1. Introduction. Within the vast literature on colorability there are only a few papers concerning the following coloring problem. By an r-coloring relative to distance k of a graph G — throughout graphs are finite, undirected, without loops and multiple edges — we mean an assignment of at most r colors to the vertices of G so that the distance between any two vertices having the same color is greater than k. Let \( \chi_k(G) \) denote the smallest number r such that G has an r-coloring relative to distance k, which we abbreviate r-coloring(k). Of course \( \chi_1(G) \) is the usual chromatic number of G. r-colorings(k) have been considered by F.Kramer and H.Kramer [16],[17],[18]; especially they calculate the numbers \( \chi_k(C_n) \) of circuit graphs \( C_n \) and characterize those graphs G which have \( \chi_k(G) = k+1 \) or \( \chi_k(G) = \chi_{k+1}(G) = k+2 \). In his forthcoming thesis C.Ivan [13] considers r-colorings(k) for cacti.

Now let \( \mathcal{G} \) be a family of graphs; then we define

\[
\chi_k(\mathcal{G}) := \sup \{ \chi_k(G) \mid G \in \mathcal{G} \}.
\]

If \( \mathcal{G} \) contains graphs with arbitrarily high maximum degree, then \( \chi_k(\mathcal{G}) = \infty \) for \( k \geq 2 \).

In order to obtain nontrivial results we consider the families \( \mathcal{G}_m \) [resp. \( \mathcal{G}_d \)] of all graphs [resp. all planar graphs] with maximum degree not exceeding d. Obviously \( \chi_k(G) = n \), if the graph G has n vertices and diameter \( d(G) \leq k \). Therefore we have

\[
(1) \quad \chi_k(\mathcal{G}) \geq n_k(\mathcal{G}),
\]

where \( n_k(\mathcal{G}) \) denotes the maximum number of vertices of those graphs in \( \mathcal{G} \) whose diameter is not greater than k. Because of (1) it seems to be suitable to collect the results (some known and some new) on the numbers \( n_k(\mathcal{G}_d) \) and \( n_k(\mathcal{G}_d) \). This is the aim of the next two sections; in section 4 we return to \( \chi_k \).

2. The numbers \( n_k(\mathcal{G}_d) \). Trivially \( n_1(\mathcal{G}_d) = d+1 \) because of \( K_{d+1} \in \mathcal{G}_d \) and \( n_k(\mathcal{G}_d) = 2k+1 \) because of \( C_{2k+1} \in \mathcal{G}_d \). Now suppose \( k > 1 \) and \( d > 2 \). We have

\[
(2) \quad n_k(\mathcal{G}_d) \leq N(d,k) := 1 + \frac{(d-1)^k - 1}{d - 2}
\]

with equality iff a (d,k)-Moore graph exists, that is only if \( k = 2 \) (see H.D.Friedman [11], R.W.Damerell [8], E.Bannai - T.Ito [4]) and even then only for \( d = 2, 3, 7 \) and possibly \( d = 57 \) (A.J.Hoffman - R.R.Singleton [12]). In any other case we have \( n_k(\mathcal{G}_d) < N(d,k) \) resp. even \( n_k(\mathcal{G}_d) < N(d,k) - 1 \), if d and N(d,k) - 1 both are odd numbers (since a graph in \( \mathcal{G}_d \) with diameter \( \leq k \) and more than \( \frac{k - 1}{d - 2} (d-1)^1 = N(d,k) - \frac{(d-1)^1 - 1}{d - 2} \) vertices is necessarily regular of degree d, if such a graph exists at all; compare also B.Elsas [9]). General lower bounds have been given by H.D.Friedman [10] and I.Korn [15], overhauling the general bounds given by B.Elsas [9]:

\[
(3) \quad n_{2h}(\mathcal{G}_d) \geq 2d(d-1)^h - 1 \quad \text{[10]}
\]

\[
(4) \quad n_{2h+1}(\mathcal{G}_d) \geq 2 \frac{2d(d-1)^{h+1}-d}{d - 2} \quad \text{[15]}
\]

But these formulas don't yield useful estimates for small values of d and k. Especially
for $k = d - 1$ S.B.Akers [2] proved

\begin{equation}
\frac{n_d}{d} \geq \binom{2d-1}{d}
\end{equation}

which is in the cases $d \leq 12$ better than (3) and (4), but apart from $d = 2, 3$ and possibly $d = 4$ by no means best possible. First we give now an improvement of both (3) and (4).

