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LIMIT THEOREMS FOR MULTIVARIATE BESSEL PROCESSES IN THE FREEZING REGIME

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ABSTRACT. Multivariate Bessel processes describe the stochastic dynamics of interacting particle systems of Calogero-Moser-Sutherland type and are related with β-Hermite and Laguerre ensembles. It was shown by Andraus, Katori, and Miyashita that for fixed starting points, these processes admit interesting limit laws when the multiplicities k tend to ∞, where in some cases the limits are described by the zeros of classical Hermite and Laguerre polynomials. In this paper we use SDEs to derive corresponding limit laws for starting points of the form $\sqrt{k}\cdot x$ for $k\to\infty$ with x in the interior of the corresponding Weyl chambers. Our limit results are a.s. locally uniform in time. Moreover, in some cases we present associated central limit theorems.

1. INTRODUCTION

The dynamics of integrable interacting particle systems of Calogero-Moser-Sutherland type on the real line $\mathbb{R}$ with N particles can be described by certain time-homogeneous diffusion processes on suitable closed subsets of $\mathbb{R}^N$. These processes are often called (multivariate) Bessel, Dunkl-Bessel, or radial Dunkl processes; for their detailed definition and properties, see [CGY, GY, R1, R2, RV1, RV2, DV, A]. These processes are classified via root systems and finitely many multiplicity parameters which act as coupling constants of interaction. In this paper, we restrict our attention to the root systems, which are mainly important for particle systems and in random matrix theory, namely those of the types A_{N-1}, B_N, and D_N, as here the number N of particles is arbitrary. Besides these root systems and a finite number of exceptional cases, there are the dihedral sytems with $N=2$ (see [De2]) as well direct products. We shall not study these cases in this paper.

To explain the results of this paper, we briefly recapitulate some well-known basic facts. In the case A_{N-1}, we have a one-dimensional multiplicity $k>0$, the processes live on the closed Weyl chamber

$$C_N^A := \{x\in\mathbb{R}^N : \quad x_1\ge x_2\ge\ldots\ge x_N\},$$

the generator of the transition semigroup is given by

$$Lf := \frac{1}{2}\Delta f + k\sum_{i=1}^N\Big(\sum_{j\neq i}\frac{1}{x_i-x_j}\Big)\frac{\partial}{\partial x_i}f, \tag{1.1}$$

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and we assume reflecting boundaries. The latter means that the domain of L may be chosen as

$$D(L) := \{f|_{C_N^A} : \ f \in C^{(2)}(\mathbb{R}^N), \ \ f \ \text{ invariant under all coordinate permutations}\}.$$

In the case B_N, we have 2 multiplicities $k_1, k_2 \geq 0$, the processes live on

$$C_N^B := \{x \in \mathbb{R}^N : \quad x_1 \geq x_2 \geq \ldots \geq x_N \geq 0\},$$

the generator of the transition semigroup is

$$Lf := \frac{1}{2}\Delta f + k_2 \sum_{i=1}^N \sum_{j\neq i} \Big(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \Big) \frac{\partial}{\partial x_i} f \quad + k_1 \sum_{i=1}^N \frac{1}{x_i} \frac{\partial}{\partial x_i} f, \tag{1.2}$$

and we again assume reflecting boundaries, i.e., the domain of L is

$$D(L) := \{f|_{C_N^B} : f \in C^{(2)}(\mathbb{R}^N), \ \ f \ \text{ invariant under all permutations of coordinates and under all sign changes in all coordinates}\}.$$

We study limit theorems for these diffusions $(X_{t,k})_{t\geq 0}$ on C_N (with $C_N = C_N^A$ or C_N^B) for the fixed times $t > 0$ in freezing regimes, where k stands for $k \geq 0$ in the A_{N-1}-case, and for (k_1, k_2) in the B_N-case. Freezing means that for fixed times $t > 0$, we consider $k \to \infty$ in the A_{N-1}-case, and in the B_N-case, the two cases $(k_1, k_2) = (\nu \cdot \beta, \beta)$ with $\nu > 0$ fixed, $\beta \to \infty$ as well as $k_2 > 0$ fixed, $k_1 \to \infty$. For these limit cases, [AM, AKM1, AKM2] present weak limit laws for $X_{t,k}$ for fixed times $t > 0$ when the processes start in the origin $0 \in C_N$ or with a fixed starting distribution independent from k. In this paper we shall derive similar limit results when the starting points of the diffusions $(X_{t,k})_{t\geq 0}$ depend on k, more precisely, if the starting points have the form $\sqrt{\kappa} \cdot x$ where κ is the parameter in the coupling constants which tends to ∞, and where x is a point in the interior of C_N. The last condition will be essential in this paper, as we shall apply an SDE approach to the limit results which works properly only in the interior of C_N as the SDEs become singular on the boundaries. It will turn out on an informal level that the limit results in [AM, AKM1, AKM2] may be seen as special cases of our results for $x = 0$, even if the case $x = 0$ is not covered by our approach.

To explain the connection of our results with [AM, AKM1, AKM2], we recapitulate some further details. The transition probabilities of the Bessel processes are given for all root systems as follows by [R1, R2, RV1, RV2]: For $t > 0$, $x \in C_N$, $A \subset C_N$ a Borel set,

$$K_t(x, A) = c_k \int_A \frac{1}{t^{\gamma+N/2}} e^{-(\|x\|^2+\|y\|^2)/(2t)} J_k(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}) \cdot w_k(y)\, dy \tag{1.3}$$

with

$$w_k^A(x) := \prod_{i<j} (x_i - x_j)^{2k}, \qquad w_k^B(x) := \prod_{i<j} (x_i^2 - x_j^2)^{2k_2} \cdot \prod_{i=1}^N x_i^{2k_1}, \tag{1.4}$$

and

$$\gamma_A(k) = kN(N-1)/2, \qquad \gamma_B(k_1, k_2) = k_2 N(N-1) + k_1 N \tag{1.5}$$

respectively. w_k is homogeneous of degree 2γ. Furthermore, $c_k > 0$ is a known normalization constant, and J_k is a multivariate Bessel function of type A_{N-1} or B_N with multiplicities k or (k_1, k_2) respectively; see e.g. [R1, R2].

We do not need much information about J_k. We only recapitulate that J_k is analytic on $\mathbb{C}^N \times \mathbb{C}^N$ with $J_k(x,y) > 0$ for $x, y \in \mathbb{R}^N$. Moreover, $J_k(x,y) = J_k(y,x)$ and $J_k(0,y) = 1$ for all $x, y \in \mathbb{C}^N$.

Therefore, if we start the process from 0, then $X_{t,k}$ has the Lebesgue density

$$\frac{c_k}{t^{\gamma+N/2}} e^{-\|y\|^2/(2t)} \cdot w_k(y)\, dy \tag{1.6}$$

on C_N for $t > 0$. In the case A_{N-1}, the density of $X_{t,k}/\sqrt{tk}$ has the form

$$\text{const.}(k) \cdot \exp\Big(k\Big(2 \sum_{i,j:i<j} \ln(y_i - y_j) - \|y\|^2/2\Big)\Big) =: \text{const.}(k) \cdot \exp\Big(k \cdot W(y)\Big)$$

which is well-known for $k = 1/2, 1, 2$ as the distribution of the eigenvalues of Gaussian orthogonal, unitary, and symplectic ensembles; see e.g. [D]. Moreover, for general $k > 0$, (1.6) appears as the distribution of the tridiagonal matrix models of Dumitriu and Edelman [DE1, DE2] for β-Hermite and β-Laguerre ensembles.

In the case A_{N-1}, it was observed in [AKM1] (see also Section 6.7 of [S]) that the maximum of W on C_N^A appears precisely for $y = \sqrt{2} \cdot z$ where $z \in C_N^A$ is the vector whose entries are the zeroes of the classical Hermite polynomial H_N where the $(H_N)_{N\geq 0}$ are orthogonal w.r.t. the density e^{-x^2}. This shows that $\frac{X_{t,k}}{\sqrt{2tk}}$ tends to z in distribution for $k \to \infty$. This means that

$$\lim_{k\to\infty} \frac{X_{t,k}}{\sqrt{2tk}} = z \tag{1.7}$$

in probability whenever the $X_{t,k}$ are defined on a common probability space. In fact, this result was proved in [AKM1] in a more general form, namely for arbitrary fixed starting distributions. Moreover, (1.7) and an associated central limit theorem for start in 0 was derived in [DE2] via the explicit tridiagonal matrix model of Dumitriu and Edelman [DE1]; see also [V2] for another elementary approach.

