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LIMIT THEOREMS AND SOFT EDGE OF FREEZING RANDOM MATRIX MODELS VIA DUAL ORTHOGONAL POLYNOMIALS

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Abstract. N-dimensional Bessel and Jacobi processes describe interacting particle systems with N particles and are related to β-Hermite, β-Laguerre, and β-Jacobi ensembles. For fixed N there exist associated limit theorems (LTs) in the freezing regime $\beta \to \infty$ in the β-Hermite and β-Laguerre case by Dumitriu and Edelman (2005) with explicit formulas for the covariance matrices Σ_N in terms of the zeros of associated orthogonal polynomials. Recently, the authors derived these LTs in a different way and computed Σ_N^{-1} with formulas for the eigenvalues and eigenvectors of Σ_N^{-1} and thus of Σ_N. In the present paper we use these data and the theory of finite dual orthogonal polynomials of de Boor and Saff to derive formulas for Σ_N from Σ_N^{-1} where, for β-Hermite and β-Laguerre ensembles, our formulas are simpler than those of Dumitriu and Edelman. We use these polynomials to derive asymptotic results for the soft edge in the freezing regime for $N \to \infty$ in terms of the Airy function. For β-Hermite ensembles, our limits are different from those of Dumitriu and Edelman.

1. Introduction

Interacting Calogero-Moser-Sutherland particle systems on $\mathbb{R}$ or $[0, \infty[$ with N particles are described via multivariate Bessel processes on closed Weyl chambers in $\mathbb{R}^N$. These have been widely studied in the mathematical and physical literature, in particular due to their connections to random matrix theory; see [Dy, Br, KO, F] for these connections and the monographs [D, Me] for the background on random matrices. These Bessel processes are classified via root systems and by coupling or multiplicity parameters k which govern the interactions; see [CDGRVY, R, RV, DF, DV] and references therein for the details. Moreover, similar systems on $[-1, 1]$ can be described via Jacobi processes on alcoves in $\mathbb{R}^N$ which have the distributions of β-Jacobi ensembles as invariant distributions; see [Dem, RR, V2].

Recently, several limit theorems were derived when one or several multiplicity parameters k tend to infinity; see [AKM1, AKM2, AV1, AV2, HV, V1, VW]. In particular, [AV1, AV2, V1, VW] contain central limit theorems for Bessel processes for $k \to \infty$, and [HV] contains a corresponding result for β-Jacobi ensembles. In the most interesting cases, the freezing limits are N-dimensional centered Gaussian distributions where the inverses Σ_N^{-1} of the covariance matrices Σ_N can be computed explicitly in terms of the zeros of classical orthogonal polynomial of order N. In

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particular, for the Bessel processes with the root systems A_{N-1} and B_N, these orthogonal polynomials are classical Hermite and Laguerre polynomials, and the associated freezing LTs for the Bessel processes with start in the origin are closely related with corresponding LTs of Dumitriu and Edelman [DE2] for β-Hermite and β-Laguerre ensembles respectively for $\beta \to \infty$. However, the statements of these LTs in [DE2] and [V1, AV1] are slightly different, as in [DE2] explicit formulas for the covariance matrices Σ_N are given instead of the inverse matrices Σ_N^{-1} in [V1, AV1]. Both types of formulas involve the zeros of the N-th Hermite or Laguerre polynomial (with a suitable parameter) respectively, but it seems to be a difficult task to verify that the approaches in [DE2] and [V1, AV1] are equivalent. In fact, for small dimensions N, this equivalence was verified numerically. This problem of seemingly different formulas in [DE2] and [V1, AV1] was one of the starting points for this paper. In fact, by [AV1], the matrices Σ_N^{-1} and thus the Σ_N can be diagonalized: Σ_N^{-1} has the eigenvalues $1, 2, \ldots, N$ in the A_{N-1}-case and $2, 4, \ldots, 2N$ in the B_N-case, and in both cases the transformation matrices can be described in terms of a finite sequence $(Q_k)_{k=0,\ldots,N-1}$ of orthogonal polynomials which are orthogonal w.r.t. the empirical measure of the zeros of the N-th associated Hermite or Laguerre polynomial respctively. We show that this diagonalization of Σ_N^{-1} also leads to an explicit three-term-recurrence relation for the sequence $(Q_k)_{k=0,\ldots,N-1}$; see the end of Section 2 in the β-Hermite case. This recurrence immediately shows that the sequences $(Q_k)_{k=0,\ldots,N-1}$ are dual in the sense of de Boor and Saff [BS] to the finite parts $(H_k)_{k=0,\ldots,N-1}$ and $(L_k^{(\alpha)})_{k=0,\ldots,N-1}$ of the associated Hermite and Laguerre polynomials respectivey. With this knowledge in mind we reprove this fact in a more elegant way in Section 4 via this duality theory; see also [VZ, I] for this duality theory. It turns out that this approach also works for the freezing limits of the β-Jacobi ensembles in [HV] where Jacobi polynomials and their zeros appear in a similar way as for the β-Hermite and β-Laguerre ensembles.

After having identified the polynomials $(Q_k)_{k=0,\ldots,N-1}$ as dual polynomials in all these 3 classical matrix ensembles, we determine new explicit formulas for the entries of Σ_N in Section 4. It turns out that our approach to Σ_N for the β-Hermite and β-Laguerre ensembles in the freezing limit leads to formulas different from [DE2], and we are not able to check equality of these formulas for arbitrary dimensions N. It seems that this equality needs some unknown connections between the zeros of the N-th Hermite or Laguerre polynomial and the corresponding polynomials of order $0, 1, 2, \ldots, N-1$. We point out that in the β-Hermite limit case, our formulas for the entries of Σ_N have the same complexity as those in [DE2], while in the Laguerre case our formulas have the same form as in the β-Hermite case while the formulas in [DE2] are considerably more complicated. Moreover, in the β-Jacobi case, our formulas for Σ_N also have the same structure while corresponding results based on tridiagonal random matrix models as in [K, KN] seem to be unknown.

In the remaining sections we use our formulas for $\Sigma_N = (\sigma_{i,j})_{i,j=1,\ldots,N}$ in order to derive limit results for $\sigma_{N,N}$ for $N \to \infty$ in the β-Hermite and β-Laguerre case which involves the Airy function Ai and the r largest zeros $a_r < a_{r-1} < \ldots < a_1 < 0$ of Ai. For a discussion of Ai we refer to [NIST, VS]. In particular, for the largest eigenvalues in the β-Hermite case we obtain the following theorem which summarizes the main results of Section 5. For the precise definition of the Bessel processes $(X_{t,k}^N)_{t\geq 0}$ of type A_{N-1} we refer to the beginning of Section 2 below.

Theorem 1.1. *Let $r \in \mathbb{N}$. For $N \geq r$ consider the Bessel processes*

$$(X_{t,k}^N)_{t\geq 0} = (X_{t,k,1}^N, \dots, X_{t,k,N}^N)_{t\geq 0}$$

of type A_{N-1} with start in $0 \in \mathbb{R}^N$. Then, for each $t > 0$,

$$\lim_{N\to\infty} \left(\lim_{k\to\infty} N^{\frac{1}{6}} \sqrt{2k} \left(\frac{X_{t,k,N-r+1}^N}{\sqrt{2kt}} - z_{N-r+1,N} \right) \right) = G_r \tag{1.1}$$

in distribution with some $\mathcal{N}(0, \sigma_{max,r}^2)$-distributed random variable G_r with variance

$$\sigma_{max,r}^2 = \int_0^\infty \frac{\mathsf{Ai}(x+a_r)^2}{\mathsf{Ai}'(a_r)^2 x} dx = \begin{cases} 0.582\dots \textit{for } r=2 \\ 0.472\dots \textit{for } r=3 \\ 0.407\dots \textit{for } r=4 \\ \dots. \end{cases}$$

where $z_{N-r+1,N}$ is the r-th largest zero of the classical Hermite polynomial H_N, which satisfies by some classical formula of Plancherel-Rotach (see e.g. [T]*)*

$$\frac{z_{N-r+1,N}}{\sqrt{2N}} = 1 - \frac{|a_r|}{2N^{\frac{2}{3}}} + O(N^{-1}) \qquad (N \to \infty). \tag{1.2}$$

Moreover, the variances $\sigma_{max,r}^2$ tend to 0 for $r \to \infty$.

For $r = 1$, this result was stated by Dumitriu and Edelman (Corollary 3.4 in [DE2]), where the result there contains a misprint and the proof is sketched only. Moreover, as the proof in [DE2] uses a different formula for $\sigma_{N,N}$, they obtain

$$\sigma_{max,r}^2 = 2\frac{\int_0^\infty \mathsf{Ai}^4(x+a_1)dx}{\left(\int_0^\infty \mathsf{Ai}^2(x+a_1)dx\right)^2} = 2\int_0^\infty \left(\frac{\mathsf{Ai}(x+a_1)}{\mathsf{Ai}'(a_1)}\right)^4 dx. \tag{1.3}$$

A numerical computation shows that the value of (1.3) seems to be equal to that in Theorem 1.1 for $r = 1$. Unfortunately, we are not able to verify this equality in an analytic way, as our suggested identity does not seem to fit to known identities for the Airy function as e.g. in [VS]. Besides this result for the largest eigenvalues in the β-Hermite case we also derive a corresponding result for the largest eigenvalues of the frozen Laguerre ensembles by the same methods. We expect that our approach will also lead to corresponding results for the smallest eigenvalues of frozen Laguerre ensembles, i.e., at the hard edge, and to the extremal eigenvalues of frozen Jacobi ensembles where then Bessel functions instead of the Airy function appear.

This paper is organized as follows: In Section 2 we recapitulate some facts on Bessel processes of type A_{N-1} and β-Hermite ensembles. In particular the LTs in the freezing limit from [DE2, V1, AV2] and the covariance matrices Σ_N and their inverses are discussed there. Moreover we shall derive the three-term-recurrence relation for $(Q_k)_{k=0,\dots,N-1}$ there via matrix analysis. Section 3 is then devoted to the corresponding known results for the β-Laguerre and β-Jacobi ensembles from [DE2, V1, AV2, HV]. In Section 4 we then discuss general dual finite orthogonal polynomials and apply this to the classical polynomials. In this way we shall obtain new formulas for the covariance matrices Σ_N for all 3 classical types of ensembles in a unifying way. These results are the applied in Section 5 for the Hermite cases, in order to determine of some entries of Σ_N for $N \to \infty$ in terms of Airy functions. Finally, in Section 6, the corresponding limits in the Laguerre cases are determined at the soft edge.

2. LTs for Hermite ensembles for $\beta \to \infty$

In this section we recapitulate some LTs for the root systems A_{N-1} for fixed $N \geq 2$ and $\beta \to \infty$ from [DE2, AV2, V1] where we add a new result in the end. Here we have a one-dimensional coupling constant $\beta = 2k \in [0, \infty[$ where the notation k is usually used in the Bessel process community and β in the random matrix community. The associated Bessel processes $(X_{t,k})_{t\geq 0}$ are Markov processes on the closed Weyl chamber

$$C_N^A := \{x \in \mathbb{R}^N : \quad x_1 \leq x_2 \leq \ldots \leq x_N\}$$

where the generator of the transition semigroup is

$$L_A f := \frac{1}{2}\Delta f + k \sum_{i=1}^{N} \Big(\sum_{j \neq i} \frac{1}{x_i - x_j} \Big) \frac{\partial}{\partial x_i} f, \tag{2.1}$$

and we assume reflecting boundaries. The transition probabilities of these processes for $t > 0$ can be expressed in terms of multivariate Bessel functions of type A_{N-1}; see [R, RV]. We here only recapitulate that under the starting condition $X_{0,k} = 0 \in C_N^A$, the random variable $X_{t,k}$ has the Lebesgue-density

$$\frac{c_k}{t^{\gamma_A + N/2}} e^{-\|y\|^2/(2t)} \cdot \prod_{i<j} (y_j - y_i)^{2k} \, dy \tag{2.2}$$

on C_N^A for $t > 0$ with the constants

$$\gamma_A := kN(N-1)/2, \qquad c_k^A := \frac{N!}{(2\pi)^{N/2}} \cdot \prod_{j=1}^{N} \frac{\Gamma(1+k)}{\Gamma(1+jk)}.$$

Up to scaling, this is simply the distribution of the ordered spectra of the β-Hermite ensembles of Dumitriu and Edelman [DE1]. Using this tridiagonal β-Hermite model, Dumitriu and Edelman (Theorem 3.1 of [DE2]) derived the following LT 2.1 for $\beta = k \to \infty$ where the data of the limits are given in terms of the ordered zeros $z_{1,N} < \ldots < z_{N,N}$ of the N-th Hermite polynomial H_N. For this we recall that, as usual (see e.g. [S]), the Hermite polynomials $(H_n)_{n\geq 0}$ are orthogonal w.r.t. the density e^{-x^2} with the three-term-recurrence relation

$$H_0 = 1, \quad H_1(x) = x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \quad (n \geq 1). \tag{2.3}$$

The Hermite polynomials, orthonormalized w.r.t. the probability measure $\pi^{-1/2}e^{-x^2}$, will be denoted by $(\tilde{H}_n)_{n\geq 0}$. By (5.5.1) of [S], we thus have

$$\tilde{H}_n(x) = \frac{1}{2^{n/2}\sqrt{n!}} H_n(x) \qquad (n \geq 0). \tag{2.4}$$

Theorem 2.1. *Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type A_{N-1} with start in $0 \in C_N^A$. Then, for each $t > 0$,*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{2k} \cdot (z_{1,N}, \ldots, z_{N,N})$$

converges for $k \to \infty$ to the centered N-dimensional normal distribution $N(0, \Sigma_N)$ with the covariance matrix $\Sigma_N = (\sigma_{i,j}^2)_{i,j=1,\dots,N}$ with

$$\sigma_{i,j}^2 = \frac{\sum_{l=0}^{N-1} \tilde{H}_l^2(z_{i,N})\tilde{H}_l^2(z_{j,N}) + \sum_{l=0}^{N-2} \tilde{H}_{l+1}(z_{i,N})\tilde{H}_l(z_{i,N})\tilde{H}_{l+1}(z_{j,N})\tilde{H}_l(z_{j,N})}{\sum_{l=0}^{N-1} \tilde{H}_l^2(z_{i,N}) \cdot \sum_{l=0}^{N-1} \tilde{H}_l^2(z_{j,N})}. \tag{2.5}$$

