A Statistical Analysis of the Roulette Martingale System: Examples, Formulas and Simulations with R[1]

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Abstract
Some gamblers use a martingale or doubling strategy as a way of improving their chances of winning. This paper derives important formulas for the martingale strategy, such as the distribution, the expected value, the standard deviation of the profit, the risk of a loss or the expected bet of one or multiple martingale rounds. A computer simulation study with R of the doubling strategy is presented. The results of doubling to gambling with a constant sized bet on simple chances (red or black numbers, even or odd numbers, and low (1–18) or high (19–36) numbers) and on single numbers (straight bets) are compared. In the long run, a loss is inevitable because of the negative expected value. The martingale strategy and the constant bet strategy on a single number are riskier than the constant bet strategy on a simple chance. This higher risk leads, however, to a higher chance of a positive profit in the short term. But on the other hand, higher risk means that the losses suffered by doublers and by single number bettors are much greater than that suffered by constant bettors.

1. Introduction

The martingale system is a popular betting strategy in roulette: Each time a gambler loses a bet, he doubles his next bet, so that the eventual win leaves him with profit equal to his original stake. However, the martingale system only works safely in casinos without table limits and where the gambler has unlimited money. Both assumptions are not very likely. Therefore, the martingale strategy is considered extremely risky. High losses are possible, although the probability of such a loss is low. Various senses of the word "martingale" are reviewed by Mansuy (2009),

2. Martingale as a two-point distribution

It is assumed that the reader knows the casino game roulette. We regard an unbiased roulette, that is, we assume that each number of the roulette wheel is equally likely. We restrict our analysis to the European version of roulette with 37 numbers (with a single zero). However, the results can be easily transferred to the American version of roulette with 38 numbers (with double zeros). Further we first assume that if zero appears all bets on simple chances (red or black, even or odd, low or high) are lost. They are not halved (à partager) or imprisoned (en prison) according to the rule of some European casinos (see also Ethier, pp. 463–465). Finally, we assume that the gambler shall risk a finite capital.

Denoting by g the profit or gain from a one-unit bet on a simple chance and by $p = 19/37$ the probability of losing the bet, the expected value for the gambler is

$$E(g) = 1\cdot(1-p) - 1\cdot p = 1 - 2p = -0.027027\,.$$

And the variance is

$$Var(g) = 1^2\cdot(1-p) + (-1)^2\cdot p - (1-2p)^2 = 4p - 4p^2 = 4p\cdot(1-p) = 0.99927\,.$$

Let N=1,2,3…,n be the number of coups or spins needed to achieve the first win in a martingale. A martingale round consists of a number of N=1,2,…,n coups of consecutive losses followed by either a win, or the loss of the total bet after n coups or n-1 doublings when the table limit has been

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reached. After a win or the total loss, the gambler starts a new martingale. Table 1 shows an illustration of the martingale system with n=10 (table limit=512 units). The player wagers on red. The probability of losing is *p=19/37*. The amount of the initial bet shall be one unit. On each loss, the bet is doubled.

Table 1: Illustration of a martingale with n=10, p=19/37 (r=red, b=black & zero)

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
coup m	black		prob1	bet	cum. bet	gain G	E(G)	E(bet)	E(m)	$p^{(m-1)}$	prob2	$E(bet_c)$
							IV × VII	IV × VI	IV × I			V × XII
1	0	r	0.48649	1	1	1	0.48649	0.48649	0.48649	1.00000	0.48711	0.48711
2	1	br	0.24982	2	3	1	0.24982	0.74945	0.49963	0.51351	0.25014	0.50027
3	2	bbr	0.12828	4	7	1	0.12828	0.89799	0.38485	0.26370	0.12845	0.51379
4	3	bbbr	0.06588	8	15	1	0.06588	0.98814	0.26350	0.13541	0.06596	0.52768
5	4	bbbbr	0.03383	16	31	1	0.03383	1.04867	0.16914	0.06954	0.03387	0.54194
6	5	bbbbbr	0.01737	32	63	1	0.01737	1.09439	0.10423	0.03571	0.01739	0.55659
7	6	bbbbbbr	0.00892	64	127	1	0.00892	1.13288	0.06244	0.01834	0.00893	0.57163
8	7	bbbbbbbr	0.00458	128	255	1	0.00458	1.16808	0.03665	0.00942	0.00459	0.58708
9	8	bbbbbbbbr	0.00235	256	511	1	0.00235	1.20201	0.02117	0.00484	0.00236	0.60295
10	9	bbbbbbbbbr	0.00121	512	1023	1	0.00121	1.23570	0.02483	0.00248	0.00121	0.61924
10	10	bbbbbbbbbb	0.00128			-1023	-1.30435	1.30435				
sum			1				**-0.30563**	**11.30815**	**2.05293**	**2.05293**	**1**	**5.50829**

Remarks: $prob1 = p^{m-1}\left(1-p\right)$; $prob2 = \dfrac{p^{m-1}}{1+p+p^2+..+p^{n-1}}$.

E(G)=expected gain, E(bet)=expected bet, E(m)=expected number of coups of a martingale round; $E(bet_c)$=expected bet per coup of a martingale round; n=10: after 10 coups the table limit of 512 has been reached.

From Table 1 follows the presentation with formulas in Exhibit 1, where p≥0.5 is the probability of losing and the gambler might bet 1 unit on the first spin on red.

