SFB 823

Nonparametric IV regression with an Archimedean dependence structure

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Nr. 41/2016

Discussion Paper

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July 22, 2016

This paper provides a characterization of the completeness of a family of distributions in terms of the copula between the random variables. We give sufficient conditions for a family of Archimedean copulas to be (boundedly) complete. Some copulas are typically excluded in nonparametric IV regression since they have non-square integrable densities. We provide conditions under which we can identify the nonparametric IV regression model if the dependence structure between the regressors and instrument variables can be described by an Archimedean copula.

Keywords: completeness, copula, identification, nonparametric IV regression model.

JEL Codes: C14

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1 Introduction

Completeness of a family of distributions is often imposed for the identification of econometric models. Examples include the nonparametric IV regression model (Newey & Powell (2003), Darolles et al. (2011)), the nonclassical measurement error model (Hu & Schennach (2008), Carroll et al. (2010), Song (2011)) and the nonparametric IV quantile regression model (Chernozhukov & Hansen (2005)). See Andrews (2011) for additional examples and references.

In this paper we provide a characterization of completeness in terms of the dependence function (copula) between random variables. We provide sufficient conditions for a family of Archimedean copulas to be (boundedly) complete. Certain Archimedean copula families do not fulfill the typical assumptions imposed in nonparametric IV models. We provide conditions such that the nonparametric IV regression model with an Archimedean dependence structure is identified.

Let X and Z be two random variables with marginal distributions F_X and F_Z, respectively, and let $\mathcal{Z}$ be the support of Z. Let $\theta \in \Theta$ be a parameter vector with parameter space Θ and let $\mathscr{H}$ be a predefined function space. We say that the family of functions $\mathscr{P} := \{\phi_{z,\theta}(x); z \in \mathcal{Z}, \theta \in \Theta\}$ is *$\mathscr{H}$-complete* (with respect to X) if for all $h(x) \in \mathscr{H}$

$$\int h(x)\phi_{z,\theta}(x)dx = 0 \text{ a.s. } [F_Z],\ \forall\theta \in \Theta \quad \Rightarrow \quad h(x) = 0 \text{ a.s. } [F_X]. \tag{1}$$

If $\phi_{z,\theta}$ does not depend on a random variable Z or parameter vector θ, we write ϕ_θ and ϕ_z, respectively.

In econometric applications, the elements of $\mathscr{P}$ are density functions. Lehmann (1986) considers $\phi_\theta(x) = f_X(x;\theta)$, where f_X is the density of X. For the identification of the nonparametric IV regression model we have $\phi_z(x) = f_{X|Z}(x|z)$, where $f_{X|Z}$ denotes the conditional density function, see Newey & Powell (2003) and Hu & Shiu (2012). Chernozhukov

& Hansen (2005) consider $\phi_z(x) = w(x,z) f_{X|Z}(x|z)$, with $w(x,z)$ a pre-defined function, for the identification of the nonparametric IV quantile regression model.

In these examples it is typically assumed that $\mathscr{H} = L^p(X) := \{h(x) : \int |h(x)|^p dF_X < \infty\}$, $p \in \{1, 2, \ldots\}$. In that case $\mathscr{P}$ is said to be L^p*-complete* if (1) holds for $p \in \{1, 2, \ldots\}$. In particular, if $p = 1$ then $\mathscr{P}$ is said to be *complete*; If $p = \infty$, then $\mathscr{P}$ is said to be *boundedly complete*. Bounded completeness is weaker than completeness. Some examples of incomplete but boundedly complete families can be found in Mattner (1993).

If $\mathscr{H} = L^p(X)$ and $\mathscr{P} = C_0(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ with compact support, then completeness of $\mathscr{P}$ follows from Duistermaat & Kolk (2010, p.37). Lehmann (1986) shows completeness if $\mathscr{P}$ is an exponential family. Mattner (1992, 1993) considers location families.

The identification of the nonparametric IV regression model requires completeness of $\{f_{X|Z}(x|z;\theta); z \in \mathcal{Z}, \theta \in \Theta\}$. The completeness condition is often accompanied by assuming that the joint distribution of X and Z, F_{XZ}, has a square integrable density. This ensures the existence of the singular value decomposition of certain integral operators, see Darolles et al. (2011, p.1546) and Horowitz (2011, p.355). It also provides a sufficient condition for F_{XZ} to have a density of the form considered in Andrews (2011), see also Lancaster (1958).