This LT was proved in [V1] by a different method which leads a explicit formula for the inverse matrix Σ_N^{-1}, but not for Σ_N. This approach was improved in [AV2] from the starting point $0 \in C_N^A$ to arbitrary starting points $x \in C_N^A$ where this LT is slightly complicated for $x \neq 0$ as the root system A_{N-1} is not reduced on $\mathbb{R}^N$ for $N \geq 2$. This means that with the vector $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^N$, the space $\mathbb{R}^N$ can be decomposed into $\mathbb{R} \cdot \mathbf{1}$ and its orthogonal complement

$$\mathbf{1}^\perp = \{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = 0\} \subset \mathbb{R}^N$$

so that the associated Weyl group, i.e. the symmetric group $\mathfrak{S}_N$ here, acts on both spaces separately. We denote the orthogonal projections from $\mathbb{R}^N$ onto $\mathbb{R} \cdot \mathbf{1}$ and $\mathbf{1}^\perp$ by $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{1}^\perp}$ respectively. In particular, for $x \in \mathbb{R}^N$, we have $\pi_{\mathbf{1}}(x) = \bar{x}\mathbf{1}$ for the center of gravity $\bar{x} = \frac{1}{N}\sum_{i=1}^N x_i$ of the particles. With these notations, the following LT is shown in [AV2]:

Theorem 2.2. *Consider the Bessel processes $(X_{t,k})_{t \geq 0}$ of type A_{N-1} on C_N^A with an arbitrary fixed starting point $x \in C_N^A$. Then, for each $t > 0$,*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{2k} \cdot (z_{1,N}, \dots, z_{N,N})$$

converges for $k \to \infty$ in distribution to the N-dimensional normal distribution $N(\pi_{\mathbf{1}}(x/\sqrt{t}), \Sigma_N)$ where the inverse $\Sigma_N^{-1} =: S_N = (s_{i,j})_{i,j=1}^N$ of the covariance matrix Σ_N is given by

$$s_{i,j} := \begin{cases} 1 + \sum_{l \neq i}(z_{i,N} - z_{l,N})^{-2} & \text{for} \quad i = j \\ -(z_{i,N} - z_{j,N})^{-2} & \text{for} \quad i \neq j. \end{cases} \tag{2.6}$$

In [AV2], the eigenvalues and eigenvectors of S_N were determined via finite orthogonal polynomials which are orthogonal w.r.t. the empirical measures

$$\mu_N := \frac{1}{N}(\delta_{z_{1,N}} + \ldots + \delta_{z_{N,N}}) \in M^1(\mathbb{R}) \tag{2.7}$$

of the zeros of H_N. For the general theory of (finite) orthogonal polynomials we refer to the monographs [C, I]. In fact, Gram-Schmidt orthonormalization of the monomials x^n, $n = 0, \dots, N-1$, leads to a unique finite sequence of orthogonal polynomials $\{Q_n^{(N)}\}_{n=0}^{N-1}$ with positive leading coefficients, $\deg[Q_n^{(N)}] = n$, and with

$$\sum_{i=1}^N Q_n^{(N)}(z_{i,N})Q_m^{(N)}(z_{i,N}) = \delta_{n,m} \qquad (n, m = 0, \dots, N-1). \tag{2.8}$$

We then have the following result by Theorem 3.1 of [AV2]:

Theorem 2.3. *For each $N \geq 2$, the matrix S_N in Theorem 2.2 has the eigenvalues $\lambda_k = k$ for $k = 1, 2, \dots, N$. Moreover, for $n = 1, \dots, N$, the vector*

$$\left(Q_{n-1}^{(N)}(z_{1,N}), \dots, Q_{n-1}^{(N)}(z_{N,N})\right)^T$$

is an eigenvector of S_N for the eigenvalue n.

The finite orthogonal polynomials $\{Q_n^{(N)}\}_{n=0}^{N-1}$ admit a three-term-recurrence relation which can be derived from the proof of Theorem 2.3 in [AV2]. This explicit relation will be essential below. It will be convenient to write down this relation for the monic orthogonal polynomials $\{\hat{Q}_n^{(N)}\}_{n=0}^{N-1}$ associated with $\{Q_n^{(N)}\}_{n=0}^{N-1}$, i.e., $Q_k^{(N)} = l_k \hat{Q}_k^{(N)}$ with the leading coefficients $l_k > 0$ of $Q_k^{(N)}$:

Proposition 2.4. *The monic orthogonal polynomials $\{\hat{Q}_n^{(N)}\}_{n=0}^{N-1}$ satisfy*

$$\hat{Q}_0^{(N)} = 1,\ \hat{Q}_1^{(N)}(x) = x,\ \hat{Q}_{k+1}^{(N)}(x) = x\hat{Q}_k^{(N)}(x) - \left(\frac{N-k}{2}\right)\hat{Q}_{k-1}^{(N)}(x) \tag{2.9}$$

for $k = 1, ..., N-2$.

Proof. For $k = 1, \ldots, N$ consider the vector $v_k = (z_{1,N}^{k-1}, ..., z_{N,N}^{k-1})^T$ as in the proof of Theorem 2 in [AV2]. Eq. (3.5) in [AV2] shows that the i-th component of the vector $(S_N - kI_N)v_k$ has the form

$$\big((S_N - kI_N)v_k\big)_i = -\left(N - \frac{k-1}{2}\right)(k-2)z_{i,N}^{k-3} + s_k(z_{i,N})$$

with some polynomial s_k of order at most $k-5$. Therefore, if we put

$$e_k := -\left(N - \frac{k-1}{2}\right)\frac{k-2}{2}, \tag{2.10}$$

we obtain

$$\begin{aligned}
&\big((S_N - kI_N)(v_k + e_k v_{k-2})\big)_i \\
&\qquad = \big((S_N - kI_N)(v_k) + (S_N - (k-2)I_N)(e_k v_{k-2}) - 2e_k v_{k-2}\big)_i \\
&\qquad = -\left(N - \frac{k-1}{2}\right)(k-2)z_{i,N}^{k-3} - 2e_k z_{i,N}^{k-3} + r_k(z_{i,N}) \\
&\qquad = r_k(z_{i,N})
\end{aligned} \tag{2.11}$$

with some polynomial r_k of degree at most $k-5$. On the other hand, by the proof of Theorem 2 in [AV2], there exist polynomials p_k of order at most $k-5$ with

$$(S_N - kI_N)(p_k(z_{1,N}), ..., p_k(z_{N,N})) = (r_k(z_{1,N}), ..., r_k(z_{N,N})). \tag{2.12}$$

(2.11) and (2.12) imply that

$$(S_N - kI_N)\left(z_{i,N}^{k-1} + e_k z_{i,N}^{k-3} - p_k(z_{i,k})\right)_{i=1,\ldots,N} = (0)_{i=1,\ldots,N}.$$

In summary, we find monic polynomials $(q_k)_{k=0,\ldots,N-1}$ with $\deg q_k = k$ and $q_k(z) = z^k + e_{k+1}z^{k-2} - p_{k+1}(z)$ such that the vector $(q_k(z_{1,N}), \ldots, q_k(z_{N,N}))^T$ is an eigenvector of the matrix S_N with the eigenvalue $k+1$. Because the eigenvectors of S_N are orthogonal, we conclude that $(q_k)_{k=0,\ldots,N-1}$ is equal to the finite monic sequence $\{\hat{Q}_k^{(N)}\}_{k=0,\ldots,N-1}$ of orthogonal polynomials w.r.t. the measure μ_N. As the measure μ_N is symmetric, the $\hat{Q}_k^{(N)}$ have a three-term recurrence of the form

$$\hat{Q}_k^{(N)}(x) = x\hat{Q}_{k-1}^{(N)}(x) - b_k\hat{Q}_{k-2}^{(N)}(x)$$

with some coefficients $b_k > 0$. This leads to

$$x^k + e_{k+1}x^{k-2} = x^k + e_k x^{k-2} - b_k x^{k-2} + \text{terms of lower degree}\ . \tag{2.13}$$

Hence, by (2.10) and (2.13) for $k = 1, ..., N-1$,

$$\begin{aligned} b_k = e_k - e_{k+1} &= \left(N - \frac{k}{2}\right)\left(\frac{k-1}{2}\right) - \left(N - \frac{k-1}{2}\right)\left(\frac{k-2}{2}\right) \\ &= \left(N - \frac{k}{2}\right)\left(\frac{k-1-(k-2)}{2}\right) - \frac{1}{2}\left(\frac{k-2}{2}\right) \\ &= \frac{1}{2}\left(\frac{2N-k-k+2}{2}\right) = \frac{N-k+1}{2}. \end{aligned}$$

This leads to the three-term-recursion in the statement. □

The three-term-recurrence (2.4) of the Hermite polynomials implies that H_n has the leading coefficient 2^n. Hence, by (2.4), the monic Hermite polynomials $(\hat{H}_n := 2^{-n}H_n)_{n\geq 0}$ satisfy the three-term-recurrence

$$H_0 = 1, \quad H_1(x) = x, x\hat{H}_n(x) = \hat{H}_{n+1}(x) + \frac{n}{2}\hat{H}_{n-1}(x) \quad (n \geq 1). \tag{2.14}$$

This recurrence is related to that in Proposition 2.4 via the theory of dual orthogonal polynomials by de Boor and Saff [BS]. We show in Section 4 that this connection between the sequences $(\hat{H}_n)_{n\geq 0}$ and $(\hat{Q}_k^{(N)})_{k=0,...,N-1}$ also holds for further random matrix models and the associated orthogonal polynomials. In this way, Proposition 2.4 can be also proved via the theory of dual orthogonal polynomials.

3. LTs for Laguerre and Jacobi ensembles for $\beta \to \infty$

In this section we recapitulate some LTs for $\beta \to \infty$ from [DE2, AV2, V1, HV] for the Bessel processes of type B_N and the Jacobi case.

We first turn to Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N. These Markov processes live in the closed Weyl chamber

$$C_N^B := \{x \in \mathbb{R}^N : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_N\},$$

and the generator of their transition semigroup is

$$L_B f := \frac{1}{2}\Delta f + k_1 \sum_{i=1}^N \frac{1}{x_i}\frac{\partial}{\partial x_i} f + k_2 \sum_{i=1}^N \Big(\sum_{j\neq i} \frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\Big)\frac{\partial}{\partial x_i} f \tag{3.1}$$

with multiplicities $k_1, k_2 \geq 0$ and reflecting boundaries. We write the multiplicities as $(k_1, k_2) = (\kappa \cdot \nu, \kappa)$ with $\nu, \kappa \geq 0$ where the parameter β from random matrix theory is $\beta = 2\kappa$. The transition probabilities of these processes for $t > 0$ are again known; see [R, RV]. In particular, under the starting condition $X_{0,k} = 0 \in C_N^A$, $X_{t,k}$ has the Lebesgue-density

$$\frac{c_k}{t^{\gamma_B + N/2}} e^{-\|y\|^2/(2t)} \cdot \prod_{i<j} (y_j^2 - y_i^2)^{2k_2} \cdot \prod_{i=1}^N y_i^{2k_1}\, dy \tag{3.2}$$

on C_N^B for $t > 0$ with some known normalizations $c_k^B > 0$ and

$$\gamma_B(k_1, k_2) = k_2 N(N-1) + k_1 N.$$

Up to scaling, these distributions belong to the ordered spectra of the β-Laguerre ensembles in [DE1]. Using their tridiagonal β-Laguerre models, Dumitriu and Edelman [DE2] derived a LT for $\beta \to \infty$ where the limits are given in terms of the ordered zeros $z_{1,N}^{(\nu-1)} \leq \cdots \leq z_{N,N}^{(\nu-1)}$ of the Laguerre polynomial $L_N^{(\nu-1)}$. We recapitulate that for $\alpha > -1$ the Laguerre polynomials $(L_n^{(\alpha)})_{n\geq 0}$ are orthogonal w.r.t. the density $e^{-x}x^\alpha$ on $]0,\infty[$ as in [S] with the three-term recurrence relation

$$\begin{aligned} &L_0^{(\alpha)} = 1, \quad L_1^{(\alpha)}(x) = -x + \alpha + 1, \\ &(n+1)L_{n+1}^{(\alpha)}(x) = (-x + 2n + \alpha + 1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \quad (n \geq 1). \end{aligned} \tag{3.3}$$

The Laguerre polynomials orthonormalized w.r.t. $\frac{1}{\Gamma(\alpha+1)}e^{-x}x^\alpha$, the Gamma distribution, will be denoted by $(\tilde{L}_n^{(\alpha)})_{n\geq 0}$. By (5.1.1) of [S], we thus have

$$\tilde{L}_n^{(\alpha)}(x) = \binom{n+\alpha}{n}^{-1/2} L_n^{(\alpha)}(x) \qquad (n \geq 0). \tag{3.4}$$

Using these notations, Dumitriu and Edelman [DE2] proved for $\nu > 0$ fixed, $X_{0,(\beta\cdot\nu/2,\beta/2)} = 0 \in C_N^B$, and $\beta \to \infty$ that with the vector $r \in C_N^B$ given by

$$(z_{1,N}^{(\nu-1)}, \ldots, z_{N,N}^{(\nu-1)}) = (r_1^2, \ldots, r_N^2), \tag{3.5}$$

the random variable

$$\frac{X_{t,(\beta\cdot\nu,\beta)}}{\sqrt{t}} - \sqrt{\beta}\cdot r$$

converges in distribution to a centered normal random variable $N(0,\Sigma_N)$ with explicit formulas for the entries of Σ_N. As these formulas are quite complicated we skip them here. Similar to the preceding Hermite case, this LT was extended in [V1] to arbitrary starting points as follows with explicit formulas for Σ_N^{-1}:

Theorem 3.1. *Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N on C_N^B for $k = (k_1,k_2) = (\kappa\cdot\nu,\kappa)$ and $\kappa,\nu > 0$ with start in $x \in C_N^B$. Then, for each $t > 0$,*

$$\frac{X_{t,(\kappa\cdot\nu,\kappa)}}{\sqrt{t}} - \sqrt{2\kappa}\cdot r$$

converges for $\kappa \to \infty$ to the centered N-dimensional distribution $N(0,\Sigma_N)$ with the regular covariance matrix Σ_N where $\Sigma_N^{-1} =: S_N = (s_{i,j})_{i,j=1,\ldots,N}$ is given by

$$s_{i,j} := \begin{cases} 1 + \frac{\nu}{r_i^2} + \sum_{l\neq i}(r_i - r_l)^{-2} + \sum_{l\neq i}(r_i + r_l)^{-2} & \text{for} \quad i = j, \\ (r_i + r_j)^{-2} - (r_i - r_j)^{-2} & \text{for} \quad i \neq j. \end{cases} \tag{3.6}$$

Again, the eigenvalues and eigenvectors of S_N were determined via finite orthogonal polynomials in [AV2]. For this we introduce the measures

$$\mu_{N,\nu} := \frac{1}{N(N+\nu-1)}(z_{1,N}^{(\nu-1)}\delta_{z_{1,N}^{(\nu-1)}} + \ldots + z_{N,N}^{(\nu-1)}\delta_{z_{N,N}^{(\nu-1)}}). \tag{3.7}$$

As

$$\sum_{k=1}^N z_{k,N}^{(\nu-1)} = N(N+\nu-1) \tag{3.8}$$

by Appendix C of [AKM2], these measures are probability measures. We now define the unique associated orthogonal polynomials $(Q_k^{(N,\nu)})_{k=0,\ldots,N-1}$ with $\deg Q_k^{(N,\nu)} = k$, positive leading coefficients, and with the normalization

$$\sum_{i=1}^N z_{i,N}^{(\nu-1)} Q_k^{(N,\nu)}(z_{i,N}^{(\nu-1)})^2 = 1 \qquad (k = 0,\ldots,N-1). \tag{3.9}$$

With this normalization, by [AV2] the matrices

$$T_N := (r_i \cdot Q_k^{(N,\nu)}(r_i^2))_{i=1,\ldots,N,k=0,\ldots,N-1} \tag{3.10}$$

are orthogonal, and moreover we have the following

Theorem 3.2. *For $N \geq 2$, the matrix S_N in Theorem 3.1 has the eigenvalues $\lambda_k = 2(k+1)$ with corresponding eigenvectors*

$$(r_1 Q_k^{(N,\nu)}(r_1^2), \ldots, r_N Q_k^{(N,\nu)}(r_N^2))^T, \ k = 0, 1, \ldots, N-1.$$

In particular, $S_N = T_N \cdot \mathrm{diag}(2, 4, \ldots, 2N) \cdot T_N^T$.

The three-term recurrence relations of the polynomials $(Q_k^{(N,\nu)})_{k=0,\ldots,N-1}$ can be determined in the same way as in the proof of Proposition 2.4. We skip this derivation here, as we shall present a more elegant proof of these relations in Section 4 via dual orthogonal polynomials.

We next turn to the β-Jacobi ensembles. In contrast with the preceding Bessel processes on noncompact spaces, we do not study the associated Jacobi processes, but turn immediately to their invariant distributions. It turns out that by [HV], it is convenient for our considerations to study these invariant distributions in a trigonometric form, that is, after performing the coordinate transformation

$$(t_1, \ldots, t_N) \longrightarrow (\cos(2t_1), \ldots, \cos(2t_N)).$$

Up to this transformation we follow [F, K, KN, Me, HV] and consider for $k_1, k_2, k_3 \geq 0$ the trigonometric β-Jacobi random matrix ensembles with the joint eigenvalue distributions $\tilde{\mu}_{(k_1,k_2,k_3)}$ given by the Lebesgue densities

$$\tilde{c}_k \cdot \prod_{1\leq i<j\leq N} (\cos(2t_j) - \cos(2t_i))^{k_3} \prod_{i=1}^{N} \Big(\sin(t_i)^{k_1} \sin(2t_i)^{k_2}\Big) \tag{3.11}$$

on the trigonometric alcoves

$$\tilde{A} := \{t \in \mathbb{R}^N | \frac{\pi}{2} \geq t_1 \geq ... \geq t_N \geq 0\}$$

with a suitable Selberg normalization $\tilde{c}_k > 0$ for $k = (k_1, k_2, k_3) \in [0, \infty[^3$; see the survey [FW] for explicit formulas. In [HV], a LT was derived which corresponds to the preceding freezing LTs for Bessel processes. We write

$$(k_1, k_2, k_3) = \kappa \cdot (a, b, 1),$$

where $a \geq 0$ $b > 0$ are fixed and κ tends to infinity. By [HV], the limit can be described via the ordered zeros of the classical Jacobi polynomials $P_N^{(\alpha,\beta)}$ with parameters

$$\alpha := a + b - 1 > -1, \quad \beta = b - 1 > -1.$$

Please notice that here, β is a parameter different from the β in random matrix theory. We recapitulate that the Jacobi polynomials $(P_n^{(\alpha,\beta)})_{n\geq 0}$ are orthogonal polynomials w.r.t. the weights $(1-x)^\alpha(1+x)^\beta$ on $]-1, 1[$; see [S]. We denote their ordered zeros by $z_1 \leq \ldots \leq z_N$ where we the suppress $\alpha, \beta > -1$. We now use the vector $z := (z_1, \ldots, z_N) \in A$. The following LT is shown in [HV]:

Theorem 3.3. *Let $a \geq 0$, $b > 0$. Let $\tilde{X}_\kappa$ be $\tilde{A}$-valued random variables with the distributions $\tilde{\mu}_{\kappa\cdot(a,b,1)}$ for $\kappa > 0$. Then, for $\kappa \to \infty$*

$$\sqrt{\kappa}(\tilde{X}_\kappa - \tilde{z}) \qquad \textit{with} \qquad \tilde{z} := (\frac{1}{2}\arccos z_1, \ldots, \frac{1}{2}\arccos z_N) \in \tilde{A}$$

converges in distribution to $N(0, \tilde{\Sigma}_N)$ where the inverse of the covariance matrix $\tilde{\Sigma}_N$ is given by $\tilde{\Sigma}_N^{-1} =: \tilde{S}_N = (\tilde{s}_{i,j})_{i,j=1,\ldots,N}$ with

$$\tilde{s}_{i,j} = \begin{cases} 4\sum_{l\neq j} \frac{1-z_j^2}{(z_j-z_l)^2} + 2(a+b)\frac{1+z_j}{1-z_j} + 2b\frac{1-z_j}{1+z_j} & \textit{for } i = j \\ \frac{-4\sqrt{(1-z_j^2)(1-z_i^2)}}{(z_i-z_j)^2} & \textit{for } i \neq j \end{cases}.$$

The eigenvalues and eigenvectors of $\tilde{S}_N$ can be determined explicitly. For this we introduce finite families of polynomials which are orthogonal w.r.t. the measures

$$\mu_{N,\alpha,\beta} := (1-z_1^2)\delta_{z_1} + \ldots + (1-z_N^2)\delta_{z_N}. \tag{3.12}$$

We consider the associated finite orthonormal polynomials $(Q_l^{(\alpha,\beta,N)})_{l=0,\ldots,N-1}$ with positive leading coefficients and the normalization

$$\sum_{i=1}^{N} Q_l^{(\alpha,\beta,N)}(z_i) Q_k^{(\alpha,\beta,N)}(z_i)(1-z_i^2) = \delta_{l,k} \qquad (k,l = 0,\ldots,N-1). \tag{3.13}$$

By [HV] we then have:

Theorem 3.4. *The matrix $\tilde{S}_N$ has the eigenvalues $\lambda_k = 2k(2N+\alpha+\beta+1-k) > 0$ ($k = 1,\ldots,N$) with the eigenvectors*

$$v_k := \left(Q_{k-1}^{(\alpha,\beta,N)}(z_1)\sqrt{1-z_1^2}, \ldots, Q_{k-1}^{(\alpha,\beta,N)}(z_N)\sqrt{1-z_N^2}\right)^T.$$

In particular, with the orthogonal matrix $T_N := (v_1, \ldots, v_N)$,

$$\tilde{S}_N = T_N \cdot \mathrm{diag}(2(2N+\alpha+\beta+1-1), \ldots, 2N(2N+\alpha+\beta+1-N)) \cdot T_N^T.$$

As mentioned previously, the three-term recurrence relations of the polynomials $(Q_k^{(\alpha,\beta,N)})_{k=0,\ldots,N-1}$ can be determined in the same way as in the proof of Proposition 2.4. A more elegant proof of these relations via dual orthogonal polynomials will be given in the following section.

4. De Boor-Saff duality and the covariance matrices

In this section we use the theory of dual orthogonal polynomials of de Boor and Saff [BS] to analyze the covariance matrices Σ_N of the LTs in the three cases of the preceding two sections. This is motivated by the observation that the finite monic orthogonal polynomials $(\hat{Q}_k^{(N)})_{k=0,\ldots,N-1}$ in Section 2 are the dual polynomials of the Hermite polynomials $(\hat{H}_k)_{k\geq 0}$ by Proposition 2.4.

To explain this we first review this theory from [VZ] and Section 2.11 of [I]. Let $(\hat{P}_n)_{n=0}^\infty$ be a sequence of monic orthogonal polynomials where the orthogonality measure is a probability measure μ on $\mathbb{R}$ which admits all moments, i.e.,

$$\int_{\mathbb{R}} \hat{P}_i(x)\hat{P}_j(x)d\mu(x) = \xi_i\delta_{ij} \quad (i,j = 0,1,2,\ldots) \tag{4.1}$$

with some constants $\xi_i > 0$ ($i \geq 0$). We also have a three-term recurrence relation

$$\hat{P}_0 = 1,\ \hat{P}_1(x) = x - a_0,\ x\hat{P}_n(x) = \hat{P}_{n+1}(x) + a_n\hat{P}_n(x) + u_n\hat{P}_{n-1}(x) \quad (n \geq 1) \tag{4.2}$$

with coefficients $a_n \in \mathbb{R}$ and $u_n > 0$. We also consider the associated orthonormal polynomials $(\tilde{P}_n := \xi_n^{-1/2}\hat{P}_n)_{n=0}^{\infty}$ with $\int_{\mathbb{R}} \tilde{P}_i(x)\tilde{P}_j(x)d\mu(x) = \delta_{ij}$. These polynomials then satisfy the three-term recurrence

$$\tilde{P}_0 = 1, \tilde{P}_1(x) = b_1^{-1}(x-a_0), x\tilde{P}_n(x) = b_{n+1}\tilde{P}_{n+1}(x)+a_n\tilde{P}_n(x)+b_n\tilde{P}_{n-1}(x) \quad (n \geq 1) \tag{4.3}$$

with $b_n = u_n\sqrt{\xi_{n-1}/\xi_n} = \sqrt{\xi_n/\xi_{n-1}}$ for $n \geq 1$. In particular we have

$$\xi_0 = 1,\ \xi_n = u_n u_{n-1}\cdots u_1 \quad \text{and} \quad b_n = \sqrt{u_n} \quad (n \geq 1). \tag{4.4}$$

Now fix $N > 0$ arbitrarily. Gauss quadrature implies that the finite set of polynomials $(\tilde{P}_n)_{n=0}^{N-1}$ obeys the discrete orthogonality relation

$$\sum_{i=1}^{N} w_i \tilde{P}_m(z_{i,N})\tilde{P}_n(z_{i,N}) = \delta_{mn}, \tag{4.5}$$

with the N ordered zeros $z_{1,N} < \ldots < z_{n,N}$ of $\tilde{P}_N$ and the Christoffel numbers

$$w_i := \frac{1}{b_N \tilde{P}_{N-1}(z_{i,N})\tilde{P}_N'(z_{i,N})} > 0 \quad (i = 1, \ldots, N) \tag{4.6}$$

which satisfy the normalization $\sum_{i=1}^{N} w_i = 1$.

Definition 4.1. Let $N > 0$. The monic polynomials $(\hat{Q}_{k,N})_{k=0}^{N-1}$ are called dual (in the de Boor-Saff sense) to $(\hat{P}_n(x))_{n=0}^{N-1}$ if they satisfy the three-term recurrence

$$\hat{Q}_{0,N} = 1,\ \hat{Q}_{1,N}(x) = x - a_{N-1}, \tag{4.7}$$

$$x\hat{Q}_{k,N}(x) = \hat{Q}_{k+1,N}(x) + a_{N-k-1}\hat{Q}_{k,N}(x) + u_{N-k}\hat{Q}_{k-1,N}(x) \quad (k = 1, \ldots, N-2).$$

This definition and Proposition 2.4 imply that the polynomials $(\hat{Q}_k^{(N)})_{k=0,\ldots,N-1}$ from Section 2 are in fact dual to the monic Hermite poynomials $\hat{H}_n$.

We now recapitulate some consequences of this duality from [VZ]:

Lemma 4.2. *The dual monic polynomials $(\hat{Q}_{k,N})_{k=0}^{N-1}$ are orthogonal w.r.t. the discrete measure*

$$\sum_{i=1}^{N} w_i^* \delta_{z_{i,N}}$$

with the dual Christoffel numbers

$$w_i^* = \frac{\tilde{P}_{N-1}(z_{i,N})}{b_N \tilde{P}_N'(z_{i,N})} > 0 \quad (i = 1, \ldots, N) \tag{4.8}$$

which again satisfy $\sum_{i=1}^{N} w_i^* = 1$.