Exhibit 1: Illustration of a martingale with formulas (r=red, b=black & zero);

coup m	colour	probability	bet	cumulative bet	gain	
1	r	$\left(1-p\right)$	2^0	2^1-1	1	
2	br	$p\cdot\left(1-p\right)$	2^1	2^2-1	1	
3	bbr	$p^2\cdot\left(1-p\right)$	2^2	2^3-1	1	
n	bb…br	$p^{n-1}(1-p)$	2^{n-1}	2^n-1	1	
n	bbb…b	p^n			$-(2^n-1)$	(r does not show up)

The sum of the probabilities in the third column is $\sum_{i=1}^{n} p^{i-1}\left(1-p\right)+p^n=(1-p^n)+p^n=1$.

The probability that the gambler will lose all *n* bets is p^n. When all bets lose, the total loss is 2^n-1.

The probability that the gambler does not lose all *n* bets is 1-p^n. In all other cases, the gambler wins one unit. Thus, the expected profit or gain per martingale is

$$E(G) = \left(1-p^n\right)\cdot 1-\left(2^n-1\right)\cdot p^n = \sum_{i=1}^{n} 1\cdot p^{i-1}\left(1-p\right)-\left(2^n-1\right)\cdot p^n = 1-(2p)^n .$$

If $p = 19/37$, then the expectation is $E(G) = 1-\left(2\cdot\frac{19}{37}\right)^{10}$ = -0.3056.

The distribution of the gain or profit G_i in the i-th martingale round follows a two-point distribution, i.e.

$P\left(G_i = -\left(2^n-1\right)\right) = p^n$ and $P\left(G_i = 1\right) = 1-p^n$

with the expected value and the variance

$E(G_i) = -\left(2^n-1\right)\cdot p^n + 1\cdot\left(1-p^n\right) = 1-\left(2p\right)^n$,

$Var(G_i) = \left(4p\right)^n - \left(2p\right)^{2n}$.

With p=19/37 and n=10, one calculates the expected value and the variance as $E(G_i) = -0.3056$ and $Var(G_i) = 1335.7$. The standard deviation is $\sigma(G_i) = 36.54$.

The variance reduces to

$Var(G_i) = 2^n - 1$ if p=0.5. If n=10 then we obtain $Var(G_i) = 2^{10} - 1 = 1023$.

After M martingale rounds, the total profit will be $W = \sum_{i=1}^{M} G_i$. Expected value and variance of the total profit are $E(W) = M\cdot\left(1-\left(2p\right)^n\right)$ and $Var(W) = M\cdot\left(\left(4p\right)^n - \left(2p\right)^{2n}\right)$.

If X_i i=1,2,…,M is a random variable with a Bernoulli distribution with $P(X_i = 0) = p^n$ and $P(X_i = 1) = 1-p^n$, then the sum of M independent Bernoulli trials $X = \sum_{i=1}^{M} X_i$ has a binomial distribution $BIN\left(M, x, 1-p^n\right)$.

The linear transformation

$W = -\left(M-X\right)\cdot\left(2^n-1\right)+X = \left(X-M\right)\cdot 2^n + M$

has the same binomial distribution. X is the number of martingale rounds $x = 0,1,2......M$ which were successful and ended with a win of one unit. If we define the number of busts Y=M-X, where the martingale rounds are counted which ended with a loss, then we obtain $W = M - 2^n\cdot Y$.

The normal distribution can be used as an approximation to the binomial distribution of X or W if the following rule of thumb holds: $M\cdot\left(1-p^n\right)\cdot p^n > 9$.

The Poisson distribution can be used as an approximation of X or W with $\lambda = M\cdot p^n$ if n is not too small.

The probability that the total profit will be less than w is $P\left(W < w\right) \approx \phi\left(\frac{w-E(W)}{\sqrt{Var(W)}}\right)$, where ϕ is the distribution function of the standard normal distribution. Specifically, the probability for a loss is given by $P\left(W < 0\right) = \phi\left(\frac{-E(W)}{\sqrt{Var(W)}}\right)$.

The probability of a loss is a function of the number of played martingales if n and p are given.

$$P(W<0)=(1-\alpha)=\phi\left(\frac{-E(W)}{\sqrt{Var(W)}}\right)=\phi\left(\frac{-M_{1-\alpha}\cdot\left(1-(2p)^n\right)}{\sqrt{M_{1-\alpha}\cdot\left((4p)^n-(2p)^{2n}\right)}}\right)=\phi\left(-u_{1-\alpha}\right),$$

where $u_{1-\alpha}$ is the (1-α)-quantile of the standard normal distribution. Solving for $M_{1-\alpha}$ yields the required number of martingale games corresponding to a given loss or win probability

$$M_{1-\alpha}=u_{1-\alpha}^2\cdot\frac{(4p)^n-(2p)^{2n}}{\left((2p)^n-1\right)^2}=u_{1-\alpha}^2\cdot CV^2$$ with the coefficient of variation $CV=\frac{\sqrt{Var(G)}}{|E(G)|}$. In particular, $CV^2=M_{0.84}$.

With $n=10$ and $p=\frac{19}{37}$ we obtain:

Loss probability 1-α	Win probability α	Quantile $u_{1-\alpha}$	Number of martingales $M_{1-\alpha}$
0.8	0.2	0.8416	10,125
0.84	0.16	1	119.56^2=14,295
0.9	0.1	1.2816	23,480
0.95	0.05	1.6449	38,680
0.99	0.01	2.3263	78,026
0.999	0.001	3.0902	136,508

E.g., we recognise that the probability of a positive profit is only 1 percent after playing 78,026 martingale rounds or 160,182 expected coups or spins (see section 3).