The square integrable assumption often requires that the copula density of X and Z is square integrable. Since a copula density that is square integrable cannot have tail dependence (see Beare (2010, Theorem 3.3)), the additional identification assumption excludes such a dependence structure. In section 2 we show that the completeness condition can be fully characterized in terms of the copula function. Furthermore, we show that the family of Archimedean copulas is boundedly complete (under certain regularity conditions). A copula which satisfies our assumptions and allows for tail dependence is for example the Clayton copula. We provide additional conditions under which we can identify the nonparametric IV regression model with an Archimedean dependence structure, see section 3. In section 4 we examine the properties of a popular estimator under such a dependence structure.

2 Characterization of completeness

Let X and Z be two random variables with marginal distributions F_X and F_Z, respectively. Let f_X and f_Z denote the density, f_{XZ} the joint density, $f_{X|Z}$ the conditional density and $c(u, v)$ the copula density function. Then the following holds a.s.$[F_Z]$

$$f_{X|Z}(x|z) = \frac{f_{XZ}(x, z)}{f_Z(z)} = c(F_X(x), F_Z(z))f_X(x).$$

This equation relates the conditional density function to the copula density. For ease of notation we omit the parameter vector θ.

The identification of the nonparametric IV regression model requires completeness of the family $\{f_{X|Z}(x|z; \theta); z \in \mathcal{Z}, \theta \in \Theta\}$. The next theorem characterizes completeness in terms of the family of copula densities of X and Z. The proof can be found in the Appendix.

Theorem 2.1. *Suppose the copula C and joint distribution function F_{XZ} of X and Z are absolutely continuous, then the family $\mathscr{F} := \{f_{X|Z}(\cdot|z; \theta); z \in \mathcal{Z}\}$ is complete if and only if the family $\mathscr{C} := \{c(u, v; \theta); v \in [0, 1], \theta \in \Theta\}$ is complete with respect to $U = F_X(X)$.*

We consider the class of strict Archimedean copulas. Let $\phi(u) : [0, 1] \to [0, \infty]$ be a continuous, strictly decreasing, convex function such that $\phi(0) = \infty$ and $\phi(1) = 0$. The copula $C(u, v)$ is called a strict Archimedean copula if

$$C(u, v) = \phi^{-1}(\phi(u) + \phi(v)).$$

The function ϕ is the called the generator of $C(u, v)$, see Nelson (2006, p.110-112).

We proof bounded completeness of a family of Archimedean copula densities under the following assumption:

Assumption 1.

A.1 $\phi(t)$, $t \in (0,1)$, *is twice continuously differentiable.*

A.2 $\phi''(t)$, $t \in (0,1)$, *is strictly positive.*

A.3 $\lim_{t\downarrow 0} \phi(t)/\phi'(t) = 0$.

Remark 2.2.

(i) Genest & MacKai (1986, Theorem 1) show that the copula $C(u,v)$ with twice differentiable generator function is absolutely continuous if and only if Assumption A3 is satisfied. Under Assumption 1, the copula $C(u,v)$ has density function

$$c(u,v) = -\frac{\phi''(C(u,v))\phi'(u)\phi'(v)}{[\phi'(C(u,v))]^3}.$$

(ii) Beare (2012) imposes similar conditions on the generator function to analyze the geometric ergodicity in Markov chains whose dependence is characterized by an Archimedean copula.

Under Assumption 1 we have, using a change of variables $\tilde{u} := -\phi(u)$ and $\tilde{v} := \phi(v)$,

$$\begin{aligned}
\int_0^1 h(u)c(u,v)du &= -\int_0^1 h(u)\frac{\phi''(C(u,v))\phi'(u)\phi'(v)}{[\phi'(C(u,v))]^3}du \\
&= \phi'(v)\int_{-\infty}^0 h(\phi^{-1}(-\tilde{u}))\frac{\phi''[\phi^{-1}(\tilde{v}-\tilde{u})]}{[\phi'(\phi^{-1}(\tilde{v}-\tilde{u}))]^3}du \\
&= \phi'(v)\{(g\times\psi)\}, \qquad (2)
\end{aligned}$$

where

$$g(t) := \begin{cases} -h(\phi^{-1}(-t)) & t \le 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \psi(t) := -\phi''[\phi^{-1}(|t|)]/[\phi'(\phi^{-1}(|t|))]^3.$$