In particular, by (4.4), the normalized dual polynomials $(\tilde{Q}_{k,N})_{k=0}^{N-1}$ with

$$\sum_{i=1}^{N} w_i^* \tilde{Q}_{m,N}(z_{i,N})\tilde{Q}_{n,N}(z_{i,N}) = \delta_{mn} \quad (m, n = 0, \ldots, N-1) \tag{4.9}$$

satisfy

$$\tilde{Q}_{k,N}(x) = \frac{\hat{Q}_{k,N}}{b_N^2 b_{N-1}^2 \cdots b_{N-k}^2}. \tag{4.10}$$

In summary we obtain from (4.10) and the three-term-recurrence in Definition 4.1:

Lemma 4.3. *The orthonormal dual polynomials* $(\tilde{Q}_{k,N})_{k=0}^{N-1}$ *satisfy the three-term-recurrence relation*

$$\tilde{Q}_{0,N} = 1,\ \tilde{Q}_{1,N}(x) = b_{N-1}^{-1}(x - a_{N-1}), \tag{4.11}$$
$$x\tilde{Q}_{k,N}(x) = b_{N-k-1}\tilde{Q}_{k+1,N}(x) + a_{N-k-1}\tilde{Q}_{k,N}(x) + b_{N-k}\tilde{Q}_{k-1,N}(x) \quad (k \le N-2).$$

Remark 4.4. The monic three-term-recurrence (4.7) is also available for $k = N-1$, i.e., we obtain a monic polynomial $\hat{Q}_{N,N}$. It can be easily seen (see [VZ] or Section 2.11 of [I]) that $\hat{Q}_{N,N} = \hat{P}_N$ holds. Moreover, if we choose $b_0 = 0$ in (4.11), then the recurrence (4.3) remains valid for $k = N-1$, arbitrary polynomials $\tilde{Q}_{k+1,N}$, and $x = z_{i,N}$ for $i = 1, \ldots, N$.

We next apply finite dual orthogonal polynomials in order to obtain additional information about the covariance matrices Σ_N in the LTs 2.2, 3.1, and 3.3. In these cases, in $S_N = \Sigma_N^{-1}$, the Hermite polynomials H_n, the Laguerre polynomials $L_n^{(\alpha)}$ with $\alpha = \nu - 1$ and the Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha = a + b - 1$, $\beta = b-1$ respectively appear. For fixed N we now study the associated orthonormal dual polynomials which we denote by $(Q_{k,N})_{k=0}^{N-1}$, $(Q_{k,N}^{(\alpha)})_{k=0}^{N-1}$, and $(Q_{k,N}^{(\alpha,\beta)})_{k=0}^{N-1}$ respectively, In all cases, let $z_{1,N} < \ldots < z_{N,N}$ be the ordered zeros of the Nth polynomial. With these notations we have:

Lemma 4.5. *In the Hermite, Laguerre and Jacobi cases, orthonormal eigenvectors of* $S_N = \Sigma_N^{-1}$ *are given by the vectors*

$$\frac{1}{\sqrt{\kappa_N}}(\sqrt{\pi(z_{1,N})}\tilde{Q}_{j-1,N}(z_{1,N}), \ldots, \sqrt{\pi(z_{N,N})}\tilde{Q}_{j-1,N}(z_{N,N}))^T,\ 1 \le j \le N, \tag{4.12}$$

where the coefficients κ_N *and functions* $\pi(x)$ *are given by*

$$\pi(x) = 1, \quad \pi^{(\alpha)}(x) = x, \quad \pi^{(\alpha,\beta)}(x) = 1 - x^2, \ \text{and} \tag{4.13}$$
$$\kappa_N = N, \quad \kappa_N^{(\alpha)} = N(N+\alpha), \quad \kappa_N^{(\alpha,\beta)} = \frac{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}$$

respectively.

Proof. We first consider the Hermite case. Here $\hat{H}_N'(x) = N\hat{H}_{N-1}(x)$ by Section 5.5 of [S]. Hence, by (4.8),

$$\sum_{i=1}^{N} \frac{1}{N}\tilde{Q}_{m,N}(z_{i,N})\tilde{Q}_{n,N}(z_{i,N}) = \delta_{mn}. \tag{4.14}$$

If we compare this with the orthogonality (2.8) of the polynomials $Q_n^{(N)}$ from Section 2, we conclude from Theorem 2.3 that the vectors

$$\frac{1}{\sqrt{N}}(\tilde{Q}_{j-1,N}(z_{1,N}), \ldots, \tilde{Q}_{j-1,N}(z_{N,N}))^T \tag{4.15}$$

for $j = 1, \ldots, N$ are orthonormal eigenvectors of S_N.

We now turn to the Laguerre case. By Section 4.6 of [I] the monic Laguerre polynomials satisfy

$$x\hat{L}_n^{(\alpha)\prime}(x) = n\hat{L}_n^{(\alpha)}(x) + n(n+\alpha)\hat{L}_{n-1}^{(\alpha)}(x). \tag{4.16}$$

In particular, $z_{i,N}^{(\alpha)}\hat{L}_N^{(\alpha)\prime}(z_{i,N}^{(\alpha)}) = N(N+\alpha)\hat{L}_{N-1}^{(\alpha)}(z_{i,N}^{(\alpha)})$. This and (4.8) yield

$$\sum_{i=1}^{N} \frac{z_{i,N}^{(\alpha)}}{N(N+\alpha)} \tilde{Q}_{m,N}^{(\alpha)}(z_{i,N}^{(\alpha)}) \cdot \tilde{Q}_{n,N}^{(\alpha)}(z_{i,N}^{(\alpha)}) = \delta_{mn}. \tag{4.17}$$

If we compare this with the orthogonality (3.9) of the polynomials $Q_k^{(N,\nu)}$ with $\alpha = \nu - 1$ in Section 3, we conclude from Theorem 3.2 that the vectors

$$\begin{aligned}
&(\sqrt{z_{1,N}^{(\alpha)}}Q_{j-1,N}^{(\alpha)}(z_{1,N}^{(\alpha)}), \ldots, \sqrt{z_{N,N}^{(\alpha)}}Q_{j-1,N}^{(\alpha)}(z_{N,N}^{(\alpha)})) \\
&\quad = \frac{1}{\sqrt{N(N+\alpha)}}(\sqrt{z_{1,N}^{(\alpha)}}\tilde{Q}_{j-1,N}^{(\alpha)}(z_{1,N}^{(\alpha)}), \ldots, \sqrt{z_{N,N}^{(\alpha)}}\tilde{Q}_{j-1,N}^{(\alpha)}(z_{N,N}^{(\alpha)})).
\end{aligned} \tag{4.18}$$

for $j = 1, \ldots, N$ are orthonormal eigenvectors of S_N.

Finally, the monic Jacobi polynomials $\hat{R}_N := \hat{P}_N^{(\alpha,\beta)}$ satisfy

$$(1 - (z_{i,N}^{(\alpha,\beta)})^2)\hat{R}_N'(z_{i,N}^{(\alpha,\beta)}) = \frac{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}\hat{R}_N(z_{i,N}^{(\alpha,\beta)}).$$

This and (4.8) show that

$$\begin{aligned}
\sum_{i=1}^{N} \frac{(1-(z_{i,N}^{(\alpha,\beta)})^2)(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)} &\tilde{Q}_{m,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)})\tilde{Q}_{n,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}) \\
&= \delta_{mn}.
\end{aligned} \tag{4.19}$$

On the other hand, in Section 3 we imposed the following condition on $Q_{m,N}^{(\alpha,\beta)}(x)$:

$$\sum_{i=1}^{N}(1 - (z_{i,N}^{(\alpha,\beta)})^2)Q_{m,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)})Q_{n,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}) = \delta_{mn}. \tag{4.20}$$

We thus conclude that the i-th component of the j-th eigenvector is equal to

$$\begin{aligned}
&\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}Q_{j-1,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}) \\
&\quad = \sqrt{\frac{(1-(z_{i,N}^{(\alpha,\beta)})^2)(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}}\tilde{Q}_{j-1,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}).
\end{aligned} \tag{4.21}$$

This completes the proof. □

Remark 4.6. Notice that the comparison of the orthogonality relations (4.14) and (2.8) in the Hermite case in the proof above yields that for all $k = 0, \ldots, N-1$, $\tilde{Q}_{k,N}(x) = \sqrt{N} \cdot Q_k^{(N)}(x)$. This leads to a new proof of Proposition 2.4.

In a similar way, the orthogonality relations (4.17) and (3.9) yield that for all $k = 0, \ldots, N-1$, and $\alpha = \nu - 1$, we have $\tilde{Q}_{k,N}^{(\alpha)} = \sqrt{N(N+\alpha)} \cdot Q_k^{(N,\nu)}$. This leads to the three-term-recurrence for the polynomials $(Q_k^{(N,\nu)})_{k=0,\ldots,N-1}$. Moreover, a corresponding result is available in the Jacobi case.

Remark 4.7. Notice that by the proof of Lemma 4.5 in the Hermite, Laguerre, and Jacobi case the dual Christoffel numbers from (4.8) have the form

$$w_i^* = \frac{\hat{P}_{N-1}(z_{i,N})}{\hat{P}_N'(z_{i,N})} = \frac{\pi(z_{i,N})}{\kappa_N} \tag{4.22}$$

with suitable constants κ_N and polynomials π of degrees 0,1,and 2 respectively. By [VZ], such simple relations for the dual Christoffel numbers are available only for the classical orthogonal polynomials. This also includes the Bessel polynomials which are limits of Jacobi polynomials; see [I, p. 124, (4.10.10) and (4.10.13)].

In the next step we use the preceding results on dual orthogonal polynomials to compute the covariance matrices Σ_N from their inverses. For this we write the recurrence (4.3) for general orthonormal polynomials $(\tilde{P}_n)_{n\geq 0}$ for $n \leq N$ at the N ordered zeros $z_{i,N}$ of $\tilde{P}_N$ as the eigenvalue equation

$$\begin{pmatrix} a_0 & b_1 & & & \\ b_1 & a_1 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & a_{N-2} & b_{N-1} \\ & & & b_{N-1} & a_{N-1} \end{pmatrix} \begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix} = z_{i,N} \begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix}$$

of an $N \times N$-dimensional matrix. The zeros $\{z_{i,N}\}_{i=1}^N$ are the eigenvalues of this symmetric matrix and are distinct; this yields that the eigenvectors of this matrix are orthogonal and unique up to a constant coefficient. On the other hand, Lemma 4.3 and Remark 4.4 show that

$$(\tilde{Q}_{N-1,N}(z_{i,N}), \ldots, \tilde{Q}_{0,N}(z_{i,N}))^T \tag{4.23}$$

is also an eigenvector of this matrix for the eigenvalue $z_{i,N}$. It follows that

$$\begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix} = c_{i,N} \begin{pmatrix} \tilde{Q}_{N-1,N}(z_{i,N}) \\ \tilde{Q}_{N-2,N}(z_{i,N}) \\ \tilde{Q}_{N-3,N}(z_{i,N}) \\ \vdots \\ \tilde{Q}_{0,N}(z_{i,N}) \end{pmatrix},$$

with a constant $c_{i,N} \neq 0$. The last row of this equation and $\tilde{Q}_{0,N}(x) = 1$ give

$$c_{i,N} = \tilde{P}_{N-1}(z_{i,N}). \tag{4.24}$$

We remark that $c_{i,N}$ usually has the sign $(-1)^{N-i}$. This follows from the well-known intelacing property of the zeros of $\tilde{P}_{N-1}(x)$ and $\tilde{P}_N(x)$ together with the assumption that the leading coefficient of $\tilde{P}_{N-1}(x)$ is positive. This assumption holds for the Hermite and Jacobi cases. The Laguerre case will be handled below.

The constants $c_{i,N}$ can be also determined from an eigenvalue equation for the inverse matrices $S_N = \Sigma_N^{-1}$ for our random matrix ensembles. In fact, as the vectors in Lemma 4.5 form an orthogonal matrix in each of the cases considered there, we see that all rows and all columns of that matrix are orthogonal. Hence,

$$\frac{\sqrt{\pi(z_{i,N})\pi(z_{k,N})}}{\kappa_N c_{i,N} c_{k,N}} \sum_{j=0}^{N-1} \tilde{P}_j(z_{i,N})\tilde{P}_j(z_{k,N}) = \delta_{i,k} \quad \text{for all} \quad 1 \leq i,k \leq N.$$

In particular, for $i = k$,

$$c_{i,N} = \pm\sqrt{\frac{\pi(z_{i,N})}{\kappa_N} \sum_{j=0}^{N-1} \tilde{P}_j^2(z_{i,N})} \quad 1 \leq i \leq N. \tag{4.25}$$

Using the sign of $c_{i,N}$ above, we conclude that in the Hermite and Jacobi cases

$$c_{i,N} = (-1)^{N-i} \sqrt{\frac{\pi(z_{i,N})}{\kappa_N} \sum_{j=0}^{N-1} \tilde{P}_j^2(z_{i,N})}. \tag{4.26}$$

In the Laguerre case, the leading coefficient of $L_{N-1}^{(\alpha)}(x)$ has the sign $(-1)^{N-1}$. In this case, we obtain

$$c_{i,N}^{(\alpha)} = (-1)^{i-1} \sqrt{\frac{\pi^{(\alpha)}(z_{i,N}^{(\alpha)})}{\kappa_N^{(\alpha)}} \sum_{j=0}^{N-1} (\tilde{L}_j^{(\alpha)}(z_{i,N}))^2}. \tag{4.27}$$

These observations now lead to the following representation of Σ_N:

Theorem 4.8. *For the Hermite and Laguerre cases, the covariance matrices $\Sigma_N = (\sigma_{i,j}^N)_{i,j=1,\ldots,N}$ are given with the notations of Lemma 4.5 and with the eigenvalues λ_k from the Theorems 2.3, 3.2, and 3.4 by*

$$\begin{aligned} \sigma_{i,j}^N &= \frac{\sqrt{\pi(z_{i,N})\pi(z_{j,N})}}{\kappa_N \tilde{P}_{N-1}(z_{i,N})\tilde{P}_{N-1}(z_{j,N})} \sum_{k=0}^{N-1} \frac{\tilde{P}_k(z_{i,N})\tilde{P}_k(z_{j,N})}{\lambda_{N-k}} \\ &= \frac{(-1)^{i+j}}{\sqrt{\sum_{k,l=0}^{N-1} \tilde{P}_k^2(z_{i,N})\tilde{P}_l^2(z_{j,N})}} \sum_{k=0}^{N-1} \frac{\tilde{P}_k(z_{i,N})\tilde{P}_k(z_{j,N})}{\lambda_{N-k}}. \end{aligned} \tag{4.28}$$

Moreover, a corresponding result holds in the trigonometric Jacobi case for the covariance matrices $\tilde{\Sigma}_N = (\tilde{\sigma}_{i,j}^N)_{i,j=1,\ldots,N}$.