**Theorem 1.** Let $h \geq 1$, $d > d_1 \geq 0$ and $k = k_1 + 2h$. Then

\begin{equation}
\frac{n_k}{d} \geq k_1 \left( \frac{2(d-d_1)(d-1)^h + d(d_1-2)}{d-2} \right)
\end{equation}

Remark: With $d_1 = k_1 = 0$ and $n_0(\frac{\varphi}{\varphi}) = 1$ one gets Friedman's formula (3) and with $d_1 = k_1 = 1$ and $n_1(\frac{\varphi}{\varphi}) = 2$ Korn's formula (4). But for $d > 4$ and suitable choice of $k_1$ and $d_1$ one gets with (6) better results than with (3) and (4).

**Proof of (6):** The construction is similar to that of Friedman and Korn. We start our construction with a graph $G_1$ of diameter $k_1$ and maximum degree $d_1$ having $N$ vertices. Now we take a rooted tree with radius $h$, whose root has valency $d - d_1$ and whose further vertices other than endvertices have valency $d$. We identify each vertex of $G_1$ with the root of a copy of such a tree thus obtaining a graph with diameter $k_1 + 2h$ and

\[ N + N(d-d_1) + N(d-d_1)(d-1) + \ldots + N(d-d_1)(d-1)^{h-1} \]

vertices. It is easy to see that the diameter remains unchanged if we take $d$ copies of this graph and identify their endvertices (see, figure 1). The resulting graph has maximum degree $d$ and

\[ N(d-d_1)(d-1)^{h-1} + d [ N + N(d-d_1) + N(d-d_1)(d-1) + \ldots + N(d-d_1)(d-1)^{h-2} ] - \]

\[ \frac{N}{d-2} \left[ 2(d-d_1)(d-1)^h + d(d_1-2) \right] \]

vertices. //

For odd diameter we get bounds sometimes better than those arising from (6) by the following formula:

\begin{equation}
\frac{n_{2k+1}}{d+1} \geq k \left( \frac{n_k}{d} \left[ n_k(\frac{\varphi_d}{\varphi}) + 1 \right] \right)
\end{equation}

**Proof.** Let $G$ be a graph with diameter $k$, maximum degree $d$ and $N$ vertices. Take $N+1$ copies of $G$ and label them $0, 1, \ldots, N$. Label the vertices of $G_1$ with the same numbers.
omitting the number \( i \), for each \( i \). Then join the vertex of \( G_i \) labelled \( i_2 \) with the vertex of \( G_{i_1} \) labelled \( i_1 \) for every pair of numbers \( i_1 + i_2 \). Thus any two of the copies of \( G \) are joined by just one edge and it's clear that the resulting graph has diameter \( 2k + 1 \), maximum degree \( d + 1 \) and \( N(N+1) \) vertices. //

The application of both (6) and (7) needs good estimations for \( n_k(\mathcal{S}_d) \) for small values of \( k \) and \( d \). Now we shall collect the results for these values.

\[
\begin{array}{cccccccccccccccc}
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 & \ldots & 11111 \\
\end{array}
\]

Fig. 2

\( d - 1 \) copies of \( C_5 \), pairwise connected by additional edges such that each pair yields a Peterson graph (see Fig. 2 for \( d = 4 \), show (8) \[ n_2(\mathcal{S}_d) \geq 5d - 5. \]

In (8) we have equality not only for \( d = 2, 3 \), as is well known, but also for \( d = 4 \); this value has been given by B. Elspas [9] together with \( n_2(\mathcal{S}_3) = 24 \), both without proof. While it is not hard to prove \( n_2(\mathcal{S}_3) = 15 \), the inequality \( n_2(\mathcal{S}_d) \geq 24 \) in [9] is erroneously based upon a graph by N.W. Green, which does not have diameter 2. Nevertheless the inequality is correct and Figure 3 displays the adjacency matrix of a graph with 24 vertices, diameter 2 and degree 5. — For \( d = 6 \) we have (9) \[ n_2(\mathcal{S}_6) \geq 32. \]

The graph which proves this inequality is built up by the two subgraphs shown in Figure 4.