We now compare (1.7) with the main results here for the case A_{N-1}. We show in Theorem 2.4 below that the Bessel processes $(X_{t,k})_{t\geq 0}$ with start in $\sqrt{k} \cdot x$ (for some point x in the interior of C_N) satisfy

$$X_{t,k}/\sqrt{k} \to \phi(t,x) \qquad \text{for} \qquad k \to \infty \tag{1.8}$$

with an error of size $O(1/\sqrt{k})$ locally uniformly in t almost surely where $\phi(t,x)$ is the solution of some (deterministic) dynamical system at time $t > 0$, where the system starts at time 0 in x. For the details we refer to Section 2.

For the root systems B_N and D_N we shall derive corresponding results.

We mention that the locally uniform convergence in t in (1.8) and the corresponding results for the further root systems ensures that we can interchange limits for $k \to \infty$ with stochastic integrals. This finally leads, in combination with the SDEs, to central limit theorems (CLTs) at least in some cases; see Section 4 below for some cases and also to [VW] for further cases.

The basis for the SDE-approach for all root systems is the following well-known result (see Lemma 3.4, Corollary 6.6, and Proposition 6.8 of [CGY]):

Theorem 1.1. *Let $k > 0$ in the A_{N-1}-case or $k_1, k_2 > 0$ in the B_N-case. Then, for each starting point $x \in C_N$ and $t > 0$, the Bessel process $(X_{t,k})_{t\geq 0}$ satisfies*

$$E\Big(\int_0^t \nabla(\ln w_k)(X_{s,k})\, ds\Big) < \infty.$$

Moreover, the initial value problem

$$X_0 = x, \qquad dX_t = dB_t + \frac{1}{2}(\nabla(\ln w_k))(X_t)\,dt \tag{1.9}$$

(with an N-dimensional Brownian motion $(B_t)_{t\geq 0}$) has a unique (strong) solution $(X_t)_{t\geq 0}$. This solution is a Bessel process as above.

Moreover, if $k \geq 1/2$ in the A_{N-1}-case or $k_1, k_2 \geq 1/2$ in the B_N-case, and if x is in the interior of C_N, then $(X_t)_{t\geq 0}$ lives on the interior on C_N, i.e. the solution does not meet the boundary almost surely.

This paper is organized as follows. In the next section we derive a strong Limit law for Bessel processes of type A_{N-1} for $k \to \infty$. Section 3 is then devoted to corresponding LLs in the case B_N for two freezing regimes which were already studied in [AKM2, AM]. For the regime $k_1 \to \infty$ and $k_2 > 0$ fixed, we use the locally uniform LL in Section 4, in order to derive an associated CLT. Finally, in Section 5 we consider the LL for the root systems of type D_N, which are related with the B_N-case for $k_2 = 0$.

2. Strong limiting law for the root system A_{N-1}

In this section we derive a locally uniform strong LL in the case A_{N-1} for $k \to \infty$ for starting points of the form $\sqrt{k} \cdot x$ with x in the interior of C_N^A. The main result corresponds to the weak LL (1.7) started at the origin.

For $k \geq 1/2$ and any starting point $x = x(k)$ in the interior of C_N^A we study the associated Bessel process $(X_{t,k})_{t\geq 0}$ of type A_{N-1} which may be seen as the unique solution of the initial value problem (1.9). In the A_{N-1} case the SDE reads

$$dX_{t,k}^i = dB_t^i + k \sum_{j \neq i} \frac{1}{X_{t,k}^i - X_{t,k}^j} dt \qquad (i = 1, \ldots, N). \tag{2.1}$$

with an N-dimensional Brownian motion $(B_t^1, \ldots, B_t^N)_{t\geq 0}$. In order to derive LLs for $X_{t,k}$, we study the renormalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ which satisfy

$$d\tilde{X}_{t,k}^i = \frac{1}{\sqrt{k}} dB_t^i + \sum_{j \neq i} \frac{1}{\tilde{X}_{t,k}^i - \tilde{X}_{t,k}^j} dt \qquad (i = 1, \ldots, N). \tag{2.2}$$

We compare (2.2) with the deterministic limit case $k = \infty$. This limit case has the following properties:

Lemma 2.1. *For $\epsilon > 0$ consider the open subset $U_\epsilon := \{x \in C_N^A : d(x, \partial C_N^A) > \epsilon\}$ (where $\mathbb{R}^N$ carries the usual Euclidean norm and $d(x,y)$ denotes the distance). Then the function*

$$H : U_\epsilon \to \mathbb{R}^N, \quad x \mapsto \Big(\sum_{j\neq 1} \frac{1}{x_1 - x_j}, \ldots, \sum_{j \neq N} \frac{1}{x_N - x_j} \Big)$$

is Lipschitz continuous on U_ϵ with Lipschitz constant $L_\epsilon > 0$, and for each starting point $x_0 \in U_\epsilon$, the solution $\phi(t, x_0)$ of the dynamical system $\frac{dx}{dt}(t) = H(x(t))$ satisfies $\phi(t, x_0) \in U_\epsilon$ for all $t \geq 0$.

Proof. For $x \in U_\epsilon$ and $i \neq j$ we have $|x_i - x_j| > \epsilon$. Hence there is a constant $C > 0$ with $|\frac{\partial H}{\partial x_i}(x)| \leq C$ for $x \in U_\epsilon$ and $i = 1, \ldots, N$ which implies the Lipschitz continuity.

For the second statement we use the new variables $y_i(t) := x_i(t) - x_{i+1}(t) > \epsilon$ for $x \in U_\epsilon$ and $i = 1, \ldots, N-1$. Then

$$\begin{aligned}\frac{dy_i}{dt}(t) =& \frac{1}{y_1 + \cdots + y_i} + \frac{1}{y_i + \cdots + y_{N-1}} \\ &+ \sum_{j=1}^{i-1}\Big(\frac{1}{y_{j+1} + \cdots + y_i} - \frac{1}{y_j + \cdots + y_{i-1}}\Big) \\ &+ \sum_{j=i+2}^{N}\Big(\frac{1}{y_i + \cdots + y_{j-2}} - \frac{1}{y_{i+1} + \cdots + y_{j-1}}\Big).\end{aligned}$$

For any $t \geq 0$ choose some $i = i(t)$ for which $y_i(t)$ is minimal, i.e., $y_j(t) \geq y_i(t)$ for all j. Notice that $i = i(t)$ is not necessarily unique. However, for each $i = i(t)$ of this kind we have

$$\frac{1}{y_{i+1}(t)} \leq \frac{1}{y_i(t)} \quad \text{and} \quad \frac{1}{y_i(t) + \cdots + y_{j-2}(t)} \geq \frac{1}{y_{i+1}(t) + \cdots + y_{j-1}(t)} \quad (j = i+3, \ldots, N)$$

and

$$\frac{1}{y_{i-1}(t)} \leq \frac{1}{y_i(t)} \quad \text{and} \quad \frac{1}{y_{j+1}(t) + \cdots + y_i(t)} \geq \frac{1}{y_j(t) + \cdots + y_{i-1}(t)} \quad (j = 1, \ldots, i-2).$$

Therefore,

$$\frac{dy_i}{dt}(t) \geq \frac{1}{y_1(t) + \cdots + y_i(t)} + \frac{1}{y_i(t) + \cdots + y_{N-1}(t)} > 0.$$

Hence, for each $t \geq 0$ there is a neighborhood on which y_i is increasing for each i, for wor which $y_i(t)$ is minimal. This means that that $s \mapsto \min_{i=1,\ldots,N-1} y_i(s)$ is increasing in this neighborhood of t. This completes the proof of the lemma. □

It seems that the dynamical system from Lemma 2.1 can be solved explicitly only for a few cases like $N = 2$ or particular starting points which are related to the zeros of the Hermite polynomial H_N. The latter is not surprising in view of the LLs of [AKM1]. To explain these solutions, we recall the following fact (see [AKM1] and Section 6.7 of [S]):

Lemma 2.2. *For $y \in C_N^A$, the following statements are equivalent:*

(1) *The function $W(x) := 2\sum_{i,j:i<j} \ln(x_i - x_j) - \|x\|^2/2$ is maximal at $y \in C_N^A$;*
(2) *For $i = 1, \ldots, N$: $\frac{1}{2}y_i = \sum_{j:j\neq i} \frac{1}{y_i - y_j}$;*
(3) *The vector*

$$z := (z_1, \ldots, z_N) := (y_1/\sqrt{2}, \ldots, y_N/\sqrt{2})$$

consists of the ordered zeroes of the classical Hermite polynomial H_N.

Part (3) of this lemma immediately leads to the following solution of the differential equation of Lemma 2.1:

Corollary 2.3. *For each $c > 0$, a particular solution of the dynamical system in Lemma 2.1 is given by $\phi(t, c \cdot z) = \sqrt{2t + c^2} \cdot z$.*

Notice that on an informal level the same statement holds also for $c = 0$.

