Proper Bounded Edge-Colorings

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Abstract

For fixed integers $k \geq 2$, and for $n$-element sets $X$ and colorings $\Delta : [X]^k \rightarrow \{0, 1, \ldots \}$ where every color class is a matching and has cardinality at most $u$, we show that there exists a totally multicolored subset $Y \subseteq X$ with

$$|Y| \geq \max \left\{ c_1 \left( \frac{n}{u} \right)^{\frac{k}{2(k+1)}}, \quad c_2 \left( \frac{n}{u} \right)^{\frac{k}{2(k+1)}} \cdot \left( \ln \left( \frac{u}{\sqrt{n}} \right) \right)^{\frac{k}{2(k+1)}} \right\}$$

where $c_1, c_2 > 0$ are constants. This lower bound is tight up to constant factors for $u = \Omega(n^{1/2+\epsilon})$ for every $\epsilon > 0$. For fixed values of $k$ we give a polynomial time algorithm for finding such a set $Y$ of guaranteed size.

1 Introduction

On each of $(\frac{3n}{3})/n$ school days, in a school attended by $3n$ students, the students are asked to line up in $n$ rows, each containing three students. In 1851, Kirkman asked for the existence of such a schedule that would allow each triple of students to form a row on exactly one of the school days, cf. [Bi 81]. This classical problem was answered completely by Ray-Chaudhuri and Wilson [RW 71] who proved that such a schedule exists for each $n \equiv 1, 3 \pmod{6}$. Here, we investigate a somewhat related combinatorial problem. Suppose that after such a schedule was prepared, the principle of the school wants (for unrevealed purposes) to select the largest group of, say, $m$ students with the property that no two triples of students form a row on the

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same day. For any schedule such an \( m \) must satisfy
\[
c_1 \cdot n^{2/5} \cdot (\ln n)^{1/5} \leq m \leq c_2 \cdot n^{2/3}
\]
where \( c_1, c_2 > 0 \) are constants. While the upper bound is straightforward, the lower bound follows from [ALR 91]. There are schedules which, up to constant factors, match the lower bound. Here, we consider the general case in which one has \( n \) students which are asked to line up in at most \( a \) rows on a day, each containing \( k \) people. Our results extend earlier work from [ALR 91] and [LRW 96] where the case \( a = n/k \) respectively \( k = 2 \) was considered. We also give a polynomial time algorithm which finds a group of \( m \) students satisfying the lower bound in (1).

It will be convenient to formulate our problem in terms of colorings.

**Definition 1** Let \( \Delta: [X]^k \rightarrow \omega \) where \( \omega = \{0, 1, \ldots\} \) be a coloring of the \( k \)-element subsets of a set \( X \). The coloring \( \Delta: [X]^k \rightarrow \omega \) with color classes \( C_0, C_1, \ldots \), i.e., \( \Delta^{-1}(i) = C_i \) for \( i \in \omega \), is called \( u \)-bounded if \( |C_i| \leq u \) for \( i = 0, 1, \ldots \). The coloring \( \Delta: [X]^k \rightarrow \omega \) is called proper if each color class \( C_i, i = 0, 1, \ldots \) is a matching, i.e., sets of the same color are pairwise disjoint, thus, \( \Delta(U) = \Delta(V) \) implies \( U \cap V = \emptyset \) for all distinct sets \( U, V \in [X]^k \). A subset \( Y \subseteq X \) is called totally multicolored if the restriction of the coloring \( \Delta \) to the set \( [Y]^k \) of all \( k \)-element subsets of \( Y \) is a one-to-one coloring.

For an \( n \)-element set \( X \), define the parameter \( f_u(n, k) \) by
\[
f_u(n, k) = \min_{\Delta} \max_{Y \subseteq X} \{|Y|; Y \text{ is totally multicolored}\},
\]
where we minimize over all proper \( u \)-bounded colorings \( \Delta: [X]^k \rightarrow \omega \) with \( |X| = n \).

The first estimates on \( f_u(n, k) \) were given by Babai, cf. [Ba 85], in connection with a Sidon-type problem. He showed for the case \( a = n/2 \) and \( k = 2 \) that
\[
c_1 \cdot n^{1/3} \leq f_{n/2}(n, 2) \leq c_2 \cdot (n \cdot \ln n)^{1/3}
\]
for constants \( c_1, c_2 > 0 \). In [ALR 91] the lower bound was improved by the factor \( \Theta((\ln n)^{1/3}) \), i.e., \( f_{n/2}(n, 2) \geq c_3 \cdot (n \cdot \ln n)^{1/3} \) where \( c_3 > 0 \) is a constant. Moreover, for fixed integers \( k \geq 2 \)
the results from [ALR 91] show that

\[ f_{n/k}(n, k) = \Theta \left( n^{k-1} \cdot (\ln n)^{1/2k-1} \right) \]

Here we will prove the following:

**Theorem 1** Let \( k \geq 2 \) be a fixed integer. There exist constants \( c_1, c_2, c_3 > 0 \) such that for

\[ 2 \leq u \leq n/k, \]

\[ \max \left\{ c_1 \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}}, \quad c_2 \cdot \left( \frac{n^k}{u} \cdot \ln \left( \frac{u}{\sqrt{n}} \right) \right)^{\frac{1}{2k-1}} \right\} \leq f_u(n, k) \leq c_3 \cdot \left( \frac{n^k}{u} \cdot \ln n \right)^{\frac{1}{2k-1}}. \quad (2) \]

Moreover, for every \( n \)-element set \( X \) and every \( u \)-bounded proper coloring \( \Delta: [X]^k \rightarrow \omega \) one can find in time \( O(u \cdot n^{2k-1}) \) a totally multicolored subset \( Y \subseteq X \) with

\[ |Y| \geq \max \left\{ c_1 \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}}, \quad c_2 \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}} \cdot \left( \ln \left( \frac{u}{\sqrt{n}} \right) \right)^{\frac{1}{2k-1}} \right\}. \]

## 2 The Existence

In this section, we will prove the existence of a totally multicolored subset as guaranteed by Theorem 1. We will use the notion of edge-colored hypergraphs. The vertices are the \( n \) students, the edges correspond to the rows, and these edges are colored by the day.