Under Assumption 1 we have that $\phi'(s) < 0$ and $\phi''(s) > 0$ for all $s \in [0, 1]$. Hence, $\psi(t) > 0$ for all $t \in \mathbb{R}$. Note that we also defined $\psi(t) > 0$ for $t < 0$. This does not affect the value of the integral (2) but is useful to prove completeness. Figure 1 illustrates $\psi(t)$ for the generator functions that occur in Example 2.5 below.

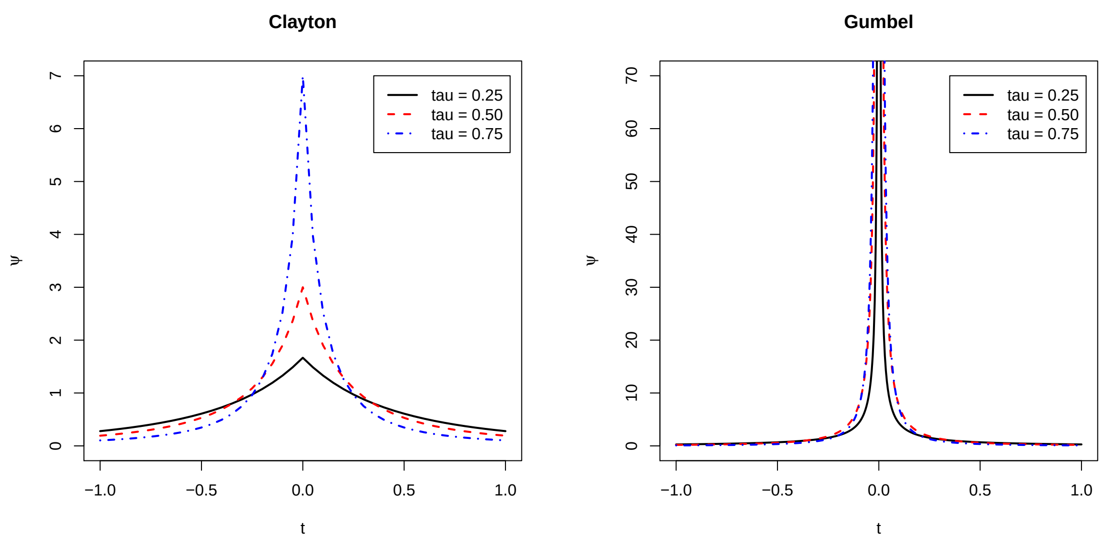

Figure 1: The function $\psi(t)$ for the Clayton and Gumbel copula. The copula parameter θ has been selected such that Kendall's τ equals 0.25, 0.5 and 0.75, respectively.

Since $\phi'(v) \in (-\infty, 0)$ for $v \in (0, 1]$, we have for $v \in (0, 1]$ that

$$\int h(u)c(u, v)du = 0 \Rightarrow h(u) = 0$$

is equivalent to

$$(g \times \psi) = 0 \Rightarrow g(\tilde{u}) = 0. \quad (3)$$

Hence, a necessary and sufficient condition for (bounded) completeness of the family $\mathscr{C}_A := \{c(u, v) = -\phi''(C(u, v))\phi'(u)\phi'(v)/[\phi'(C(u, v))]^3\}$ is (bounded) completeness of the location family $\mathscr{L}_A := \{\psi(\tilde{v} - t), t \in \mathbb{R}; \tilde{v} \in (0, \infty)\}$.

A necessary and sufficient condition for the bounded completeness of location families, is that the characteristic function of ψ does not contain any zeros. In the remainder of the paper, we assume therefore that $\int_0^\infty |\psi(t)|dt < \infty$.