Figure 1 shows the distribution of the total profit after M=10,000 martingale rounds (more detailed results are listed in the Appendix). The expected profit is E(W)=-3056.27, and the standard deviation is $\sigma_W=3655.66$. The probability of a positive profit is 18.28 percent if calculated with the binomial distribution. The approximation with the normal distribution yields about 20 percent. From Fig. 1 or more accurately from the table in the Appendix, we can observe that the probability of a loss of 10,480 units is about 1.5 percent. The probability that the loss is 10,480 units or higher is 3.55 percent. On the other side, the probability of a (positive) gain of 3,856 units or higher is only about 3 percent.

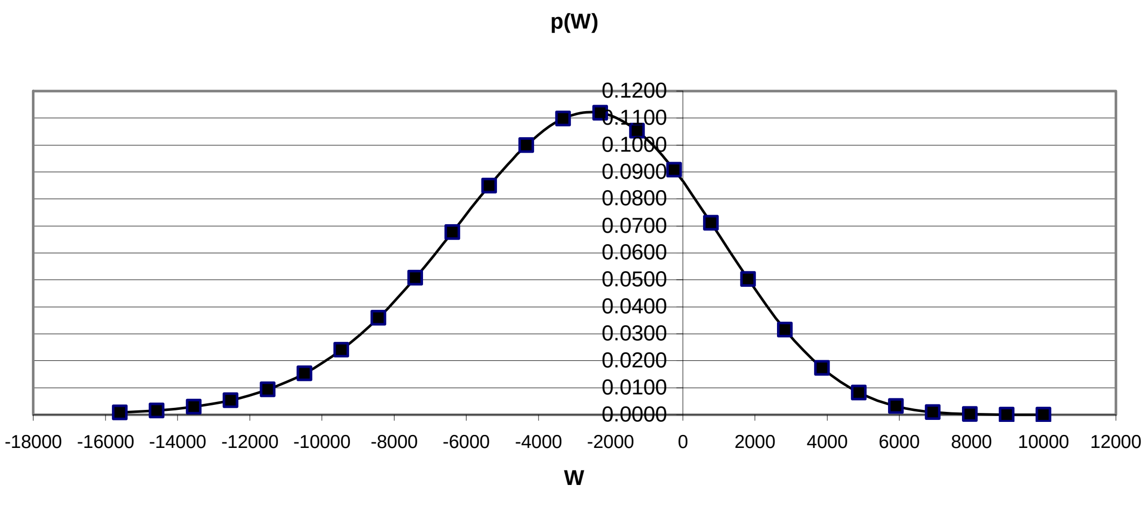

Figure 1: Profit distribution after 10,000 martingale rounds

Exhibit 2 shows the results for only 20 martingale rounds or about 41 coups. The win probability is very high. But the possible loss of 1,004 units considerably exceeds the possible total profit of only 20 units. Gamblers should be aware that the martingale play is a very risky strategy which could produce extremely high losses.

Exhibit 2: Results of playing 20 martingales

M	20	E(G)	-0.30562581
n	10	Var(G)	1335.6552
		S(G)	36.5466168
p	0.51351351		
$1-p^n$	0.99872497		
p^n	0.00127503	E(W)	-6.11251612
		Var(W)	26713.104
$\lambda = M \cdot p^n$	0.0255005	S(G)	163.441439

x (successes)	y (busts)	W	Binomial	Cum. Binomial	Poisson
20	0	20	0.9748	0.9748	0.9748
19	1	-1004	0.0249	0.9997	0.0249
18	2	-2028	0.0003	1.0000	0.0003
17	3	-3052	0.0000	1.0000	0.0000
.	.	.	.	.	.

3. Expected number of coups and expected gain and bet of a martingale round

The gambler shall put his bet always on red. If, e.g., red appears after two black colours (bbr), the martingale ends after 3 coups. The probability for this event is $p^2 \cdot (1-p)$, and the total amount bet is $2^3 - 1 = 1 + 2 + 4 = 7$. The maximum number of coups is n because of the table limit.

From Table 1 and Exhibit 2, we conclude that the expected number of coups is given by

$$E(m) = \sum_{i=1}^{n-1} i \cdot p^{i-1}(1-p) + n \cdot \left(p^{n-1}(1-p) + p^n\right) = \sum_{i=1}^{n} i \cdot p^{i-1}(1-p) + n \cdot p^n = \frac{1-p^n}{1-p}.$$

Note that $\sum_{0}^{n-1} p^i = \frac{1-p^n}{1-p}$.

After calculating the second moment $E(m^2)$, we find for the variance

$$Var(m) = \frac{p - p^{2n}}{(1-p)^2} - (2n-1) \cdot \frac{p^n}{1-p}.$$

If n is large, we can use the following approximations for the expected value and the variance (parameters of the geometric distribution):

$$E(m) = \frac{1}{1-p},$$

$$Var(m) = \frac{p}{(1-p)^2}.$$

Table 2 shows the parameters of the number of coups m within a martingale game as a function of the maximum rounds n due to the table limit. (p=19/37). Without a table limit, the expected value is 2.0555 and the variance is 2.17.

Table 2: Parameters of the number of coups

n	Expectation	Variance	Standard deviation
1	1	0	0
2	1.514	0.250	0.500
3	1.777	0.701	0.837
4	1.913	1.149	1.072
5	1.982	1.504	1.226
6	2.018	1.754	1.324
7	2.036	1.918	1.385
8	2.046	2.021	1.421
9	2.050	2.083	1.443
10	2.053	2.120	1.456

Let $m_i = 1, 2, ..n$ be the number of coups of the i-th martingale round. Then the total number of coups of a roulette game is $N = \sum_{i=1}^{M} m_i$ with $E(N) = M \cdot E(m)$ and $Var(N) = M \cdot Var(m)$. E(N) is the number of martingale rounds multiplied by the expected number of coups within a martingale round.