Let \( G = (V, E) \) be a hypergraph with vertex set \( V \) and edge set \( E \). For a vertex \( v \in V \), let \( d(v) \) denote the *degree* of \( v \) in \( G \), i.e., the number of edges \( E \in E \) containing \( v \). Let \( d = \sum_{v \in V} d(v) / |V| \) denote the *average degree* of \( G \). If for some fixed \( k \) we have \( |E| = k \) for each edge \( E \in E \), then \( G \) is called *\( k \)-uniform*. A 2-cycle in \( G \) is an (unordered) pair \( E, E' \in E \) of distinct edges which intersect in at least two vertices. The *independence number* \( \alpha(G) \) is the largest size of a subset \( I \subseteq V \) such that the induced hypergraph contains no edges, i.e., \( E \not\subseteq I \) for every edge \( E \in E \).

### Lower Bounds

It turns out that the independence number is important in our considerations. Some of our arguments are based on a result of Ajtai, Komlós, Pintz, Spencer and Szemerédi, [AKPSS 82]. Here, we will use a modified version proved in [DLR 95].
Theorem 2 Let $G$ be a $k$-uniform hypergraph on $n$ vertices. Assume that

(i) $G$ contains no 2-cycles, and

(ii) the average degree satisfies $d \leq t^k$ where $t \geq t_0(k),$

then for some positive constant $c = c(k)$,

$$\alpha(G) \geq c \cdot \frac{n}{t} \cdot (\ln t)^{\frac{1}{t}}.$$  \hspace{1cm} (3)

Now we are ready to prove the lower bounds given in Theorem 1.

Proof: We start by showing the two lower bounds in (2). Let $\Delta: [X]^k \to \omega$ be a $u$-bounded proper coloring where $|X| = n$. We construct a $2k$-uniform hypergraph $H = (X, E)$ on $X$ where $E \in E \subseteq [X]^{2k}$ if there exist two distinct $k$-element sets $S, T \in [X]^k$, $S, T \subseteq E$, so that $\Delta(S) = \Delta(T)$. As the number of $k$-element sets of the same color is at most $u$, the number of edges in $H$ satisfies

$$|E| = \sum_{i \in \omega} \left( \Delta^{-1}(i) \right) \leq \left( \frac{n}{k} \right) \cdot \left( \frac{u}{2} \right).$$  \hspace{1cm} (4)

Observe that, if $I \subseteq X$ is an independent set of $H$, then $I$ is totally multicolored with respect to the coloring $\Delta$. Concerning the first lower bound, it is enough to show that $H$ contains an independent set of size $c_1 \cdot (n^k/u)^{1/(2k-1)}$. To see this, pick every vertex in $X$ at random independently of the other vertices with probability

$$p = (n^{k-1} \cdot u)^{-\frac{1}{2k-2}}.$$  \hspace{1cm} (5)

By Chernoff’s inequality, there exists a subset $Y \subseteq X$ of cardinality at least

$$(1 - o(1)) \cdot p \cdot n = (1 - o(1)) \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-2}},$$

and by Markov’s inequality, the on $Y$ induced subhypergraph $H_0 = (Y, E \cap [Y]^{2k})$ of $H$ contains at most

$$2 \cdot p^{2k} \cdot |E| \leq 2 \cdot p^{2k} \cdot \left( \frac{n}{2} \right) \cdot \left( \frac{u}{2} \right) \leq \frac{1}{2} \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-2}}$$

edges since $k \geq 2$. By deleting one vertex from each edge in $[Y]^{2k} \cap E$, we obtain a subset $Y' \subseteq Y$ with $|Y'| \geq |Y|/2 \geq (1/2 - o(1)) \cdot p \cdot n$. Clearly, $Y'$ is an independent set in $H$, and hence $Y'$ is totally multicolored with respect to $\Delta$, i.e., $f_a(n, k) = \Omega((n^k/u)^{1/(2k-1)})$. 


If \( u = \sqrt{n} \cdot \omega(n) \), where \( \omega(n) \to \infty \) with \( n \to \infty \), we can improve the lower bound
\[
 f_n(u, k) \geq c_1 \cdot (n^k/u)^{1/(2k-1)}
\]
by a logarithmic factor. Let \( \Delta: [X]^k \to \omega \) be a \( u \)-bounded proper coloring. Consider the \( 2k \)-uniform hypergraph \( \mathcal{H} \) with vertex set \( X \) and with the set \( E \) of edges defined in the same way as above. Again, we want to show a large lower bound on the independence number of \( \mathcal{H} \). Our strategy will be to find a random subset \( Y \subseteq X \) such that the induced hypergraph has only a few \( 2 \)-cycles. By deleting these \( 2 \)-cycles the desired result will follow with Theorem 2.

Throughout this proof, let \( c_1, c_2, \ldots \) denote positive constants. Recall that the number of edges of \( \mathcal{H} \) satisfies inequality (4). For \( j = 2, 3, \ldots, 2k-1 \), let \( \nu_j \) denote the number of \( (2, j) \)-cycles in \( \mathcal{H} \), i.e., the number of pairs \( \{ E, E' \} \in [E]^2 \) of edges which intersect in exactly \( j \) vertices. First, we estimate the total number \( \nu_j \) of \( (2, j) \)-cycles in the hypergraph \( \mathcal{H} \). We fix an edge \( E \in \mathcal{E} \). The number of unordered pairs \( \{U, V\} \) of distinct sets \( U, V \in [X]^k \) with \( \Delta(U) = \Delta(V) \) and \( |(U \cup V) \cap E| = j \) and \( 1 \leq U \cap E, |V \cap E| \leq j - 1 \) is bounded from above by
\[
 \sum_{i=[j/2]}^{j-1} \binom{2k}{i} \cdot \binom{n-2k}{k-i} \cdot \binom{2k-j}{j-i} \leq c_1 \cdot n^{k-\lfloor j/2 \rfloor}, \tag{6}
\]
as either \( |U \cap E| \geq \lceil j/2 \rceil \) or \( |V \cap E| \geq \lceil j/2 \rceil \), and every color class is a matching.