Proof. In all cases,

$$T_N^T \Sigma_N T_N = \operatorname{diag}(\lambda_1^{-1}, \ldots, \lambda_N^{-1}), \tag{4.29}$$

where the orthogonal matrix T has entries

$$\begin{aligned} [T_N]_{i,j} = Q_{j-1}^{(N)}(z_{i,N}) &= \sqrt{\frac{\pi(z_{i,N})}{\kappa_N}} \tilde{Q}_{j-1,N}(z_{i,N}) \\ &= \frac{1}{c_{i,N}} \sqrt{\frac{\pi(z_{i,N})}{\kappa_N}} \tilde{P}_{N-j}(z_{i,N}). \end{aligned} \tag{4.30}$$

Hence,

$$\sigma_{i,j}^N = \frac{\sqrt{\pi(z_{i,N})\pi(z_{j,N})}}{\kappa_N c_{i,N} c_{j,N}} \sum_{k=0}^{N-1} \frac{\tilde{P}_{N-1-k}(z_{i,N})\tilde{P}_{N-1-k}(z_{j,N})}{\lambda_{k+1}}. \tag{4.31}$$

The substitution $N-1-k \to k$ and (4.24), (4.26), and (4.27) yield the result. □

Remark 4.9. The formulas for the entries of the covariance matrices Σ_N in (4.28) should be compared with the corresponding results of Dumitriu and Edelman [DE2] for the Hermite and Laguerre ensembles. In the Hermite case, the entries of Σ_N in (4.28) must be equal to entries in (2.5) in Theorem 2.1. Unfortunately, we are not able to verify the equivalence of (4.28) and (2.5) for arbitrary dimensions N. For small N, we checked the equality by a numerical computation. In our opinion, our representation in (4.28) seems to be slightly nicer that the formula (2.5) of Dumitriu and Edelman [DE2] in the Hermite case.

In the Laguerre case we have a corresponding picture. However, here our formula (4.28) has the same structure as in the Hermite case, while the corresponding formula in [DE2] is much more involved.

In the Jacobi case, there do not exist formulas for the entries of Σ_N in the literature as far as we are aware.

All preceding results for β-Jacobi ensembles were stated in trigonometric coordinates as only in this case the eigenvalues and eigenvectors of the (inverse) covariance matrices of the limit are known; see Theorem 3.4. On the other hand, all trigonometric results above can be easily transfered to classical β-Jacobi ensembles. We briefly collect these results here. We follow [F, K, KN, Me, HV] and consider the β-Jacobi random matrix ensembles for $k_1, k_2, k_3 \geq 0$ with the joint eigenvalue distributions $\mu_{(k_1,k_2,k_3)}$ with the densities

$$c_{k_1,k_2,k_3} \prod_{1\leq i<j\leq N} (x_j - x_i)^{k_3} \prod_{i=1}^{N} (1-x_i)^{\frac{k_1+k_2}{2}-\frac{1}{2}} (1+x_i)^{\frac{k_2}{2}-\frac{1}{2}} \tag{4.32}$$

on the alcoves $A := \{x \in \mathbb{R}^N : -1 \leq x_1 \leq ... \leq x_N \leq 1\}$ with some Selberg constant $c_{k_1,k_2,k_3} > 0$. As in Section 3 we write $(k_1, k_2, k_3) = \kappa \cdot (a, b, 1)$ with $a \geq 0$ $b > 0$ fixed and $\kappa \to \infty$. We put $\alpha := a + b - 1 > -1$, $\beta = b - 1 > -1$, and consider the vector $z := (z_1, \ldots, z_N) \in A$ consisting of the ordered zeros of the Jacobi polynomial $P_N^{(\alpha,\beta)}$). Using the transformation

$$T : \tilde{A} \longrightarrow A, \quad T(t_1, \ldots, t_N) := (\cos(2t_1), \ldots, \cos(2t_N)),$$

the LT 3.3 then reads as follows by [HV].

Theorem 4.10. *Let $a \geq 0$ and $b > 0$. Let X_κ be random variables with the distributions $\mu_{\kappa\cdot(a,b,1)}$ as above. Then $\sqrt{\kappa}(X_\kappa - z)$ converges for $\kappa \to \infty$ to the normal distribution $N(0, \Sigma_N)$ with some regular covariance matrix Σ_N whose inverse $\Sigma_N^{-1} =: S_N = (s_{i,j})_{i,j=1,\ldots,N}$ is given by*

$$s_{i,j} = \begin{cases} \sum_{l=1,\ldots,N; l\neq j} \frac{1}{(z_j - z_l)^2} + \frac{a+b}{2} \frac{1}{(1-z_j)^2} + \frac{b}{2} \frac{1}{(1+z_j)^2} & \text{for } i = j \\ \frac{-1}{(z_i - z_j)^2} & \text{for } i \neq j \end{cases}.$$

The inverse covariance matrices $\tilde{\Sigma}_N^{-1}$ and Σ_N^{-1} from the LTs 3.3 and 4.10 are related by $\tilde{S} = DSD$ with the diagonal matrix

$$D = \operatorname{diag}\left(-2\sqrt{1 - z_{1,N}^2}, \ldots, -2\sqrt{1 - z_{N,N}^2}\right)$$

by [HV]. Hence, Theorem 4.8 means in the non-trigonometric Jacobi case:

Theorem 4.11. *The covariance matrix $\Sigma_N = (\sigma_{i,j}^N)_{i,j=1,\ldots,N}$ in Theorem 4.10 has entries*

$$\sigma_{i,j}^N = \frac{(-1)^{i+j} 4\sqrt{1 - z_{i,N}^2}\sqrt{1 - z_{j,N}^2}}{\sqrt{\sum_{k,l=0}^{N-1} (\tilde{P}_k^{(\alpha,\beta)}(z_{i,N}) \tilde{P}_l^{(\alpha,\beta)}(z_{j,N}))^2}} \sum_{k=0}^{N-1} \frac{\tilde{P}_k^{(\alpha,\beta)}(z_{i,N}) \tilde{P}_k^{(\alpha,\beta)}(z_{j,N})}{\lambda_{N-k}}. \tag{4.33}$$

5. Limit results for the largest eigenvalue for $N \to \infty$ in the Hermite case

In this chapter we discuss the soft edge statistics in the Hermite case in the freezing regime. This means that we analyze the limit behaviour of the largest eigenvalue in the freezing regime in Theorem 2.3 for $N \to \infty$. This will be done on the basis of Theorem 4.8. We remark that this problem is also discussed in [DE2] through (2.5). We show that our approach via (4.28) leads to a limit with a different form from that in [DE2].

As in Section 2, let now $z_{1,N} < ..., z_{N,N}$ be the the ordered zeros of the Hermite polynomial H_N. Moreover, for each N, let $(Q_{k,N})_{k=0,..,N-1}$ be the dual polynomials associated with $(H_k)_{k=0,...,N}$ normalized as in (2.8). This means that $T_N := (Q_{j-1,N}(z_{i,N}))_{i,j=1,...,N}$ is an orthogonal matrix with $T_N^T \Sigma_N T_N = \mathrm{diag}(1, ..., \frac{1}{N})$ as in the proof of Theorem 4.8. These polynomials satisfy the three-term-recurrence

$$xQ_{k,N}(x) = \sqrt{\frac{N-k-1}{2}}Q_{k+1,N}(x) + \sqrt{\frac{N-k}{2}}Q_{k-1,N}(x) \quad (k \le N) \tag{5.1}$$

with the initial conditions $Q_{-1,N} = 0$ and $Q_{0,N} = \frac{1}{\sqrt{N}}$.

We now derive a limit result for $N \to \infty$ which involves the Airy function Ai. For this we recapitulate some well known facts about Ai; see e.g. Section 9 of [NIST] or the monograph [VS]. Ai is the unique solution of

$$y''(z) = z \cdot y(z) \quad (z \in \mathbb{R}) \quad \text{with} \quad \lim_{z\to\infty} y(z) = 0 \tag{5.2}$$

and with $y(0) = \frac{1}{3^{2/3}\Gamma(2/3)} = 0.355028\ldots$. The Airy function Ai has a unique largest zero at $a_1 = -2.338\ldots$ with $\mathsf{Ai}(z) > 0$ for $z > a_1$. Moreover, Ai has infinitely many isolated, simple zeros in $]-\infty, a_1]$. For $r \in \mathbb{N}$, the r-th largest zero a_r of Ai satisfies

$$a_r \simeq -\Big(\frac{3\pi}{2}(r-1/4)\Big)^{2/3} \qquad \text{for} \quad r \to \infty. \tag{5.3}$$

In addition, we have the asymptotic behavior as $z \to -\infty$

$$\mathsf{Ai}(-z) \simeq \frac{1}{\sqrt{\pi}z^{1/4}} \cos\Big(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\Big), \tag{5.4}$$

as well as

$$\mathsf{Ai}'(a_r) \simeq \frac{(-1)^{r-1}}{\sqrt{\pi}} \Big(\frac{3\pi}{2}(r-1/4)\Big)^{1/6} \qquad \text{for} \quad r \to \infty. \tag{5.5}$$

The following theorem is the central step for our limit results for $N \to \infty$:

Theorem 5.1. *Consider the functions*

$$f_N(y) := N^{\frac{1}{6}} Q_{\lfloor N^{\frac{1}{3}} y \rfloor, N}(z_{N,N}) \quad \text{for} \quad y \in [0, N^{\frac{2}{3}}[$$

and $f_N(y) = 0$ otherwise. Then $(f_N)_{N\ge1}$ tends for $N \to \infty$ locally uniformly to

$$f(y) = \frac{\mathsf{Ai}(y + a_1)}{\mathsf{Ai}'(a_1)} \qquad \text{for} \quad y \in [0, \infty[.$$

We split the proof into three lemmas and use the abbreviation $q_k := Q_{k,N}(z_{N,N})$ where we suppress the dependence on N. We start with the following result:

Lemma 5.2. *The functions f_N satisfy for $y \in [0, N^{\frac{2}{3}}[$ the equation*

$$f_N(y) = \int_0^y \int_0^s (t - |a_1|) f_N(t)\, dt\, ds + y + \mathrm{err}(y, N).$$

The error term $\mathrm{err}(y, N)$ *is specified in Eq. (5.17) at the end of the proof.*

Proof. Let $y \geq 0$. We divide the recurrence (5.1) with $x := z_{N,N}$ by $\sqrt{N}$ and get

$$\sqrt{1-\frac{k+1}{N}}q_{k+1} = \frac{2z_{N,N}}{\sqrt{2N}}q_k - \sqrt{1-\frac{k}{N}}q_{k-1} \quad (k \leq N) \tag{5.6}$$

with $q_{-1} = 0$, $q_0 = N^{-1/2}$. We next observe that by the Lagrange remainder in Taylor's formula, for $k = 0, ..., \lfloor yN^{\frac{1}{3}} \rfloor$,

$$\sqrt{1-\frac{k}{N}} = 1 - \frac{k}{2N} - \frac{1}{8(1-\xi_k)^{\frac{3}{2}}}\left(\frac{k}{N}\right)^2 \quad \text{with} \quad \xi_k \in (0, \frac{k}{N}). \tag{5.7}$$

We now define

$$\alpha(k, N) := \frac{1}{8(1-\xi_k)^{\frac{3}{2}}}\left(\frac{k}{N}\right)^2$$

and conclude from (5.7) that for $k = 0, ..., \lfloor yN^{\frac{1}{3}} \rfloor$

$$0 < \alpha(k, N) < \left(\frac{N}{N-k}\right)^{\frac{3}{2}}\left(\frac{k}{N}\right)^2. \tag{5.8}$$

Moreover, we obtain from a sharp Plancherel-Rotach theorem of Ricci [Ri] that

$$\frac{z_{N,N}}{\sqrt{2N}} = 1 - \frac{|a_1|}{2N^{\frac{2}{3}}} + O(N^{-1}). \tag{5.9}$$

Using (5.9) we rewrite the recurrence (5.6) as

$$\begin{aligned} &q_{k+1} - q_k - (q_k - q_{k-1}) \\ =&\frac{k+1}{2N}q_{k+1} - \frac{|a_1|}{N^{\frac{2}{3}}}q_k + \frac{k}{2N}q_{k-1} + \alpha(k+1, N)q_{k+1} + \alpha(k, N)q_{k-1} + O(N^{-1})q_k. \end{aligned} \tag{5.10}$$

Summation over $k = 0, ..., l$ now yields

$$\begin{aligned} q_{l+1} - q_l - \frac{1}{\sqrt{N}} = q_{l+1} - q_l - (q_0 - q_{-1}) &= \sum_{k=0}^{l}\Big(q_{k+1} - q_k - (q_k - q_{k-1})\Big) \\ &= \sum_{k=0}^{l}\Big(\frac{k+1}{2N}q_{k+1} - \frac{|a_1|}{N^{\frac{2}{3}}}q_k + \frac{k}{2N}q_{k-1}\Big) + \\ &\quad + \sum_{k=0}^{l}\Big(\alpha(k+1, N)q_{k+1} + \alpha(k, N)q_{k-1} + O(N^{-1})q_k\Big). \end{aligned}$$