The expected total amount bet within a martingale round is given by (see Exhibit 1)

$$E(bet) = \sum_{i=1}^{n} \left(2^i - 1\right) \cdot p^{i-1}(1-p) + \left(2^n - 1\right) \cdot p^n = \frac{1-(2p)^n}{1-2p} = \frac{E(G)}{E(g)}.$$

If p=0.5, we get

$$E(bet) = \lim_{p \to 0.5} \frac{1-(2p)^n}{1-2p} = n.$$

Ethier (2010, p. 279) remarks that the ratio $\frac{E(G)}{E(bet)} = E(g) = (1-2p)$ corresponds to the expected profit from a single-unit bet. This is not coincidental. He shows that all systems have this property (see Ethier, 2010, p. 298 ff). “All betting systems lead ultimately to the same mathematical expectation of gain per unit amount wagered” (Epstein, 2009, p. 52).

The variance is given by

$$Var(bet) = p^n \cdot \left(\frac{3 \cdot 2^{(2n)}}{4p-1} + \frac{2^{(n+1)}}{1-2p} \right) + \frac{(2p+1)}{(2p-1)(4p-1)} - \left(\frac{1-(2p)^n}{1-2p} \right)^2 \quad p \neq 0.5$$

or

$$Var(bet) = 3 \cdot 2^n - n^2 - 2n - 3 \qquad \text{if } p = 0.5.$$

Since a martingale consists on average of $E(m) = \frac{1-p^n}{1-p}$ coups, we can conclude that the expected value of a bet per coup is

$$E(bet_c) = \frac{1-p}{1-p^n} \cdot \frac{1-(2p)^n}{1-2p} = \frac{E(G)}{E(g) \cdot E(m)} \qquad p \neq 0.5.$$

Table 3: Parameters of the bet per martingale round as a function of n

	p=19/37			p=0.5		
n	Expected value	Variance	Standard deviation	Expected value	Variance	Standard deviation
1	1.000	0.000	0.000	1	0	0.000
2	2.027	0.999	1.000	2	1	1.000
3	3.082	6.158	2.482	3	6	2.449
4	4.165	22.140	4.705	4	21	4.583
5	5.278	62.813	7.925	5	58	7.616
6	6.420	156.854	12.524	6	141	11.874
7	7.594	363.378	19.062	7	318	17.833
8	8.799	804.021	28.355	8	685	26.173
9	10.037	1728.871	41.580	9	1	37.868
10	11.308	3651.859	60.431	10	2949	54.305

An alternative approach to calculating the expected value $E(bet_c)$, the second moment $E(bet_c^2)$, and thereby the variance $Var(bet_c)$ uses a modification of the geometric probability distribution of the bet per coup which is seen in the following scheme in Exhibit 3, where n is the maximum number of coups.
The expected value and the second moment of this distribution are given by

$$E(bet_c) = \sum_{i=0}^{n-1} 2^i \frac{p^i}{\sum_{i=0}^{n-1} p^i} = \sum_{i=0}^{n-1} 2^i p^i \frac{1-p}{1-p^n} = \frac{1-p}{1-p^n} \cdot \frac{1-(2p)^n}{1-2p},$$

$$E(bet_c^2) = \sum_{i=0}^{n-1} (2^i)^2 p^i \frac{1-p}{1-p^n} = \frac{1-p}{1-p^n} \frac{1-(4p)^n}{1-4p},$$

$$Var(bet_c) = E(bet_c^2) - \left(E(bet_c)\right)^2 .$$

Exhibit 3: Derivation of the expected bet per coup

coup i	bet_c	probability	probability without table limit since $\lim_{i\to\infty} \sum_{i=0}^{n-1} p^i = \frac{1}{1-p}$
1	2^0	$\frac{1}{1+p+p^2+..+p^{n-1}}$	$1-p$
2	2^1	$\frac{p}{1+p+p^2+..+p^{n-1}}$	$p\cdot(1-p)$
3	2^2	$\frac{p^2}{1+p+p^2+..+p^{n-1}}$	$p^2\cdot(1-p)$
n	2^{n-1}	$\frac{p^{n-1}}{1+p+p^2+..+p^{n-1}}$	$p^{n-1}\cdot(1-p)$

Simplifications of the above formulas arise if p=0.5:

$$E(bet_c) = \frac{2^{n-1}}{2^n - 1} \cdot n \approx \frac{n}{2}$$

$$E(bet_c^2) = 2^{n-1}$$

$$Var(bet_c) = 2^{n-1} - \left(\frac{2^{n-1}}{2^n - 1} \cdot n \right)^2 .$$

4. Roulette simulations with R and its results

A simulation with R was carried out for 20,529 coups wagering on a simple chance. We chose this number because we wanted to simulate about 10,000 martingale rounds. The simulation was repeated 1,000 times. The initial bet was 1 unit on red. The probability of losing was 19/37. After each loss, the bet was doubled until reaching the table limit of 512 units. Table 4 shows important parameters (mean, standard deviation, and percentiles). The series length shows the maximum number of times the colour red appeared in a row. Schilling (2012) provides approximation formulas for the longest run of red or black. Other simulations of a roulette wheel can be found, e.g., in Turner (1998), Croucher (2005) or Kendall (2018).

