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A SERIAL VERSION OF HODGES AND LEHMANN'S "6/π RESULT"

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Abstract

While the asymptotic relative efficiency (ARE) of Wilcoxon rank-based tests for location and regression with respect to their parametric Student competitors can be arbitrarily large, Hodges and Lehmann (1961) have shown that the ARE of the same Wilcoxon tests with respect to their van der Waerden or normal-score counterparts is bounded from above by $6/\pi \approx 1.910$, and that this bound is sharp. We extend this result to the serial case, showing that, when testing against linear (ARMA) serial dependence, the ARE of the Spearman-Wald-Wolfowitz and Kendall rank-based autocorrelations with respect to the van der Waerden or normal-score ones admits a sharp upper bound of $(6/\pi)^2 \approx 3.648$.

Key words: Asymptotic relative efficiency, rank-based tests, Wilcoxon test, van der Waerden test, Spearman autocorrelations, Kendall autocorrelations, linear serial rank statistics

1. Introduction

The Pitman asymptotic relative efficiency $\text{ARE}_f(\phi_1/\phi_2)$ under density f of a test ϕ_1 with respect to a test ϕ_2 is defined as the limit, when it exists, as n_1 tends to infinity, of the ratio $n_2^f(n_1)/n_1$ of the number $n_2^f(n_1)$ of observations it takes for the test ϕ_2, under density f, to match the local performance of the test ϕ_1 based on n_1 observations. That concept was first proposed by Pitman in the unpublished lecture notes [30] he prepared for a 1948-49 course at Columbia University. The first published rigorous treatment of the subject was by Noether [27] in 1955. A similar definition applies to point estimation; see, for instance, Hallin (2012) for a more precise definition. An in-depth treatment of the concept can be found in

Chapter 10 of Serfling [33], Chapter 14 of van der Vaart [34], or in the monograph by Nikitin [26].

The study of AREs of rank tests and R-estimators with respect to each other or with respect to their classical Gaussian counterparts has produced a number of interesting and sometimes quite surprising results. Considering the van der Waerden or normal-score two-sample location rank test ϕ_{vdW} and its classical normal-theory competitor, the two-sample Student test $\phi_{\mathcal{N}}$, Chernoff and Savage in 1958 established the rather striking fact that, under any density f satisfying very mild regularity assumptions,

$$\mathrm{ARE}_f(\phi_{\mathrm{vdW}}/\phi_{\mathcal{N}}) \geq 1, \tag{1.1}$$

with equality holding at the Gaussian density $f = \phi$ only. That result implies that rank-based tests based on Gaussian scores (that is, the two-sample rank-based tests for location, but also the one-sample signed-rank ones, traditionally associated with the names of van der Waerden, Fraser, Fisher, Yates, Terry and/or Hoeffding—for simplicity, in the sequel, we uniformly call them van der Waerden tests—asymptotically outperform the corresponding everyday practice Student t-test; see [2]. That result readily extends to regression models with independent noise.

Another celebrated bound is the one obtained in 1956 by Hodges and Lehmann, who proved that, denoting by ϕ_{W} the Wilcoxon test (same location and regression problems as above),

$$\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathcal{N}}) \geq 0.864, \tag{1.2}$$

which implies that the price to be paid for using rank- or signed-rank tests of the Wilcoxon type (that is, logistic score-based rank tests) instead of the traditional Student ones never exceeds 13.5% of the total number of observations. That bound moreover is sharp, being reached under the Epanechnikov density f. On the other hand, the benefits of considering Wilcoxon rather than Student can be arbitrarily large, as it is easily shown that the supremum over f of $\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathcal{N}})$ is infinite; see [20].

Both (1.1) and (1.2) created quite a surprise in the statistical community of the late fifties, and helped dispelling the wrong idea, by then quite widespread, that rank-based methods, although convenient and robust, could not be expected to compete with the efficiency of traditional parametric procedures.