We proof the following result:

Theorem 2.3. *Let $\mathscr{C}_A$ be a family of strict Archimedean copula densities satisfying Assumption 1 for all $\theta \in \Theta$. Then the family $\mathscr{C}_A$ is boundedly complete with respect to U if and only if*

$$\left|\frac{1}{\phi'(1)}\right| > \xi^2 \hat{k}(\xi), \quad \textit{for all } \xi \in \mathbb{R}$$

where $\hat{k}(\xi) = \int_0^\infty \phi^{-1}(t)\cos(\xi t)dt$.

Remark 2.4. Note that $\frac{1}{\phi'(1)} = \frac{\partial}{\partial t}\phi^{-1}(t)\big|_{t=0}$. Therefore, the bounded completeness of a family of Archimedean copulas depends solely on the behavior of the inverse generator function $\phi^{-1}(t)$.

We verify Assumption 1 for several copula families.

Example 2.5.

1. Consider the Clayton family with generator function

$$\phi(u) = \frac{1}{\theta}(u^{-\theta} - 1), \qquad \theta \in [-1, \infty)\backslash\{0\}.$$

   For $\theta \in [-1, 0)$ the generator function ϕ is not strict. For $\theta \in (0, \infty)$ Assumption 1 holds. We verified the condition in Theorem 2.3 numerically for fixed $\theta \in (0, \infty)$. The lower tail dependence parameter of the Clayton copula is given by $\lambda_L = 2^{1/\theta}$ (see Nelson (2006, p.215)). Hence, the family $\mathscr{C}_1 = \{c(u, v; \theta) : \ \phi_\theta(u) = \frac{1}{\theta}(u^{-\theta} - 1), \ \theta \in (0, \infty)\}$, is boundedly complete and allows for copulas with lower tail dependence.

2. Consider the Gumbel family with generator function

$$\phi(u) = (-\ln t)^\theta, \qquad \theta \in [1, \infty).$$

Assumption 1 is satisfied for all $\theta \in [1, \infty)$. However, $\int_0^\infty |\psi(t)| dt$ is not finite. Therefore, we cannot use Theorem 2.3.

The identification of the nonparametric IV regression model requires that the family $\mathscr{C}_A$ is complete. We proof completeness under the following additional conditions:

Assumption 2.

B.1 There exist finite M and r such that for all t the following holds $[F_v]$-almost surely

$$\psi(t+w) \leq M(1+|t|^2)^r \psi(w), \qquad w \in \mathbb{R}. \tag{4}$$

B.2 Let A be some finite set. Then $\hat{\psi}(t)$ is infinity often differentiable in $\mathbb{R} \backslash A$.

Remark 2.6.

(i) Mattner (1992) uses these conditions to proof completeness of location families. d'Haultfoeuille (2011) uses these conditions for the identification of nonparametric instrument variable models (see Remark 3.1 below).

(ii) A sufficient condition for Assumption B.1 is the existence of some real m, M and $r > 0$ such that

$$0 < m \leq \psi(t)(1+|t|)^r \leq M < \infty, \tag{5}$$

see Mattner (1992, Proposition 1.2).

From Mattner (1992, Theorem 1.1) we obtain

Theorem 2.7. *Under Assumption 1 and 2, the family of strict Archimedean copula densities $\mathscr{C}_A$ is complete.*

We verify Assumption 2 for the copulas of Example 2.5.

Example 2.8.

1. Consider the family of Clayton copulas with $\phi(u) = \frac{1}{\theta}(u^{-\theta} - 1)$, $\theta \in (0, \infty)$. Then, $-\phi''[\phi^{-1}(t)]/(\phi'[\phi^{-1}(t)])^3 = (\theta + 1)(t\theta + 1)^{-(2\theta+1)/\theta}$. Hence, for $\theta \geq 1$ choose $r \leq (2\theta + 1)/\theta$. Then $\psi(t)(1 + |t|)^r$ is bounded. Hence, condition (5) is satisfied and from Theorem 2.7 we have that the family of Clayton copulas with $\theta \in [1, \infty)$ is complete provided Assumption B.2. is satisfied.

   To verify Assumption B.2, we have to show that $\hat{\psi}(t)$ is an analytic function on $\mathbb{R}\backslash A$. The Paley-Wiener theorems provide conditions on $\psi(t)$ such that $\hat{\psi}$ is analytic, see e.g. Strichartz (2003, p.119-125).