Table 4: Simulation results

	mean	sd	0%	25%	50%	75%	100%	skew	kurtos
total profit W	-2948.206	3693.94	-17753	-5385	-3181.5	-262	6995	-0.168	-0.014
bet per coup	5.51415	0.28	4.729699	5.317843	5.504676	5.694895	6.471723		
max. series length	14.694	1.90	11	13	14	16	24		
no. of martingales	9998.387	69.24	9808	9953	9998.5	10045	10194		

The probability of a negative total profit is 80 percent.

The calculated and simulated values of the total profit and the bet per coup are more or less identical as the following formulas show. Expected value and variance of the total profit after 10,000 martingale rounds or after 20,529 coups are

$$E(W) = M \cdot \left(1 - (2p)^n\right) = 10{,}000 \cdot \left(1 - \left(2 \cdot \frac{19}{37}\right)^{10}\right) = \text{-3,056.26}$$

and $\sigma(W) = \sqrt{M \cdot \left((4p)^n - (2p)^{2n}\right)} = \sqrt{10{,}000 \cdot \left(\left(4 \cdot \frac{19}{37}\right)^{10} - \left(2 \cdot \frac{19}{37}\right)^{20}\right)} = 3654.66 .$

The expected bet per coup is

$$E(bet_c) = \frac{1-p}{1-p^n} \cdot \frac{1-(2p)^n}{1-2p} = \frac{1 - \frac{19}{37}}{1 - \left(\frac{19}{37}\right)^{10}} \cdot \frac{1 - \left(2 \cdot \frac{19}{37}\right)^{10}}{1 - 2 \cdot \frac{19}{37}} = 5.508 .$$

Using the normal approximation, we get the probability of a negative total profit

$$P(W < 0) = \phi\left(\frac{3056}{3655}\right) \approx \phi(0.84) \approx 0.8 .$$

The distribution of the outcome is skewed to the left even after more than 20,000 coups (see Fig. 2). In one simulation, the colour red appeared 24 times in a row. The series length record was registered in 1943, when the colour red came up 32 times in a row (*www.casino-games-online.biz/roulette/record-series.html*).

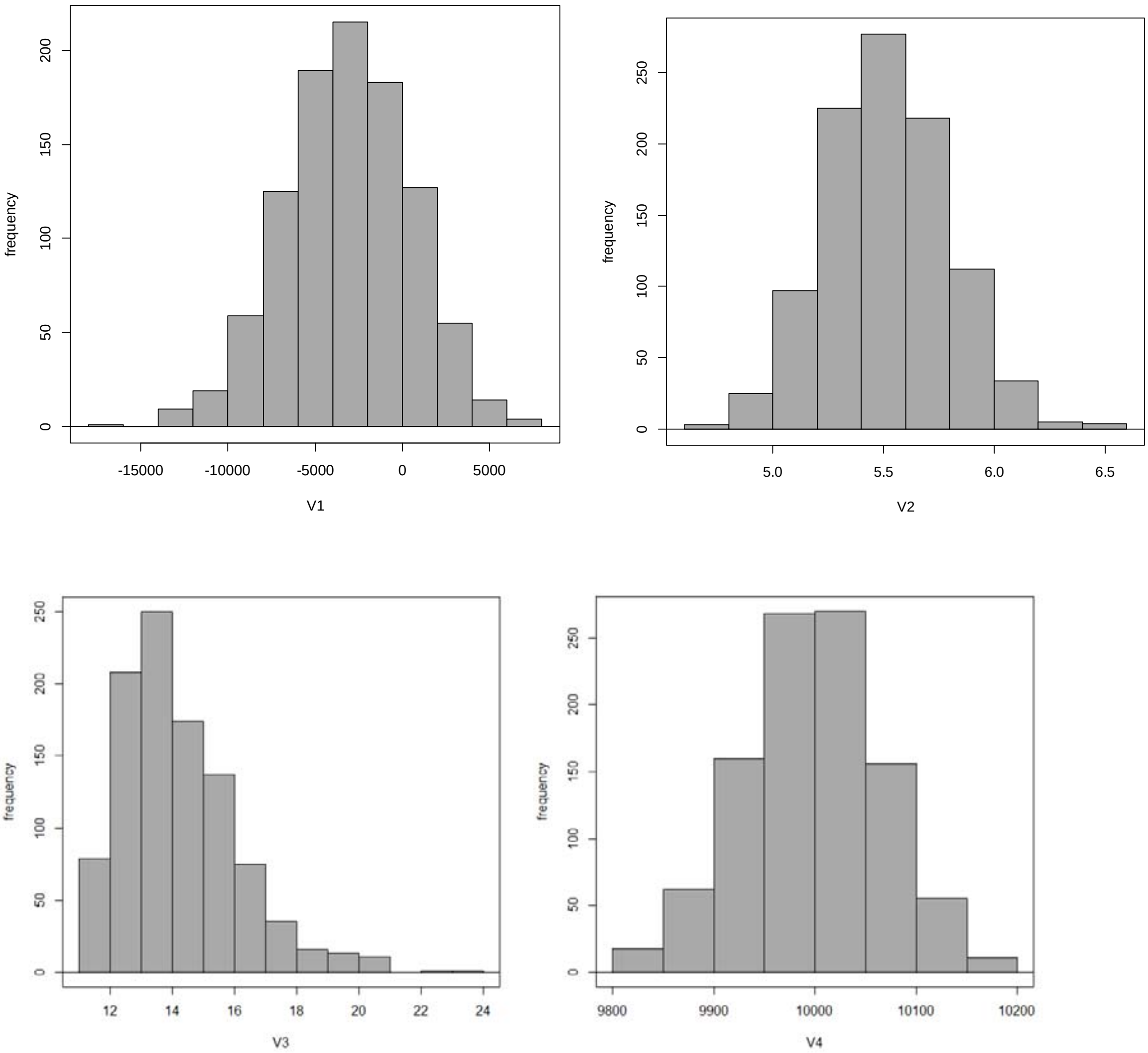

Figure 2: Histograms of the output or total profit (V1), the bet per coup (V2), the maximum series length of red (V3), and the number of martingales (V4)