A second summation over $l = 0, ..., \lfloor yN^{\frac{1}{3}} \rfloor - 1$ now leads to

$$\begin{aligned} q_{\lfloor yN^{\frac{1}{3}} \rfloor} - \frac{\lfloor yN^{\frac{1}{3}} \rfloor + 1}{\sqrt{N}} &= \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1}\left(q_l - q_{l-1} - \frac{1}{\sqrt{N}}\right) = \\ &= \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1}\sum_{k=0}^{l}\left(\frac{k+1}{2N}q_{k+1} - \frac{|a_1|}{N^{\frac{2}{3}}}q_k + \frac{k}{2N}q_{k-1}\right) + \rho(y, N) \end{aligned}$$

with

$$\rho(y, N) := \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1}\sum_{k=0}^{l}\Big(\alpha(k+1, N)q_{k+1} + \alpha(k, N)q_{k-1} + O(N^{-1})q_k\Big). \tag{5.11}$$

If we multiply this by $N^{\frac{1}{6}}$ we get

$$
\begin{aligned}
&f_N(y) - \frac{\lfloor yN^{\frac{1}{3}}\rfloor + 1}{N^{\frac{1}{3}}} - N^{\frac{1}{6}}\rho(y,N) \\
&= \frac{1}{N^{\frac{1}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor - 1} \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{l} \left(\frac{k+1}{2N^{\frac{1}{3}}} N^{\frac{1}{6}} q_{k+1} - |a_1| N^{\frac{1}{6}} q_k + \frac{k}{2N^{\frac{1}{3}}} N^{\frac{1}{6}} q_{k-1} \right) \\
&= \frac{1}{N^{\frac{1}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor - 1} \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{l} \left(N^{\frac{1}{6}} q_k \left(\frac{k+\frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right) \right) + \\
&\quad + \frac{1}{N^{\frac{2}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor - 1} \left(\frac{l+1}{2N^{\frac{1}{3}}} N^{\frac{1}{6}} \left(q_{l+1} - q_l\right) \right).
\end{aligned}
\tag{5.12}
$$

Notice that the last equation was obtained from the shifts $k+1 \mapsto k$ and $k-1 \mapsto k$. We now compare the r.h.s. of (5.12) with

$$
\int_0^y \int_0^s (x - |a_1|) f_N(t) dtds. \tag{5.13}
$$

For this we use the functions

$$
g_N(t) := \sum_{k=0}^{N-1} t_{k,N} \mathbf{1}_{\left[\frac{k}{N^{\frac{1}{3}}}, \frac{k+1}{N^{\frac{1}{3}}}\right]}(t) \quad \text{with} \quad t_{k,N} := N^{\frac{1}{6}} q_k \left(\frac{k+\frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right).
$$

An elementary calculation yields

$$
\begin{aligned}
&\int_0^y \int_0^s g_N(t)\, dtds = \int_0^y \int_0^s \sum_{k=0}^{N-1} t_{k,N} \mathbf{1}_{\left[\frac{k}{N^{\frac{1}{3}}}, \frac{k+1}{N^{\frac{1}{3}}}\right]}(t)\, dt\, ds \\
&\quad = \frac{1}{2} \left(\frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}}\rfloor}{N^{\frac{1}{3}}} \right)^2 t_{\lfloor yN^{\frac{1}{3}}\rfloor, N} + \frac{1}{N^{\frac{2}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor - 1} (\lfloor yN^{\frac{1}{3}}\rfloor - k) t_{k,N} \\
&\qquad + \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}}\rfloor - \frac{1}{2}}{N^{\frac{1}{3}}} \cdot \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor - 1} t_{k,N}.
\end{aligned}
$$

Moreover,

$$
\sum_{l=0}^{L} \sum_{k=0}^{l} t_{k,N} = \sum_{k=0}^{L} (L-k+1) t_{k,N} \qquad (L \in \mathbb{N}). \tag{5.14}
$$

Hence,

$$
\begin{aligned}
&\int_0^y \int_0^s g_N(t) dtds - \frac{1}{N^{\frac{2}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor - 1} \sum_{k=0}^{l} t_{k,N} \qquad (5.15) \\
&= \frac{1}{2} \left(\frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}}\rfloor}{N^{\frac{1}{3}}} \right)^2 t_{\lfloor yN^{\frac{1}{3}}\rfloor, N} + \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}}\rfloor - \frac{1}{2}}{N^{\frac{1}{3}}} \cdot \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor - 1} t_{k,N}.
\end{aligned}
$$

(5.12), (5.13), and (5.15) now show that

$$f_N(y) = \int_0^y \int_0^s f_N(t) \left(\frac{\lfloor tN^{\frac{1}{3}} \rfloor + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right) dtds + \frac{\lfloor yN^{\frac{1}{3}} \rfloor + 1}{N^{\frac{1}{3}}} + \widetilde{\mathrm{err}}(N,y) \quad (5.16)$$

with the error term

$$\begin{aligned}\widetilde{\mathrm{err}}(N,y) := N^{\frac{1}{6}} \rho(y,N) &+ \frac{1}{N^{\frac{2}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \frac{l+1}{2N^{\frac{1}{3}}} (q_{l+1} - q_l) \\ &- \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor - \frac{1}{2}}{N^{\frac{1}{3}}} \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} N^{\frac{1}{6}} q_k \left(\frac{k + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right) \\ &- \frac{1}{2} \left(\frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor}{N^{\frac{1}{3}}} \right)^2 N^{\frac{1}{6}} q_{\lfloor yN^{\frac{1}{3}} \rfloor} \left(\frac{\lfloor yN^{\frac{1}{3}} \rfloor + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right).\end{aligned}$$

As

$$\frac{\lfloor yN^{\frac{1}{3}} \rfloor + 1}{N^{\frac{1}{3}}} = y + O(N^{-\frac{1}{3}})$$

and

$$\frac{\lfloor tN^{\frac{1}{3}} \rfloor + \frac{1}{2}}{N^{\frac{1}{3}}} = t + \frac{\lfloor tN^{\frac{1}{3}} \rfloor - tN^{\frac{1}{3}} + \frac{1}{2}}{N^{\frac{1}{3}}} = t + O(N^{-\frac{1}{3}}),$$

we get

$$f_N(y) = \int_0^y \int_0^s (t - |a_1|) f_N(t) dtds + y + \mathrm{err}(y,N)$$

with the error term

$$\begin{aligned}\mathrm{err}(N,y) = N^{\frac{1}{6}} \rho(y,N) &+ \frac{1}{N^{\frac{2}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \frac{l+1}{2N^{\frac{1}{3}}} (q_{l+1} - q_l) \\ &- \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor - \frac{1}{2}}{N^{\frac{1}{3}}} \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} N^{\frac{1}{6}} q_k \left(\frac{k + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right) \\ &- \frac{1}{2} \left(\frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor}{N^{\frac{1}{3}}} \right)^2 N^{\frac{1}{6}} q_{\lfloor yN^{\frac{1}{3}} \rfloor} \left(\frac{\lfloor yN^{\frac{1}{3}} \rfloor + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right) \\ &+ \frac{\lfloor yN^{\frac{1}{3}} \rfloor - yN^{\frac{1}{3}} + 1}{N^{\frac{1}{3}}} + \int_0^y \int_0^s \frac{\lfloor tN^{\frac{1}{3}} \rfloor - tN^{\frac{1}{3}} + \frac{1}{2}}{N^{\frac{1}{3}}} f_N(t) dtds \qquad (5.17)\end{aligned}$$

□

Lemma 5.3. *The error term in (5.17) satisfies* $\mathrm{err}(N,y) = O(N^{-\frac{1}{3}})$ *locally uniformly in* $y \in [0,\infty[$.

Proof. Fix some $M > 0$ and consider $y \in [0,M]$. We recapitulate that the matrices $T_N = (Q_{k-1,N}(z_{i,N}))_{k,i=1,\ldots,N}$ are orthogonal which implies that for all $N \in \mathbb{N}$

$$1 = \sum_{k=0}^{N-1} (Q_{k,N}(z_{N,N}))^2 = \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{N-1} (N^{\frac{1}{6}} Q_{k,N}(z_{N,N}))^2 = \int_0^\infty f_N^2(t) dt. \qquad (5.18)$$

We next prove

$$\int_0^y f_N(t)dt = O(1) \quad \text{for} \quad N \to \infty. \tag{5.19}$$

For this we recall that by the definition of f_N

$$f_N(t) = \sum_{k=0}^{N-1} N^{\frac{1}{6}} Q_{k,N}(z_{N,N}) \mathbf{1}_{\left[\frac{k}{N^{1/3}}, \frac{k+1}{N^{1/3}}\right]}(t)$$

with $Q_{k,N}(z_{N,N}) > 0$ for all k. This follows from the fact that the polynomials $Q_{k,N}$ have a positive leading coefficient and are orthogonal w.r.t. some measure with support $\{z_{1,N}, \dots, z_{N,N}\}$ which implies that all their zeros are contained in $]z_{1,N}, z_{N,N}[$; see e.g. [C]. We thus see that $f_N(t) \geq 0$ for $t \geq 0$. Hence, for $y \in [0, M]$,

$$\begin{aligned}\int_0^y f_N(t)dt \leq \int_0^M f_N(t)dt &= \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor MN^{\frac{1}{3}} \rfloor -1} N^{\frac{1}{6}} q_k + \frac{MN^{\frac{1}{3}} - \lfloor MN^{\frac{1}{3}} \rfloor}{N^{\frac{1}{3}}} N^{\frac{1}{6}} q_{\lfloor MN^{\frac{1}{3}} \rfloor} \\ &\leq \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor MN^{\frac{1}{3}} \rfloor} N^{\frac{1}{6}} q_k.\end{aligned}$$

Hölder's inequality and (5.18) now imply that for $y \in [0, M]$ and $N \in \mathbb{N}$,

$$\begin{aligned}\int_0^y f_N(t)dt &\leq \frac{1}{N^{\frac{1}{3}}} \left(\sum_{k=0}^{\lfloor MN^{\frac{1}{3}} \rfloor} q_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\lfloor MN^{\frac{1}{3}} \rfloor} N^{\frac{1}{3}} \right)^{\frac{1}{2}} \\ &\leq \frac{1}{N^{\frac{1}{3}}} \sqrt{N^{\frac{1}{3}} (\lfloor MN^{\frac{1}{3}} \rfloor + 1)} \leq \sqrt{M + \frac{2}{N^{\frac{1}{3}}}} \leq \sqrt{M+2}.\end{aligned} \tag{5.20}$$

This shows (5.19). In an analogous way we prove that for $y \in [0, M]$ and $\theta \in [0, 1]$,

$$\frac{1}{N^{\frac{1}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor -1} \frac{l+\theta}{N^{\frac{1}{3}}} N^{\frac{1}{6}} q_l = O(1). \tag{5.21}$$

For this we observe that

$$\sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor -1} \frac{l+\theta}{N^{\frac{1}{3}}} N^{\frac{1}{6}} q_l \leq \sum_{l=0}^{\lfloor MN^{\frac{1}{3}} \rfloor -1} \frac{MN^{\frac{1}{3}}+1}{N^{\frac{1}{3}}} N^{\frac{1}{6}} q_l \leq (M+1) \sum_{l=0}^{\lfloor MN^{\frac{1}{3}} \rfloor -1} N^{\frac{1}{6}} q_l.$$

This together with (5.19) shows (5.21).

Moreover, (5.20) leads to the following estimate for the last term in (5.17):

$$\begin{aligned}\left| \int_0^y \int_0^s \frac{\lfloor tN^{\frac{1}{3}} \rfloor - tN^{\frac{1}{3}} + \frac{1}{2}}{N^{\frac{1}{3}}} f_N(t)dtds \right| &\leq \frac{1}{2N^{\frac{1}{3}}} \int_0^y \sqrt{M+2}ds \\ &\leq \frac{M\sqrt{M+2}}{N^{\frac{1}{3}}} = O(N^{-\frac{1}{3}}).\end{aligned} \tag{5.22}$$

We now turn to the estimation of $N^{\frac{1}{6}}\rho(y, N)$. For $y \in [0, M]$ and $k = 0, ..., \lfloor yN^{\frac{1}{3}} \rfloor$ we obtain that $N/(N-k)$ remains bounded for large N. Therefore, (5.8) implies readily that $\alpha(k,N)=O(N^{-\frac{4}{3}})$ and thus, by (5.11),

$$\begin{aligned} &|N^{\frac{1}{6}}\rho(y,N)| \qquad (5.23)\\ &\le \sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\Big(|O(N^{-\frac{4}{3}})|N^{\frac{1}{6}}q_{k+1}+|O(N^{-\frac{4}{3}})|N^{\frac{1}{6}}q_{k-1}+|O(N^{-1})|N^{\frac{1}{6}}q_k\Big)\\ &\le \sum_{l=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\Big(|O(N^{-\frac{7}{6}})|q_{k+1}+|O(N^{-\frac{7}{6}})|q_{k-1}+|O(N^{-\frac{5}{6}})|q_k\Big).\end{aligned}$$

If we use the summation formula (5.14) and Hölder's inequality, we see that the third summand on the r.h.s. of (5.23) satisfies

$$\begin{aligned} &|O(N^{-\frac{5}{6}})|\sum_{l=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}q_k=|O(N^{-5/6})|\sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1}(\lfloor MN^{\frac{1}{3}}\rfloor-k)q_k\\ \le&|O(N^{-\frac{5}{6}})|\left(\sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1}q_k^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1}\underbrace{(\lfloor MN^{\frac{1}{3}}\rfloor-k)^2}_{\le M^2N^{\frac{2}{3}}}\right)^{\frac{1}{2}}\\ \le&|O(N^{-\frac{5}{6}})|\left(\lfloor MN^{\frac{1}{3}}\rfloor M^2N^{\frac{2}{3}}\right)^{\frac{1}{2}}=O(N^{-\frac{1}{3}}).\end{aligned}$$