   Define $q(t) := -\phi''[\phi^{-1}(t)]/(\phi'[\phi^{-1}(t)])^3$, $t \in [0, \infty)$. Then

$$\begin{aligned} \int_0^\infty |q(t)|^2 \, dt &= \int_0^\infty |(\theta + 1)(t\theta + 1)^{-(2\theta+1)/\theta}|^2 dt \\ &= \left[-\frac{(\theta + 1)^2}{3\theta + 2} (t\theta + 1)^{-\frac{3\theta+2}{\theta}} \right]_0^\infty < \infty \end{aligned}$$

   for all $\theta \in [1, \infty)$. Hence, from Strichartz (2003, Theorem 7.2.4) we have that $\hat{q}(\xi) := \int_0^\infty F(t) \exp(\xi t) dt$ is analytic. Therefore, $\hat{\psi}(\xi) = \hat{q}(\xi) + \hat{q}(-\xi)$ is analytic as well.

2. We already showed that the family of Gumbel copulas is not boundedly complete and, hence, not complete.

3 Identification

In this section we consider the identification of the function $h(\cdot)$ in the nonparametric instrumental variable regression model

$$Y = h(X) + \varepsilon, \qquad E(\varepsilon|Z) = 0 \tag{6}$$

where Y is the dependent variable, X an endogenous regressor, ε an disturbance and Z an instrument variable.

Newey & Powell (2003) show that the function $h(\cdot)$ is identified if and only if the family of densities $\{f_{X|Z}(\cdot)\}$ is complete. Darolles et al. (2011, p.1546), Horowitz (2011, p.355), among others, also assume that the joint density f_{XZ} of X and Z is square integrable. If X and Z are uniformly distributed (e.g. after a transformation), then it follows from

$$f_{XZ}(x,z) = c(F(x), F(z))f_X(x)f_Z(z)$$

that the copula density is square integrable. Since a copula density that is square integrable cannot have tail dependence (see Beare (2010, Theorem 3.3)), this assumption excludes the dependence structures of Example 2.5.

We make the following assumption:

Assumption 3. *The copula of X and Z is an Archimedean copula*

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v))$$

where ϕ satisfies Assumption 1 and 2.

Remark 3.1.

(i) Bücher et al. (2012) developed statistical tests for the hypothese that the copula of X and Z is an Archimedean copula.

(ii) d'Haultfoeuille (2011) also analyses the identification of (6). Instead of Assumption 3 he assumes that the regressor and instrument variable are related through

$$X = \mu(\nu(Z) + \zeta)$$

where μ and ν are functions and ζ a disturbance.

The following result follows than immediately from Theorem 2.7 and Newey & Powell (2003, Proposition 2.1).

Corollary 3.2. *The model* (6) *is identified under Assumption 3.*

4 Estimation

We consider the the estimation of $h(\cdot)$ in the non-parametric IV regression model (6). To illustrate the effect of a non-square integrable density, we simulate the following data generating process (DGP)

$$\begin{aligned} Y &= h(X) + \varepsilon \\ X &= 0.1\Phi^{-1}(V) + \zeta \\ Z &= \Phi^{-1}(U) \end{aligned} \tag{7}$$

where $(U\ V) \sim C(u, v)$ and $\varepsilon = -0.5\zeta + \nu$, $\nu \sim N(0, (0.05)^2)$ and $\zeta \sim N(0, (0.27)^2)$. The function $h(X)$ is equal to X^2. Note that the DGP is closely related to the one considered in Darolles et al. (2011).

The function $C(u, v)$ is the Gaussian copula with correlation parameter ρ or an Archimedean copula with parameter θ. We choose the parameter θ such that the linear correlation is equal to ρ.

We estimate the function $h(\cdot)$ using the estimation method proposed in Horowitz (2011). Note that the data generating process does not satisfy the assumptions made in Horowitz (2011) if the copula density is not square integrable. We used Legendre-type polynomials as basis functions. All estimates are based on 3 basis functions.

We perform 10000 replications of n = 1000 observations of the DGP (7) and calculate for each run the sum of squared errors (SSE)

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{h}(x_i))^2.$$

where $\hat{h}$ denotes the nonparametric estimate of $h(\cdot)$.