Each vertex of the graph on the left hand — the graph of the dodecahedron together with its ten diagonals — has to be joined with two vertices of the graph on the right hand as indicated

Fig. 4

by numbers. (The construction of this graph has been inspired by a 5-valent graph with girth 5 and 30 vertices given by N. Robertson [19]).
Next we consider \( k = 3 \). \( n_3(\mathcal{G}_3) = 20 \) has been proved by B. Elspas [9]. We draw this graph (which possibly is unique) in a somewhat different manner (figure 5) to exhibit its relationship to the Petersen graph.

From (5) we have \( n_3(\mathcal{G}_4) \geq 35 \), which is best possible up to now.

Further we have

\[
(10) \quad n_k(\mathcal{G}_d) \geq 2 \left( \frac{d-1}{d} \right)^k - 1 \quad \text{for } d = 3, 4, 6 \; \text{; } d-1 \text{ a prime power},
\]

since the corresponding \( d \)-regular graphs with girth \( 2k \) given by F. Károly [14], W.G. Brown [7] and C.T. Benson [5] also have diameter \( k \) (compare also R. Singleton [20]).

In the case \( k = 6 \) this fact is not explicitly mentioned by Benson [5], but easy to prove by counting vertices: A \( d \)-regular graph with girth \( 2k \) and diameter \( > k \) would have necessarily more than

\[
1 + d + d(d-1) + \ldots + d(d-1)^{k-2} + (d-1)^k = 2 \left( \frac{d-1}{d} \right)^k - 1
\]

vertices. — For \( k = 3 \) (10) yields \( n_3(\mathcal{G}_d) \geq 2d^2 - 2d + 2 \), for \( d-1 \) a prime power, and concerning the other cases we have Elspas' result [9]

\[
(11) \quad n_3(\mathcal{G}_d) \geq 2d^2 - 3d + 1.
\]

Finally a graph showing

\[
(12) \quad n_3(\mathcal{G}_3) = 48
\]

is given in figure 6.

Table 1 summarizes the results on \( n_k(\mathcal{G}_d) \) for small values of \( k \) and \( d \). The number in brackets indicates the formula from which this lower bound results.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( d )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>24</td>
<td>36 (3)</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>20 (3)</td>
<td>52 (3)</td>
<td>104 (10)</td>
<td>186 (10)</td>
<td>300 (11)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9 (10)</td>
<td>44 (10)</td>
<td>160 (10)</td>
<td>424 (10)</td>
<td>936 (10)</td>
<td>1812</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11 (12)</td>
<td>92 (12)</td>
<td>484 (7)</td>
<td>1704 (7)</td>
<td>4686 (7)</td>
<td>1088 (7)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>13 (10)</td>
<td>188 (10)</td>
<td>1456 (10)</td>
<td>8824 (10)</td>
<td>23436 (10)</td>
<td>65317</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>126</td>
<td>728</td>
<td>2730</td>
<td>7812</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. The numbers \( n_k(\overline{F_d}) \). Trivial upper bounds for \( n_k(\overline{F_d}) \) we get from the previous section since

\[
(13) \quad n_k(\overline{F_d}) \leq n_k(\overline{F_d}) .
\]

Thus we have \( N(d,k) \) as an upper bound and since every planar graph contains vertices of degree \(< 5\) one may improve this bound for \( d > 5 \) immediately to

\[
(14) \quad n_k(\overline{F_d}) \leq 1 + 5 \frac{(d-1)^k - 1}{d - 2} \quad (d > 5) .
\]

Although this is a rather rough bound, it seems to be hard to give general improvements.

