frac{1 + (-1)^N}{2} \right\} \left[ -\frac{1}{45\epsilon^2(N + 1)^2} + \frac{1}{\epsilon} \left[ \frac{P_1}{1800(N - 1)N(N + 1)^2} \right. \right. \\
+ S_1(N) \right. \\
+ \frac{S_1(N)}{90(N + 1)} \right\} + \frac{\eta^3}{420(N + 1)} \left[ -S_3(N) - S_1 \left( \frac{1}{1 - \eta}, N \right) S_{1,1} (1 - \eta, 1, N) \right. \\
+ S_{1,2} \left( \frac{1}{1 - \eta}, 1 - \eta, N \right) - S_{1,2} \left( 1 - \eta, \frac{1}{1 - \eta}, N \right) + S_{1,1,1} \left( 1 - \eta, 1, \frac{1}{1 - \eta}, N \right) \right] \\
\left. \left. + S_{1,1,1} \left( 1 - \eta, \frac{1}{1 - \eta}, 1, N \right) + \frac{P_2}{9072000(N - 1)^2N^2(N + 1)^3(2N - 3)} \right\} \right. \\
\]
\[
\begin{align*}
\frac{2^{2N}(2N)}{N}P_5 & \quad - \frac{2^{2N}(2N)}{N}P_5 \\
\frac{105(N + 1)^2(2N - 3)(2N - 1)}{105(N + 1)^2(2N - 3)(2N - 1)} & \quad + \frac{53760(\eta - 1)(N + 1)^2(2N - 3)(2N - 1)}{403200(N + 1)^2(2N - 3)} \\
\sum_{i_1=1}^{N} \frac{2^{2i_1}(1 - \eta)^{-i_1}}{(2i_1)} [S_{1,1,1}(1 - \eta, i_1) - S_2(1 - \eta, i_1)] & \quad - \frac{P_4S_1(N)}{105(N + 1)^2(2N - 3)} \\
\frac{(1 - \eta)^{-N}\eta P_6}{26880(\eta - 1)(N + 1)^2(2N - 3)(2N - 1)} & \quad \left[ S_2(1 - \eta, N) - S_{1,1}(1 - \eta, 1, N) \right] \\
\frac{1}{360(N + 1)} [S_2(N) + S_2(N)] + \ln^2(\eta) & \quad \left[ \frac{41\eta^2 + 38\eta - 75}{53760(N + 1)} \right] \quad S_2(N) \quad \frac{1}{1 - \eta}N \quad \eta^3 \quad \eta^3 \\
+ \frac{2^{2N}(2N)}{N}P_5 & \quad + \frac{105\eta - 1)(N + 1)^2(2N - 3)(2N - 1)}{105\eta - 1)(N + 1)^2(2N - 3)(2N - 1)} \\
\frac{53760(\eta - 1)(N + 1)^2(2N - 3)(2N - 1)}{53760(\eta - 1)(N + 1)^2(2N - 3)(2N - 1)} & \quad + \ln(\eta) \left[ \frac{\eta^3}{420(N + 1)} \right] \\
\frac{P_3}{2^{2N-8}(2N)} & \quad \frac{P_3}{2^{2N-8}(2N)} \\
\frac{105(N + 1)^2(2N - 3)(2N - 1)}{2^{2N-8}(2N)} & \quad + \frac{105(\eta - 1)(N + 1)^2(2N - 3)(2N - 1)}{105(\eta - 1)(N + 1)^2(2N - 3)(2N - 1)} \\
\sum_{i_1=1}^{N} \frac{2^{2i_1}(1 - \eta)^{-i_1}S_1(1 - \eta, i_1)}{(2i_1)} & \quad - \frac{(1 - \eta)^{-N}\eta P_6S_1(1 - \eta, N)}{26880(\eta - 1)(N + 1)^2(2N - 3)(2N - 1)} \\
\frac{2^{2N-8}(2N)}{\sqrt{\eta}(N + 1)^2(2N - 3)(2N - 1)} & \quad \frac{P_5}{105} - \frac{(105\eta^3 + 35\eta^2 + 63\eta - 75)}{13440(\sqrt{\eta}(N + 1))} \\
\times [H_{-1,0,0}\left(\sqrt{\eta}\right) + H_{1,0,0}\left(\sqrt{\eta}\right)] & \quad \left[ \frac{\zeta_2}{120(N + 1)} \right],
\end{align*}
\]

where

\[
\begin{align*}
P_1 & = (30\eta^2 + 25\eta - 8)N^3 - (30\eta^2 + 25\eta + 12)N + 20N^2 + 40 \\
P_2 & = -4 \left(70875\eta^3 + 100395\eta^2 + 77400\eta - 49057\right)N^6 + 6(94500\eta^3 + 149895\eta^2 + 98400\eta - 55777)N^6 + 212625\eta^3 + 60660\eta^2 + 150075\eta - 190856)N^5 \\
& - 3 \left(259875\eta^3 + 352080\eta^2 + 174225\eta - 145828\right)N^4 + (70875\eta^3 + 340920\eta^2 - 29475\eta + 770788)N^3 + 3 \left(70875\eta^3 + 52290\eta^2 - 22575\eta - 504674\right)N^2 + 2520(75\eta + 52)N + 302400 \\
P_3 & = -315\eta^3 (4N^2 - 4N - 3) + \eta^2 (-468N^2 + 198N + 756) + \eta (-656N^2 + 328N + 609) + 450 (2N^2 - N - 3) \\
P_4 & = 1575\eta^3 (4N^2 - 4N - 3) + 30\eta^2 (224N^2 - 106N - 345) + 25\eta (224N^2 - 112N - 261) - 448(3 - 2N)^2 \\
P_5 & = 105\eta^3 (8N^3 - 12N^2 - 2N + 3) - 35\eta^2 (4N^2 - 8N + 3) + 189\eta(2N - 3)
\end{align*}
\]
The ladder-type diagram $D_7$ is symmetric under $m_1 \rightarrow m_2$ and only one mass assignment has to be considered. It evaluates to

\[
D_7(z) = (m_1^2)^{-3+3/2\varepsilon}\left\{ -\frac{1}{\varepsilon} \frac{1}{24\eta^2} + \frac{Q_2}{8640\eta^3 z} + \frac{(\eta^3 + 1)(z^2 - 1)}{180\eta^3 z} \cdot G\left(\left\{ \frac{1}{1 - \tau}, \frac{1}{1 - \tau} \right\}, z\right) \right. \\
+ G\left(\left\{ \frac{1}{1 - \tau}, \frac{1}{\eta \tau} \right\}, z\right) - \frac{1}{\eta^3} \frac{1}{180} (\eta^3 + 1) z \cdot G\left(\left\{ \frac{1}{\eta \tau}, \frac{1}{1 - \tau} \right\}, z\right) + G\left(\left\{ \frac{1}{\eta \tau}, \frac{1}{1 - \eta} \right\}, z\right) \\
- \sqrt{1 - z} \frac{1}{\eta^3 \sqrt{z}} \frac{1}{720} (\eta + 1) Q_3 \cdot G\left(\left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{\sqrt{1 - \tau + \eta \tau}}, \frac{1}{1 - \tau} \right\}, z\right) + G\left(\left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{\sqrt{1 - \tau + \eta \tau}}, \frac{1}{1 - \eta} \right\}, z\right) \\
+ \frac{1}{(\eta - 1)\eta^3 z} \frac{1}{720} Q_4 \cdot G\left(\left\{ \frac{1}{1 - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z\right) + G\left(\left\{ \frac{1}{1 - \tau + \eta \tau}, \frac{1}{1 - \eta} \right\}, z\right) \\
- \sqrt{1 - z} (\eta - 1)^2 (27 + 2(5\eta + 27) z + (27\eta^2 - 10\eta - 81) z^2) \\
\times G\left(\left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{\sqrt{1 - \tau + \eta \tau}}, \frac{1}{1 - \eta} \right\}, z\right) + G\left(\left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{\sqrt{1 - \tau + \eta \tau}}, \frac{1}{\eta \tau} \right\}, z\right) \\
- \frac{1}{(\eta - 1)\eta^3 z} \frac{Q_6}{720} \cdot G\left(\left\{ \frac{1}{-\eta - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z\right) + G\left(\left\{ \frac{1}{-\eta - \tau + \eta \tau}, \frac{1}{1 - \eta} \right\}, z\right) \\
- \sqrt{1 - z} (\eta - 1)^2 (10\eta(z - 1) z - 27 z^2 + 27\eta^2 (3z^2 - 2z - 1)) \\
\times G\left(\left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{-\eta - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z\right) + G\left(\left\{ \frac{\sqrt{1 - \tau} \sqrt{\tau}}{-\eta - \tau + \eta \tau}, \frac{1}{\eta \tau} \right\}, z\right) \\
+ \frac{1}{\eta^3} \frac{1}{180} (\eta^3 + 1) \cdot G\left(\left\{ \frac{1}{1 - \tau}, \frac{1}{1 - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z\right) + G\left(\left\{ \frac{1}{1 - \tau}, \frac{1}{1 - \tau + \eta \tau}, \frac{1}{1 - \eta} \right\}, z\right) \\
+ \frac{1}{(1 - \eta)} \frac{1}{180} \cdot G\left(\left\{ \frac{1}{1 - \tau}, \frac{1}{1 - \tau + \eta \tau}, \frac{1}{1 - \tau} \right\}, z\right) + G\left(\left\{ \frac{1}{1 - \tau}, \frac{1}{1 - \tau + \eta \tau}, \frac{1}{1 - \eta} \right\}, z\right)
\right\}.
\]

Figure 13: $D_7$, both mass assignments $m_a = m_2, m_b = m_1$ and $m_a = m_1, m_b = m_2$ yield the same result due to symmetry reasons.
The Mellin space-expression for this diagram reads

\[
\times G\left(\{\sqrt{1-\tau}, \sqrt{1-\tau+\eta \tau}\}, z\right) - \frac{1}{\eta^3} \frac{(\eta - 1)^2\left(81\eta^2 + 10\eta - 27\right)}{1440} \times G\left(\{\sqrt{1-\tau}, \sqrt{1-\tau+\eta \tau}\}, z\right) + \frac{(\eta - 1)}{90} G\left(\{\frac{1}{\tau}, \frac{1}{\eta - 1 - \tau + \eta \tau}\}, z\right) G\left(\{\frac{1}{\tau}, \eta\}, z\right)
\]

\[
- \frac{(\eta + 1)Q_7\left(\{\frac{1}{\tau}, \frac{1}{\tau}\}, z\right) - 3\sqrt{1-z}(\eta - 1)^2(\eta + 1)(4z^2 - 2z - 1)}{40\eta^3 \sqrt{z}} G\left(\{\sqrt{1-\tau}, \sqrt{1-\tau+\eta \tau}\}, z\right)
\]

\[
+ \left[ - \frac{16\eta + \eta^2 + 8z + \eta^3(8z - 27)}{1440\eta^3} \left(\eta^3 + 1\right) G\left(\{\frac{1}{1 - \tau}, \frac{1}{\tau}\}, z\right) G\left(\{\frac{1}{1 - \tau}, \frac{1}{\tau + \eta \tau}\}, \eta\right)
\]

\[
+ \frac{(\eta - 1)^2(\eta + 1) + \eta^3}{5} G\left(\{\sqrt{1-\tau}, \sqrt{1-\tau+\eta \tau}\}, z\right) - \frac{(\eta^3 + 1)}{180\eta^3} G\left(\{\frac{1}{1 - \tau}, \frac{1}{\tau}, \frac{1}{\tau}\}, \eta\right)
\]

\[
+ \frac{(27\eta^2 + 10\eta + 27)}{2880\eta^3/2} G\left(\{\sqrt{1-\tau}, \frac{1}{\tau}, \frac{1}{\tau}\}, \eta\right) + \left[ \frac{(\eta^3 + 1)}{180\eta^3} - \frac{(\eta^3 + 1)}{180\eta^3} G\left(\{\frac{1}{1 - \tau}, \frac{1}{\tau}, \eta\}, z\right) \right] \zeta_2
\]

\[
- \frac{1}{90} G\left(\{\frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}\}, \eta\right) \right) .
\]

\[
Q_1 = 27\eta^4(6z^4 - 11z^3 + 3z^2 + 3z - 6) - 2\eta^3(22z^3 - 66z^2 - 15z + 63) + \eta^2(-324z^2 + 682z^2 - 426z^2 - 30z + 84) + \eta(-44z^3 + 132z^2 + 30z + 204) + 27z(6z^3 - 11z^2 + 3z + 3),
\]

\[
Q_2 = -27\eta^3z(24z^4 - 46z^3 + 11z^2 + 16z - 24) + \eta^2(648z^3 - 1370z^2 + 873z^3 - 12z^2 - 18z - 180) + \eta(648z^5 - 1370z^4 + 873z^3 - 12z^2 + 18z - 180) - 27z^2(24z^3 - 46z^2 + 11z + 16),
\]

\[
Q_3 = 20\eta(z - 1)z + 27(2z^2 - 2z - 1) + 27\eta^2(2z^2 - 2z - 1),
\]

\[
Q_4 = 4\eta^4(1 - 2z)z + 4\eta^5z^2 + \eta^3z(4z + 3) + 4(z^2 - 1) - 8\eta(z^2 + z - 1) + \eta^2(4z^2 - 15z - 4),
\]

\[
Q_5 = 9(z - 1)^3(3z + 2) + 9\eta^2(z - 1)^3(3z + 2) + 2\eta(5z^4 - 33z^3 + 42z^2 + 14z - 43),
\]

\[
Q_6 = -4z^2 + 4\eta z(2z - 1) - \eta^2 z(4z + 3) + \eta^3(-4z^2 + 15z + 4) - 4\eta^5(z^2 - 1) + 8\eta^4(z^2 + z - 1),
\]

\[
Q_7 = 9z(3z^3 - 7z^2 + 3z + 3) + 9\eta^2z(3z^3 - 7z^2 + 3z + 3) + 2\eta(5z^4 - 33z^3 + 42z^2 + 14z + 11).
\]

The Mellin space-expression for this diagram reads

\[
D_7(N) = (m_1^2)^{-3+3/2N} \left[ \frac{1 + (-1)^N}{2} \right] \left\{ - \frac{\eta + 1}{24\pi^2(N + 1)} \right\}
\]

\[
+ \left[ - \frac{32 + 32\eta^3 + 11\eta N(N^2 + 3N + 2) + 11\eta^2 N(N^2 + 3N + 2)}{5760\eta^3 N(N + 1)^2(N + 2)} \right] - \frac{(\eta - 1)^3}{\eta^3(N + 1)^2(N + 2)} \sum_{i=1}^{N} \frac{2^{2i}(-\eta)^i}{(2i)} - \frac{1}{45 (\eta - 1)\eta^3(N + 1)^2(N + 2)} \sum_{i=1}^{N} 2^{2i}(-\eta)^i
\]

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\[-3 \cdot \frac{2^{-2N-7}(2N)}{N} (\eta + 1) N \frac{P_5}{\eta^2 (N + 1)^2} + \frac{(\eta - 1)^N}{11520(\eta - 1)\eta^2 N(N + 1)^2(N + 2)} P_6^N \]
\[+ \frac{1}{360(N + 1)} S_1 \left( \frac{1}{1 - \eta}, N \right) + \frac{1}{360\eta^3 (N + 1)} S_1 \left( \frac{\eta}{\eta - 1}, N \right) \]
\[- \frac{1}{45} P_{10} (\eta - 1)\eta (N + 1)(N + 2) \sum_{i=1}^{N} \frac{2^{2i} (1 - \eta)^{-i} 1}{(2^i)_{i}} \ln^2(\eta) \]
\[+ \int \left[ \frac{P_7}{5760\eta^3 (N + 1)^2(N + 2)} - \frac{(\eta - 1) P_{11} 2^{-2N-7}(2N)}{45\eta^3 (N + 1)^2(N + 2)} \right] \ln(\eta) \]
\[+ \frac{1}{90} \left( \frac{1}{\eta} \right)^3 N + \frac{1}{180\eta^3 (N + 1)(N + 2)} S_1 (N) - \frac{(\eta - 1)^N}{180\eta^3 (N + 1)} \]
\[+ \frac{1}{5760} (\eta - 1)\eta^3 N(N + 1)(N + 2) \int \left( \frac{1}{\eta - 1}, N \right) \]
\[+ \frac{(\eta - 1)^N}{5760\eta^3 (N + 1)^2(N + 2)} P_6 S_1 \left( \frac{\eta - 1}{\eta}, N \right) \]
\[+ \frac{1}{180(N + 1)} \int \left( \frac{1}{1 - \eta}, N \right) \]
\[+ \frac{1}{180\eta^3 (N + 1)} S_1 \left( \frac{\eta}{\eta - 1}, N \right) \]
\[- \frac{1}{45} P_{10} (\eta - 1)\eta (N + 1)(N + 2) \sum_{i=1}^{N} \frac{2^{2i} (1 - \eta)^{-i} 1}{(2^i)_{i}} \ln(\eta) \]
\[+ \frac{1}{\eta^{3/2}} \left[ \frac{(27\eta^2 + 10\eta + 27)}{1440(N + 1)} - \frac{2^{-2N-6}(2N)}{45(N + 1)^2(N + 2)} \right] \times \left[ H_{-1,0,0} (\sqrt{\eta}) + H_{1,0,0} (\sqrt{\eta}) \right] \]
\[+ \frac{1}{45} P_{10} 2^{-2N-6}(2N) \frac{P_3}{45(N + 1)^2(N + 2)} \]
\[+ \frac{1}{45\eta^3 (N + 1)^2(N + 2)} + \frac{1}{11520(\eta - 1)\eta^3 (N + 1)^2(N + 2)} \]
\[+ \frac{1}{180\eta^3 (N + 1)^2(N + 2)} S_2 (N) - \frac{1}{S_1^2 (N)} \]
\[+ \frac{1}{180(N + 1)} \frac{S_1 (1 - \eta, 1, N) - S_2 (1 - \eta, i_1)}{(2^i)_{i}} \]

\[\times \int \left[ (\eta + 1) P_5 \right] \]
\[- \frac{(\eta + 1) P_5}{2880\eta^3 (N + 1)^2(N + 2)} + \frac{(\eta + 1) P_5 S_1 (N)}{5760\eta^3 (N + 1)^2(N + 2)} \]
\[+ \frac{(\eta + 1)}{180\eta^3 (N + 1)^2(N + 2)} \left[ S_2 (N) - \frac{1}{S_1^2 (N)} \right] + \frac{(\eta + 1)}{180\eta^3 (N + 1)} \frac{S_3 (N)}{180\eta^3 (N + 1)} \]
\[+ \frac{(\eta + 1)}{5760(\eta - 1)\eta^3 (N + 1)^2(N + 2)} \left[ S_1 (1 - \eta, 1, N) - S_2 (1 - \eta, N) \right] \]
\[+ \frac{(\eta - 1)^N}{5760(\eta - 1)\eta^3 (N + 1)^2(N + 2)} \left[ S_1 (\frac{\eta - 1}{\eta}, 1, N) - S_2 (\frac{\eta - 1}{\eta}, N) \right] \]
\[+ \frac{1}{180(N + 1)} \left[ S_1 \left( \frac{1}{1 - \eta}, N \right) \right] S_1 (1 - \eta, 1, N) - S_1 (1 - \eta, 1, N) \]
This expression contains the polynomials

\[ P_1 = 27\eta^2 (2N^2 + 4N + 3) - 10\eta (2N + 1) + 27 (2N^2 + 4N + 3) , \]  
\[ P_2 = 81\eta^2 - 10\eta (2N + 1) + 27 (4N^2 + 8N + 3) , \]  
\[ P_3 = 27\eta^2 (4N^2 + 8N + 3) + 2\eta (6N^2 + 73N + 115) + 27 (4N^2 + 8N + 3) , \]  
\[ P_4 = 27\eta^2 (2N^2 + 4N + 3) + 2\eta (27N^2 + 44N - 5) + 27 (4N^2 + 8N + 3) , \]  
\[ P_5 = 64\eta^3 - 64\eta^2 (N + 1) + 5\eta N(N + 1) - N (54N^2 + 103N + 17) , \]  
\[ P_6 = \eta^3 N (54N^2 + 103N + 17) - 5\eta^2 N(N + 1) + 64\eta (N + 1) - 64 , \]  
\[ P_7 = -27\eta^3 (4N^2 + 8N + 3) + \eta^2 (196N^2 + 586N + 449) + \eta (164N^2 + 494N + 271) + 27 (4N^2 + 8N + 3) , \]  
\[ P_8 = \eta^3 (27N^3 + 81N^2 + 54N - 32) - \eta^2 N (N^2 + 3N + 2) + 16\eta N (N^2 + 3N + 2) - 32 , \]  
\[ P_9 = 27\eta^2 N (4N^2 + 8N + 3) + 2\eta (27N^3 + 76N^2 + 78N + 60) + 27N (4N^2 + 8N + 3) , \]  
\[ P_{10} = 27 (4N^2 + 8N + 3)\eta^2 - 10(2N + 1)\eta + 81 , \]  
\[ P_{11} = 27\eta^2 (4N^2 + 8N + 3) + 2\eta (54N^2 + 98N - 5) + 27 (4N^2 + 8N + 3) , \]  
\[ P_{12} = 27\eta^3 (4N^2 + 8N + 3) + \eta^2 (71 - 20N) + \eta (71 - 20N) + 27 (4N^2 + 8N + 3) . \]  

Finally we turn to the diagrams \( D_{8a,b} \). In \( z \)-space they contain contributions which have to be

Figure 14: \( D_{8a} \) with \( m_a = m_2, m_b = m_1 \) and \( D_{8b} \) with \( m_a = m_1, m_b = m_2 \) respectively.

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regularized as in Eq. 405. For $D_{8a}$ this contribution is given by

$$
D_{8a}^{(+)} = (m_1^2)^{\epsilon/2} (m_2^2)^{-3+\epsilon} \left\{ \frac{1}{\epsilon} \frac{1}{90(-1+z)} + \frac{1}{450(-1+z)} + \frac{1}{180(1-z)} G \left[ \left\{ \frac{1}{1-\tau} \right\}, z \right] \right.
$$

$$
+ \left[ \frac{(-1+\eta)(25\sqrt{z}+(-100+63\eta)\sqrt{1-z}(1-z)z)}{3360\eta(1-z)^{3/2}} \right]
$$

$$
(1+\eta) \left( 25\sqrt{z} + (-100 + 63\eta) \sqrt{1-z} \sqrt{(1-z)z} \right) G \left[ \left\{ \frac{1}{\epsilon} \right\}, \eta \right]
$$

$$
- \frac{6720\eta(1-z)^{3/2}}{6720\eta(1-z)^{3/2}} \times G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] - \frac{(-1+\eta)^2 (25\sqrt{z} + (-100 + 63\eta)(1-z)\sqrt{z})}{26880\eta(1-z)^{3/2}}
$$

$$
\times G \left[ \left\{ \frac{1}{\tau} \right\}, \eta \right] G \left[ \left\{ \frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, z \right\} \right]
$$

$$
(1+\eta) \left( 25\sqrt{z} + (-100 + 63\eta) \sqrt{1-z} \sqrt{-(1+z)z} \right)
$$

$$
+ \frac{6720\eta(1-z)^{3/2}}{6720\eta(1-z)^{3/2}} \times \left[ G \left[ \left\{ \frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{1-\tau} \right\}, z \right] + G \left[ \left\{ \frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{1-\tau} \right\}, z \right] \right] \right\}.
$$

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The regular contribution to $D_{8a}(z)$ reads

$$
D_{8a}^{\text{Reg}} = (m_1^2)^{\epsilon/2} (m_2^2)^{-3+\epsilon} \left\{ -\frac{1}{45\epsilon^2} - \frac{1}{225\epsilon} + \frac{7}{20250} - \frac{\zeta_2}{120} \right\} \delta(1-z)
$$

$$
- \frac{1}{45\epsilon^2} + \frac{1}{\epsilon} \left[ \frac{\eta}{72} + \frac{1}{450} \left( 8 - \frac{5}{z} \right) + \frac{1}{90} G \left[ \left\{ \frac{1}{1-\tau} \right\}, z \right] - G \left[ \left\{ \frac{1}{\epsilon} \right\}, z \right] \right]
$$

$$
+ \frac{Q_1}{18144000\eta z} - \frac{\zeta_2}{120} + G \left[ \left\{ \frac{1}{1-\tau} \right\}, z \right] \left( \frac{Q_2}{201600(-1+\eta)\eta} + \frac{\eta^3(-1+z)G \left[ \left\{ \frac{1}{\epsilon} \right\}, \eta \right]}{420z} \right) + \frac{Q_3}{201600(-1+\eta)\eta z} G \left[ \left\{ \frac{1}{\epsilon} \right\}, z \right]
$$

$$
+ \frac{(-1+\eta)\sqrt{1-z}Q_4}{3360\eta^{3/2}} G \left[ \left\{ \frac{1}{\tau} \right\}, z \right] - \frac{(3\eta^3(-1+z) + 7z)}{1260z} G \left[ \left\{ \frac{1}{1-\tau}, \frac{1}{1-\tau} \right\}, z \right] - \frac{\eta^3(-1+z)}{420z} G \left[ \left\{ \frac{1}{1-\tau}, \frac{1}{1-\tau} \right\}, z \right]
$$

$$
- \frac{\eta(25 - 6\eta + 105\eta^2)}{26880} G \left[ \left\{ \frac{1}{\tau}, \frac{1}{\tau} \right\}, \eta \right] - \frac{1}{180} G \left[ \left\{ \frac{1}{\tau}, \frac{1}{\tau} \right\}, z \right]
$$

$$
+ \frac{(1+\eta)\sqrt{1-z}Q_5}{6720\eta^{3/2}} G \left[ \left\{ \sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau} \right\}, z \right] + G \left[ \left\{ \sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau} \right\}, z \right] \right\}.
$$
\[-\left(-1 + \eta\right)(-75 + 63\eta + 35\eta^2 + 105\eta^3) \frac{840\eta}{G}\left[\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}}{\eta}, z\right\}\right] + G\left[\left\{\frac{1}{\tau}, \eta\right\}\left(\frac{Q_5}{40320(-1 + \eta)\eta} - \frac{(1 + \eta)\sqrt{1-z}Q_4}{6720\eta\sqrt{z}}\right)\left[\left\{\left\{\frac{1}{\tau}, \frac{1}{1-\tau}\right\}, z\right\}\right]\right] - \frac{\eta Q_6}{3360(-1 + \eta)\eta} G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] - \frac{(1 + \eta)^2\sqrt{1-z}Q_4}{26880\eta\sqrt{z}} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] + \frac{(1 + \eta)(-75 + 63\eta + 35\eta^2 + 105\eta^3)}{1680\eta} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] + \frac{\eta Q_6}{3360(-1 + \eta)\eta} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] + \frac{(1 + \eta)^2\sqrt{1-z}Q_4}{26880\eta\sqrt{z}} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] + \frac{(1 + \eta)(-75 + 63\eta + 35\eta^2 + 105\eta^3)}{53760\sqrt{\eta}} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] + \frac{(1 + \eta)(-75 + 63\eta + 35\eta^2 + 105\eta^3)}{1680\eta} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] + \frac{(1 + \eta)(-75 + 63\eta + 35\eta^2 + 105\eta^3)}{6720\eta} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right] + \frac{(1 + \eta)(-75 + 63\eta + 35\eta^2 + 105\eta^3)}{450\eta^2 (280 + 30z)} \times G\left[\left\{\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta - \tau + \eta\tau}, \frac{1}{\tau}\right\}, z\right\}\right]\right],

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with

\[Q_1 = \frac{16875z^2 (2 + 3z - 26z^2 + 12z^3) + 23625\eta^4 z (-12 + 14z + z^2 - 22z^3 + 12z^4) - 12600\eta^3 (-12 - 3z + 25z^2 + 5z^3 - 35z^4 + 15z^5) + 450\eta^2 (280 + 30z)}{450\eta^2 (280 + 30z)}\]
\[ Q_2 = 525\eta^3 (-1 + z)^2 (-1 + z + 3z^2 + \eta^2 (-1332 + 1845z + 270z^2 + 2470z^3 - 1650z^4) + 175\eta^4 (-13 + 9z + 12z^2 - 11z^3 + 3z^4) + 375 (1 + 2z^2 - 7z^3 + 3z^4) - 5\eta^3 (4 + 315z + 210z^2 - 350z^3 + 126z^4) - 3\eta^2 (-619 + 500z + 690z^2 - 985z^3 + 315z^4) \]  
\[ Q_3 = 375z^3 (2 - 7z + 3z^2) + 525\eta^5 z^2 (3 - 5z^2 + 3z^3) + 5\eta^4 z (-96 + 315z + 420z^2 - 385z^3 + 105z^4) - 5\eta^3 z (-376 + 315z + 210z^2 - 350z^3 + 126z^4) + \eta^2 (-1120 - 728z + 1845z^2 + 270z^3 + 2470z^4 - 1650z^5) - \eta (-1120 + 672z + 1500z^2 + 2070z^3 - 2955z^4 + 945z^5) \]  
\[ Q_4 = 75(1 - 2z)z + 63\eta z(-1 + 2z) + 35\eta^2 z(-1 + 2z) + 105\eta^3 (-1 - z + 2z^2) \]  
\[ Q_5 = -75z^2 (2 - 7z + 3z^2) + 3\eta z (100 + 138z - 197z^2 + 63z^3) - 35\eta^4 (-3 + 9z + 12z^2 - 11z^3 + 3z^4) - 105\eta^5 (-3 + 3z - 5z^3 + 3z^4) + \eta^3 (-395 + 315z + 210z^2 - 350z^3 + 126z^4) + \eta^2 (-25 - 369z - 54z^2 - 494z^3 + 330z^4) \]  
\[ Q_6 = -8\eta^4 (-1 + z) - 25z + 26\eta z + 8\eta^3 (-2 + 3z) + \eta^2 (8 + 15z) \]  
In Mellin space one obtains

\[ D_{8a}(N) = (m_1^2)^{\varepsilon/2} (m_2^2)^{-3+\varepsilon} \left[ \frac{1 + (-1)^N}{2} \right] \left\{ -\frac{N + 2}{45\varepsilon^2 (N + 1)} + \frac{1}{\varepsilon} \left[ \frac{(N + 2)S_1(N)}{90(N + 1)} - \frac{8N^3 + (4 - 25\eta)N^2 - (25\eta + 24)N + 20}{1800N(N + 1)^2} \right] \right. \]

\[-\frac{(7N(N^2 + 3N + 2) - 3\eta^3)S_1^2(N)}{2520N(N + 1)^2} - \frac{P_4 S_1(N)}{403200\eta(N + 1)^2} \]

\[-\frac{2^{-2N - 7} \binom{N}{2} P_2}{105\eta(N + 1)^2} + \frac{2^{-2N - 8} \binom{N}{2} P_2}{105\sqrt{\eta}(N + 1)^2} \times \left[ H_{-1,0,0} (\sqrt{\eta}) + H_{1,0,0} (\sqrt{\eta}) \right] \]

\[ + \frac{2^{-2N} \binom{N}{2} P_2}{53760(\eta - 1)\eta(N + 1)^2} \sum_{i_1 = 1}^N 2^{2i_1} \left( \frac{-\eta}{1 + \eta} \right)^{i_1} \left( \frac{1 + \eta}{\eta} \right)^{i_1} S_2 \left( \frac{-1 + \eta}{\eta}, i_1 \right) - S_{1,1} \left( \frac{-1 + \eta}{\eta}, 1, i_1 \right) \]

\[-\frac{\ln^2(\eta)}{840N(N + 1)^2} - \frac{(\eta - 1)^{-N - 1} \eta^N P_1}{N(N + 1)^2} \frac{1}{53760} - \frac{2^{-2N - 10} \binom{N}{2} P_2}{105(\eta - 1)(N + 1)^2} \]

\[-\frac{2^{-2N - 10} \binom{N}{2} P_2}{105(\eta - 1)\eta(N + 1)^2} \sum_{i_1 = 1}^N 2^{2i_1} (-1 + \eta)^{-i_1} \eta^i \left[ \frac{2^{2i_1}}{2^{2i_1}} \right] \]

\[-\frac{2^{-2N - 10} \binom{N}{2} P_2}{105(\eta - 1)\eta(N + 1)^2} + \frac{P_3}{9072000\eta N^2(N + 1)^3} \]

\[-\frac{(3\eta^3 + 7N(N^2 + 3N + 2))S_2(N)}{2520N(N + 1)^2} + \ln(\eta) \left\{ -\frac{S_1(N) \eta^3}{420N(N + 1)^2} \right. \]

\[-\frac{2^{-2N - 8} \binom{N}{2} P_2}{105\eta(N + 1)^2} + 2^{-2N - 9} \binom{N}{2} P_2 \]

\[ \times \sum_{i_1 = 1}^N 2^{2i_1} (-1 + \eta)^{-i_1} \eta^i S_1 \left( \frac{-1 + \eta}{\eta}, i_1 \right) + \left( \eta - 1 \right)^{-N - 1} \eta^N P_1 \frac{1}{26880N(N + 1)^2} S_1 \left( \frac{\eta - 1}{\eta}, N \right) \]
and the regular contribution is given by

\[
D_{8b}^{\text{Reg}}(z) = (m_1^2)^{-3+\epsilon} (m_2^2)^{\epsilon/2} \left\{ \frac{7}{20250} - \frac{1}{45\epsilon^2} - \frac{1}{225\epsilon} - \frac{\zeta_2}{120} \right\} \delta(1-z)
\]

and the regular contribution is given by

\[
D_{8b}^{\text{Reg}}(z) = (m_1^2)^{-3+\epsilon} (m_2^2)^{\epsilon/2} \left\{ \frac{7}{20250} - \frac{1}{45\epsilon^2} - \frac{1}{225\epsilon} - \frac{\zeta_2}{120} \right\} \delta(1-z)
\]

and the regular contribution is given by

\[
D_{8b}^{\text{Reg}}(z) = (m_1^2)^{-3+\epsilon} (m_2^2)^{\epsilon/2} \left\{ \frac{7}{20250} - \frac{1}{45\epsilon^2} - \frac{1}{225\epsilon} - \frac{\zeta_2}{120} \right\} \delta(1-z)
\]
\[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ \frac{1}{72\eta} + \frac{1}{450} (8 - \frac{z}{\tau}) + \frac{1}{90} G\left(\left\{\frac{1}{\tau}, \frac{1}{\varepsilon}\right\}, \frac{3}{\tau} \right) \right] + \frac{G\left(\left\{\frac{1}{\tau}, \frac{1}{\varepsilon}\right\}, \frac{3}{\tau} \right)}{201600(\eta - 1)\eta^3 z} + \frac{Q_4}{1814400\eta^3 z} + \frac{Q_2}{(\eta - 1)\eta^3 201600} G\left(\left\{\frac{1}{\tau}, \frac{1}{\varepsilon}\right\}, z\right) \]

\[-\frac{\sqrt{1 - z}(\eta + 1)Q_5}{\eta^3 \sqrt{z}} \left[ G\left(\left\{\frac{\sqrt{1 - z}}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + G\left(\left\{\frac{\sqrt{1 - z}}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \right] \]

\[+ \frac{1}{(\eta - 1)\eta^3 z} (8 - 8z + 26\eta^3 z - 25\eta^4 z + 8\eta(3z - 2) + \eta^2(15z + 8)) \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[+ \frac{1}{\eta^3 420} G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[+ \frac{1}{\eta^3 420} (\eta - 1) \left[ G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \right] \]

\[+ \frac{1}{\eta^3} \left(\frac{Q_1}{40320(\eta - 1)\eta^3} - \frac{1}{\eta^3 420} (z - 1) G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) - \frac{\sqrt{1 - z}(\eta + 1)Q_5}{\eta^3 \sqrt{z}} \right] \]

\[\times G\left(\left\{\frac{\sqrt{1 - z}}{1 - \tau}, \frac{\sqrt{1 - z}}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + G\left(\left\{\frac{\sqrt{1 - z}}{1 - \tau}, \frac{\sqrt{1 - z}}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[+ \frac{Q_1}{40320(\eta - 1)\eta^3} - \frac{1}{\eta^3 420} (z - 1) G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) - \frac{\sqrt{1 - z}(\eta + 1)Q_5}{\eta^3 \sqrt{z}} \]

\[\times G\left(\left\{\frac{\sqrt{1 - z}}{1 - \tau}, \frac{\sqrt{1 - z}}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + G\left(\left\{\frac{\sqrt{1 - z}}{1 - \tau}, \frac{\sqrt{1 - z}}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[+ \frac{\sqrt{1 - z}(\eta - 1)Q_5}{3360(\eta - 1)\eta^3 \sqrt{z}} G\left(\left\{\frac{\sqrt{1 - z}}{1 - \tau}, \frac{\sqrt{1 - z}}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + \frac{(3 - 3z - 27z^2)}{1260(1 - \tau)\eta^3} \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) + \frac{(3 - 3z - 7z^2)}{1260(1 - \tau)\eta^3} \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]

\[\times G\left(\left\{\frac{1}{\tau}, \frac{1}{1 - \tau}, \frac{1}{1 - \tau}, \frac{1}{\tau}\right\}, \frac{3}{\tau \eta^3} \right) \]
Finally one obtains the $N$-space representation

\[ D_{88}(N) = \left( m_1^2 \right)^{-3+\epsilon} \left( m_2^2 \right)^{\epsilon/2} \left[ \frac{1 + (-1)^N}{2} \right] \left\{ -\frac{N + 2}{4\epsilon^2(N + 1)} \right\} \]

\[ + \frac{1}{\epsilon} \left[ \frac{(N + 2) S_1(N)}{90(N + 1)} - \frac{4\eta(2N^3 + N^2 - 6N + 5) - 25N(N + 1)}{1800\eta N(N + 1)^2} \right] \]

\[ - \frac{7\eta^3 N(N^2 + 3N + 2) - 3}{2520\eta^3 N(N + 1)^2} S_1^2(N) - \frac{P_3 S_1(N)}{403200\eta^2(N + 1)^2} \]

\[ - \frac{2^{-2N-7}(2^N N)}{105\eta^2(N + 1)^2} + \frac{2^{-2N-8}(2^N N)}{105\eta^{5/2}(N + 1)^2} \times \left[ H_{-1,0,0}(\sqrt{\eta}) + H_{1,0,0}(\sqrt{\eta}) \right] \]

\[ + \frac{2^{-2N}(2^N N)}{53760(\eta - 1)\eta(N + 1)^2} \sum_{i_1=1}^{N} 2^{2i_1}(1 - \eta)^{\text{even}} S_{1,1}(1 - \eta, 1, i_1) - S_2(1 - \eta, i_1) \]

\[ + \frac{\ln^2(\eta)}{53760\eta^2 N(N + 1)^2} \left[ (-1)^N(\eta - 1)^{-N-1} P_1 + \frac{2^{-2N}(2^N N)}{107520(\eta - 1)\eta(N + 1)^2} \right] \]