If we keep in mind that $q_0=\frac{1}{\sqrt{N}}$ we can estimate the other two sums in the same way. In summary, we conclude for the first term in (5.17) that

$$N^{\frac{1}{6}}\rho(y,N)=O(N^{-\frac{1}{3}}).$$

Furthermore, the second term in (5.17) can be estimated by a corresponding bound by (5.21) with $\theta=1/2$ and 1 and with an index shift together with $Q_{0,N}=\frac{1}{\sqrt{N}}$. Moreover, the third term in (5.17) can be estimated in the same way by splitting the sum there and using (5.21) for the first and (5.18) for the second sum. Finally, the fourth and fifth term in (5.17) obviously have order $O(N^{-\frac{1}{3}})$, while this follows for the last term from (5.22). This completes the proof. □

We now complete the proof of Theorem 5.1 by proving the following

Lemma 5.4. *For $N\to\infty$, $|f_N(y)-f(y)|=O(N^{-\frac{1}{3}})$ locally uniformly for $y\in[0,\infty[$.*

Proof. Again, fix $M>0$, let $y\in[0,M]$, and assume that $N^{\frac{2}{3}}>M$. The ODE (5.2) yields that the function $f(y)=\frac{\mathsf{Ai}(y+a_1)}{\mathsf{Ai}'(a_1)}$ satisfies

$$f''(y)=(y+a_1)f(y)\quad\text{with}\quad f(0)=0,\quad f'(0)=1. \tag{5.24}$$

This ODE leads to the integral equation

$$f(y)=\int_0^y\int_0^s(t-|a_1|)f(t)dtds+y=\int_0^y(t-|a_1|)(y-t)f(t)\,dt+y. \tag{5.25}$$

Notice that the second equation in (5.25) follows by partial integration. Moreover, by Lemma 5.2,

$$f_N(y)=\int_0^y(t-|a_1|)(y-t)f_N(t)\,dt+y+\mathrm{err}(y,N).$$

We thus obtain

$$\begin{aligned}|f(y)-f_N(y)| &= \left|\int_0^y (t-|a_1|)(y-t)(f(t)-f_N(t))dt - \operatorname{err}(y,N)\right| \\ &\le \int_0^y |t-|a_1||\cdot|y-t|\cdot|f(t)-f_N(t)|dt + |\operatorname{err}(y,N)|\end{aligned}$$

where we know from Lemma 5.3 that there exist a constant $M' = M'(M) > 0$ with

$$|\operatorname{err}(y,N)| \le \frac{M'}{N^{\frac{1}{3}}} \quad \text{for} \quad y \in [0,M]$$

and N sufficently large.

As $t \mapsto |(t-|a_1|)(y-t)|$ is the absolute value of a second-order polynomial, we find a constant $M'' > 0$ with $|(t-|a_1|)(y-t)| < M''$ for all $t \in [0,y]$ and $y \in [0,M]$. Hence,

$$|f(y)-f_N(y)| \le \int_0^y M''|f(t)-f_N(t)|dt + \frac{M'}{N^{\frac{1}{3}}}.$$

The Lemma of Gronwall now implies our claim that

$$|f(y)-f_N(y)| \le \frac{M'}{N^{\frac{1}{3}}}e^{M''y} \le \frac{M'}{N^{\frac{1}{3}}}e^{M''M} = O(N^{-\frac{1}{3}}).$$

□

We now apply Lemma 5.4 to the (N,N)-entries of the covariance matrices Σ_N for β-Hermite ensembles in the freezing regime as in Theorem 4.8 for $N \to \infty$.

Theorem 5.5. *Consider the covariance matrices* $\Sigma_N =: (\sigma_{i,j})_{i,j=1,\dots,N}$ *of β-Hermite ensembles in the freezing regime. Then*

$$\lim_{N\to\infty} N^{\frac{1}{3}}\sigma_{N,N} = \int_0^\infty \frac{\mathsf{Ai}(x+a_1)^2}{\mathsf{Ai}'(a_1)^2 x}dx = 0.834\dots$$

Proof. We recapitulate that $\Sigma_N = T_N \operatorname{diag}(1, 1/2, ..., 1/N)T_N^T$. Therefore,

$$\sigma_{N,N} = \sum_{k=1}^N \frac{1}{k}(Q_{k-1}^N(z_{N,N}))^2 = \frac{1}{N^{\frac{2}{3}}}\sum_{k=0}^{N-1}\frac{N^{\frac{1}{3}}}{k+1}\left(N^{\frac{1}{6}}Q_k^N(z_{N,N})\right)^2.$$

Define the functions

$$h_N(y) := \sum_{k=0}^{N-1}\frac{N^{\frac{1}{3}}}{k+1}\mathbf{1}_{\left[\frac{k}{N^{\frac{1}{3}}},\frac{k+1}{N^{\frac{1}{3}}}\right)}(y),$$

which are approximations of the function $y \mapsto \frac{1}{y}$ with

$$0 \le \frac{1}{y} - h_N(y) \le \frac{N^{\frac{1}{3}}}{k(k+1)} \le \frac{1}{y}\frac{1}{k} \quad \text{for} \quad k = \lfloor yN^{\frac{1}{3}}\rfloor,\ y > 0. \tag{5.26}$$

With this notation we have

$$N^{\frac{1}{3}}\sigma_{N,N} = \frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{N-1}\frac{N^{\frac{1}{3}}}{k+1}\left(N^{\frac{1}{6}}Q_k^N(z_{N,N})\right)^2 = \int_0^\infty (f_N(y))^2 h_N(y)dy.$$

The statement of the theorem is now equivalent to

$$\lim_{N\to\infty}\int_0^\infty (f_N(y))^2 h_N(y)dy = \int_0^\infty \frac{\mathsf{Ai}(x+a_1)^2}{\mathsf{Ai}'(a_1)^2 y}dy = \int_0^\infty \frac{f(y)^2}{y}dy.$$

To show this, we shall show that

$$\lim_{N\to\infty}\int_0^1 f_N(y)^2 h_N(y)\,dy = \int_0^1 \frac{f(y)^2}{y}\,dy \tag{5.27}$$

and

$$\lim_{N\to\infty}\int_1^\infty f_N(y)^2 h_N(y)\,dy = \int_1^\infty \frac{f(y)^2}{y}\,dy. \tag{5.28}$$

For this we first recapitulate from (5.18) that

$$\int_0^\infty (f_N(y))^2\,dy = 1. \tag{5.29}$$

Furthermore, as $f''(y) = (y-|a_1|)f(y)$, we know that

$$\int_0^\infty f(y)^2\,dy = -\left[|a_1|f(y)^2 + f'(y)^2\right]_{y=0}^\infty = 1. \tag{5.30}$$

We next observe that Theorem 5.1 implies that the measures $f_N^2 d\lambda$ with Lebesgue densities f_N^2 converge in a vague way to the measure $f^2 d\lambda$ on $[0,\infty[$. As all these measures are probability measures by (5.29) and (5.30), we conclude from a standard result in probability (see e.g. [Bi]) that these measures converge even weakly, i.e, for all bounded continuous functions $g:[0,\infty[\to\mathbb{R}$ we have

$$\lim_{N\to\infty}\int_0^\infty g(y)f_N(y)^2\,dy = \int_0^\infty g(y)f(y)^2\,dy. \tag{5.31}$$

Moreover, as all these probability measures have Lebesgue densities, we again conclude from a standard result in probability (see e.g. [Bi]) that (5.31) remains correct on $[1,\infty[$, i.e., for the bounded continuous function $g(y):=\frac{1}{y}$ on $[1,\infty[$ we have

$$\lim_{N\to\infty}\int_1^\infty \frac{f_N(y)^2}{y}\,dy = \int_1^\infty \frac{f(y)^2}{y}\,dy =: R. \tag{5.32}$$

On the other hand, (5.26) shows that for any $\varepsilon>0$ there is some sufficiently large $N(\varepsilon)$ with

$$\left|\frac{1}{y} - h_N(y)\right| \le \frac{\varepsilon}{y} \quad \text{for} \quad y\ge 1,\ N\ge N(\varepsilon).$$

Therefore,

$$\int_1^\infty \left|\frac{1}{y} - h_N(y)\right| f_N(y)^2\,dy \le \varepsilon\int_1^\infty \frac{1}{y}\, f_N(y)^2\,dy, \tag{5.33}$$

where, by (5.32), the r.h.s. converges for $N\to\infty$ to εR. (5.33), (5.32), and the triangle inequality now readily lead to (5.28).

We finally check (5.27). We recall that

$$\lim_{N\to\infty} f_N(y)^2 h_N(y) = \frac{(f(y))^2}{y} \quad \text{for} \quad y\in[0,1[.$$

Notice that this formula also holds for $y=0$, as f is analytic in 0 with $f(0)=0$. Moreover, (5.26), the fact that $h_N(y)\le N^{1/3}$, and Lemma 5.4 show that for N

sufficiently large

$$\begin{aligned}|f_N(y)^2 h_N(y)| &\le |f_N(y)^2 - f(y)^2| h_N(y) + f(y)^2 h_N(y)\\ &\le (f_N(y)+f(y))|f_N(y)-f(y)|N^{\frac{1}{3}} + \frac{f(y)^2}{y}\\ &\le (1+2f(y))O(1) + \frac{f(y)^2}{y}.\end{aligned}$$

As this is a bounded continuous function for $y \in [0,1]$, we conclude from dominated convergence that (5.27) holds. This completes the proof. □

If we combine Theorem 5.5 with LT 2.2, we finally obtain:

Theorem 5.6. *Consider the Bessel processes*

$$(X_{t,k}^N)_{t\ge 0} = (X_{t,k,1}^N, \dots, X_{t,k,N}^N)_{t\ge 0}$$

of type A_{N-1} on C_N^A with start in $0 \in C_N^A$. Then, for each $t > 0$,

$$\lim_{N\to\infty}\left(\lim_{k\to\infty} N^{\frac{1}{6}}\sqrt{2k}\left(\frac{X_{t,k,N}^N}{\sqrt{2kt}} - z_{N,N}\right)\right) = G \tag{5.34}$$

in distribution with some $\mathcal{N}(0,\sigma_{max}^2)$-distributed random variable G with variance

$$\sigma_{max}^2 := \int_0^\infty \frac{\mathsf{Ai}(x+a_1)^2}{(\mathsf{Ai}'(a_1))^2 x}dx = 0.834... \tag{5.35}$$

Remarks. (1) If we combine Theorem 5.6 with the formula of Plancherel-Rotach

$$\frac{z_{N,N}}{\sqrt{2N}} = 1 - \frac{|a_1|}{2N^{\frac{2}{3}}} + r_N \quad \text{with} \quad r_N = O(N^{-1}),$$

we can state (5.34) as

$$\lim_{N\to\infty}\left(\lim_{k\to\infty}\left(N^{\frac{2}{3}}\left(\frac{X_{t,k,N}^N}{\sqrt{tN}} - 2\sqrt{k}\right) + 2\sqrt{k}(|a_1| - N^{\frac{2}{3}} r_N)\right)\right) = G. \tag{5.36}$$

Please notice that in this limit the term $2\sqrt{k}N^{\frac{2}{3}}r_N$ cannot be neglected.

(2) Theorem 5.6 was stated by Dumitriu and Edelman (Corollary 3.4 in [DE2]), where their expression contains a misprint and the proof is sketched only. Moreover, the proof in [DE2] is based on the representation of the covariance matrix Σ_N in Theorem 2.1, which is also due to [DE2]. This representation of Σ_N with essentially the same proof as above leads also to Theorem 5.6 where then with the aid of (5.30) one obtains the formula

$$\sigma_{max}^2 = 2\frac{\int_0^\infty \mathsf{Ai}^4(x+a_1)dx}{\left(\int_0^\infty \mathsf{Ai}^2(x+a_1)dx\right)^2} = 2\int_0^\infty \left(\frac{\mathsf{Ai}(x+a_1)}{\mathsf{Ai}'(a_1)}\right)^4 dx. \tag{5.37}$$

A numerical computation shows that the value of (5.37) seems to be equal to that in (5.35). Unfortunately, we are not able to verify this equality in an analytic way, as our suggested identity does not fit to identities for integrals of the Airy function in the literature as e.g. in [VS].

(3) In [RRV], Ramirez, Rider, and Virag study the largest eigenvalues of β-Hermite ensembles where they first take the limit $N \to \infty$ and then $\beta \to \infty$, i.e., $k \to \infty$ here. From the results in [RRV] one obtains that

$$\lim_{k\to\infty}\left(\lim_{N\to\infty}\left(N^{\frac{2}{3}}\left(\frac{X_{t,k,N}^N}{\sqrt{tN}}-2\sqrt{k}\right)+2\sqrt{k}|a_1|\right)\right)=G. \tag{5.38}$$

in distribution where G is $\mathcal{N}(0,\sigma_{max}^2)$-distributed with σ_{max}^2 as in (5.37).

Remark 5.7. Clearly, the preceding limit results for the largest particle in the Hermite case can be transfered to the smallest particle by symmetry.

We next use the following formula of Plancherel-Rotach

$$\frac{z_{N-r+1,N}}{\sqrt{2N}}=1-\frac{|a_r|}{2N^{\frac{2}{3}}}+O(N^{-1}), \tag{5.39}$$

where a_r is the r-th largest zero of the Airy function; this formula is derived from the well-known relationship between Hermite and Laguerre polynomials [NIST]

$$H_{2n}(x)=(-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2),$$
$$H_{2n+1}(x)=(-1)^n 2^{2n+1} n! L_n^{(1/2)}(x^2),$$

and a corresponding Plancherel-Rotach formula for the Laguerre zeros given by (5) in Tricomi [T]. This leads to the following result for the r-th largest particle.

Theorem 5.8. *For $r \in \mathbb{N}$ consider the functions*

$$f_N(y) := N^{\frac{1}{6}} Q_{\lfloor N^{\frac{1}{3}} y\rfloor, N}(z_{N-r+1,N}) \quad \textit{for} \quad y \in [0, N^{\frac{2}{3}}[$$

and $f_N(y)=0$ otherwise. Then $(f_N)_{N\ge1}$ tends for $N\to\infty$ locally uniformly to

$$f(y) := \frac{\mathsf{Ai}(y+a_r)}{\mathsf{Ai}'(a_r)} \qquad \textit{for} \quad y \in [0,\infty[.$$

Moreover, the covariance matrices $\Sigma_N =: (\sigma_{i,j})_{i,j=1,\ldots,N}$ of the freezing β-Hermite ensembles satisfy

$$\lim_{N\to\infty} N^{\frac{1}{3}} \sigma_{N-r+1,N-r+1} = \sigma_{max,r}^2,$$

with $\sigma_{max,r}^2$ as specified in Theorem 1.1.