If \( U \cap E = \emptyset \) or \( V \cap E = \emptyset \), but \( |(U \cup V) \cap E| = j \), then the number of such pairs \( \{U, V\} \) is at most
\[
 \binom{2k}{j} \cdot \binom{n-2k}{k-j} \cdot (u-1) \leq c_2 \cdot n^{k-j} \cdot u. \tag{7}
\]

Now, (4), (6) and (7) imply that
\[
 \nu_j \leq |\mathcal{E}| \cdot \left( c_1 \cdot n^{k-\lfloor j/2 \rfloor} + c_2 \cdot n^{k-j} \cdot u \right) \leq c_3 \cdot u \cdot \left( n^{2k-\lfloor j/2 \rfloor} + n^{2k-j} \cdot u \right).
\]

As \( u \leq n/k \) and \( j \geq 2 \), we have \( n^{2k-\lfloor j/2 \rfloor} \geq n^{2k-j} \cdot u \), hence
\[
 \nu_j \leq c_4 \cdot u \cdot n^{2k-\lfloor j/2 \rfloor}. \tag{8}
\]

With foresight we use a slightly larger value than in (5) for the probability \( p \) of picking vertices, namely, we set
\[
 p = \left( \frac{1}{n^{k-1} \cdot u} \right)^{1/(2k-1)} \cdot \left( \frac{u}{\sqrt{n}} \right)^{1/(4+1)(2k-1)}. \tag{9}
\]
Let $Y$ be a random subset of $X$ obtained by choosing vertices $v \in X$ with probability $p$ independently of the other vertices. The expected size $E(|Y|)$ of $Y$ is given by

$$E(|Y|) = p \cdot n = \left( \frac{n^k}{u} \right)^{\frac{j-2k+1}{2k-1}} \cdot \left( \frac{n}{\sqrt{n}} \right)^{\frac{1}{(2k+1)(2k-1)}}.$$ 

Let $\nu_j(Y)$, for $j = 2, 3, \ldots, 2k - 1$, be random variables counting the number of $(2, j)$-cycles contained in $Y$. The random variable $\mu_2(Y) = \sum_{j=2}^{2k-1} \nu_j(Y)$ counts the total number of $2$-cycles of the subhypergraph induced on $Y$. Let $E(\mu_2(Y))$ and $E(\nu_j(Y))$ denote the corresponding expected values.

We infer for $j = 2, 3, \ldots, 2k - 1$ that

$$E(\nu_j(Y)) \leq p^{4k-j} \cdot c_4 \cdot u \cdot n^{2k-[j/2]}$$

$$= pn \cdot c_4 \cdot u \cdot n^{2k-1} \cdot \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)}.$$ 

Thus,

$$E(\nu_j(Y)) \leq \begin{cases} 
    pn \cdot c_4 \cdot u \cdot n^{2k-1} \cdot \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)} & \text{if } j \text{ is even} \\
    pn \cdot c_4 \cdot u \cdot n^{2k-1} \cdot \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)} & \text{if } j \text{ is odd}.
\end{cases}$$

Recall that $u = \sqrt{n} \cdot \omega(n) \leq n/k$ with $\omega(n) \to \infty$ with $n \to \infty$, hence, $\omega(n) = O(\sqrt{n})$. Then, for $j$ even,

$$E(\nu_j(Y)) \leq pn \cdot c_4 \cdot u \cdot n^{2k-1} \cdot \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)}$$

$$\leq pn \cdot c_4 \cdot \omega(n) \cdot \left( \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)} \right)$$

$$= o(pn).$$

For $j$ odd, we obtain

$$E(\nu_j(Y)) \leq pn \cdot c_4 \cdot u \cdot n^{2k-1} \cdot \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)}$$

$$= pn \cdot c_4 \cdot \omega(n) \cdot \left( \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)} \right)$$

$$\leq pn \cdot c_4 \cdot \omega(n) \cdot \left( \frac{j-2k+1}{2k-1} \cdot \frac{4k-j-1}{2k-1} \cdot \frac{k-1}{2k+1} \cdot \frac{(4k-j-1)!}{j!} \cdot \frac{1}{2k-1} \cdot \frac{1}{(2k+1)(2k-1)} \right)$$

$$= o(pn)$$

as $\omega(n) = O(\sqrt{n})$. 

(9)
Hence, by (9) and (10) we conclude \( E(\mu_2(Y)) = \sum_{j=2}^{2k-1} E(\nu_j(Y)) = o(p \cdot n) \). Using Chernoff’s and Markov’s inequality, we infer that there exists a subset \( Y \subseteq X \) with \( |Y| = c_5pn \), such that the induced hypergraph \( \mathcal{H}_0 = (Y, \mathcal{E} \cap [Y]^{2k}) \) contains at most \( c_6p^{2k}|\mathcal{E}| \) edges, and has only \( o(pn) \) 2-cycles. We omit one vertex from each 2-cycle in \( \mathcal{H}_0 \). The resulting induced subhypergraph \( \mathcal{H}_1 \) has \( (c_5-o(1)) \cdot pn \) vertices, contains no 2-cycles anymore, and by (4) has average degree at most

\[
d \leq t^{2k-1} = \frac{2k \cdot c_6 \cdot p^{2k} \cdot |\mathcal{E}|}{(c_5-o(1)) \cdot pn} \leq c_7 \cdot p^{2k-1} \cdot n^{k-1} \cdot u, \]

i.e., \( t \leq c_8 \cdot p \cdot (n^{k-1} \cdot u)^{\frac{1}{2k-1}} = c_8 \cdot (\frac{u}{\sqrt{n}})^{\frac{1}{k+1}} \cdot (\frac{1}{2k-1}). \) As \( u/\sqrt{n} \rightarrow \infty \) with \( n \rightarrow \infty \) we can apply Theorem 2 to the subhypergraph \( \mathcal{H}_1 \) which yields

\[
o(\mathcal{H}) \geq o(\mathcal{H}_1) \geq c \cdot \frac{(c_5-o(1)) \cdot p \cdot n}{c_8 \cdot p \cdot (n^{k-1} \cdot u)^{\frac{1}{2k-1}}} \cdot \left[ \ln \left( c_8 \cdot \left( \frac{u}{\sqrt{n}} \right)^{\frac{1}{k+1}} \right) \right]^{\frac{1}{2k-1}}
\]

\[
\geq c' \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}} \cdot \left( \ln \left( \frac{u}{\sqrt{n}} \right) \right)^{\frac{1}{2k-1}},
\]

i.e., \( f_u(n, k) = \Omega((n^k/u)^{1/(2k-1)} \cdot (\ln n)^{1/(2k-1)}). \)

**Upper Bounds**

Next, we will show the upper bound in (2) generalizing some arguments from [Ba 85].