We now turn to the main result of this section, a locally uniform strong LL with a quite strong order of convergence:

Theorem 2.4. *Let x be a point in the interior of C_N^A, and let $y \in \mathbb{R}^N$. Let $k_0 \geq 1/2$ with $\sqrt{k} \cdot x + y$ in the interior of C_N^A for $k \geq k_0$.*

For $k \geq k_0$ consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type A_{N-1} on C_N^A, started at $\sqrt{k} \cdot x + y$, and which satisfy the SDEs

$$dX_{t,k}^i = dB_t^i + k \sum_{j\neq i} \frac{1}{X_{t,k}^i - X_{t,k}^j} dt \qquad (i = 1, \ldots, N).$$

Then, for all $t > 0$,

$$\sup_{0\leq s\leq t, k\geq k_0} \|X_{s,k} - \sqrt{k}\phi(s,x)\| < \infty$$

almost surely. In particular,

$$X_{t,k}/\sqrt{k} \to \phi(t,x) \qquad \text{for} \qquad k \to \infty$$

locally uniformly in t almost surely and thus locally uniformly in t in probability.

Proof. Recall that the processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ satisfy

$$\tilde{X}_{t,k}^i = \frac{1}{\sqrt{k}}(y_i + B_t^i) + x_i + \int_0^t \sum_{j\neq i} \frac{1}{\tilde{X}_{s,k}^i - \tilde{X}_{s,k}^j} ds \qquad (i = 1, \ldots, N).$$

We compare the solutions of these SDEs with the solution $Y_t = \phi(t,x)$ $(t \geq 0)$ of the deterministic equation

$$Y_t^i = x_i + \int_0^t \sum_{j\neq i} \frac{1}{Y_s^i - Y_s^j} ds \qquad (i = 1, \ldots, N)$$

of Lemma 2.1. For both equations we perform Picard iterations as follows. We set the starting points at

$$\tilde{X}_{t,k,0} := Y_{t,0} := x$$

and, for $m \geq 0$, we set the recursions

$$\tilde{X}_{t,k,m+1}^i := \frac{1}{\sqrt{k}}(y_i + B_t^i) + x_i + \int_0^t \sum_{j\neq i} \frac{1}{\tilde{X}_{s,k,m}^i - \tilde{X}_{s,k,m}^j} ds \qquad (i = 1, \ldots, N)$$

and

$$Y_{t,m+1}^i := x_i + \int_0^t \sum_{j\neq i} \frac{1}{Y_{s,m}^i - Y_{s,m}^j} ds \qquad (i = 1, \ldots, N).$$

For given points x, y and given k_0 as in the statement, we find $\epsilon > 0$ small enough that $x + y/\sqrt{k} \in U_\epsilon$ for $k \geq k_0$ where U_ϵ is given as in Lemma 2.1. Consider the stopping times

$$T_{\epsilon,k} := \inf\{t > 0 : \ \tilde{X}_{t,k} \notin U_\epsilon\}.$$

We study the stopped maximal differences

$$D_{t,k,m,\epsilon} := \sup_{s\in[0,t\wedge T_{\epsilon,k}]} \|\tilde{X}_{s,k,m} - Y_{s,m}\| \qquad (t \geq 0, m \geq 0)$$

with $D_{t,k,0,\epsilon} = 0$. Using the Lipschitz constants $L_\epsilon > 0$ on U_ϵ as in Lemma 2.1, we obtain

$$D_{t,k,m+1,\epsilon} \leq \frac{1}{\sqrt{k}}(\|y\| + \sup_{s\in[0,t]} \|B_s\|) + L_\epsilon \cdot \sup_{s\in[0,t]} \int_0^s D_{u,k,m,\epsilon}\, du.$$

Induction on m shows that for all m,

$$D_{t,k,m,\epsilon} \leq \frac{1}{\sqrt{k}}(\|y\| + \sup_{s\in[0,t]} \|B_s\|) \cdot \Big(\sum_{l=0}^{m-1} (L_\epsilon t)^l \frac{1}{l!}\Big)$$
$$\leq \frac{1}{\sqrt{k}}(\|y\| + \sup_{s\in[0,t]} \|B_s\|) \cdot e^{L_\epsilon t}. \tag{2.3}$$

On the other hand, it is well known from the classical theory of SDEs (see, for example, Theorems 7 and 8 of Section V.3 of [P]) that under a Lipschitz condition, for $m \to \infty$,

$$\sup_{s\in[0,t\wedge T_{\epsilon,k}]} \|\tilde{X}_{s,k,m} - \tilde{X}_{s,k}\| \to 0 \quad \text{and} \quad \sup_{s\in[0,t\wedge T_{\epsilon,k}]} \|Y_{s,m} - Y_s\| \to 0$$

in probability. This means that some subsequence converges almost surely. We conclude from (2.3) that

$$\sup_{s\in[0,t\wedge T_{\epsilon,k}]} \|\tilde{X}_{s,k} - Y_s\| \leq \frac{1}{\sqrt{k}} \cdot C_t \cdot e^{L_\epsilon t} \tag{2.4}$$

almost surely where $C_t := \|y\| + \sup_{s\in[0,t]} \|B_s\|$ is a random variable which is almost surely finite.

We now consider the events $\Omega_M := \{\omega :\ C_t(\omega) \leq M\}$ for $M \in \mathbb{N}$ which satisfy $P(\Omega_M) \to 1$ for $M \to \infty$. For given x, t, M and ϵ we enlarge $k_0 := k_0(x, y, t, M, \epsilon)$ such that in addition,

$$\frac{1}{\sqrt{k_0}} M \cdot e^{L_\epsilon t} < \frac{d(x, \partial C_N^A) - \epsilon}{2}.$$

Then, for $k \geq k_0$ and $\omega \in \Omega_M$,

$$\sup_{s\in[0,t\wedge T_{\epsilon,k}(\omega)]} \|\tilde{X}_{s,k}(\omega) - Y_s\| < \frac{d(x, \partial C_N^A) - \epsilon}{2}.$$

As $d(Y_s, \partial C_N^A) \geq d(x, \partial C_N^A)$ for $s \geq 0$ by Lemma 2.1, we see that for $s \in [0, t \wedge T_{\epsilon,k}(\omega)]$,

$$d(\tilde{X}_{s,k}(\omega), \partial C_N^A) \geq \frac{d(x, \partial C_N^A) + \epsilon}{2} > \epsilon. \tag{2.5}$$

Because the paths of the Bessel processes we consider are almost surely continuous, we conclude that $T_{\epsilon,k}(\omega) = \infty$ and thus $\tilde{X}_{s,k}(\omega) \in U_\epsilon$ for all $s \in [0, t]$, $\omega \in \Omega_M$ and $k \geq k_0$. Hence, for $\omega \in \Omega_M$ and $k \geq k_0$,

$$\sup_{s\in[0,t]} \|\tilde{X}_{s,k}(\omega) - Y_s\| \leq \frac{1}{\sqrt{k_0}} M \cdot e^{L_\epsilon t}. \tag{2.6}$$

As $P(\Omega_M) \to 1$ for $M \to \infty$, the first statement of the theorem is clear, and the second statement follows immediately from taking the limit $k_0 \to \infty$, which forces $k \to \infty$. □

Remark 2.5. Theorem 2.4 can be easily generalized to the case where the points $x, y \in \mathbb{R}^N$ are independent random variables X, Y which are also independent of the Brownian motion $(B_t)_{t\geq 0}$.

In fact, if X has values in an open subset U_ϵ of C_N^A as described in Lemma 2.1, and if there exists $k_0 > 0$ such that $\sqrt{k}X + Y$ has values in C_N^A for $k \geq k_0$, then the proof of Theorem 2.4 still holds, that is, we obtain that the Bessel processes $(X_{t,k})_{t\geq 0}$ with $X_{0,k} = \sqrt{k}X + Y$ satisfy

$$sup_{s\in[0,t]}\|X_{s,k}(\omega) - \sqrt{k}\cdot\phi(s,X)\| < \infty \qquad \text{a.s..} \tag{2.7}$$

Moreover, if $\mathbf{P}(X \in \partial C_N^A) = 0$, and if $\sqrt{k}X + Y$ has values in C_N^A for $k \geq k_0$ given $k_0 > 0$, then

$$k^\alpha(X_{t,k}/\sqrt{k} - \phi(t,X)) \to 0$$

for all $\alpha < 1/2$ in probability. This also follows immediately from Theorem 2.4 and the fact that $\mathbf{P}(X \in \partial C_N^A) = 0$ implies that $\mathbf{P}(X \in U_{1/n}) \to 1$ for $n \to \infty$.

We also remark that the limiting laws 3.5, 3.7, and 5.5 below for the root systems B_N and D_N and fixed starting points can be also extended to random starting points in the same way.

