Central limit theorems for multivariate Bessel processes in the freezing regime

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CENTRAL LIMIT THEOREMS FOR MULTIVARIATE BESSEL PROCESSES IN THE FREEZING REGIME

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Abstract. Multivariate Bessel processes $(X_{t,k})_{t\geq 0}$ are classified via associated root systems and multiplicity constants $k \geq 0$. They describe the dynamics of interacting particle systems of Calogero-Moser-Sutherland type. Recently, Andraus, Katori, and Miyashita derived some weak laws of large numbers for $X_{t,k}$ for fixed times $t > 0$ and $k \to \infty$.

In this paper we derive associated central limit theorems for the root systems of types A, B and D in an elementary way. In most cases, the limits will be normal distributions, but in the B-case there are freezing limits where distributions associated with the root system A or one-sided normal distributions on half-spaces appear. Our results are connected to central limit theorems of Dumitriu and Edelman for β-Hermite and β-Laguerre ensembles.

1. Introduction

The dynamics of integrable interacting particle systems of Calogero-Moser-Sutherland type on the real line $\mathbb{R}$ with N particles can be described by certain time-homogeneous diffusion processes on suitable closed subsets of $\mathbb{R}^N$. These processes are often called (multivariate or interacting) Bessel- or Dunkl-Bessel processes; for the general background we refer to [CGY], [GY], [R1], [R2], [RV1], [RV2] as well as to [An], [DF], [DV]. These processes are classified via root systems and a finite number of multiplicity parameters which govern the interactions. We here consider the root systems of types A_{N-1}, B_N, and D_N.

Let us consider some details of the case A_{N-1} first. Here we have a multiplicity $k \in [0, \infty[$, the processes $(X_{t,k})_{t\geq 0}$ live on the closed Weyl chamber

$$C_N^A := \{x \in \mathbb{R}^N : \quad x_1 \geq x_2 \geq \ldots \geq x_N\},$$

the generator of the transition semigroup is

$$Lf := \frac{1}{2}\Delta f + k \sum_{i=1}^{N} \Big(\sum_{j \neq i} \frac{1}{x_i - x_j} \Big) \frac{\partial}{\partial x_i} f, \tag{1.1}$$

and we assume reflecting boundaries, i.e., the domain of the operator L is

$$D(L) := \{f|_{C_N} : f \in C^{(2)}(\mathbb{R}^N), f \text{ invariant under all permutations of coordinates}\}.$$

We are interested in limit theorems for $(X_{t,k})_{t\geq 0}$ for fixed $t > 0$ in freezing regimes, i.e., for $k \to \infty$. For this we recall that by [R1], [R2], [RV1], [RV2], the

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transition probabilities are given for $t>0$, $x\in C_N$, $A\subset C_N$ a Borel set, by

$$K_t(x,A)=c_k^A\int_A \frac{1}{t^{\gamma_A+N/2}}e^{-(\|x\|^2+\|y\|^2)/(2t)}J_k^A(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}})\cdot w_k^A(y)\,dy \tag{1.2}$$

with $\|.\|$ the usual euclidean norm on $\mathbb{R}^N$,

$$w_k^A(x):=\prod_{i<j}(x_i-x_j)^{2k}, \qquad \gamma_A=kN(N-1)/2, \tag{1.3}$$

and the Macdonald-Mehta-Opdam constant

$$c_k^A:=\Bigl(\int_{C_N^A}e^{-\|y\|^2/2}\cdot\prod_{i<j}(y_i-y_j)^{2k}\,dy\Bigr)^{-1}=\frac{N!}{(2\pi)^{N/2}}\cdot\prod_{j=1}^N\frac{\Gamma(1+k)}{\Gamma(1+jk)}; \tag{1.4}$$

see [O] or [Me]. Notice that w_k^A is homogeneous of degree $2\gamma_A$. Moreover, J_k^A is a multivariate Bessel function of type A with multiplicity k; see e.g. [R1], [R2] and references there. For the moment, we do not need much informations about J_k^A. We only recapitulate that J_k^A is analytic on $\mathbb{C}^N\times\mathbb{C}^N$ with $J_k^A(x,y)>0$ for $x,y\in\mathbb{R}^N$, and with $J_k^A(x,y)=J_k^A(y,x)$ and $J_k^A(0,y)=1$ for $x,y\in\mathbb{C}^N$.

If we start in $0\in\mathbb{R}^N$, then $X_{t,k}$ has the density

$$\frac{c_k}{t^{\gamma+N/2}}e^{-\|y\|^2/(2t)}\cdot w_k(y)\,dy \tag{1.5}$$

on C_N^A for $t>0$, i.e., $X_{t,k}/\sqrt{tk}$ has a density of the form

$$const.(k)\cdot exp\Bigl(k\Bigl(2\sum_{i,j:i<j}\ln(y_i-y_j)-\|y\|^2/2\Bigr)\Bigr)=:const.(k)\cdot exp\Bigl(k\cdot W_A(y)\Bigr)$$

which is in particular well-known for $k=1/2,1,2$ as the distribution of the ordered eigenvalues of Gaussian orthogonal, unitary, and symplectic ensembles; see e.g. [D]. For general $k>0$ it is known from the tridiagonal β-Hermite ensembles of [DE1]. It is well-known (see [AKM1] and also Section 6.7 of [S]) that W_A is maximal on C_N^A precisely for $y=\sqrt{2}\cdot\mathbf{z}$ where $\mathbf{z}\in C_N^A$ is the vector with the zeros of the classical Hermite polynomial H_N as entries where the $(H_N)_{N\geq 0}$ are orthogonal w.r.t. the density e^{-x^2}. A saddle point argument thus immediately yields that

$$\lim_{k\to\infty}\frac{X_{t,k}}{\sqrt{2tk}}=\mathbf{z} \qquad (k\to\infty) \tag{1.6}$$

in distribution and thus in probability whenever the $X_{t,k}$ are defined on a common probability space. It was shown in [AKM1] that this even holds when we start in any fixed point $x\in C_N^A$ or even with some more or less arbitrary starting distribution.

We prove a corresponding central limit theorem in an elementary way in Section 2. This CLT was derived in [DE2] by other methods via an interpretation through tridiagonal matrix models. We prove that for starting in $0\in C_N^A$ and any $t>0$, $X_{t,k}-\sqrt{2kt}\cdot\mathbf{z}$ converges in distribution to some centered N-dimensional normal distribution with some covariance matrix which again contains the zeros of H_N as major ingredients; see Theorem 2.2 below. The proof is based on the explicit density of $X_{t,k}$ above and elementary calculations which involve the zeros of H_N. As a byproduct of the CLT we automatically get some determinantal formula for the zeros of H_N which is possibly new.

We also derive corresponding CLTs for the Bessel processes associated with the root systems B_N and D_N. In the B-case, the multiplicity $k = (k_1, k_2)$ is 2-dimensional. Motivated by the LLNs in [AKM2] and [AM] for the B-cases, we study central limit theorems for several freezing cases. We in particular study the case $(k_1, k_2) = (c \cdot \beta, \beta)$ with $c > 0$ fixed and $\beta \to \infty$ in Section 3, but we also shall study the case where $k_2 > 0$ is fixed and $k_1 \to \infty$ in Section 4 as well as the case $k_1 > 0$ fixed and $k_2 \to \infty$ in Section 6. It will turn out that in the first case we obtain again a classical normal distribution in the limit where the covariance matrix is formed in terms of the zeros of classical Laguerre polynomials, where the index of the polynomials depends on k_2. In the second regime, the limit distribution has a density of type A as in (1.5). In the third case we shall get some one-sided normal distributions which live on a certain halfspace. Furthermore, Section 5 will be devoted to the root system D_N. We point out that some of the limit results are available for arbitrary, fixed starting points and not just for the case with start in 0.

The Bessel processes are diffusions on Weyl chambers which satisfy some stochastic differential equations; see [GY], [CGY]. These SDEs are used in [AV1] to derive locally uniform strong laws of large numbers for $X_{t,k}$ for $k \to \infty$ with strong rates of convergence, whenever the processes start in points of the form $\sqrt{k} \cdot x$ where x is some point in the interior of the Weyl chamber. It was possible to derive a CLT in the B-case for a particular freezing regime for these starting points in [AV1]. Further CLTs for starting points of the form $\sqrt{k} \cdot x$ are given in [VW].

2. A central limit theorem for the root system A_{N-1}

In this section we derive a CLT for Bessel processes of type A for $k \to \infty$ with an N-dimensional normal distribution as limit. The centerings as well as the entries of the covariance matrices of the limit will be described in terms of the zeros of the classical Hermite polynomials H_N. This connection is based on the following fact on these zeros, which is originally due to Stieltjes:

Lemma 2.1. *For $y \in C_N^A$, the following statements are equivalent:*

(1) *The function $W_A(x) := 2 \sum_{i,j:i<j} \ln(x_i - x_j) - \|x\|^2/2$ is maximal at $y \in C_N^A$;*
(2) *For $i = 1, \ldots, N$: $\frac{1}{2} y_i = \sum_{j:j \neq i} \frac{1}{y_i - y_j}$;*
(3) *The vector*

$$\mathbf{z} := (z_1, \ldots, z_N) := (y_1/\sqrt{2}, \ldots, y_N/\sqrt{2})$$

consists of the ordered zeros of the classical Hermite polynomial H_N.