Finally one obtains the $N$-space representation.
\[
\sum_{i_1=1}^{N} (-1)^{i_1} 2^{i_1} (-1 + \eta)^{-i_1} \left( \frac{1}{\binom{2i_1}{i_1}} \right) + \frac{P_5}{9072000\eta^2 N^2 (N + 1)^3} - \frac{7N(N^2 + 3N + 2)\eta^3 + 3}{2520\eta^3 N(N + 1)^2} S_2(N) + \ln(\eta) \left[ -\frac{P_3}{8064\eta^2 (N + 1)^2} + \frac{2^{-2N-8}(2N)^P_2}{105\eta^2 (N + 1)^2} \right.
\]
\[
+ \left. \frac{2^{-2N-9}(2N)^P_2}{105(\eta - 1)\eta(N + 1)^2} \right] \sum_{i_1=1}^{N} (-1)^{i_1} 2^{i_1} (-1 + \eta)^{-i_1} \binom{2i_1}{i_1} S_1(1 - \eta, N) - \frac{S_1(N)}{420\eta^3 N(N + 1)^2} \right) + \frac{(-1)^N(\eta - 1)^{-N-1} P_1}{26880\eta^3 N(N + 1)^2} S_2(1 - \eta, N)
\]

\[
- S_{1,1}(1 - \eta, 1, N) - \frac{(N + 2)}{120(N + 1)} \zeta_2 \right) \right)}.
\]

Here we used the polynomials

\[
P_1 = 25\eta^4 N \left( 4N^3 + 20N^2 + 31N + 15 \right) - 2\eta^3 N \left( 88N^2 + 245N + 157 \right) + 3\eta^2 N(N + 1) + 64\eta(N + 1) - 64,
\]

\[
P_2 = 25\eta^3 \left( 8N^3 + 36N^2 + 46N + 15 \right) - 63\eta^2 \left( 4N^2 + 8N + 3 \right) - 35\eta(2N + 1) + 192,
\]

\[
P_3 = 75\eta^3 \left( 8N^3 + 36N^2 + 46N + 15 \right) - 6\eta^2 \left( 76N^2 + 52N - 93 \right) - \eta(538N + 547) + 192,
\]

\[
P_4 = 375\eta^3 \left( 8N^3 + 36N^2 + 46N + 15 \right) + \eta^2 \left( -3176N^2 - 2008N + 5478 \right) + 5\eta(22N + 13) + 960,
\]

\[
P_5 = 16875\eta^3 N^2 \left( 8N^4 + 44N^3 + 82N^2 + 61N + 15 \right) - 4\eta^2 \left( 58616N^5 + 203774N^4 + 241285N^3 + 101167N^2 - 32760N - 12600 \right) + 225\eta N \left( 478N^3 + 945N^2 + 747N + 280 \right) + 10800N \left( 9N^2 + 16N + 7 \right).
\]

With the exception of $D_1$ and $D_3$ in $z$-space the scalar $A_{gg,Q}$ diagrams are not expressible within the class of the usual harmonic polylogarithms [180], but generalizations thereof occur. These are given in terms of iterated integrals over the following letters

\[
\left\{ \frac{d\tau}{1 - \tau}, \frac{d\tau}{\tau}, \frac{d\tau\sqrt{\tau}}{1 - \tau}, \frac{d\tau\sqrt{1 - \tau\sqrt{\tau}}}{\eta\tau - \tau + 1}, \frac{d\tau\sqrt{1 - \tau\sqrt{\tau}}}{\eta\tau - \eta + \tau}, \frac{d\tau\sqrt{1 - \tau\sqrt{\tau}}}{\eta\tau - \eta - \tau} \right\}.
\]

In Mellin-space all scalar $A_{gg,Q}$-diagrams are expressible in terms of $\ln(\eta)$, the harmonic polylogarithms $H_{-1,0,0}(\sqrt{\eta}), H_{1,0,0}(\sqrt{\eta})$, alternating harmonic sums, $\eta$-dependant generalized harmonic sums and $\eta$-dependant finite binomial sums. For fixed values of the Mellin variable $N$, these $\eta$-dependant sums turn into rational functions in $\eta$. Thus for fixed Mellin moments, all diagrams are given in terms of the $\ln(\eta)$ and the combination $H_{-1,0,0}(\sqrt{\eta}) + H_{1,0,0}(\sqrt{\eta})$ with rational coefficients in $\eta$.

The summands of many of these sums diverge for $\eta \to 1$ due to factors as $(1 - \eta)^{-i}$, where $j$ is a summation index which assumes positive integer values. Furthermore also contributions
\( (1 - \eta)^{-N} \) emerge. Physically the limit \( \eta \to 1 \) represents the equal mass case \( m_1 = m_2 \) and thus the diagrams are expected to be divergent in this limit. Due to the many individually divergent terms this is highly non-trivial to prove for general values of \( N \). However, evaluating a series of Mellin moments \( N = 2 \ldots 30 \), yields convergent results for \( \eta = 1 \), which agree with the results given in Ref. [289] previously. This indicates that these apparent divergences are just a relic of this specific representation which has been applied. The diagrams \( (D_{2a}, D_{2b}), (D_{4a}, D_{4b}), (D_{5a}, D_{5b}), (D_{6a}, D_{6b}) \) and \( (D_{8a}, D_{8b}) \) have all been computed independently. One notes, that as expected the respective \( z \)- and Mellin-space results can be translated into each other by interchanging the masses \( m_1 \leftrightarrow m_2, \eta \to 1/\eta \). Furthermore the results for the mass-symmetric diagrams \( D_1, D_3 \) and \( D_7 \) turn are invariant under this interchange, which constitutes further evidence on the correctness of these results.

For all scalar \( A_{ggQ}^{(3)} \)-topologies series expansions up to \( O(\eta^3 \ln^3(\eta)) \) for a series of fixed Mellin moments \( (N = 2, 4, 6) \) have been computed using the code Q2e/Exp [185, 186]. All the general \( N \) and general-\( \eta \) results agree with these expansions.
7 Integration via hyperlogarithms

In this Section we explore the possibilities to apply parametric integration via hyperlogarithmic functions to massive Feynman parameter integrals with operator insertions. This method is designed for performing convergent Feynman parameter integrals in space-time dimensions $D \in \mathbb{N}$. Most Feynman integrals carry infrared, ultraviolet or collinear divergences which require regularization. In order to compute divergent diagrams in dimensional regularization [250], the $\varepsilon$-expansion has to be performed before the actual integration. Special methods to achieve this have been proposed, e.g. sector decomposition [290–292], which, however, is not very well suited for this application as it usually leads to a very large number of terms and is based on variable transformations which may easily result in more complicated structures of the integrands. In Refs. [293, 294] a method to extract the $\varepsilon$-pole terms prior to the integration via partial integrations has been presented. This method may, however, lead to huge increase in the size of the expressions for the case of many distinct singularities in the integration domain. Here we restrict ourselves to diagrams which are convergent in $D = 4$ dimensions and do therefore not require dimensional regularization \(^{10}\). This applies in particular to very complicated topologies for which the present method turns out to be very useful [151, 296, 297]. For some diagrams the method requires a mapping of the integration variables to obtain linearity of the denominator functions in the integration variable [270, 296].

7.1 The $\alpha$-parametrization.

We consider massive Feynman diagrams at $l = 3$ loops with operator insertions. The Feynman rules of Appendix B.2 are applied, a Schwinger parameter $\alpha_i$ is attached to each propagator and the momentum integration is performed according to [208]. The Feynman parameter integrals of a graph $G$ are then expressed as integrals over the Schwinger parameters [298]:

$$I_G = \frac{\Gamma(a - lD/2)}{\prod_j \Gamma(a_j)} \int_0^\infty \prod_j \alpha_j^{a_j - 1} OP_i(\alpha_i, N) \frac{\delta \left(1 - \sum_{l \in v} \alpha_l\right)}{\Psi_G^{D/2} M_G^{a - lD/2}} \, d\alpha_i, \quad (545)$$

where the $a_i$ denote the powers of the different propagators, $a = \sum_{i \in \text{edges}} a_i$ and $v$ an arbitrary subset of the edges $E$ of $G$ according to the Cheng-Wu-theorem [299, 300]. While $M_G$ equals the sum of all Schwinger parameters which are attached to a massive line, Since we consider massive Feynman diagrams whose external momentum is strictly on-shell in our case the second Symanzik polynomial $F$ just into $F = U_G M_G$, the graph polynomial $\Psi_G$ and the operator insertion $OP_i(\alpha_i, N)$ obey nice graph theoretical descriptions. For a graph with $n_v$ vertices and $n_e$ edges we define the $n_e \times n_v$ graph incidence matrix

$$ (\varepsilon)_{e,v} = \begin{cases} 1, & \text{if the edge } e \text{ starts at vertex } v \\ -1, & \text{if the edge } e \text{ ends at vertex } v \\ 0, & \text{if the edge } e \text{ is not connected to vertex } v. \end{cases} \quad (546)$$

We choose $\varepsilon_G$ as the $n_e \times (n_v - 1)$-matrix obtained from (546) by removing one arbitrary column. $\varepsilon_G$ is thus not uniquely defined and depends on the direction of the edges and the choice of the

\(^{10}\)Note that convergence is in general not a sufficient condition for not requiring any regularization [295].
The first Symanzik polynomial \( \Psi_G \) reads

\[
\Psi_G = - \det(M_G) .
\]

Although the matrix \( M_G \) is not uniquely defined \( \Psi_G \) is independent of the possible choices \( M_G \). The first Symanzik polynomial (or first graph polynomial) is always linear in all \( \alpha \)-parameters and of homogeneous degree \( d \).

If \( I,J,K \) are sets of edges in the graph \( G \) with \( |I| = |J| \) we define the Dodgson polynomials as

\[
\Psi_{G,I}^{I,J} = \pm \det(M_G(I,J))_{\alpha_e=0} \quad \text{for all } \epsilon \in K ,
\]

with \( M_G(I,J) \) being the matrix \( M_G \) after removing all rows corresponding to the edges in \( I \) and all columns corresponding to the edges in \( J \). If \( K \) is empty we omit it and write \( \Psi_{G}^{I,J} \). We consider the graph \( \tilde{G} \) where the external line of \( G \) has been closed. Figure 15 shows the the functions \( OP_i(\alpha_i, N) \) expressed in terms of Dodgson polynomials for the different operators studied in the following. The Dodgson polynomials \( \Psi_{G,I}^{I,J} \) are only defined up to a sign which generally

\[
\begin{align*}
OP_1(\alpha_i, N) &= \left( \frac{\Psi_{G,i}^{i,L+1}}{\Psi_G} \right)^N \\
OP_2(\alpha_i, N) &= \frac{1}{(\Psi_G)^N} \sum_{m=0}^{N} \left( \Psi_{G,i}^{i,L+1} \Psi_G \right)^m \left( \Psi_{G,j}^{j,L+1} \right)^{N-m} \\
OP_3(\alpha_i, N) &= \frac{1}{(\Psi_G)^N} \sum_{n=0}^{N-3} \sum_{m=0}^{N-2} \left( \Psi_{G,i}^{i,L+1} \right)^m \left( \Psi_{G,j}^{j,L+1} \right)^{N-n-2} \\
&\quad \times \left[ C_1 \left( \Psi_{G,i}^{i,L+1} + \Psi_{G,j}^{j,L+1} \right)^{n-m-1} + C_2 \left( \Psi_{G,i}^{i,L+1} + \Psi_{G,k}^{k,L+1} \right)^{n-m-1} \right]
\end{align*}
\]
Figure 16: This ladder graph is two-edge reducible concerning the pairs of edges 1, 8, 2, 5 and 3, 4 depends on the orientation of the edges in $\varepsilon_G$ and also on the columns that were removed to define $M_G$. For the present paper we were able to choose $\Psi_{G,K}^{I,J} = \det M_G(I,J)|_{\alpha_e=0}$ for all $e\in K$ when the directions of the edges were chosen according to the Feynman rules in Appendix C.

The first graph polynomial obeys the following symmetry property which we use frequently in the present computation: If a graph is two-edge-reducible, i.e. the graph can be split into two non-connected components by removing two lines, the first Symanzik polynomial is a function of the sum of the two corresponding $\alpha$-parameters only [187].

Let us consider the graph in Figure 16. It is two-edge-reducible concerning the edges 1; 8, 2; 5 and 3; 4 and its first Symanzik polynomial is given by

$$G = 8(1 + 6)(2 + 5) + (1 + 6)(2 + 5)(3 + 4) + (7 + 8)(1 + 6)(3 + 4).$$

(550)

. In many cases the mass polynomial $M_G$ shares the same symmetry. For the graph in Figure 16 we have

$$M_G = x_1 + x_2 + x_3,$$

(551)

It is thus advantageous to introduce new variables $x_1 = \alpha_1 + \alpha_6$, $x_2 = \alpha_2 + \alpha_5$ and $x_3 = \alpha_3 + \alpha_4$ and write

$$\Psi_G = \alpha_8(\alpha_1 + \alpha_6)(\alpha_2 + \alpha_5) + (\alpha_1 + \alpha_6)(\alpha_2 + \alpha_5)(\alpha_3 + \alpha_4) + (\alpha_7 + \alpha_8)(\alpha_1 + \alpha_6)(\alpha_3 + \alpha_4) + \alpha_8\alpha_7(\alpha_1 + \alpha_6) + \alpha_7(\alpha_2 + \alpha_5)(\alpha_3 + \alpha_4) + \alpha_8\alpha_7(\alpha_2 + \alpha_5) + \alpha_8\alpha_7(\alpha_3 + \alpha_4).$$

(550)

The integrals then obey the following representation

$$I_G = \frac{\Gamma(a - lD/2)}{\prod_j \Gamma(a_j)} \int_0^\infty \prod_j a_j^{a_j - 1} OP_i(\alpha_1, \alpha_2, \alpha_3, \alpha_7, \alpha_8, x_1, x_2, x_3, N) \Psi_G(x_1, x_2, x_3, \alpha_7, \alpha_8)^{D/2} M_G^{a_l - lD/2} \delta \left(1 - \sum_{l\in V} \alpha_l\right) d\alpha_i.$$ 

(554)

### 7.2 Linear reduction

The integration method presented in this Section only works as long as all the denominators factor into linear polynomials in the present integration variable at every integration step. In order to verify if this condition holds for a given parametric integral a linear reduction algorithm has been presented in [159,187]. In our case we had to modify this algorithm slightly due to the Heaviside functions appearing in (554).

We start with the set of denominators of our integrand $S_0 = \{f_1, \cdots, f_k\}$. If all $s \in S_0$ are linear in $x_i$ performing one single integration step $\int_a^b$ over the variable $x_i$ yields the denominators
\[ S[x] \{ f_1 | x = a, \cdots, f_k | x = a, f_1 | x = b, \cdots, f_k | x = b, [f_1, f_2]_x, \cdots, [f_1, f_k]_x, \cdots, [f_k, f_1]_x, \cdots, [f_k, f_{k-1}] \} \]

where \( f_i | x = \infty \) is defined as the leading coefficient of \( f_i \) in \( x \) and

\[ [a_1 + b_1 x, a_2 + b_2 x]_x = b_1 a_2 - b_1 a_1 . \]

If \( S_0 \) contains one or more non-linear polynomials in \( x \) we set \( S[x] = \emptyset \).

Following this procedure iteratively for all integration steps one obtains a superset of the possible denominators appearing at all integration steps. The number of possible denominators is reduced by using the fact that our integrals are independent of the respective integration order. One may thus restrict the set of possible denominators at a specific integration step to the intersection of all denominator sets of all compatible integration orders up to this integration. A compatible integration order in this sense is one, where at each integration step only linear polynomials in the respective integration variable are encountered. \(^{11}\)

### 7.3 Parametric integration

Let \( \sigma \) be a set of distinct points in \( \mathbb{C} \) and \( \bar{a} \) a word where each letter corresponds to an element in \( \sigma \). The elements in \( \sigma \) may be constants or rational functions of further parameters. Then the hyperlogarithm \([301]\) \( L(\bar{a}, z) : \mathbb{C} \setminus \sigma \rightarrow \mathbb{C} \) is a function defined by

\[ L(\{0, \cdots, 0\}, z) = \begin{cases} \frac{1}{n!} \ln(z)^n & \text{for } n \text{ times} \\ \end{cases} \]

and otherwise recursively by

\[ L(\{a\}, z) = \int_0^z \frac{1}{z_1 - a} dz_1 \]

\[ L(\bar{a}, z) = L(\{a_1, \bar{a}\}, z) = \int_0^z \frac{1}{z_1 - a_1} L(\{\bar{a}\}, z_1) dz_1 . \]

The weight \( w \) of a hyperlogarithm is defined as the number of letters in \( \bar{a} \). Being an iterated integral, the hyperlogarithm satisfies a shuffle product \([166–168, 172, 174, 180]\)

\[ L(\bar{a}_1, z) L(\bar{a}_2, z) = L(\bar{a}_1 \sqcup \bar{a}_2, z) . \]

For example one has

\[ L(\{a, b\}, z) L(\{c, d\}, z) = L(\{a, b, c, d\}, z) + L(\{a, c, b, d\}, z) + L(\{a, c, d, b\}, z) + L(\{c, a, b, d\}, z) + L(\{c, a, d, b\}, z) . \]

\(^{11}\)Further methods to restrict the number of spurious elements in this superset of denominators are outlined in Ref. [187].
Derivatives with respect to the argument $z$ follow from (559)

$$ \frac{d}{dz} L(\{a_1, \bar{a}\}, z) = \frac{1}{z - a_1} L(\{\bar{a}\}, z) . $$

(562)

Using these properties one may construct primitives over expressions containing rational functions and hyperlogarithms in $z$ if all denominators factor into linear terms in $z$. A graph $G$ is called linear reducible if there exists an integration order, for which this condition holds at every integration step. If and for which integration order a given graph is linear reducible can be checked a priori applying reduction algorithms presented in [159,187]. Using the shuffle product and a partial fraction decomposition one obtains expressions of the form

$$ I(b,n) = \int dx (x + b)^n L(\{a_1, \bar{a}\}, x) . $$

(563)

If $n = -1$ the integral is identified with $L(\{-b, a_1, \bar{a}\}, x)$. Otherwise we apply the integration by parts relation

$$ I(b,n) = \frac{(x + b)^{n+1}}{n + 1} L(\{a_1, \bar{a}\}, x) - \int dx (x + b)^{n+1} \frac{1}{(n + 1)(x - a_1)} L(\{\bar{a}\}, x) , $$

(564)

where the last term is simpler as the weight of the hyperlogarithm is reduced by one.

In order to evaluate the primitives at the respective integration limits $x \to 0, x \to \infty$, we need the respective series expansions. A hyperlogarithm of weight $w$ satisfies sum representations of the form

$$ L(\{a_1, \cdots, a_n\}, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{w} c_{i,j}^{(0)} \ln^i(z) z^j , $$

(565)

$$ L(\{a_1, \cdots, a_n\}, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{w} c_{i,j}^{(\infty)} \ln^i(z) z^{-j} . $$

(566)

We follow [159] and define the restricted regularization $RReg_{z \to \{0, \infty\}}$ as the constant parts of the generalized series expansion, which are given by

$$ RReg_{z \to 0} L(\{a_1, \cdots, a_n\}, z) = c_{0,0}^{(0)} = 0 $$

(567)

$$ RReg_{z \to \infty} L(\{a_1, \cdots, a_n\}, z) = c_{0,0}^{(\infty)} . $$

(568)

The restricted regularization with respect to inner variables is defined analogously. Using the restricted regularization we define the regularized integration as

$$ \int_{\text{Reg}(0)}^{z} f(y) dy = F(z) - RReg_{y \to 0} F(y) . $$

(569)

The series expansions are constructed by first differentiating w.r.t. the argument of the hyperlogarithm, performing the (simpler) series expansion of the derivative (which is of a lower weight) and then undoing the differentiation by finding a primitive and fixing the respective integration constant. With $\text{Ser}_{y \to \infty}^{(k)}$ the series operator up to $O\left(y^{-k} \ln^w(y)\right)$ one obtains

$$ \text{Ser}_{z \to \infty}^{(k)} L(\{a_1, \bar{a}\}, z) = \int_{\text{Reg}(0)}^{z} \text{Ser}_{z \to \infty}^{(k+1)} \frac{d}{dz} L(\{a_1, \bar{a}\}, z) + RReg_{z \to \infty} L(\{a_1, \bar{a}\}, z) \]
To prepare the next integration step, the constants logarithm always yields expressions which contain only hyperlogarithms of lower weight. Note that taking the derivative with respect to the argument or an inner variable of the hyperlogarithm are computed by taking the derivative under the integral

\[
\frac{\partial}{\partial t} L(\{a_1(t), \ldots, a_n(t)\}, z) = \int_z^{z_1} \int_{\text{Reg}(0)}^z dz_1 \int_{\text{Reg}(0)}^{z_1} dz_2 \cdots \int_{\text{Reg}(0)}^{z_{n-1}} dz_n \prod_{i=0}^n \frac{\partial}{\partial t} \frac{1}{z_i - a_i(t)} .
\]  

(573)

Note that taking the derivative with respect to the argument or an inner variable of the hyperlogarithm always yields expressions which contain only hyperlogarithms of lower weight. To prepare the next integration step, the constants

\[
c^{(\infty)}_{0,0} (\{a_1, \ldots, a_n\}) = \text{RReg}_{y \to \infty} L(\{a_1, \ldots, a_n\}, y)
\]

(574)

have to be rewritten in terms of hyperlogarithms, such that the next integration variable does not appear in the respective alphabet. These constants are rewritten bydifferentiating, rewriting the (now weight reduced) expression and then undoing the differentiation again. Consider for example \(c^{(\infty)}_{0,0} (-x, -1) = \text{RReg}_{y \to \infty} L(\{-x, -1\}, y)\). From

\[
\text{RReg}_{y \to \infty} \frac{\partial}{\partial x} L(\{-x, -1\}, y) = \text{RReg}_{y \to \infty} L(\{-x\}, y) \frac{y+1}{x-1} - \frac{y+1}{x-1}(x+y)
\]

(575)

follows, that

\[
c^{(\infty)}_{0,0} (-x, -1) = \int_0^x \text{RReg}_{y \to \infty} \frac{\partial}{\partial x'} L(\{-x', -1\}, y) + \text{RReg}_{x \to 0} \text{RReg}_{y \to \infty} L(\{-x, -1\}, y)
\]

\[
= \int_0^x dx \frac{L(\{0\}, x')}{x'-1} + \text{RReg}_{y \to \infty} L(\{0, -1\}, y)
\]

(576)

\[
= -L(\{1, 0\}, x) + \zeta_2 .
\]

(577)

Special care has to be taken when evaluating constants \(c^{(\infty)}_{0,0} (a_1, \ldots, a_n)\) which contain letters of the form \(x^{-i} f(x)\) with \(f(x) \neq 0\) as \(x \to 0\) or trailing letters of the form \(x^i f(x)\) with \(f(x)\)}
convergent as \( x \to 0 \). In all other cases \( \text{RReg}_{x \to 0} L (a_1, \cdots, a_n, y) \) is just obtained by taking the limit \( x \to 0 \) under the integral. In the first case the limit \( x \to 0 \) does not commute with the limit \( y \to \infty \). If a hyperlogarithm does not have any trailing zero in its word we may substitute the integration variables \( z_i \to az_i \) in (559) to obtain

\[
L (\{a_1, \cdots, a_n\}, z) = L (\{aa_1, \cdots, aa_n\}, az) . \tag{578}
\]

In other cases trailing zeros have to be removed by means of the shuffle algebra first, e.g.

\[
L (\{a_1, 0, 0\}, z) = L (\{a_1, z\} L (\{0, 0\}, z) - L (\{0\}, z) L (\{0, a_1\}, z)
+ L (\{aa_1, 0, 0\}, az) , \tag{579}
\]

after using (578), (558) and (560). Applying (578), resp. (579), we obtain

\[
c^{(\infty)}_{0,0} (x^{-i} f_1 (x), \cdots, f_n (x)) = \text{RReg}_{y \to \infty} L (x^{-i} f_1 (x), \cdots, f_n (x), y) \\
= \text{RReg}_{y \to \infty} L (f_1 (x), \cdots, x^i f_n (x), yx^i) \\
= \text{RReg}_{y \to \infty} \left[ \text{Ser}^{(0)}_{z \to \infty} L (f_1 (x), \cdots, x^i f_n (x), z) \right]_{z = yx^i} \\
= \left[ \text{Ser}^{(0)}_{z \to \infty} L (f_1 (x), \cdots, x^i f_n (x), z) \right]_{z = x^i} . \tag{580}
\]

By definition \( \text{Ser}^{(0)}_{z \to \infty} L (f_1 (x), \cdots, x^i f_n (x), z) \) does only depend logarithmically on the variable \( z = yx^i \) and the operation \( \text{RReg}_{y \to \infty} \) in the second last step is easily performed.

In the case of trailing letters of the shape \( x^i f (x) \) with \( f (x) \) convergent as \( x \to 0 \), the limit \( x \to 0 \) does not commute with the implicit limits contained in the definition of the hyperlogarithm. Here we apply the identity

\[
\text{RReg}_{x \to 0} L (\{x^{i1} f_1 (x), \cdots, x^{in} f_n (x)\}, y) = \text{RReg}_{x \to 0} L (\{x^{i1-1} f_1 (x), \cdots, x^{in-1} f_n (x)\}, \frac{y}{x}) \\
= \text{Ser}^{(0)}_{y \to \infty} \text{RReg}_{x \to 0} L (\{x^{i1-1} f_1 (x), \cdots, x^{in-1} f_n (x)\}, y) \tag{581}
\]

on the parts containing the respective letters, if necessary repeatedly. The identity (581) is derived by considering the change of integration variables \( z_i \to \frac{2}{x} \) in (559).

As an example we consider

\[
\text{RReg}_{x \to 0} L (\{-2, -\frac{x}{2}\}, y) = \text{RReg}_{x \to 0} \int_0^y \frac{dz_1}{z_1 + 2} \text{RReg}_{x \to 0} L (\{-\frac{1}{2}, \frac{z_1}{x}\}) \\
= \int_0^y \frac{dz_1}{z_1 + 2} \left[ \text{Ser}^{(0)}_{z_1 \to \infty} L (\{-\frac{1}{2}, z_1\}) \right] \\
= \int_0^y \frac{dz_1}{z_1 + 2} \ln 2 + L (\{0\}, z_1) \\
= L (\{-2\}, y) \ln 2 + L (\{-2, 0\}, y) . \tag{582}
\]

We repeat the previous steps for all further integration variables until we have rewritten all constants in a way suitable for the following integration step. These can then be performed analogously.
7.4 Treatment of local operator insertions

In all the different operators we encounter polynomials with symbolic integer exponents, which cannot be treated directly with the method of parametric integration. We thus introduce a generating function [151], by

\[ OP_i(\alpha_i, t) = \sum_{n=0}^{\infty} t^n OP_i(\alpha_i, n) . \]  

The function \( OP_i(\alpha_i, N) \) can be re-gained by taking the \( N \)th coefficient of the generating function. We obtain the following generating functions for the three different operators in Figure 15:

\[
OP_1(\alpha_i, N) = \frac{\Psi_G}{\Psi_G - t\Psi_{\alpha_i}^{i+1}} \]

\[
OP_2(\alpha_i, N) = \sum_{N=0}^{\infty} \frac{1}{\Psi_G^N} \sum_{m=0}^{N} \left( \Psi_G^{i+1} \right)^{N-m} \left( \Psi_G^{j+1} \right)^m 
= \sum_{N=0}^{\infty} t^N \frac{\Psi_G^{i+1}}{\Psi_G^{i+1} - \Psi_G^{j+1}} \left( \frac{\Psi_G^{i+1}}{\Psi_G^{i+1} - \Psi_G^{j+1}} \right) \]

\[
OP_3(\alpha_i, N) = \sum_{N=0}^{\infty} \frac{1}{\Psi_G^N} \sum_{m=0}^{N} \sum_{n-m+1}^{N-2} \left( \Psi_G^{i+1} \right)^m \left( \Psi_G^{j+1} \right)^{N-n-2} \]

\[
\times \left[ C_1 \left( \Psi_G^{i+1} + \Psi_G^{j+1} \right)^{n-m-1} + C_2 \left( \Psi_G^{i+1} + \Psi_G^{k+1} \right)^{n-m-1} \right] \]

\[
= \left( \frac{\Psi_G}{\Psi_G - t\Psi_G^{i+1}} \right) \left( \frac{\Psi_G}{\Psi_G - t\Psi_G^{j+1}} \right) \left[ C_1 \left( \Psi_G^{i+1} + \Psi_G^{j+1} \right)^{n-m-1} + C_2 \left( \Psi_G^{i+1} + \Psi_G^{k+1} \right)^{n-m-1} \right] \]

For fixed values of \( N \) all massive 3-loop QCD two-point functions with operators insertions are linear reducible. If we introduce one of the generating functions (584-586) this changes drastically. Some diagrams remain linear reducible, others could be transformed into linear reducible diagrams via a variable transformation and in other cases we could not find a way to restore linear reducibility.

After evaluating all \( \alpha \)-parameter integrals we obtain the generating expression

\[
\hat{I}_1(x) = \frac{1}{(1+N)(2+N)x} \left[ 2L_{-1}(x) - 2(-1+2x)L_1(x) - 4L_{-1,1}(x) \right] \zeta_3 
- 3L_{-1,0,0,1}(x) + 2L_{-1,0,1,1}(x) - 2xL_{0,0,1,1}(x) + 3xL_{0,1,0,1}(x) 
\]
for the Benz–diagram in Figure 17. One applies the package HarmonicSums [175,177–179] to obtain the Nth coefficient. It is given by

$$I_1(N) = \frac{1}{(N+1)(N+2)(N+3)} \left\{ \frac{P_1}{(1+N)^3(2+N)^3(3+N)^3} - \frac{2((-1)^N N + (-1)^N N)}{(1+N)} - \frac{2((-1)^N S_{-3}}{N(1+N)} - \frac{N}{6(1+N)} S_3 - \frac{1}{24} S_4^4 - \frac{1}{4} S_4 \\
- \frac{2((7+22N+10N^2)}{2(1+N)^2(2+N)} S_2 - \frac{19}{8} S_2^2 - \frac{1+4N+2N^2}{2(1+N)^2(2+N)} S_1^2 + \frac{9}{4} S_2 S_1 - \frac{(-9+4N)}{3(1+N)} S_3 \\
-2((-1)^N S_{-2,1} + \frac{(-1+6N)}{(1+N)} S_{2,1} + \frac{P_2}{(1+N)^3(2+N)^2(3+N)^2 S_1} \\
+4\zeta_3 S_1 - \frac{(-2+7N)}{2(1+N)} S_2 S_1 + \frac{13}{3} S_3 S_1 - 7S_{2,1} S_1 - 7S_{3,1} + 10S_{2,1,1} \right\},$$  

(588)

with the polynomials

$$P_1 = 648 + 1512N + 1458N^2 + 744N^3 + 212N^4 + 32N^5 + 2N^6$$

(589)

$$P_2 = 54 + 207N + 246N^2 + 130N^3 + 32N^4 + 3N^5.$$ 

(590)

The corresponding x-space representations needed in the analysis of experimental data is obtained by the inverse Mellin transform for the corresponding structure functions represented in N-space, including their QCD-evolution in analytic form. The transform is obtained by a single numerical integral around all the singularities of the complex N-space representation [302–305]. The latter is obtained for large values of N by the asymptotic representation. For all other values, outside the singularities, one uses the algebraic shift relations of the corresponding sums from N+1 → N. The latter relations are given in Appendix D. In the following we list also the asymptotic expansions for the individual diagrams. It will turn out later that their derivation needs the application of more and more sophisticated techniques corresponding to the growing complexity of the respective sums. Using the corresponding algorithms in HarmonicSums [175,177–179] one obtains following asymptotic representation for \(|N| \to \infty\)

$$I_1^{asy}(N) \approx \left( \frac{1}{24N^3} - \frac{1}{4N^4} + \frac{25}{24N^5} - \frac{15}{4N^6} + \frac{301}{24N^7} - \frac{161}{4N^8} + \frac{3025}{24N^9} - \frac{1555}{4N^{10}} \right) \ln^4(\tilde{N})$$

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The same diagram with the operator on a different quark line cannot be represented in

\[
I_2(N) = \frac{1}{(N+1)(N+2)(N+3)} \left\{ \frac{2}{(N+1)^3(N+2)} \right\} - \frac{4}{N+1} \left( \frac{2}{N+1} \right) \zeta_3
\]

\[
+ \frac{1}{2(N+1)(N+2)} S_1^2 - \frac{1}{2} S_3^3 + \frac{(-1 + 9N + 4N^2)}{2(N+1)^2(N+2)} S_2 - \frac{5}{2}(N+2) S_2^2
\]

where \( N = N \exp(\gamma_E) \), with \( \gamma_E \) the Euler-Mascheroni constant.

Figure 18: The 3-loop Benz diagram for \( I_2(N) \), Eq. (592).

The same diagram with the with the operator on a different quark line cannot be represented in terms of harmonic sums, but requires also generalized harmonic sums, see Appendix D.3
Figure 19: The 3-loop Benz diagram for $I_3(N)$, Eq. (594).