3. Strong limiting laws for the root system B_N

In this section we derive LLs in the case B_N for the two freezing regimes from the introduction for starting points in the interior of C_N^B. In both cases we consider $k = (k_1, k_2)$ with $k_1, k_2 > 0$, and study Bessel process $(X_{t,k})_{t\geq 0}$ which are solutions of (1.9). In the B-case, the SDE (1.9) reads

$$dX_{t,k}^i = dB_t^i + k_2\sum_{j\neq i}\Big(\frac{1}{X_{t,k}^i - X_{t,k}^j} + \frac{1}{X_{t,k}^i + X_{t,k}^j}\Big)dt + \frac{k_1}{X_{t,k}^i}dt \tag{3.1}$$

for $i = 1,\ldots,N$ with an N-dimensional Brownian motion $(B_t^1,\ldots,B_t^N)_{t\geq 0}$.

The two freezing regimes have to be handled differently from the previous, A_{N-1}, case. We start with the case $(k_1,k_2) = (\nu\cdot\beta,\beta)$ with $\nu > 0$ fixed and $\beta \to \infty$ which was studied in [AKM2, AM] for the case of a fixed starting distribution on C_N^B. Similar to the A_{N-1} case, we study the renormalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{\beta})_{t\geq 0}$ which satisfy

$$d\tilde{X}_{t,k}^i = \frac{1}{\sqrt{\beta}}dB_t^i + \sum_{j\neq i}\Big(\frac{1}{\tilde{X}_{t,k}^i - \tilde{X}_{t,k}^j} + \frac{1}{\tilde{X}_{t,k}^i + \tilde{X}_{t,k}^j}\Big)dt + \frac{\nu}{\tilde{X}_{t,k}^i}dt \tag{3.2}$$

for $i = 1,\ldots,N$. We again compare $\tilde{X}_{t,k}$ with the solution of a deterministic dynamical system.

Lemma 3.1. *Let $\nu > 0$. For $\epsilon > 0$ consider the open subset*

$$U_\epsilon := \{x \in C_N^B : x_N > \frac{\epsilon\nu}{N-1}, \quad \text{and} \quad x_i - x_{i+1} > \epsilon \quad \text{for} \quad i = 1,\ldots,N-1\}.$$

Then $\cup_{\epsilon>0}U_\epsilon$ is the interior of C_N^B, and the function

$$H : U_\epsilon \to \mathbb{R}^N, \quad x \mapsto \begin{pmatrix} \sum_{j\neq 1}\Big(\frac{1}{x_1-x_j} + \frac{1}{x_1+x_j}\Big) + \frac{\nu}{x_1} \\ \vdots \\ \sum_{j\neq N}\Big(\frac{1}{x_N-x_j} + \frac{1}{x_N+x_j}\Big) + \frac{\nu}{x_N}\end{pmatrix}$$

is Lipschitz continuous on U_ϵ with Lipschitz constant $L_\epsilon > 0$. Moreover, for each starting point $x_0 \in U_\epsilon$, the solution $\phi(t,x_0)$ of the dynamical system $\frac{dx}{dt}(t) = H(x(t))$ satisfies $\phi(t,x_0) \in U_\epsilon$ for all $t \geq 0$.

Proof. There exits a constant $\tilde{\epsilon} > 0$ such that for all $x \in U_\epsilon$ and $i \neq j$ we have $|x_i \pm x_j| > \tilde{\epsilon}$ and $x_i > \tilde{\epsilon}$. Hence there is a constant $C > 0$ with $|\frac{\partial H}{\partial x_i}(x)| \le C$ for $x \in U_\epsilon$ and $i = 1, \dots, N$ which implies the Lipschitz continuity.

We now proceed as in the proof of Lemma 3.1 and use the new variables $y_i(t) := x_i(t) - x_{i+1}(t) > \epsilon$ for $i = 1, \dots, N-1$ as well as $y_N(t) := \frac{N-1}{\nu} \cdot x_N(t)$. For any $t \ge 0$ we choose $i = i(t)$ for which $y_i(t)$ is minimal, i.e., $y_j(t) \ge y_i(t)$ for all j. If $i \in \{1, \dots, N-1\}$, then the estimations in the proof of Lemma 3.1 immediately imply $\frac{dy_i}{dt}(t) \ge 0$, as the right hand side of the dynamical system here is clearly greater than the right hand side of the system in Lemma 2.1.

Moreover, for $i = N$ (that is, if $y_N(t) = \min_{1 \le j \le N} y_j(t)$),

$$\begin{aligned}\frac{dy_N}{dt}(t) &= \frac{N-1}{\nu}\Big[\sum_{j \neq N}\Big(\frac{1}{x_N(t) - x_j(t)} + \frac{1}{x_N(t) + x_j(t)}\Big) + \frac{\nu}{x_N(t)}\Big] \\ &\ge \frac{N-1}{\nu}\Big[\frac{\nu}{x_N(t)} - \frac{N-1}{x_{N-1} - x_N(t)}\Big] \;=\; \frac{(N-1)^2}{\nu}\Big(\frac{1}{y_N(t)} - \frac{1}{y_{N-1}(t)}\Big) \ge 0.\end{aligned}$$

In summary, we see that $\min_{i=1,\dots,N} y_i(t)$ is increasing in t. This completes the proof. □

As in the A_{N-1} case, it seems difficult to solve the dynamical systems of Lemma 3.1 except for a few cases like $N = 1$ or particular starting points which are related to the zeros of certain Laguerre polynomials. The latter is not surprising in view of the LLs of [AKM2]. To explain this, we recapitulate the following fact; see [AKM2] and Section 6.7 of [S] and notice that our parameters (β, ν) correspond to the parameters $(\beta/2, \nu + 1/2)$ in [AKM2]:

Lemma 3.2. *Let $\nu > 0$. For $y \in C_N^B$, the following statements are equivalent:*

(1) *The function*

$$W(x) := 2\sum_{i<j} \ln(x_i^2 - x_j^2) + 2\nu \sum_i \ln x_i - \|x\|^2/2$$

is maximal at $y \in C_N^B$;

(2) *For $i = 1, \dots, N$,*

$$\frac{1}{2}y_i = \sum_{j:j\neq i} \frac{2y_i}{y_i^2 - y_j^2} + \frac{\nu}{y_i} = \sum_{j:j\neq i}\Big(\frac{1}{y_i - y_j} + \frac{1}{y_i + y_j}\Big) + \frac{\nu}{y_i};$$

(3) *If $z_1^{(\nu-1)}, \dots, z_N^{(\nu-1)}$ are the ordered zeros of the classical Laguerre polynomial $L_N^{(\nu-1)}$ (where the $L_N^{(\nu-1)}$ are orthogonal w.r.t. the density $e^{-x} \cdot x^{\nu-1}$), then*

$$2(z_1^{(\nu-1)}, \dots, z_N^{(\nu-1)}) = (y_1^2, \dots, y_N^2). \tag{3.3}$$

Remark 3.3. Using the known explicit representation

$$L_N^{(\alpha)}(x) := \sum_{k=0}^{N} \binom{N+\alpha}{N-k} \frac{(-x)^k}{k!}$$

of the Laguerre polynomials according to (5.1.6) of [S], we can form the polynomial $L_N^{(-1)}$ of order $N \ge 1$ where, by (5.2.1) of [S],

$$L_N^{(-1)}(x) = -\frac{x}{N} L_{N-1}^{(1)}(x). \tag{3.4}$$

Continuity arguments thus show that the equivalence of (2) and (3) in Lemma 3.2 remains valid also for $\nu = 0$ and $N \geq 1$ by using the N different zeros $z_1 > \ldots > z_N = 0$ of $L_N^{(-1)}$.

Notice that Lemma 3.1 and thus the following results cannot be applied directly to the case $\nu = 0$. On the other hand, the root system B_N for $\nu = 0$ is closely related to the root sytem D_N, and thus results for $\nu = 0$ can be derived via the corresponding results for D_N; see Section 5 below.

Parts (2) and (3) of Lemma 3.2 lead to the following explicit solution of the differential equation of Lemma 3.1:

Corollary 3.4. *Let $\nu > 0$ and $y \in C_N^B$ the vector in Eq. (3.3). Then for each $c > 0$, a solution of the dynamical system in Lemma 3.1 is given by $\phi(t, c\cdot y) = \sqrt{t + c^2}\cdot y$.*

Notice that on an informal level, Corollary 3.4 holds also for $c = 0$.