Table 1 shows some summary statistics of the SSE. The average SSE is relatively large due to a couple of outliers. For the 0.9-quantile we have that the Clayton copula is somewhat higher and the Gumbel copula is somewhat lower compared to the Gaussian copula.

	$n = 1000,\ \rho = 0.5$			$n = 1000,\ \rho = 0.9$		
copula	Gaussian	Clayton	Gumbel	Gaussian	Clayton	Gumbel
parameter	0.50	1.00	1.50	0.90	4.97	3.48
upper tail	0.00	0.00	0.41	0.00	0.00	0.78
lower tail	0.00	0.50	0.00	0.00	0.87	0.00
q(0.10)	17.144	17.247	16.645	17.481	17.688	17.438
q(0.25)	22.153	22.776	21.446	19.756	20.197	19.738
q(0.50)	34.602	36.818	32.805	23.041	24.548	23.191
q(0.75)	92.282	102.583	76.733	28.300	33.519	28.675
q(0.90)	520.389	584.254	402.426	38.250	61.728	39.535

Table 1: Simulated sum of squared errors of data generating process (7) with $\rho = 0.5$ and 0.9. Function $h(\cdot)$ nonparametrically estimated using the estimation procedure of Horowitz (2011). 10000 replications.

5 conclusion

In this paper we provide a characterization of completeness in terms of the dependence function (copula) between random variables. Simulations shown that the quantile of some Archimedean copulas are higher compared to the Gausian copula.

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A Proofs

Proof of Theorem 2.1: For all $\theta \in \Theta$ the following holds a.s. $[F_Z]$

$$\begin{aligned}\int_{-\infty}^{\infty} h(x) f_{X|Z}(x|z;\theta) dx &= \int_{-\infty}^{\infty} h(x) \frac{f_{X,Z}(x,z;\theta)}{f_Z(z)} dx \\ &= \int_{-\infty}^{\infty} h(x) f_X(x) c(F_X(x), F_Z(z);\theta) dx.\end{aligned}$$

"$\Rightarrow$" Suppose $\mathscr{F}$ is complete. Then

$$\int h(x) f_X(x) c(F_X(x), F_Z(z);\theta) dx = 0 \text{ a.s } [F_Z],\ \forall \theta \in \Theta \quad \Rightarrow \quad h(x) = 0 \text{ a.s } [F_X]. \tag{8}$$

We have for all $k(u) \in L^p(U)$

$$\int_0^1 k(u) c(u,v;\theta) du = 0 \text{ a.s.}[F_V] \iff \int k(F_X(x)) f_X(x) c(F_X(x), F_Z(z);\theta) dx = 0 \text{ a.s } [F_Z]$$

From (8) we have $k(F_X(x)) = 0$ a.s.$[F_X]$ or $k(u) = 0$ a.s.$[F_U]$.

"$\Leftarrow$" Suppose the family $\mathscr{C}$ is complete. For any $z \in \mathcal{Z}$ we have

$$\begin{aligned}\int h(x) f_{X|Z}(x|z;\theta) dx &= \int h(x) f_X(x) c(F_X(x), F_Z(z);\theta) dx \\ &= \int h(F_X^{-1}(u)) f_X(F_X^{-1}(u)) c(u,v;\theta) \frac{1}{f_X(F_X^{-1}(u))} du\end{aligned}$$

where we used that the transformation $u = F_X(x)$ is piecewise regular. Since $\mathscr{C}$ is complete, we have $h(F_X^{-1}(u)) = 0$ a.s. and hence $h(x) = 0$ a.s.$[F_X]$. □

Proof of Theorem 2.3

From (3) we have that a necessary and sufficient condition for the bounded completeness of $\mathscr{C}_A$ is the bounded completeness of the location family $\mathscr{L}_A$. From an adaptation of the proof of **?**, Theorem 2.4 (see d'Haultfoeuille (2011)) we have that a necessary and sufficient condition for bounded completeness is that the Fourier transform of $\hat{\psi}(\xi)$ of $\psi(\xi)$ does not contain any zeros.