$-3S_3 - \frac{3(N+2)}{2}S_4 - \frac{(5+3N)}{N+1}S_{2,1} - \frac{N^2 - 3}{(N+1)^3(N+2)}S_1 + 4(N+2)S_1\zeta_3$

$-\frac{7}{2}S_1S_2 - 2(N+2)S_1S_{2,1} + 2(N+2)S_{3,1} + 2^{4+N}S_{1,2}\left(\frac{1}{2}, 1\right)$

$+ 4(N+2)S_{2,1,1} + 2^{3+N}S_{1,1,1}\left(\frac{1}{2}, 1, 1\right)$ \quad (592)

Additionally terms of $O(2^N)$ are observed, which cancel in the asymptotic expansion

$I_2^{asy}(N) = \left(\frac{1}{2N^3} + \frac{3}{N^4} - \frac{25}{2N^5} + \frac{45}{N^6} - \frac{301}{2N^7} + \frac{483}{N^8} - \frac{3025}{2N^9} + \frac{4665}{N^{10}}\right)\ln^3(\tilde{N})$

$+ \left(\frac{-19}{4N^2} + \frac{297}{8N^5} - \frac{196}{N^6} + \frac{72289}{80N^7} - \frac{163837}{40N^8} + \frac{6772187}{336N^9} - \frac{6652459}{56N^{10}}\right)\ln^2(\tilde{N})$

$+ \left(\frac{-2}{N^3} + \frac{14}{N^4} - \frac{72N^5}{72N^6} + \frac{33071}{72N^7} - \frac{17131999}{7200N^8} + \frac{22857919}{1800N^9} - \frac{1113784177}{14700N^{10}}\right)\ln(\tilde{N})$

$+ \left(\frac{19063098643}{35280N^{10}}\right)\ln(\tilde{N}) - \frac{4}{N^3} + \frac{35}{N^4} - \frac{4181}{2N^5} - \frac{432N^6}{108N^5} - \frac{24331}{432N^6} + \frac{16232209}{12000N^7}$

$- \frac{663086111}{72000N^8} + \frac{1575813188009}{16464000N^9} - \frac{483184825009}{592704N^{10}}$

$+ \left(\frac{-7}{2N^3} + \frac{21}{N^4} - \frac{175}{2N^5} + \frac{315}{N^6} - \frac{2107}{2N^7} + \frac{3381}{N^8} - \frac{21175}{2N^9} + \frac{32655}{N^{10}}\right)\ln(\tilde{N})$

$+ \frac{3}{N^3} - \frac{133}{4N^4} + \frac{4819}{24N^5} - \frac{1945}{2N^6} - \frac{347613}{80N^7} - \frac{783477}{40N^8} - \frac{490035913}{5040N^9} - \frac{97672721}{168N^{10}}\zeta_2$

$+ \left(\frac{3}{N^3} - \frac{18}{N^4} - \frac{75}{N^5} - \frac{270}{N^6} + \frac{903}{N^7} - \frac{2898}{N^8} + \frac{9075}{N^9} - \frac{27990}{N^{10}}\right)\zeta_3 + \left(\frac{27}{10N^2}\right)^2$

$- \frac{54}{5N^3} - \frac{351}{10N^4} - \frac{108}{N^5} - \frac{3267}{10N^6} + \frac{4914}{5N^7} + \frac{29511}{10N^8} - \frac{8856}{N^9} + \frac{265707}{10N^{10}}\zeta_2$

$+ O\left(\frac{\ln^3(\tilde{N})}{N^{11}}\right), \quad (593)$

however, and we obtain a regular representation for $|N| \to \infty$. Considering the same diagram with the operator on the lower fermionic line demonstrates how strongly the complexity of the
The result depends on the location of the local operator insertion. The diagram in Figure 19 yields

\[ I_3(N) = \frac{1}{(N+1)(N+2)^2} \left\{ \frac{4}{(N+1)^2(N+2)} - \frac{4S_1}{(N+2)} + 4S_2 \right\}, \tag{594} \]

with the asymptotic representation

\[ I_3^{asy}(N) = \left( -\frac{4}{N^4} + \frac{28}{N^5} - \frac{124}{N^6} + \frac{444}{N^7} - \frac{1404}{N^8} + \frac{4092}{N^9} - \frac{11260}{N^{10}} \right) \ln(N) \]

\[ + \frac{20}{N^4} + \frac{181}{3N^5} - \frac{133}{N^6} + \frac{2009}{10N^7} - \frac{1297}{30N^9} + \frac{728377}{630N^{10}} \]

\[ + \left( \frac{4}{N^3} - \frac{20}{N^4} - \frac{68}{N^5} - \frac{196}{N^6} + \frac{516}{N^7} - \frac{1284}{N^8} + \frac{3076}{N^9} - \frac{7172}{N^{10}} \right) \zeta_2 + O\left( \frac{\ln(N)}{N^{11}} \right). \tag{595} \]

Figure 20: The 3-loop Benz diagram for \( I_4(N) \), Eq. (596).

Also diagrams which do not contribute in the direct QCD computation but which are typical for master integrals after an integration by parts reduction \([156,157,306-309] \) like the diagrams in Figures 20 and 21 are computable with the method of hyperlogarithms.

Diagram 4 can be expressed completely within the functional class of generalized harmonic sums

\[ I_4(N) = \frac{1}{(N+1)(N+2)} P_3 \left\{ \frac{P_3}{(N+1)(N+2)} \zeta_3 \right\} \]

\[ + \frac{1}{N+2} S_{-3} + \frac{(-1)^N}{2(N+2)} S_{1}^3 - \frac{(-1)^N(3+2N)}{2(N+1)^2(N+2)} S_2 + \frac{5(-1)^N}{2} S_2^2 \]

\[ + \frac{(-1)^N(3+2N)}{2(N+1)^2(N+2)} S_2^3 - \frac{(-1)^N}{2} S_2 S_1^2 + \frac{3(-1)^N(4+3N)}{(N+1)(N+2)} S_3 + 3(-1)^N S_4 \]

\[ + \frac{2}{(N+2)} S_{-2,1} + 2(-1)^N \zeta_3 S_1(2) + \frac{2(-1)^N(3+N)}{(N+1)(N+2)} S_{2,1} - 12(-1)^N S_1 \zeta_3 \]

\[ + \frac{(-1)^N(5+7N)}{2(N+1)(N+2)} S_1 S_2 + 3(-1)^N S_1 S_3 + 4(-1)^N S_{2,1} S_1 - 4(-1)^N S_{3,1} \]

\[ - 4 \left\{ \left(-1\right)^N 2^{2+N} - 3\left(-2\right)^N N + 3\left(-1\right)^N 2^{1+N} N \right\} S_{1,2} \left( \frac{1}{2}, 1 \right) \]

\[ + \frac{2 \left(-1\right)^N 2^{2+N} - 13\left(-2\right)^N N + 5\left(-1\right)^N 2^{1+N} N}{(N+1)(N+2)} S_{1,1,1} \left( \frac{1}{2}, 1, 1 \right) \]

\[ - 2\left(-1\right)^N S_{1,1,2} \left( 2, \frac{1}{2}, 1 \right) + \left(-1\right)^N S_{1,1,1,1} \left( 2, \frac{1}{2}, 1, 1 \right) - 5\left(-1\right)^N S_{2,1,1} \right\}, \tag{596} \]
\[ P_3 = 2 \left( 1 - 13(-1)^N + (-1)^N 2^{3+N} + N - 7(-1)^N N + 3(-1)^N 2^{1+N} N \right). \]

For \(|N| \to \infty\) \(I_4\) obeys the expansion

\[
I_4^{\text{asy}}(N) = (-1)^N \left\{ \begin{array}{c}
-\frac{1793}{2N^{10}} + \frac{769}{2N^9} - \frac{321}{2N^8} + \frac{129}{2N^7} - \frac{49}{2N^6} - \frac{17}{2N^5} - \frac{5}{2N^4} + \frac{1}{2N^3} \\
+ \left[ -\frac{3}{2N^3} + \frac{21}{2N^4} - \frac{363}{8N^5} + \frac{1323}{8N^6} - \frac{9389}{16N^7} + \frac{183573}{80N^8} - \frac{538097}{48N^9} + \frac{12345081}{1680N^{10}} \right] \\
\times \ln^2(\bar{N}) + \left[ -\frac{1}{2N^2} + \frac{3}{2N^3} + \frac{7}{2N^4} + \frac{15}{2N^5} + \frac{31}{2N^6} + \frac{63}{2N^7} - \frac{127}{2N^8} \\
+ \frac{255}{2N^9} - \frac{511}{2N^{10}} \right] \ln(\bar{N}) + \left[ -\frac{5}{N^3} + \frac{285}{8N^4} - \frac{3887}{24N^5} + \frac{181091}{288N^6} - \frac{1151603}{480N^7} \right. \\
+ \frac{792381}{720N^8} - \frac{14793223}{280N^9} + \frac{217689527539}{604800N^{10}} \right] \zeta_2 + \left[ -\frac{1}{N^2} + \frac{3}{N^3} - \frac{7}{N^4} + \frac{15}{N^5} - \frac{31}{N^6} \\
\frac{3}{N^7} - \frac{127}{N^8} + \frac{255}{N^9} - \frac{511}{N^{10}} \right] \ln(\bar{N}) + \left[ -\frac{3}{N^3} + \frac{67}{12N^4} - \frac{59}{4N^5} + \frac{1363}{40N^6} - \frac{2949}{40N^7} \right. \\
+ \frac{388153}{2520N^8} - \frac{53027}{168N^9} + \frac{460691}{720N^{10}} \right] \zeta_3 + \left[ -\frac{12}{5N^2} + \frac{36}{5N^3} - \frac{84}{5N^4} + \frac{36}{5N^5} - \frac{372}{5N^6} + \frac{756}{5N^7} \right. \\
- \frac{1524}{5N^8} + \frac{612}{N^9} - \frac{6132}{5N^{10}} \right] \zeta_2^2 + \left[ -\frac{4}{N^3} + \frac{49}{4N^4} + \frac{181}{216N^5} + \frac{27119}{144N^6} - \frac{4022219}{27000N^7} \right. \\
+ \frac{1251907}{125N^8} - \frac{10792338497459}{148176000N^9} + \frac{18342053050631}{29635200N^{10}} \right] \right\} + O \left( \frac{\ln^3(\bar{N})}{N^{11}} \right). \]  

(598)

Figure 21: The 3-loop Benz diagram for \(I_5(N)\), Eq. (599).

Although diagrams 4 and 5 are topologically very similar one obtains a more simple sum structure in the latter case.

\[
I_5(N) = \frac{(-1)^N}{(N+1)(N+2)} \left\{ -\frac{2}{(1+N)} + \frac{2 + (-1)^N(2 + N)}{1 + N} \zeta_3 + \frac{3}{(N+1)^2} S_2 + \frac{5}{2} S_2^2 + \frac{3}{2} S_4 \right\}.
\]
\[ \frac{2}{(N+1)^2}S_{2,1} - \frac{2}{(N+1)^3}S_1 - 4\zeta_3 S_1 + 2S_{2,1}S_1 - 2S_{3,1} - 4S_{2,1,1} \right\}. \] (599)

The asymptotic representation for this diagram reads

\[ I_6^{\text{asy}}(N) = (-1)^N \left\{ \left( \frac{2}{N^3} - \frac{15}{2N^4} + \frac{166}{9N^5} - \frac{445}{12N^6} - \frac{59153}{900N^7} - \frac{7987}{75N^8} + \frac{1185269}{7350N^9} \right) - \frac{227247}{980N^{10}} \ln(N) + \left( -\frac{3}{N^3} + \frac{27}{2N^4} - \frac{41}{N^5} + \frac{105}{2N^6} - \frac{2449}{10N^7} + \frac{5397}{10N^8} - \frac{40158}{35N^9} \right) \right\} + \frac{16686}{7N^{10}} \zeta_2 + \left( \frac{4}{N^3} - \frac{25}{2N^4} + \frac{2885}{108N^5} - \frac{883}{18N^6} + \frac{381781}{4500N^7} - \frac{1312181}{9000N^8} \right) + \frac{475694037}{18522000N^9} - \frac{386004953}{823200N^{10}} \right\} + \left( \frac{27}{10N^2} + \frac{81}{10N^3} - \frac{189}{10N^4} + \frac{81}{2N^5} - \frac{837}{10N^6} \right) + \frac{1701}{10N^7} - \frac{3429}{10N^8} - \frac{1377}{2N^9} + \frac{13797}{10N^{10}} \right\} \zeta_2 \right\} + \left( \frac{2}{N^2} - \frac{4}{N^3} + \frac{6}{N^4} + \frac{8}{N^5} - \frac{10}{N^6} \right) \zeta_3 + O\left( \frac{\ln(N)}{N^{11}} \right). \] (600)

![Figure 22: The 3-loop Benz diagram for $I_6(N)$, Eq. (601).](image)

As a final Benz–diagram we consider the graph in Figure 22, which is consists of two color contributions due to the Feynman rule for the operator on the 4–vertex, see Appendix D. The complete expression is then given by

\[ I_6(N) = C_1 \left\{ \frac{P_4}{(N+1)^4(N+2)^3(N+3)} - (-1)^N \frac{P_5}{(N+1)^5(N+2)^3(N+3)} + 10S_{-5} \right. \]
\[ + \frac{P_5}{2(N+1)^3(N+2)^3(N+3)^2} S_{-3} + \frac{P_7}{2(N+1)^2(N+2)^2(N+3)^2} S_{-3} + \frac{4}{N+3} S_1 S_{-3} \]
\[ - \frac{P_5}{(N+1)^2(N+2)^3(N+3)^2} + \frac{P_5}{(N+1)^2(N+2)^2(N+3)^2} S_{-3} + \frac{5}{N+3} S_4 - S_5 - 2S_{-4,1} \]
\[ + \left( \frac{3(-1)^NP_9}{(N+1)^2(N+2)^3(N+3)^2} + \frac{P_{10}}{(N+1)^2(N+2)^2(N+3)^2} \right) S_3 - \frac{2}{(N+3)^2} S_{-2,1} \]
\[ - 8S_{-2,3} + \left[ (-1)^N \frac{2P_9}{(N+1)^2(N+2)^2(N+3)^2} - 4S_{-2} - 4S_2 - \frac{2(N+2)}{N+3} S_1 (2) \right] \]
\[
\begin{align*}
&\quad + \frac{2P_{11}}{(N + 1)^2(N + 2)^2(N + 3)^2} + 2^{N+2} \frac{P_{12}}{(N + 1)^2(N + 2)^2(N + 3)^2} \right) \zeta_3 - 5S_{2,-3} \\
&\quad + \left[ \frac{17 + 23N + 9N^2 + N^3}{(N + 1)(N + 2)(N + 3)^2} - (-1)^N \frac{58 + 84N + 43N^2 + 10N^3 + N^4}{(N + 1)^2(N + 2)^2(N + 3)^2} \right] S_{2,1} \\
&\quad + 2 \frac{P_{15}}{(N + 1)^3(N + 2)^3(N + 3)^2} S_2 - 2 \frac{P_{14}}{(N + 1)^3(N + 2)^3(N + 3)^2} S_2 \\
&\quad - S_3S_2 - 2S_{-2,1}S_2 + 2S_{2,1}S_2 + (-1)^N \frac{2(7 + 6N + N^2)(9 + 10N + 3N^2)}{(N + 1)^4(N + 2)^4(N + 3)^2} S_1 \\
&\quad + \left[ \frac{P_{16}}{(N + 1)^4(N + 2)^4(N + 3)^2} + (-1)^N \frac{P_{17}}{(N + 1)^5(2 + N)^5(3 + N)} \right] - 10S_{-5} \\
&\quad + \frac{38 + 45N + 16N^2 + N^3}{(N + 1)^2(2 + N)^2(3 + N)} S_{-3} - \frac{4S_1}{3 + N} S_{-3} - 3S_2S_{-3} + S_5 + 2S_{-4,1} + 8S_{-2,3} \\
&\quad + \left\{-\frac{1}{2(3 + N)} + \frac{(-1)^N}{(2 + N)(3 + N)} \right\} S_2^2 + S_3S_2 + 2S_{-2,1}S_2 + 5S_{2,-3} \\
&\quad + 2 \left[ \frac{11 + 15N + 7N^2 + N^3}{(1 + N)^2(2 + N)^2(3 + N)} - (-1)^N \frac{23 + 28N + 10N^2 + N^3}{(1 + N)^2(2 + N)^2(3 + N)} + 2S_{-2} \\
&\quad + \left( -\frac{1}{3 + N} - (-1)^N \frac{1}{(2 + N)(3 + N)} \right) S_1 + S_2 \right) \zeta_3 - \frac{(-1)^N}{2(2 + N)(3 + N)} S_1^2 S_2 \\
&\quad - \frac{2(-1)^N(5 + 6N + 2N^2)}{(1 + N)^2(2 + N)^3(3 + N)} S_2^2 + \frac{2}{(2 + N)(3 + N)} S_{-2,1} \\
&\quad + \left( \frac{9 + 10N + 3N^2}{(1 + N)^2(2 + N)^2(3 + N)} - \frac{3(-1)^N(23 + 28N + 10N^2 + N^3)}{(1 + N)^2(2 + N)^2(3 + N)} \right) S_3 \\
&\quad + \left( -\frac{3}{2(3 + N)} + \frac{3(-1)^N}{2(2 + N)(3 + N)} \right) S_4 \\
&\quad - \frac{(-1)^N(-8 - 7N + N^3)}{(1 + N)^3(2 + N)^3(3 + N)} S_2 + \frac{17 + 27N + 15N^2 + 3N^3}{(1 + N)^3(2 + N)^3(3 + N)} S_2
\end{align*}
\]
\[
\begin{aligned}
&\frac{-2(17 + 27N + 15N^2 + 3N^3)}{(1 + N)^3(2 + N)^3(3 + N)} S_{-2} + \frac{2S_2}{3 + N} S_{-2} - 2S_3 S_{-2} - 2S_{2,1} S_{-2} \\
&+ \frac{(-1)^N (23 + 28N + 10N^2 + N^3)}{(1 + N)^2(2 + N)^2(3 + N)} S_{2,1} \\
&- \frac{4(-1)^N (3 + 2N)(3 + 3N + N^2)}{(1 + N)^4(2 + N)^4(3 + N)} S_1 - \frac{(-1)^N (23 + 28N + 10N^2 + N^3)}{(1 + N)^2(2 + N)^2(3 + N)} S_1 S_2 \\
&+ \left( -\frac{1}{3 + N} - \frac{3(-1)^N}{(2 + N)(3 + N)} \right) S_3 S_1 + \frac{(-1)^N}{(2 + N)(3 + N)} S_1 S_{2,1} \\
&+ \left( \frac{1}{3 + N} + \frac{5(-1)^N}{(2 + N)(3 + N)} \right) S_{3,1} + 2S_{2,1,-2} - \frac{5(-1)^N}{(2 + N)(3 + N)} S_{2,1,1} \right \} .
\end{aligned}
\]  

(601)

Here \(C_1\) and \(C_2\) represent the group-theoretic color factors. Since we consider only scalar graphs we leave them unspecified here. The polynomials in (601) read

\[
\begin{align*}
P_1 &= -70 - 108N - 18N^2 + 49N^3 + 30N^4 + 5N^5 \quad (602) \\
P_2 &= -70 - 104N - 3N^2 + 70N^3 + 43N^4 + 8N^5 \quad (603) \\
P_3 &= 47 + 98N + 81N^2 + 30N^3 + 4N^4 \quad (604) \\
P_4 &= 61 + 136N + 123N^2 + 55N^3 + 12N^4 + N^5 \quad (605) \\
P_5 &= 112 + 168N + 89N^2 + 18N^3 + N^4 \quad (606) \\
P_6 &= 58 + 84N + 43N^2 + 10N^3 + N^4 \quad (607) \\
P_{10} &= 48 + 213N + 274N^2 + 150N^3 + 36N^4 + 3N^5 \quad (608) \\
P_{11} &= -126 - 284N - 259N^2 - 116N^3 - 25N^4 - 2N^5 \quad (609) \\
P_{12} &= 16 + 60N + 80N^2 + 47N^3 + 12N^4 + N^5 \quad (610) \\
P_{13} &= 51 + 103N + 81N^2 + 29N^3 + 4N^4 \quad (611) \\
P_{14} &= 325 + 758N + 669N^2 + 262N^3 + 38N^4 \quad (612) \\
P_{15} &= 160 + 391N + 396N^2 + 204N^3 + 52N^4 + 5N^5 \quad (613) \\
P_{16} &= 142 + 370N + 388N^2 + 203N^3 + 52N^4 + 5N^5 \quad (614) \\
P_{17} &= 142 + 374N + 403N^2 + 224N^3 + 65N^4 + 8N^5 \quad (615)
\end{align*}
\]

The asymptotic expansion of \(I_6\) is given by

\[
I_6^{\text{asy}}(N) = C_1 \left\{ \left[ \frac{1}{4N^2} - \frac{19}{12N^3} + \frac{15}{2N^4} - \frac{1889}{60N^5} + \frac{247}{2N^6} - \frac{38935}{84N^7} + \frac{3371}{2N^8} - \frac{359009}{60N^9} + \frac{41679}{2N^{10}} \right] \right. \\
\times \ln^3(\bar{N}) + \left[ \frac{1}{8N^2} + \frac{23}{12N^3} - \frac{223}{12N^4} + \frac{45229}{400N^5} - \frac{280379}{480N^6} + \frac{66622583}{23520N^7} - \frac{23133233}{1680N^8} \right. \\
+ \frac{724473271}{10080N^9} - \frac{2931192779}{6720N^{10}} \left\} \ln^2(\bar{N}) + \left( -\frac{1}{N} \right)^N \left[ \frac{1}{N^7} - \frac{21}{N^8} + \frac{242}{N^9} - \frac{1998}{N^{10}} \right] \\
- \left[ \frac{7}{8N^2} + \frac{95}{18N^3} - \frac{3371}{288N^4} - \frac{69017}{2000N^5} + \frac{8462677}{14400N^6} - \frac{7789424551}{1646400N^7} + \frac{323933401}{9800N^8} \\
- \frac{247879811629}{1058400N^9} + \frac{3111216830509}{1693440N^{10}} \right] \ln(\bar{N}) + \left[ \frac{24}{5} + \frac{43}{10N} - \frac{129}{10N^2} + \frac{387}{10N^3} \right]
\]
\]

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\[
- \frac{1161}{10N^4} + \frac{3483}{10N^5} - \frac{10449}{10N^6} + \frac{31347}{10N^7} - \frac{94041}{10N^8} + \frac{282123}{10N^9} - \frac{846369}{10N^{10}} \zeta_2
+ (-1)^N \left[ \frac{1}{N^4} + \frac{1}{4N^5} - \frac{884}{9N^6} + \frac{14041}{24N^7} - \frac{1768501}{600N^8} + \frac{657507}{50N^9} - \frac{262301037}{4900N^{10}} \right]
+ \left[ (-1)^N \left[ -\frac{3}{N^3} + \frac{27}{N^4} - \frac{159}{N^5} + \frac{765}{N^6} - \frac{3249}{N^7} + \frac{12663}{N^8} - \frac{64443}{N^9} + \frac{163377}{N^{10}} \right] - \frac{3}{2N^2} \right]
+ \frac{19}{2N^3} \left[ N^4 + \frac{10N^5}{2N^3} - \frac{741}{14N^7} + \frac{38935}{10N^8} - \frac{10113}{10N^9} - \frac{359009}{-125037} \zeta_3 \right] + \frac{1}{2N^2}
+ \left[ (-1)^N \left[ -\frac{5}{2N^4} + \frac{295}{12N^5} - \frac{605}{4N^6} + \frac{89029}{120N^7} - \frac{127147}{40N^8} + \frac{31520947}{2520N^9} - \frac{2616665}{56N^{10}} \right]
+ \left[ (-1)^N \left[ -\frac{1}{N^3} + \frac{9}{N^4} - \frac{53}{N^5} + \frac{255}{N^6} - \frac{1083}{N^7} + \frac{4221}{N^8} - \frac{15481}{N^9} + \frac{54459}{N^{10}} \right] + \frac{7}{4N^2} \right]
- \frac{133}{12N^3} + \frac{105}{2N^4} - \frac{13223}{60N^5} + \frac{1729}{2N^6} - \frac{38935}{12N^7} + \frac{23597}{2N^8} - \frac{2513063}{60N^9} + \frac{291753}{2N^{10}} \ln(\bar{N})

+ 3\zeta_3 - \frac{11}{8N^2} + \frac{569}{36N^3} - \frac{1225}{12N^4} + \frac{216201}{400N^5} - \frac{1261231}{480N^6} + \frac{125654423}{1008N^7} - \frac{306787391}{5040N^8}
+ \frac{9847032577}{30240N^9} - \frac{13758651023}{6720N^{10}} \right]
+ \zeta_5 - \frac{31}{16N^2} + \frac{2153}{216N^3} - \frac{5735}{128N^4} + \frac{40340069}{18000N^5}
- \frac{542992637}{43200N^6} + \frac{659641453013}{8643600N^7} - \frac{7397109902939}{148176000N^8} + \frac{96209004292051}{2667168000N^9}
- \frac{330634683598931}{111132000N^{10}} \right)
+ C_2 \left\{ (-1)^N \left[ -\frac{2}{N^3} + \frac{15}{N^4} + \frac{226}{3N^5} - \frac{950}{3N^6} + \frac{18049}{15N^7} - \frac{12859}{3N^8} + \frac{511284}{35N^9} - \frac{337628}{7N^{10}} \right]
\times \ln^2(\bar{N}) + (-1)^N \left[ \frac{4}{N^3} - \frac{20}{N^4} + \frac{581}{9N^5} - \frac{2879}{18N^6} + \frac{132043}{450N^7} - \frac{39521}{180N^8} - \frac{1617779}{1225N^9}
- \frac{41782189}{4410N^{10}} \right] \ln(\bar{N}) + \left[ (-1)^N \left[ -\frac{19}{10N^2} + \frac{19}{2N^3} - \frac{361}{10N^4} + \frac{247}{2N^5} - \frac{409}{10N^6} + \frac{2527}{2N^7}
- \frac{39121}{10N^8} + \frac{23959}{2N^9} - \frac{364249}{10N^{10}} \right] - \frac{8}{5N} + \frac{24}{5N^2} - \frac{72}{5N^3} + \frac{216}{5N^4} - \frac{648}{5N^5} + \frac{1944}{5N^6}
- \frac{5832}{5N^7} + \frac{17496}{5N^8} - \frac{52488}{5N^9} + \frac{157464}{5N^{10}} \right] \zeta_2^2 + (-1)^N \left[ \frac{5}{N^3} - \frac{227}{8N^4} + \frac{3259}{27N^5} - \frac{395983}{864N^6}
+ \frac{1296603}{800N^7} - \frac{488729}{90N^8} + \frac{6474303661}{3704400N^9} - \frac{40570237223}{740880N^{10}} \right] + (-1)^N \left[ -\frac{1}{2N^2} + \frac{5}{2N^3} \right] \right\}
\]
One of the diagrams that could first be evaluated using this method is the ladder diagram in figure 23. Very recent implementations in Sigma [236, 237] do now allow to solve the physical graphs of this kind using integration by parts and differential equation techniques [310].

The generating function is given in terms of hyperlogarithms over the alphabet \( \{1/2, 0, 1, -2\} \),

\[
\hat{I}_\tau(x) = \left[ \frac{1 + x}{x^3} \right] L_{-1}(x) - \frac{2x - 1}{x^3} L_{1/2}(x) - \frac{3(1-x)}{x^3} L_1(x) - \frac{1 - 2x + x^2}{(1-x)x^3} L_{0,-1}(x)
\]

\[
+ \frac{1 - 2x^2}{x^3} L_{0,1/2}(x) - \frac{3 - 4x - 3x^2 + 3x^3}{(1-x)x^3} L_{0,1}(x) - \frac{1 - 2x^2}{x^3} L_{1,1/2}(x)
\]

\[
+ \frac{(1-x)(2+3x)}{x^3} L_{1,1}(x) \right] \zeta_3
\]

\[
+ \left( \frac{1}{2x^3} \right) \left[ 3L_{-1,0,0,1}(x) - 2L_{-1,0,1,1}(x) - 3L_{1,0,0,1}(x) \right] + \frac{1}{x^2} \left[ 6L_{0,0,1,1}(x) - 4L_{0,1,0,1}(x) \right]
\]

Again this series is regular as \( |N| \to \infty \).

Figure 23: The ladder diagram for \( I_\tau(N) \), Eq. (618).
\[-L_{0,1,1,1}(x) = \frac{(-1 + 2x)}{2x^3} \left[ 3L_{1/2,0,0,1}(x) - L_{1/2,0,1,1}(x) - 3L_{1/2,1,0,1}(x) + L_{1/2,1,1,1}(x) \right] \]

\[-\frac{3}{2x^2}L_{1,0,1,1}(x) + \frac{2}{x^2}L_{1,1,0,1}(x) - \frac{(-1 + x)}{2x^3}L_{1,1,1,1}(x) + \frac{2}{x^2} \left[ L_{0,1,1} - L_{0,1,0} \right] \]

\[\frac{5}{-1 + x}L_{0,0,0,1,1}(x) - \frac{5}{2(-1 + x)}L_{0,0,1,0,1}(x) + \frac{3(3 + x)}{2(-1 + x)}xL_{0,0,1,1,1}(x) \]

\[-\frac{(-1 + 2x^2)}{2x^3} \left[ 3L_{0,1/2,0,0,1}(x) - 2L_{0,1/2,0,1,1}(x) \right] \]

As in generally observed expressions stemming from generating functions over this alphabet require harmonic sums, alternating harmonic sums and additional generalized harmonic sums for their representation. The \( N \) space result is given by

\[I_7(N) = \frac{P_1}{2(1 + N)^5(2 + N)^5(3 + N)^5} + \frac{P_2}{(1 + N)^2(2 + N)^2(3 + N)^2} \zeta_3 \]

\[+ \frac{(-1)^N(65 + 101N + 56N^2 + 13N^3 + N^4)}{2(1 + N)^2(2 + N)^2(3 + N)^2}S_{-3} + \frac{(-24 - 5N + 2N^2)}{12(2 + N)^2(3 + N)^2}S_1^3 \]

\[-\frac{1}{2(1 + N)(3 + N)}S_2^2 + \frac{1}{(2 + N)(3 + N)}S_3^2S_2 \]

\[+ \frac{314 + 631N + 578N^2 + 288N^3 + 68N^4 + 5N^5}{4(1 + N)^3(2 + N)^2(3 + N)^2}S_1^2 - \frac{3}{2}S_5 \]

\[-\frac{(399 + 2069N + 2774N^2 + 1510N^3 + 349N^4 + 27N^5)}{6(1 + N)^2(2 + N)^2(3 + N)^2}S_3 - 2S_{-2, -3} \]

\[-2\zeta_3S_{-2} - S_{-2, 1}S_{-2} + \frac{(-1)^N(65 + 101N + 56N^2 + 13N^3 + N^4)}{(1 + N)^2(2 + N)^2(3 + N)^2}S_{-2, 1} \]

\[+ \frac{(59 + 42N + 6N^2)}{2(1 + N)(2 + N)(3 + N)}S_4 + \frac{(5 + N)}{(1 + N)(3 + N)}\zeta_3S_1(2) \]

\[\frac{752 + 2087N + 2490N^2 + 1580N^3 + 558N^4 + 105N^5 + 8N^6}{4(1 + N)^3(2 + N)^2(3 + N)^2}S_2 - \zeta_3S_2 \]

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\[- \frac{3}{2} S_3 S_2 - 2 S_{2,1} S_2 + \frac{(99 + 225 N + 190 N^2 + 65 N^3 + 7 N^4)}{2(1 + N)^2(2 + N)^2(3 + N)} S_{2,1} P_3 (1 + N)^4 (2 + N)^4 (3 + N)^4 \]
\[- \frac{1}{(1 + N)(2 + N)(3 + N)} S_1 - \frac{(1 + N)}{(1 + N)(2 + N)(3 + N)} S_{2,1} S_1 S_3 S_1 - \frac{(53 + 29 N)}{2(1 + N)(2 + N)(3 + N)} S_{3,1} - \frac{(79 - 40 N + N^2)}{2(1 + N)(2 + N)(3 + N)} S_{3,1} - \frac{(7 - 2 N^2)}{(1 + N)(2 + N)(3 + N)} S_{2,1,1} S_{1,2} \left( \frac{1}{2}, 1 \right) - \frac{(7 - 2 N^2)}{(1 + N)(2 + N)(3 + N)} S_{2,1,1} S_{1,1} \left( \frac{1}{2}, 1, 1 \right) \]
\[- \frac{(5 + N)}{(1 + N)(3 + N)} S_{1,1,2} \left( 2, \frac{1}{2}, 1 \right) - \frac{(5 + N)}{2(1 + N)(3 + N)} S_{1,1,1,1} \left( 2, \frac{1}{2}, 1, 1 \right), \]

with
\[ P_1 = -31104 - 159408 N - 353808 N^2 - 446652 N^3 - 353808 N^4 - \ln(N^2) - 13044 N^7 - 1584 N^8 - 84 N^9, \]
\[ P_2 = -105 + 65(-1)^N + 7 \times 2^{1+N} - 150 N + 101(-1)^N N + 39 \times 2^{-1+N} N - 73 N^2 + 56(-1)^N N^2 - 33 \times 2^{1+N} N^2 - 12 N^3 + 13(-1)^N N^3 + 2^{2+N} N^3 + (-1)^N N^4 - 2^{1+N} N^4, \]
\[ P_3 = 5436 + 29004 N + 67285 N^2 + 89175 N^3 + 74616 N^4 + 41120 N^5 + 15107 N^6 + 3659 N^7 + 562 N^8 + 50 N^9 + 2 N^{10}, \]

In order to show that the $2^N$-terms cancel as $N \to \infty$, we calculate the asymptotic representation of $I_7$:

\[ I_7^{asy}(N) = \left[ \frac{1115231}{20 N^{10}} - \frac{74121}{4 N^9} + \frac{122951}{20 N^8} - \frac{40677}{20 N^7} + \frac{13391}{20 N^6} - \frac{40677}{4 N^5} + \frac{13391}{20 N^4} - \frac{417}{20 N^3} + \frac{101}{20 N^2} \right] \frac{1}{N^2} \ln(\tilde{N}) \]
\[ + \left[ \frac{95855}{2 N^{10}} + \frac{31525}{2 N^9} - \frac{10295}{2 N^8} + \frac{3325}{2 N^7} - \frac{1055}{2 N^6} + \frac{325}{2 N^5} - \frac{95}{2 N^4} + \frac{25}{2 N^3} - \frac{5}{N^2} \right] \ln^2(\tilde{N}) \]
\[ + \left[ \frac{19171}{N^{10}} - \frac{6305}{N^9} + \frac{2059}{N^8} - \frac{665}{N^7} + \frac{211}{N^6} - \frac{65}{N^5} + \frac{19}{N^4} - \frac{5}{N^3} + \frac{1}{N^2} \right] \ln(\tilde{N}) + \frac{1}{N^2} \ln(\tilde{N}) + \frac{29993001621}{302400 N^{10}} - \frac{4402272031}{302400 N^9} + \frac{22261739}{840 N^8} - \frac{78507473}{14112 N^7} + \frac{180961}{144 N^6} \]
\[- \frac{1}{N^2} \ln(\tilde{N}) + \frac{29993001621}{302400 N^{10}} - \frac{4402272031}{302400 N^9} + \frac{22261739}{840 N^8} - \frac{78507473}{14112 N^7} + \frac{180961}{144 N^6} \]
\[- \frac{1}{N^2} \ln(\tilde{N}) + \frac{29993001621}{302400 N^{10}} - \frac{4402272031}{302400 N^9} + \frac{22261739}{840 N^8} - \frac{78507473}{14112 N^7} + \frac{180961}{144 N^6} \]

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Figure 24: Diagram 8. For this diagram linear reducibility can only be gained through a suitable variable transformation.

\[
\begin{align*}
&+ \left( \frac{249223}{6N^{10}} + \frac{145015}{12N^{10}} + \frac{10295}{3N^8} + \frac{11305}{12N^7} + \frac{1477}{6N^6} + \frac{715}{12N^5} + \frac{38}{3N^4} - \frac{25}{12N^3} + \frac{1}{6N^2} \right) \ln^3(N) \\
&+ \left( \frac{193493767}{10080N^{10}} + \frac{210658237}{10080N^9} - \frac{21541697}{2520N^8} + \frac{243269}{96N^7} - \frac{30539}{48N^6} + \frac{2123}{16N^5} - \frac{59}{3N^4} \right) \ln^2(N) \\
&+ \left( \frac{5}{8N^3} + \frac{1}{2N^2} \right) \ln(N) \\
&+ \left( \frac{2207364771673}{4233600N^{10}} + \frac{1390655509}{352800N^9} + \frac{285594061}{22050N^8} - \frac{67234111}{14400N^7} + \frac{8617073}{7200N^6} - \frac{35209}{144N^5} + \frac{116}{3N^4} - \frac{119}{24N^3} + \frac{1}{N^2} \right) \ln(N) \\
&+ \left( \frac{1344226725047831}{889056000N^{10}} - \frac{165849841805771}{889056000N^9} + \frac{808151260279}{27783000N^8} - \frac{708430537}{120960N^7} + \frac{304474703}{216000N^6} - \frac{606811}{1728N^5} + \frac{1867}{144N^4} + \frac{1}{N^3} + O\left( \frac{1}{N^2} \right) \right),
\end{align*}
\]

which displays a convergent behavior.