Chernoff-Savage and Hodges-Lehmann inequalities since then have been extended to a variety of more general settings. In the elliptical context, optimal rank-based procedures for location (one and m-sample case), regression, VARMA models, and scatter (one and m-sample case) have been constructed in a series of papers by Hallin and Paindaveine ([8], [9], [10], [11], and [13]), based on a multivariate concept of signed ranks. The Gaussian competitors here are of the Hotelling, Fisher, correlogram-based portmanteau, Durbin-Watson or Lagrange multiplier forms. For all those tests, Chernoff-Savage result similar to (1.1) have been established (see also [28, 29]). Hodges-Lehmann results also have been obtained, with bounds that, quite interestingly, depend on the dimension of the observation space: see [8].

Another type of extension is into the direction of time series and serial statistics. Hallin [6] extended Chernoff and Savage's result (2.4) to the serial context by showing that the serial van der Waerden rank tests also uniformly dominate their Gaussian competitors (of the correlogram-based portmanteau, Durbin-Watson or Lagrange multiplier forms). Similarly, Hallin and Tribel [19] proved that the 0.864 upper bound in (2.5) no longer holds for the AREs of the Wilcoxon serial rank test with respect to their Gaussian competitors, and is to be replaced by a slightly lower 0.854 one.

Bounds on AREs typically are obtained via variational techniques. More precisely, given a rank-based test ϕ_J, one obtains bounds on $\mathrm{ARE}_f(\phi_J/\phi_{\mathcal{N}})$ by minimizing some integral over all densities satisfying specific moment and integrability constraints. In most cases, there exists a non degenerate distribution f_0 achieving the infimum, which therefore is a minimum and a sharp lower bound. In the sequel, however, we also call sharp a bound that is attained only as a supremum or an infimum with respect to some sequence of densities.

Now, taking AREs with respect to Gaussian procedures as the t-test is not always the best way to evaluate the asymptotic performance of a test. Such AREs indeed require the Gaussian procedure to be valid under the density f under consideration, a condition which places restrictions on f that may not be satisfied. When the Gaussian tests are no longer valid, one rather may choose to consider AREs of the form

$$\mathrm{ARE}_f(\phi_J/\phi_K) \tag{1.3}$$

comparing the asymptotic performances (under f) of two rank-based tests ϕ_J and ϕ_K, based on score functions J and K, respectively. Being distribution-free, rank-based procedures indeed do not impose any validity conditions on f, so that, contrary to $\mathrm{ARE}_f(\phi_J/\phi_{\mathcal{N}})$, $\mathrm{ARE}_f(\phi_J/\phi_K)$ exists for any f (satisfying the mild requirements for AREs to exist); see, for instance, [17] and [18], where rank-based inference is performed in linear models with stable errors under which Students tests are not valid.

When studying the extrema with respect to f of $\mathrm{ARE}_f(\phi_J/\phi_K)$, however, one is faced with a problem involving the ratio of two integrals, while the simpler case of $\mathrm{ARE}_f(\phi_J/\phi_{\mathcal{N}})$ only involves a single integral. The resulting variational problem then often has trivial solutions, in the sense that there exists no nondegenerate distribution at which the supremum/infimum is attained. In general, though, one can construct sequences of densities f_i under which those extremal values are obtained as limits for $i \to \infty$. In such cases, we still call them *sharp*.

The first result about AREs of the form (1.3) was obtained in 1961 by Hodges and Lehmann, who in [21] show that

$$0 \leq \mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}}) \leq 6/\pi \approx 1.910 \tag{1.4}$$

for all f in some class $\mathcal{F}$ of density functions satisfying weak differentiability conditions. As anticipated, the $6/\pi$ bound is not attained. However, it is sharp in the sense indicated above, as Hodges and Lehmann exhibit a parametric family of densities $x \mapsto f_\alpha(x)$ (with parameter α ranging between 0 and $+\infty$) for which the function $\alpha \mapsto \mathrm{ARE}_{f_\alpha}(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}})$ achieves any value in the interval $(0, 6/\pi)$. In case f has finite second-order moments, of course, one has that

$$\mathrm{ARE}_f(\phi_{\mathrm{vdW}}/\phi_{\mathcal{N}}) = \mathrm{ARE}_f(\phi_{\mathrm{vdW}}/\phi_{\mathrm{W}}) \times \mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_{\mathcal{N}});$$

Hodges and Lehmann's "$6/\pi$ result" thus implies that the ARE of the van der Waerden tests with respect to the Student ones, which by the Chernoff-Savage result is larger than or equal to one, actually can be quite high, and even arbitrarily large.