We have

$$\begin{aligned}
\hat{\psi}(\xi) &= \int_{-\infty}^{\infty} \psi(t) \exp(-i\xi t) dt \\
&= -\int_{-\infty}^{\infty} \frac{\phi''(\phi^{-1}(|t|))}{[\phi'(\phi^{-1}(|t|)]^3} \exp(-i\xi t) dt \\
&= -\{\hat{q}(\xi) + \hat{q}(-\xi)\}
\end{aligned}$$

where

$$\hat{q}(\xi) := \int_0^{\infty} \frac{\phi''[\phi^{-1}(t)]}{[\phi'(\phi^{-1}(t)]^3} \exp(-i\xi t) dt.$$

Using a change of variables $t = \phi(u)$ we obtain

$$\begin{aligned}
\hat{q}(\xi) &= \int_0^{\infty} \frac{\phi''(\phi^{-1}(t))}{[\phi'(\phi^{-1}(t))]^3} \exp(-i\xi t) dt \\
&= -\int_0^1 \frac{\phi''(u)}{[\phi'(u)]^2} \exp(-i\xi\phi(u)) du \\
&= -\int_0^1 \frac{\partial}{\partial u}\left(-\frac{1}{\phi'(u)}\right) \exp(-i\xi\phi(u)) du \\
&= -\left\{\left[-\frac{\exp(-i\xi\phi(u))}{\phi'(u)}\right]_0^1 - \int_0^1 \left[-\frac{1}{\phi'(u)}(-i\xi\phi'(u)) \exp(-i\xi\phi(u))\right] du\right\} \\
&= -\left\{\left[-\frac{\exp(-i\xi\phi(u))}{\phi'(u)}\right]_0^1 - i\xi \int_0^1 \exp(-i\xi\phi(u)) du\right\}
\end{aligned}$$

Note that $\phi(1) = 0$, $\exp(i\xi\phi(u))$ is bounded and $\lim_{t\downarrow 0} \phi'(t) = -\infty$ implies that

$$\left[-\frac{\exp(-i\xi\phi(u))}{\phi'(u)} \right]_0^1 = -\frac{1}{\phi'(1)}.$$

Furthermore,

$$\begin{aligned}
\int_0^1 \exp(-i\xi\phi(u))du &= [u\exp(-i\xi\phi(u))]_0^1 - \int_0^1 u(-i\xi\phi'(u))\exp(-i\xi\phi(u))du \\
&= 1 + i\xi \int_0^1 u\phi'(u)\exp(-i\xi\phi(u))du \\
&= 1 - i\xi \int_0^\infty \phi^{-1}(t)\phi'(\phi^{-1}(t))\exp(-i\xi t)\frac{1}{\phi'(\phi^{-1}(t))}dt \\
&= 1 - i\xi \int_0^\infty \phi^{-1}(t)\exp(-i\xi t)dt.
\end{aligned}$$

Also

$$\hat{q}(\xi) = \frac{1}{\phi'(1)} + i\xi - (i\xi)^2 \int_0^\infty \phi^{-1}(t)\exp(-i\xi t)dt.$$

Finally

$$\begin{aligned}
\hat{\psi}(\xi) &= -\{\hat{q}(\xi) + \hat{q}(-\xi)\} \\
&= -\left\{ \frac{2}{\phi'(1)} + \xi^2 \int_{-\infty}^\infty \phi^{-1}(|t|)\exp(-i\xi t)dt \right\} \\
&= -\left\{ \frac{2}{\phi'(1)} + 2\xi^2\hat{k}(\xi) \right\}.
\end{aligned}$$

where

$$\hat{k}(\xi) := \int_0^\infty \phi^{-1}(t)\cos(\xi t)dt.$$

Note that $\phi'(1) < 0$. Since $\phi(\cdot)$ is convex and strictly decreasing we have that ϕ^{-1} is convex. The convexity of ϕ^{-1} implies that its Fourier-cosine transform $\hat{k}(\xi)$ is positive (see Tuck (2006)).

For $\xi = 0$ we have then that $\hat{\psi}(\xi) = -\frac{2}{\phi'(1)} > 0$. Therefore, the characteristic function $\hat{\psi}(\xi)$ does not contain any zeros if and only if

$$\left|\frac{1}{\phi'(1)}\right| > \xi^2 \hat{k}(\xi)$$

for all $\xi \in \mathbb{R}$. □