Furthermore, the vector $\mathbf{z}$ of (3) satisfies for $t > 0$,

$$-\frac{\|\mathbf{z}\|^2}{2t} + 2\sum_{i<j} \ln(z_i - z_j) = -\frac{N(N-1)}{2}(1 - \ln t) + \sum_{j=1}^{N} j \ln j. \tag{2.1}$$

Proof. For the equivalence of (1)-(3) see [AKM1] or Section 6.7 of [S]. For (2.1) we refer to appendix D and the comments between Eqs. (58) and (59) in [AKM1]. □

Using the zeros of H_N we now turn to the main result of this section:

Theorem 2.2. *Consider the Bessel processes* $(X_{t,k})_{t\geq 0}$ *of type* A_{N-1} *on* C_N^A *for* $k \geq 0$ *with start in* $0 \in C_N^A$ *. Then*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{2k}\cdot \mathbf{z}$$

converges for $k \to \infty$ *to the centered* N*-dimensional distribution* $N(0, t\cdot\Sigma)$ *with the regular covariance matrix* Σ *with* $\Sigma^{-1} = (s_{i,j})_{i,j=1,\ldots,N}$ *with*

$$s_{i,j} := \begin{cases} 1 + \sum_{l\neq i}(z_i - z_l)^{-2} & \textit{for} \quad i = j \\ -(z_i - z_j)^{-2} & \textit{for} \quad i \neq j \end{cases} . \tag{2.2}$$

Proof. We first observe that by the formulas of the transition kernels K_t of our Bessel processes in (1.2), the K_t admit the same space-time-scaling as Brownian motions, i.e., for all $t > 0$, $x \in C_N^A$, and all Borel sets $A \subset C_N^A$, $K_t(x, A) = K_1(\sqrt{t}\cdot x, \sqrt{t}\cdot A)$. We thus may assume that $t = 1$ in the proof.

$X_{1,k}$ has the density

$$c_k^A e^{-\|y\|^2/2} \cdot exp\Big(2k \sum_{i<j} \ln(y_i - y_j)\Big)$$

on C_N^A. Hence, $X_{1,k} - \sqrt{2k}\cdot\mathbf{z}$ has the Lebesgue density

$$\begin{aligned} f_k^A(y) &:= c_k^A \cdot exp\Big(-\|y + \sqrt{2k}\cdot \mathbf{z}\|^2/2 + 2k\sum_{i<j} \ln\big(y_i - y_j + \sqrt{2k}(z_i - z_j)\big)\Big) \quad (2.3)\\ &= c_k^A \cdot exp\Big(-\|y\|^2/2 - \sqrt{2k}\langle y, \mathbf{z}\rangle - k\|\mathbf{z}\|^2 + 2k\sum_{i<j} \ln(\sqrt{2k}(z_i - z_j))\Big)\times \\ &\quad \times exp\Big(2k\sum_{i<j} \ln\big(1 + \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)}\big)\Big) \end{aligned}$$

on the shifted cone $C_N^A - \sqrt{2k}\cdot\mathbf{z}$ with $f_k^A(y) = 0$ otherwise on $\mathbb{R}^N$. We now split this formula into two parts

$$f_k^A(y) = \tilde{c}_k \cdot h_k(y),$$

where h_k depends on y and the remainder $\tilde{c}_k$ is constant w.r.t. y. This constant term is

$$\begin{aligned} \tilde{c}_k :=& c_k^A e^{-k\|\mathbf{z}\|^2} \cdot exp\Big(2k\sum_{i<j} \ln(\sqrt{2k}(z_i - z_j))\Big) \\ =& c_k^A exp\Big(-k\big(\|\mathbf{z}\|^2 - 2\sum_{i<j} \ln(z_i - z_j)\big)\Big)\cdot (2k)^{kN(N-1)/2} \\ =& c_k^A exp\Big(-k\frac{N(N-1)}{2}(1 + \ln 2) + k\sum_{j=1}^N j\ln j\Big)\cdot(2k)^{kN(N-1)/2} \\ =& c_k^A (k/e)^{kN(N-1)/2}\cdot \prod_{j=1}^N j^{kj}. \end{aligned}$$

Notice that the third $=$ above follows from (2.1) for $t = 1/2$. Hence, by (1.4),

$$\tilde{c}_k(x) = \frac{N!}{(2\pi)^{N/2}}\cdot \prod_{j=1}^N \frac{\Gamma(1+k)}{\Gamma(1+jk)}\cdot (k/e)^{kN(N-1)/2}\cdot\prod_{j=1}^N j^{kj}. \tag{2.4}$$

Stirling's formula $\Gamma(k+1) \sim \sqrt{2\pi k}(k/e)^k$ and elementary calculations now lead to

$$\lim_{k\to\infty} \tilde{c}_k = \frac{\sqrt{N!}}{(2\pi)^{N/2}}. \tag{2.5}$$

We next turn to the factor $h_k(y)$; it is given by

$$h_k(y) := exp\Big(-\|y\|^2/2 - \sqrt{2k}\langle y, \mathbf{z}\rangle + 2k\sum_{i<j} \ln\big(1 + \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)}\big)\Big)$$

By the power series of $\ln(1+x)$,

$$\ln\big(1 + \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)}\big) = \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)} - \frac{(y_i - y_j)^2}{4k(z_i - z_j)^2} + O(k^{-3/2}). \tag{2.6}$$

Furthermore, by part (2) of Lemma 2.1,

$$-\sqrt{2k}\langle y, \mathbf{z}\rangle + \sqrt{2k}\sum_{i<j} \frac{y_i - y_j}{z_i - z_j} = \sqrt{2k}\sum_{i=1}^N y_i\big(-z_i + \sum_{j:\, j\neq i} \frac{1}{z_i - z_j}\big) = 0. \tag{2.7}$$

Therefore,

$$h_k(y) = exp\Big(-\|y\|^2/2 - \frac{1}{2}\sum_{i<j} \frac{(y_i - y_j)^2}{(z_i - z_j)^2} + O(k^{-1/2})\Big). \tag{2.8}$$

Now let $f \in C_b(\mathbb{R}^N)$ be a bounded continuous function. We conclude from (2.3), (2.5),(2.8) that

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}^N} f(y)\cdot f_k^A(y)\, dy &= \lim_{k\to\infty} \tilde{c}_k \int_{\mathbb{R}^N} f(y)\cdot h_k(y)\, dy \\ &= \frac{\sqrt{N!}}{(2\pi)^{N/2}} \cdot \int_{\mathbb{R}^N} f(y) e^{-\|y\|^2/2} exp\Big(-\frac{1}{2}\sum_{i<j}\frac{(y_i - y_j)^2}{(z_i - z_j)^2}\Big)\, dy.\end{aligned} \tag{2.9}$$

For this we have to check that we may apply dominated convergence. For this we again consider the Taylor polynomial of $\ln(1+x)$ and notice that by the Lagrange remainder,

$$\ln\big(1 + \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)}\big) = \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)} - \frac{(y_i - y_j)^2}{4k(z_i - z_j)^2} \cdot w \tag{2.10}$$

with some $w \in [0,1]$. This implies readily that we could apply dominated convergence in (2.9).

On the other hand, Eq. (2.9) says that the probability measures with the densities f_k^A tend weakly to the measure with Lebesgue density

$$\frac{\sqrt{N!}}{(2\pi)^{N/2}} \cdot e^{-\|y\|^2/2} \cdot exp\Big(-\frac{1}{2}\sum_{i<j}\frac{(y_i - y_j)^2}{(z_i - z_j)^2}\Big), \tag{2.11}$$

which must a probability measure by weak convergence. On the other hand, this measure is just the normal distribution claimed in the theorem possibly up to the correct normalization constant. But, as both measures are probability measures, the normalizations in (2.11) are necessarily the correct ones. This completes the proof. □

Notice that the final arguments in the proof about the correct normalizations above automatically lead to the following remarkable result for the zeros of the Hermite polynomial H_N:

Corollary 2.3. *For each $N \in \mathbb{N}$ consider the ordered zeros $z_1 \geq \ldots \geq z_N$ of the N-th Hermite polynomial H_N. Form the matrix $S := (s_{i,j})_{i,j=1,\ldots,N}$ with*

$$s_{i,j} := \begin{cases} 1 + \sum_{l \neq i}(z_i - z_l)^{-2} & \text{for} \quad i = j \\ -(z_i - z_j)^{-2} & \text{for} \quad i \neq j \end{cases}. \tag{2.12}$$

Then det $S = N!$.

For details on the eigenvalues and eigenvectors of the matrix S we refer to [AV2].

Remark 2.4. As mentioned in the introduction, Theorem 2.2 was derived in [DE2] via the tridiagonal matrix models for β-Hermite ensembles of Dumitriu and Edelman in [DE1]. In fact, in Theorem 3.1 of [DE2], Dumitriu and Edelman obtain the CLT above with a direct, but quite complicated formula for the covariance matrix Σ of the limit in terms of the zeros of H_N combined with the Hermite polynomials H_l $(l = 1, \ldots, N)$. It is quite unclear how the expression for Σ in [DE2] corresponds to our formula for Σ^{-1} above, i.e., the equality of these matrices may be seen as a further corollary from Theorem 2.2.

Remark 2.5. One might try to extend the preceding proof to the case where the processes $(X_{t,k})_{t \geq 0}$ start in some $x \in C_N^A$ with $x \neq 0$. If x has the form $x = c(1, \ldots, 1)$, then $\frac{X_{t,k}}{\sqrt{t}} - \sqrt{2k} \cdot \mathbf{z} - x$ again tends to $N(0, t \cdot \Sigma)$ in distribution with Σ as above in the theorem.

This follows easily from Theorem 2.2 and the fact that the kernels K_t are partially translation invariant in the sense that

$$K_t(x + c(1, \ldots, 1), A + c(1, \ldots, 1)) = K_t(x, A) \qquad \text{for } c, t > 0,\ x \in C_N^A,\ A \subset C_N^A. \tag{2.13}$$

This invariance is a consequence from the well-known fact that the Bessel functions J_k^A satisfy

$$J_k^A(x, y) = e^{N\bar{x}\cdot\bar{y}} \cdot J_k^A(\pi_N(x), \pi_N(y)) \qquad (x, y \in \mathbb{R}^N) \tag{2.14}$$

with $\bar{x} := \frac{1}{N}(x_1 + \ldots + x_N)$ and with the orthogonal projection π_N from $\mathbb{R}^N$ onto the orthogonal complement $\mathbf{1}^\perp := (1, \ldots, 1)^\perp$ of $\mathbb{R} \cdot (1, \ldots, 1)$ w.r.t. the standard scalar product on $\mathbb{R}^N$. This factorization (2.14) follows from the fact that the non-reduced root system A_{N-1} on $\mathbb{R}^N$ may be regarded as a root system on the $n-1$-dimensional space $\mathbf{1}^\perp$ where the diagonal $\mathbb{R}\cdot\mathbf{1}$ remains unchanged. The theory of Dunkl kernels and Bessel functions then easily leads to (2.14); see e.g. the survey [R2].