The V-type diagram in Figure 24 has the color structure

\[ I_8 = C_1 I_{8a} + C_2 I_{8b} \tag{623} \]

Here we leave the group theoretic factors \( C_1 \) and \( C_2 \) again unspecified for the scalar case. Because the two contributions \( I_{8a} \) and \( I_{8b} \) are of a very different complexity, we will treat them individually from now on.

The respective parametric integrals are given by

\[
\begin{align*}
I_{8a} &= \int_0^\infty dx_1dx_2d\alpha_1d\alpha_2d\alpha_3d\alpha_4d\alpha_7 \sum_{j_1=0}^N \sum_{j_2=j_1+1}^{N+1} (-T_2)^{j_1} (T_1)^{N+1-j_2} (T_1 + T_3)^{j_2-j_1-1} U^{N+2} M \\
&\times \theta(x_1 - \alpha_1) \theta(x_2 - \alpha_4) \tag{624} \\
I_{8b} &= \int_0^\infty dx_1dx_2d\alpha_1d\alpha_2d\alpha_3d\alpha_4d\alpha_7 \sum_{j_1=0}^N \sum_{j_2=j_1+1}^{N+1} (-T_2)^{j_1} (T_1)^{N+1-j_2} (T_1 - T_4)^{j_2-j_1-1} U^{N+2} M \\
&\theta(x_1 - \alpha_1) \theta(x_2 - \alpha_4) \tag{625},
\end{align*}
\]

where

\[
\begin{align*}
x_1 &= \alpha_1 + \alpha_6, \\
x_2 &= \alpha_4 + \alpha_5. \tag{626}
\end{align*}
\]

The different graph polynomials in Eqs. (624) and (625) read

\[
M = x_1 + x_2 + \alpha_7
\]
generalized harmonic sums with the asymptotic representation are all of the form
\[ U = -\alpha_3\alpha_2\alpha_7 - \alpha_2\alpha_7x_2 - \alpha_2x_2 + \alpha_2x_2x_1 - \alpha_3\alpha_2\alpha_1 - \alpha_3\alpha_\xi \]
\[ T_1 = -\alpha_3\alpha_7x_1 + \alpha_3\alpha_2\alpha_7 - \alpha_2\alpha_3\alpha_1 - \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_7x_2 + \alpha_2x_2x_1 - \alpha_2x_2\alpha_1 + \alpha_3\alpha_2\alpha_1 + \alpha_3\alpha_2\alpha_1 + \alpha_3\alpha_\xi \]
\[ T_2 = -\frac{3\alpha_7\alpha_4\alpha_2 - \alpha_3\alpha_2\alpha_7 + \alpha_2\alpha_3\alpha_1 + \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_7x_2 - \alpha_2x_2x_1 + \alpha_2x_2\alpha_1 - \alpha_3\alpha_2\alpha_1 - \alpha_3\alpha_\xi}{x_1 - \alpha_7x_2x_1 + \alpha_\xi x_1 - \alpha_3x_2x_1 + \alpha_3\alpha_4x_1 - \alpha_3\alpha_\xi} \]
\[ T_3 = \alpha_7x_2\alpha_1 + \alpha_3\alpha_2\alpha_1 + \alpha_3\alpha_\xi - \alpha_3\alpha_4\xi \]
\[ T_4 = -\alpha_3\alpha_2\alpha_1 + \alpha_2\alpha_4\xi + \alpha_7\alpha_4\alpha_2 + \alpha_2\alpha_4\alpha_1 . \]  

The first color contribution yields an expression in hyperlogarithms over the alphabet \( a_i \in \{0, -1, 1\} \), which is thus expressible within the class of harmonic polylogarithms

\[ I_{8a}(x) = \frac{4}{x^2(x + 1)} \left\{ -[L_{0,1}(x) + L_{0,-1}(x)]\zeta_3 - 4L_{0,-1,-1,0,-1}(x) - 2L_{0,-1,0,-1,-1}(x) + 2L_{0,-1,0,0,-1}(x) + 6L_{0,0,-1,-1,-1}(x) - 4L_{0,1,0,-1,-1}(x) + 2L_{0,1,0,0,-1}(x) \right\} . \]

The factors in the denominators of the rational functions correspond to the same alphabet and are all of the form \( \frac{1}{x-a_i} \). As generally observed this yields a representation within the class of generalized harmonic sums

\[ I_{8a}(N) = (-1)^N \left[ -\frac{12}{(N+1)^3(N+2)^3} S_1^2 + \frac{8}{(N+1)^2(N+2)^2} (2S_1S_2 - S_{2,1}) \right] \]
\[ -\frac{8}{(N+1)^3(N+2)^2} S_1 + 8S_3S_2 + 16S_2S_2 + 8S_{-2}S_{-2} + 8S_5 - 8S_{2,3} + 24S_{4,1} \]
\[ -8S_{-1,-2} - 24S_{2,1} - 24S_{3,1,1} + \frac{4}{(N+1)^3(N+2)^3} \left( 10N^3 + 43N^2 + 65N + 35 \right) S_2 \]
\[ + \frac{8(2N^2 + 3)}{(N+1)^2(N+2)^2} [S_{-3} - 2S_{-2,1}] \]
\[ + 4 \left[ (-1)^N \left( \frac{2N^2 + 6N + 5}{(N+1)^2(N+2)^2} + S_2 + S_{-2} \right) - \frac{2N^2 + 3}{(N+1)^2(N+2)^2} \right] \zeta_3 , \]

with the asymptotic representation

\[ I_{8a}^{asy}(N) \propto \left[ (-1)^N \left( -\frac{16}{N} + \frac{40}{N^2} + \frac{296}{3N^3} + \frac{240}{N^4} - \frac{8632}{15N^5} + \frac{1360}{N^6} - \frac{66536}{21N^7} + \frac{72800}{N^8} - \frac{247672}{15N^9} + \frac{37008}{N^{10}} \right) \right] \]
\[ \times \ln(N) + (-1)^N \left[ -\frac{16}{N} - \frac{2}{N^2} + \frac{538}{9N^3} - \frac{721}{3N^4} + \frac{18996}{25N^5} - \frac{6514}{3N^6} + \frac{12902497}{2205N^7} + \frac{954313}{63N^8} \right] \]
\[ \times \left( \frac{627}{N^2} - \frac{30}{N^3} + 111 \frac{360}{N^4} + \frac{1079}{N^5} \right) \zeta_2 + (-1)^N \left( \frac{6}{N^2} - \frac{30}{N^3} + \frac{111}{N^4} - \frac{360}{N^5} + \frac{1079}{N^6} \right) \]

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\[-\frac{3060}{N^7} + \frac{8317}{N^8} - \frac{21840}{N^9} + \frac{278631}{5N^{10}} \ln^2(\tilde{N}) + \left(\frac{10}{N^2} - \frac{20}{N^3} + \frac{11}{6N^4} + \frac{485}{3N^5} - \frac{15469}{18N^6} \right) + \frac{19465}{6N^7} - \frac{13226411}{1260N^8} + \frac{216849}{7N^9} - \frac{920336}{105N^{10}} \ln(\tilde{N}) + \frac{1}{N^2} + \frac{62}{3N^3} - \frac{7457}{72N^4} + \frac{31339}{90N^5} \right] \\
- \frac{5369077}{5400N^6} + \frac{6553031}{2520N^7} - \frac{341676107}{529200N^8} + \frac{820719223}{529200N^9} - \frac{19478383749}{529200N^{10}} - 5\zeta_5 \right] + \left( -1 \right)^N \left( \frac{12}{N} - \frac{30}{N^2} + \frac{74}{N^3} - \frac{180}{N^4} + \frac{2158}{5N^5} - \frac{1020}{N^6} + \frac{16634}{7N^7} - \frac{5460}{N^8} + \frac{61918}{5N^9} \right) \\
- \frac{27756}{N^{10}} \zeta_3 + O \left( \frac{\ln^2(\tilde{N})}{N^{11}} \right). \tag{630} \]

### 7.5 Obtaining linear reducibility via transformations of the integration variables

If a diagram is not linearly reducible this does generally mean that it is not calculable using hyperlogarithms. In some cases linear reducibility may be obtained by a suitable transformation of integration variables. For the integral (625) no completely linear reducible integration order exists a priori. In the last integration step quadratic denominators appear. Since these non-linear denominators are not of a higher polynomial degree than 2 and appear in the last integration step only it is, however, possible to remap the tracing variable to gain linear reducibility, e.g:

\[
\int_0^\infty dy \frac{L \{ \cdots \}, y}{y^2 + y(2 + t) + 1} = \int_0^\infty dy \frac{L \{ \cdots \}, y}{(y + 1 + t/2 + \sqrt{t^2 + 4t/2}) (y + 1 + t/2 - \sqrt{t^2 + 4t/2})}. \tag{631} \]

Applying the transformation

\[
t = \frac{4x^2}{1 - x^2} \tag{632} \]

yields

\[
\int_0^\infty dy \frac{(x^2 - 1)^2}{(y(x^2 - 1) - 1 - 3x^2 + 2x)(y(x^2 - 1) - 1 - 3x^2 - 2x)} \tag{633} \frac{L \{ \cdots \}, y}{L \{ \cdots \}, y}.
\]

As a result, the final expression will consist of hyperlogarithms of argument

\[
x = \sqrt{\frac{t}{t + 4}}. \tag{634} \]

In cases that no variable transformation is needed to obtain linear reducibility the generating function is given in terms of a linear combination of hyperlogarithms with argument $t$. In this case obtaining the $N$-space representation is achieved by evaluating the Cauchy-products between the corresponding sum representations of the rational prefactors and the hyperlogarithms [179]. If transformations like (631) have been applied the $N$th coefficient is extracted by generating difference equations which are then solved by using the package Sigma [236, 237]. The second method is more time consuming but has the advantage of working in much more general setups.
The generating expression in terms of hyperlogarithms is rather lengthy and will not be presented here. It is however given completely in terms of hyperlogarithms with argument $x$ over the alphabet

$$\left\{1, 0, -1, -4; \frac{1}{2}; -\frac{1}{3}; -\frac{i}{\sqrt{3}}; \frac{i}{\sqrt{3}}; -\frac{1}{\sqrt{5}}; \frac{1}{\sqrt{5}}\right\}.$$

Due to the more complex dependence on the generating parameter $t$, the $N$-space results also contain classes of sums beyond harmonic and alternating sums. This class is given by finite binomial and inverse binomial sums over harmonic or alternating harmonic sums [311].

The complete Mellin space representation of $I_{sb}$ is then given by

$$I_{sb} = \frac{-2(3N+2)}{(N+1)^5(N+2)^2} + \frac{2(4N^3 + 35N^2 + 82N + 58)}{(N+1)^3(N+2)^3} \left[ S_2 + 3S_{-2} \right]$$

$$- \frac{4(N^3 + 8N^2 + 23N + 20)}{(N+1)^2(N+2)^2} S_3 - \frac{4(N^3 + 8N^2 + 27N + 26)}{(N+1)^2(N+2)^2} S_{-3}$$

$$- \frac{8(N^2 + 6N + 7)}{(N+1)^3(N+2)^2} S_{-2,1} + 2^{N+2} \frac{(2N^3 + 12N^2 + 31N + 26)}{(N+1)^2(N+2)^2} \left[ S_{1,2} \left( \frac{1}{2}, 1, N \right) \right.$$

$$+ 3S_{1,2} \left( \frac{1}{2}, -1, N \right) \left] + \frac{(-1)^N}{2N} \left\{ - \frac{3(4N^2 + 6N - 3)}{(N+1)(N+2)(2N+1)} \sum_{i=1}^{N} (-2)^i \left( \frac{2i}{i} \right) S_{1,2} \left( \frac{1}{2}, 1, i \right) \right.$$}

$$- \frac{9(4N^2 + 6N - 3)}{(N+1)(N+2)(2N+1)} \sum_{i=1}^{N} (-2)^i \left( \frac{2i}{i} \right) S_{1,2} \left( \frac{1}{2}, 1, i \right)$$

$$+ \frac{(N+1)}{(N+2)(2N+1)} \left[ \right.$$}

$$- \frac{N}{2} \sum_{i=1}^{N} \frac{(-1)^i \left( \frac{2i}{i} \right) S_3(i)}{i^3} - 2 \sum_{i=1}^{N} \frac{(-1)^i \left( \frac{2i}{i} \right) S_3(i)}{i^2} - 3 \sum_{i=1}^{N} \frac{(-2)^i \left( \frac{2i}{i} \right) S_3(i)}{i} + \frac{9(N+1)}{2} \sum_{i=1}^{N} \frac{(-2)^i \left( \frac{2i}{i} \right) S_2(i)}{i}$$

$$+ 2 \sum_{i=1}^{N} \frac{(-1)^i \left( \frac{2i}{i} \right) S_3(i)}{i} + 3 \sum_{i=1}^{N} \frac{(-1)^i \left( \frac{2i}{i} \right) S_{-2}(i)}{i} + 6 \sum_{i=1}^{N} \frac{(-1)^i \left( \frac{2i}{i} \right) S_{-2}(i)}{i}$$

$$\left\} \right.$$}

$$+ \frac{(-1)^N}{2N} \left\{ - \frac{8(N^3 + 6N^2 + 11N + 7)}{3(N+1)^2(N+2)^2} S_1^3 + \left( \frac{-4N^3 - 7N^2 + 6N + 10}{(N+1)^2(N+2)^3} \right) S_1^2$$

$$+ \left( \frac{2(16N^3 + 107N^2 + 222N + 146)}{(N+1)^4(N+2)^3} - \frac{12(N^3 + 6N^2 + 11N + 7)}{(N+1)^2(N+2)^2} \right) S_1^1$$

$$+ 4S_2 + 6S_{-2} + 10S_4 + \frac{2(3N+2)}{(N+1)^5(N+2)^2} - \frac{8(5N^3 + 24N^2 + 37N + 20)}{3(N+1)^2(N+2)^2} S_3$$

$$- 8(5N+12)S_5 + 8S_{-4} - 10(N+N)S_{-5}$$

$$+ \left[ \frac{8(2N^3 + 10N^2 + 16N + 9)}{(N+1)^2(N+2)^2} - 2(5N+12)S_2 - 6(5N+12)S_{-2} \right] S_{-3}$$

$$+ \left[ \frac{-36N^3 - 165N^2 - 270N - 158}{(N+1)^3(N+2)^3} - 2(5N+12)S_3 - 4S_{2,1} \right] S_2$$

$$+ \frac{4(N^3 + 6N^2 + 11N + 7)}{(N+1)^2(N+2)^2} S_{2,1} + 2(5N+12)S_{2,3} + 2(5N+16)S_{2,-3} - 12S_{3,1}$$

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\[ +16(N + 2)S_{4,1} + \frac{16(N^3 + 6N^2 + 11N + 7)}{(N + 1)^2(N + 2)^2} S_{-2,1} + \left[ -\frac{2(4N^3 + 7N^2 - 6N - 10)}{(N + 1)^3(N + 2)^3} \right] S_{-2} \]

\[ -\frac{16(N^3 + 6N^2 + 11N + 7)}{(N + 1)^2(N + 2)^2} S_1 + 4S_2 + 6(N + 4)S_3 + 8(N + 2)S_{-2,1} \left] S_{-2} \right. \]

\[ +2NS_{-2,3} + 2(23N + 60)S_{-2,-3} + 4S_{2,1,1} - 16S_{2,1,-2} + 8S_{2,2,1} \]

\[ +6(N + 4)S_{3,1,1} - 8(N + 2)S_{-2,1,-2} - 16S_{2,1,1,1} \]

\[ -2(3N + 8) \left[ S_{1,2} \left( \frac{1}{2}, 1, N \right) + 3S_{1,2} \left( \frac{1}{2}, -1, N \right) \right] S_{2}(-2) \]

\[ +2(3N + 8) \left[ -3S_{1,4} \left( \frac{1}{2}, 2, N \right) - S_{1,4} \left( \frac{1}{2}, -2, N \right) + S_{1,2,2} \left( \frac{1}{2}, 1, -2, N \right) \right. \]

\[ +3S_{1,2,2} \left( \frac{1}{2}, -1, -2, N \right) + S_{1,2,2} \left( \frac{1}{2}, -2, 1, N \right) + 3S_{1,2,2} \left( \frac{1}{2}, -2, -1, N \right) \]

\[ -6(3N + 8) \sum_{i=1}^{N} (-2)^i \left( \begin{array}{c} 2i \\ i \end{array} \right) \left[ S_{1,2} \left( \frac{1}{2}, 1, i \right) + 3S_{1,2} \left( \frac{1}{2}, -1, i \right) \right] \sum_{j=1}^{i} \frac{1}{(2j)} j^2 \]

\[ +36 \sum_{i=1}^{N} (-2)^i \left( \begin{array}{c} 2i \\ i \end{array} \right) S_{1,2} \left( \frac{1}{2}, 1, i \right) \sum_{j=1}^{i} \frac{1}{(2j)} j \]

\[ +108 \sum_{i=1}^{N} (-2)^i \left( \begin{array}{c} 2i \\ i \end{array} \right) S_{1,2} \left( \frac{1}{2}, -1, i \right) \sum_{j=1}^{i} \frac{1}{(2j)} j \]

\[ +6(3N + 8) \left[ \sum_{i=1}^{N} (-2)^i \left( \begin{array}{c} 2i \\ i \end{array} \right) S_{1,2} \left( \frac{1}{2}, 1, i \right) \right] \]

\[ +3 \sum_{i=1}^{N} (-2)^i \left( \begin{array}{c} 2i \\ i \end{array} \right) S_{1,2} \left( \frac{1}{2}, -1, i \right) \sum_{i=1}^{N} \frac{1}{(2i)} i^2 \]

\[ -36 \left[ \sum_{i=1}^{N} (-2)^i \left( \begin{array}{c} 2i \\ i \end{array} \right) S_{1,2} \left( \frac{1}{2}, 1, i \right) + 3 \sum_{i=1}^{N} (-2)^i \left( \begin{array}{c} 2i \\ i \end{array} \right) S_{1,2} \left( \frac{1}{2}, -1, i \right) \right] \sum_{i=1}^{N} \frac{1}{(2i)} i \]

\[ +\frac{3}{2} (3N + 8) \left[ \sum_{j=1}^{N} \frac{i}{j} \left( \begin{array}{c} 2j \\ j \end{array} \right) S_{1}(j) \right] \sum_{i=1}^{N} \frac{i}{j} \left( \begin{array}{c} 2j \\ j \end{array} \right) S_{2}(j) \]

\[ +3 \sum_{j=1}^{N} \frac{(2i)}{(i)} (1 + i) \]

\[ +3 \sum_{j=1}^{N} \frac{(2i)}{(i)} (1 + i) \]

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\[ 
+2(3N + 8) \left[ \sum_{i=1}^{N} \frac{(-1)^{j}(2j)}{j} \frac{S_{2}(j)}{2^{i}} \left(\frac{1}{i} + i\right) \right] + \frac{3}{2} \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{-2}(j)}{2^{i}} \left(\frac{1}{i} + i\right) \\
+6(3N + 8) \sum_{i=1}^{N} \frac{(-1)^{j}(2j)}{j} \frac{S_{-2}(j)}{2^{i}} \left(\frac{1}{i} + i\right) + 2(3N + 5) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{2}(j)}{2^{i}} \left(\frac{1}{i} + 2i\right) \\
+4(3N + 5) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{1}(j)}{2^{i}} \left(\frac{1}{i} + 2i\right) - 3(3N + 5) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{2}(j)}{2^{i}} \left(\frac{1}{i} + 2i\right) \\
-9(3N + 5) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{2}(j)}{2^{i}} \left(\frac{1}{i} + 2i\right) - 4(3N + 5) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{-2}(j)}{2^{i}} \left(\frac{1}{i} + 2i\right) \\
-6(3N + 5) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{-2}(j)}{2^{i}} \left(\frac{1}{i} + 2i\right) - 12(3N + 5) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{2}(j)}{2^{i}} \left(\frac{1}{i} + 2i\right) \\
+(-3N - 8) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{1}(j)}{2^{i}} \left(\frac{1}{i} + i\right) - 2(3N + 8) \sum_{i=1}^{N} \frac{(2j)}{j} \frac{S_{2}(j)}{2^{i}} \left(\frac{1}{i} + i\right) \right] \\
+ \left\{ (-1)^{N} \sum_{i=1}^{N} \left( \frac{1}{N^{2} + 1} \frac{1}{(N - 1)N^{2}} \sum_{i=1}^{N} (-2)^{i} \frac{2^{i}}{i} \right) \\
-6(3N - 1) \sum_{i=1}^{N} \left( \frac{-2)^{i} \frac{2^{i}}{i} \right) + 36 \sum_{i=1}^{N} \left( \frac{-2)^{i} \frac{2^{i}}{i} \right) \\
-36S_{1}(-2, N) + 8(3N - 1)S_{2}(-2, N) + \frac{4(N^{2} - N + 1)}{(N - 1)N^{2}} + 4S_{2} \right\} \\
\]
\[-4(2N - 1)S_{-2} + \frac{4(N^2 - 3N + 1)}{(N - 1)N^2} - \frac{2^{N+3}(N^2 - N + 1)}{(N - 1)N^2}\right\} \zeta_3. \quad (636)\]

Here as in all other \(N\)-space representations the code \texttt{Sigma} has been used to solve the difference equations and we obtain a representation in which all occurring sums are transcendental to each other. Some of the individual sums contributing to (636) diverge \(\propto 8^N, 4^N\) and \(2^N\) for large values of \(N\). Performing the asymptotic expansion yields that the divergences \(\propto 8^N\) and \(\propto 4^N\) cancel while this is not the case for the divergences \(2^N\).

The asymptotic expansion in this case takes the following form

\[I_{sb}(N) \propto 2^N I_{sb,1}(N) + I_{sb,2}(N), \quad (637)\]

with

\[I_{sb,1}(N) \approx \left[ -\frac{112}{9N^3} + \frac{7568}{81N^4} - \frac{27280}{81N^5} + \frac{2256112}{2187N^6} - \frac{52719920}{19683N^7} + \frac{373195088}{59049N^8} \right] \zeta_3 \quad (638)\]

\[I_{sb,2}(N) \approx \left[ \frac{1}{N^4} - \frac{12}{N^5} + \frac{91}{6N^6} - \frac{574}{N^7} + \frac{3451}{N^8} \right] \zeta_2 + 2^{-N} \left[ -\frac{3}{2N^2} + \frac{1}{2N^3} + \frac{6}{N^4} - \frac{35}{2N^5} + \frac{17}{N^6} + \frac{79}{2N^7} - \frac{152}{N^8} \right] \ln^2(2) \]

\[+ \left[ -\frac{3}{N^2} + \frac{1}{N^3} - \frac{12}{N^4} - \frac{35}{N^5} + \frac{34}{N^6} + \frac{79}{N^7} - \frac{304}{N^8} \right] \left( \text{Li}_2 \left( -\frac{1}{2} \right) + \frac{1}{2} \zeta_2 \right) \]

\[+ \left[ -\frac{3}{2N^2} + \frac{1}{2N^3} + \frac{6}{N^4} - \frac{35}{2N^5} + \frac{17}{N^6} + \frac{79}{2N^7} - \frac{152}{N^8} \right] \zeta_2 \]

\[+ \left[ \frac{2}{N^2} - \frac{6}{N^3} \right] \ln^2(\tilde{N}) + \ln^3(\tilde{N}) + \left[ -\frac{4}{3N^2} + \frac{52}{9N^3} - \frac{56}{3N^4} \right] \zeta_3 + (-1)^N \left[ -\frac{74}{9N^3} + \frac{133}{3N^4} - \frac{4103}{25N^5} + \frac{15439}{30N^6} \right] \ln(\tilde{N}) \]

\[+ \left[ -\frac{1308}{N^8} \right] \zeta_2 + \frac{4}{N^2} - \frac{436}{27N^3} + \frac{29}{N^4} - \frac{32}{375N^5} - \frac{8489}{36N^6} + \frac{8193131}{6860N^7} - \frac{778753}{180N^8} \ln(\tilde{N}) \]

\[+ A_1 + A_2 N + \left[ -\frac{8}{N} + \frac{21}{9N^2} - \frac{520}{9N^3} + \frac{476}{3N^4} - \frac{21473}{50N^5} + \frac{68569}{60N^6} - \frac{26328833}{8820N^7} \right] \zeta_2 + 2^{-N} \left[ \frac{3}{2N} - \frac{1}{N^2} - \frac{15}{N^3} - \frac{121}{N^4} + \frac{1023}{N^5} - \frac{9721}{N^6} \right] \]

\[+ \left[ -\frac{3}{2N} + \frac{11}{2N^2} - \frac{55}{2N^3} + \frac{602}{3N^4} - \frac{50497}{30N^5} + \frac{239851}{15N^6} \right] \ln^2(2) \]

\[+ \left[ -\frac{3}{2N} - \frac{1}{N^2} + \frac{15}{N^3} - \frac{121}{N^4} \right] \]
\[
\begin{align*}
&+ \frac{1023}{N^5} - \frac{9721}{N^6} + \frac{104415}{N^7} - \frac{1259161}{N^8} \right] \zeta_2^2 + \left[ -\frac{3}{N} + \frac{11}{2N^2} - \frac{55}{2N^3} + \frac{602}{3N^4} \\
&- \frac{50497}{30N^5} + \frac{239851}{15N^6} - \frac{36068621}{210N^7} + \frac{43495976}{21N^8} \right] \zeta_2 + \left( \text{Li}_2 \left( -\frac{1}{2} \right) + \frac{1}{2} \zeta_2 \right) \\
&\times \left[ -\frac{3}{N} - \frac{2}{N^2} + \frac{2}{N^3} - \frac{4}{N^4} + \frac{20}{N^5} - \frac{2046}{N^6} + \frac{19442}{N^7} - \frac{208830}{N^8} + \frac{2518322}{N^9} \right] \zeta_2 \\
&- \frac{6}{N} + \frac{11}{N^2} - \frac{55}{N^3} + \frac{1204}{3N^4} - \frac{50497}{15N^5} + \frac{479702}{15N^6} - \frac{36068621}{105N^7} + \frac{86991952}{21N^8} \right] \zeta_2 \\
&+ \left[ -\frac{2}{N} + \frac{10}{3N^2} - \frac{46}{9N^3} + \frac{20}{3N^4} - \frac{242}{45N^5} - \frac{20}{3N^6} + \frac{3194}{63N^7} - \frac{180}{N^8} \right] \zeta_3 \\
&+ \frac{6}{N^2} - \frac{1732}{81N^3} + \frac{793}{12N^4} - \frac{1217029}{5625N^5} + \frac{130343}{180N^6} - \frac{10153834441}{4321800N^7} + \frac{1632850801}{226800N^8} \right] \\
&+ \frac{4}{N^5} - \frac{62}{N^6} + \frac{1759}{3N^7} - \frac{4530}{N^8} \\
&+ \left\{ -\frac{2}{N^2} - \frac{6}{N^3} - \frac{8}{N^4} - \frac{2}{N^5} + \frac{8}{N^6} - \frac{10}{N^7} - \frac{72}{N^8} \right. \\
&\left. + (-1)^N \left[ \frac{10}{3} \zeta_2 - \frac{4\pi}{\sqrt{3}} \left( \frac{1}{N} \right)^{5/2} + \frac{2}{N^2} + \frac{10}{3N^3} + \frac{4}{N^4} + \frac{62}{15N^5} + \frac{4}{N^6} + \frac{82}{21N^7} + \frac{4}{N^8} \right] \right. \\
&\left. + \left( -\frac{1}{4} \right)^N \sqrt{\pi} \left[ -\frac{64}{3} \left( \frac{1}{N} \right)^{5/2} + \frac{232}{9} \left( \frac{1}{N} \right)^{7/2} - \frac{6697}{54} \left( \frac{1}{N} \right)^{9/2} \right] \right. \\
&\left. + \frac{65167}{144} \left( \frac{1}{N} \right)^{11/2} - \frac{30311555}{13824} \left( \frac{1}{N} \right)^{13/2} + \frac{394221963}{331776} \left( \frac{1}{N} \right)^{15/2} \right\} \zeta_3 \ .
\end{align*}
\]

Here the constants }$A_1$ and }$A_2$ are
\[
A_1 \approx 6.233478110414834, \quad A_2 < 10^{-15}.
\] (640)

It has, however, been observed, that in the physical case including the numerator structures the }$2^N$-divergences cancel as well, cf. [310].

### 7.6 Fixed Mellin moments

The strategy to obtain general }$N$ representations via a generating function breaks down in some cases, e.g. for the diagrams in figure 25, because the respective integrals are not linearly reducible. If one is not interested in the general }$N$ representation but values for fixed integer values of }$N$ the integrals for all diagrams at 3–loop level are linearly reducible, however. In theory it would thus be possible to compute a large amount of Mellin moments, for all practical purposes one is, however, constrained by the available computing power and memory sizes. Here the resource demands rise exponentially in }$N$ due to the larger rational expressions and the intensive use of partial fraction decompositions in the algorithm. As an example we computed a series of Mellin moments for the most complicated class of diagrams contributing to the 3–loop OMEs. For the
diagrams in figure 25 the computation of the 9th Mellin moment took about 8 hours of CPU
time and about 35 Gigabyte of memory. In comparison the FORM based code MATAD [272] has
similar resource demands on CPU time and memory storage but allows the computation of a
few higher Mellin moments.

Figure 25: Crossed-box topologies with local operator insertions.

Table 2: Moments of the finite crossed-box graphs (a–c) shown in Figure 25.
8 Conclusions

For a precise analysis of the deeply inelastic scattering data recorded at HERA within one percent accuracy, a precise understanding of the heavy quark flavor contributions to the structure function $F_2$ at three loop level is essential. At $O(\alpha_s^3)$ the basic description of a single heavy quark to the unpolarized Wilson coefficients $C_{(q,g)^2}^{NS,PS,S}$ in the asymptotic region $Q^2 \gg m^2$ via massive operator matrix elements is known. It relies on a factorization theorem [128, 139] which allows to represent the heavy flavor Wilson coefficients at $Q^2 \gtrsim 10m^2$ [128] as a convolution of the light flavor Wilson coefficients which are known up to NNLO [104, 161] and the process independent massive operator matrix elements. This kinematic domain thereby covers the most interesting kinematic region a very important kinematic region of the deep-inelastic scattering experiments at HERA [129–132]. At $O(\alpha_s^3)$ a series of fixed Mellin moments $N = 2, \ldots, 10(12, 14)$ for all relevant OMEs are known [139]. The main challenge for a complete understanding of the single heavy quark contributions is thus to obtain these operator matrix elements for general values of the Mellin variable $N$. Generally this thesis aims at extending the present understanding of the $O(\alpha_s^3)$ heavy flavor contributions to DIS at two fronts:

Firstly, there is no strong hierarchy between the charm- and bottom-quark masses and thus there exists a kinematic region where a description based on one single heavy and $N_F$ light fermion flavors does not suffice. We aim to extend the present description to a scenario with two heavy quark flavors, to work out the respective renormalization prescriptions and to lay out the groundwork for a complete calculation of the respective contributions.

Secondly, despite recent success [156–158], a general algorithm to evaluate the respective Feynman diagrams remains unknown and one is thus forced to restrict oneself to subclasses and subtopologies of the complete set of Feynman diagrams. Therefore new gauge-invariant color contributions to different OMEs are calculated and new algorithms are explored in order to break the ground for a systematic evaluation of further topologies.

Section 3 presents the computation of the OME $A_{qg,Q}$. This completes the $O(C_{A,F}T_F^2N_F)$ contributions to operator matrix elements contributing to the structure function $F_2$ [140], which constitutes the first complete color factors of the general $N$ computation. The computation relied on representations within the functional class of generalized hypergeometric functions and their respective sum representations. These allow for a Laurent series expansion in the dimensional regularization parameter $\varepsilon$ and were afterwards evaluated using advanced summation algorithms encoded in the packages Sigma [236,237], SumProduction and EvaluateMultiSums [238,239]. Furthermore, the method by representations in terms of special functions encouraged an automatized approach, with which the subtopologies which contain two completely massless bubble diagrams contributing to all relevant OMEs for the structure function $F_2$ and the variable flavor number scheme (VFNS) were solved.

In the case of a single heavy quark flavor all logarithmic contributions $\propto \ln(Q^2/m^2)$ to the heavy flavor Wilson coefficients are a direct consequence of the factorization relation and the renormalization prescription. They have been determined for the massive operator matrix elements $A_{gg,Q}^{NS}, A_{qg,Q}^{PS}, A_{qg,Q}^{PS}, A_{Qg,Q}, A_{gg,Q}$ and $A_{gg,Q}$ and the associated heavy flavor Wilson coefficients [220] in Section 4. The computation of the OME $A_{gg,Q}$ completed the Wilson coefficient $L_{g,2}^S$, which has been presented together with the completed Wilson coefficient $L_{g,2}^PS$ in this Section. Quantitatively the contributions of these two Wilson coefficients, however, turn out to be exceeded by other contributions to the structure function $F_2$.

The renormalization prescription for massive operator matrix elements which receive contributions from two different masses has been worked out in the Section 5. Here we apply an on-shell-
scheme for the mass renormalization. In order to renormalize the coupling constant we use the background field mechanism to obtain an intermediary MOM-scheme. Finally, we perform the transformation into the $\overline{\text{MS}}$-scheme, where we assume the decoupling of the heavy quark contribution. We present all renormalization formulas for the respective two flavor contributions $\tilde{A}_{ij}$ in the on shell scheme for the masses and the $\overline{\text{MS}}$-scheme for the coupling constant. Furthermore we provide the respective two-mass transformations to the $\overline{\text{MS}}$, respectively MOM-scheme.

In Section 6 we extended the present description of the massive contributions to the VFNS and the structure function $F_i$. This yields five OMEs $A_{qq,Q}^\text{NS}(m_1^2,m_2^2,\mu^2)$, $A_{Qg}^\text{PS}(m_1^2,m_2^2,\mu^2)$, $A_{Qg}(m_1^2,m_2^2,\mu^2)$, $A_{gg,Q}(m_1^2,m_2^2,\mu^2)$ and $A_{gg,Q}(m_1^2,m_2^2,\mu^2)$ which receive genuine contributions from Feynman diagrams with two heavy fermionic lines. We present the computation of a series of fixed Mellin moments $N = 2, 4, 6$ up to $O(\eta^3 \ln^3(\eta))$, with $\eta = m_2^2/m_1^2$ for all these OMEs. The respective Feynman diagrams were generated using QGRAF. Projection operators, cf. Ref. [139], were applied to map the diagrams with operator insertions onto pure tadpole diagrams. These were expanded in the mass ratio $\eta$ and evaluated using the codes Q2e, EXP and MATAD. We computed the complete two-flavor contributions to the OME $A_{qq,Q}^\text{NS}$ and the transversity OME $A_{Qg,Q}^\text{NS,trans}$. The computation was performed using Mellin-Barnes representations and yielded a structure in which the $\eta$ dependence completely factors of the $N$-dependent parts. This property allows to expand the respective generalized hypergeometric functions using HypExp and to obtain the general $N$ solution. The same approach is used to evaluate the respective contributions to the single-mass case with $m_1 = m_2$. For this case also the $O(C_F T_F^2)$-contribution to the OME $A_{Qg,Q}^\text{PS}$ could be evaluated. For other OMEs a larger entanglement between the Mellin variable $N$ and the mass-ratio occurs. This demands more sophisticated approaches which have been described in Section 6.5. Here all scalar topologies contributing to the OME $A_{gg,Q}$ have been computed. The method is based on various mappings of the integration variables in order to obtain representations in which only one single Feynman parameter integral depends on the Mellin variable $N$ and the Mellin Barnes variable $\xi$. These integral representations allow for a splitting of the integration domain which ensures the convergence of the associated associated contour integrals. Collecting the residues yielded sums which were evaluated using the packages Sigma [236,237] and EvaluateMultiSums. The remaining integral was remapped to obtain the $x$-space representations for the respective scalar diagrams. The Mellin space representations were obtained in a final step using difference equation methods encoded in HarmonicSums [175,177–179] and Sigma [236,237] to solve these equations. The method required new iterated integrals and sums in intermediary and final steps. On the integral side advanced methods to rewrite these integrals in suitable representations were developed.