In this paper, we discuss (Section 2) extensions of (1.4) to a broader class of densities (thereby recovering the main results from [3]), and (Section 3) to the case of serial rank statistics and alternatives of serial dependence, where we show that the serial counterpart of the $6/\pi \approx 1.910$ bound, for the Spearman-Wald-Wolfowitz or Kendall autocorrelation coefficients with respect to the van der Waerden ones, turns out to be $(6/\pi)^2 \approx 3.648$.

2. Asymptotic relative efficiencies of rank-based procedures for location and regression

The asymptotic behavior under local alternatives of rank-based test statistics, in general, is obtained via an application of Le Cam's Third Lemma (a method that goes back to Hájek and Šidák [5]; see, for instance, Chapter 13 of [34]). In one- and multisample location and regression models (this includes ANOVA etc.), the asymptotic distribution under error density f of a rank test statistic based on the score-generating function J (satisfying some regularity conditions) depends on quantities of the form

$$\mathcal{K}(J) := \int_0^1 J^2(u)\mathrm{d}u \quad \text{and} \quad \mathcal{K}(J, f) := \int_0^1 J(u)\varphi_f(F^{-1}(u))\mathrm{d}u$$

where, assuming that f, with distribution function F, admits a weak derivative f' (differentiability in quadratic mean of $f^{1/2}$ is the standard assumption here, see Chapter 7 of [34]; but absolute continuity of f in the traditional sense, with a.e. derivative f', is sufficient in the present context) and, letting $\varphi_f := -f'/f$, has finite Fisher information for location $\mathcal{I}(f) := \int_0^1 \varphi_f^2(F^{-1}(u))du$. Denote by $\mathcal{F}$ the class of such densities.

When testing for location, the ARE, under a density $f \in \mathcal{F}$, of a rank-based test ϕ_{J_1} based on the square-summable score-generating function J_1 with respect to another rank-based test ϕ_{J_2} based on the square-summable score-generating function J_2 then takes the form

$$\mathrm{ARE}_f\left(\phi_{J_1}/\phi_{J_2}\right) = \frac{\mathcal{K}(J_2)}{\mathcal{K}(J_1)}\left(\frac{\int_0^1 J_1(u)\varphi_f(F^{-1}(u))\mathrm{d}u}{\int_0^1 J_2(u)\varphi_f(F^{-1}(u))\mathrm{d}u}\right)^2, \tag{2.1}$$

provided that the scores are monotone, or the difference between two monotone functions. Those ARE values readily extend to two- and m-sample testing and R-estimation problems, ANOVA and regression, and, in a time-series context, under slightly more restrictive assumptions on the scores, to the partly rank-based tests and R-estimators considered by Koul and Saleh [23] and [24].

Our first result is a sharp bound on the quantities in (2.1) (these bounds include those in the right-hand side of (1.4)) wherein we exploit the simplicity of the Wilcoxon scores $J_{\mathrm{W}}(x) = (x - \frac{1}{2})$ which serves as a natural reference basis for ARE comparisons. In the sequel, we write ϕ_{W} instead of $\phi_{J_{\mathrm{W}}}$, ϕ_{vdW} instead of $\phi_{J_{\mathrm{vdW}}}$, etc.

Proposition 2.1. *Suppose that $f \in \mathcal{F}$ is a symmetric probability distribution function. Let J be a score function which is skew-symmetric about $1/2$ on $[0, 1]$. If J is differentiable at $1/2$ and*

(i) convex on $(1/2, 1)$, then

$$\mathrm{ARE}_f(\phi_{\mathrm{W}}/\phi_J) \leq 12\mathcal{K}(J)/(J'(1/2))^2; \tag{2.2}$$

(ii) concave on $(1/2, 1)$*, then*

$$\mathrm{ARE}_f(\phi_J/\phi_{\mathrm{W}}) \leq (J'(1/2))^2/12\mathcal{K}(J). \tag{2.3}$$

Both bounds moreover are sharp.