**Proof:** Let \( X \) be an \( n \)-element set where without loss of generality \( n \) is divisible by \( k \). Set \( m = \lceil c \cdot n^k/u \rceil \), where \( c > 0 \) is a constant. Let \( M_1, M_2, \ldots, M_m \) be random matchings, chosen uniformly and independently from the set of all matchings of size \( u \) on \( X \). We define a coloring \( \Delta: [X]^k \rightarrow \omega \) in rounds as follows: in round \( j = 1, 2, \ldots, m \), we color every \( k \)-element set in \( M_j \) which has not been colored before, by color \( j \). Let \( C_j \) be the set of all \( k \)-element subsets of \( X \) which are colored in some round \( i = 1, 2, \ldots, j-1 \). In round \( m+1 \) we color the remaining \( k \)-elements sets in \( [X]^k \setminus C_{m+1} \) in an arbitrary way, such that each color class is a matching of size at most \( u \). Let \( Y \subseteq V \) be a fixed subset with \( |Y| = x \), where \( x = o(n/u^{1/k}) \). We will prove that for \( x \geq C \cdot (n^k/u \cdot \ln u)^{1/(2k-1)} \) with probability approaching to 1 the set \( Y \) is not
totally multicolored where $C > 0$ is a sufficiently large constant. This will give the desired result. We split the proof into several claims.

First, we give an upper bound on the probability that a certain number of $k$-element subsets of $Y$ is colored in round $j$.

**Claim 1** For $j = 1, 2, \ldots, m$ and $t = 0, 1, \ldots,$

$$
\text{Prob} \left[ |M_j \cap [Y]^k| \geq t \right] \leq \left( \frac{u \cdot x^k}{n^k} \right)^t .
$$

**Proof:** The left hand side of (11) does not depend on the particular choice of $Y$. Thus, assume that the matching $M_j$ is fixed. The set $Y$ can be chosen in $\binom{n}{x}$ ways. If $|M_j \cap [Y]^k| \geq t$, then from $M_j$ we can choose $t$ edges in $\binom{n}{x}$ ways, and the remaining elements of $Y$ can be chosen in at most $\left( \frac{(n-k)^t}{(n-x)^t} \right)$ ways, hence

$$
\text{Prob} \left[ |M_j \cap [Y]^k| \geq t \right] \leq \frac{(\binom{n}{x}) \cdot (\binom{n-k}{x-t})}{\binom{n}{x}} \leq \left( \frac{u \cdot x^k}{n^k} \right)^t .
$$

□

Now, we estimate the probability that a certain number of $k$-element subsets of $Y$ is colored in some round $i \leq m$.

**Claim 2** For $t = 0, 1, \ldots$ and for positive integers $n$,

$$
\text{Prob} \left[ |C_{m+1} \cap [Y]^k| \geq t \right] \leq \left( \frac{e \cdot (t+m) \cdot u \cdot x^k}{t \cdot n^k} \right)^t .
$$

**Proof:** For $j = 1, 2, \ldots, m$, consider the events $|M_j \cap [Y]^k| \geq t$. As the matchings are chosen independently of each other, these events are independent. By Claim 1 we have

$$
\text{Prob} \left[ |M_j \cap [Y]^k| \geq t_j \right] \leq \left( \frac{u \cdot x^k}{n^k} \right)^{t_j} .
$$

Since $|C_{m+1} \cap [Y]^k| \leq \sum_{j=1}^{m} |M_j \cap [Y]^k|$ we infer, using $\binom{n}{k} \leq (e \cdot n/k)^k$, that

$$
\text{Prob} \left[ |C_{m+1} \cap [Y]^k| \geq t \right] \leq \text{Prob} \left[ \sum_{j=1}^{m} |M_j \cap [Y]^k| \geq t \right]
$$
\[ \leq \sum_{(t_j)_{j=1}^m, t_j \geq 0, \sum_{j=1}^m t_j = t} \prod_{j=1}^m \text{Prob} \left[ |M_j \cap |Y|^k| \geq t_j \right] \]
\[ \leq \sum_{(t_j)_{j=1}^m, t_j \geq 0, \sum_{j=1}^m t_j = t} \prod_{j=1}^m \left( \frac{a \cdot x^k}{n^k} \right)^{t_j} \]
\[ = \sum_{(t_j)_{j=1}^m, t_j \geq 0, \sum_{j=1}^m t_j = t} \left( \frac{e \cdot (t + m)}{t} \right)^t \cdot \left( \frac{a \cdot x^k}{n^k} \right)^t \]
\[ = \left( \frac{e \cdot (t + m) \cdot a \cdot x^k}{t \cdot n^k} \right)^t. \]

For \( i = 1, 2, \ldots, m+1 \), let \( E_i \) denote the event \(|C_i \cap |Y|^k| \leq [e_1 \cdot x^k]| \) where \( e_1 > 0 \) is a constant with \( 3e \leq e_1 \leq 1/2 \cdot 1/k! \). Note that if \( E_i \) does not hold for some \( i \), then also \( E_{m+1} \) does not hold.