We now turn to the first main result of this section, a locally uniform strong LL which is analog to Theorem 2.4:

Theorem 3.5. *Let $\nu > 0$. Let x be a point in the interior of C_N^B, and let $y \in \mathbb{R}^N$. Let $\beta_0 \geq 1/2$ with $\sqrt{\beta}\cdot x + y$ in the interior of C_N^B for $\beta \geq \beta_0$.*

For $\beta \geq \beta_0$, consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B with $k = (k_1, k_2) = (\beta\cdot\nu, \beta)$, which start in $\sqrt{\beta}\cdot x + y$. Then, for all $t > 0$,

$$\sup_{0\leq s\leq t, \beta\geq\beta_0} \|X_{s,k} - \sqrt{\beta}\phi(s, x)\| < \infty \qquad a.s..$$

In particular,

$$X_{t,(\nu\cdot\beta,\beta)}/\sqrt{\beta} \to \phi(t, x) \qquad \text{for} \qquad \beta \to \infty$$

locally uniformly in t a.s. and thus locally uniformly in t in probability.

Proof. The proof is analog to that of Theorem 2.4; we only sketch the most important steps. Recall that $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{\beta})_{t\geq 0}$ satisfies

$$\tilde{X}^i_{t,k} = \frac{1}{\sqrt{\beta}}(y_i + B^i_t) + x_i + \int_0^t \Big(\sum_{j\neq i}\Big(\frac{1}{\tilde{X}^i_{s,k} - \tilde{X}^j_{s,k}} + \frac{1}{\tilde{X}^i_{s,k} + \tilde{X}^j_{s,k}}\Big) + \frac{\nu}{\tilde{X}^i_{s,k}}\Big) ds$$

for $i = 1, \ldots, N$. We compare $\tilde{X}_{t,k}$ with the solution $Y_t = \phi(t, x)$ of

$$Y^i_t = x_i + \int_0^t \Big(\sum_{j\neq i}\Big(\frac{1}{Y^i_s - Y^j_s} + \frac{1}{Y^i_s + Y^j_s}\Big) + \frac{\nu}{Y^i_s}\Big) ds$$

for $i = 1, \ldots, N$ of Lemma 3.1.

For both equations we perform Picard iterations as in the proof of Theorem 2.4. If we use Lemma 3.1 instead of 2.1, we obtain that for each $t > 0$, a suitable $\epsilon > 0$ with $x + y/\sqrt{\beta} \in U_\epsilon$ for $\beta \geq \beta_0$, and the stopping times

$$T_{\epsilon,k} := \inf\{t > 0 : \tilde{X}_{t,k} \notin U_\epsilon\},$$

we have

$$\sup_{s\in[0,t\wedge T_{\epsilon,k}]} \|\tilde{X}_{s,k} - Y_s\| \leq \frac{1}{\sqrt{\beta}}\cdot C\cdot e^{L_\epsilon t} \tag{3.5}$$

with suitable Lipschitz constants on U_ϵ and the almost-surely finite random variable $C := \|y\| + \sup_{s\in[0,t]} \|B_s\|$.

We now use the modified distance

$$d(x, \partial C_N^B) := \max\Big\{\frac{(N-1)x_N}{\nu},\ x_1 - x_2, \ldots,\ x_{N-1} - x_N\Big\}$$

of $x \in U_\epsilon$ from ∂C_N^B (which fits to the definition of U_ϵ in Lemma 3.1) instead of the usual distance in the proof of Theorem 2.4. Using (3.5) we then complete the proof precisely as in the A_{N-1} case. □

We now turn to the second freezing regime with $k_1 \to \infty$ and $k_2 > 0$ fixed. We study the normalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k_1})_{t\geq 0}$ with

$$d\tilde{X}_{t,k}^i = \frac{1}{\sqrt{k_1}} dB_t^i + \frac{k_2}{k_1} \sum_{j\neq i} \Big(\frac{1}{\tilde{X}_{t,k}^i - \tilde{X}_{t,k}^j} + \frac{1}{\tilde{X}_{t,k}^i + \tilde{X}_{t,k}^j} \Big) dt + \frac{1}{\tilde{X}_{t,k}^i} dt \tag{3.6}$$

for $i = 1, \ldots, N$. We again compare $\tilde{X}_{t,k}$ with the solutions of a deterministic dynamical system which is much easier than in the previous cases.

Lemma 3.6. *Let $k_2 > 0$. For $\epsilon > 0$ consider the open sets $U_\epsilon := \{x \in C_N^B : d(x, \partial C_N^B) > \epsilon\}$. Then the function*

$$H : U_\epsilon \to \mathbb{R}^N, \quad x \mapsto (1/x_1, \ldots, 1/x_N)$$

is Lipschitz continuous on U_ϵ with constant ϵ^{-2}. Moreover, for each starting point $x_0 \in U_\epsilon$, the solution $\phi(t, x_0)$ of the dynamical system $\frac{dx}{dt}(t) = H(x(t))$ is given by

$$\phi(t, x_0) = \Big(\sqrt{2t + x_{0,1}^2}, \ldots, \sqrt{2t + x_{0,N}^2}\Big)$$

with $\phi(t, x_0)$ in the interior of C_N^B for $t \geq 0$.

The proof of this lemma is straightforward. Notice that on an informal level, the dynamical system of Lemma 3.6 has the solution $\phi(t, x_0)$ for all starting points $x_0 \in C_N^B$. We now turn to the strong limiting law.

Theorem 3.7. *Let $k_2 > 0$. Let x be a point in the interior of C_N^B, and let $y \in \mathbb{R}^N$. Let $k_0 \geq 1/2$ large enough such that $\sqrt{k_1} \cdot x + y$ is in the interior of C_N^B for $k_1 \geq k_0$.*

For $k_1 \geq k_0$, consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N with $k = (k_1, k_2)$, which start in $\sqrt{k_1} \cdot x + y$. Then, for all $t > 0$,

$$\sup_{0\leq s\leq t, k_1 \geq k_0} \|X_{t,k} - \sqrt{k_1}\phi(t, x)\| < \infty \qquad a.s..$$

In particular,

$$X_{t,(k_1,k_2)}/\sqrt{k_1} \to \phi(t, x) \qquad \text{for} \qquad k_1 \to \infty$$

locally uniformly in t a.s. and thus locally uniformly in t in probability.

Proof. The proof is analog to that of Theorems 2.4 and 3.5; we only sketch the main steps and describe the differences. Notice that $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k_1})_{t\geq 0}$ satisfies

$$\tilde{X}_{t,k}^i = \frac{1}{\sqrt{k_1}}(y_i + B_t^i) + \frac{k_2}{k_1} \int_0^t \sum_{j\neq i} \Big(\frac{1}{\tilde{X}_{s,k}^i - \tilde{X}_{s,k}^j} + \frac{1}{\tilde{X}_{s,k}^i + \tilde{X}_{s,k}^j} \Big) ds + x_i + \int_0^t \frac{1}{\tilde{X}_{s,k}^i} ds$$

for $i = 1, \ldots, N$. We compare $\tilde{X}_{t,k}$ with the solution $Y_t = \phi(t, x)$ of

$$Y_t^i = x_i + \int_0^t \frac{1}{Y_s^i} ds \qquad (i = 1, \ldots, N)$$

of Lemma 3.6.

For both equations we perform Picard iterations as in the proof of Theorem 2.4. We notice that for any given time $t \geq 0$ and given k_0 as above, we can find a small $\epsilon > 0$ such that the deterministic solution $\phi(s, x + y/\sqrt{k_1})$ of Lemma 3.6 is contained in U_ϵ for all $k_1 \geq k_0$ and all $s \in [0, t]$. If we consider the stopping times

$$T_{\epsilon,k} := \inf\{t > 0 : \tilde{X}_{t,k} \notin U_\epsilon\}$$

we obtain as in the proof of Theorem 2.4 that

$$\sup_{s \in [0, t \wedge T_{\epsilon,k}]} \|\tilde{X}_{s,k} - Y_s\| \leq \frac{1}{\sqrt{k_1}} \cdot C \cdot e^{t/\epsilon^2} \tag{3.7}$$

with the a.s. finite random variable $C := \|y\| + \sup_{s \in [0,t]} \|B_s\|$.

We complete this proof by following the steps in the proof of Theorem 2.4. □

4. A central limit theorem for the root system B

In this section we show that the locally uniform limit law in Theorem 3.7 above can be used to derive a central limit theorem. This result generalizes the case B_1 for classical one-dimensional Bessel processes where this is a classical and well-known result; see Remark 4.2 below.