Proof. We have $\mathcal{K}(J_{\mathrm{W}}) = 1/12$ so that, by the symmetry assumption on f, (2.1) takes the form

$$\mathrm{ARE}(\phi_J/\phi_{\mathrm{W}}) = \frac{1}{12\mathcal{K}(J)}\left(\frac{\int_{1/2}^1 J(u)\varphi_f(F^{-1}(u))\mathrm{d}u}{\int_{1/2}^1 (u-1/2)\varphi_f(F^{-1}(u))\mathrm{d}u}\right)^2.$$

If J is concave (resp., convex) on $(1/2, 1)$ then $J(x) \leq J'(1/2)(x - \frac{1}{2})$ (resp., $J(x) \geq J'(1/2)(x - \frac{1}{2})$) for all $\frac{1}{2} \leq x \leq 1$; the result follows. To see that those bounds are sharp, note that, although no non-degenerate density f is achieving equality in (2.2) and (2.3), sequences f_i, $i = 1, 2, \ldots$ of densities such that $\mathrm{ARE}_{f_i}(\phi_{J_{\mathrm{W}}}/\phi_J)$ converges to the bound as $i \to \infty$ do exist : it suffices to construct a sequence of symmetric distributions $f_i(x)$ which put most of their weight at round $x = 1/2$ and such that with $\lim_{i\to\infty} F_i(x) = 1/2$. An example of such a sequence is $f(x; \alpha_i)$ with $\alpha_i > 0$ and $\lim_{i\to\infty} \alpha_i = 0$, where $f(x; \alpha)$ denotes the symmetric α-stable density with tail index α (see [17, Figure 1] for several numeric illustrations). □

Applying Proposition 2.1 to the score functions $J_{\mathrm{vdW}}(x) = \Phi^{-1}(x)$ (the van der Waerden score function) and $J_{\mathrm{Cauchy}}(x) = \sin(2\pi(x - \frac{1}{2}))$ (the Cauchy score function) yields the following corollary.

Corollary 2.1. *For all symmetric probability densities* $f \in \mathcal{F}$*,*

$$ARE_f(\phi_{\mathrm{W}}/\phi_{\mathrm{vdW}}) \leq 6/\pi \quad \textit{and} \quad ARE_f(\phi_{\mathrm{Cauchy}}/\phi_{\mathrm{W}}) \leq 2\pi^2/3, \tag{2.4}$$

and thus

$$ARE_f(\phi_{\mathrm{Cauchy}}/\phi_{\mathrm{vdW}}) \leq 4\pi. \tag{2.5}$$

Those bounds moreover are sharp.

Proof. The van der Waerden score is convex and skew-symmetric about 1/2, with

$$\mathcal{K}(J_{\mathrm{vdW}}) = 1 \quad \text{and} \quad J'_{\mathrm{vdW}}(u) = \sqrt{2\pi}\exp((\Phi^{-1}(u))^2/2),$$

so that $J'_{\mathrm{vdW}}(1/2) = \sqrt{2\pi}$. The Cauchy score is concave and skew-symmetric about 1/2, with

$$\mathcal{K}(J_{\mathrm{Cauchy}}) = 1/2 \quad \text{and} \quad J'_{\mathrm{Cauchy}}(u) = 2\pi \cos(2\pi(u - 1/2)),$$

so that $J'_{\mathrm{Cauchy}}(1/2) = 2\pi$. The conclusion follows. □

Both Proposition 2.1 and its Corollary 3.1 are already available in Gastwirth [3]; Gastwirth's s proof, however, relies on the following assumption.