The next simulation presents one possible trend of the profit W playing around 100,000 martingales (see Figure 3). The loss of the player with an initial wealth of zero and an initial bet of one will

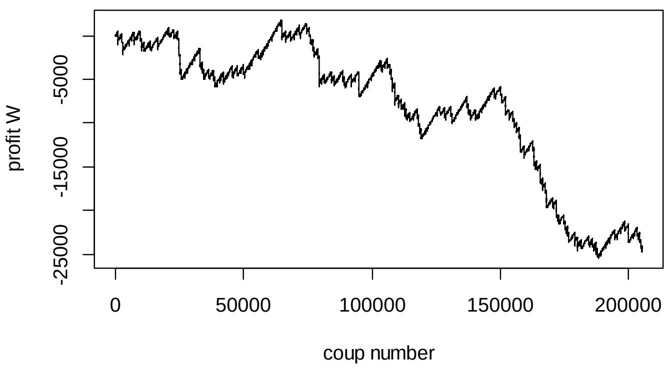

Figure 3: Trend of the total profit W of 205,301 simulated coups (roughly 100,000 martingales)

amount to 24,097 after 205,301 coups. The highest profit of the series is 1,773, and the highest loss is 25,437.

In the case of the absence of a table limit, the player would win all martingales in this simulation if he were able to bet a maximum of 262,144 units (see Table 5).

Table 5: Distribution of the bets of 200,305 coups in the absence of a table limit

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
bet	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072	262144
freq.	99687	51238	26338	13601	6983	3599	1901	964	491	245	121	63	33	17	10	4	3	2	1

5. Comparison of different roulette strategies

We will investigate how well a player using a doubling strategy would compete compared to a player betting a comparable amount on a simple chance and a player betting a comparable amount on single number each time (plein). See also Turner (1998) or Croucher (2005).

Assumptions:

a) Martingale player: Initial bet is one unit, doubling after each loss up to a table limit of 512 (a maximum of 9 doublings). He plays 10,000 martingale rounds or expected 20,529 coups. We know from the results of the previous chapter that the expected bet per coup is $E(bet_c) = 5.508$.

b) Simple chance player: He plays 20,529 times and wagers 5.508 units each time on a simple chance.

c) Plein player: He plays 20,529 times and wagers 5.508 units each time on a single number (plein).

Expected values and standard deviations of the three strategies are given in Table 6.

Table 6: Parameters of the total profit after 10,000 martingale rounds or 20,529 coups

	E(W)	$\sigma(W)$
Simple chance	-3,056 $= -0.027 \cdot 5.508 \cdot 20529$	$788.9 = 5.508 \cdot \sqrt{0.9993} \cdot \sqrt{20529}$
Single number	-3,056 $= -0.027 \cdot 5.508 \cdot 20529$	$4{,}607.1 = 5.508 \cdot \sqrt{34.08} \cdot \sqrt{20529}$
Martingale n=10	-3,056	3,654.7

Figure 4 shows the density and distribution functions of the total profit for the chosen strategies using the normal approximation. The riskiest strategy with the highest standard deviation is betting on a single number. The highest risk yields also the highest probability of about 25 percent for a positive total profit. The selected martingale strategy is comparable to the single number strategy with slightly less risk. The probability of a positive profit is around 20 percent. With the simple chance strategy, it is practically impossible to have a positive profit after 20,529 coups. High risk increases the probability of a positive profit, but it also increases the risk of severe losses, as can been seen clearly in Figure 4.

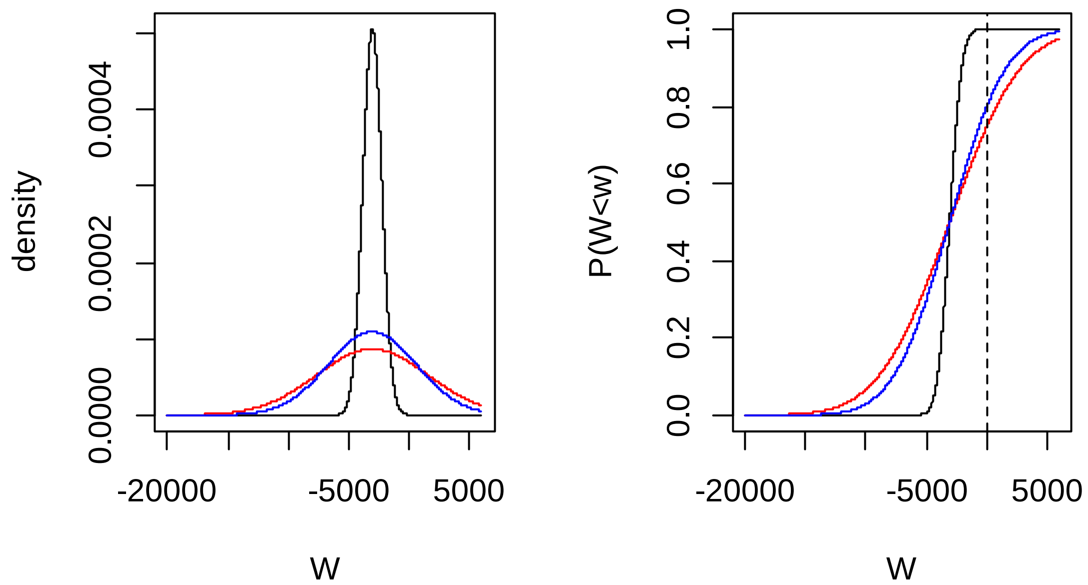

Figure 4: Density and distribution functions of the total profit for different roulette strategies after 20,529 coups (red: single number, blue: martingale, black: simple chance)