Theorem 4.1. *Let $k_2 > 0$. Let x be a point in the interior of C_N^B, and let $y \in \mathbb{R}^N$. Let $k_0 \geq 1/2$ large enough that $\sqrt{k_1} \cdot x + y$ is in the interior of C_N^B for $k_1 \geq k_0$. For $k_1 \geq k_0$, consider the Bessel processes $(X_{t,k})_{t \geq 0}$ of type B with $k = (k_1, k_2)$, which start in $\sqrt{k_1} \cdot x + y$. Then, for all $t > 0$,*

$$X_{t,(k_1,k_2)} - \sqrt{k_1} \cdot \left(\sqrt{2t + x_1^2}, \ldots, \sqrt{2t + x_N^2}\right)$$

tends in distribution for $k_1 \to \infty$ to the normal distribution

$$N\left(0, \operatorname{diag}\left(\frac{t^2 + tx_1^2}{2t + x_1^2}, \ldots, \frac{t^2 + tx_N^2}{2t + x_N^2}\right)\right).$$

Proof. Consider the process $\big(Z_{t,k} := ((X_{t,k}^1)^2, \ldots, (X_{t,k}^N)^2)\big)_{t \geq 0}$. The Itô formula and the SDE for $(X_{t,k}^i)_{t \geq 0}$ show for $i = 1, \ldots, N$ that

$$\begin{aligned} dZ_{t,k}^i &= 2X_{t,k}^i \, dX_{t,k}^i \; + \; dt \\ &= 2X_{t,k}^i \, dB_t^i \; + \; 2k_2 \sum_{j \neq i} \Big(\frac{X_{t,k}^i}{X_{t,k}^i - X_{t,k}^j} + \frac{X_{t,k}^i}{X_{t,k}^i + X_{t,k}^j} \Big) dt \; + \; (2k_1 + 1) dt. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{\sqrt{k_1}} \Big(Z_{t,k}^i - Z_{0,k}^i - (2k_1 + 1)t \Big) = \\ &\quad = 2 \int_0^t \frac{X_{s,k}^i}{\sqrt{k_1}} \, dB_s^i \; + \; \frac{2k_2}{\sqrt{k_1}} \int_0^t \sum_{j \neq i} \Big(\frac{X_{s,k}^i}{X_{s,k}^i - X_{s,k}^j} + \frac{X_{s,k}^i}{X_{s,k}^i + X_{s,k}^j} \Big) ds. \\ &\quad = 2 \int_0^t \frac{X_{s,k}^i}{\sqrt{k_1}} \, dB_s^i \; + \; \frac{4k_2}{\sqrt{k_1}} \int_0^t \sum_{j \neq i} \frac{(X_{s,k}^i)^2}{(X_{s,k}^i)^2 - (X_{s,k}^j)^2} ds. \end{aligned} \tag{4.1}$$

As $X_{s,k}^i/\sqrt{k_1} \to \phi(s, x)$ locally uniformly in probability by Theorem 3.7 with the function ϕ from Lemma 3.6, we obtain from standard results on stochastic integrals (see e.g. Section II.4 of [P]) that

$$\int_0^t \frac{X_{s,k}^i}{\sqrt{k_1}}\, dB_s^i \longrightarrow \int_0^t \phi(s,x)\, dB_s^i$$

locally uniformly in t in probability. Moreover, by the same argument, the integrand of the second integral of the r.h.s. of (4.1) converges also to a finite, continuous deterministic function, that is, the second summand of the r.h.s. of (4.1) converges to 0 locally uniformly in t in probability. Hence, using the initial condition, we see that

$$\frac{1}{\sqrt{k_1}}(Z_{t,k}^i - (x_i\sqrt{k_1} + y_i)^2 - (2k_1+1)t) \longrightarrow 2\cdot \int_0^t \sqrt{2s + x_i^2}\, dB_s^i$$

in probability for $k \to \infty$ and $i = 1, \ldots, N$. As the limits are $N(0, 4t^2 + 4tx_i^2)$-distributed and independent for $i = 1, \ldots, N$ we conclude that

$$\frac{1}{\sqrt{k_1}}\Big(Z_{t,k}^1 - x_1^2 k_1 - 2\sqrt{k_1}x_1y_1 - 2k_1t, \ldots, Z_{t,k}^N - x_N^2 k_1 - 2\sqrt{k_1}x_Ny_N - 2k_1t\Big) \quad (4.2)$$

tends in distribution to the N-dimensional normal distribution

$$N(0, \mathrm{diag}(4t^2 + 4tx_1^2, \ldots, 4t^2 + 4tx_N^2)). \quad (4.3)$$

In order to obtain a CLT for the original variables $X_{t,k}^i$, we use the definition of $Z_{t,k}$ and observe that

$$\begin{aligned}
&\frac{1}{\sqrt{k_1}}\Big((X_{t,k}^i)^2 - k_1(x_i^2 + 2t + 2x_iy_i/\sqrt{k_1})\Big)\\
&\quad = \Big(X_{t,k}^i - \sqrt{k_1}\cdot\sqrt{x_i^2 + 2t + 2x_iy_i/\sqrt{k_1}}\Big)\\
&\quad\quad \times \frac{1}{\sqrt{k_1}}\Big(X_{t,k}^i + \sqrt{k_1}\cdot\sqrt{x_i^2 + 2t + 2x_iy_i/\sqrt{k_1}}\Big)
\end{aligned}$$

where the second factor in the r.h.s. tends in probability to $2\sqrt{x_i^2 + 2t}$ for $k \to \infty$ and $i = 1, \ldots, N$ by Theorem 3.7. This, the CLT for $Z_{t,k}$ in the first part of the proof, and Slutsky's lemma applied to the quotient

$$\frac{(Z_{t,k}^1 - x_1^2k_1 - 2\sqrt{k_1}x_1y_1 - 2k_1t)/\sqrt{k_1}}{\Big(X_{t,k}^i + \sqrt{k_1}\cdot\sqrt{x_i^2 + 2t + 2x_iy_i/\sqrt{k_1}}\Big)/\sqrt{k_1}} = X_{t,k}^i - \sqrt{k_1}\cdot\sqrt{x_i^2 + 2t + 2x_iy_i/\sqrt{k_1}}$$

yield that

$$\Big(X_{t,k}^1 - \sqrt{k_1}\sqrt{x_1^2 + 2t}, \ldots, X_{t,k}^N - \sqrt{k_1}\sqrt{x_N^2 + 2t}\Big)$$

converges in distribution to the normal distribution given in the statement. □

We illustrate this theorem for the case $N = 3$ in Figure 1, where the limiting distribution is compared with numerical simulations of the process for several values of k_1 and t. As expected from the theorem, the (centered) distribution of each particle approaches the limiting normal distribution as k_1 grows, but there is a clear bias in the numerical results. By observing the plots corresponding to $t = 10$ (plots b), d), and f) in the figure), it is apparent that a larger value of k_1 is necessary to approach the limiting distribution at larger times. This means that the bias is not an effect of the starting position of the process, but rather an accumulating effect of the second term in the last line of (4.1), which represents the repulsion between particles. Indeed, the rightmost particle ($X_{t,k}^1$) is pushed to the right and

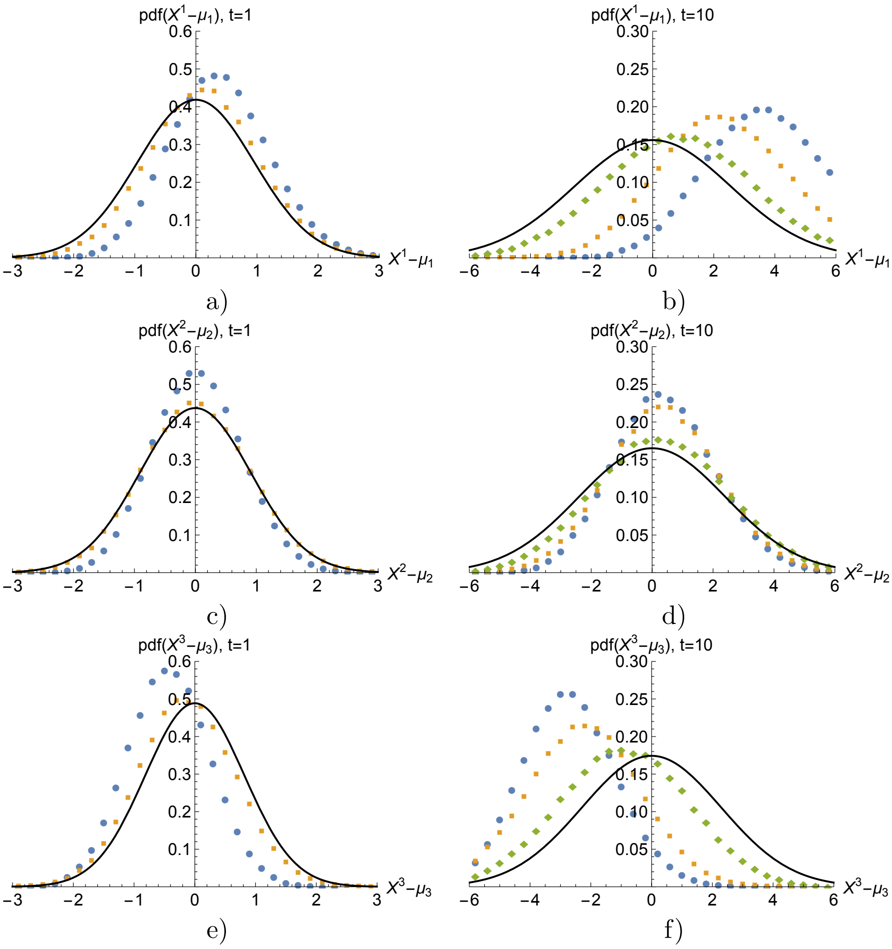

Figure 1.
Illustration of Theorem 4.1 for $N = 3$ and $k_2 = 1$ with $(x_1, x_2, x_3) = (3, 2, 1)$. The i-th row corresponds to $X^i_{t,k}$, while the first column corresponds to $t = 1$ and the second column corresponds to $t = 10$. In each plot, the solid lines correspond to the limiting normal distribution, while the blue circles, yellow squares and green rhombi correspond to the distribution of $X^i_{t,k} - \sqrt{k_1(2t - x_i^2)}$ obtained in numerical simulations for $k_1 = 5$, 50, and 500, respectively.