For \( k = 1 \) we have

\[
(15) \quad n_1(\overline{F_d}) = \begin{cases} d + 1 & \text{for } d \leq 3 \\ 4 & \text{for } d > 3 \end{cases}
\]

because of the nonplanarity of \( K_{d+1} \) for \( d > 4 \). Of course \( n_k(\overline{F_2}) = n_k(\overline{F_2}) = 2k + 1 \) since \( C_{2k+1} \in \overline{F_2} \). For \( k = 2 \) we prove:

\textbf{Theorem 2.}

\[
(16) \quad \left\lfloor \frac{3d}{2} \right\rfloor + 1 \leq n_2(\overline{F_d}) ,
\]

\[
(17) \quad \frac{3d}{2} + 8 \leq n_2(\overline{F_d}) \quad \text{for } d \geq 22 .
\]

Inequality (16) is proved by the graph of figure 7, where dotted lines may be added in the cases \( d \geq 4 \) in order to obtain a 3-connected graph, if desired.

In order to prove the second inequality we give a preparatory lemma.

\textbf{Lemma.} Let \( G \) be outerplanar and let be given a plane embedding of \( G \) with straight edges such that all vertices of \( G \) are situated on a circle \( C \) (this is always possible). Let \( A, B \) be two sets of vertices of \( G \) with the following properties: \( A \) and \( B \) are separable by some straight line, \( |A| \geq 4 \), \( |B| \geq 4 \) and any pair \( a, b \) of vertices \( a \in A, b \in B \) has distance at most 2 in \( G \). Then there exists a vertex \( x \) in \( G \) dominating both \( A \) and \( B \).

(We say that a vertex \( x \) dominates \( A \) if each vertex \( a \in A \), \( a \neq x \), is adjacent to \( x \).)

\textbf{Proof.} Because of the separability we may assume that \( a_1, \ldots, a_m, b_1, \ldots, b_n \) is a labelling of \( A \cup B \) in counterclockwise order on \( C \).

\textbf{Case 1:} There exist \( i \in \{2, \ldots, m-1\} \) and \( j \in \{2, \ldots, n-1\} \) such that \( a_i \sim b_j \). *)

Now the edge \( (a_i, b_j) \) separates \( a_1 \) and \( b_1 \); to ensure \( \text{dist}(a_1, b_1) \leq 2 \) we must have \( a_1 \sim a_i \sim b_j \) or \( a_1 \sim b_j \sim b_1 \), say \( a_1 \sim a_i \sim b_j \). Considering further pairs of vertices we see that \( a_1 \) dominates \( A \cup B \).

*) \( a \sim b \) denotes adjacency of \( a \) and \( b \).
Case 2: None of \( a_2, \ldots, a_{m-1} \) is adjacent to any of the vertices \( b_2, \ldots, b_{n-1} \). Then there exists \( x \notin \{a_2, \ldots, a_{m-1}, b_2, \ldots, b_{n-1} \} \) such that \( a_2 \sim x \sim b_2 \). By arguments similar to those above we see that \( x \) is a dominating vertex. //

Proof of (17): Let \( G \in \overline{G}_m \) have diameter 2.

Case 1: There exists a separating set \( T \) of at most three vertices \( a_1 \). Because of \( \text{diam} \ G = 2 \), \( T \) is a dominating set in \( G \). Let \( R, S \) be the two sets of vertices separated by \( T = \{a_1, a_2, a_3\} \) and \( r = |R|, s = |S|, n = |\text{vert} \ G| \). By \( R_1 \), resp. \( R_1^c \), we denote the set of vertices of \( R \) having in \( T \) only \( a_1 \), resp. \( a_1 \) as neighbour, likewise \( S_1 \) and \( S_1^c \); we assume \( R_1 := |R_1| \) and so on. The numbers \( r_{123} \) and \( s_{123} \) of vertices of \( R \) and \( S \) adjacent to all three vertices of \( T \) is 0 or 1.

Case 1.1: Each vertex of \( R \cup S \) is adjacent to at least two vertices of \( T \). Then \( 2(r+s) \leq 3d \) and thus \( n = r + s \leq 3 + \frac{3d}{2} \).

Case 1.2: There exist vertices in \( R \cup S \) adjacent to just one vertex in \( T \), say \( r_1 \neq 0 \). Then \( r_{23} \leq 2 \) and \( s_{23} \leq s_{3} = 0 \) and
\[
(*) \quad r_1 + r_{12} + r_{13} + r_{123} + s \leq d
\]
since any vertex of \( S \) must be adjacent to \( a_1 \).