It turns out that with high probability \( E_{m+1} \) holds, i.e., only at most the constant fraction \( e_1 \) of all \( k \)-element subsets of \( Y \) is colored before round \( m+1 \):

**Claim 3** For large enough positive integers \( n \),

\[ \text{Prob} \left[ E_{m+1} \right] \geq 1 - 2^{-e_1 \cdot x^k}. \]

**Proof:** Set \( t = [e_1 \cdot x^k] \). Since \( x = o \left( n^{1/k} \right) \) we have \( t = o(n^{k}/n) \). For \( n \) large enough, with \( m = \lceil e \cdot n^k/u \rceil \), and as \( e \cdot c / c_1 \leq 1/3 \), the quotient \( \frac{e \cdot (t+m) \cdot a \cdot x^k}{t \cdot n^k} \) is less than \( 1/2 \), hence with (12) we have

\[ \text{Prob} \left[ E_{m+1} \right] \geq 1 - \text{Prob} \left[ |C_{m+1} \cap |Y|^k| \geq t \right] \geq 1 - 2^{-t} \geq 1 - 2^{-e_1 \cdot x^k}. \]

We define another random variable \( Y_j \) by \( Y_j = \left[ |M_j|^2 \cap |Y|^k \right] \) for \( j = 1, 2, \ldots, m \). Then \( Y_j \) counts the number of pairs of distinct \( k \)-element subsets of \( Y \) which are colored in round
j. For \( j = 1, 2, \ldots, m \), we want to determine the probability \( \text{Prob} [Y_j = 0] \). However, we do not know how many \( k \)-element sets of \( Y \) were already colored in some round \( i < j \). Therefore, we condition on the event that only at most the fraction \( c_1 \) of all \( k \)-element subsets of \( Y \) has been colored before round \( j \).

For a random variable \( Z \) let \( E(Z) \) denote the expected value of \( Z \).

**Claim 4** For some constant \( c_2 > 0 \), and sufficiently large positive integers \( n \), and for \( j = 1, 2, \ldots, m \),

\[
E(Y_j | E_j) > c_2 \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}.
\]

**Proof:** As \( E_j \) holds, we have for some constant \( c_1' > 0 \) that

\[
| [Y]^k \setminus C_j | \geq \binom{x}{k} - c_1 \cdot x^k \geq c_1' \cdot x^k.
\]

For each set \( S \in [Y]^k \) there are less than \( k \cdot \left( \frac{n-1}{k-1} \right) \) \( k \)-element subsets of \( Y \) which are not disjoint from \( S \). Hence, for some constant \( c_2 > 0 \) and \( n \) large enough, the number of (unordered) pairs \( \{S,T\} \in [[Y]^k \setminus C_j]^2 \) of sets with \( S \cap T = \emptyset \) is at least

\[
\frac{1}{2} \cdot c_1' \cdot x^k \cdot \left( c_1' \cdot x^k - k \cdot \binom{x-1}{k-1} \right) \geq c_2 \cdot x^{2k}.
\]

For given disjoint \( k \)-element sets \( S, T \in [X]^k \), the probability that both sets are in \( M_j \) is given by

\[
\text{Prob} [S, T \in M_j] = \frac{u \cdot (u-1)}{\binom{n}{k} \cdot \binom{n-k}{k}} \geq \frac{u^2}{n^{2k}}.
\]

Hence, by (13) and (14) for the conditional expected value \( E(Y_j | E_j) \) we have

\[
E(Y_j | E_j) \geq c_2 \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}.
\]

\( \square \)

**Claim 5** For \( j = 1, 2, \ldots, m \), and large positive integers \( n \),

\[
\text{Prob} [Y_j = 1 \mid E_j] \geq (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}.
\]
Proof: For \( t = 1, 2, \ldots \), we claim that
\[
\text{Prob}[Y_j \geq t \mid E_j] \leq \left( \frac{u \cdot x^k}{n^k} \right)^{\left\lceil \frac{\sqrt{2t + 1}}{2} \right\rceil}.
\] (15)

Namely, \( t \) pairwise distinct two-element sets span a set of cardinality at least \( \left\lceil \sqrt{2t + 1} \right\rceil \), i.e., \( Y_j \geq t \) implies \( |M_j \cap |Y]_k| \geq \left\lceil \sqrt{2t + 1} \right\rceil \). By Claim 1 this shows inequality (15):
\[
\text{Prob}[Y_j \geq t \mid E_j] \leq \text{Prob}\left[|M_j \cap |Y]_k| \geq \left\lceil \sqrt{2t + 1} \right\rceil \right] \leq \left( \frac{u \cdot x^k}{n^k} \right)^{\left\lceil \frac{\sqrt{2t + 1}}{2} \right\rceil}.
\]

For \( i = 0, 1, \ldots \), set \( p_i = \text{Prob}[Y_j = i \mid E_j] \). Then we infer from (15), using \( x = o\left(n/u^{1/k}\right) \), that
\[
E(Y_j \mid E_j) = \sum_{i \geq 0} i \cdot p_i \leq p_1 + \sum_{i \geq 2} i \cdot \left( \frac{u \cdot x^k}{n^k} \right)^{\left\lceil \frac{\sqrt{2t + 1}}{2} \right\rceil} = p_1 + O\left( \left( \frac{u \cdot x^k}{n^k} \right)^{\frac{3}{2}} \right) = p_1 + o\left( \left( \frac{u^2 \cdot x^{2k}}{n^{2k}} \right) \right).
\]

By Claim 4 we infer that \( p_1 \geq (c_2 - o(1)) \cdot u^2 \cdot x^{2k}/n^{2k} \). \( \square \)

Finally, for \( j = 1, 2, \ldots, m \) let \( A_j \) denote the event \( (Y_j = 0 \text{ and } E_j) \).