The CLT for arbitrary starting points will be derived in [AV2] by using some limit result J_k^A for $k \to \infty$ from [AM].

3. A central limit theorem for the root system B_N

In this section we derive a first CLT for Bessel processes of type B. We recapitulate that in the case B_N, we have 2 multiplicities $k_1, k_2 > 0$, the processes live on

$$C_N^B := \{x \in \mathbb{R}^N : \quad x_1 \geq x_2 \geq \ldots \geq x_N \geq 0\},$$

the generator of the transition semigroup is

$$Lf := \frac{1}{2}\Delta f + k_2 \sum_{i=1}^{N}\sum_{j\neq i}\Big(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\Big)\frac{\partial}{\partial x_i} f \quad + k_1 \sum_{i=1}^{N} \frac{1}{x_i}\frac{\partial}{\partial x_i} f, \tag{3.1}$$

and we again assume reflecting boundaries. Similar to the A-case in the introduction, we have the transition probabilities

$$K_{t,k}(x,A) = c_k^B \int_A \frac{1}{t^{\gamma_B + N/2}} e^{-(\|x\|^2+\|y\|^2)/(2t)} J_k^B(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}) \cdot w_k^B(y)\, dy \tag{3.2}$$

with

$$w_k^B(x) := \prod_{i<j}(x_i^2 - x_j^2)^{2k_2} \cdot \prod_{i=1}^{N} x_i^{2k_1}, \tag{3.3}$$

$\gamma_B = k_2 N(N-1) + k_1 N$, and with the Macdonald-Mehta-Opdam-type normalization

$$\begin{aligned} c_k^B :=& \Big(\int_{C_N^B} e^{-\|y\|^2/2} w_k^B(y)\, dy\Big)^{-1} \\ =& \frac{N!}{2^{N(k_1+(N-1)k_2-1/2)}} \cdot \prod_{j=1}^{N} \frac{\Gamma(1+k_2)}{\Gamma(1+jk_2)\Gamma(\frac{1}{2}+k_1+(j-1)k_2)}; \end{aligned} \tag{3.4}$$

see [O]. Again w_k^B is homogeneous of degree $2\gamma_B$, and J_k^B is a multivariate Bessel function of type B with multiplicities $k := (k_1, k_2)$.

We now study CLTs fo several freezing regimes in this section as well as in Sections 4 and 6. We here start with the case $(k_1, k_2) = (\nu\cdot\beta, \beta)$ with $\nu > 0$ fixed and $\beta \to \infty$. Laws of large numbers in this case can be found in in [AKM2], [AV1] where in the limit now the zeros of the classical Laguerre polynomials $L_N^{(\nu-1)}$ appear. We recapitulate that the $L_N^{(\nu-1)}$ are orthogonal w.r.t. the density $e^{-x}\cdot x^{\nu-1}$ on $]0,\infty[$ for $\nu > 0$. We need the following known facts about the zeros of $L_N^{(\nu-1)}$.

Lemma 3.1. *Let $\nu > 0$. For $r \in C_N^B$, the following statements are equivalent:*

(1) *The function*

$$W_B(y) := 2\sum_{i<j}\ln(y_i^2 - y_j^2) + 2\nu\sum_i \ln y_i - \|y\|^2/2$$

is maximal at $r \in C_N^B$;

(2) *For $i = 1, \ldots, N$, $r = (r_1, \ldots, r_N)$ satisfies*

$$\frac{1}{2} r_i = \sum_{j:j\neq i} \frac{2r_i}{r_i^2 - r_j^2} + \frac{\nu}{r_i} = \sum_{j:j\neq i}\Big(\frac{1}{r_i - r_j} + \frac{1}{r_i + r_j}\Big) + \frac{\nu}{r_i};$$

(3) *If $z_1^{(\nu-1)} \geq \ldots \geq z_N^{(\nu-1)}$ are the ordered zeros of $L_N^{(\nu-1)}$, then*

$$2(z_1^{(\nu-1)}, \ldots, z_N^{(\nu-1)}) = (r_1^2, \ldots, r_N^2). \tag{3.5}$$

The vector r of (1)-(3) satisfies

$$-\frac{1}{2}\|r\|^2+\nu\sum_{j=1}^N \ln r_j^2 + 2\sum_{i<j}\ln(r_i^2-r_j^2) = \qquad (3.6)$$
$$= N(N+\nu-1)(-1+\ln 2)+\sum_{j=1}^N j\ln j+\sum_{j=1}^N(\nu+j-1)\ln(\nu+j-1)$$

Proof. For the equivalence of (1)-(3) we refer to [AKM2]; see in particular Appendix C there. Moreover, this equivalence is more or less also contained in Section 6.7 of [S]. For the proof of (3.6) we also refer to [AKM2]. In fact, one has to compare Eq. (12) with the comments on between (75) and (76) there. Please notice that the definitions of (β,ν) here are slightly different from that in [AKM2]; in β there is a multiplicative factor 2, and in ν there is a shift by 1/2. □

In order to handle arbitrary starting points x, we need the following asymptotic result for the Bessel functions of type B; see Lemma 5 of [AKM2] and notice that our notions of ν,β are different from [AKM2] as described above:

Lemma 3.2. *For all $x,y\in C_N^B$ and $\nu>0$,*

$$\lim_{\beta\to\infty} J^B_{(\nu\cdot\beta,\beta)}(\sqrt{\beta}\cdot x,y)=exp\Big(\frac{\|x\|^2\|y\|^2}{4N(\nu+N-1)}\Big).$$

This limit holds locally uniformly in x,y.

We now turn to the main result of this section:

Theorem 3.3. *Fix some starting point x in the Weyl chamber C_N^B, and consider the associated Bessel processes $(X_{t,k})_{t\ge0}$ of type B_N on C_N^B for $k=(k_1,k_2)$. Then, for the vector $r\in C_N^B$ of Lemma 3.1,*

$$\frac{X_{t,(\nu\cdot\beta,\beta)}}{\sqrt{t}}-\sqrt{\beta}\cdot r$$

converges for $\beta\to\infty$ to the centered N-dimensional distribution $N(0,t\cdot\Sigma)$ with the regular covariance matrix Σ with $\Sigma^{-1}=(s_{i,j})_{i,j=1,\ldots,N}$ with

$$s_{i,j}:=\begin{cases} 1+\frac{2\nu}{r_i^2}+2\sum_{l\ne i}(r_i-r_l)^{-2}+2\sum_{l\ne i}(r_i+r_l)^{-2} & \textit{for}\quad i=j\\ 2(r_i+r_j)^{-2}-2(r_i-r_j)^{-2} & \textit{for}\quad i\ne j\end{cases}. \qquad (3.7)$$

Proof. As in the A-case we may assume that $t=1$ without loss of generality. Let $k=(\nu\cdot\beta,\beta)$. Taking the starting point x into account, $X_{1,k}$ has the density

$$c_k^B e^{-\|x\|^2/2-\|y\|^2/2}\cdot J_k^B(x,y)\cdot exp\Big(2\beta\sum_{i<j}\ln(y_i^2-y_j^2)+2\nu\beta\sum_{i=1}^N\ln y_i\Big)$$

on C_N^B. Hence, $X_{1,k} - \sqrt{\beta}\cdot r$ has the density

$$f_\beta^B(y) := c_k^B e^{-\|x\|^2/2} J_k^B(x, y+\sqrt{\beta}\cdot r) e^{-\|y+\sqrt{\beta}\cdot r\|^2/2} \times \tag{3.8}$$

$$\times exp\Big(2\beta\sum_{i<j}\ln\big((y_i+\sqrt{\beta}\cdot r_i)^2-(y_j+\sqrt{\beta}\cdot r_j)^2\big)+2\nu\beta\sum_{i=1}^N \ln(y_i+\sqrt{\beta}\cdot r_i)\Big)$$

$$= c_k^B e^{-\|x\|^2/2} J_k^B(x, y+\sqrt{\beta}\cdot r) e^{-\|y\|^2/2} e^{-\beta\|r\|^2/2} e^{-\sqrt{\beta}\langle y,r\rangle}\times$$

$$\times\, exp\Big(2\beta\sum_{i<j}\ln\big(1+\frac{y_i-y_j}{\sqrt{\beta}(r_i-r_j)}\big)+2\beta\sum_{i<j}\ln\big(1+\frac{y_i+y_j}{\sqrt{\beta}(r_i+r_j)}\big)\Big)\times$$

$$\times\, exp\Big(2\nu\beta\sum_{i=1}^N\ln(1+\frac{y_i}{\sqrt{\beta}r_i})\Big) exp\Big(2\nu\beta\sum_{i=1}^N\ln(\sqrt{\beta}r_i)\Big)\times$$

$$\times\, exp\Big(2\beta\sum_{i<j}\ln(\sqrt{\beta}(r_i-r_j))+2\beta\sum_{i<j}\ln(\sqrt{\beta}(r_i+r_j))\Big)$$

on the shifted cone $C_N^B - \sqrt{\beta}\cdot r$ with $f_\beta^B(y) = 0$ otherwise on $\mathbb{R}^N$. We now split this formula into two parts

$$f_\beta^B(y) = \tilde{c}_\beta \cdot h_\beta(y),$$

where $h_\beta(y)$ depends on y, and where the remainder $\tilde{c}_\beta$ is constant w.r.t. y. The part depending on y is given by

$$h_\beta(y) := exp\Big(-\|y\|^2/2 - \sqrt{\beta}\langle y,r\rangle + 2\beta\sum_{i<j}\ln\big(1+\frac{y_i-y_j}{\sqrt{\beta}(r_i-r_j)}\big)\Big)\times$$

$$\times\, exp\Big(2\beta\sum_{i<j}\ln\big(1+\frac{y_i+y_j}{\sqrt{\beta}(r_i+r_j)}\big)+2\nu\beta\sum_{i=1}^N\ln(1+\frac{y_i}{\sqrt{\beta}r_i})\Big)\cdot J_k^B(x, y+\sqrt{\beta}\cdot r)$$

for $y \in C_N^B - \sqrt{\beta}\cdot r$, and by $h_\beta(y) = 0$ otherwise. By the power series of $\ln(1+x)$,

$$\ln\big(1+\frac{y_i\pm y_j}{\sqrt{\beta}(r_i\pm r_j)}\big) = \frac{y_i\pm y_j}{\sqrt{\beta}(r_i\pm r_j)} - \frac{(r_i\pm r_j)^2}{2\beta(r_i\pm r_j)^2} + O(\beta^{-3/2}) \tag{3.9}$$

and

$$\ln(1+\frac{y_i}{\sqrt{\beta}r_i}) = \frac{y_i}{\sqrt{\beta}r_i} - \frac{y_i^2}{2\beta r_i^2} + O(\beta^{-3/2}). \tag{3.10}$$