the leftmost particle $(X^3_{t,k})$ is pushed to the left. In the case of $X^2_{t,k}$, the bias is much smaller because the repulsion from $X^1_{t,k}$ and $X^3_{t,k}$ cancel each other partially, leading to a much faster convergence to the limiting distribution.

Remark 4.2. We briefly discuss the CLT 4.1 for $N = 1$ which is the case where 4.1 is reduced to a known classical CLT for classical one-dimensional Bessel processes. To explain this fix $\tilde{x} \in]0, \infty[$. Consider independent one-dimensional Brownian motions $(B^l_t)_{t\geq 0}$ starting in 0 for $l \in \mathbb{N}$. It is well-known (see e.g. Sections VI.3

and XI.1 of [RY]) that for $d \in \mathbb{N}$ the sums of shifted squares

$$\Big(S_{t,d} := \sum_{l=1}^{d} (B_t^l + \tilde{x})^2\Big)_{t\geq 0}$$

are squares of classical one-dimensional Bessel processes, i.e., of Bessel processes of type B_1 with multiplicity $k_1 = (d-1)/2$ (k_2 is irrelevant here).

Now fix $t > 0$. Then $S_{t,d}$ is a sum of d iid random variables with mean $t + \tilde{x}^2$ and variance $2t^2 + 4t\tilde{x}^2$. Therefore, by the classical CLT for sums of iid random variables,

$$\frac{S_{t,d} - d(t + \tilde{x}^2)}{\sqrt{d}} \to N(0, 2t^2 + 4t\tilde{x}^2) \tag{4.4}$$

for $d \to \infty$ in distribution.

This CLT corresponds perfectly with the convergence of (4.2) to the distribution (4.3) if one takes into account that we have $k_1 = (d-1)/2$ which implies that the point x in Theorem 4.1 is related to $\tilde{x}$ by $d\tilde{x}^2 = kx^2$, i.e., $\tilde{x}^2 = \frac{d-1}{2d}x^2$. We notice that this approach to the one-dimensional CLT (4.4) also works for any real parameter $d \in [1, \infty[$ and also for the starting point $\tilde{x} = 0$.

We also notice that for the cases $k_2 = 1/2, 1, 2$, the central limit theorem is related to a central limit theorem for Wishart distributions on the cones $\Pi_N(\mathbb{F})$ of all $N \times N$-dimensional, positive semidefinite matrices over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and the skew-field of quaternions $\mathbb{H}$ respectively. We discuss this extension of the preceding remark briefly.

Remark 4.3. Fix one of the (skew-)fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as before with the real dimension $d = 1, 2, 4$ respectively. For integers $p \in \mathbb{N}$ consider the vector space $M_{p,N}(\mathbb{F})$ of all $p \times N$-matrices over $\mathbb{F}$ with the real dimension dpN. Choose the standard basis there with d basis vectors in each entry, and consider the dpN-dimensional associated Brownian motion $(B_t^p)_{t\geq 0}$ on $M_{p,N}(\mathbb{F})$. If we write $A^* := \overline{A}^T \in M_{N,p}(\mathbb{F})$ for matrices $A \in M_{p,N}(\mathbb{F})$ with the usual conjugation on $\mathbb{F}$, the process $(Z_t^p := (B_t^p)^* B_t^p)_{t\geq 0}$ becomes a Wishart process on the cone $\Pi_N(\mathbb{F})$ of all $N \times N$ positive semidefinite matrices over $\mathbb{F}$ with shape parameter p; see [Bru, DDMY] for details on Wishart processes.

Let $\sigma_N : \Pi_N(\mathbb{F}) \to C_N^B$ be the mapping which relates to each matrix in $\Pi_N(\mathbb{F})$ its ordered spectrum. Then, it is well-known that $(\sqrt{\sigma_N(Z_t^p)})_{t\geq 0}$ is a Bessel process on C_N^B of type B_N with multiplicity $(k_1, k_2) := ((p - N + 1) \cdot d/2, d/2)$ where the symbol $\sqrt{.}$ means taking square roots in each component; see e.g. [BF, R3] for details.

We thus conclude that the CLT 4.1 for $k_2 = 1/2, 1, 2$ corresponds to a CLT for Wishart distributions on $\Pi_N(\mathbb{F})$ with fixed time parameters where the shape parameters p tend to ∞. Notice that the distributions $\mu_t^p := P_{Z_t^p} \in M^1(\Pi_N(\mathbb{F}))$ of Z_t^p satisfy $\mu_t^{p_1} * \mu_t^{p_2} = \mu_t^{p_1+p_2}$ for $p_1, p_2 \in \mathbb{N}$ with the usual convolution of measures on the vector space of all $N \times N$ Hermitian matrices over $\mathbb{F}$ by the very construction of the random variables Z_t^p. Moreover, this convolution relation holds for all real parameters p which are sufficiently large. We thus may apply the classical law of large numbers and CLT for sums of iid random variables on finitely dimensional vector spaces to obtain LLs and a CLT for Wishart distributions for $p \to \infty$. If one computes the mean vectors and covariance matrices for Z_t^p one obtains readily that this classical CLT for $p \to \infty$ on the level of Hermitian matrices corresponds to Theorem 4.1 on the Weyl chamber C_N^B.

We also remark that in this setting there are related LLNs and CLTs for radial random walks $(X_n^p)_{n\geq 0}$ on the vector space $M_{p,N}(\mathbb{F})$ when the dimension parameter p as well as the time parameter n tend to ∞ in a coupled way; see [G, RV3, V1]. We also mention that the CLT 4.1 has some relations with limit theorems of Bougerol [B] for noncompact Grassmann manifolds over $\mathbb{F}$ when the dimensions tend to infinity.

The strong LLs 2.4, 3.5, and 5.5 also admit central limit theorems similar to Theorem 4.1. These results, whose proofs are also based on these strong LLNs, are more complicated and will be presented in [VW]. To get some impression, we fix a root system, a multiplicity k (which might be 2-dimensional in the case B_N), and the corresponding Bessel processes $(X_{t,k})_{t\geq 0}$. For each function $F \in C^{(2)}(\mathbb{R}^N)$ we obtain from the Itô formula and the general SDE (1.9) that

$$\begin{aligned} dF(X_{t,k}) &= \nabla F(X_{t,k})\, dX_{t,k} + \frac{1}{2}\Delta F(X_{t,k})\, dt \\ &= \nabla F(X_{t,k})\, dB_t + \frac{1}{2}\Big((\nabla F \cdot \nabla(\ln w_k))(X_{t,k}) + \Delta F(X_{t,k})\Big) dt \end{aligned}$$

where $\nabla(\ln w_k))$ has the form $k \cdot H(x)$ for the root systems of type A_{N-1}, D_N, and the form $k_1 \cdot H_1(x) + k_2 \cdot H_2(x)$ for the root system B_N with suitable functions H, H_1, H_2. We now search for $F \in C^{(2)}(\mathbb{R}^N)$, for which

$$(\nabla F \cdot \nabla(\ln w_k))(x)$$

is independent of $x \in \mathbb{R}^N$. Similar to the proof of Theorem 4.1 we then obtain a CLT for $F(X_{t,k})$ for starting points in the interior when the multiplicity or a part of it tends to infinity.