In Section 7 the application of parametric integration methods via hyperlogarithms [159,187] to massive Feynman diagrams with operator insertions, which are convergent in $D = 4$ space-time dimensions has been explored. This approach turns out to be relatively efficient to compute fixed Mellin moments. In order to obtain general $N$ representations for scalar diagrams of more evolved topologies, generating functions were constructed. These allowed for an evaluation using the methods of hyperlogarithms and in a final step the general $N$-results were reconstructed from these generating functions using methods encoded in Sigma [236,237] and HarmonicSums [175,177–179]. The method allowed to evaluate various new scalar topologies for the first time. While some diagrams could be made accessible to this method by a suitable transformation of the integration variables, it turns out that not all scalar topologies are solvable by this approach. This is due to the emergence of non-linearity of the denominator polynomials, which occur for some topologies.
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\section*{A Conventions}

We use natural units
\begin{equation}
    \hbar = 1, \quad c = 1, \quad \varepsilon_0 = 1,
\end{equation}
where $\hbar$ denotes Planck’s constant, $c$ the vacuum speed of light and $\varepsilon_0$ the permittivity of vacuum. The electromagnetic fine-structure constant $\alpha$ is given by \cite{312}
\begin{equation}
    \alpha = \alpha' (\mu^2 = 0) = \frac{e^2}{4\pi \varepsilon_0 \hbar c} = \frac{e^2}{4\pi} \approx \frac{1}{137.03599911(46)}. \tag{A.2}
\end{equation}
\noindent In this convention, energies and momenta are given in the same units, electron volt (eV).

The space–time dimension is taken to be $D = 4 + \varepsilon$ and the metric tensor $g_{\mu\nu}$ in Minkowski-space is defined as
\begin{equation}
    g_{00} = 1, \quad g_{ii} = -1, \quad i = 1 \ldots D - 1, \quad g_{ij} = 0, \quad i \neq j. \tag{A.3}
\end{equation}
Einstein’s summation convention is used in form of
\begin{equation}
    x_\mu y^\mu := \sum_{\mu=0}^{D-1} x_\mu y^\mu. \tag{A.4}
\end{equation}

Minkowski-space vectors are represented by
\begin{equation}
    x = (x_0, \vec{x}). \tag{A.5}
\end{equation}
If not stated otherwise, Greek indices refer to the $D$–component space–time vector and Latin ones to the $D - 1$ spatial components only. The Minkowski product of two vectors is defined by
\begin{equation}
    p.q = p_0 q_0 - \sum_{i=1}^{D-1} p_i q_i. \tag{A.6}
\end{equation}

The Dirac–matrices $\gamma_\mu$ are taken to be of dimension $D$ and obey the anti–commutation relation
\begin{equation}
    \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{A.7}
\end{equation}
It follows that
\begin{align}
    \gamma_\mu \gamma^\mu &= D \tag{A.8} \\
    Tr (\gamma_\mu \gamma_\nu) &= 4g_{\mu\nu} \tag{A.9} \\
    Tr (\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) &= 4[g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}] \tag{A.10}
\end{align}

The Feynman–dagger for a $D$-momentum $p$ is defined by
\begin{equation}
    \hat{p} := \gamma_\mu p^\mu. \tag{A.11}
\end{equation}

The conjugate of a bi–spinor $u$ of a particle is given by
\begin{equation}
    \bar{u} = u^\dagger \gamma_0, \tag{A.12}
\end{equation}
where \( \dag \) denote Hermitian and \(*\) complex conjugation, respectively. The bi–spinors \( u \) and \( v \) fulfill the free Dirac–equation,

\[
\begin{align*}
\dot{\psi} - m u(p) &= 0, \quad \bar{\psi}(p)(\dot{\psi} - m) = 0 \quad (A.13) \\
\dot{\psi} + m v(p) &= 0, \quad \bar{\psi}(p)(\dot{\psi} + m) = 0 \quad (A.14)
\end{align*}
\]

Bi–spinors and polarization vectors are normalized to

\[
\begin{align*}
\sum_{\sigma} u(p, \sigma) \bar{\psi}(p, \sigma) &= \dot{\psi} + m \quad (A.15) \\
\sum_{\sigma} v(p, \sigma) \bar{\psi}(p, \sigma) &= \dot{\psi} - m \quad (A.16) \\
\sum_{\lambda} e^{\mu}(k, \lambda) e^{\nu}(k, \lambda) &= -g^{\mu\nu} \quad (A.17)
\end{align*}
\]

where \( \lambda \) and \( \sigma \) represent the spin.

The commonly used caret “\(^\wedge\)" to signify an operator, e.g. \( \dot{O} \), is omitted if confusion is not to be expected.

The gauge symmetry group of QCD is the Lie–Group \( SU(3)_c \). We consider the general case of \( SU(N_c) \). The non–commutative generators are denoted by \( t^a \), where \( a \) runs from 1 to \( N^2_c - 1 \). The generators can be represented by Hermitian, traceless matrices, [201]. The structure constants \( f^{abc} \) and \( d^{abc} \) of \( SU(N_c) \) are defined via the commutation and anti–commutation relations of its generators, [204,313],

\[
\begin{align*}
[t^a, t^b] &= i f^{abc} t^c \quad (A.18) \\
\{t^a, t^b\} &= d^{abc} t^c + \frac{1}{N_c} \delta_{ab} \quad (A.19)
\end{align*}
\]

The indices of the color matrices, in a given representation, are denoted by \( i, j, k, l, \ldots \) The color invariants most commonly encountered are

\[
\begin{align*}
\delta_{ab} C_A &= f^{acd} f^{bcd} \quad (A.20) \\
\delta_{ij} C_F &= t^a_{il} t^a_{lj} \quad (A.21) \\
\delta_{ab} T_F &= t^a_{ik} t^b_{ki} \quad (A.22)
\end{align*}
\]

These constants evaluate to

\[
C_A = N_c, \quad C_F = \frac{N^2_c - 1}{2 N_c}, \quad T_F = \frac{1}{2}, \quad (A.23)
\]

in \( SU(N_c) \). At higher loops, more irreducible color–invariants emerge. At 3–loop order, one additionally obtains

\[
d^{aib} d_{abc} = (N^2_c - 1)(N^2_c - 4)/N_c \quad (A.24)
\]

In case of \( SU(3)_c \), \( C_A = 3 \), \( C_F = 4/3 \), \( d^{abc} d_{abc} = 40/3 \) holds.

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B Feynman Rules

B.1 QCD-Feynman rules

For the QCD Feynman rules, Figure 26, we follow Ref. [139, 204], cf. also Refs. [314, 315]. D–
dimensional momenta are denoted by $p_i$ and Lorentz-indices by Greek letters. Color indices are
$a, b, ...$ and $i, j$ are indices of the color matrices. Solid directed lines represent fermions, curvy
lines gluons and dashed lines ghosts. Arrows denote the direction of the momenta. A factor
$(-1)$ has to be included for each closed fermion– or ghost loop.

\begin{equation}
ig_s \gamma_\mu t^a_{ji}
\end{equation}

\begin{equation}
-g_s f^{abc} [ (p_1 - p_2)_\rho g_{\mu \nu} + (p_2 - p_3)_\rho g_{\nu \rho} + (p_3 - p_1)_\rho g_{\mu \nu} ]
\end{equation}

\begin{equation}
-g_s f^{abc} p_\mu
\end{equation}

\begin{equation}
-ig^2 \sum \left\{ f^{abc} f^{cde} [ g_{\mu \nu} g_{\alpha \sigma} - g_{\mu \alpha} g_{\nu \sigma} ] + f^{ace} f^{bde} [ g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\rho \nu} ] + f^{ade} f^{cbe} [ g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\rho \nu} ] \right\}
\end{equation}

\begin{equation}
\frac{i}{p^2 - m^2 + i0} \delta_{ij}
\end{equation}

\begin{equation}
\frac{i}{p^2 + m^2} (- g_{\mu \nu} + \xi p_\mu p_\nu / (p^2 + i0)) \delta_{ab}
\end{equation}

\begin{equation}
\frac{i}{p^2 + i0} \delta_{ab}
\end{equation}

Figure 26: Feynman rules of QCD.
The Feynman rules for the quarkonic composite operators are given in Figure 27. Up to $O(g^2)$ they can be found in Ref. [316] and also in [317]. Note that the $O(g)$ term in the former reference contains a typographical error. In Ref. [139] these rules were checked and agree up to normalization factors, which may be due to different conventions. There also the new rule with three external gluons was given. The terms $\gamma_+^i$ refer to the unpolarized (+) and polarized (−) case, respectively. Gluon momenta are taken to be incoming.

\[
\delta^{ij}\Delta \gamma_\pm(\Delta \cdot p)^{N-1}, \quad N \geq 1
\]

\[
g_i^{\mu a} \Delta^\mu \Delta^\pm \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^{j} (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2
\]

\[
g_i^2 \Delta^\mu \Delta^\nu \Delta^\rho \gamma_\pm \sum_{j=0}^{N-3} \sum_{l+1}^{N-2} (\Delta p_3)^{l} (\Delta p_1)^{N-l-2} \left[ (t^{\alpha \beta \gamma})_{jl}(\Delta p_1 + \Delta p_3)^{l-j-1} + (t^{\beta \gamma \alpha})_{jl}(\Delta p_1 + \Delta p_3)^{l-j-1} \right], \quad N \geq 3
\]

\[
g_i^3 \Delta^\mu \Delta^\nu \Delta^\rho \Delta^\sigma \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=0}^{N-3} \sum_{m=0}^{N-2} (\Delta p_3)^{l} (\Delta p_1)^{m-l-1} \left[ (t^{\alpha \beta \gamma \delta})_{jml}(\Delta p_1 + \Delta p_3 + \Delta p_4)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} \right], \quad N \geq 4
\]

\[
\gamma_+ = 1, \quad \gamma_- = \gamma_5. \quad \text{For transversity, one has to replace:} \quad \Delta \gamma_\pm \rightarrow \sigma^\mu \nu \Delta_\nu.
\]

Figure 27: Feynman rules for quarkonic composite operators. $\Delta$ denotes a light-like 4-vector, $\Delta^2 = 0$; $N$ is a suitably large positive integer, Ref. [139]
B.2 Scalar Feynman Rules

Additionally to complete QCD diagrams, in Sections 6.5 and 7 we calculated massive scalar Feynman integrals with a local operator insertion. The corresponding scalar Feynman rules read:

\[ \frac{1}{p^2 - m^2} \]  
\[ \frac{1}{p^2} \]  
\[ g \]  
\[ (\Delta_p)^N \]  
\[ g \sum_{j=0}^{N} (\Delta.p_1)^j(\Delta.p_2)^{N-j} \]

\[ a) \quad g^2 \sum_{j=0}^{N} \sum_{l=0}^{N-j} (\Delta.p_2)^j(\Delta.p_1)^{N-l-j}(\Delta.p_1 + \Delta.p_2)^l \]
\[ b) \quad g^2 \sum_{j=0}^{N} \sum_{l=0}^{N-j} (\Delta.p_2)^j(\Delta.p_1)^{N-l-j}(\Delta.p_1 + \Delta.p_3)^l \]
C  D-dimensional integrals

In the calculation of the D-dimensional loop integrals [250,318,319] with $D = 4 + \varepsilon$, we perform first a Wick-rotation to Euclidean momenta

$$
\prod_{i=1}^{M} \int \frac{d^D k_i}{(2\pi)^D} f\left(k_i\right) \prod_{i=1}^{M} \left(\frac{1}{k_i^2 - m_i^2}\right)^{a_i} = (-1)^{-\sum_{i=1}^{M} a_i} \prod_{i=1}^{M} \int \frac{d^D k_i}{(2\pi)^D} f\left(-k_i\right) \times \prod_{i=1}^{M} \left(\frac{1}{-k_i^2 - m_i^2}\right)^{a_i},
$$

with $\forall a_i \in \mathbb{N}$. One obtains the following Euclidean integrals, where, cf. [204],

$$
k^2_E = k_0^2 + \tilde{k}^2,
$$

$$
\int \frac{d^D k}{(2\pi)^D} \left(\frac{k^2}{k^2 - \varphi}\right)^m = (-1)^{(r-m)} \int \frac{d^D k_E}{(2\pi)^D} \left(\frac{k_E^2}{k_E^2 + \varphi}\right)^m = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(r+D/2)}{\Gamma(D/2)} \frac{\varphi^{r-m+D/2}}{(m-r-D/2)}
$$

(C.8)

and

(C.9)

$$
\int d^D k \ k^2 f(k^2) = \frac{g^{\alpha_1\alpha_2}}{D} \int d^D k \ k^2 f(k^2)
$$

$$
\int d^D k \ k_1^\alpha k_2^\beta k_3^\gamma k_4^\delta f(k^2) = \frac{g^{\alpha_1\alpha_2}g^{\alpha_3\alpha_4} + g^{\alpha_1\alpha_3}g^{\alpha_2\alpha_4} + g^{\alpha_1\alpha_4}g^{\alpha_2\alpha_3}}{D^2 + 2D}
$$

$$
\times \int d^D k \ k^4 f(k^2)
$$

$$
\int d^D k \ k_1^\alpha k_2^\alpha k_3^\alpha k_4^\alpha k_5^\alpha k_6^\alpha f(k^2) = \frac{1}{D^6 + 6D^2 + 8D} \int d^D k \ k^6 f(k^2),
$$

(C.10)

(C.11)

with

$$
Q(g^{\alpha_1\alpha_2}) = g^{\alpha_1\alpha_2} \left[ g^{\alpha_3\alpha_4}g^{\alpha_5\alpha_6} + g^{\alpha_3\alpha_5}g^{\alpha_4\alpha_6} + g^{\alpha_3\alpha_6}g^{\alpha_4\alpha_5} \right] + g^{\alpha_1\alpha_3} \left[ g^{\alpha_2\alpha_4}g^{\alpha_5\alpha_6} + g^{\alpha_2\alpha_5}g^{\alpha_4\alpha_6} + g^{\alpha_2\alpha_6}g^{\alpha_4\alpha_5} \right] + g^{\alpha_1\alpha_4} \left[ g^{\alpha_2\alpha_3}g^{\alpha_5\alpha_6} + g^{\alpha_2\alpha_5}g^{\alpha_3\alpha_6} + g^{\alpha_2\alpha_6}g^{\alpha_3\alpha_4} \right] + g^{\alpha_1\alpha_5} \left[ g^{\alpha_2\alpha_3}g^{\alpha_4\alpha_6} + g^{\alpha_2\alpha_4}g^{\alpha_3\alpha_6} + g^{\alpha_2\alpha_6}g^{\alpha_3\alpha_4} \right] + g^{\alpha_1\alpha_6} \left[ g^{\alpha_2\alpha_3}g^{\alpha_4\alpha_5} + g^{\alpha_2\alpha_5}g^{\alpha_3\alpha_4} + g^{\alpha_2\alpha_4}g^{\alpha_3\alpha_5} \right]
$$

(C.12)

and

$$
\int d^D k \ \prod_{i=1}^{2M+1} k_i^\alpha f(k^2) = 0.
$$

(C.13)

For each loop integral a universal factor

$$
S_\varepsilon = \exp \left[ (\gamma_E - \ln(4\pi)) \frac{\varepsilon}{2} \right]
$$

(C.14)

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emerges, where $\gamma_E$ denotes the Euler–Mascheroni constant

$$\gamma_E = \lim_{k \to \infty} \left[ \sum_{l=1}^{k} \frac{1}{l} - \ln(l) \right].$$  \hspace{1cm} (C.15)$$

The factors $S_\varepsilon$ are kept separately and are not expanded in $\varepsilon$. In the $\overline{\text{MS}}$-scheme [88] they are set to $S_\varepsilon = 1$ at the end of the calculation. The $\Gamma$-function obeys the relation

$$\Gamma(1 + \varepsilon) = \exp \left[ -\gamma_E \varepsilon + \sum_{n=2}^{\infty} \frac{(-\varepsilon)^n}{n} \zeta_n \right],$$  \hspace{1cm} (C.16)$$

with

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad n \in \mathbb{N}, \quad n \geq 2,$$

the Riemann $\zeta$-function at integer arguments, $n \geq 2$.

We apply the following Feynman parametrization to combine denominators

$$\frac{1}{A_1 \ldots A_n} = \Gamma(n) \int_{0}^{1} dx_1 \ldots \int_{0}^{1} dx_n \delta \left( \sum_{k=1}^{n} x_k - 1 \right) \frac{1}{(x_1 A_1 + \ldots + x_n A_n)^n}$$  \hspace{1cm} (C.18)$$

resp.

$$\frac{1}{A_1^{a_1} \ldots A_n^{a_n}} = \Gamma \left( \sum_{k=1}^{n} a_k \right) \frac{1}{\prod_{k=1}^{n} \Gamma(a_k)} \int_{0}^{1} dx_1 \ldots \int_{0}^{1} dx_n \delta \left( \sum_{k=1}^{n} x_k - 1 \right) \prod_{k=1}^{n} x_k^{a_k-1} \frac{1}{(x_1 A_1 + \ldots + x_n A_n) \left( \sum_{k=1}^{n} a_k \right)},$$  \hspace{1cm} (C.19)$$

with $\forall a_i \in \mathbb{N}$. The integral over the $\delta$–distribution yields

$$\int_{0}^{1} dx_1 \delta \left( \sum_{k=1}^{n} x_k - 1 \right) = \int_{-\infty}^{+\infty} dx_1 \delta \left( \sum_{k=1}^{n} x_k - 1 \right) \theta \left( 1 - \sum_{k=1}^{n} x_k \right) \prod_{m=1}^{n} \theta(x_m)$$

$$= \theta \left( 1 - \sum_{k=1,k \neq l}^{n} x_k \right) \prod_{m=1,m \neq l}^{n} \theta(x_m),$$  \hspace{1cm} (C.20)$$

where $\theta(z)$ denotes the Heaviside function

$$\theta(z) = \begin{cases} 
1, & z \geq 0 \\
0, & z < 0
\end{cases}.$$  \hspace{1cm} (C.21)
D Special functions

D.1 The Euler Integrals

The $\Gamma$-function, cf. [320, 321], is analytic in the whole complex plane except at the non-positive integers, where it possesses single poles. Euler’s infinite product defines

$$\Gamma(z) = z \exp(\gamma_E z) \prod_{i=1}^{\infty} \left(1 + \frac{z}{i}\right) \exp\left(-\frac{z}{i}\right).$$

(D.22)

The residues of the $\Gamma$-function at its poles are given by

$$\text{Res}[\Gamma(z)]_{z=-N} = \frac{(-1)^N}{N!}, \quad N \in \mathbb{N} \cup \{0\}.$$

(D.23)

In case of $\text{Re}(z) > 0$, the $\Gamma$-function can be expressed by Euler’s integral

$$\Gamma(z) = \int_0^\infty dt \exp(-t) t^{z-1},$$

(D.24)

from which one infers the well known functional equation of the $\Gamma$-function

$$\Gamma(z+1) = z\Gamma(z), \quad z \neq -n, n \in \mathbb{N} \cup \{0\}.$$

(D.25)

Eq. (D.25) may be used to continue the $\Gamma$-function analytically. Around $z = 1$, the following series expansion is obtained

$$\Gamma(1 - \varepsilon) = \exp(\varepsilon \gamma_E) \exp\left\{\sum_{i=2}^{\infty} \zeta_i \frac{\varepsilon^i}{i}\right\}, |\varepsilon| < 1.$$

(D.26)

Here and in (D.22), $\gamma_E$ denotes the Euler-Mascheroni constant, cf. (C.15), and $\zeta_k$ Riemann’s $\zeta$–function for integer arguments $k$, cf. (C.17). A shorthand notation for rational functions of $\Gamma$–functions is

$$\Gamma\left[\begin{array}{c} a_1, \ldots, a_i \\ b_1, \ldots, b_j \end{array}\right] := \frac{\Gamma(a_1)\ldots\Gamma(a_i)}{\Gamma(b_1)\ldots\Gamma(b_j)}.$$

(D.27)

A further useful identity involving the $\Gamma$-function is given by

$$\Gamma(\varepsilon - N) = (-1)^N \frac{\Gamma(\varepsilon)\Gamma(1 - \varepsilon)}{\Gamma(N + 1 - \varepsilon)}.$$

(D.28)

Functions closely related to the $\Gamma$-function are the Euler Beta-function $B(A, C)$ function, the $\psi(x)$–, and the $\beta(x)$–functions.

The Beta-function can be defined by Eq. (D.27)

$$B(A, C) = \Gamma\left[\begin{array}{c} A, C \\ A + C \end{array}\right].$$

(D.29)

If $\text{Re}(A), \text{Re}(C) > 0$, the following integral representation is valid

$$B(A, C) = \int_0^1 dx \ x^{A-1}(1 - x)^{C-1}.$$

(D.30)
For arbitrary values of $A$ and $C$, (D.30) can be continued analytically outside of the respective singularities using Eqs. (D.22, D.29). Its expansion around singularities can be performed via Eqs. (D.23, D.26). The $\psi$-function and $\beta(x)$ are defined as logarithmic derivatives of the $\Gamma$-function via

\[
\psi(x) = \frac{1}{\Gamma(x)} \frac{d}{dx} \Gamma(x) , \quad \beta(x) = \frac{1}{2} \left[ \psi \left( \frac{x+1}{2} \right) - \psi \left( \frac{x}{2} \right) \right] .
\]

**D.2 Polylogarithms**

The polylogarithms [283–285] are defined on the unit disk in the complex plane by

\[
\text{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k} .
\]

This definition can be uniquely extended to the whole complex plane by a analytic continuation. In the special case $z = 1$ (D.33) becomes the sum representation of the Riemann $\zeta$–function

\[
\text{Li}_k(1) = \zeta_k , k \geq 2 .
\]

The polylogarithm (D.33) furthermore has the representation

\[
\text{Li}_0(z) = \frac{z}{1 - z}
\]

and

\[
\text{Li}_{k+1}(z) = \int_0^z \frac{\text{Li}_k(t)}{t} \, dt , k \geq 0 .
\]

When considering the combination $\text{Li}_k(-z) + \text{Li}_k(z)$ all odd powers in definition (D.33) cancel and one obtains the identity

\[
\text{Li}_k(-z) + \text{Li}_k(z) = 2^{1-k}\text{Li}_k(z^2) .
\]

**D.2.1 Harmonic polylogarithms**

A more general form of the iterated integrals are the harmonic polylogarithms [180]. A harmonic polylogarithm of weight $w$ is uniquely defined by

\[
H_{\{\}}(x) = 1 , \quad H_{\{0,\ldots,0\}}(x) = \frac{1}{n!} \ln^n(x) , \quad H_{\{a,\overline{a}\}}(x) = \int_0^1 dx \, f_a(x) H_{\overline{a}}(x) .
\]

and otherwise

\[
H_{a,w}(x) = \int_0^1 dx \, f_a(x) H_{w}(x) .
\]
Here \( a \in \{-1, 0, 1\} \) and \( f_a \) is defined by

\[
f_1 = \frac{1}{1-x}, \quad f_0 = \frac{1}{x}, \quad f_{-1} = \frac{1}{1+x}. \tag{D.41}
\]

All these iterated integral structures are associated to specific sum structures in two different ways:

They emerge in the coefficients \( c_n, d_n \) of the series expansions around 0 and \( \infty \), e.g.

\[
L_{\overline{a}}(z) = \sum_{k=0}^{w(v)} \sum_{n=0}^{\infty} c_n z^n,
\]

\[
L_{\overline{a}}(z) = \sum_{k=0}^{w(v)} \sum_{n=0}^{\infty} d_n z^{-n}. \tag{D.43}
\]

Furthermore, the same sum structures are observed in the Mellin transforms of the same iterated integrals.

### D.3 Sum structures

We distinguish the following classes of sums [286, 311]:

Harmonic sums have been observed in physical results for some time by now. Many of their algebraic [166–168], differential and structural relations [169–171] have been studied. They are defined by

\[
S_{a_1, \ldots, a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\text{sign}(a_1))^{n_1}}{n_1^{a_1}} \frac{(\text{sign}(a_2))^{n_2}}{n_2^{a_2}} \cdots \frac{(\text{sign}(a_m))^{n_m}}{n_m^{a_m}},
\]

\( N \in \mathbb{N}, \ \forall \ l \ a_l \in \mathbb{Z} \setminus 0 \),

\( S_{\emptyset} = 1 \). \tag{D.44}

A first generalization, the generalized harmonic sums are obtained by allowing for additional weights in the numerators [174, 175]:

\[
S_{a_1, \ldots, a_m}(x_1, \ldots, x_m, N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(x_1)^{n_1}}{n_1^{a_1}} \frac{(x_2)^{n_2}}{n_2^{a_2}} \cdots \frac{(x_m)^{n_m}}{n_m^{a_m}},
\]

\( N \in \mathbb{N}, \ \forall \ l \ a_l \in \mathbb{N} \),

\( S_{\emptyset}(\emptyset) = 1 \). \tag{D.47}

Furthermore, in recent calculations finite sums weighted by binomials \( \binom{2i}{j} \) in both, numerators and denominators have been frequently observed [287, 296]. They are of a similar form as

\[
\sum_{i=1}^{N} \binom{2i}{i} (-1)^i \frac{1}{j} \binom{2j}{j} S_{1,2}(\frac{1}{2}, -1; j)
\]

\[
= \int_0^1 dx \frac{x^{N-1}}{x-1} \sqrt{\frac{x}{8+x}} [H^{*}_{w_{17},-1,0}(x) - 2H^{*}_{w_{18},-1,0}(x)]
\]

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\[ + \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^N - 1}{x + 1} \sqrt{\frac{x}{8 + x}} [H_{12}^*(x) - 2H_{13}^*(x)] + c_3 \int_0^1 dx \frac{(-8x)^N - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1 - x}}, \]

where in this example

\[ w_{12} = \frac{1}{\sqrt{x(8 - x)}}, \quad w_{13} = \frac{1}{(2 - x)\sqrt{x(8 - x)}}, \]
\[ w_{17} = \frac{1}{\sqrt{x(8 + x)}}, \quad w_{18} = \frac{1}{(2 + x)\sqrt{x(8 + x)}}. \]  

(D.49)

Here the \( H_{xy}^* \) represent integrals over the word \( w_{xy} \) on the interval \([x, 1]\).
E The logarithmic contributions to the heavy flavor Wilson coefficients

In this Appendix we present all logarithmic contributions to asymptotic heavy flavor Wilson coefficients \( H_{q,2}^{PS} \) and \( H_{g,2}^{S} \) with respect to the single-mass case, see also Ref. [220].

\[
H_{q,2}^{PS} = \frac{1}{2} [1 + (-1)^N]
\]

\[
\times \left\{ a_s^2 \left[ C_F T_F \left[ -\frac{4L_1^M(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{(4S_1^N - 12S_2)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right]
\]
\[
+ \frac{4L_2^M(N^2 + N + 2)^2}{8P_7(N-1)N^2(N+1)^2(N+2)} - \frac{32(-1)^NP_{45}}{3(N-1)N^2(N+1)^3(N+2)^3}
\]
\[
+ \frac{3(N-1)N^4(N+1)^4(N+2)^3}{8P_5S_1} + \frac{(N-1)N^3(N+1)^3(N+2)^2}{8P_57} + L_Q \left[ \frac{64S_{-2}}{(N-1)N(N+1)(N+2)} \right]
\]
\[
- \frac{8(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)L_M}{(N-1)N^3(N+1)^3(N+2)^2} \right\}
\]

\[
+ a_t^2 \left[ C_F T_F \left[ L_Q^3 \left[ \frac{8(N^2 + N + 2)^2(3N^2 + 3N + 2)}{3(N-1)N^3(N+1)^3(N+2)} - \frac{32(N^2 + N + 2)^2S_1}{3(N-1)N^2(N+1)^2(N+2)} \right]
\]
\[
+ \frac{3(N-1)N^4(N+1)^4(N+2)^3}{8P_62S_1} + \frac{(N-1)N^3(N+1)^3(N+2)^2}{4P_{173}S_7^2}
\]
\[
+ \frac{16(N^2 + N - 22)S_3(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{(128S_{-3} - 256S_{-2,1} - 384\zeta_3)(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)}
\]
\[
- \frac{2(N-1)N^3(N+1)^3(N+2)^2}{64(-1)^NP_{38}^{100}} + \frac{15(N-2)(N-1)^4N^3(N+1)^6(N+2)^4(N+3)^3}{128(-1)^NP_{55}S_1}
\]
\[
+ \frac{3(N-1)^3N^5(N+1)^6(N+2)^4(N+3)^3}{8P_{70}S_1} + \frac{16(N^2 + N + 2)^2S_1}{512S_{-2}S_1} - \frac{4(N^2 + N + 2)^2(3N^2 + 3N + 2)}{(N-1)N^3(N+1)^3(N+2)^2}
\]
\[
+ L_M^2 \left[ \frac{16(N^2 + N + 2)^2S_1}{(N-1)N^2(N+1)^2(N+2)} - \frac{4(N^2 + N + 2)^2(3N^2 + 3N + 2)}{(N-1)N^3(N+1)^3(N+2)^2} \right]
\]
\[
+ L_M \left[ \frac{32(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)S_1}{(N-1)N^3(N+1)^3(N+2)^2}
\]
\[
- \frac{8(N^2 + 5N + 2)(3N^2 + 3N + 2)(5N^3 + 7N^2 + 4N + 4)}{(N-1)N^3(N+1)^4(N+2)}
\]
\[
+ \frac{32P_{90}S_{-2}}{(N-2)(N-1)N^3(N+1)^3(N+2)(N+3)} \right]\]
\[
\begin{align*}
&+ \frac{64 P_{37} S_1}{3(N-1)N^3(N+1)^3(N+2)^2} + \frac{16 P_{61} S_2}{9(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[ \frac{-64 S_9}{9} - \frac{32 S_2 S_1 - \frac{32}{3} \zeta S_1 + \frac{160 S_3}{9} + \frac{128 \zeta_3}{9}}{8} \right] \\
&+ N_F T_F^2 C_F \left[ \frac{32(N^2 + N + 2)^2 L_Q^3}{9(N-1)N^2(N+1)^2(N+2)} - \frac{32 P_{86} L_Q^3}{9(N-1)N^3(N+1)^3(N+2)^2} \\
&\quad + \left[ \frac{-16 S_9^2 - \frac{16 S_2}{3}}{(N-1)N^2(N+1)^2(N+2)^2} \right] \left( \frac{N^2 + N + 2}{8} \right) + \frac{32(8N^3 + 13N^2 + 27N + 16) S_1(N^2 + N + 2)}{9(N-1)N^2(N+1)^3(N+2)} \\
&- \frac{256(-1)^N P_{45}}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{27(N-1)N^4(N+1)^4(N+2)^4}{32 P_{84}} \\
&+ \frac{512 S_{-2}}{3(N-1)N(N+1)(N+2)} L_Q - \frac{9(N-1)N^2(N+1)^2(N+2)}{16} \frac{\zeta_2}{S_1} \\
&+ \frac{32 P_{64}}{(N-1)N^2(N+1)^3(N+2)^2} - \frac{32(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \\
&+ \frac{L_M^2}{- \frac{16 S_9^2 - \frac{80 S_2}{3}}{(N-1)N^2(N+1)^2(N+2)^2} \left( \frac{N^2 + N + 2}{8} \right) - \frac{27(N-1)N^4(N+1)^4(N+2)^3}{32 P_{78}} \\
&+ \frac{9(N-1)N^3(N+1)^3(N+2)^2}{32 P_{63} S_1} \frac{64(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4) S_2}{3(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[ \frac{64 S_9^2 + \frac{16 S_2}{3} \zeta_2 + \frac{32 \zeta_3}{9}}{8} \right] \\
&+ C_A C_F T_F \left[ L_Q^3 \frac{-16 S_1(N^2 + N + 2)^2}{3(N-1)N^2(N+1)^2(N+2)} \\
&\quad - \frac{8(11N^4 + 22N^3 - 23N^2 - 34N - 12)(N^2 + N + 2)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2} \right] \\
&+ L_Q^2 \left[ \frac{-16(5N^2 - 1)S_1(N^2 + N + 2)^2}{(N-1)^2N^3(N+1)^3(N+2)^2} + \frac{(16S_9^2 - 16S_7 - 32S_{-2})(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right] \\
&\quad + \frac{16(1 \cdot 5(N^5 + 9N^4 + 23N^3 + 36N^2 + 32N + 8)(N^2 + N + 2)}{(N-1)N^3(N+1)^4(N+2)^3} \\
&\quad + \frac{8 P_{85}}{9(N-1)^2N^4(N+1)^4(N+2)^3} \right]\right]
\end{align*}
\]
\[
\frac{32(-1)^N P_7 S_1}{3(N - 1) N^3(N + 1)^3(N + 2)^3} + \frac{8 P_{ss} S_1}{9(N - 1) N^2(N + 1)^4(N + 2)^3} + \frac{16 P_{7} S_{-2}}{3(N - 1)^2 N^3(N + 1)^3(N + 2)^2} + \frac{4(N^2 + N + 2) P_{43} S_{1}^2}{(N^2 + N + 2) \zeta_3} + \frac{8(N^2 + N + 2)}{(N - 1) N^3(N + 1)^3(N + 2)^2} \zeta_2 + 8(P_{5,9} + \frac{4}{9} (N^2 + N + 2) \zeta_2 P_{34}^4 S_1)
\]

\[
\frac{8 P_{39} S_1}{3(N - 1)^2 N^2(N + 1)^3(N + 2)^2} - \frac{8(N^2 + N + 2) P_{70} S_1}{(N - 1) N^2(N + 1)^4(N + 2)^3} + L_M \left[ \frac{16 S_1 (N^2 + N + 2)^2}{3(N - 1) N^2(N + 1)^4(N + 2)^2} \right] + \frac{4 P_{ss} S_2}{3(N - 1)^2 N^2(N + 1)^3(N + 2)^2}
\]

\[
\frac{8(N^2 + N + 2)(N^4 + 2N^3 + 7N^2 + 22N + 20) S_{-2}}{(N - 1) N^2(N + 1)^4(N + 2)^3} + L_M \left[ \frac{(16 S_2 + 32 S_{-2}) (N^2 + N + 2)^2}{(N - 1) N^2(N + 1)^2(N + 2)} + \frac{8 P_{34} S_1 (N^2 + N + 2)}{3(N - 1)^2 N^2(N + 1)^3(N + 2)^2} \right] - \frac{16(N^2 + N + 2) P S_3}{3(N - 1)^2 N^2(N + 1)^3(N + 2)^2}
\]

\[
32(-1)^N (N^2 + N + 2)(N^4 + 2N^3 + 7N^2 + 22N + 20) S_{-2}
\]

\[
- \frac{8 P_{83}}{9(N - 1)^2 N^4(N + 1)^4(N + 2)^3} + \frac{(N^2 - N - 4)(N^2 + N + 2)}{(N - 1) N^2(N + 1)^4(N + 2)^2} \left[ -64(-1)^N S_1 S_{-2} - 32(-1)^N S_{-3} + 64 S_{-2.1} - 32(-1)^N S_1 \zeta_2 - 24(-1)^N \zeta_3 \right]
\]

\[
+ \frac{(N^2 + N + 2)^2}{(N - 1) N^2(N + 1)^2(N + 2)} \left[ \frac{2}{3} S_1^2 - 20 S_2 S_1^2 - 32(-1)^N S_{-3} S_1 + \left( 64 S_{-2.1} - \frac{160 S_3}{3} \right) S_1 \right] - \frac{8}{3} \left( -7 + 9(-1)^N \zeta_3 \right) S_1 - 2 S_2^2 + S_{-2}(-32(-1)^N S_1^2 - 32(-1)^N S_2) - 36 S_4 - 16(-1)^N S_{-4}
\]

\[
+ 16 S_{3.1} + 32 S_{-2.2} + 32 S_{-3.1} + 16 S_{2.1.1} - 64 S_{-2.1.1} + \left( -4(-1 + 4(-1)^N) S_2 \right) - 4(-1 + 4(-1)^N) S_2 - 8(1 + 2(-1)^N) S_{-2} \zeta_2 + L_M \left[ \frac{(N^2 + N + 2)^2}{(N - 1) N^2(N + 1)^2(N + 2)} \left[ -\frac{8}{3} S_1^3 + 40 S_2 S_1 + 32(1 + (-1)^N) S_{-2} S_1 + 16(-1)^N S_{-3} - 32 S_{2.1} + 12(-9 + (-1)^N) \zeta_3 \right]
\]

\[
+ \frac{4(17 N^4 - 6 N^3 + 41 N^2 - 16 N - 12) S_1^2 (N^2 + N + 2)}{3(N - 1)^2 N^2(N + 1)^2(N + 2)} + \frac{4 P_{56} S_2 (N^2 + N + 2)}{3(N - 1)^2 N^2(N + 1)^3(N + 2)^2}
\]