Furthermore, by part (2) of Lemma 3.1,

$$-\sqrt{\beta}\langle y,r\rangle + 2\sqrt{\beta}\sum_{i<j}\frac{y_i-y_j}{r_i-r_j} + 2\sqrt{\beta}\sum_{i<j}\frac{y_i+y_j}{r_i+r_j} + 2\nu\sqrt{\beta}\sum_{i=1}^N\frac{y_i}{r_i} \tag{3.11}$$

$$= \sqrt{\beta}\sum_{i=1}^N y_i\Big(-r_i + \sum_{j:\, j\neq i}\frac{1}{r_i-r_j} + \sum_{j:\, j\neq i}\frac{1}{r_i+r_j} + \frac{2\nu}{r_i}\Big) = 0.$$

Therefore, by (3.9)-(3.11),

$$h_\beta(y) = exp\Big(-\|y\|^2/2 - \sum_{i<j}\frac{(y_i-y_j)^2}{(r_i-r_j)^2} - \sum_{i<j}\frac{(y_i+y_j)^2}{(r_i+r_j)^2}\Big)\times \tag{3.12}$$

$$\times\, exp\Big(-\nu\sum_{i=1}^N y_i^2/r_i^2 + O(\beta^{-1/2})\Big)\cdot J_k^B(x, y+\sqrt{\beta}\cdot r).$$

We next observe that by Lemma 3.2,

$$\lim_{\beta\to\infty} J^B_{(\nu\cdot\beta,\beta)}(x, y+\sqrt{\beta}\cdot r) = \lim_{\beta\to\infty} J^B_{(\nu\cdot\beta,\beta)}(x, \sqrt{\beta}(r+y/\sqrt{\beta})) = exp\Big(\frac{\|x\|^2\|r\|^2}{4N(\nu+N-1)}\Big) =: d_\nu(x). \tag{3.13}$$

In summary,

$$\lim_{\beta\to\infty} h_\beta(y) = d_\nu(x) exp\Big(-\frac{\|y\|^2}{2} - \sum_{i<j}\frac{(y_i-y_j)^2}{(r_i-r_j)^2} - \sum_{i<j}\frac{(y_i+y_j)^2}{(r_i+r_j)^2} - \nu\sum_{i=1}^N \frac{y_i^2}{r_i^2}\Big). \tag{3.14}$$

We next study the second factor in the density $f^B_\beta(y)$ in (3.8) which is independent of y. This constant is given by

$$\begin{aligned}
\tilde c_\beta :=& c_k^B e^{-\|x\|^2/2-\beta\|r\|^2/2} exp\Big(2\nu\beta\sum_{i=1}^N \ln(\sqrt{\beta}\cdot r_i)\Big)\times \\
&\times exp\Big(2\beta\sum_{i<j}\ln(\sqrt{\beta}(r_i-r_j)) + 2\beta\sum_{i<j}\ln(\sqrt{\beta}(r_i+r_j))\Big) \\
=& c_k^B exp\Big(\beta\Big(-\frac{\|r\|^2}{2} + 2\nu\sum_{i=1}^N \ln r_i + 2\sum_{i<j}(\ln(r_i-r_j)+\ln(r_i+r_j))\Big)\Big)\times \\
&\times \beta^{\nu\beta N+\beta N(N-1)}\cdot e^{-\|x\|^2/2} \\
=& c_k^B exp\Big(\beta\Big(N(N+\nu-1)(-1+\ln 2) + \sum_{j=1}^N j\ln j + \sum_{j=1}^N(\nu+j-1)\ln(\nu+j-1)\Big)\Big)\times \\
&\times \beta^{\nu\beta N+\beta N(N-1)}\cdot e^{-\|x\|^2/2}.
\end{aligned} \tag{3.15}$$

Notice that the last equation above follows from (3.6). We next study the constant c_k^B. We conclude from (3.4), Stirling's formula $\Gamma(k+1)\sim\sqrt{2\pi k}(k/e)^k$, from

$$\frac{\Gamma(k+1/2)}{\Gamma(k+1)} \sim \frac{1}{\sqrt{k}}$$

for $k\to\infty$, and from a longer elementary calculation that

$$\begin{aligned}
c_k^B \sim & \frac{exp(\beta N(N+\nu-1))\cdot\sqrt{N!}}{2^{N\nu\beta+N(N-1)\beta-N/2}\cdot(2\pi)^{N/2}}\times \\
&\times \frac{1}{\prod_{j=1}^N j^{j\beta}\cdot\prod_{j=1}^N(\nu-1+j)^{\beta(\nu-1+j)}\beta^{N(N-1)\beta-N/2+\beta\nu N}}.
\end{aligned}$$

If we plug this into (3.15), we see that

$$\lim_{\beta\to\infty}\tilde c_\beta = \frac{e^{-\|x\|^2/2}2^{N/2}\cdot\sqrt{N!}}{(2\pi)^{N/2}}. \tag{3.16}$$

Now let $f \in C_b(\mathbb{R}^N)$ be a bounded continuous function. We shall conclude from (3.8), (3.16), and (3.14) that

$$\lim_{\beta\to\infty} \int_{\mathbb{R}^N} f(y)\cdot f_\beta^B(y)\, dy = \lim_{\beta\to\infty} \Big(\tilde{c}_\beta \int_{\mathbb{R}^N} f(y)\cdot h_\beta(y)\, dy\Big) \tag{3.17}$$
$$= \frac{e^{-\|x\|^2/2} d_\nu(x) 2^{N/2}\cdot \sqrt{N!}}{(2\pi)^{N/2}} \cdot$$
$$\cdot \int_{\mathbb{R}^N} f(y)\, exp\Big(-\Big(\frac{\|y\|^2}{2} + \sum_{i<j}\frac{(y_i-y_j)^2}{(r_i-r_j)^2} + \sum_{i<j}\frac{(y_i+y_j)^2}{(r_i+r_j)^2} + \nu\sum_{i=1}^N \frac{y_i^2}{r_i^2}\Big)\Big)\, dy.$$

For this we have to check that we may apply dominated convergence. For this we again consider the Taylor polynomial of $\ln(1+x)$ and notice that by the Lagrange remainder,

$$\ln\big(1+\frac{y_i \pm y_j}{\sqrt{\beta}(z_i\pm z_j)}\big) = \frac{y_i \pm y_j}{\sqrt{\beta}(z_i\pm z_j)} - \frac{(y_i\pm y_j)^2}{2\beta(z_i\pm z_j)^2}\cdot w_\pm \tag{3.18}$$

with some $w_\pm \in [0,1]$. Moreover, by the same reason,

$$\ln(1+\frac{y_i}{\sqrt{\beta}r_i}) = \frac{y_i}{\sqrt{\beta}r_i} - \frac{y_i^2}{2\beta r_i^2}\cdot w. \tag{3.19}$$

with some $w\in[0,1]$. This implies that for all $k>0$ and y,

$$0 \le h_k(y) \le J_{(\nu\beta,\beta)}^B(x, y+\sqrt{\beta}\cdot r)\cdot e^{-\|y\|^2/2}. \tag{3.20}$$

We next consider $J_{(\nu\beta,\beta)}^B$. For this we recapitulate from [RV2] that for all root systems and all multiplicities $k\ge 0$, the associated Bessel functions J satisfy

$$0 < J(a,b) \le exp(\|a\|\cdot\|b\|) \qquad \text{for all} \qquad a,b\in\mathbb{R}^N.$$

In particular, for all ν,β and all $x,y\in C_N^B$,

$$J_{(\nu\beta,\beta)}^B(x, y+\sqrt{\beta}\cdot r) \le exp(\|x\|\cdot(\|y\|+\sqrt{\beta}\cdot\|r\|)).$$

This shows that

$$J_{(\nu\beta,\beta)}(x, y+\sqrt{\beta}\cdot r) \le e^{2\|x\|\cdot\|y\|} \qquad \text{for } \beta>0,\ y \text{ with } \ \|y\|\ge\sqrt{\beta}\cdot\|r\|. \tag{3.21}$$

On the other hand, if $\|y\| \le \sqrt{\beta}\cdot\|r\|$, then $y/\sqrt{\beta}+r$ is contained in some fixed compactum $C\subset\mathbb{R}^N$. We thus obtain from Lemma 3.2 that

$$\sup_{y\in\mathbb{R}^N,\, \beta\ge 0:\, \|y\|\le\sqrt{\beta}\cdot\|r\|} J_{(\nu\beta,\beta)}^B(x, y+\sqrt{\beta}\cdot r) =$$
$$= \sup_{y\in\mathbb{R}^N,\, \beta\ge 0:\, \|y\|\le\sqrt{\beta}\cdot\|r\|} J_{(\nu\beta,\beta)}^B(x, \sqrt{\beta}(y/\sqrt{\beta}+r))$$

is bounded. This estimation, (3.21), and (3.20) readily imply that the dominated convergence theorem in (3.17) works as claimed.