Case 1.2.1: \( r_2 + r_{23} + r_3 \leq d \). Then \( n = r + s \leq 3 + \frac{3d}{2} \).

Case 1.2.2: \( r_2 + r_{23} + r_3 > d \). Then \( r_2 + r_3 > \frac{d}{2} - 2 \); since \( d > 16 \) we may assume \( r_2 \geq 4 \) and so \( r_{12} \leq 2 \).

If also \( r_3 = 0 \), then \( r_{12} \leq 2 \), \( s = s_{123} = 1 \) and \( r_1 + r_2 + r_3 \leq d + 5 \) in view of the lemma: Assume \( r_1 + r_2 + r_3 > d + 5 \); since \( r_1 \leq d - 1 \) at least two of these numbers are greater than 3, say \( r_1 \) and \( r_2 \), taking \( A = R_1 \) and \( B = R_2 \cup R_3 \) we see that there exist a dominating vertex of degree of \( r_1 + r_2 + r_3 - 1 \), which is impossible. Thus
\[
n = r_1 + r_2 + r_3 + r_{12} + r_{13} + r_{123} + s \leq 3 + d + 5 + 2 + 2 + 1 + 1 = d + 16 \leq \frac{3d}{2} + 5
\]
for \( d \geq 22 \).

Now assume \( r_3 = 0 \). Then \( r_2 > \frac{d}{2} - 2 \) and with the help of the lemma \( r_1 + r_{12} \leq \frac{d}{2} + 2 \) (take \( A = R_2 \) and \( B = R_1 \cup R_3 \)), thus \( n \leq \frac{3d}{2} + 5 \) since similar to (*)
\[
r_2 + r_1 + r_{12} + r_{13} + r_{123} + s \leq d.
\]

Case 2: Any separating set of vertices has at least 4 vertices. Then \( G \) cannot contain vertices of degree \( \leq 3 \). Let \( x \) be a vertex of minimum degree \( k \) (\( k = 4 \) or \( 5 \)) and \( y_1, \ldots, y_k \) its neighbours labelled according to their plane cyclical order. Any further vertex of \( G \) is adjacent to at least one of \( y_1, \ldots, y_k \). \( y_i \) cannot be adjacent to \( y_j \) unless \( j = i \pm 1 \) (mod \( k \)). Otherwise \( \{x, y_i, y_j\} \) would be a separating set.

Case 2.1: Two of \( y_1, \ldots, y_k \), say \( y_j, y_j \), that are not cyclically neighboured, have a common neighbour \( z, x \). Omitting \( y_i \) and adding edges \( z, y_{i+1} \) (mod \( k \)) so for \( y_i \) we get a graph \( G' \in \overline{G}_d \) with \( \text{diam} \ G' = 2 \) having the separating set \( \{y_j, y_1, z\} \) and so \( n \leq 1 + \frac{3(d+1)}{2} + 5 \) according to case 1.

Case 2.2: Any further vertex belongs to some set \( R_1 \) of vertices adjacent to \( y_i \) only or to some set \( R_{i+1} \) of vertices adjacent to both \( y_i \) and \( y_{i+1} \) (mod \( k \)). We have \( |R_{i+1}| = 1 \), otherwise we would have a separating triple. Now with \( A_1 = R_1 \cup R_{i+1} \cup R_{i+1} \) and \( B = R_{i+2} \cup R_{i+2} \cup R_{i+3} \cup \ldots \cup R_{i+1} \) (mod \( k \)) we may apply the lemma. Thus either there exists \( i \) such that both \( |A_1| = 4 \) and \( |B_1| = 4 \) and then according to the lemma \( |A_1| + |B_1| = d \) and so \( n \leq d + 2 + 5 + 1 = d + 8 \), or we have for each \( i \) either \( |A_1| = 3 \) or \( |B_1| = 3 \). But then there is at most one \( j \) such that \( |A_1| = 4 \), on the other hand we have \( |R_i, j| \leq d - 1 \) and so \( n \leq d - 1 + 2 + 3 + 1 + k + 1 \leq d + 12 \). //
For $3 \leq d \leq 5$ we have

\[(18) \quad n_2(\overline{G}_d) = 7, \quad n_2(\overline{G}_4) = 9, \quad n_2(\overline{G}_5) = 10.\]