Claim 6 For some constant \( c_3 > 0 \), and large enough positive integers \( n \),
\[
\text{Prob}[A_1 \land \ldots \land A_m] \leq \exp\left(-c_3 \cdot u \cdot \frac{x^{2k}}{n^k}\right).
\]

Proof: Notice that
\[
\text{Prob}[A_1 \land \ldots \land A_m] = \text{Prob}[A_1] \cdot \prod_{i=2}^{m} \text{Prob}[A_i \mid A_1 \land \ldots \land A_{i-1}].
\] (16)

By Claim 5 we have
\[
\text{Prob}[A_1] \leq \text{Prob}(Y_1 = 0 \mid E_1) \leq \text{Prob}(Y_1 \neq 1 \mid E_1) \leq 1 - (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}},
\] (17)
while for \( i \geq 2 \) we infer

\[
\text{Prob} [A_i \mid A_1 \land \ldots \land A_{i-1}] \leq \text{Prob} [Y_i = 0 \mid A_1 \land \ldots \land A_{i-1}]
\]
\[
\leq \text{Prob} [Y_i = 0 \mid E_i]
\]
\[
\leq \text{Prob} [Y_i \neq 1 \mid E_i]
\]
\[
\leq 1 - (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}.
\]

Using \((1 - x)^m \leq \exp(-m \cdot x)\) where \( m = \lceil e \cdot n^k / u \rceil \), inequalities (17), (18) together with (16) imply

\[
\text{Prob} [A_1 \land A_2 \land \ldots \land A_m] \leq \left(1 - (c_2 - o(1)) \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}\right)^m
\]
\[
\leq \exp \left(- (c_2 - o(1)) \cdot m \cdot \frac{u^2 \cdot x^{2k}}{n^{2k}}\right)
\]
\[
\leq \exp \left(- c \cdot (c_2 - o(1)) \cdot \frac{u \cdot x^{2k}}{n^k}\right)
\]
\[
\leq \exp \left(- c_3 \cdot \frac{u \cdot x^{2k}}{n^k}\right).
\]

\[\Box\]

**Claim 7** For large enough positive integers \( n \), the probability that there exists a totally multicolored \( x \)-element subset is at most

\[
\binom{n}{x} \cdot \left(\exp \left(- c_3 \cdot \frac{u \cdot x^{2k}}{n^k}\right) + 2^{-c_1 \cdot x^k}\right).
\]

**Proof:** If \( Y \) is totally multicolored, then \( Y_1 = Y_2 = \ldots = Y_m = 0 \). Thus, either \( A_1 \land A_2 \land \ldots \land A_m \) holds or some \( E_i \), hence, \( E_{m+1} \) fails. As there are exactly \( \binom{n}{x} \) \( x \)-element sets \( Y \), by combining the estimates from Claim 3 and Claim 6 we obtain (19).

\[\Box\]

We want to show that for \( n \to \infty \) expression (19) tends to 0 for \( x \geq C \cdot \left(\frac{n^k}{u}\right)^{1/(2k-1)} \cdot (\ln n)^{1/(2k-1)} \) where \( C > 0 \) is a big enough constant. Namely,

\[
\binom{n}{x} \cdot 2^{-c_1 \cdot x^k} \leq \left(\frac{e \cdot n}{x}\right)^x \cdot 2^{-c_1 \cdot x^k}
\]
\[
\leq \exp \left(x \cdot \ln \frac{n}{x} - c_1 \cdot \ln 2 \cdot x^k\right)
\]
\[
= o(1)
\]
for \( x \geq C \cdot (\ln n)^{1/(k-1)} \) where \( C > 0 \) is a large enough constant.

Moreover, we have

\[
\binom{n}{x} \cdot \exp\left(-c_3 \cdot \frac{u \cdot 2^k}{n^k}\right) \leq \left(\frac{e \cdot n}{x}\right)^x \cdot \exp\left(-c_3 \cdot \frac{u \cdot 2^k}{n^k}\right)
\]
\[
\leq \exp\left(2x \cdot \ln n - c_3 \cdot \frac{u \cdot 2^k}{n^k}\right)
\]
\[
\leq \exp\left(2C - c_3 \cdot C^{2k} \cdot \left(\frac{n^k}{u}\right)^{1/(2k-1)} \cdot (\ln n)^{2k/(2k-1)}\right)
\]
\[
= o(1)
\]

provided \( C^{2k-1} > 2/c_3 \) and \( n \) is large enough. Thus, expression (19) tends to 0 with \( n \to \infty \).

For \( n \leq m_0 \) one can obtain asymptotically the same upper bound by taking an appropriately large constant \( C > 0 \).

\[\square\]

3 \hspace{1em} An Algorithm

Here, we show that one can find in time \( O(u \cdot n^{2k-1}) \) a totally multicolored subset as guaranteed by Theorem 1. The algorithm follows the probabilistic arguments given before. It is based on recent results of Fundia [Fu 96] and from [BL 96].

Proof: Let \( k \geq 2 \) be a fixed integer and let \( \Delta: [X]^k \to \omega \) with \( |X| = n \) be a proper \( u \)-bounded coloring. First, we order the set \([X]^k\) of \( k \)-element subsets according to their color. This can be done in time \( O(n^k \cdot \ln n) \). Then, by examining all \( k \)-element sets in \([X]^k\) we form a \( 2k \)-uniform hypergraph \( H = (X, E) \), \( E \subseteq [X]^{2k} \), where \( E \in E \) if there exist two distinct \( k \)-element sets \( S, T \in [X]^k \) with \( S \cup T = E \) and \( \Delta(S) = \Delta(T) \). By (4), we have \( |\mathcal{E}| = O(n^k \cdot u) \), hence constructing the hypergraph \( H \) can be done in time \( O(n^k \cdot u + n^k \cdot \ln n) \). We use the following algorithmic version of Turán’s theorem, cf. [BL 96]. The existence result was given by Spencer [Sp 72].