On the other hand, Eq. (3.17) says that the probability measures with the densities f_β^A tend weakly to the measure with Lebesgue density

$$\frac{e^{-\|x\|^2/2} d_\nu(x) 2^{N/2}\cdot \sqrt{N!}}{(2\pi)^{N/2}} \cdot exp\Big(-\Big(\frac{\|y\|^2}{2} + \sum_{i<j}\frac{(y_i-y_j)^2}{(r_i-r_j)^2} + \sum_{i<j}\frac{(y_i+y_j)^2}{(r_i+r_j)^2} + \nu\sum_{i=1}^N \frac{y_i^2}{r_i^2}\Big)\Big), \tag{3.22}$$

which is a probability measure by our results. This measure is necessarily the normal distribution claimed in the theorem with the correct normalization. This completes the proof. □

Notice that the final arguments in the proof above for $x = 0$ about the correct normalizations above automatically lead to the following remarkable result on the zeros of the Laguerre polynomial $L_N^{(\nu-1)}$:

Corollary 3.4. *For $N \in \mathbb{N}$ and $\nu > 0$ consider the ordered zeros $z_1^{(\nu-1)} \geq \ldots \geq z_N^{(\nu-1)} > 0$ of the Laguerre polynomial $L_N^{(\nu-1)}$. Let $r_i := \sqrt{2z_i^{(\nu-1)}}$ for $i = 1, \ldots, N$, and form the matrix $S := (s_{i,j})_{i,j=1,\ldots,N}$ with*

$$s_{i,j} := \begin{cases} 1 + \frac{2\nu}{r_i^2} + 2\sum_{l\neq i}(r_i - r_l)^{-2} + 2\sum_{l\neq i}(r_i + r_l)^{-2} & \text{for} \quad i = j \\ 2(r_i + r_j)^{-2} - 2(r_i - r_j)^{-2} & \text{for} \quad i \neq j \end{cases} . \tag{3.23}$$

Then $\det S = N! \cdot 2^N$.

For further details on the matrix S we refer to [AV2].

Remark 3.5. In the case that the processes start in the origin, Theorem 3.3 was derived in [DE2] via the tridiagonal matrix models for β-Laguerre ensembles of Dumitriu and Edelman where in Theorem 4.1 of [DE2] a direct, quite complicated formula for the covariance matrix Σ of the limit in terms of the zeros of $L_N^{(\nu-1)}$ and the Laguerre polynomials $L_l^{(\nu-1)}$ $(l = 1, \ldots, N)$ is given. As in the Hermite case in Section 2 it is unclear how the expression for Σ in [DE2] corresponds to our formula for Σ^{-1} above.

Remark 3.6. If we analyze the constants in the end of the proof of Theorem 3.3 for arbitrary $x \in C_N^B$, we obtain that

$$e^{-\|x\|^2/2} \cdot exp\Big(\frac{\|x\|^2\|r\|^2}{4N(N+\nu-1)}\Big) = 1$$

with the vector r from Lemma 3.1. This means that $\|r\|^2 = 2N(N+\nu-1)$. In fact, if we translate this equation via (3.5) into a corresponding formula for the zeros of $L_N^{(\nu-1)}$, then we just obtain Eq. (C.10) in [AKM2].

Theorem 3.3 can be easily extended from fixed starting points $x \in C_N^B$ to arbitrary starting distributions $\mu \in M^1(C_N^B)$:

Corollary 3.7. *Let $\mu \in M^1(C_N^B)$ be an arbitrary starting distribution and $\nu > 0$. Consider the Bessel processes $(X_{t,(\nu\beta,\beta)})_{t\geq 0}$ of type B on C_N^B with starting distribution μ. Then, for each $t > 0$ and with the vector $r \in C_N^B$ and the normal distribution $N(0, t\cdot\Sigma)$ of Theorem 3.3,*

$$\frac{X_{t,(\nu\beta,\beta)}}{\sqrt{t}} - \sqrt{\beta}\cdot r \to N(0, t\cdot\Sigma) \quad \textit{in distribution for} \quad \beta \to \infty.$$

Proof. Let $f \in C_b(\mathbb{R}^N)$ be a bounded continuous function, and let $t > 0$ fixed. Using the kernels $K_{t,\beta}$ of (3.2), we obtain for the distributions $P_\beta \in M^1(\mathbb{R}^N)$ of

$X_{t,(\nu\beta,\beta)} - \sqrt{\beta t}\cdot r$ that

$$\int_{\mathbb{R}^N} f\,dP_\beta = \int_{C_N^B}\Big(\int_{C_N^B} f(y-\sqrt{\beta t}\cdot r)\;K_{t,(\nu\beta,\beta)}(x,dy)\Big)d\mu(x)$$
$$=: \int_{C_N^B}(T_\beta f)(x)\,d\mu(x)$$

where $(T_\beta f)(x)\to\int_{\mathbb{R}^N} f\,dN(0,t\cdot\Sigma)$ holds for $\beta\to\infty$ and all $x\in C_N^B$ by Theorem 3.3. As $\|T_\beta f\|_\infty\le\|f\|_\infty$, dominated convergence shows that

$$\int_{\mathbb{R}^N} f\,dP_\beta \to \int_{\mathbb{R}^N} f\,dN(0,t\cdot\Sigma)$$

for all $f\in C_b(\mathbb{R}^N)$. This proves the claim. □

4. A second central limit theorem for the root system B_N

In this section we derive a second CLT for Bessel processes of type B with parameters $k=(k_1,k_2)$. We here fix k_2 and consider the case $k_1\to\infty$. We proceed similar to the preceding section for the first CLT in the case B and include the case that we have an arbitary fixed starting point $x\in C_N^B$.

In order to handle arbitrary starting points x, we need the following asymptotic estimation for the Bessel functions of type B. For the proof we refer to Lemma 7 in [AKM2] where a slightly different notation with $(\nu\beta,\beta)=(k_1,k_2)$ is used, and where in the right hand side limit in Lemma 7 in [AKM2] the terms $\sqrt{2}$ have to be replaced by 2:

Lemma 4.1. *Let $k_2>0$. Then,*

$$\lim_{k_1\to\infty} J^B_{(k_1,k_2)}(\sqrt{k_1}x,y) = J^A_{k_2}(x^2/2,y^2/2)$$

locally uniformly in $x,y\in C_N^B$ where we use the notation $x^2 =: (x_1^2,...,x_N^2)\in C_N^B$.

We notice that Lemma 4.1 was derived for $x,y\in i\cdot\mathbb{R}^N$ with precise estimates for the rate of convergence in [RV3].

We now turn to the main result of this section, a CLT where the limit distribution is a distribution of type A as studied in Section 2:

Theorem 4.2. *For any fixed starting point $x\in C_N^B$ consider the Bessel processes $(X_{t,(k_1,k_2)})_{t\ge0}$ of type B_N on C_N^B. Consider the vector $\mathbf{1} := (1,...,1)\in\mathbb{R}^N$. Then, for all $k_2>0$ and $t>0$,*

$$X_{t,(k_1,k_2)} - \sqrt{2tk_1}\cdot\mathbf{1}$$

converges for $k_1\to\infty$ in distribution to $X^A_{t/2,k_2}$, where $(X^A_{s,k_2})_{s\ge0}$ is a Bessel process of type A starting in the origin.

Proof. As in the proofs of the preceding CLTs we may assume that $t=1$ by the same self-similarity property of all involved processes. We recapitulate that $X_{1,(k_1,k_2)}$ has the density

$$c^B_{(k_1,k_2)}e^{-\|x\|^2/2-\|y\|^2/2}\cdot J^B_{(k_1,k_2)}(x,y)\cdot\prod_{i<j}(y_i^2-y_j^2)^{2k_2}\cdot exp\left(2k_1\sum_{i=1}^N \ln y_i\right)$$

for y in the interior of C_N^B, where this density is equal to 0 elsewhere on $\mathbb{R}^N$, and where we use the notations from Section 3. Hence, $X_{1,(k_1,k_2)} - \sqrt{2k_1}\cdot \mathbf{1}$ has the density

$$\begin{aligned} f^B_{(k_1,k_2)}(y) :&= c^B_{(k_1,k_2)} \cdot e^{-\|x\|^2/2}\cdot e^{\frac{-\|y+\sqrt{2k_1}\cdot\mathbf{1}\|^2}{2}} \cdot J^B_{(k_1,k_2)}(x, y+\sqrt{2k_1}\cdot\mathbf{1})\cdot \\ &\cdot \prod_{i<j}((y_i+\sqrt{2k_1})^2-(y_j+\sqrt{2k_1})^2)^{2k_2}\cdot exp\left(2k_1\sum_{i=1}^N \ln\left(y_i+\sqrt{2k_1}\right)\right) \\ &=: c(k_1,k_2,x,N)\cdot\prod_{i<j}(y_i-y_j)^{2k_2}\tilde f_{k_1,k_2,x}(y) \end{aligned}$$

with

$$\begin{aligned} \tilde f_{k_1,k_2,x}(y) :=& \prod_{i<j}\left(\frac{y_i+y_j+2\sqrt{2k_1}}{2\sqrt{2k_1}}\right)^{2k_2}\cdot J^B_{(k_1,k_2)}(x,y+\sqrt{2k_1}\cdot\mathbf{1})\times \\ &\times exp\left(-\|y\|^2/2-\sqrt{2k_1}\cdot\sum_{i=1}^N y_i+2k_1\sum_{i=1}^N\ln(1+\frac{y_i}{\sqrt{2k_1}})\right) \end{aligned}$$

on the shifted cone $C_N^B - \sqrt{2k_1}\cdot\mathbf{1}$, where $f^B_{(k_1,k_2)}(y)=0$ otherwise on $\mathbb{R}^N$. Here $c(k_1,k_2,x,N)>0$ is some constant which may be computed explicitly.