Graphs showing $\equiv$ are given in figure 8, the proofs of $\equiv$ are elementary, but tedious; we omit details.

\[\text{Fig. 8}\]

It is worth noting that the first graph of figure 8 is not 3-connected. Indeed for the class $\mathcal{P}_3$ of 3-valent, 3-connected planar graphs (i.e. the graphs of simple 3-polytopes) we have $n_2(\mathcal{P}_3) = 6$.

Figure 9 shows

\[(19) \quad n_2(\overline{G}_6) \geq 11, \quad n_2(\overline{G}_7) \geq 12\]
and we conjecture:

Conjecture. $n(\overline{G}_d) = d + 5$ for $d = 6, 7$

\[n(\overline{G}_d) = \left\lfloor \frac{3d}{2} \right\rfloor + 1 \text{ for } d \geq 8\]

\[\text{Fig. 9}\]

Finally we give by some easy constructions shown for $r = 2$ in figures 10-12 general lower bounds and it seems very likely that these bounds are close by the exact values.

\[(20) \quad n_{2r+1}(\overline{G}_d) \geq 3(d-1)^r + \frac{(d-1)^r - 1}{d-2} \text{ for } d = 3, 4 \quad (\text{see figure 10}).\]

\[\text{Fig. 10}\]

\[(21) \quad n_{2r+1}(\overline{G}_d) \geq (4d-2)(d-1)^{r-1} \text{ for } d > 4 \quad (\text{see figure 11}).\]

\[\text{Fig. 11}\]

\[(22) \quad n_{2r}(\overline{G}_d) \geq \frac{1}{d-2}[(d+2)(d-1)^r - 4] \quad (\text{see figure 12}).\]

In general these inequalities will not be best possible, for instance we have

\[(23) \quad n_3(\overline{G}_3) \geq 12 \quad (\text{see figure 13}).\]
4. The coloring numbers \( \chi_k(\mathcal{F}_d) \) and \( \chi_k(\mathcal{G}_d) \). As noted in section 1 we have \( \chi_k(\mathcal{G}_d) \equiv \eta_k(\mathcal{G}_d) \) and \( \chi_k(\mathcal{F}_d) \equiv n_k(\mathcal{F}_d) \). For \( d = 2 \) we have equality in both cases, so we restrict our attention in the following to \( d \geq 3 \). And as just used in [1] the problem of finding an \( r \)-coloring(k) of a graph \( G \) may be reduced to the problem of finding an ordinary coloring by considering the \( k \)-th power of \( G \): \( \chi_k(G) \) equals the ordinary chromatic number \( \chi(G^k) \) of \( G^k \).

If we define the clique number \( \delta_k(G) \) relative to distance \( k \) to be the maximum number of vertices of subgraphs \( G' \) of \( G \) with \( \text{diam} G' \leq k \), then we have similarly \( \delta_k(G) = \delta(G^k) \), where \( \delta \) denotes the usual clique number. Of course \( \chi_k(G) \equiv \delta_k(G) \). If \( G \) has maximum degree \( d \), then \( G^k \) has maximum degree \( N(d,k) - 1 \). Thus according to a wellknown theorem \( \chi(G^k) \equiv N(d,k) \) for every \( G \in \mathcal{F}_d \) and we have

\[
(24) \quad n_k(\mathcal{F}_d) = \delta_k(\mathcal{F}_d) = \chi_k(\mathcal{F}_d) = N(d,k) \quad (*)
\]

whenever a \((d,k)\)-Mooregraph exists (this cases are listed in section 2, now including \( k = 1 \)), while in any other case we get using a theorem of Brooks [5]

\[
(25) \quad n_k(\mathcal{G}_d) \leq \delta_k(\mathcal{G}_d) \leq \chi_k(\mathcal{G}_d) < N(d,k).
\]