Next, we compare only 100 martingale rounds (same assumptions as above) with 205 coups betting 5.508 units on a single number. We calculate the following parameters:

	Martingale betting	Straight betting
Expectation	-30.56	-30.52
Standard deviation	365.41	460.41

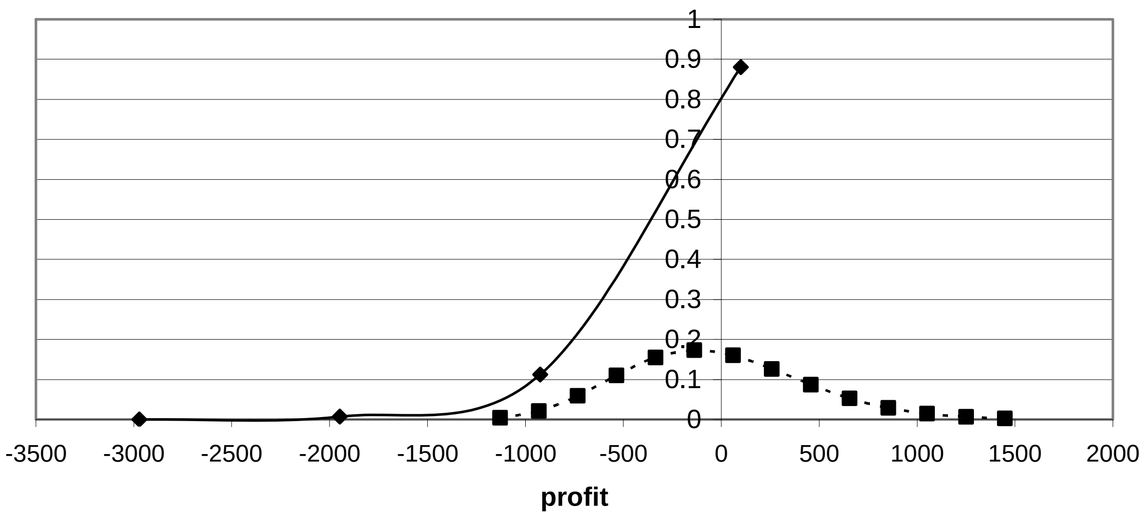

Figure 5: Distributions of the total profit after 205 coups (martingale betting vs. single number betting)

The probability that the martingale player wins 100 units is 88 percent. The probability of a positive profit of a player betting on straight (single number) is only 48 percent. In contrast to the long-term view, the probability of a positive profit is here lower for the single number player. But betting on straight avoids extreme losses and often provides higher earnings compared to martingale betting (see Figure 5).

The standard deviation does not reflect sufficiently the extreme risk of a martingale strategy. The maximum loss of the straight betting strategy is around 1,000, whereas the maximum loss of the martingale strategy is three times as high.

6. American roulette and consideration of en prison and la partage rules

The probabilities in American and European roulette are different because American roulette has an extra green number (the double zero, 00). The probability of losing one bet on a simple chance is $p = \frac{20}{38} \approx 0.5263$. The expected gain decreases in this case form E(g)= -0.027 to E(g)= -0.0526. However, all our derived formulas of the martingale strategy can be further used if we replace $p = \frac{19}{37}$ by $p = \frac{20}{38}$.

Expected value and variance of the American roulette with absence of the special zero rules are

$$E(g) = 1 \cdot \frac{18}{38} - 1 \cdot \frac{20}{38} = -\frac{1}{19} = -0.052632,$$

$$Var(g) = 1 - \left(-\frac{1}{19}\right)^2 = \frac{360}{361},$$

$$\sigma_g = 0.998614.$$

The case is more complicated if we consider European roulette casinos using the “en prison” rule or the “la partage” rule.

With the “la partage” rule, the player loses half the bet on a simple chance when the zero turns up. Expected value and variance are no longer calculated by using a two-point distribution:

$$E(g) = 1 \cdot \frac{18}{37} - 1 \cdot \frac{18}{37} - \frac{1}{2} \cdot \frac{1}{37} = -\frac{1}{74} = -0.013514,$$

$$E(g^2) = 1^2 \cdot \frac{18}{37} + (-1)^2 \cdot \frac{18}{37} + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{37} = \frac{145}{148}.$$

And the variance and the standard deviation are

$$Var(g) = \frac{145}{148} - \left(\frac{1}{74}\right)^2 = \frac{1341}{1369},$$

$$\sigma_g = 0.989721.$$

In order to use above martingale formulas, we propose using $p = \frac{18.75}{37}$, being aware that the results are now approximations. In this case, we obtain

$$E(g)=1\cdot\frac{18.25}{37}-1\cdot\frac{18.75}{37}=-\frac{1}{74}=-0.013514,$$
$$Var(g)=1-\left(-\frac{1}{74}\right)^2=\frac{5475}{5476},$$
$$\sigma_g=0.9999.$$

With the “en prison” rule, the player leaves the bet (en prison = in prison) for the next spin of the roulette wheel. If the subsequent spin is again zero, then the whole bet is lost. Otherwise the player's money is returned.