\[
+ \frac{8(31 N^2 + 31 N + 74) S_2 (N^2 + N + 2)}{3(N - 1)^2 N^2(N + 1)^2(N + 2)} + \frac{16(7 N^2 + 7 N + 10) S_{-3}(N^2 + N + 2)}{(N - 1) N^2(N + 1)^2(N + 2)}
\]

\[
- \frac{128(N^2 + N + 1) S_{-2.1} (N^2 + N + 2)}{(N - 1) N^2(N + 1)^2(N + 2)} + \frac{(N^2 - N - 4)(N^2 + N + 2) 32(-1)^N S_{-2}}{(N - 1) N^2(N + 1)^3(N + 2)^2}
\]
\[
\begin{aligned}
& 64(-1)^{N}P_{34} \quad 8P_{96} \\
& \quad 9(N - 1)^{N}P_{46}S_{1} \quad 27(N - 1)^{2}N^{4}(N + 1)^{5}(N + 2)^{4} \\
& + \quad 3(N - 1)^{N}P_{46}S_{1} \quad 8P_{96}S_{1} \\
& + \quad 16P_{88}S_{-2} \quad \left\{ \frac{9(N - 1)^{2}N^{4}(N + 1)^{4}(N + 2)^{3}}{(N - 1)^{N}P_{46}(N + 1)^{3}(N + 2)^{2}} \right\} \bigg] \\
& + a_{N}^{P}(3) + C_{2q}^{P}(3)(N_{F} + 1) \bigg), \\
\end{aligned}
\]

with the polynomials

\begin{align}
P_{34} &= N^{6} + 6N^{5} + 7N^{4} + 4N^{3} + 18N^{2} + 16N - 8 \\
P_{44} &= 2N^{6} + 7N^{5} + 31N^{4} + 82N^{3} + 86N^{2} + 32N + 8 \\
P_{45} &= 4N^{6} + 22N^{5} + 48N^{4} + 53N^{3} + 45N^{2} + 36N + 8 \\
P_{46} &= 5N^{6} + 29N^{5} + 78N^{4} + 118N^{3} + 114N^{2} + 72N + 16 \\
P_{47} &= 5N^{6} + 135N^{5} + 327N^{4} + 329N^{3} + 220N^{2} - 176N - 120 \\
P_{48} &= 8N^{6} + 29N^{5} + 84N^{4} + 193N^{3} + 162N^{2} + 124N + 24 \\
P_{49} &= 11N^{6} + 6N^{5} + 75N^{4} + 68N^{3} - 200N^{2} - 80N - 24 \\
P_{50} &= 11N^{6} + 29N^{5} - 7N^{4} - 25N^{3} - 56N^{2} - 72N - 24 \\
P_{51} &= 16N^{6} + 35N^{5} + 33N^{4} - 11N^{3} - 41N^{2} - 36N - 12 \\
P_{52} &= 17N^{6} - 57N^{5} - 213N^{4} - 175N^{3} - 140N^{2} + 64N + 72 \\
P_{53} &= 17N^{6} + 27N^{5} + 75N^{4} + 149N^{3} - 20N^{2} - 80N - 24 \\
P_{54} &= 17N^{6} + 51N^{5} + 51N^{4} + 89N^{3} + 40N^{2} - 80N - 24 \\
P_{55} &= 38N^{6} + 108N^{5} + 151N^{4} + 106N^{3} + 21N^{2} - 28N - 12 \\
P_{56} &= 73N^{6} + 189N^{5} + 45N^{4} + 31N^{3} - 238N^{2} - 412N - 120 \\
P_{57} &= N^{7} - 15N^{6} + 58N^{5} - 92N^{4} - 76N^{3} - 48N - 16 \\
P_{58} &= 2N^{7} + 14N^{6} + 37N^{5} + 102N^{4} + 155N^{3} + 158N^{2} + 132N + 40 \\
P_{59} &= 3N^{7} - 15N^{6} - 153N^{5} - 577N^{4} - 854N^{3} - 652N^{2} - 408N - 128 \\
P_{60} &= 5N^{7} + 19N^{6} + 61N^{5} + 197N^{4} + 266N^{3} + 212N^{2} + 136N + 32 \\
P_{61} &= 5N^{7} + 37N^{6} + 188N^{5} + 643N^{4} + 925N^{3} + 742N^{2} + 460N + 120 \\
P_{62} &= 7N^{7} + 21N^{6} + 5N^{5} - 117N^{4} - 244N^{3} - 232N^{2} - 192N - 80 \\
P_{63} &= 8N^{7} + 37N^{6} + 68N^{5} - 11N^{4} - 86N^{3} - 56N^{2} - 104N - 48 \\
P_{64} &= 8N^{7} + 37N^{6} + 83N^{5} + 85N^{4} + 61N^{3} + 58N^{2} - 20N - 24 \\
P_{65} &= 9N^{7} + 37N^{6} - 103N^{5} - 575N^{4} - 998N^{3} - 948N^{2} - 696N - 256 \\
P_{66} &= 11N^{7} + 37N^{6} + 53N^{5} + 7N^{4} - 68N^{3} - 56N^{2} - 80N - 48 \\
P_{67} &= 25N^{7} + 91N^{6} + 101N^{5} - 195N^{4} - 546N^{3} - 556N^{2} - 520N - 224 \\
P_{68} &= 62N^{7} + 329N^{6} + 986N^{5} + 1790N^{4} + 2242N^{3} + 1653N^{2} + 650N + 96 \\
P_{69} &= N^{8} + 8N^{7} + 8N^{6} - 14N^{5} - 53N^{4} - 82N^{3} + 60N^{2} + 104N + 96 \\
P_{70} &= 2N^{8} + 22N^{7} + 117N^{6} + 386N^{5} + 759N^{4} + 810N^{3} + 396N^{2} + 72N + 32 \\
P_{71} &= 6N^{8} - 42N^{7} - 241N^{6} - 579N^{5} - 307N^{4} + 477N^{3} + 602N^{2} + 492N + 168 \\
P_{72} &= 10N^{8} + 71N^{7} + 244N^{6} + 497N^{5} + 698N^{4} + 720N^{3} + 512N^{2} + 248N + 48 \\
P_{73} &= 19N^{9} + 86N^{8} + 144N^{7} - 38N^{6} - 535N^{5} - 1016N^{4} - 1180N^{3} - 872N^{2} \\
&\quad - 416N - 96 \\
P_{74} &= N^{10} + 15N^{9} + 105N^{8} + 361N^{7} + 660N^{6} + 828N^{5} + 814N^{4} + 384N^{3} \\
&\quad - 112N^{2} - 128N - 32 \\
\end{align}
\]
\[ P_{75} = 6N^{10} + 49N^9 + 197N^8 + 472N^7 + 833N^6 + 1469N^5 + 2142N^4 + 1904N^3 + 1040N^2 + 432N + 96 \]
\[ P_{76} = 11N^{10} + 123N^9 + 541N^8 + 1273N^7 + 1806N^6 + 1672N^5 + 1006N^4 + 320N^3 - 16N^2 - 64N - 32 \]
\[ P_{77} = 19N^{10} + 143N^9 + 412N^8 + 426N^7 - N^6 + 159N^5 + 1066N^4 + 1552N^3 + 1456N^2 + 848N + 224 \]
\[ P_{78} = 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 + 672N + 288 \]
\[ P_{79} = 60N^{10} + 397N^9 + 1073N^8 + 1111N^7 + 623N^6 + 4328N^5 + 12432N^4 + 15944N^3 + 12704N^2 + 6816N + 1728 \]
\[ P_{80} = 67N^{10} + 383N^9 + 867N^8 + 696N^7 - 755N^6 - 2391N^5 - 3027N^4 - 2744N^3 - 1256N^2 - 48N + 144 \]
\[ P_{81} = 77N^{10} + 646N^9 + 2553N^8 + 6903N^7 + 14498N^6 + 22898N^5 + 24861N^4 + 17068N^3 + 7040N^2 + 1760N + 192 \]
\[ P_{82} = 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 + 1864N^2 + 1248N + 432 \]
\[ P_{83} = 118N^{10} + 675N^9 + 1588N^8 + 1652N^7 + 326N^6 + 357N^5 + 876N^4 + 1672N^3 + 3440N^2 + 2544N + 576 \]
\[ P_{84} = 127N^{10} + 740N^9 + 1737N^8 + 1308N^7 - 1592N^6 - 2226N^5 + 1386N^4 + 3064N^3 + 3040N^2 + 2496N + 864 \]
\[ P_{85} = 151N^{10} + 708N^9 + 1156N^8 + 464N^7 - 967N^6 + 372N^5 + 3672N^4 + 5236N^3 + 6152N^2 + 3792N + 864 \]
\[ P_{86} = 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 + 9608N^3 + 6512N^2 + 2432N + 384 \]
\[ P_{87} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2 - 208N - 32 \]
\[ P_{88} = 118N^{11} + 529N^{10} + 1264N^9 + 3846N^8 + 11353N^7 + 23684N^6 + 32793N^5 + 31801N^4 + 22836N^3 + 10448N^2 + 2592N + 432 \]
\[ P_{89} = 127N^{11} + 820N^{10} + 2197N^9 + 1890N^8 - 1847N^7 - 1960N^6 + 3843N^5 + 9730N^4 + 13632N^3 + 10688N^2 + 4944N + 864 \]
\[ P_{90} = 136N^{11} + 1039N^{10} + 3100N^9 + 3534N^8 - 1295N^7 - 6352N^6 - 8421N^5 - 11729N^4 - 7644N^3 + 1376N^2 + 1920N + 144 \]
\[ P_{91} = 7N^{12} + 47N^{11} + 123N^{10} + 76N^9 - 598N^8 - 2178N^7 - 3626N^6 - 3933N^5 - 3254N^4 - 1608N^3 - 144N^2 + 112N + 32 \]
\[ P_{92} = 37N^{12} + 305N^{11} + 1017N^{10} + 1462N^9 + 592N^8 + 408N^7 + 4064N^6 + 9645N^5 + 12222N^4 + 10280N^3 + 6064N^2 + 2192N + 352 \]
\[ P_{93} = 242N^{12} + 1853N^{11} + 6173N^{10} + 12711N^9 + 18608N^8 + 17040N^7 - 302N^6 - 24986N^5 - 32225N^4 - 20010N^3 - 7904N^2 - 2016N - 288 \]
\[ P_{94} = 5N^{13} + 27N^{12} - 97N^{11} - 1410N^{10} - 5754N^9 - 12428N^8 - 16530N^7 - 14531N^6 - 7956N^5 - 1038N^4 + 2176N^3 + 1632N^2 + 448N + 32 \]
\[ P_{95} = 119N^{13} + 1897N^{12} + 12595N^{11} + 48221N^{10} + 124877N^9 + 239946N^8 \]
\[ H = S^+ + 1 \]
\[ \mu^2 = 1245, \quad \eta^2 = 1790 \]
\[ N_1^2 = 30N^6 + 397N^15 + 56250N^{12} + 90805N^{11} + 43917N^{10} - 38450N^9 - 42314N^8 - 169217N^7 + 616623N^6 - 992860N^5 - 964980N^4 - 697072N^3 - 376464N^2 - 23544N - 3888 \]
\[ N_2^2 = 12N^{17} + 162N^{16} + 1030N^{15} + 4188N^{14} + 11527N^{13} + 19051N^{12} + 11176N^{11} - 17182N^{10} - 36527N^9 - 27469N^8 - 11770N^7 + 5554N^6 + 32640N^5 + 4656N^4 + 34528N^3 + 14816N^2 + 3584N + 384 \]
\[ N_3^2 = \frac{1245N^{18} + 19980N^{17} + 133282N^{16} + 461805N^{15} + 78716N^{14} + 185392N^{13} - 1368400N^{12} - 225082N^{11} + 697863N^{10} + 1314336N^9 + 5808466N^8 - 11433627N^7 - 19928573N^6 - 12013164N^5 + 1462668N^4 + 8209584N^3 + 6906384N^2 + 2980800N + 544320}{E.108} \]

The Wilson coefficient \( H^{S}_{g,2} \), except for the constant contribution \( a_{Qg}^{(3)} \), has a similar structure. It is given by:

\[ H^{S}_{g,2} = \frac{1}{2} [1 + (-1)^N] \]
\[ \times \left\{ a_{\pi} T_F \left\{ -\gamma_{qg} L_Q - \frac{4(N^3 - 4N^2 - N - 2)}{N^2(N + 1)(N + 2)} + \gamma_{qg}^0 S_1 + \gamma_{qg}^0 L_M \right\} \right. \]
\[ \left. + a_{\pi}^2 \left\{ \frac{4}{3} \gamma_{qg} L_M^2 - \frac{4}{3} \gamma_{qg}^0 L_Q L_M + \left[ \frac{4}{3} \gamma_{qg}^0 S_1 - \frac{16(N^3 - 4N^2 - N - 2)}{3N^2(N + 1)(N + 2)} \right] L_M \right\} \right. \]
\[ \left. + C_F T_F \left[ \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} + 2\gamma_{qg}^0 S_1 \right] L_Q + \left[ \frac{4P_{111}}{N^3(N + 1)^3(N + 2)} \right] \right. \]
\[ \left. + \frac{4(3N^4 + 2N^3 - 9N^2 - 16N - 12)S_1}{N^2(N + 1)^2(N + 2)} + L_M \left[ -\frac{4(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} \right] \right. \]
\[ \left. - 4\gamma_{qg} S_1 \right. \]
\[ \left. + \gamma_{qg}^0 (4S_2 - 4S_1^2) \right\} L_Q - \frac{2(9N^4 + 6N^3 - 15N^2 - 28N - 20)S_1^2}{N^2(N + 1)^2(N + 2)} \]
\[ + \frac{16(-1)^NP_{252}}{15(N - 2)(N - 1)^2N^2(N + 1)^4(N + 2)^4(N + 3)} + L_M^2 \left[ \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} \right] \]
\[ + \frac{16(-1)^NP_{106}}{15(N - 1)^2N^4(N + 1)^4(N + 2)^4(N + 3)^3} + \frac{4P_{111}}{N^3(N + 1)^3(N + 2)} \left[ \frac{4P_{111}}{N^3(N + 1)^3(N + 2)} \right] \]
\[ + L_M \left[ \frac{4P_{111}}{N^3(N + 1)^3(N + 2)} - \frac{4(3N^4 + 2N^3 - 9N^2 - 16N - 12)S_1}{N^2(N + 1)^2(N + 2)} + \gamma_{qg}^0 \left[ 4S_1^2 - 4S_2 \right] \right] \]
\[
2(11N^4 + 42N^3 + 43N^2 - 4N - 12)S_2 \quad + \quad 16P_{103}S_{-2} \quad + \quad (N - 2)^2N(1)^2(N + 2)(N + 3)
\]
\[
+ \gamma_{qq}^0 \left[ 2S_1^3 - 2S_2S_1 - 4S_{21} \right] + \frac{128S_{12}S_{-2} + 64S_{-2} - 128S_{-21}}{N(N + 1)(N + 2)} \quad + \quad (N - 1) \left[ 48 \zeta_3 - 16 \zeta_3 \right] \frac{N}{N(N + 1)}
\]
\[
+ C_A T_F \left[ \left[ \frac{16(N^2 + N + 1)(N^2 + N + 2)}{(N - 1)N^2(N + 1)^2(N + 2)^2} + 2\gamma_{qq}^0 S_1 \right] L_Q^2 \quad + \quad \left[ - \frac{32(N^3 + 4N^2 + 7N + 5)}{(N + 1)^3(N + 2)^3} \right] \right]
\]
\[
+ \frac{8P_{160}}{(N - 1)^3(N + 1)^2(N + 2)^3} + \frac{16(N^2 + N + 1)(N^2 - 4N - 1)}{(N - 1)N(N + 1)^2(N + 2)} \quad + \quad 32(-1)^N(N^3 + 4N^2 + 7N + 5)
\]
\[
+ \frac{16P_{160}S_1}{(N - 1)^3(N + 1)^2(N + 2)^3} + \frac{4(2N^5 - 3N^4 - 3N^3 - 5N^2 - 33N - 6)S_1^2}{N(N + 1)^2(N + 2)^2}
\]
\[
+ \frac{16(-1)^NP_{167}}{(N - 1)^3(N + 1)^2(N + 2)^3} - \frac{8P_{239}}{3(N - 1)^3(N + 1)^4(N + 2)^4}
\]
\[
+ \frac{32(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N + 1)^3(N + 2)^3} + L_M^2 \left[ - \frac{16(N^2 + N + 1)(N^2 + N + 2)}{(N - 1)N^2(N + 1)^2(N + 2)^2} + 2\gamma_{qq}^0 S_1 \right]
\]
\[
+ \frac{4P_{105}S_2}{(N - 1)^3(N + 1)^2(N + 2)^3} + \frac{8(3N^2 + 3N + 2)S_3}{N(N + 1)^2(N + 2)} + L_M \left[ \frac{32(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N + 1)^3(N + 2)^3} \right]
\]
\[
+ \frac{8P_{161}}{(N - 1)^3(N + 1)^2(N + 2)^3} - \frac{32(2N + 3)S_1}{(N + 1)^2(N + 2)^2} - 2\gamma_{qq}^0 \left[ - 2S_1^2 - 2S_2 - 4S_{-2} \right]
\]
\[
+ \frac{16(N^5 - 4N^4 - 5N^3 + 3N^2 + 14N + 12)S_{-2}}{(N - 1)N(N + 1)^2(N + 2)^2} + \frac{16(N^2 + N + 4)S_{-3}}{N(N + 1)(N + 2)}
\]
\[
+ \frac{(N - 1) \left[ 16S_{-2}S_1 - 16S_{-21} \right]}{N(N + 1)} + \left[ \frac{(N^2 - N - 4)16(-1)^NS_{-2}}{(N + 1)^2(N + 2)^2} - \frac{6(5N^2 + 5N - 6)\zeta_3}{N(N + 1)(N + 2)} \right]
\]
\[
+ \gamma_{qq}^0 \left[ - 8S_1S_2 - 4(-1)^NS_{-2}S_1 - 2(-1)^NS_{-3} + 4S_{21} - \frac{3}{2}(-1)^N\zeta_3 \right] \quad \right]
\]
\[
+ a_3 \left\{ \frac{16}{9} \gamma_{qq}^0 L_M^2 \quad - \quad \frac{16}{9} \gamma_{qq}^0 L_Q L_M \quad + \quad \frac{16}{9} \gamma_{qq}^0 S_1 = - \frac{64(N^3 - 4N^2 - N - 2)}{9N^2(N + 1)(N + 2)} \right\} L_M^2 \quad - \quad \frac{16\gamma_{qq}^0 S_3}{9}
\]
\[
+ C_A T_F \left[ \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{8\gamma_{qq}^0 S_1}{9} \right] L_Q^3 \quad + \quad \left[ - \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N + 1)^3(N + 2)^3} \right] \right]
\]
\[
+ \frac{8P_{194}}{9(N - 1)^3(N + 1)^3(N + 2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N - 1)N^2(N + 1)^2(N + 2)}
\]
\[
+ \frac{L_M \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{8\gamma_{qq}^0 S_1}{3} \right] + \gamma_{qq}^0 \left[ \frac{4}{3}S_1^2 + \frac{4S_2}{3} + \frac{8}{3}S_{-2} \right]}{L_Q^2}
\]
\[
+ \frac{- \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1^2}{9(N - 1)N^2(N + 1)^2(N + 2)} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N + 1)^3(N + 2)^3} \right]}
\]
\[
- \frac{32P_{186}S_1}{27(N - 1)N^2(N + 1)^3(N + 2)^3} + \frac{64(-1)^NP_{170}}{9(N - 1)N^2(N + 1)^4(N + 2)^4}
\]
\[
- \frac{16P_{233}}{27(N - 1)N^2(N + 1)^4(N + 2)^4} + \frac{L_M^2 \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{8\gamma_{qq}^0 S_1}{3} \right]}{169}
\]
\[
\begin{align*}
&+ \frac{32(8N^4 + 13N^3 - 22N^2 - 9N - 26)S_2}{9(N - 1)N(N + 1)(N + 2)^2} + \frac{64(8N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}}{9(N - 1)N(N + 1)^2(N + 2)^2} \\
&+ \frac{128(N^2 + N - 1)S_3}{9N(N + 1)(N + 2)} + \gamma_{99}^0 \left[ - \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N + 1)^3(N + 2)^3} + \frac{32P_{195}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \\
&- \frac{64(N - 1)(N^3 + 9N^2 + 7N + 3)S_1}{9(N - 1)N(N + 1)(N + 2)} + \gamma_{99}^0 \left[ - \frac{8}{3}S_1 - \frac{8}{3}S_2 + \frac{16}{3}S_{-2} \right] \right] - \frac{128(N^2 + N + 3)S_{-3}}{3N(N + 1)(N + 2)} \\
&+ \gamma_{99}^0 \left[ \frac{8}{9}S_1^2 - 8S_2S_1 + \frac{32}{3}S_{2,1} \right] + \frac{256S_{-2,1}}{3N(N + 1)(N + 2)} + \frac{(N - 1) \left[ \frac{64}{3}S_{-2}S_1 - 32\zeta_3 \right]}{N(N + 1)} \\
&+ \frac{8P_{152}S_1^2}{8P_{264}} + \frac{32}{3} \left( \frac{2N^4 + 3N^3 + 10N^2 + 37N + 35}{(N + 1)^3(N + 2)^3} \right) \\
&+ \frac{32(9N^5 - 4N^4 + N^3 + 92N^2 + 42N + 28)\zeta_4}{9} \left( \frac{N - 1)^2(N^2 + 1)^2(N + 2)^2}{(N + 1)^3(N + 2)^3} \right) \\
&+ \frac{32(5N^4 + 8N^3 + 17N^2 + 43N + 20)\zeta_2}{9} \left( \frac{N - 1)^2(N^2 + 1)^2(N + 2)^2}{(N + 1)^3(N + 2)^3} \right) \\
&- \frac{56\gamma_{99}^0 S_1}{9} + \frac{8P_{192}S_2}{3(N - 1)^3(N + 1)^3(N + 2)^3} - \frac{32(3N^3 - 12N^2 - 27N - 2)S_1S_2}{3N(N + 1)(N + 2)^2} \\
&+ \frac{256(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{L_M^2 \left[ \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N + 1)^3(N + 2)^3} \right]}{9(N - 1)N(N + 1)^2(N + 2)^2} \\
&- \frac{8P_{199}}{9(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{32(8N^5 + 9N^4 - 57N^3 - 31N^2 + 25N - 26)S_1}{9(N - 1)N(N + 1)^2(N + 2)^2} \\
&+ \gamma_{99}^0 \left[ - \frac{20}{3}S_1^2 - 4S_2 - 8S_{-2} \right] + \frac{128(-1)^N(N^4 + 2N^3 + 7N^2 + 22N + 20)S_{-2}}{3(N + 1)^3(N + 2)^3} \\
&+ \frac{(N^2 - N - 4)}{(N + 1)^2(N + 2)^2} \left[ - \frac{256}{3}(-1)^N S_1 S_{-2} - \frac{128}{3}(-1)^N S_{-3} + \frac{256}{3}S_{-2,1} - \frac{128}{3}(-1)^N S_1 \zeta_2 \right] \\
&- \frac{32(-1)^N \zeta_4}{9} \left[ \frac{2}{3}S_1^2 + \frac{20}{3}S_2S_2 + \frac{32}{3}(-1)^N S_{-3}S_1 + \frac{160S_3}{9} - \frac{64}{3}S_{-2,1} \right] + S_1 + \frac{2}{3}S_2^2 \\
&+ \frac{8(-2 + 9(-1)^N \zeta_3S_1 + \frac{32}{3}(-1)^N S_1^2 + \frac{32}{3}(-1)^N S_2}{16} + \frac{16}{3}S_4 \\
&+ \frac{16}{3}(-1)^N S_{-4} + \frac{16}{3}S_{-3,1} - \frac{32}{3}S_{-2,2} - \frac{32}{3}S_{-3,1} - \frac{16}{3}S_{2,1,1} + \frac{64}{3}S_{-2,1,1} + \frac{2}{3}(-3 + 8(-1)^N S_1^2} \\
&+ \frac{2}{3}(-3 + 8(-1)^N S_2 + \frac{4}{3}(1 + 4(-1)^N S_{-2}) \zeta_2 \right] + \frac{L_M \left[ \frac{64(N^5 + 9N^4 + 3N^3 + N^2 + 26N - 4)S_1^2}{9(N - 1)N(N + 1)^3(N + 2)^2} \right]}{9(N - 1)N(N + 1)^2(N + 2)^2} \\
&+ \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N + 1)^3(N + 2)^3} + \frac{16P_{198}S_1}{27(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{64(N - 1)S_{-2}S_1}{3N(N + 1)} \\
&- \frac{64(-1)^NP_{157}}{9(N - 1)N^2(N + 1)^3(N + 2)^2} + \frac{16P_{242}}{27(N - 1)N^4(N + 1)^4(N + 2)^4} + \frac{64(7N^2 + 7N + 8)S_3}{9N(N + 1)(N + 2)} \\
&- \frac{64P_{104}S_2}{64(2N^5 + 21N^4 + 36N^3 - 7N^2 - 68N - 56)S_{-2}} \\
&- \frac{9(N - 1)N^2(N + 1)^2(N + 2)^2}{9(N - 1)N(N + 1)^2(N + 2)^2}
\end{align*}
\]
\[-\frac{128S_{-3}}{3N(N+1)(N+2)} - \frac{128S_{-2,1}}{3(N+2)} + \frac{(N^2 - N - 4)\frac{128}{9}(-1)^NS_{-2}}{(N+1)^2(N+2)^2} - \frac{16(3N^2 + 3N - 2)\zeta_3}{N(N+1)(N+2)}
\]
\[+\gamma_9^0 \left[ \frac{8}{9}S_1^3 - \frac{40}{3}S_2S_1 - \frac{32}{3}(-1)^NS_{-2}S_1 - \frac{16}{3}(-1)^NS_{-3} - 4(-1)^N\zeta_3 \right] \]
\[+CA_{TF}N_F \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{9}\gamma_9^0 S_1 \right] L_3^2 + \left[ \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right] \]
\[+\frac{8}{9}P_{94} \left[ \frac{9(N-1)N^3(N+1)^3(N+2)^3}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1^2}{9(N-1)N(N+1)(N+2)^2} + \frac{16P_{187}S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^NP_{170}}{9(N-1)N^2(N+1)^4(N+2)^2} \]
\[+\frac{8P_{248}}{9(N-1)N^4(N+1)^4(N+2)^4} + \frac{9(N-1)N(N+1)(N+2)^2}{64(8N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}} + \frac{128(N^2 + N + 3)S_{-3}}{9(N-1)N(N+1)^2(N+2)^2} \]
\[+16\gamma_9^0 \left[ \frac{8}{9}S_1^3 - 8S_2S_1 + \frac{32}{3}S_{2,1} \right] + \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1)\left[ \frac{64}{9}S_{-2}S_1 - 32\zeta_3 \right]}{N(N+1)} \right] L_Q \]
\[+\frac{16(N^3 + 8N^2 + 11N + 2)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{192}S_1^2}{3N(N+1)^3(N+2)^3} + \frac{4}{9}\left( \frac{N-1)N^3(N+1)^3(N+2)^3}{27N-1)N^2(N+1)^3(N+2)^3} \right) \]
\[+\frac{16P_{261}}{3(N-1)N^5(N+1)^5(N+2)^5} + \frac{16(9N^5 - 14N^4 - 19N^3 + 52N^2 + 12N + 8)\zeta_3}{9(N-1)N^2(N+1)^2(N+2)^2} \]
\[+\frac{16P_{166}S_1}{3N(N+1)^4(N+2)^4} + \frac{16\left( 5N^4 + 32N^3 + 47N^2 + 28N + 20 \right)\zeta_2 S_1}{9N(N+1)^2(N+2)^2} \]
\[+L^3 \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{3N(N+1)^3(N+2)^3} + \frac{80S_3}{3(N-1)N^3(N+1)^3(N+2)^3} \right] \]
\[+\frac{128(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8P_{172}}{32(5N^4 + 20N^3 + 47N^2 + 58N + 20)S_1}{9N(N+1)^2(N+2)^2} \]
\[+\frac{32(5N^4 + 20N^3 + 47N^2 + 58N + 20)S_1}{9N(N+1)^2(N+2)^2} + \frac{\gamma_9^0 \left[ \frac{4}{3}S_1^2 - \frac{4S_2}{3} - \frac{8}{3}S_{-2} \right]}{9N(N+1)(N+2)^2} \]
\[+\frac{(N^4 + 2N^3 + 7N^2 + 22N + 20)\left[ \frac{64}{3}(-1)^NS_{-2} + \frac{32}{3}(-1)^N\zeta_2 \right]}{(N+1)^3(N+2)^3} \]
\[+\frac{(N^2 - N - 4)\left[ \frac{128}{3}(-1)^NS_{-2} + \frac{64}{3}(-1)^NS_{-3} + \frac{128}{3}S_{-2} - \frac{64}{3}(-1)^NS_{-2} \right]}{(N+1)^3(N+2)^2} \]
\[+\gamma_9^0 \left[ \frac{1}{9}S_1^4 + \frac{10}{3}S_2S_1^2 + \frac{16}{3}(-1)^NS_{-3}S_1 + \left[ \frac{80S_3}{9} - \frac{32}{3}S_{-2,1} \right] S_1 + \frac{4}{9}(-7 + 9(-1)^N)\zeta_3 S_1 + \frac{1}{3}S_2^2 \right] \]
\[+S_{-2} \left[ \frac{16}{3}(-1)^NS_1^2 + \frac{16}{3}(-1)^NS_2 \right] + 6S_4 + \frac{8}{3}(-1)^NS_{-4} - \frac{8}{3}S_{-3,1} - \frac{16}{3}S_{-2,2} - \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1} \]