It was noticed by J. Woerner that in all cases, a non-trivial example of a function F with the desired properties is given by $F(x) := \|x\|_2^2$. This can be checked easily for all root systems. This observation leads readily to the following CLT:

Proposition 4.4. *Consider Bessel processes $(X_{t,k})_{t\geq 0}$ of types A_{N-1}, B_N, or D_N as above in the strong LLs 2.4, 3.5, and 5.5 with the starting points given therein. Let $\gamma > 0$ be such that the weight function w_k is homogeneous of degree 2γ (see (1.5) for the cases A_{N-1}, B_N and the beginning of the next section for the case D_N for precise formulas).*

Then, for each $t > 0$ and for all multiplicities k with $\gamma \to \infty$,

$$\|X_{t,k}\|_2 - \sqrt{\gamma + (N-1)/2} \cdot \sqrt{2t + \|x\|_2^2}$$

converges in distribution to

$$N\Big(0, \frac{t^2 + t\|x\|_2^2}{2t + \|x\|_2^2}\Big)$$

Proof. The proof can be carried out by using the function $F(x) := \|x\|_2^2$ as explained above.

We give a second proof. It is well-known by [RV1] that $(\|X_{t,k}\|_2)_{t\geq 0}$ is a classical one-dimensional Bessel process of type B_1 with multiplicity $\gamma + (N-1)/2$. If we apply Theorem 4.1 to this case, the statement follows. □

Besides of the CLT 4.4 there exist further CLTs. For instance, for the case A_{N-1}, Eq. (2.1) implies that the center of gravity is

$$\frac{1}{N}\sum_{i=1}^{N} dX_{t,k}^{i} = \frac{1}{N}\sum_{i=1}^{N} dB_t^i,$$

i.e., it is a Brownian motion up to scaling. Also for the case A_1 with 2 particles, a CLT can be derived in a simple way.

5. Strong limiting law for the root system D_N

We next briefly a study limit theorem for Bessel processes of type D_N. We recapitulate that the root system is given here by

$$D_N = \{\pm e_1 \pm e_j : \quad 1 \le i < j \le N\}$$

with associated closed Weyl chamber

$$C_N := C_N^D = \{x \in \mathbb{R}^N : \quad x_1 \ge \ldots \ge x_{N-1} \ge |x_N|\}.$$

C_N^D may be seen as a doubling of C_N^B w.r.t. the last coordinate. We have a one-dimensional multiplicity $k \ge 0$. The weight function from (1.3) is given by

$$w_k(x) := w_k^D(x) := \prod_{i<j}(x_i^2 - x_j^2)^{2k},$$

the associated constant γ by $\gamma_D := kN(N-1)$, and the generator of the transition semigroup by

$$Lf := \frac{1}{2}\Delta f + k\sum_{i=1}^{N}\sum_{j\ne i}\Big(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\Big)\frac{\partial}{\partial x_i} f, \tag{5.1}$$

c.f. (1.1) and (1.2) for the cases A_{N-1} and B_N. For further details on the D_N case we refer to [De1].

Consider the Bessel process $(X_{t,k})_{t\ge 0}$ of type D_N which starts at time 0 from the origin, $0 \in C_N^D$. In this case, the SDE (1.9) reads as (3.1) with $k_1 := 0$, $k_2 := k$. Moreover, by (1.3), the random variable $X_{t,k}/\sqrt{kt}$ ($t > 0$) has the Lebesgue density

$$\mathrm{const}(k) \cdot \exp\Big(k\Big(-\|y\|_2^2 + 2\sum_{i<j}\ln(y_1^2 - y_j^2)\Big)\Big) =: \mathrm{const}(k) \cdot \exp(k \cdot W_D(y)) \tag{5.2}$$

on C_N^D. Similar to the results above in the cases A_{N-1} and B_N, we have:

Lemma 5.1. *For $y \in C_N^D$, the following statements are equivalent:*

(1) *The function $W_D(x) := 2\sum_{i<j}\ln(x_i^2 - x_j^2) - \|x\|^2/2$ is maximal at $y \in C_N^B$;*

(2) *$y_N = 0$, and for $i = 1, \ldots, N-1$,*

$$4\sum_{j:j\ne i}\frac{1}{y_i^2 - y_j^2} = 1;$$

(3) *If $z_1^{(1)} > \ldots > z_{N-1}^{(1)} > 0$ are the $N-1$ ordered zeros of the classical Laguerre polynomial $L_{N-1}^{(1)}$, then*

$$2(z_1^{(1)}, \ldots, z_{N-1}^{(1)}, 0) = (y_1^2, \ldots, y_N^2). \tag{5.3}$$

Proof. Clearly $W_D(y)$ tends to $-\infty$ for $y \in C_N^D$ with $\|y\| \to \infty$ and for the case where y tends to some point in ∂C_N^D. This shows that W_D admits a global maximum on C_N^D which is in the interior of C_N^D. Each candidate for a maximum satisfies

$$-y_i + 4\sum_{j:j\neq i} \frac{y_i}{y_i^2 - y_j^2} = 0 \qquad (i = 1,\ldots,N). \tag{5.4}$$

Using $y \in C_N^D$ we see easily that $y_N = 0$ (as otherwise the l.h.s. of (5.4) is negative for $i = N$). Moreover, as $y \notin \partial C_N^D$, we have $y_i > 0$ for $i = 1,\ldots,N-1$, that is, we obtain the condition in (2). If we have the equivalence of (2) and (3), we see that we only have one candidate for a maximum. This shows that (1) and (2) are equivalent. Finally, the equivalence of (2) and (3) is shown in Remark 3.3. □

Lemma 5.1 and the explicit densities (5.2) of $X_{t,k}/\sqrt{kt}$ immediately imply the following weak limiting law for $X_{t,k}$ for $k \to \infty$ for start in 0 which is analog to the LLs in [AKM1, AKM2, AM]:

Corollary 5.2. *Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D_N which start at time 0 in the origin $0 \in C_N^D$. Then, for each $t > 0$, $X_{t,k}/\sqrt{kt} \to y$ in probability for $k \to \infty$, where $y \in C_N^D$ is the vector in Lemma 5.1.*

The LLs for a starting point in the interior of the Weyl chamber as in Theorems 2.4, 3.5, and 3.7 can be also derived for the root system D_N. For this, we again compare $\tilde{X}_{t,k} := X_{t,k}/\sqrt{k}$ with solutions of a deterministic dynamical system.

Lemma 5.3. *For $\epsilon > 0$ consider the open subsets $U_\epsilon := \{x \in C_N^D : d(x, \partial C_N^D) > \epsilon\}$. Then the function*

$$H : U_\epsilon \to \mathbb{R}^N, \quad x \mapsto \begin{pmatrix} \sum_{j\neq 1}\left(\frac{1}{x_1 - x_j} + \frac{1}{x_1 + x_j}\right) \\ \vdots \\ \sum_{j\neq N}\left(\frac{1}{x_N - x_j} + \frac{1}{x_N + x_j}\right)\end{pmatrix}$$

is Lipschitz continuous on U_ϵ with Lipschitz constant $L_\epsilon > 0$. Moreover, for each starting point $x_0 \in U_\epsilon$, the solution $\phi(t, x_0)$ of the dynamical system $\frac{dx}{dt}(t) = H(x(t))$ satisfies $\phi(t, x_0) \in U_\epsilon$ for all $t \geq 0$.

Proof. The proof is completely analog, but slightly simpler than that of Lemma 3.1. We skip the details. □

Parts (2) and (3) of Lemma 5.1 lead to the following explicit solution of the differential equation of Lemma 5.3:

Corollary 5.4. *Let $y \in C_N^D$ be the vector in Eq. (5.3). Then for each $c > 0$, a solution of the dynamical system in Lemma 5.3 is given by $\phi(t, c\cdot y) = \sqrt{t + c^2}\cdot y$.*

Theorem 5.5. *Let x be a point in the interior of C_N^D, and $y \in \mathbb{R}^N$. Let $k \geq 1/2$ with $\sqrt{k}\cdot x + y$ in the interior of C_N^B for $k \geq k_0$. For $k \geq k_0$, consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D_N started from $\sqrt{k}\cdot x + y$. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq k_0} \|X_{s,k} - \sqrt{k}\phi(s,x)\| < \infty$$

almost surely. In particular,

$$X_{t,k}/\sqrt{k} \to \phi(t,x) \qquad \textit{for} \qquad k \to \infty$$

locally uniformly in t almost surely and thus locally uniformly in t in probability.

Proof. The proof is analog to that of Theorem 2.4. We skip the details. □

Remark 5.6. Let $(X_{t,k}^D)_{t\geq 0}$ be a Bessel process of type D with multiplicity $k \geq 0$ on the chamber C_N^D. Then the process $(X_{t,k}^B)_{t\geq 0}$ with

$$X_{t,k}^{B,i} := X_{t,k}^{D,i} \quad (i = 1, \ldots, N-1), \quad X_{t,k}^{B,N} := |X_{t,k}^{D,N}|$$

is a Bessel process of type B with the multiplicity $(k_1, k_2) := (0, k)$. This follows easily from a comparison of the corresponding generators.

We thus conclude from Theorem 5.5 that the strong LL 3.5 remains valid also for $\nu = 0$.

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