Lemma 1 Let \( \mathcal{G} = (V, \mathcal{E}) \) be a \( k \)-uniform hypergraph on \( n \) vertices with average degree \( d^{k-1} \geq 1 \). Then, one can find in time \( O(|V| + |\mathcal{E}|) \) an independent set \( I \subseteq V \) with

\[
|I| \geq \frac{k-1}{k} \cdot \frac{n}{d}.
\]

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Proof: We sketch the arguments. We use the method of conditional probabilities, cf. [AS 92]. Let \( V = \{v_1, v_2, \ldots, v_n\} \). Every vertex \( v_i \) will be assigned a probability \( p_i \in [0,1] \), \( i = 1, 2, \ldots, n \). Define a potential by

\[
V(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i - \sum_{E \in \mathcal{E}, v_i \in E} \prod_{v_i} p_i.
\]

The choice \( p_1 = p_2 = \ldots = p_n = p = 1/d \) gives the initial value of the potential

\[
V(p, \ldots, p) = p \cdot n - p^k \cdot \frac{n \cdot d^{k-1}}{k} = \frac{k-1}{k} \cdot \frac{n}{d}.
\]

In each step \( i, i = 1, 2, \ldots, n \), one after the other, we choose either \( p_i = 0 \) or \( p_i = 1 \) in order to maximize the current value of \( V(p_1, p_2, \ldots, p_n) \). As \( V(p_1, p_2, \ldots, p_n) \) is linear in each \( p_i \), for \( i = 1 \), for example, either \( V(p_1, \ldots, p_n) \leq V(1, p_2, \ldots, p_n) \) or \( V(p_1, \ldots, p_n) \leq V(0, p_2, \ldots, p_n) \).

If \( V(p_1, \ldots, p_n) \leq V(1, p_2, \ldots, p_n) \), we set \( p_1 = 1 \), else let \( p_1 = 0 \). Iterating this, we obtain finally \( p_1, p_2, \ldots, p_n \in \{0, 1\} \).

By our strategy, we infer \( V(p_1, p_2, \ldots, p_n) \geq V(p, p, \ldots, p) \). For \( V' = \{v_i \in V \mid p_i = 1\} \) we have

\[
|V'| = \sum_{i=1}^{n} p_i = V(p_1, p_2, \ldots, p_n) + \sum_{E \in \mathcal{E}, v_i \in E} \prod_{v_i} p_i.
\]

We can assume that \( V' \) is independent as otherwise we omit one vertex from each edge contained in \( V' \) and the value of \( V(p_1, p_2, \ldots, p_n) \) will not decrease. Thus, \( |V'| \geq V(p, p, \ldots, p) = \frac{k-1}{k} \cdot \frac{n}{d} \) and \( V' \) is an independent set. The running time is \( O(|V| + |\mathcal{E}|) \).

By (4) the average degree \( d \) of \( \mathcal{H} \) satisfies \( d^{k-1} \leq 2k \cdot |\mathcal{E}|/|X| \leq c_1 \cdot n^{k-1} \cdot u \). By Lemma 1 we can find in time \( O(|X| + |\mathcal{E}|) = O(n^k \cdot u) \) an independent set in \( \mathcal{H} = (X, \mathcal{E}) \) of size at least

\[
\frac{k-1}{k} \cdot \frac{n}{d} \geq c' \cdot \frac{n}{(n^{k-1} \cdot u)^{\frac{1}{2k-1}}} = c' \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}}
\]

where \( c' > 0 \) is a constant. With the sorting procedure in the beginning, this part of the algorithm can be done in time \( O(n^k \cdot u + n^k \cdot \ln n) \).

Now, assume that \( u = \sqrt{n} \cdot \omega(n) \) where \( \omega(n) \to \infty \) with \( n \to \infty \). Again we consider the hypergraph \( \mathcal{H} = (X, \mathcal{E}) \). First, we construct the sets \( C_{2,j} \) of \((2,j)\)-cycles in \( \mathcal{H} \), \( j = 2, 3, \ldots, 2k - 1 \). Using that the \( k \)-element sets are sorted according to their color, and that
sets of the same color are pairwise disjoint, and using the considerations leading to (8), all 2-cycles in \( H \) can be constructed in time \( O(|C_{2,j}|) = O(n^{2k-\lceil j/2 \rceil}) \).

We use the following lemma.

**Lemma 2** Let \( k \geq 3 \) be an integer. Let \( G = (V,\mathcal{E}) \) be a \( k \)-uniform hypergraph with \( |V| = n \). Let \( G \) contain \( \nu_j(G) \) many \((2, j)\)-cycles which can be determined all in time \( O(\nu_j(G)) \), \( j = 2, 3, \ldots, k-1 \). Then, for every real \( p \) with \( 0 \leq p \leq 1 \), one can find in time \( O(|V| + |\mathcal{E}| + \sum_{j=2}^{k-1} \nu_j(G)) \) an induced subhypergraph \( G' = (V', \mathcal{E}') \) such that

\[
|V'| \geq p/3 \cdot |V| \\
|\mathcal{E}'| \leq 3 \cdot p^k \cdot |\mathcal{E}| \\
\nu_j(G') \leq 3k \cdot p^{2k-j} \cdot \nu_j(G)
\]

for \( j = 2, 3, \ldots, k-1 \).

**Proof:** As in the proof of Lemma 1, we use the method of conditional probabilities. Let \( C_{2,j} \) be the set of all \((2, j)\)-cycles in \( G \), \( j = 2, 3, \ldots, k-1 \).

Let \( V = \{v_1, v_2, \ldots, v_n\} \). If \( pn < 3.9 \), any two-element subset \( V' \subseteq V \) gives the desired subhypergraph, thus let \( pn \geq 3.9 \). Every vertex \( v_i \) will be assigned a probability \( p_i \in [0,1] \), \( i = 1, 2, \ldots, n \). Define a potential \( V(p_1, p_2, \ldots, p_n) \) by

\[
V(p_1, p_2, \ldots, p_n) = 3^{pn/3} \cdot \prod_{i=1}^{n} \left( 1 - \frac{2}{3} \cdot p_i \right) + \\
\quad + \sum_{E \in \mathcal{E}} \prod_{e \in E} p_i + \sum_{j=2}^{k-1} \frac{1}{3} \cdot \frac{\sum_{C \in C_{2,j}} \prod_{e \in C} p_i}{3k \cdot p^{2k-j} \cdot \nu_j(G)}.
\]

With \( p_1 = p_2 = \ldots = p_n = p \) in the beginning, for \( pn/3 \geq 1.3 \) we have

\[
V(p, \ldots, p) = 3^{pn/3} \cdot \left( 1 - \frac{2}{3} \cdot p \right)^n + \frac{p^k \cdot |\mathcal{E}|}{3 \cdot p^k \cdot |\mathcal{E}|} + \sum_{j=2}^{k-1} \frac{p^{2k-j} \cdot \nu_j(G)}{3k \cdot p^{2k-j} \cdot \nu_j(G)}
\]

\[
\leq \left( \frac{3}{e^2} \right)^{pn/3} + \frac{2}{3}
\]

\[
< 1.
\]

Step by step, we decide which choice of \( p_i \in \{0,1\} \) minimizes the current value of \( V(p_1, p_2, \ldots, p_n) \).