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\[
\begin{align*}
&\frac{32}{3} S_{-2,1,1} + \left[ \frac{4}{3} (1 - 1 + 2(-1)^N) S_1^2 + \frac{4}{3} (1 - 1 + 2(-1)^N) S_2 + \frac{8}{3} (-1)^N S_{-2} \right] \zeta_2 \\
+ & L_M \left[ - \frac{16(10N^4 + 43N^3 + 106N^2 + 131N + 46) S_1^2}{9(N + 1)^2(N + 2)^2} + \frac{16P_{40} S_1}{27N(N + 1)^4(N + 2)^3} \\
- & \frac{64(-1)^N(7N^5 + 43N^4 + 117N^3 + 166N^2 + 107N + 16)}{9(N + 1)^4(N + 2)^4} + \frac{8P_{246}}{27(N - 1)^4(N + 1)^4(N + 2)^4} \\
- & \frac{16P_{22} S_2}{9(N + 1)N^2(1 + 1)^2(N + 2)^2} - \frac{64(5N^2 + 8N + 10) S_2}{9(N + 1)(N + 2)} + \frac{N^2 - N - 4}{(N + 1)^2(N + 2)^2} \\
+ & \frac{8\zeta_2}{9} \left[ \frac{8}{9} S_1^3 - 8 S_2 S_1 - \frac{16}{3} (-1)^N S_{-2} S_1 - \frac{40S_3}{9} \right] \left[ - \frac{8}{3} (2 + (-1)^N) S_{-3} - \frac{16}{3} S_{2,1} + \frac{16}{3} S_{-2,1} \right] \right] \\
+ & C_F T_F \left[ - \frac{(15N^4 + 6N^3 - 25N^2 - 32N - 28) S_1^4}{3N^2(N + 1)^2(N + 2)} + \frac{2P_{159} S_1^2}{N^3(N + 1)^3(N + 2)} \\
- & \frac{4(3N^5 - 47N^4 - 147N^3 - 93N^2 - 8N + 12) S_1^3}{3N^3(N + 1)^2(N + 2)} - \frac{2(5N^4 - 14N^3 + 53N^2 + 120N + 28) S_2 S_1^2}{N^2(N + 1)^2(N + 2)} \\
- & \frac{8(3N^4 + 90N^3 + 83N^2 - 44N - 4) S_3 S_1}{3N^3(N + 1)^2(N + 2)} - \frac{2P_{211} S_1}{N^5(N + 1)^5(N + 2)} \cdot \frac{4P_{118} S_2 S_1}{N^3(N + 1)^3(N + 2)} \\
- & \frac{8P_{32} \zeta_2 S_1}{N^3(N + 1)^3(N + 2)} - \frac{(11N^4 + 142N^3 + 147N^2 - 32N - 12) S_2^3}{N^2(N + 1)^2(N + 2)} - \frac{P_{247}}{N^6(N + 1)^6(N + 2)} \\
+ & \frac{16(-1)^N(N^2 + N + 2) \zeta_2}{N(N + 1)^4(N + 2)} + \frac{2}{3} \left( \frac{N^2 + N + 2}{N(N + 1)^4(N + 2)} \right)^2 \left( \frac{153N^4 + 306N^3 + 165N^2 + 12 + N + 4) \zeta_3}{N^3(N + 1)^3(N + 2)} \right) \\
+ & L_Q \left[ - \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^4(N + 2)^2} - \frac{16(N^2 + N + 2) S_1(3N^2 + 3N + 2)}{3N^2(N + 1)^4(N + 2)} \right] \\
- & \frac{8\zeta_2}{3} \left[ \frac{8}{3} \zeta_2 S_1^2 \right] + L_M \left[ - \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^3(N + 1)^4(N + 2)} + \frac{16(N^2 + N + 2) S_1(3N^2 + 3N + 2)}{3N^2(N + 1)^4(N + 2)} \right] \\
- & \frac{8\zeta_2}{3} \left[ \frac{8}{3} \zeta_2 S_1^2 \right] + L_M \left[ - \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^3(N + 1)^4(N + 2)} + \frac{16(N^2 + N + 2) S_1(3N^2 + 3N + 2)}{3N^2(N + 1)^4(N + 2)} \right] \\
- & \frac{8\zeta_2}{3} \left[ \frac{8}{3} \zeta_2 S_1^2 \right] + L_M \left[ - \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^3(N + 1)^4(N + 2)} + \frac{16(N^2 + N + 2) S_1(3N^2 + 3N + 2)}{3N^2(N + 1)^4(N + 2)} \right] \\
- & \frac{8\zeta_2}{3} \left[ \frac{8}{3} \zeta_2 S_1^2 \right] + L_M \left[ - \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^3(N + 1)^4(N + 2)} + \frac{16(N^2 + N + 2) S_1(3N^2 + 3N + 2)}{3N^2(N + 1)^4(N + 2)} \right]
\end{align*}
\]
\[
\begin{align*}
&- \frac{32(N^2 + N + 2)S^2}{N(N + 1)^2(N + 2)} + \frac{\tilde{S}^0_{qg}}{N^2(N + 1)^2(N + 2)} \left[ 8S_1^3 - 16S_2S_1 - 16S_{-2}S_1 - 8S_3 - 8S_{-3} + 16S_{-2,1} \right] \\
&+ \frac{P_{90c_2}}{2N^4(N + 1)^4(N + 2)} + \frac{16(N^2 + N + 2)S_{-2}c_2}{N^2(N + 1)^2(N + 2)} + \frac{96\tilde{S}^0_{qg} \log(2)c_2}{N^2(N + 1)^2(N + 2)} \\
&+ \frac{(N^2 + N + 2)(3N^2 + 3N + 2)\left[ 6S_4 - 16S_{-1,1} + 32S_{2,1,1} + 8S_2c_2 - \frac{16}{9}S_1c_3 \right]}{N^2(N + 1)^2(N + 2)} + \frac{\tilde{S}^0_{qg}}{N^2(N + 1)^2(N + 2)} \left[ \frac{1}{3}S_1^5 \right] \\
&= \frac{-2}{3}S_2S_3 + \left( -\frac{16}{3}S_3 - 16S_{2,1} \right)S_1^2 - \frac{8}{3}c_3S_3^2 + \left[ -3S_2^2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1} \right]S_1 \\
&+ \frac{8}{3}S_2S_3 + \left[ -4S_1^2 + 8S_2S_1 + 8S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1} \right]c_2 \\
&+ LQ \left[ \frac{16(3N^4 - 13N^2 - 18N - 12)S_1^3}{N^2(N + 1)^2(N + 2)} - \frac{2P_{136}S_1^2}{N^3(N + 1)^3(N + 2)} + \frac{64(-1)^N(N^2 + N + 2)S_1}{N^2(N^2 + N + 2)} \right] \\
&= \frac{2P_{139}S_1}{N^4(N + 1)^3} - \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)S_2S_1}{N^2(N^2 + N + 2)} + \frac{64(2N^3 + 2N^2 - 3N - 10)S_{-2}S_1}{N^3(N + 1)^2(N + 2)} \\
&- \frac{32(-1)^NP_{260}}{N^3(N + 1)^3(N + 2)} + \frac{5(N - 2)(N - 1)^25(N^3(N + 1)^3(N + 2)^3(N + 3)^3)}{N^2(N + 1)^2(N + 2)} \\
&- \frac{48(8N^4 + 10N^3 + 9N^2 - 8N + 12)c_3}{N^2(N + 1)^2(N + 2)} + \frac{L^2}{N^3(N + 1)^3(N + 2)} \left[ \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{N^3(N + 1)^3(N + 2)} \right] + \frac{2P_{147}S_2}{N^3(N + 1)^3(N + 2)} \\
&- \frac{16(3N^4 + 10N^3 + 15N^2 + 16N - 12)c_3}{N^2(N + 1)^2(N + 2)} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)c_3}{N^3(N + 1)^4(N + 2)} \\
&- \frac{64P_{212}S_{-2}}{N^3(N + 1)^3(N + 2)^3(N + 3)} - \frac{64(N^2 + 3N + 4)c_3}{N(N + 1)^2(N + 2)} \\
&+ \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)c_3}{N^2(N + 1)^2(N + 2)} + \frac{L_M}{N^3(N + 1)^4(N + 2)} \left[ \frac{64(N^3 + 5N^2 + 6N + 4)c_3}{N^2(N + 1)^2(N + 2)} \right] + \frac{2P_{178}}{N^3(N + 1)^3(N + 2)} \\
&- \frac{4P_{136}S_1}{N^3(N + 1)^3(N + 2)} + \frac{64(-1)^N(N^2 + N + 2)}{N^3(N + 1)^3(N + 2)} + \frac{2P_{178}}{N^4(N + 1)^4(N + 2)} \\
&+ \frac{32(N^2 + N + 2)(3N^2 + 3N + 2)c_3}{N^2(N + 1)^2(N + 2)} + \frac{64(N^2 + N + 2)c_3}{N^3(N + 1)^3(N + 2)} + \frac{\tilde{S}^0_{qg}}{N^2(N + 1)^2(N + 2)} \left[ -16S_1^3 + 32S_2S_1 + 32S_{-2}S_1 \right] \\
&+ \frac{16S_3 + 16S_{-3} - 32S_{-2,1}}{N^2(N + 1)^2(N + 2)} + \frac{128(-1)^N(N^2 + 2N + 4)c_3}{N^2(N + 1)^2(N + 2)} + \frac{\tilde{S}^0_{qg}}{N^2(N + 1)^2(N + 2)} \left[ -8S_1^4 + 24S_2S_1^2 - 48S_{-3}S_1 \right] \\
&+ \left[ 16S_{2,1} - 32S_{-2,1} \right]S_1 - 48c_3S_1 - 16S_2^2 - 32S_{-2}^2 + S_{-2} \left[ 32S_1^2 - 32S_2 \right] - 32S_4 - 80S_{-4} - 32S_{3,1} \\
&+ 32S_{-2,2} + 64S_{-3,1} \right] + \frac{L_M}{N^2} \left[ \frac{16(3N^4 - 13N^2 - 18N - 12)c_3}{N^2(N + 1)^2(N + 2)} + \frac{2P_{136}S_1^2}{N^3(N + 1)^3(N + 2)} \right] \\
&- \frac{64(-1)^N(N^2 + N + 2)c_3}{N(N + 1)^4(N + 2)} + \frac{2P_{136}S_1}{N^4(N + 1)^3(N + 2)} + \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)c_3}{N^2(N + 1)^2(N + 2)} \\
&- \frac{64(2N^3 + 2N^2 + 3N - 10)c_3}{N^2(N + 1)^2(N + 2)} + \frac{32(-1)^NP_{260}}{N^3(N + 1)^3(N + 2)^3(N + 3)^3} \\
&- \frac{5(N - 1)^2N^5(N + 1)^5(N + 2)^3(N + 3)^3}{N^2(N + 1)^2(N + 2)} + \frac{9(N^4 + 10N^3 + 9N^2 - 8N + 12)^c_3}{N^2(N + 1)^2(N + 2)} \\
&= 173
\end{align*}
\]
\[
\begin{align*}
&\frac{2P_{147}S_2}{N^3(N + 1)^3(N + 2)} + \frac{16(3N^4 + 10N^3 + 15N^2 + 16N - 12)S_3}{N^2(N + 1)^2(N + 2)} - \\
&\frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_{-2}}{(N + 1)^3(N + 2)^3} + \frac{64P_{212}S_{-2}}{(N - 2)N^3(N + 1)^3(N + 2)^3(N + 3)} - \\
&\frac{64(N^2 + 3N + 4)S_{-3}}{N(N + 1)^2(N + 2)} - \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)S_{2,1}}{N^2(N + 1)^2(N + 2)} \quad \frac{128(N - 1)(N^2 + 2N + 4)S_{-2,1}}{N^2(N + 1)^2(N + 2)} - \\
&\frac{\gamma_{0q}}{8}S_1^4 - 24S_2S_1^2 + 48S_{-3}S_1 + \left[32S_{-2,1} - 16S_{2,1}\right]S_1 + 48\zeta_3S_1 + 16S_2^2 + 32S_{-2}^2 - \\
&\left.S_{-2}\left[32S_2 - 32S_2^2\right] + 32S_4 + 80S_{-4} + 32S_{3,1} - 32S_{-2,2} - 64S_{-3,1}\right]\]
\end{align*}
\]
\[
+C_4^2 T_F \left[ \frac{P_{123}S_1^4}{9(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{4P_{173}S_1^3}{9(N - 1)N^2(N + 1)^3(N + 2)^3} - \frac{4P_{121}S_1^2}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \right] + \\
\frac{3P_{213}S_1^2\zeta_2}{2(N - 2)(55N^5 + 347N^4 + 379N^3 + 213N^2 + 326N + 120)S_2S_1^2} - \\
\frac{4P_{217}S_1\zeta_2}{3(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{4P_{262}S_1}{3(N - 1)^2N^3(N + 1)^3(N + 2)^3} - \\
\frac{4P_{193}S_2S_1}{(N - 1)^2N^3(N + 1)^3(N + 2)^3} - \frac{16P_{144}S_3S_1}{9(N - 1)^2N^3(N + 1)^5(N + 2)^5} - \\
\frac{2P_{213}\zeta_2}{3(N - 1)N^3(N + 1)^4(N + 2)^4} - \frac{2P_{251}\zeta_2}{9(N - 1)^2N^4(N + 1)^4(N + 2)^4} - \\
\frac{4(11N^4 + 22N^3 - 35N^2 - 46N - 24)P_{261}}{3(N - 1)^2N^6(N + 1)^6(N + 2)^6} - \frac{4\zeta_3P_{148}S_1}{9(N - 1)N^2(N + 1)^2(N + 2)^2} - \\
\frac{4(11N^4 + 22N^3 - 35N^2 - 46N - 24)(9N^5 - 14N^4 - 19N^3 + 52N^2 + 12N + 8)\zeta_3}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} - \\
\frac{L_Q^3}{-\frac{8\gamma_{0q}S_1^2}{3} + \frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{9(N - 1)N^2(N + 1)^2(N + 2)^2} -} + \\
\frac{16(N^2 + N + 1)(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} - \\
\frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} + \\
\frac{16(N^2 + N + 1)(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} - \\
\frac{2(11N^4 + 22N^3 - 35N^2 - 46N - 24)P_{191}S_2}{3(N - 1)^2N^4(N + 1)^4(N + 2)^4} - \\
\frac{32(11N^4 + 22N^3 - 35N^2 - 46N - 24)(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} - \\
\frac{16(-1)^N(N^4 + 2N^3 + 7N^2 + 22N + 20)(11N^4 + 22N^3 - 35N^2 - 46N - 24)S_{-2}}{3(N - 1)^2N^4(N + 1)^4(N + 2)^4} - \\
\frac{-32(N^2 + N + 1)(N^2 + N + 2)\zeta_2}{(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{L_M^3}{\frac{4P_{127}S_1^2}{3(N - 1)N^2(N + 1)^2(N + 2)^2}}.
\]
\[-64(-1)^N(N^3 + 4N^2 + 7N + 5)S_1 + \frac{8P_{216}S_1}{(N + 1)^3(N + 2)^3} \]
\[-3(N - 1)N^3(N + 1)^3(N + 2)^4 + \frac{9(N - 1)^2N^4(N + 1)^3(N + 2)^4}{16P_{241}} \]
\[-4(N^2 + N + 2)(11N^4 + 22N^3 - 83N^2 - 94N - 72)S_2 \]
\[-8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_{-2} \]
\[+ \frac{4P_{441S_1^2}}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \]
\[+ \frac{4S_3 + 4S_{-3} - 8S_{-2,1}}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \]
\[+ \gamma_0 \left[ 4S_1^3 - 12S_2S_1 - 16S_{-2}S_1 - 4S_4 - 4S_{-3} + 8S_{-2,1} \right] \]
\[+ \frac{5N^5 - 131N^3 - 58N^2 + 232N + 96}{(N - 1)N(N + 1)^2(N + 2)^3} \left[ \frac{32}{3}(-1)^N S_1 S_{-2} + \frac{16}{3}(-1)^N S_1 \zeta_2 \right] \]
\[+ \frac{(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{(N - 1)N^2(N + 1)^2(N + 2)^2} \left[ \frac{1}{3}S_2^2 + \frac{16}{3}(-1)^N S_{-2}S_2 \right] \]
\[+ \frac{6S_4 + \frac{8}{3}(-1)^NS_{-4} - \frac{8}{3}S_{3,1} - \frac{16}{3}S_{-2,2} - \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1} + \frac{32}{3}S_{-2,1,1}}{(N - 1)N(N + 1)^3(N + 2)^3} \]
\[+ \frac{\left( \frac{8}{3}(-1)^N S_2 + \frac{8}{3}(-1)^N S_{-2} \zeta_2 \right)}{(N - 1)N^2(N + 1)^2(N + 2)^2} \]
\[+ \frac{\left( 4(-1)^N \zeta_3 + \frac{16}{3}(-1)^N S_{-3} - \frac{32}{3}S_{-2,1} \right)}{(N - 1)N^2(N + 1)(N + 2)^2} \]
\[+ \frac{\left( 8(47N^4 - 133N^3 + 11N^2 - 155N - 58)S_3^3 \right)}{9(N - 1)N(N + 1)^2(N + 2)} + \frac{8P_{223S_1^2}}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} \]
\[- \frac{32(N - 2)(N + 3)S_{-2}S_2^2}{N(N + 1)(N + 2)} - \frac{32(-1)^NP_{197}S_1}{3(N - 1)N^3(N + 1)^4(N + 2)^4} + \frac{8P_{257}S_1}{27(N - 1)^2N^4(N + 1)^4(N + 2)^4} \]
\[- \frac{8P_{137S_2S_1}}{(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{32(N^2 + N + 6)S_3S_1}{N(N + 1)(N + 2)} - \frac{32(N - 1)N^2(N + 1)^2(N + 2)^2}{16P_{142S_2S_1}} \]
\[+ \frac{32(9N^2 + 9N + 22)S_{-3}S_1^2}{N(N + 1)(N + 2)} - \frac{64(3N^2 + 3N + 10)S_{-2,1}S_1}{N(N + 1)(N + 2)} - \frac{384\zeta_3S_1}{N(N + 1)(N + 2)} \]
\[- \frac{8}{(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{16(-1)^NP_{256}}{9(N - 1)^2N^4(N + 1)^6(N + 2)^6} \]
\[
\begin{align*}
+4P_{269} & \quad \frac{27(N-1)^2N^5(N+1)^3(N+2)^5}{8P_{310}S_3} - \frac{8P_{219}S_2}{16P_{229}S_{-2}} \\
-9(N-1)N^2(N+1)^2(N+2)^2 & \quad \frac{9(N-1)N^2N^3(N+1)^3(N+2)^3}{16P_{229}S_{-2}} \\
(4N^3 + 4N^2 + 7N + 5)(-64(-1)^NS_1^2 + 64(-1)^NS_2 + 192(-1)^NS_{-2}) & \quad \left(\frac{N+1}{3}(N+2)^3\right) \\
+16P_{133}S_{-3} & \quad + \frac{32P_{112}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
+\gamma_{99}^0 \left[-2S_1^4 + 16S_2S_1^2 - 2S_2^2 - 12S_2S_1 - 4S_1 - 44S_{-4} - \frac{88}{3}S_{2,1} - 16S_{3,1} + 56S_{-2,2}ight] & \quad + 64S_{-3,1} - 96S_{-2,1,1} \right] + \gamma_{99}^0 \left[-\frac{1}{3}S_1^3 - 10S_2S_1^2 - 16(-1)^NS_{-3}S_1^2 + \left[32S_{-2,1} - \frac{80S_3}{3}\right]S_1^2 \\
-\frac{4}{3}(-7 + 9(-1)^N)S_1^3 & \quad + \left[-S_2^2 - 18S_1 + 8S_3 + 16S_{-2,2} + 16S_{-3,1} + 8S_{2,1,1} \right] \\
-32S_{-2,1,1}S_1 & \quad + S_{-2} \left[-16(-1)^NS_1^2 - 16(-1)^NS_2S_1 + \left[-4(-1 + 2(-1)^N)S_1^2 - 8(-1)^NS_2S_1 \right] \\
-4(1 + 2(-1)^N)S_{-2}S_1 & \quad + \frac{11S_2}{3} - 2S_3 - 2S_{-3} + 4S_{-2,1} \right] \right]_2 + L_M \left[\frac{128(-1)^N P_{171}S_1}{3(N-1)N^2(N+1)^4(N+2)^4} \right] \\
-8(11N^4 + 26N^3 - 139N^2 - 218N + 8)S_1^3 & \quad + \frac{4P_{221}S_1^2}{8P_{254}S_1} + \frac{9(N-1)^2N^3(N+1)^3(N+2)^3}{8P_{124}S_1} \\
-27(N-1)^2N^4(N+1)^4(N+2)^4 & \quad - \frac{32(2N^5 - 23N^4 - 32N^3 + 13N^2 + 4N - 12)S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{2}{(N-1)N^2(N+1)^2(N+2)^2} \right]_2 \\
-9(N-1)^3N^5(N+1)^5(N+2)^5 & \quad - \frac{4P_{222}S_2}{5P_{145}S_3} + \frac{9(N-1)^2N^3(N+1)^3(N+2)^3}{16P_{202}S_{-2}} \\
+9(N-1)^2N^3(N+1)^3(N+2)^3 & \quad + \frac{16(-1)^NP_{129}S_{-2}}{(N-1)^3(N+2)^3} \\
-9(N-1)N^2(N+1)^2(N+2)^2 & \quad \left[\frac{9(N-1)N^2N^3(N+1)^3(N+2)^3}{16P_{202}S_{-2}} + \frac{16(N^2 + N + 2)(11N^4 + 22N^3 + 13N^2 + 2N + 24)}{3(N-1)N^2(N+1)^2(N+2)^2} \right] \\
+\frac{3(N-1)N^2(N+1)^2(N+2)^2}{(N-1)N^2(N+1)^2(N+2)^2} & \quad \left[\frac{8}{3}(-1)^NS_{-3} - 2(-1)^NS_3 \right] \\
\left[11N^5 + 34N^4 - 49N^3 - 24N^2 - 68N - 48 \right] & \quad - \frac{16}{9}(-1)^NS_1S_{-2} \\
+\gamma_{99}^0 \left[2S_1^4 + 32S_2S_1^2 + 8(8 + (-1)^N)S_{-3}S_1 + [40S_3 - 16S_{2,1} - 80S_{-2,1}]S_1 \right] \\
+6(-5 + (-1)^N)S_1S_4 + 2S_2^2 + 12S_{-2}S_2 + \left[8(3 + 2(-1)^N)S_1^2 + 16S_2 \right] + 4S_4 + 44S_{-4} + 16S_{3,1} \\
-56S_{-2,2} & \quad - 64S_{-3,1} + 96S_{-2,1,1} + \frac{16P_{125}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \right] \\
+C_F T^2 \left[\frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^4 + 6N^3 - N^2 - 4N + 12)}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{16}{9}\gamma_{99}^0 S_1 \right] L^3 \right]
\end{align*}
\]
\[ + \left[ \frac{16L_M}{(N - 1)N^3(N + 1)^3(N + 2)^2} - \frac{4P_{232}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right] + \frac{16P_{275}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{\gamma_0}{\gamma_0} \left[ \frac{20S_2}{3} - 4S_1^2 \right] \right] L^2 + \left[ \frac{16P_{181}S_1^2}{64(N - 1)N^3(N + 1)^3(N + 2)^2} \right]

\[ - \frac{16P_{183}S_2}{9(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{45(N - 2)(N - 1)^2N^3(N + 1)^4(N + 2)^4(N + 3)^3}{3(N - 1)N^3(N + 1)^3(N + 2)^2} \]

\[ + \frac{45(N - 1)^2N^5(N + 1)^4(N + 2)^4(N + 3)^3}{3(N - 1)N^3(N + 1)^3(N + 2)^2} + L_M \left[ \frac{8P_{231}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{32P_{173}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right]

\[ + \frac{16P_{183}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \] + L_M \left[ \frac{8P_{231}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{32P_{173}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right]

\[ + \frac{16P_{234}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \] + L_M \left[ \frac{8P_{231}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{32P_{173}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right]

\[ + \frac{16P_{183}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \] + L_M \left[ \frac{8P_{173}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{32P_{173}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right]

\[ + \frac{16P_{234}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \] + L_M \left[ \frac{8P_{231}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{32P_{173}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right]
\[
+ C_F N_F T_F^2 \left[ \frac{16}{9} \gamma_{gq}^0 S_1 + \frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^4 + 6N^3 - N^2 - 4N + 12)}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \right] L_Q^3 \\
+ \left[ - \frac{4P_{232}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{16P_{277} S_1}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \right] + L_M \left[ - \frac{8(N^2 + N + 2)(3N^2 + 3N + 2)}{3N^2(N + 1)^2(N + 2)} - \frac{8}{3} \gamma_{gq}^0 S_1 + \frac{20S_2}{3} - 4S_1^2 \right] L_Q^3 \\
+ \left[ - \frac{16P_{181} S_1^2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} - \frac{8P_{229} S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right] + 45(N - 2)(N - 1)^2 N^3(N + 1)^4(N + 2)^4(N + 3)^3 + \frac{45(N - 1)^2 N^5(N + 1)^5(N + 2)^4(N + 3)^3}{(N + 3)^3} \\
+ L_M \left[ \frac{8(N^2 + N + 2)(57N^4 + 72N^3 + 29N^2 - 22N - 24)}{9N^3(N + 1)^3(N + 2)} + \frac{8}{3} \gamma_{gq} \left[ 8S_2^2 - 8S_2 \right] \right] \\
- \frac{16(N^2 + N + 2)(29N^2 + 29N - 6) S_1}{9N^2(N + 1)^2(N + 2)} - \frac{256(N^2 + N + 1) S_3}{3N(N + 1)(N + 2)} + \frac{16P_{183} S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \\
+ \left[ - \frac{64P_{163} S_{-2}}{3(N - 1)N^6(N + 1)^6(N + 2)^3} + \frac{512}{3} \gamma_{gq} \left[ 8S_3 - 8S_2 S_1 - \frac{32}{3} S_{2,1} \right] \right] \\
+ \frac{512}{3} S_{-2} + \frac{256}{3} S_{-3} - \frac{512}{3} S_{-2,1} + \frac{64(N - 1) \zeta_3}{N(N + 1)} \right] L_Q - \frac{2}{9} \frac{P_{235} \zeta_2}{(N - 1)N^4(N + 1)^4(N + 2)^3} \\
- \frac{8(N^4 - 5N^3 - 32N^2 - 18N - 4) S_1^2}{3N^2(N + 1)^2(N + 2)} - \frac{8}{9} \frac{(N^2 + N + 2) \zeta_3}{N(N + 1)N^3(N + 1)^3(N + 2)^2} P_{108} \\
- \frac{3(N - 1)N^6(N + 1)^6(N + 2)^3}{16(2N^5 - 4N^4 - 11N^3 - 19N^2 - 44N - 12) S_1} \\
- \frac{16}{9} \frac{(5N^3 + 11N^2 + 28N + 12) \zeta_2}{N^2(N + 1)(N + 2)} S_1 + L_M^3 \left[ \frac{8(N^2 + N + 2) P_{108}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{8}{9} \gamma_{gq} S_1 \right] \\
- \frac{8P_{229} S_2}{3(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{(3N + 2) \left[ - \frac{16}{9} S_3^2 - \frac{16}{3} S_2 S_1 \right]}{N^2(N + 2)} \\
- \frac{4P_{228}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{32(5N^3 + 8N^2 + 19N + 6) S_1}{9N^2(N + 1)(N + 2)^3} + \frac{8}{3} \gamma_{gq} \left[ \frac{4}{3} S_1^2 + \frac{4}{3} S_2 \right] \right] \\
- \frac{16P_{169} S_3}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{64(N^2 - 3N - 2) S_{2,1}}{3N^2(N + 1)(N + 2)} \\
+ L_M \left[ \frac{8(49N^4 + 122N^3 + 213N^2 + 164N + 12) S_1^2}{9N^2(N + 1)(N + 2)^2} - \frac{8P_{258}}{9(N - 1)N^5(N + 1)^5(N + 2)^4} \right] \\
+ \left[ - \frac{8P_{146} S_1}{9N^3(N + 1)^3(N + 2)} - \frac{8P_{184} S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{16}{3} S_1 S_2 - 16S_3 + \frac{16}{3} S_{2,1} \right] \\
+ \gamma_{gq} \left[ \frac{1}{9} S_1^4 - \frac{3}{2} S_2 S_1^2 - \frac{4}{3} \zeta_2 S_1^2 + \left[ - \frac{8}{9} S_3 - \frac{16}{3} S_{2,1} \right] S_1 - \frac{8}{9} \zeta_3 S_1 - \frac{1}{3} S_2^2 + 2S_4 - \frac{16}{3} S_{4,1} + \frac{32}{3} S_{2,1,1} \right] \\
+ C_A C_F T_F \left[ \frac{2P_{134^4}}{9(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{4P_{180} S_3}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \right]
\]
\[
\frac{4P_{245}S_1^2}{3(N-1)N^4(N+1)^3(N+2)^4} - \frac{8}{3} \frac{(19N^5 - 11N^4 - 8N^3 - 49N^2 - 17N + 18)\zeta_2}{(N-1)N^2(N+1)(N+2)^2} S_1^2 \\
+ \frac{4P_{143}S_2^2}{3(N-1)N^2(N+1)^3(N+2)^2} - \frac{2}{9} \frac{(N-1)N^2(N+1)(N+2)^2}{P_{155}S_1\zeta_3} \\
+ \frac{9}{4} \frac{(N-1)N^3(N+1)^3(N+2)^3}{P_{188}S_1\zeta_2} + \frac{4P_{263}S_1}{3(N-1)N^6(N+1)^5(N+2)^5} \\
+ \frac{3(N-1)N^3(N+1)^3(N+2)^3}{4P_{200}S_1\zeta_2} + \frac{(9(N-1)N^2(N+1)^2(N+2)^2}{8P_{183}S_3S_1} \\
+ \frac{3(N-1)N^3(N+1)^3(N+2)^3}{16(11N^5 + 45N^4 - 3N^3 - 145N^2 - 176N - 20)S_{2,1}S_1} - \frac{2P_{101}S_2^2}{(N+1)^3(N+2)^3} \\
- \frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N+1)^3(N+2)^3} + \frac{8(-1)^N\zeta_2}{N^2(N+1)^4(N+2)^5} P_{115} S_1 - \frac{2}{9} \frac{(N-1)N^2(N+1)^3(N+2)^2}{P_{188}S_3} \\
+ \frac{1}{18(N-1)N^3(N+1)^4(N+2)^2} + \frac{3(N-1)N^6(N+1)^6(N+2)^5}{P_{273}} \\
+ L_3^M \left[ \frac{16}{3} \gamma_{qq}^0 S_1^2 - \frac{8(N^2 + 2N + 2)(N^2 + N + 6)(7N^2 + 7N + 4)}{9(N-1)N^2(N+1)^2(N+2)^2} S_1 \\
- \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)}{9(N-1)N^3(N+1)(N+2)^2} \\
+ \frac{8(N^2 + 2N + 2)(13N^4 + 26N^3 - 43N^2 - 56N - 12)}{9(N-1)N^2(N+1)^2(N+2)^2} S_1 \\
- \frac{4(N^2 + N + 2)(3N^2 + 3N + 2)(11N^4 + 22N^3 - 23N^2 - 34N - 12)}{9(N-1)N^3(N+1)(N+2)^2} \right] \\
+ L_3^Q \left[ -\frac{8}{3} \gamma_{qq}^0 S_1^2 \\
- \frac{4\zeta_2P_{107}S_2}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4P_{243}S_2}{3(N-1)N^4(N+1)^4(N+2)^4} + \frac{4P_{188}S_3}{9(N-1)N^3(N+1)^3(N+2)^2} \\
+ \frac{4(N^2 + N + 2)(19N^4 + 38N^3 - 22N^2 - 41N - 30)S_4}{N^2(N+1)^2(N+2)^2} + \frac{16(-1)^NP_{116}S_{-2}}{16(N^2 - 3N - 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)S_{2,1}} \\
- \frac{3(N-1)N^3(N+1)^2(N+2)^2}{8(N^2 + N + 2)(31N^4 + 62N^3 - 73N^2 - 104N - 60)S_{3,1}} \\
- \frac{3(N-1)N^2(N+1)^2(N+2)^2}{8(N^2 + N + 2)(31N^4 + 62N^3 - 73N^2 - 104N - 60)S_{3,1}} \right] \\
+ L_2^Q \left[ -\frac{8(10N^5 + 6N^4 + 5N^3 - 38N^2 - 17N + 2)S_1^2}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3(N+2)^3} \\
- \frac{4P_{204}S_1}{9(N-1)N^3(N+1)^3(N+2)^3} - \frac{16(-1)^N(3N^3 + 11N^3 + 19N^2 + 15N + 2)}{N(N+1)^3(N+2)^3} S_1 \\
+ \frac{P_{238}}{9(N-1)N^4(N+1)^4(N+2)^4} + L_M \left[ \frac{22(N^2 + N + 2)(3N^2 + 3N + 2)}{3N^2(N+1)^2(N+2)^2} \right] \\
+ \frac{22}{3} \gamma_{qq}^0 S_1 + \frac{8(N^2 + N + 2)(23N^4 + 46N^3 - 50N^2 - 73N - 18)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} + \gamma_{qq}^0 \left[ 4S_1^3 - 12S_2S_1 \right] \\
+ 4S_3 + 6S_{-2} + 4S_{-3} - 8S_{-2,1} \right] + L_M^2 \left[ \frac{8P_{121}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} \right] \\
+ \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3(N+2)^3} - \frac{8P_{203}S_1}{9(N-1)N^3(N+1)^3(N+2)^3} \right]
\]
\[ -\frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^2 + 3N + 2)}{(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{(N^2 + N + 10)(-64S_{-2}S_1^2 - 32S_{-3}S_1)}{N(N + 1)(N + 2)} \\
- \frac{4P_{207}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{(N^3 + 4N^2 + 7N + 5)(128(-1)^N S_2 - 128(-1)^N S_2^2)}{(N + 1)^3(N + 2)^3} \\
+ \frac{3(N - 1)N^2(N + 1)^2(N + 2)^2}{16P_{119}S_{-3}} + \frac{16(-1)^N(3N^5 - 6N^4 - 61N^3 - 124N^2 - 96N - 16)S_{-2}}{N(N + 1)^3(N + 2)^3} \\
- \frac{3(N - 1)N^2(N + 1)^2(N + 2)^2}{16P_{226}S_{-2}} + \frac{3(N - 2)(N - 1)N^3(N + 1)^3(N + 2)^3}{32P_{226}S_{-2}} + \frac{3(N - 2)(N - 1)N^3(N + 1)^3(N + 2)^3(N + 3)}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \\
+ \frac{3(N - 1)N^2(N + 1)^2(N + 2)^2}{16P_{120}S_{-2,1}} - \frac{3N^2(N + 1)^2(N + 2)}{16P_{120}S_{-2,1}} \\
+ L_M \left[ \frac{16(10N^5 + 40N^4 + 121N^3 + 161N^2 + 52N + 12)S_1^2}{3N^2(N + 1)^2(N + 2)^2} + \frac{4P_{205}S_1}{9(N - 1)N^3(N + 1)^3(N + 2)^3} \right] \\
- 128(-1)^N(3N^3 + 4N^2 + 7N + 5)S_1 + \frac{32(-1)^N(3N^3 + 11N^3 + 19N^2 + 15N + 2)}{(N + 1)^3(N + 2)^3} \\
- \frac{2P_{236}}{9(N - 1)N^4(N + 1)^3(N + 2)^3} - \frac{16(N^2 + N + 2)(4N^2 + 4N - 1)S_2}{N^2(N + 1)^2(N + 2)^2} + \frac{80S_3^2 + 8S_2S_1 - 8S_3}{64} \\
- \frac{12S_{-2} - 8S_{-3} + 16S_{-2,1}}{N^2(N + 1)^2(N + 2)^2} + \frac{2S_{-2}S_{-3} + 3S_{-2,1}}{N^2(N + 1)^2(N + 2)^2} \\
+ \frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)^2} \left[ 8(-1)^N S_{-3}S_3 - 6(-1)^N \zeta_3 \right] + \frac{\zeta_3}{40S_2S_1^2 + 16(-1)^N S_2S_1^2} \\
+ 8(-1)^N S_{-3}S_3 - 16S_{-2,1}S_1 + 6(-1)^N \zeta_3 S_1 - 8S_2^2 + 24S_{-2,2} + 8S_4 + 40S_{-4} \\
+ 32S_{3,1} - 16S_{-3,1} - 32S_{-2,1,1} \right] + L_M \left[ \frac{3N^5 + 2N^4 - 61N^3 - 112N^2 - 56N - 24}{N^2(N + 1)^2(N + 2)^2} \right] \\
+ \frac{32(-1)^N(15N^5 + 97N^4 + 260N^3 + 328N^2 + 158N - 4)S_1}{N(N + 1)^4(N + 2)^4} \\
- \frac{2P_{205}S_1^2}{9(N - 1)N^4(N + 1)^3(N + 2)^4} + \frac{8P_{114}S_2S_1}{2P_{205}S_1} \\
- \frac{9(N - 1)N^4(N + 1)^3(N + 2)^4}{16P_{114}S_2S_1} + \frac{3(N - 1)N^2(N + 1)^2(N + 2)^2}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \\
+ \frac{(3N^3 + 2N^2 - 47N - 62)S_{-2}S_1}{N(N + 1)^2(N + 2)^2} - \frac{\zeta_3 P_{150}}{6(N - 1)N^2(N + 1)^2(N + 2)^2} \\
- \frac{16(-1)^N P_{267}}{5(N - 2)(N - 1)^2N^3(N + 1)^5(N + 2)^5(N + 3)^3 + 45(N - 1)^2N^5(N + 1)^5(N + 2)^5(N + 3)^3} \\
+ \frac{2P_{210}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{(N^3 + 4N^2 + 7N + 5)(128(-1)^N S_2^2 - 128(-1)^N S_2)}{(N + 1)^4(N + 2)^3} \\
+ \frac{2P_{210}S_2^3}{16P_{210}S_2S_3} + \frac{16(-1)^N(3N^5 - 6N^4 - 61N^3 - 124N^2 - 96N - 16)S_{-2}}{N(N + 1)^3(N + 2)^3} \\
- \frac{(N - 2)N^3(N + 1)^3(N + 2)^3(N + 3)}{16P_{214}S_{-2}} + \frac{16(3N^4 + 4N^3 - 9N^2 - 14N + 8)S_{-3}}{N^2(N + 1)^2(N + 2)^2} \\
+ \frac{32(N^2 + N + 2)(10N^4 + 20N^3 + 5N^2 - 5N + 6)S_{2,1}}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \\
+ \frac{16(3N^4 + 10N^3 + 43N^2 + 44N - 20)S_{-2,1}}{N^2(N + 1)^2(N + 2)^2} \\
+ \frac{181}{181}
with the polynomials

\[ P_{101} = N^6 - 81N^5 - 264N^4 - 185N^3 - 307N^2 - 256N - 204 \]

\[ P_{102} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \]

\[ P_{103} = N^6 + 7N^5 - 7N^4 - 39N^3 + 14N^2 + 40N + 48 \]

\[ P_{104} = N^6 + 21N^5 + 57N^4 + 31N^3 + 26N^2 + 20N + 24 \]

\[ P_{105} = 2N^6 - 7N^5 - 41N^4 - 31N^3 - 29N^2 - 22N - 16 \]

\[ P_{106} = 2N^6 - 7N^5 - 24N^4 - 35N^3 - 44N^2 - 44N - 16 \]

\[ P_{107} = 3N^6 + 5N^5 + 27N^4 + 35N^3 + 6N^2 + 12N + 8 \]

\[ P_{108} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \]

\[ P_{109} = 3N^6 + 9N^5 + 2N^4 - 11N^3 - 23N^2 - 16N - 12 \]

\[ P_{110} = 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 \]

\[ P_{111} = 4N^6 + 5N^5 - 10N^4 - 39N^3 - 40N^2 - 24N - 8 \]

\[ P_{112} = 6N^6 - 12N^5 + 17N^4 + 106N^3 + 127N^2 + 104N + 84 \]

\[ P_{113} = 6N^6 + 18N^5 + 7N^4 - 16N^3 - 31N^2 - 20N - 12 \]

\[ P_{114} = 7N^6 - 93N^5 - 327N^4 - 287N^3 - 316N^2 - 112N - 24 \]

\[ P_{115} = 7N^6 - 20N^5 - 176N^4 - 335N^3 - 276N^2 - 116N - 16 \]

\[ P_{116} = 7N^6 - 19N^5 - 171N^4 - 325N^3 - 264N^2 - 108N - 16 \]

\[ P_{117} = 7N^6 + 21N^5 + 5N^4 - 25N^3 - 204N^2 - 188N - 192 \]

\[ P_{118} = 8N^6 + 13N^5 - 111N^4 - 193N^3 - 89N^2 - 56N - 20 \]

\[ P_{119} = 9N^6 + 21N^5 + 11N^4 - 5N^3 - 104N^2 - 76N - 144 \]

\[ P_{120} = 9N^6 + 39N^5 + 3N^4 + 25N^3 + 94N^2 + 44N + 312 \]

\[ P_{121} = 10N^6 + 18N^5 - 111N^4 - 164N^3 - 61N^2 - 16N + 36 \]

\[ P_{122} = 10N^6 + 63N^5 + 105N^4 + 31N^3 + 17N^2 + 14N + 48 \]

\[ P_{123} = 11N^6 - 15N^5 - 327N^4 - 181N^3 + 292N^2 - 20N - 48 \]

\[ P_{124} = 11N^6 + 15N^5 - 285N^4 - 319N^3 - 254N^2 - 368N + 240 \]

\[ P_{125} = 11N^6 + 33N^5 - 189N^4 - 361N^3 - 194N^2 - 92N - 72 \]

\[ P_{126} = 11N^6 + 33N^5 - 114N^4 - 247N^3 - 263N^2 - 176N + 2 \]

\[ P_{127} = 11N^6 + 33N^5 - 87N^4 - 85N^3 + 4N^2 - 116N + 48 \]

\[ P_{128} = 11N^6 + 35N^5 + 59N^4 + 57N^3 - 3N^2 - 68N + 40 \]

\[ P_{129} = 11N^6 + 47N^5 + 7N^4 + 9N^3 + 90N^2 + 28N + 96 \]
\[ P_{130} = 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48 \] (E.139)
\[ P_{131} = 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48 \] (E.140)
\[ P_{132} = 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10 \] (E.141)
\[ P_{133} = 16N^6 + 78N^5 - 23N^4 - 228N^3 - 503N^2 - 408N - 228 \] (E.142)
\[ P_{134} = 17N^6 + 111N^5 + 234N^4 + 203N^3 - 89N^2 - 296N - 36 \] (E.143)
\[ P_{135} = 22N^6 + 69N^5 + 71N^4 + 23N^3 - 57N^2 - 68N + 84 \] (E.144)
\[ P_{136} = 23N^6 - 7N^5 - 237N^4 - 593N^3 - 678N^2 - 548N - 200 \] (E.145)
\[ P_{137} = 23N^6 + 9N^5 - 71N^4 - 53N^3 - 184N^2 - 92N - 16 \] (E.146)
\[ P_{138} = 25N^6 + 35N^5 - 55N^4 - 243N^3 - 286N^2 - 204N - 72 \] (E.147)
\[ P_{139} = 29N^6 + 91N^5 + 235N^4 + 405N^3 + 272N^2 + 288N + 120 \] (E.148)
\[ P_{140} = 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 \] (E.149)
\[ P_{141} = 35N^6 - 15N^5 - 183N^4 - 133N^3 - 356N^2 - 164N - 48 \] (E.150)
\[ P_{142} = 35N^6 - 15N^5 - 101N^4 + 31N^3 + 54N^2 + 164N + 120 \] (E.151)
\[ P_{143} = 44N^6 + 96N^5 + 369N^4 + 290N^3 - 695N^2 - 428N - 108 \] (E.152)
\[ P_{144} = 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 \] (E.153)
\[ P_{145} = 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 \] (E.154)
\[ P_{146} = 57N^6 + 161N^5 - 25N^4 - 193N^3 - 172N^2 - 36N + 48 \] (E.155)
\[ P_{147} = 65N^6 + 199N^5 + 197N^4 - 143N^3 - 330N^2 - 316N - 120 \] (E.156)
\[ P_{148} = 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 \] (E.157)
\[ P_{149} = 80N^6 + 60N^5 + 9N^4 + 230N^3 + 901N^2 + 988N + 1188 \] (E.158)
\[ P_{150} = 81N^6 + 211N^5 - 23N^4 - 355N^3 - 334N^2 - 4N - 344 \] (E.159)
\[ P_{151} = 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 \] (E.160)
\[ P_{152} = 130N^6 + 865N^5 + 2316N^4 + 3811N^3 + 4434N^2 + 2884N + 536 \] (E.161)
\[ P_{153} = 133N^6 + 699N^5 + 1395N^4 + 217N^3 - 880N^2 + 164N + 288 \] (E.162)
\[ P_{154} = 155N^6 + 369N^5 + 211N^4 - 65N^3 - 1002N^2 - 556N - 1416 \] (E.163)
\[ P_{155} = 215N^6 + 429N^5 + 891N^4 + 491N^3 - 2486N^2 - 1436N - 408 \] (E.164)
\[ P_{156} = 3N^7 + 28N^6 + 665N^5 + 90N^4 + 107N^3 + 78N^2 + 36N + 8 \] (E.165)
\[ P_{157} = 9N^7 + 71N^6 + 214N^5 + 320N^4 + 275N^3 + 215N^2 + 160N + 32 \] (E.166)
\[ P_{158} = 21N^7 + 120N^6 - 128N^5 - 1038N^4 - 89N^3 + 2382N^2 + 1636N - 600 \] (E.167)
\[ P_{159} = 81N^7 + 247N^6 + 291N^5 + 277N^4 + 108N^3 - 56N^2 + 20N + 24 \] (E.168)
\[ P_{160} = N^8 + 5N^7 + 10N^6 + 27N^5 + 65N^4 + 112N^3 + 124N^2 + 80N + 32 \] (E.169)
\[ P_{161} = N^8 + 5N^7 + 14N^6 + 23N^5 + 25N^4 + 52N^3 + 56N^2 + 48N + 16 \] (E.170)
\[ P_{162} = N^8 + 8N^7 - 2N^6 - 60N^5 - 23N^4 + 108N^3 + 96N^2 + 16N + 48 \] (E.171)
\[ P_{163} = N^8 + 8N^7 - 2N^6 - 60N^5 + N^4 + 156N^3 + 24N^2 - 80N - 240 \] (E.172)
\[ P_{164} = N^8 + 22N^7 + 111N^6 + 211N^5 + 42N^4 - 281N^3 - 406N^2 - 204N - 72 \] (E.173)
\[ P_{165} = 2N^8 + N^7 - 6N^6 + 26N^5 + 64N^4 + 51N^3 + 54N^2 + 28N + 8 \] (E.174)
\[ P_{166} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \] (E.175)
\[ P_{167} = 2N^8 + 44N^7 + 211N^6 + 485N^5 + 654N^4 + 581N^3 + 391N^2 + 192N + 32 \] (E.176)
\[ P_{168} = 3N^8 + 41N^7 + 136N^6 + 233N^5 + 331N^4 + 360N^3 + 208N^2 + 80N + 16 \] (E.177)
\[ P_{169} = 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 \] (E.178)
\[ P_{170} = 5N^8 - 8N^7 - 137N^6 - 436N^5 - 713N^4 - 672N^3 - 407N^2 - 192N - 32 \] (E.179)