As in the proofs of the preceding CLTs we conclude from the Taylor expansion of $\ln(1+z)$ that

$$\ln\left(1+\frac{y_i}{\sqrt{2k_1}}\right)=\frac{y_i}{\sqrt{2k_1}}-\frac{y_i^2}{4k_1}+O(k_1^{-3/2}).$$

This shows that

$$exp\left(-\sqrt{2k_1}\cdot\sum_{i=1}^N y_i+2k_1\sum_{i=1}^N\ln(1+\frac{y_i}{\sqrt{2k_1}})\right)=exp\left(-\sum_{i=1}^N\frac{y_i^2}{2}+O(k_1^{-1/2})\right). \tag{4.1}$$

Moreover, we observe from Lemma 4.1 and Eq. (2.14) that

$$\begin{aligned} \lim_{k_1\to\infty} J^B_{(k_1,k_2)}(x,y+\sqrt{2k_1}\cdot\mathbf{1}) &= \lim_{k_1\to\infty} J^B_{(k_1,k_2)}(x,\sqrt{2k_1}(\mathbf{1}+y/\sqrt{2k_1})) \\ &= J^A_{k_2}(x^2,\frac{1}{2}\cdot\mathbf{1}) = e^{\|x\|^2/2}. \end{aligned} \tag{4.2}$$

In summary, we obtain for all $x\in C_N^B$ and all y in the limit set C_N^A of the shifted cones $C_N^B-\sqrt{2k_1}\cdot\mathbf{1}$ for $k_1\to\infty$ that

$$\lim_{k_1\to\infty}\tilde f_{k_1,k_2,x}(y)=e^{\|x\|^2/2-\|y\|^2}.$$

We next check with dominated convergence that for any $h\in C_b(\mathbb{R}^n)$,

$$\begin{aligned} \lim_{k_1\to\infty}\int_{C_N^B-\sqrt{2k_1}\cdot\mathbf{1}} & h(y)\cdot\prod_{i<j}(y_i-y_j)^{2k_2}\cdot\tilde f_{k_1,k_2,x}(y)\,dy \\ &= \int_{C_N^A} h(y)\cdot\prod_{i<j}(y_i-y_j)^{2k_2}\cdot e^{\|x\|^2/2-\|y\|^2}\,dy. \end{aligned} \tag{4.3}$$

In order to check the assumptions of dominated convergence, we first again notice that for all $y \in C_N^B$ by Lagrange remainder there exists $w \in [0,1]$ such that

$$\ln\left(1+\frac{y_i}{\sqrt{2k_1}}\right) = \frac{y_i}{\sqrt{2k_1}} - \frac{y_i^2}{4k_1}\cdot w \tag{4.4}$$

which ensures that

$$exp\left(-\|y\|^2/2 - \sqrt{2k_1}\cdot\sum_{i=1}^N y_i + 2k_1\sum_{i=1}^N \ln(1+\frac{y_i}{\sqrt{2k_1}})\right)$$

can be bounded above by $e^{-\|y\|^2/2}$. Moreover, as in Section 3, we know from [RV2] that the Bessel functions $J_{(k_1,k_2)}^B$ admit the estimate

$$J_{(k_1,k_2)}^B(x, y+\sqrt{2k_1}\cdot\mathbf{1}) \le \exp(\|x\|\cdot(\|y\|+\sqrt{2k_1}\cdot\|\mathbf{1}\|)). \tag{4.5}$$

This shows that

$$J_{(k_1,k_2)}^B(x, y+\sqrt{2k_1}\cdot\mathbf{1}) \le e^{2\|x\|\cdot\|y\|} \qquad \text{for } k_1>0,\ y \text{ with } \ \|y\| \ge \sqrt{2k_1}\cdot\|\mathbf{1}\|. \tag{4.6}$$

On the other hand, if $\|y\| \le \sqrt{2k_1}\cdot\|\mathbf{1}\|$, then $y/\sqrt{2k_1}+\mathbf{1}$ is contained in some fixed compactum $C\subset\mathbb{R}^N$. We thus obtain from Lemma 4.1 that

$$\sup_{y\in\mathbb{R}^N, k_1>0:\ \|y\|\le\sqrt{2k_1}\cdot\|\mathbf{1}\|} J_{(k_1,k_2)}^B(x, \sqrt{2k_1}(y/\sqrt{2k_1}+\mathbf{1}))$$

is finite. Furthermore,

$$\prod_{i<j}\left(\frac{y_i+y_j+2\sqrt{2k_1}}{2\sqrt{2k_1}}\right)^{2k_2}$$

can be estimated from above by some polynomial in y independent of $k_1 \ge 1$, and the additional factor $\prod_{i<j}(y_i-y_j)^{2k_2}$ is also polynomially growing in y. Taking all upper estimates into account, we find an upper bound of the form $e^{-\|y\|^2/2}P(y)$ with some polynomial P. This ensures by dominated convergence that (4.3) is in fact correct for all $h\in C_b(\mathbb{R}^n)$.

As the right hand side density

$$\prod_{i<j}(y_i-y_j)^{2k_2}\cdot e^{\|x\|^2/2-\|y\|^2} \qquad (y\in C_N^A)$$

in (4.3) is the density of the distribution of $X_{1/2,k_2}^A$ (with start in the origin) up to normalization constants depending on k_2, x, N, we conclude for $h=1$, that also the constants converge as needed for the theorem, and that the theorem holds. □

Remark 4.3. We expect that the CLT 4.2 can be derived for Bessel processes $(X_{t,(k_1,k_2)})_{t\ge0}$ of type B with start in the origin $x=0$ also from the tridiagonal matrix models of Dumitriu and Edelman [DE1] similar to the proofs of the CLTs in [DE2] which correspond to the CLTs in Sections 1 and 2 above for $x=0$.

Remark 4.4. The CLT 4.2 for the starting point 0 should be compared with the CLT 4.1 in [AV1] where also Bessel processes $(X_{t,(k_1,k_2)})_{t\ge0}$ of type B are studied with k_2 fixed, $k_1\to\infty$, where the starting points have the form $\sqrt{k_1}\cdot x$ with some fixed point x in the interior on C_N^B. In Theorem 4.1 of [AV1], the limit distribution is an N-dimensional normal distribution with a covariance matrix depending on x. This limit is quite different from the limit in Theorem 4.2 above for $x=0$. This means that the assertion of Theorem 4.1 of [AV1] cannot be extended "continuously" from the interior of C_N^B to the origin. This is also clear by simple geometric considerations about the support of the limit measure. It might be interesting to study CLTs similar to Theorem 4.1 of [AV1] with starting points of the form $\sqrt{k_1}\cdot x$ with x on the boundary of C_N^B, but $x\neq 0$.

For particular values of k_2, namely $k=1/2,1,2$, Theorem 4.2 above has some matrix-theoretic, or "geometric", background:

Remark 4.5. Fix one of the (skew-)fields $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$ with the real dimension $d=1,2,4$ respectively. For integers $p\in\mathbb{N}$ consider the vector spaces $M_{p,N}(\mathbb{F})$ of all $p\times N$-matrices over $\mathbb{F}$ with the real dimension dpN. Choose the standard bases on these vector spaces such that we have d basis vectors in each entry. Consider the dpN-dimensional associated Brownian motion $(B_t^p)_{t\geq 0}$ on $M_{p,N}(\mathbb{F})$ starting in the origin. If we write $A^*:=\overline{A}^T\in M_{N,p}(\mathbb{F})$ for matrices $A\in M_{p,N}(\mathbb{F})$ with the usual conjugation on $\mathbb{F}$, then the process $(Z_t^p:=(B_t^p)^*B_t^p)_{t\geq 0}$ becomes a Wishart process on the closed cone $\Pi_N(\mathbb{F})$ of all $N\times N$ positive semidefinite matrices over $\mathbb{F}$ with shape parameter p; see [Bru, DDMY] and references there for details on Wishart processes.

Consider the spectral mapping $\sigma_N:\Pi_N(\mathbb{F})\to C_N^B$ which assigns to each matrix in $\Pi_N(\mathbb{F})$ its ordered spectrum. It is well-known that then $(\sqrt{\sigma_N(Z_t^p)})_{t\geq 0}$ is a Bessel process on C_N^B of type B_N with multiplicities

$$(k_1,k_2):=((p-N+1)\cdot d/2, d/2)$$

where the symbol $\sqrt{.}$ means taking square roots in each component; see e.g. [BF, R3] for details.