Proof. The proof is essentially the same as for Theorems 5.1 and 5.5 where now f_N and f are now those of Theorem 5.8. In particular, for f we now have

$$f''(y)=(y+a_r)f(y) \quad \text{with} \quad f(0)=0, \quad f'(0)=1.$$

Moreover, a_1 has to be replaced by a_r, and (5.9) by (5.39). We notice that now $f_N(t)\ge 0$ for $t\ge 0$ does not hold; we here however still can estimate

$$\left|\int_0^y f_N(t)dt\right| \le \int_0^y |f_N(t)|dt$$

with the triangle inequality. We then can proceed precisely as in (5.20). □

Remark 5.9. For the first few values of r, the integral

$$\int_0^\infty \frac{\mathsf{Ai}(x+a_r)^2}{\mathsf{Ai}'(a_r)^2 x}dx = \int_{a_r}^\infty \frac{\mathsf{Ai}(x)^2}{\mathsf{Ai}'(a_r)^2(x-a_r)}dx$$

seems to be decreasing in r. This is indeed the case as $r \to \infty$. For this we decompose the last integral into the regions $[a_r, a_{r-1}[$, $[a_{r-1}, a_1[$ and $[a_1, \infty[$, and estimate it in each case. Let us first note that by (5.5),

$$\mathsf{Ai}'(a_r)^{-2} = \Big(\frac{2\pi^2}{3}\Big)^{1/3} r^{-1/3} + O(r^{-4/3}).$$

Now, for the first region we use (5.4), (5.3), the substitution $y = 2(-x)^{3/2}/3\pi - r + 3/4$, and obtain for $r \to \infty$ that

$$\int_{a_r}^{a_{r-1}} \frac{\mathsf{Ai}(x)^2}{(x-a_r)} dx = \Big(\frac{2}{3\pi}\Big)^{4/3} r^{-1/3} \int_{-1/2}^{1/2} \frac{3\cos(\pi y)^2}{1-2y} dy + O(r^{-4/3}).$$

Because the Airy function has a global maximum, (5.3) leads to

$$\begin{aligned}\int_{a_{r-1}}^{a_1} \frac{\mathsf{Ai}(x)^2}{(x-a_r)} dx \le \log \frac{a_1 - a_r}{a_{r-1} - a_r} &= \log\Big(r\Big(1 + \Big(\frac{2}{3\pi}\Big)^{2/3} a_1 r^{-2/3} + O(r^{-5/3})\Big)\Big) \\ &= \log r + O(r^{-2/3}).\end{aligned}$$

Finally, Theorem 5.5 and $a_r < a_1$ yield the bound

$$\int_{a_1}^{\infty} \frac{\mathsf{Ai}(x)^2}{(x-a_r)} dx \le \int_{a_1}^{\infty} \frac{\mathsf{Ai}(x)^2}{(x-a_1)} dx < 1.$$

Putting everything together we see that for a sufficiently large r there exists a constant $C > 0$ such that

$$\int_0^{\infty} \frac{\mathsf{Ai}(x+a_r)^2}{\mathsf{Ai}'(a_r)^2 x} dx \le C r^{-1/3} \log r.$$

This stresses the fact that $r \to \infty$ means that we go from the edge into the bulk, where repulsion interactions are stronger, i.e., all variances there are much smaller than at the edge.

6. Limit results for the largest eigenvalue in the Laguerre case

We now discuss the soft edge statistics in the freezing Laguerre case similar to Section 5 for the Hermite case. This means that we analyze the limit behaviour of the largest eigenvalue in the freezing regime in Theorems 3.1 and 3.2 for $N \to \infty$ on the basis of Theorem 4.8. We here again use the ordered zeros $z_{1,N}^{(\alpha)} < ..., z_{N,N}^{(\alpha)}$ of the N-th Laguerre polynomial $L_N^{(\alpha)}$ as in Section 3. Moreover, for each N, let $(Q_{k,N}^{(\alpha)})_{k=0,..,N-1}$ be the dual polynomials associated with $(L_k^{(\alpha)})_{k=0,...,N}$ normalized as in (3.9). This means that the matrices

$$T_N := \Big(\sqrt{z_{i,N}^{(\alpha)}}\, Q_{j-1,N}^{(\alpha)}(z_{i,N}^{(\alpha)})\Big)_{i,j=1,...,N}$$

are orthogonal with with $T_N^T \Sigma_N T_N = \mathrm{diag}(\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2N})$ as in the proof of Theorem 4.8. The $Q_{k,N}^{(\alpha)}$ have the three-term-recurrence

$$\begin{aligned} xQ_{k,N}^{(\alpha)}(x) = &\sqrt{(N-k)(N-k+\alpha)}\, Q_{k-1,N}^{(\alpha)}(x) + (2(N-k)+\alpha-1) Q_{k,N}^{(\alpha)} \\ &+ \sqrt{(N-k-1)(N-k-1+\alpha)}\, Q_{k+1,N}^{(\alpha)}(x) \quad (k \le N) \end{aligned} \tag{6.1}$$

with the initial conditions $Q_{-1,N}^{(\alpha)} = 0$ and $Q_{0,N}^{(\alpha)} = \frac{1}{\sqrt{N(N+\alpha)}}$.

In the Laguerre case we have the following analogue of Theorem 5.1:

Theorem 6.1. *Let $\alpha > -1$ and define the functions*

$$f_N(y) := N^{\frac{1}{6}} \sqrt{z_{N,N}^{(\alpha)}} Q_{\lfloor N^{\frac{1}{3}} y \rfloor, N}(z_{N,N}^{(\alpha)}) \quad \text{for} \quad y \in [0, N^{\frac{2}{3}}[$$

and $f_N(y) = 0$ otherwise. Then $(f_N)_{N \geq 1}$ tends for $N \to \infty$ locally uniformly to

$$f(y) = \frac{2^{\frac{1}{3}} \mathrm{Ai}(2^{\frac{2}{3}} y + a_1)}{\mathrm{Ai}'(a_1)} \quad (y \in [0, \infty[). \tag{6.2}$$

Proof. We put $q_k^{(\alpha)} := Q_{k,N}^{(\alpha)}(z_{N,N}^{(\alpha)})$ for $k = 0, \ldots, N-1$. The Landau symbol O will be always used for $N \to \infty$ and will be locally uniform w.r.t. $y \in [0, \infty[$.

The sharp Plancherel-Rotach formula for the zeros $z_{N,N}^{(\alpha)}$ in Theorem 1.2 of [G] yields

$$\frac{z_{N,N}^{(\alpha)}}{4N} = 1 + \frac{a_1}{(2N)^{\frac{2}{3}}} + O(N^{-1}). \tag{6.3}$$

We will also use the Taylor expansion

$$\sqrt{1 + \frac{\alpha}{N-k}} = 1 + \frac{\alpha}{2(N-k)} + O(N^{-2}) \qquad \text{for} \quad 0 \leq k \leq y N^{\frac{1}{3}}. \tag{6.4}$$

The recurrence relation (6.1) for $x = z_{N,N}^{(\alpha)}$ and a division by N yield

$$\begin{aligned}
&\left(\frac{z_{N,N}^{(\alpha)}}{N} - 2\left(1 - \frac{k}{N}\right) + \frac{\alpha - 1}{N} \right) q_k^{(\alpha)} = \\
&\left(1 - \frac{k+1}{N}\right) \sqrt{1 + \frac{\alpha}{N-k-1}} q_{k+1}^{(\alpha)} + \left(1 - \frac{k}{N}\right) \sqrt{1 + \frac{\alpha}{N-k}} q_{k-1}^{(\alpha)}.
\end{aligned}$$

Using (6.3) and (6.4), we obtain

$$\begin{aligned}
&q_k^{(\alpha)} \left(2 + \frac{2^{\frac{4}{3}} a_1}{N^{\frac{2}{3}}} + \frac{2k}{N} + O(N^{-1}) \right) \\
=& q_{k-1}^{(\alpha)} \left(1 - \frac{k}{N} + \frac{\alpha}{2(N-k)} (1 - \frac{k}{N}) + O(N^{-1})(1 - \frac{k}{N}) \right) \\
&+ q_{k+1}^{(\alpha)} \left(1 - \frac{k+1}{N} + \frac{\alpha}{2(N-k-1)} (1 - \frac{k+1}{N}) + O(N^{-1})(1 - \frac{k+1}{N}) \right) \\
=& q_{k-1}^{(\alpha)} \left(1 - \frac{k}{N} + O(N^{-1}) \right) + q_{k+1}^{(\alpha)} \left(1 - \frac{k+1}{N} + O(N^{-1}) \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
q_{k+1}^{(\alpha)} + q_{k-1}^{(\alpha)} - 2 q_k^{(\alpha)} =& \frac{k+1}{N} q_{k+1}^{(\alpha)} + \frac{k}{N} q_{k-1}^{(\alpha)} + \left(\frac{2^{\frac{4}{3}} a_1}{N^{\frac{2}{3}}} + \frac{2k}{N} \right) q_k^{(\alpha)} \\
&+ O(N^{-1}) q_{k+1}^{(\alpha)} + O(N^{-1}) q_k^{(\alpha)} + O(N^{-1}) q_{k-1}^{(\alpha)}.
\end{aligned} \tag{6.5}$$

Eq. (6.5) is very similar to Eq. (5.10), so we skip some details as the calculation below will be very similar to Section 5. We sum (6.5) over $k = 0, \ldots, l$ and then over $l = 0, \ldots, \lfloor y N^{\frac{1}{3}} \rfloor - 1$. After multiplying by $N^{\frac{1}{6}} \sqrt{z_{N,N}^{(\alpha)}}$ we obtain from (6.3) that the

LHS is equal to

$$\begin{aligned}f_N(y)-\frac{(1+\lfloor yN^{\frac{1}{3}}\rfloor)N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}}{\sqrt{N(N+\alpha)}}&=f_N(y)-\frac{(1+\lfloor yN^{\frac{1}{3}}\rfloor)N^{\frac{1}{6}}2\sqrt{N}(1+O(N^{-2/3}))}{\sqrt{N(N+\alpha)}}\\&=f_N(y)-2y+O(N^{-1/3})\end{aligned}$$

and the RHS to

$$\begin{aligned}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\Bigg(&\frac{k+1}{N}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_{k+1}^{(\alpha)}+\frac{k}{N}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_{k-1}^{(\alpha)}\\&+N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}\left(\frac{2^{\frac{4}{3}}a_1}{N^{\frac{2}{3}}}+\frac{2k}{N}\right)q_k^{(\alpha)}\Bigg)+O(N^{-1/3}).\end{aligned}\tag{6.6}$$

Note that in the second case, we used the estimation

$$\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\left(O(N^{-1})q_{k+1}^{(\alpha)}+O(N^{-1})q_k^{(\alpha)}+O(N^{-1})q_{k-1}^{(\alpha)}\right)=O(N^{-\frac{1}{3}})$$

which can be proved precisely as Eq. (5.23) in the proof of Lemma 5.3. We next use the index shifts $k+1\mapsto k$ and $k-1\mapsto k$ in (6.6) and obtain that the RHS above is

$$\begin{aligned}&\frac{1}{N^{\frac{1}{3}}}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{l}\left(N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\left(\frac{k+1}{N^{\frac{1}{3}}}+\frac{k}{N^{\frac{1}{3}}}+2^{\frac{4}{3}}a_1+\frac{2k}{N^{\frac{1}{3}}}\right)\right)\\&+\frac{1}{N^{\frac{2}{3}}}\sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor}\frac{k}{N^{\frac{1}{3}}}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}-\frac{1}{N^{\frac{2}{3}}}\sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\frac{k+1}{N^{\frac{1}{3}}}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\\&=\frac{1}{N^{\frac{1}{3}}}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{l}\left(N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\left(\frac{k+1}{N^{\frac{1}{3}}}+\frac{k}{N^{\frac{1}{3}}}+2^{\frac{4}{3}}a_1+\frac{2k}{N^{\frac{1}{3}}}\right)\right)+O(N^{-\frac{1}{3}}),\end{aligned}$$

where for the last equation an analogue estimation to that in (5.19) has been used.

In summary we have proved that

$$f_N(y)-2y+O(N^{-1/3})=\frac{1}{N^{\frac{2}{3}}}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\left(N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\left(\frac{4k+1}{N^{\frac{1}{3}}}+2^{\frac{4}{3}}a_1\right)\right)$$

If we use (5.15), we see that this leads to the integral equation

$$f_N(y)=\int_0^y\int_0^s f_N(s)(4t+2^{\frac{4}{3}}a_1)dtds+2y+O(N^{-\frac{1}{3}}).$$

As the function f defined in (6.2) satisfies $f(0)=0$, $f'(0)=2$ and $f''(x)=(4x+2^{\frac{4}{3}}a_1)f(x)$, we obtain from the Lemma of Gronwall (see also Lemma 5.4) that

$$|f_N(y)-f(y)|=O(N^{-1/3}).$$

□

We now apply Theorem 6.1 to the (N,N)-entries of the covariance matrices Σ_N for β-Laguerre ensembles in the freezing regime in Theorem 4.8 for $N\to\infty$.

Corollary 6.2. *Consider the covariance matrices* $\Sigma_N =: (\sigma_{i,j})_{i,j=1,\ldots,N}$ *of* β*-Laguerre ensembles in the freezing regime. Then*

$$\lim_{N\to\infty} N^{\frac{1}{3}}\sigma_{N,N} = \frac{1}{2}\int_0^\infty \frac{\mathsf{Ai}(x+a_1)^2}{\mathsf{Ai}'(a_1)^2 x}dx = 0.417\ldots$$

Proof. The proof is completely analog to the proof of Theorem 5.5. □

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