We set \( p_1 = 1 \), if \( V(1, p_2, \ldots, p_n) \leq V(0, p_2, \ldots, p_n) \), else we set \( p_1 = 0 \). Iterating this for all vertices \( v_1, v_2, \ldots, v_n \), we obtain finally \( p_1, p_2, \ldots, p_n \in \{0,1\} \).
We have chosen the \( p_i \)'s to minimize the potential, thus, \( V(p_1, p_2, \ldots, p_n) < 1 \). The set \( V' = \{ v_i \in V \mid p_i = 1 \} \) yields the desired induced subhypergraph as otherwise \( V(p_1, p_2, \ldots, p_n) > 1 \).

The whole computation can be done in time \( O(|V| + |E| + \sum_{j=2}^{k-1} \nu_j(G)) \).

We apply Lemma 2 to the hypergraph \( \mathcal{H} = (X, \mathcal{E}) \) with

\[
p = \left( \frac{1}{n^{k-1} \cdot u} \right)^{\frac{1}{k-1}} \left( \frac{u}{\sqrt{n}} \right)^{\frac{1}{k-1} \cdot \left( \frac{1}{2^k - 1} \right)},
\]

and we obtain in time \( O(|X| + |E| + \sum_{j=2}^{k-1} \nu_j(\mathcal{H})) = O(u \cdot n^{2k-1}) \) an induced subhypergraph \( \mathcal{H}' = (X', \mathcal{E}') \) of \( \mathcal{H} \) with \( |X'| \geq pn/3 \), and, \( |\mathcal{E}'| \leq 3p^{2k} \cdot |\mathcal{E}| \) and, using the considerations (9), (10) the 2-cycles of \( \mathcal{H}' \) satisfy \( \sum_{j=2}^{k-1} \nu_j(\mathcal{H}') \leq pn/6 \) for \( n \) large enough. Then, in time at most \( O(u \cdot n^{2k-1}) \) we can determine all 2-cycles in \( \mathcal{H}' \) and delete from \( \mathcal{H}' \) one vertex from each 2-cycle. The resulting induced hypergraph \( \mathcal{H}'' \) on at least \( pn/6 \) vertices contains at most \( c \cdot p^{2k} \cdot n^k \cdot u \) edges, thus, has average degree \( d^{2k-1} \leq c' \cdot p^{2k-1} \cdot n^k \cdot u \). Then, we apply the following result from [BL 96] which gives an algorithmic version of the existence result from [DLR 95] and extends an algorithm of Fundia [Fu 96].

**Theorem 3** Let \( k \geq 3 \) be a fixed integer. Let \( \mathcal{G} = (V, \mathcal{E}) \) be a \( k \)-uniform hypergraph on \( n \) vertices with average degree at most \( t^{k-1} \). If \( \mathcal{G} \) does not contain any 2-cycles, then one can find for every fixed \( \delta > 0 \) in time \( O(n \cdot t^{k-1} + n^3/t^{3-\delta}) \) an independent set of size at least \( c(k, \delta) \cdot n/t \cdot (\ln t)^{1/(k-1)} \).

We apply Theorem 3 to \( \mathcal{H}'' \) and in time \( O \left( p^{2k} \cdot n^k \cdot u + n^3 \left/ \left( p \cdot n^{\frac{k-1}{2k-1}} \cdot u^{\frac{k-1}{2k-1}} \right)^{3-\delta} \right. \right) \)

\[ = o \left( n^{2k-1} \cdot u \right) \], where \( \delta < 3 \), we obtain an independent set in \( \mathcal{H}'' \) hence in \( \mathcal{H} \) of size at least

\[ c_2 \cdot \left( \frac{n^k}{u} \right)^{\frac{1}{2k-1}} \cdot \left( \ln \left( \frac{u}{\sqrt{n}} \right) \right)^{\frac{1}{2k-1}}. \]

The corresponding vertices form a totally multicolored set of size as desired.

4 Concluding Remarks

The running time of the algorithm can be reduced slightly as follows. Similarly as in Lemma 2, we choose first a subhypergraph \( \mathcal{H}' = (X', \mathcal{E}') \) of \( \mathcal{H} = (X, \mathcal{E}) \), where we do not control
the 2-cycles, but where $|X'| = p_1 n/3$ and $|E'| \leq 3p_1^{2k} \cdot |E|$. Then, $\mathcal{H}'$ contains at most $O(u \cdot (p_1 \cdot n)^{2k-[j/2]}-1)$ many $(2,j)$-cycles. The value of $p_1 > 0$ should be chosen as small as possible such that for some constant $\gamma > 0$ and $j = 2, 3, \ldots, 2k - 1$, cf. [DLR 95] or [BL 96]:

$$v \cdot (p_1 n)^{2k-[j/2]} - 1 = O \left( p_1 n \cdot \left( p_1 \cdot n^{2^{k-1}-1} \cdot u^{3^{k-1}} \right)^{4^{k-1}-j-\gamma} \right).$$

For this subhypergraph we apply Lemma 2 with a different parameter $p_2$ with $p \approx p_1 \cdot p_2$ and proceed as before where the value of $p$ is given by (20). Thus, we save some time by controlling the 2-cycles later. However, more interesting might be to find the real growth rate of $f_u(n,k)$ and a corresponding fast algorithm. It might be also of some interest to give explicitly a coloring which yields our, or possibly better upper bounds, on $f_u(n,k)$.

References