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\[ P_{171} = 7N^8 + 40N^7 + 110N^6 + 193N^5 + 261N^4 + 313N^3 + 260N^2 + 96N + 16 \]  
(E.180)

\[ P_{172} = 9N^8 + 54N^7 + 80N^6 - 110N^5 - 645N^4 - 1168N^3 - 1132N^2 - 672N - 160 \]  
(E.181)

\[ P_{173} = 10N^8 + 46N^7 + 87N^6 + 85N^5 - 75N^4 - 251N^3 - 274N^2 - 132N - 72 \]  
(E.182)

\[ P_{174} = 11N^8 + 74N^7 + 213N^6 + 281N^5 - 30N^4 - 427N^3 - 446N^2 - 180N - 72 \]  
(E.183)

\[ P_{175} = 15N^8 + 36N^7 + 50N^6 - 252N^5 + 357N^4 + 152N^3 - 68N^2 + 88N + 48 \]  
(E.184)

\[ P_{176} = 18N^8 + 101N^7 + 128N^6 + 208N^5 + 190N^4 - 769N^3 - 1200N^2 - 212N - 48 \]  
(E.185)

\[ P_{177} = 19N^8 + 70N^7 + 63N^6 - 41N^5 - 192N^4 - 221N^3 - 142N^2 - 60N - 72 \]  
(E.186)

\[ P_{178} = 21N^8 + 42N^7 - 38N^6 - 360N^5 - 631N^4 - 730N^3 - 472N^2 - 216N - 48 \]  
(E.187)

\[ P_{179} = 23N^8 + 2N^7 - 135N^6 + 29N^5 + 210N^4 - 151N^3 - 350N^2 - 132N - 72 \]  
(E.188)

\[ P_{180} = 27N^8 - 36N^7 - 956N^6 + 1724N^5 + 187N^4 + 1288N^3 + 70N^2 - 224N - 72 \]  
(E.189)

\[ P_{181} = 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 - 373N^3 - 218N^2 - 60N - 72 \]  
(E.190)

\[ P_{182} = 41N^8 + 5N^7 - 195N^6 - 97N^5 + 326N^4 + 424N^3 + 208N^2 + 72N + 16 \]  
(E.191)

\[ P_{183} = 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 + 469N^3 - 290N^2 - 60N - 72 \]  
(E.192)

\[ P_{184} = 79N^8 + 196N^7 + 132N^6 + 274N^5 + 465N^4 + 82N^3 + 332N^2 + 456N + 288 \]  
(E.193)

\[ P_{185} = 105N^8 + 978N^7 + 1688N^6 - 1330N^5 - 5245N^4 + 4672N^3 - 2212N^2 - 544N - 288 \]  
(E.194)

\[ P_{186} = 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 - 859N^3 - 778N^2 - 172N + 72 \]  
(E.195)

\[ P_{187} = 170N^8 + 369N^7 - 521N^6 - 1393N^5 + 761N^4 + 952N^3 + 544N^2 + 32N + 144 \]  
(E.196)

\[ P_{188} = 264N^8 + 1407N^7 + 2246N^6 + 1746N^5 + 804N^4 - 1069N^3 - 674N^2 - 92N - 24 \]  
(E.197)

\[ P_{189} = 283N^8 + 838N^7 + 1482N^6 + 628N^5 - 1497N^4 - 1130N^3 + 772N^2 + 456N + 288 \]  
(E.198)

\[ P_{190} = 633N^8 + 2532N^7 + 5036N^6 + 6142N^5 + 4275N^4 + 1118N^3 - 176N^2 - 184N - 48 \]  
(E.199)

\[ P_{191} = N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 \]  
(E.200)

\[ P_{192} = 4N^9 + 53N^8 + 193N^7 + 233N^6 + 87N^5 + 554N^4 + 1172N^3 + 904N^2 + 512N + 128 \]  
(E.201)

\[ P_{193} = 6N^9 + 93N^8 + 57N^7 + 1296N^6 + 586N^5 + 359N^4 + 2000N^3 + 1996N^2 + 1488N + 384 \]  
(E.202)

\[ P_{194} = 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 \]  
(E.203)

\[ P_{195} = 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 \]  
(E.204)

\[ P_{196} = 11N^9 + 78N^8 + 214N^7 + 335N^6 + 383N^5 + 571N^4 + 916N^3 + 876N^2 + 480N + 96 \]  
(E.205)

\[ P_{197} = 35N^9 + 150N^8 + 232N^7 + 137N^6 + 119N^5 + 661N^4 + 1174N^3 + 876N^2 + 480N + 96 \]  
(E.206)

\[ P_{198} = 37N^9 + 210N^8 - 52N^7 - 2738N^6 + 7249N^5 + 9368N^4 - 8216N^3 + 5888N^2 + 2448N - 576 \]  
(E.207)

\[ P_{199} = 45N^9 + 270N^8 + 820N^7 + 1478N^6 + 1683N^5 + 1996N^4 + 2356N^3 + 2328N^2 + 1408N + 288 \]  
(E.208)

\[ P_{200} = 57N^9 + 624N^8 + 1756N^7 + 1092N^6 - 1803N^5 - 1512N^4 + 966N^3 + 1116N^2 + 920N + 528 \]  
(E.209)

\[ P_{201} = 69N^9 + 366N^8 + 1124N^7 + 1966N^6 + 2523N^5 + 5228N^4 + 7340N^3 + 5352N^2 + 3008N + 672 \]  
(E.210)

\[ P_{202} = 94N^9 + 597N^8 + 1616N^7 + 2410N^6 + 1841N^5 + 1165N^4 + 2191N^3 + 3802N^2 + 2916N + 648 \]  
(E.211)

\[ P_{203} = 121N^9 + 696N^8 + 1535N^7 + 1585N^6 + 416N^5 - 749N^4 - 836N^3 + 16N^2 \]  
(E.212)
\[ P_{204} = 197N^9 + 1242N^8 + 2938N^7 + 3524N^6 + 2713N^5 + 2234N^4 + 3680N^3 + 6176N^2 + 4080N + 864 \] (E.212)

\[ P_{205} = 439N^9 + 2634N^8 + 6008N^7 + 6694N^6 + 3545N^5 + 736N^4 + 2008N^3 + 6208N^2 + 5136N + 1152 \] (E.213)

\[ P_{206} = 538N^9 + 3333N^8 + 7802N^7 + 7630N^6 + 458N^5 - 1415N^4 + 7786N^3 + 12340N^2 + 5592N + 864 \] (E.214)

\[ P_{207} = 664N^9 + 3861N^8 + 9038N^7 + 11830N^6 + 9344N^5 + 3793N^4 + 3874N^3 + 11044N^2 + 9624N + 2592 \] (E.215)

\[ P_{208} = 891N^9 + 4455N^8 + 16078N^7 + 28774N^6 + 37047N^5 + 45835N^4 + 42192N^3 + 28888N^2 + 10640N + 1776 \] (E.216)

\[ P_{209} = 923N^9 + 5208N^8 + 11824N^7 + 12854N^6 + 2185N^5 - 7030N^4 + 1436N^3 + 15032N^2 + 12864N + 3456 \] (E.217)

\[ P_{210} = 965N^9 + 4884N^8 + 10816N^7 + 20810N^6 + 36895N^5 + 40442N^4 + 27692N^3 + 22712N^2 + 14496N + 3456 \] (E.218)

\[ P_{211} = 2N^{10} - 46N^9 - 98N^8 + 282N^7 + 1063N^6 + 1569N^5 + 1275N^4 + 403N^3 - 94N^2 - 108N - 24 \] (E.219)

\[ P_{212} = 2N^{10} + 12N^9 + 24N^8 + 11N^7 - 48N^6 - 151N^5 - 282N^4 - 480N^3 - 664N^2 - 576N - 288 \] (E.220)

\[ P_{213} = 11N^{10} + 44N^9 + 74N^8 + 196N^7 + 31N^6 - 1426N^5 - 3044N^4 - 2762N^3 - 1476N^2 - 480N - 96 \] (E.221)

\[ P_{214} = 11N^{10} + 76N^9 + 138N^8 - 204N^7 - 1041N^6 - 988N^5 + 752N^4 + 1896N^3 + 944N^2 - 384N - 576 \] (E.222)

\[ P_{215} = 37N^{10} + 392N^9 + 2106N^8 + 6514N^7 + 9211N^6 + 1258N^5 - 9218N^4 - 6116N^3 - 72N^2 - 1296N - 576 \] (E.223)

\[ P_{216} = 85N^{10} + 425N^9 + 902N^8 + 932N^7 - 521N^6 - 685N^5 + 2022N^4 + 2928N^3 + 968N^2 - 1296N - 576 \] (E.224)

\[ P_{217} = 103N^{10} + 575N^9 + 1124N^8 - 334N^7 - 1505N^6 + 3755N^5 + 4926N^4 + 36N^3 - 472N^2 - 2160N - 864 \] (E.225)

\[ P_{218} = 118N^{10} + 425N^9 + 197N^8 + 86N^7 + 1240N^6 + 2489N^5 + 4401N^4 + 3480N^3 + 524N^2 - 1728N - 864 \] (E.226)

\[ P_{219} = 118N^{10} + 557N^9 + 461N^8 - 94N^7 + 1300N^6 + 3521N^5 + 4509N^4 + 1920N^3 - 1132N^2 - 2376N - 1008 \] (E.227)

\[ P_{220} = 127N^{10} + 536N^9 + 611N^8 + 602N^7 + 1474N^6 + 2099N^5 + 798N^4 - 2301N^3 - 4486N^2 - 3708N - 936 \] (E.228)

\[ P_{221} = 170N^{10} + 883N^9 + 2041N^8 + 2998N^7 - 448N^6 - 5465N^5 + 129N^4 + 6624N^3 + 1132N^2 - 2016N - 864 \] (E.229)

\[ P_{222} = 170N^{10} + 1213N^9 + 3235N^8 + 2794N^7 - 2692N^6 - 3767N^5 - 1293N^4 - 1632N^3 - 5324N^2 - 6240N - 2016 \] (E.230)

\[ P_{223} = 226N^{10} + 317N^9 - 811N^8 + 662N^7 + 4552N^6 + 3857N^5 + 3933N^4 + 2364N^3 + 236N^2 - 1656N - 720 \] (E.231)
\[ P_{224} = 489N^{10} + 2934N^9 + 9364N^8 + 18830N^7 + 18627N^6 + 124N^5 - 19856N^4 - 19296N^3 - 10640N^2 - 2880N - 1152 \] (E.233)

\[ P_{225} = 3N^{11} + 42N^{10} + 144N^9 + 74N^8 - 459N^7 - 1060N^6 - 1152N^5 - 1424N^4 - 1688N^3 - 1232N^2 - 736N - 192 \] (E.234)

\[ P_{226} = 11N^{11} + 37N^{10} - 27N^9 - 118N^8 + 21N^7 - 249N^6 - 1097N^5 - 1138N^4 + 552N^3 + 3448N^2 + 3456N + 2016 \] (E.235)

\[ P_{227} = 21N^{11} + 231N^{10} + 1334N^9 + 4086N^8 + 6277N^7 + 1775N^6 - 9488N^5 - 18076N^4 - 18208N^3 - 11344N^2 - 5568N - 1728 \] (E.236)

\[ P_{228} = 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4 + 7328N^3 + 5456N^2 + 3456N + 1152 \] (E.237)

\[ P_{229} = 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4 + 12976N^3 + 8016N^2 + 4608N + 1728 \] (E.238)

\[ P_{230} = 51N^{11} + 269N^{10} + 46N^9 - 1934N^8 - 3973N^7 - 875N^6 + 7364N^5 + 14972N^4 + 16768N^3 + 10896N^2 + 5376N + 1728 \] (E.239)

\[ P_{231} = 51N^{11} + 357N^{10} + 1238N^9 + 2586N^8 + 2755N^7 - 1435N^6 - 9212N^5 - 15028N^4 - 15280N^3 - 9808N^2 - 5184N - 1728 \] (E.240)

\[ P_{232} = 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 - 4645N^6 - 8936N^5 - 11980N^4 - 12352N^3 - 8272N^2 - 4800N - 1728 \] (E.241)

\[ P_{233} = 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 - 20743N^6 - 24563N^5 - 16702N^4 - 6840N^3 + 120N^2 + 2432N + 960 \] (E.242)

\[ P_{234} = 243N^{11} + 1701N^{10} + 5378N^9 + 10350N^8 + 11479N^7 + 1193N^6 - 14684N^5 - 20572N^4 - 16288N^3 - 8944N^2 - 4992N - 1728 \] (E.243)

\[ P_{235} = 333N^{11} + 2331N^{10} + 6556N^9 + 9270N^8 + 5081N^7 - 6701N^6 - 17554N^5 + 20036N^4 - 15680N^3 - 9200N^2 - 5664N - 1728 \] (E.244)

\[ P_{236} = 753N^{11} + 4809N^{10} + 13174N^9 + 20466N^8 + 17717N^7 + 6829N^6 + 3908N^5 + 15304N^4 + 25408N^3 + 20272N^2 + 8448N + 1152 \] (E.245)

\[ P_{237} = 837N^{11} + 7775N^{10} + 30120N^9 + 68575N^8 + 119176N^7 + 191350N^6 + 262979N^5 + 258308N^4 + 163106N^3 + 63360N^2 + 14848N + 1536 \] (E.246)

\[ P_{238} = 1017N^{11} + 6195N^{10} + 14050N^9 + 12738N^8 - 2023N^7 - 5093N^6 + 27548N^5 + 69760N^4 + 80752N^3 + 54064N^2 + 20928N + 3456 \] (E.247)

\[ P_{239} = 3N^{12} + 21N^{11} + 17N^{10} - 202N^9 - 842N^8 - 1924N^7 - 3378N^6 - 5059N^5 - 6008N^4 - 4860N^3 - 2536N^2 - 960N - 192 \] (E.248)

\[ P_{240} = 9N^{12} + 63N^{11} + 38N^{10} - 414N^9 - 1035N^8 - 1341N^7 - 1511N^6 - 2972N^5 - 6011N^4 - 8038N^3 - 6892N^2 - 3432N - 864 \] (E.249)

\[ P_{241} = 9N^{12} + 63N^{11} + 71N^{10} - 381N^9 - 1536N^8 - 2529N^7 - 1946N^6 - 1331N^5 - 2096N^4 - 4036N^3 - 4144N^2 - 2304N - 576 \] (E.250)

\[ P_{242} = 39N^{12} + 585N^{11} + 2938N^{10} + 7136N^9 + 9083N^8 + 7745N^7 + 14668N^6 + 38246N^5 + 59856N^4 + 55560N^3 + 32144N^2 + 12480N + 2304 \] (E.251)

\[ P_{243} = 48N^{12} + 459N^{11} + 2322N^{10} + 8290N^9 + 20159N^8 + 30862N^7 + 28247N^6 + 16109N^5 + 9312N^4 + 7488N^3 + 4064N^2 + 1328N + 192 \] (E.252)

\[ P_{244} = 61N^{12} + 302N^{11} + 531N^{10} + 348N^9 - 349N^8 - 786N^7 + 457N^6 + 2524N^5 \] 186
\[ P_{265} = 75N^{16} + 1245N^{15} + 829N^{14} + 27609N^{13} + 43437N^{12} + 14221N^{11} - 5995N^{10} + 182937N^9 + 488696N^8 + 296818N^7 - 452292N^6 - 730430N^5 - 186180N^4 + 259728N^3 + 241056N^2 + 116640N + 25920 \] (E.273)

\[ P_{266} = 115N^{16} + 1838N^{15} + 11829N^{14} + 36114N^{13} + 30900N^{12} - 133946N^{11} - 454068N^{10} - 457420N^9 + 249211N^8 + 864716N^7 + 312979N^6 - 634466N^5 - 587862N^4 - 19556N^3 + 104832N^2 + 9504N + 1728 \] (E.274)

\[ P_{267} = 185N^{16} + 2988N^{15} + 19694N^{14} + 62954N^{13} + 64470N^{12} - 207876N^{11} - 792388N^{10} - 861230N^9 + 437231N^8 + 1750616N^7 + 869954N^6 - 1016136N^5 - 1130122N^4 - 96596N^3 + 199872N^2 + 31104N + 1728 \] (E.275)

\[ P_{268} = 939N^{16} + 10527N^{15} + 37207N^{14} + 18679N^{13} - 202006N^{12} - 617170N^{11} - 930025N^{10} - 882917N^9 - 157123N^8 + 1388549N^7 + 2739376N^6 + 2837500N^5 + 2088640N^4 + 1259669N^3 + 622464N^2 + 211392N + 34560 \] (E.276)

\[ P_{269} = 1155N^{16} + 12417N^{15} + 37693N^{14} - 12293N^{13} - 285754N^{12} - 613900N^{11} - 571735N^{10} - 134309N^9 + 778901N^8 + 2698745N^7 + 4995724N^6 + 5915740N^5 + 4978144N^4 + 3161840N^3 + 1498752N^2 + 497808N + 76032 \] (E.277)

\[ P_{270} = 1665N^{16} + 33005N^{15} + 287646N^{14} + 1402624N^{13} + 4031902N^{12} + 6199846N^{11} + 1054640N^{10} - 1668628N^9 - 3727255N^8 - 38892027N^7 - 17387942N^6 + 3962700N^5 + 1562580N^4 + 26960688N^3 + 27379296N^2 + 12985920N + 2323800 \] (E.278)

\[ P_{271} = 87N^{17} + 1099N^{16} + 6055N^{15} + 19019N^{14} + 37119N^{13} + 45159N^{12} + 29583N^{11} - 2639N^{10} - 30218N^9 - 40778N^8 - 39994N^7 - 35844N^6 - 30808N^5 - 30384N^4 - 28256N^3 - 16064N^2 - 5248N - 768 \] (E.279)

\[ P_{272} = 829N^{17} + 13413N^{16} + 83461N^{15} + 226391N^{14} + 55508N^{13} - 1239070N^{12} - 2862466N^{11} - 1217372N^{10} + 3372689N^9 + 2779147N^8 - 2705687N^7 + 171733N^6 + 8617302N^5 + 5817902N^4 - 3127236N^3 - 3652560N^2 - 336096N - 25920 \] (E.280)

\[ P_{273} = 1407N^{17} + 18107N^{16} + 103463N^{15} + 347083N^{14} + 760095N^{13} + 1142715N^{12} + 1220067N^{11} + 983393N^{10} + 702746N^9 + 533822N^8 + 337702N^7 - 3552N^6 - 300296N^5 - 332160N^4 - 188128N^3 - 63232N^2 - 13184N - 1536 \] (E.281)

\[ P_{274} = 95N^{18} + 3940N^{17} + 48989N^{16} + 308380N^{15} + 1166094N^{14} + 2843192N^{13} + 4428234N^{12} + 3171928N^{11} - 4692053N^{10} - 19875244N^9 - 34305831N^8 - 34774388N^7 - 16392680N^6 + 11584912N^5 + 30493776N^4 + 29700864N^3 + 18783360N^2 + 8294400N + 1866240 \] (E.282)

\[ P_{275} = 325N^{18} + 4280N^{17} + 17759N^{16} - 14880N^{15} - 412326N^{14} - 1696848N^{13} - 3216546N^{12} - 1169232N^{11} + 8956857N^{10} + 23914216N^9 + 31536899N^8 + 25361392N^7 + 9982840N^6 - 10154128N^5 - 26098704N^4 - 26761536N^3 - 17642880N^2 - 8087040N - 1866240 \] (E.283)

\[ P_{276} = 500N^{18} + 8215N^{17} + 56287N^{16} + 208100N^{15} + 361782N^{14} + 98826N^{13} - 759348N^{12} - 495786N^{11} + 3942186N^{10} + 11896133N^9 + 16709737N^8 + 13315736N^7 + 3779660N^6 - 7306454N^5 - 14232852N^4 - 13254768N^3 - 8367840N^2 - 3771360N - 855360 \] (E.284)

\[ P_{277} = 150N^{19} + 2815N^{18} + 42285N^{17} + 131358N^{16} + 511310N^{15} + 1515954N^{14} + 3372978N^{13} + 5213980N^{12} + 4715522N^{11} + 980739N^{10} - 2709391N^9 - 3741506N^8 - 4630558N^7 - 5623132N^6 + 3419632N^4 + 5238496N^3 + 3231936N^2 \] (E.285)
\[ P_{278} = 5410N^{19} + 98215N^{18} + 764965N^{17} + 3280996N^{16} + 8031920N^{15} + 8939378N^{14} \\
-7608074N^{13} - 44964380N^{12} - 74768226N^{11} - 57879177N^{10} - 5243187N^9 + 1374588N^8 \\
-28158216N^7 - 49672024N^6 + 14757808N^5 + 94650144N^4 + 100507392N^3 \\
+53764992N^2 + 15655680N + 1866240 \]  

\[ P_{279} = 7060N^{20} + 123495N^{19} + 898682N^{18} + 3394183N^{17} + 6222824N^{16} + 376386N^{15} \\
-22032204N^{14} - 39912378N^{13} - 13976964N^{12} + 31985011N^{11} + 4994394N^{10} \\
-91499501N^9 - 97243208N^8 + 54501988N^7 + 183103272N^6 + 127073120N^5 \\
-20272608N^4 - 88410816N^3 - 62225280N^2 - 21772800N - 3110400 \]  

The logarithmic contributions to all heavy flavor Wilson coefficients at 3–loop level are expressible within the class of alternating harmonic sums only.
Fixed Mellin moments for $m_1 \neq m_2$

In this Section we present the Mellin moments $N = 4$ and $6$ to the constant parts of the contributions with two massive lines of unequal mass to the OMEs $A_{qq, Q}^{NS}$, $A_{PS, Qq}^{NS}$, $A_{Qg}$, $A_{gq, Q}$ and $A_{gg, Q}$. We present expansions in the mass ratio $\eta = \frac{m_2}{m_1}$ up to $O(x^3 \ln^3(x))$. In order to obtain a compact notation we furthermore use the abbreviations

$$L_1 = \ln \left( \frac{m_1^2}{\mu^2} \right), \quad L_2 = \ln \left( \frac{m_2^2}{\mu^2} \right), \quad L_\eta = \ln(\eta).$$

\[ (F.289) \]

$$\tilde{a}_q^{NS, (3)}_{qq, Q}$$:

\[ F.289 \]

$$\tilde{a}_q^{NS, (3)}_{qq, Q} (N = 4) = C_F T_F^2 \left\{ -\frac{10048}{225} L_\eta - \frac{1689287216}{4219543125} - \frac{7300016}{13395375} L_\eta \right\} \eta^3 + \left( -\frac{70417954}{5788125} - \frac{628}{525} L_\eta - \frac{332212}{55125} L_\eta \right) \eta^2 + \left( -\frac{118064}{3375} - \frac{5024}{225} L_\eta \right) \eta + \left( -\frac{53084}{2025} L_2 - \frac{1256}{45} L_1 + \frac{2512}{135} \right) \eta + O(\eta^4 L_\eta^3)$$

\[ (F.290) \]

$$\tilde{a}_q^{NS, (3)}_{qq, Q} (N = 6) = C_F T_F^2 \left\{ -\frac{90752}{297675} L_\eta - \frac{16883116384}{29536801875} - \frac{66203584}{93767625} L_\eta \right\} \eta^3 + \left( -\frac{636004196}{40516875} - \frac{5672}{3675} L_\eta - \frac{3000488}{385875} L_\eta \right) \eta^2 + \left( -\frac{1066336}{23625} - \frac{45376}{1575} L_\eta \right) \eta + \left( -\frac{3424952}{99225} - \frac{11344}{315} L_2 + \frac{202733427313}{1093955625} + \frac{45376}{2835} \right) \eta + \left( -\frac{520819486}{3472875} L_1 - \frac{700881658}{10418625} L_2 - \frac{3424952}{99225} L_2 L_1 - \frac{22688}{945} L_1 L_2 \right) \eta + O(\eta^4 L_\eta^3)$$

\[ (F.291) \]

$$\tilde{a}_q^{PS, (3)}$$:

$$\tilde{a}_q^{PS, (3)} (N = 4) = C_F T_F^2 \left\{ \frac{190292193776}{1123242379875} + \frac{8509216}{324168075} L_\eta - \frac{1472}{93555} L_\eta \right\} \eta^3 + \left( -\frac{76}{315} L_\eta - \frac{89252}{99225} L_\eta \right) \eta^2 + \left( -\frac{5008}{945} - \frac{32}{9} L_\eta \right) \eta + \left( -\frac{2236}{2025} - \frac{968}{225} \right) L_2$$

\[ (F.291) \]
\(\tilde{a}_{qq}^{(3)}(N = 6) = C_F T_F^2 \left\{ \left( -5 + \frac{3}{4} L_1^3 \right) \zeta_2 - \frac{3}{4} \frac{L_1}{L_1 - \frac{L_1^2}{L_2}} + \frac{2}{15} L_2 + \frac{3}{10} L_3 - \frac{2}{10} \right\} + O(\eta^4 L_\eta^3) \) (F.292)

\(\tilde{a}_{qg}^{(3)}(N = 4) = C_A T_F^2 \left\{ \left( 2 + \frac{1}{4} L_1^3 \right) \zeta_2 - \frac{1}{4} \frac{L_1}{L_1 - \frac{L_1^2}{L_2}} + \frac{1}{5} L_2 + \frac{1}{3} L_3 - \frac{1}{3} \right\} + O(\eta^4 L_\eta^3) \) (F.293)

\(\tilde{a}_{qg}^{(3)}(N = 4) = C_F T_F^2 \left\{ \left( 2 + \frac{1}{4} L_1^3 \right) \zeta_2 - \frac{1}{4} \frac{L_1}{L_1 - \frac{L_1^2}{L_2}} + \frac{1}{5} L_2 + \frac{1}{3} L_3 - \frac{1}{3} \right\} + O(\eta^4 L_\eta^3) \) (F.294)
\[ \tilde{a}^{(3)}_{g,q}(N = 6) \] = 
\[ C_{A T}^2 \left\{ \left( -\frac{84840004938801319}{1381947564807810000} - \frac{2287164970759}{15339633390000} \right) \eta^3 \right. \]
\[ + \left( \frac{105157957}{360186750} \right) L_n + \frac{755537213056}{624023544375} \eta \left( \frac{49373}{103950} \right) L_n^2 \eta^2 + \left( \frac{832369820129}{29172150000} \right) \right) \]
\[ + \left( \frac{1406143531}{38915000} \right) L_n - \frac{112669}{1323000} L_n^2 \eta \left( \frac{1316809}{79380} + \frac{39248}{2205} \right) (L_2 + L_1) \right] \zeta_2 \]
\[ + \left( \frac{11771644229}{308962000} \right) L_2 + \left( \frac{206404}{19845} \right) L_3 \eta - \frac{197648}{19845} \eta \left( \frac{83755534727}{1166886000} \right) \]
\[ + \left( \frac{11119228}{4862025} \right) \eta \left( \frac{111848}{46305} \right) \eta + \frac{112669}{3969} L_2 L_1 + \frac{162074}{19845} L_3^2 \]
\[ + \frac{69882273800453}{735138180000} \right\} \]
\[ + C_{F T}^2 \left\{ \left( \frac{990283034941336}{2467763508585375} \right) + \frac{1255768040}{2191376187} L_n + \frac{63929464}{42567525} L_n^2 \right] \eta^3 \]
\[ + \left( \frac{11478584}{3361743} \right) L_n + \frac{524351089261}{231525} \eta \left( \frac{88972}{40425} \right) L_n^2 \eta^2 + \left( \frac{32427817736}{2552563125} \right) \right) \]
\[ + \left( \frac{64271512}{24310125} \right) L_n + \frac{376216}{231525} L_n^2 \eta \left( \frac{4784009}{4862025} - \frac{55924}{15435} \right) (L_2 + L_1) \right] \zeta_2 \]
\[ + \left( \frac{1786067629}{302410100} \right) L_2 - \frac{615164}{138915} L_3^2 + \frac{223696}{19845} \eta \left( \frac{24797875607}{204250500} \right) L_1 \]
\[ + \frac{111848}{46305} \eta \left( \frac{111848}{46305} \right) \eta + \frac{11119228}{4862025} L_2 L_1 - \frac{55924}{19845} L_3^2 \]
\[ - \frac{3161811182177}{142943535000} \right\} + O (\eta^4 L_n^3) \]  
\hspace{0.5cm} (F.295) 

\[ \tilde{a}^{(3)}_{g q, Q} \] :

\[ \tilde{a}^{(3)}_{g q, Q}(N = 4) \] = 
\[ C_{F T}^2 \left\{ \left( \frac{261938336}{1406514375} \right) + \frac{1027136}{4465125} L_n + \frac{1408}{14175} L_n^2 \right] \eta^3 \right. \]
\[ + \left( \frac{88}{175} L_n^2 + \frac{46552}{18375} L_n \right) \eta^2 + \left( \frac{16544}{1125} + \frac{704}{75} \right) L_n \eta + \left( \frac{2504}{675} \right) \right) \]
\[ + \left( \frac{176}{15} (L_2 + L_1) \right) \zeta_2 + \left( \frac{176}{45} L_2^2 L_2 + \frac{704}{135} \right) \eta \left( \frac{436138}{10125} \right) L_1 + \frac{54446}{3375} L_2 \]
\[ + \left( \frac{2504}{675} L_2 L_1 + \frac{352}{45} L_1 L_2^2 + \frac{704}{135} L_2^3 + \frac{2504}{675} (L_2^2 + L_1^2) + \frac{176}{27} L_3^2 \right) \right] \frac{18480197}{455625} \}
\[ + O (\eta^4 L_n^3) \]  
\hspace{0.5cm} (F.296) 

\[ \tilde{a}^{(3)}_{g q, Q}(N = 6) \] = 
\[ C_{F T}^2 \left\{ \left( \frac{1047753344}{98456000625} + \frac{4108544}{31255875} L_n + \frac{5632}{99225} L_n^2 \right) \eta^3 \right. \]
\[ + \left( \frac{39469936}{13505625} \right) \right) \]
\[ + O (\eta^4 L_n^3) \]  
\hspace{0.5cm} (F.297)
(F.297)

\[ a_{g_0Q}^{(3)} : \]

\[ a_{g_0Q}^{(3)} (N = 4) = C_A T_F^2 \left\{ \left( \frac{311441927}{1687817250} - \frac{1293167}{5358150} L_\eta + \frac{1681}{34020} L_\eta^2 \right) \eta^2 + \left( -\frac{205123}{99225} L_\eta \right) \right\} \]

\[ + \left( \frac{28979}{675} - \frac{1558}{45} (L_2 + L_1) \right) \eta^3 \]

\[ + \left( \frac{28979}{675} - \frac{1558}{45} (L_2 + L_1) \right) \zeta_2 \]

\[ + O(\eta^4 L_\eta^3) \]
\[
- \frac{3604631677201}{13127467500} - \frac{120697}{2205} \left( L_2^2 + L_1^2 \right)
\]

\[+ C_F T_F^2 \left\{ \left( - \frac{91864096}{5907360375} + \frac{979936}{3750705} L_1 - \frac{130048}{297675} L_2 \right) \eta^2 + \left( - \frac{41008}{2701125} \right) \right\} \]

\[+ \frac{10816}{46305} \left( L_1 - \frac{384}{1225} L_2 \right) \eta^2 + \left( \frac{2133088}{826875} + \frac{3616}{3375} L_1 - \frac{1216}{11025} L_2 \right) \eta \]

\[+ \left( \frac{197492}{231525} + \frac{968}{2205} \left( L_2 + L_1 \right) \right) \zeta_2 + \frac{413083}{2431025} L_2 + \frac{10648}{19845} L_3^1 - \frac{3872}{2835} \zeta_3 \]

\[+ \frac{281801489}{72930375} L_1 + \frac{1936}{6615} L_1 L_2^1 + \frac{968}{2835} L_2^3 - \frac{1936}{6615} L_1^2 L_2 + \frac{81176}{99225} L_2 L_1 \]

\[+ \frac{14596284331}{5105126250} + \frac{604598}{694575} \left( L_2^2 + L_1^2 \right) \right\} + O \left( \eta^4 L_3^3 \right) \quad \text{(F.299)}
References


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