The difficulty to prove further restrictions on \( \chi_k(\mathcal{G}_d) \) becomes evident if we now consider the case of planar graphs. Again we have \( \chi_k(\mathcal{G}_d) - n_k(\mathcal{G}_d) = 2k + 1 \) and for \( d \geq 3 \) similarly to (25)

\[
(26) \quad n_k(\mathcal{F}_d) \leq \delta_k(\mathcal{F}_d) \leq \chi_k(\mathcal{F}_d) < N(d,k)
\]

for any \( d \geq 3 \) and \( k \geq 2 \), since all the Moore graphs in question are not planar. For \( k = 1 \) we know \( \chi_1(\mathcal{F}_d) = 4 \) for \( 3 \leq d \leq 5 \) (see J.K.Aarts - J.de Groot [1]), but the question whether \( \chi_1(\mathcal{G}_d) = 4 \) holds for all \( d \geq 3 \) is precisely the famous and long standing four color problem, which just has been solved by K.Appel and W.Haken with a proof that is very long and depends heavily on extensive use of computers (see K.Appel - W.Haken [3]).

In order to stimulate further research we venture a challenging conjecture:

**Conjecture:** For any \( d \geq 3 \), \( k \geq 1 \)

\[
\begin{align*}
n_k(\mathcal{F}_d) &= \delta_k(\mathcal{F}_d) = \chi_k(\mathcal{F}_d) \quad \text{and} \\
n_k(\mathcal{G}_d) &= \delta_k(\mathcal{G}_d) = \chi_k(\mathcal{G}_d)
\end{align*}
\]

As noted above one cannot expect a general answer but it would be interesting to settle some cases. As a first step in this direction we prove \( \chi_2(\mathcal{F}_3) \equiv 8 \) and it remains open whether \( \chi_2(\mathcal{F}_3) = 7 \) or \( \chi_2(\mathcal{F}_3) = 8 \).

**Theorem 3.** \( \chi_2(\mathcal{F}_3) \equiv 8 \).

*) where \( \delta_k(\mathcal{G}) := \sup \{ \delta_k(G) \mid G \in \mathcal{G} \} \).
Proof. Let $G$ be a graph of $\overline{K}_3$ with $\chi_2(G) = 9$ and minimum number of vertices. We prove by contradiction that such a graph cannot exist. In order to do this we first deduce some properties of $G$.

(a) $G$ is regular of degree 3 and does not contain 3-circuits or pairs of 4-circuits with an edge in common.

Otherwise let $v$ be a vertex of degree $< 3$ or a vertex of some 3-circuit or a vertex of an edge belonging to two 4-circuits. The antistar $G'$ of $v$ in $G$ is 8-colorable(2) by minimality of $G$. But this coloring can be extended to $G$ since $v$ has at most 7 neighbours of first and second order, a contradiction. /

(b) $G$ is 3-connected.

Clearly $G$ is connected. Assume that $G$ is not 3-connected and let $e_1 = (v'_1, v''_1)$ and $e_2 = (v'_2, v''_2)$ be two edges separating $G$ into two components $G'$ and $G''$ with $v'_1 \in G'$ and $v''_1 \in G''$ [omit $e_2$ in the case of 1-connectedness *]. We are able to color $G'$ rel. to distance two with 8 colors — this coloring may be described by a function $f : \text{vert } G' \rightarrow \{1, 2, \ldots, 8\}$ such that $f(v'_1), f(v''_2) \in \{1, 2\}$ and none of the neighbours of $v'_1$ and $v''_2$ has color 3 or 4 (since there are at most 4 neighbours). Likewise we color $G''$ such that $f(v''_1), f(v''_2) \in \{3, 4\}$ and none of the neighbours of $v''_1, v''_2$ has color 1 or 2. Obviously both colorings may be fitted together to yield an 8-coloring(2) of $G$. /

It is worth noting that in so far we didn't make use of the planarity of $G$.

(c) $G$ cannot contain 4-circuits.

Assume that $x_1, \ldots, x_6$ are the vertices of some 4-circuit $C$ of $G$. Because of (a) and (b) each $x_i$ has a neighbour $y_i \notin C$ and all $y_1$ are different and nonadjacent ($y_1 \not\sim y_3$ and $y_2 \not\sim y_4$) involve together with (b) the planarity of $G$).