Expected value and standard deviation are (derivation see Ethier, 2010, p. 464, Feldman/Fox, 1991, p. 109)

$$E(g)=-0.013701\approx\frac{1}{73},$$
$$\sigma_g=0.993220.$$

In order to use above martingale formulas for approximation results, we should put $p=\frac{18.5}{36.5}$, where we obtain

$$E(g)=1\cdot\frac{18}{36.5}-1\cdot\frac{18.5}{36.5}=-\frac{1}{73}=-0.01370,$$
$$Var(g)=1-\left(-\frac{1}{73}\right)^2=\frac{5328}{5329},$$
$$\sigma_g=0.9999.$$

A more sophisticated approach based on the appearance of zeros and colours is found in a publication of Schneider (1997, p. 68) with

$$1-p=\frac{\frac{18}{37}}{1-\frac{1}{37}\cdot\frac{18}{37}-\left(\frac{1}{37}\right)^2\cdot\left(\frac{18}{37}\right)^2}=\frac{911754}{1849195}\approx 0.49305.$$

In this case, we should put $p=\frac{937441}{1849195}\approx 0.50695$ in order to use the above martingale formulas.

7. Conclusion

Methods for teaching introductory statistics are often considered ineffective because they do not show a clear context between statistics and their use in the real world. A nice and instructive example of illustrating statistical distributions in statistics courses is the application of the roulette martingale strategy.

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Appendix: Profit distribution after 10,000 martingale rounds

M=10,000 martingales. n=10. p=19/37

M	10,000	EG	-0.30562581
n	10	Var(G)	1335,6552
		S(G)	36.5466168
p	0.51351351		
$1\text{-}p^n$	0.99872497		
P^n	0.00127503	E(W)	-3056,25806
		Var(W)	13,356,552
lambda	12.750252	S(G)	3654,66168
rule npq>9	12.7339951	12.750252	

y (busts)	W	Binomial	Cum. Binomial Distribution	Poisson	Cum. Normal Distribution
0	10,000	0	0	0	0.0002
1	8,976	0	0	0	0.0005
2	7,952	0.0002	0.0003	0.0002	0.0013
3	6,928	0.001	0.0013	0.001	0.0031
4	5,904	0.0032	0.0045	0.0032	0.0071
5	4,880	0.0081	0.0126	0.0081	0.0149
6	3,856	0.0173	0.0299	0.0173	0.0293
7	2,832	0.0315	0.0614	0.0315	0.0536
8	1,808	0.0502	0.1116	0.0503	0.0916
9	784	0.0712	0.1828	0.0712	0.1467
10	-240	0.0908	0.2736	0.0908	0.2205
11	-1,264	0.1053	0.3789	0.1052	0.3119
12	-2,288	0.1119	0.4908	0.1118	0.4168
13	-3,312	0.1097	0.6005	0.1097	0.5279
14	-4,336	0.0999	0.7004	0.0999	0.6369
15	-5,360	0.0849	0.7854	0.0849	0.7358
16	-6,384	0.0677	0.8531	0.0677	0.8187
17	-7,408	0.0507	0.9038	0.0507	0.8831
18	-8,432	0.0359	0.9397	0.0359	0.9293
19	-9,456	0.0241	0.9638	0.0241	0.96
20	-10,480	0.0154	0.9792	0.0154	0.9789
21	-11,504	0.0093	0.9885	0.0093	0.9896
22	-12,528	0.0054	0.9939	0.0054	0.9952
23	-13,552	0.003	0.9969	0.003	0.998
24	-14,576	0.0016	0.9985	0.0016	0.9992
25	-15,600	0.0008	0.9993	0.0008	0.9997

P(W<0)=0.7985; P(W>0)=0.2015

M=20 martingales, n=10, p=19/37

M	20	EG	-0.30562581
n	10	Var(G)	1335.6552
		S(G)	36.5466168
p	0.51351351		
$1-p^n$	0.99872497		
p^n	0.00127503	E(W)	-6.11251612
		Var(W)	26713.104
lambda	0.0255005	S(G)	163.441439

x (success)	y (busts)	W	Binomial	Cum. Binomial	Poisson
20	0	20	0.9748	0.9748	0.9748
19	1	-1004	0.0249	0.9997	0.0249
18	2	-2028	0.0003	1.0000	0.0003
17	3	-3052	0.0000	1.0000	0.0000
16	4	-4076	0.0000	1.0000	0.0000
15	5	-5100	0.0000	1.0000	0.0000
14	6	-6124	0.0000	1.0000	0.0000
13	7	-7148	0.0000	1.0000	0.0000
12	8	-8172	0.0000	1.0000	0.0000
11	9	-9196	0.0000	1.0000	0.0000
10	10	-10220	0.0000	1.0000	0.0000
9	11	-11244	0.0000	1.0000	0.0000
8	12	-12268	0.0000	1.0000	0.0000
7	13	-13292	0.0000	1.0000	0.0000
6	14	-14316	0.0000	1.0000	0.0000
5	15	-15340	0.0000	1.0000	0.0000
4	16	-16364	0.0000	1.0000	0.0000
3	17	-17388	0.0000	1.0000	0.0000
2	18	-18412	0.0000	1.0000	0.0000
1	19	-19436	0.0000	1.0000	0.0000
0	20	-20460	0.0000	1.0000	0.0000