We thus conclude that Theorem 4.2 for $k_2=1/2,1,2$ corresponds to a CLT for Wishart distributions on $\Pi_N(\mathbb{F})$ with fixed time parameters where the shape parameters p tend to ∞. To explain this CLT on the level of matrices, we recapitulate that the distributions $\mu_t^p:=P_{Z_t^p}\in M^1(\Pi_N(\mathbb{F}))$ of Z_t^p satisfy $\mu_t^{p_1}*\mu_t^{p_2}=\mu_t^{p_1+p_2}$ for $p_1,p_2\in\mathbb{N}$ with the usual convolution of measures on the vector space $\mathbb{H}_N(\mathbb{F})$ of all $N\times N$ Hermitian matrices over $\mathbb{F}$ by the very construction of the random variables Z_t^p. Moreover, this convolution relation even remains valid for all $p\in]0,\infty[$ which are sufficiently large. We thus may apply the classical LLNs and CLT for sums of iid random variables on finitely dimensional vector spaces to obtain LLNs and a CLT for Wishart distributions for $p\to\infty$. A computation in this setting for the CLT shows that here the centering (on the vector space $\mathbb{H}_N(\mathbb{F})$) is performed with a multiple of the identity matrix. Also the covariance matrices the associated centered limit normal distributions on $\mathbb{H}_N(\mathbb{F})$ can be determined explicitely. For this we notice that the Gaussian limit on $\mathbb{H}_N(\mathbb{F})$ (before or after centering) is invariant under all conjugations in $U_N(\mathbb{F})$ (acting on $\mathbb{H}_N(\mathbb{F})$) as this is the case for the Wishart distributions above by their construction. Moreover, it is clear by Theorem 4.2 that the image measures of the centered limit normal distributions on $\mathbb{H}_N(\mathbb{F})$ under the spectral map $\sigma_N:\mathbb{H}_N(\mathbb{F})\to C_N^A$ are just the distributions with densities

$$c(k,t,N)\cdot e^{-\|y\|/t}\cdot w_k^A(y)\,dy \tag{4.7}$$

of type A on C_N^A as in Theorem 4.2 above with some normalization constants $c(k,t,N)>0$. As these measures are just the spectral distributions of Gaussian orthogonal/unitary/symplectic ensembles respectively up to time scaling, it follows that up to this scaling, the centered Gaussian limits on $\mathbb{H}_N(\mathbb{F})$ are just equal to the distributions of Gaussian orthogonal/unitary/symplectic ensembles.

In summary, for $k_2 = 1/2, 1, 2$, Theorem 4.2 corresponds to a CLT for Wishart distributions on $\mathbb{H}_N(\mathbb{F})$ where the shape parameters tend to ∞.

We finally remark that in the setting of Remark 4.5, Theorem 4.2 for $k_2 = 1/2, 1, 2$ is also related to limit theorems for radial random walks $(X_n^p)_{n\geq 0}$ on the vector spaces $M_{p,N}(\mathbb{F})$ and their projections to $\Pi_N(\mathbb{F})$ and C_N^B, when the dimension parameter p as well as the time parameter n tend to ∞ in a coupled way; see [G, RV4, V].

As in the end of Section 3, Theorem 4.2 can be easily extended from arbitrary, but fixed starting points to arbitrary starting distributions on C_N^B. As the proof is the same as for Corollary 3.7, we omit the proof.

Corollary 4.6. *Let $\mu \in M^1(C_N^B)$ be an arbitrary starting distribution. Consider the Bessel processes $(X_{t,(k_1,k_2)})_{t\geq 0}$ of type B on C_N^B with starting distribution μ. Then, for all $k_2 > 0$ and $t > 0$,*

$$X_{t,(k_1,k_2)} - \sqrt{2tk_1} \cdot \mathbf{1}$$

converges for $k_1 \to \infty$ in distribution to $X^A_{t/2,k_2}$, where $(X^A_{s,k_2})_{s\geq 0}$ is a Bessel process of type A starting in the origin.

5. A central limit theorem for the root system D_N

We next briefly study limit theorems for Bessel processes of type D_N. We recapitulate that the root system is given here by

$$D_N = \{\pm e_i \pm e_j : \quad 1 \leq i < j \leq N\}$$

with associated closed Weyl chamber

$$C_N := C_N^D = \{x \in \mathbb{R}^N : \quad x_1 \geq \ldots \geq x_{N-1} \geq |x_N|\}.$$

C_N^D may be seen as a doubling of C_N^B w.r.t. the last coordinate. We have a one-dimensional multiplicity $k \geq 0$. The generator of the transition semigroup of the Bessel process $(X_{t,k})_{t\geq 0}$ of type D is

$$Lf := \frac{1}{2}\Delta f + k \sum_{i=1}^{N} \sum_{j\neq i} \Big(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \Big) \frac{\partial}{\partial x_i} f, \tag{5.1}$$

where we again assume reflecting boundaries. Similar to the preceding cases, we have the transition probabilities

$$K_{t,k}(x, A) = c_k^D \int_A \frac{1}{t^{\gamma_D + N/2}} e^{-(\|x\|^2 + \|y\|^2)/(2t)} J_k^D(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}) \cdot w_k^D(y)\, dy \tag{5.2}$$

with

$$w_k^D(x) := \prod_{i<j} (x_i^2 - x_j^2)^{2k}, \quad \gamma_D := kN(N-1);$$

see [Dem] for further details on this root system. Furthermore, using the normalization (3.4) for $k_2 = k, k_1 = 0$, we see that the normalization constant c_k^D is given by

$$c_k^D := \Big(\int_{C_N^D} e^{-\|y\|^2/2} w_k^D(y)\, dy\Big)^{-1} \tag{5.3}$$
$$= \frac{N!}{2^{N(N-1)k-N/2+1}} \cdot \prod_{j=1}^N \frac{\Gamma(1+k)}{\Gamma(1+jk)\Gamma(\frac{1}{2}+(j-1)k)}.$$

We now proceed similarly to Section 4 of [AV1]. Using the known explicit representation

$$L_N^{(\alpha)}(x) := \sum_{k=0}^N \binom{N+\alpha}{N-k} \frac{(-x)^k}{k!}$$

of the Laguerre polynomials according to (5.1.6) of [S], we can form the polynomial $L_N^{(-1)}$ of order $N \ge 1$ where, by (5.2.1) of [S],

$$L_N^{(-1)}(x) = -\frac{x}{N} L_{N-1}^{(1)}(x). \tag{5.4}$$

Continuity arguments thus show that the equivalence of (2) and (3) of Lemma 3.1 remains valid also for $\nu = 0$ by using the N different zeros $z_1 > \ldots > z_N = 0$ of $L_N^{(-1)}$. With these notations we have the following fact which is similar to the results above in the cases A and B; see Section 4 of [AV1]:

Lemma 5.1. *For $r \in C_N^D$, the following statements are equivalent:*

(1) *The function $W_D(y) := 2\sum_{i<j} \ln(y_i^2 - y_j^2) - \|y\|^2/2$ is maximal in $r \in C_N^B$;*
(2) *$r_N = 0$, and for $i = 1, \ldots, N-1$,*

$$4 \sum_{j: j\ne i} \frac{1}{r_i^2 - r_j^2} = 1;$$

(3) *If $z_1^{(1)} > \ldots > z_{N-1}^{(1)} > 0$ are the $N-1$ ordered zeros of the classical Laguerre polynomial $L_{N-1}^{(1)}$, then*

$$2 \cdot (z_1^{(1)}, \ldots, z_{N-1}^{(1)}, 0) = (r_1^2, \ldots, r_N^2). \tag{5.5}$$

We now turn to the main result of this section:

Theorem 5.2. *Consider the Bessel processes $(X_{t,k})_{t\ge 0}$ of type D_N on C_N^D with multiplicity $k > 0$ which start in 0. Then, for the vector $r \in C_N^D$ of Lemma 5.1,*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{k} \cdot r$$

converges for $k \to \infty$ to the centered N-dimensional distribution $N(0, t\cdot\Sigma)$ with the regular covariance matrix Σ with $\Sigma^{-1} = (s_{i,j})_{i,j=1,\ldots,N}$ with

$$s_{i,j} := \begin{cases} 1 + 2\sum_{l\ne i}(r_i - r_l)^{-2} + 2\sum_{l\ne i}(r_i + r_l)^{-2} & \text{for } i = j \\ 2(r_i + r_j)^{-2} - 2(r_i - r_j)^{-2} & \text{for } i \ne j \end{cases}. \tag{5.6}$$

Proof. As in the preceding cases we assume that $t = 1$ without loss of generality. $X_{1,k}$ has the density

$$c_k^D e^{-\|y\|^2/2} \cdot exp\Big(2k \sum_{i<j} \ln(y_i^2 - y_j^2)\Big)$$

on C_N^D. Hence, $X_{1,k} - \sqrt{k} \cdot r$ has the density

$$\begin{aligned} f_k^D(y) :=& c_k^D e^{-\|y+\sqrt{k}\cdot r\|^2/2} exp\Big(2k\sum_{i<j} \ln\big((y_i+\sqrt{k}\cdot r_i)^2 - (y_j+\sqrt{k}\cdot r_j)^2\big)\Big) \\ =& c_k^D exp\Big(2k\sum_{i<j}\ln\big(1+\frac{y_i-y_j}{\sqrt{k}(r_i-r_j)}\big) + 2k\sum_{i<j}\ln\big(1+\frac{y_i+y_j}{\sqrt{k}(r_i+r_j)}\big)\Big)\times \\ & e^{-\|y\|^2/2} e^{-k\|r\|^2/2} e^{-\sqrt{k}\langle y,r\rangle} exp\Big(2k\sum_{i<j}\Big(\ln(\sqrt{k}(r_i-r_j)) + \ln(\sqrt{k}(r_i+r_j))\Big)\Big) \end{aligned}$$

on the shifted cone $C_N^D - \sqrt{k}\cdot r$, with $f_k^D(y) = 0$ elsewhere on $\mathbb{R}^N$. We again split $f_k^D(y)$ into

$$f_k^D(y) = \tilde{c}_k \cdot h_k(y),$$

where $h_k(y)$ depends on y, and $\tilde{c}_k$ is constant w.r.t. y. The part depending on y is given by

$$h_k(y) := exp\Big(-\|y\|^2/2-\sqrt{k}\langle y,r\rangle+2k\sum_{i<j}\Big(\ln\big(1+\frac{y_i-y_j}{\sqrt{k}(r_i-r_j)}\big)+\ln\big(1+\frac{y_i+y_j}{\sqrt{k}(r_i+r_j)}\big)\Big)\Big).$$

The expansion of $\ln(1+x)$ together with Lemma 5.1 yields as in the proof of Theorem 3.3 that

$$\lim_{k\to\infty} h_k(y) = exp\Big(-\frac{\|y\|^2}{2} - \sum_{i<j}\frac{(y_i-y_j)^2}{(r_i-r_j)^2} - \sum_{i<j}\frac{(y_i+y_j)^2}{(r_i+r_j)^2}\Big). \tag{5.7}$$