Omitting $C$ and the edges incident with $C$ we get a graph $G'$ (see figure 13) which has an 8-coloring(2). We try to extend this coloring to $G'$. Consider one fixed $x_i$; coloring this $x_i$ we have to avoid the colors of five vertices of $G'$. Thus in view of $G'$ we can assign to each $x_i$ a set $A_i$ of at least three admissible colors. Now it is possible to choose for all $x_i$ different admissible colors provided that not all $A_i$ consist of the same three colors, say the colors 6, 7, 8. In that case we change the coloring of $y_1$ in $G'$. First note that the colors assigned to $y_1, y_2, y_4, u, v$ (see figure 14) all are different, otherwise at least four colors would be admissible for $x_i$. Further at least one of the colors $f(y_2), f(y_6), 6, 7, 8$ does not occur within the colors of the (at most four) neighbours of second order of $y_1$ (among which may be some $y_i$). Recoloring $y_1$ with that color $x_i$ has four admissible colors or an admissible triple different from that of $x_i$, which remains unchanged. After recoloring we have the general case of above. /

In the last step of the proof we show that $G$ cannot contain 5-circuits. Since every 3-connected, planar graph contains n-circuits with $n < 6$ this proves the nonexistence of $G$.

*) In the case of cubic graphs edge-connectivity coincides with vertex-connectivity.
The procedure in this last step is the same as in the proof of (c). Let \( x_1, \ldots, x_5 \) be the vertices of a 5-circuit \( C \) of \( G \); each \( x_i \) has a neighbour \( y_i \not\in C \) and all \( y_i \) are different and nonadjacent. Let be given an 8-coloring(2) of the antistar \( G' \) of \( C \) in \( G \). As in (c) denote by \( A_i \) the set of colors admissible for \( x_i \) (in view of \( G' \)). In any way we have up to permutation of vertices or colors one of the following cases.

**Case 1:** \( f(y_1) = f(y_2) = f(y_3) \). Then \( |A_0| \geq 5 \), \( |A_1| \geq 4 \) and \( |A_2| \geq 4 \) and the \( A_i \) have in either case a transversal, which means we can assign to all \( x_i \) different admissible colors.

In the following we consider only the "critical cases" where such a transversal does not necessarily exist and we indicate which vertex of \( G' \) should be recolored in that case.

**Case 2:** \( f(y_1) = f(y_2) = f(y_4) \). The critical case is \( \bigcup_{i=1}^{5} A_i = 4 \). In that case recolor \( y_4 \)!

**Case 3:** \( f(y_1) = f(y_2) \), but none of the cases above. Then \( |A_1| \geq 4 \) and \( |A_2| \geq 4 \). The critical case is \( A_0 \cup A_2 \cup A_5 \subseteq A_1 = A_2 \). Then \( f(y_3) \neq f(y_5) \not\in A_1 \) and say \( f(y_3) \neq f(y_4) \) (if \( f(y_3) \) and \( f(y_5) \) cannot both equal \( f(y_4) \), otherwise case 1). Then recolor \( y_3 \)!

**Case 4:** \( f(y_2) = f(y_5) \), but none of the cases above. Then \( f(y_3) \neq f(y_6) \), \( |A_1| \geq 4 \) and the critical case is: \( |A_1| = 4 \) and \( A_2 \cup A_3 \cup A_4 \cup A_5 \subseteq A_1 \). Then not both \( f(y_3) \), \( f(y_6) \not\in A_1 \), say \( f(y_3) \not\in A_1 \). Recolor \( y_3 \)!

**Case 5:** All colors \( f(y_i) \) are pairwise different. Then we have two critical cases:

\[ \bigcup_{i=1}^{5} A_i \leq 4 \] or some four of the \( A_i \) consist of the same triple of colors.

If in that case for some \( i \) \( y_i \not\in A_{i+2} \) or \( y_i \not\in A_{i-2} \) (\( i \mod 5 \)), say \( y_i \not\in A_2 \), then recolor \( y_5 \)!

Otherwise necessarily all \( A_i \) consist of the same triple of colors and recoloring any of \( y_i \) reduces also to one of the cases above. //

**References.**