Moreover, as in the proofs of Theorems 3.3 and 2.2, we see that for all $f \in C_b(\mathbb{R}^N)$,

$$\begin{aligned} &\lim_{k\to\infty}\int_{\mathbb{R}^N} f(y)\cdot h_k(y)\,dy = \qquad (5.8)\\ &= \int_{\mathbb{R}^N} f(y)\, exp\Big(-\Big(\frac{\|y\|^2}{2} + \sum_{i<j}\frac{(y_i-y_j)^2}{(r_i-r_j)^2} + \sum_{i<j}\frac{(y_i+y_j)^2}{(r_i+r_j)^2}\Big)\Big)\,dy. \end{aligned}$$

This implies the theorem as in the proof of Theorem 3.3. □

Let $(X_{t,k}^D)_{t\ge 0}$ be a Bessel process of type D with multiplicity $k \ge 0$ on the chamber C_N^D. Then the process $(X_{t,k}^B)_{t\ge0}$ with

$$X_{t,k}^{B,i} := X_{t,k}^{D,i} \quad (i=1,\dots,N-1), \quad X_{t,k}^{B,N} := |X_{t,k}^{D,N}|$$

is a Bessel process of type B with the multiplicity $(k_1,k_2) := (0,k)$. This follows easily from a comparison of the corresponding generators and was also observed in [AV1]. The central limit theorem 5.2 for $(X_{t,k}^D)_{t\ge0}$ thus leads to the following "one-sided CLT" for Bessel processes of type B with the multiplicities $(0,k)$ for $k\to\infty$:

Corollary 5.3. *Consider the Bessel processes $(X_{t,k})_{t\ge0}$ of type B_N on C_N^B with multiplicities $(0,k)$ which start in 0. Then, for the vector r of Lemma 5.1 on the boundary of C_N^B,*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{k}\cdot r$$

converges for $k\to\infty$ in distribution to the "one-sided normal distribution" on the half space

$$H_N := \{x\in\mathbb{R}^N : x_N \ge 0\}$$

which appears as image of the centered N-dimensional distribution $N(0, t\cdot\Sigma)$ with the regular covariance matrix Σ as in Theorem 5.2 under the mapping

$$\mathbb{R}^N \longrightarrow H_N, \quad (x_1,\ldots,x_N)\mapsto (x_1,\ldots,x_{N-1},|x_N|).$$

6. A third central limit theorem for the root system B_N

In this section we again study Bessel processes of type B with multiplicities $k=(k_1,k_2)$. We here fix $k_1>0$ and consider $k_2\to\infty$. As far as we know, this limit was not considered in the literature up to now. It will turn out that this case is closely related to the limits in the case D above and in particular to the B-case for the multiplicities $(0,k_2)$ for $k_2\to\infty$ in Corollary 5.3.

We start with a law of large numbers which corresponds to limit results of [AKM1], [AKM2] for the root systems of type A and B. For this we fix $k_1>0$ and consider the densities $f_{t,k}$ of $X_{t,k}/\sqrt{tk_2}$ of the Bessel processes $(X_{t,k})_{t\ge 0}$ of type B_N on C_N^B where we suppose that the processes start in the origin. Then, by the scaling properties of the $X_{t,k}$ and by (3.2), these densities are independent of $t>0$ and have the form

$$f_{1,k}(x)=const(k)\cdot\prod_{i=1}^N x_i^{2k_1}\cdot exp\big(k_2W(x)\big)$$

with some normalization constant $const(k)>0$ (see Section 2 for the details), and with

$$W(x)=-\|x\|^2/2+2\sum_{i<j}\ln(x_i^2-x_j^2) \qquad \text{for} \quad \mathrm{x}\in C_N^B.$$

We know from Lemma 5.1 that on C_N^B, the function W admits a unique maximum which is located at $r=(r_1,\ldots,r_N)\in C_N^B$ with

$$(r_1^2,\ldots,r_N^2)=2\cdot(z_1^{(1)},\ldots,z_{N-1}^{(1)},0)$$

for the $N-1$ ordered zeros $z_1^{(1)}>\ldots>z_{N-1}^{(1)}>0$ of $L_{N-1}^{(1)}$. This optimal point r is in the support of the measure $\mathbf{1}_{C_N^B}\cdot\prod_{i=1}^N x_i^{2k_1}\,dx$. Some standard arguments from analysis now readily lead to the following limit law.

Proposition 6.1. *For each $k_1>0$ and $t>0$, $X_{t,(k_1,k_2)}/\sqrt{tk_2}$ converges for $k_2\to\infty$ in distribution to the point measure δ_r for $r\in C_N^B$ as above.*

We here skip details of the proof, as this proposition follows immediately from the following associated CLT which is analog to the CLTs in the previous sections and in particular to Corollary 5.3. Notice that the proof of Theorem 6.2 does not rely on Proposition 6.1.

Theorem 6.2. *Consider the Bessel processes $(X_{t,k})_{t\ge0}$ of type B_N on C_N^B with multiplicities (k_1,k_2) which start in 0. Fix $k_1\ge 0$ and $t>0$. Then, for $k_2\to\infty$,*

$$\frac{X_{t,(k_1,k_2)}}{\sqrt{t}}-\sqrt{k_2}\cdot r$$

converges in distribution to the "one-sided normal distribution" on the half space $H_N:=\{x\in\mathbb{R}^N: x_N\ge 0\}$ which appears in Corollary 5.3.

Proof. As in the preceding cases we assume $t = 1$. The case $k_1 = 0$ is shown in Corollary 5.3.

The case $k_1 > 0$ can be proved similarly as in Theorem 5.2. In fact, the shifted random variables $X_{1,(k_1,k_2)} - \sqrt{k_2} \cdot r$ have densities of the form

$$\begin{aligned} f_{(k_1,k_2)}(y) :=& \tilde{c}(k_1, k_2) \cdot \prod_{i=1}^{N} (y_i + \sqrt{k_2} \cdot r_i)^{2k_1} \cdot e^{-\|y+\sqrt{k_2}\cdot r\|^2/2} \\ & \times exp\Big(2k_2 \sum_{i<j} \ln\big((y_i + \sqrt{k_2} \cdot r_i)^2 - (y_j + \sqrt{k_2} \cdot r_j)^2\big)\Big) \\ =& c(k_1, k_2) \cdot \prod_{i=1}^{N} \left(\frac{y_i + \sqrt{k_2} \cdot r_i}{\sqrt{k_2} \cdot r_i} \right)^{2k_1} \cdot h_{k_2}(y) \end{aligned} \tag{6.1}$$

on the shifted cone $C_N^B - \sqrt{k_2} \cdot r$ with

$$\begin{aligned} h_{k_2}(y) := exp\Big(& - \|y\|^2/2 - \sqrt{k_2} \langle y, r \rangle + \\ & +2k_2 \sum_{i<j} \Big(\ln\big(1 + \frac{y_i - y_j}{\sqrt{k_2}(r_i - r_j)}\big) + \ln\big(1 + \frac{y_i + y_j}{\sqrt{k_2}(r_i + r_j)}\big)\Big)\Big) \end{aligned}$$

where $f_{(k_1,k_2)}(y) = 0$ otherwise on $\mathbb{R}^N$ and where $\tilde{c}(k_1, k_2), c(k_1, k_2) > 0$ are suitable constants. As in Theorem 5.2 we obtain that

$$\lim_{k_2\to\infty} h_{k_2}(y) = exp\Big(- \frac{\|y\|^2}{2} - \sum_{i<j} \frac{(y_i - y_j)^2}{(r_i - r_j)^2} - \sum_{i<j} \frac{(y_i + y_j)^2}{(r_i + r_j)^2} \Big). \tag{6.2}$$

On the other hand,

$$\left(\prod_{i=1}^{N} \frac{y_i + \sqrt{k_2} \cdot r_i}{\sqrt{k_2} \cdot r_i} \right)^{2k_1}$$

tends to 1 for $k_2 \to \infty$ and is bounded by an expression of the form

$$c(k_1) + d(k_1)(y_1 y_2 \ldots y_N)^{2k_1}$$

for all y independent of $k_2 \geq 1$ with suitable constants $c(k_1), d(k_1) > 0$. These (in y) polynomial bounds do not affect the fact that we still may apply dominated convergence theorem like in the proofs of Theorems 3.3, 2.2, and 5.2 due to the Gaussian bounds for $h_{k_2}(y)$ in the proofs there. Moreover, as the shifted cones $C_N^B - \sqrt{k_2} \cdot r$ converge to $\mathbb{R}^{N-1} \times [0, \infty[$ for $k_2 \to \infty$ due to $r_N = 0$, we thus conclude that for all $f \in C_b(\mathbb{R}^N)$,

$$\begin{aligned} \lim_{k_2\to\infty} \int_{C_N^B - \sqrt{k_2}\cdot r} f(y) \cdot \left(\prod_{i=1}^{N} \frac{y_i + \sqrt{k_2} \cdot r_i}{\sqrt{k_2} \cdot r_i} \right)^{2k_1} \cdot h_{k_2}(y)\, dy = \\ = \int_{\mathbb{R}^{N-1}\times[0,\infty[} f(y)\, exp\Big(- \Big(\frac{\|y\|^2}{2} + \sum_{i<j} \frac{(y_i - y_j)^2}{(r_i - r_j)^2} + \sum_{i<j} \frac{(y_i + y_j)^2}{(r_i + r_j)^2} \Big)\Big)\, dy. \end{aligned} \tag{6.3}$$

If we look into this formula for $f \equiv 1$, we obtain that the corresponding normalization constants converge as needed for our CLT, and the theorem follows. □

Remark 6.3. It can be easily seen that the limit relation for Bessel functions of type B in Lemma 3.2 is also available for $\nu = 0$ and $N \geq 2$. This shows that Theorem 6.2 can be extended from the starting point 0 to an arbitrary starting point $x \in C_N^B$. The details of the proof are the same as in the proofs of Theorems 3.3 and 4.2.

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