ASSUMPTION A. The score function J and the density f are weakly differentiable and such that

$$\lim_{|x|\to\infty} J(F(x))f(x) = 0.$$

That assumption is not required for the derivation of the ARE bounds (2.2) and (2.3); however, it guarantees that integration by parts is permitted in (2.1), from which we immediately obtain the neat formula

$$\mathrm{ARE}_f\,(\phi_{J_1}/\phi_{J_2}) = \frac{\mathcal{K}(J_2)}{\mathcal{K}(J_1)} \left(\frac{\mathrm{E}\left[(J_1 \circ F)'(X)\right]}{\mathrm{E}\left[(J_2 \circ F)'(X)\right]} \right)^2. \tag{2.6}$$

Introducing the constants (which may be infinite)

$$\kappa^+(J, f) := \sup_x (J \circ F)'(x) \quad \text{and} \quad \kappa^-(J, f) := \inf_x (J \circ F)'(x),$$

one then immediately obtains from (2.6) the double inequality

$$\frac{\kappa^-(J_1, f)}{\kappa^+(J_2, f)} \le \frac{\mathcal{K}(J_1)}{\mathcal{K}(J_2)}\,\mathrm{ARE}_f\,(\phi_{J_1}/\phi_{J_2}) \le \frac{\kappa^+(J_1, f)}{\kappa^-(J_2, f)}. \tag{2.7}$$

Remarkably the bounds contained in (2.7) are optimal in many cases. (For instance, (2.7) contains the bounds given in Proposition 2.1.) More importantly, equation (2.7) provides insight onto how one can easily construct ARE inequalities by restricting f to subclasses of densities over which the constants $\kappa^\pm$ satisfy some adequate conditions. For example, from (2.7), we get

$$12\kappa^-(\Phi^{-1}, f) \le \mathrm{ARE}_f\,(\phi_{J_{vdW}}/\phi_{J_W}) \le 12\kappa^+(\Phi^{-1}, f)$$

so that, considering the class of densities f for which $\kappa^-(\Phi^{-1}, f) \ge c$ for some constant $c > 0$, we get the new ARE bound

$$\mathrm{ARE}_f\,(\phi_{vdW}/\phi_W) \ge 12c.$$

Note that the condition $\kappa^{-}(\Phi^{-1}, f) \geq c > 0$ is crucial for obtaining convergence in Mallows distance of normalized sums of independent copies of X with density f towards the Gaussian, see [22].

Finally we mention several other extensions that are of interest. Denoting by G_1 and G_2 the distribution functions associated with the symmetric densities g_1 and g_2, let $J_1(u) = \varphi_{g_1}(G_1^{-1}(u))$ and $J_2(u) = \varphi_{g_2}(G_2^{-1}(u))$. Restrictions on the convexity/concavity of the function $u \mapsto G_1^{-1} \circ F(u)$ can clearly be used to obtain better bounds on $\mathrm{ARE}_f\,(\phi_{J_1}/\phi_{J_2})$. More generally, stochastic ordering considerations (involving f, g_1, and g_2) will lead to restricted ARE bounds as in, for instance, [25] and [1].

3. Extending Hodges and Lehmann's "6/π result" to the serial case

Until the early eighties, and despite some forerunning time-series applications such as (as early as 1943) Wald and Wolfowitz [35], rank-based methods have been essentially limited to statistical models involving univariate independent observations. Therefore, traditional ARE bounds (Hodges and Lehmann [20, 21], Chernoff-Savage [2] or Gastwirth [3]), as well as classical monographs (Hájek and Šidák [5], Randles and Wolfe [32], Puri and Sen [31]) mainly deal with univariate location and single-output linear models with independent observations. The situation since then has changed, and rank-based procedures nowadays have been proposed for a much broader class of statistical models, including time series problems, where serial dependencies are the main features under study. It is therefore of interest to reconsider classical results on ARE bounds in that serial dependence context.

In this section, we focus on the linear rank statistics of the serial type involving two square-integrable score functions. Those statistics enjoy optimality properties in the context of linear time series (ARMA models; see [16] for details). Once adequately standardized, those statistics yield the so-called *rank-based autocorrelation coefficients*. Denote by $R_1^{(n)}, \ldots, R_n^{(n)}$ the ranks in a triangular array $X_1^{(n)}, \ldots, X_n^{(n)}$ of observations. *Rank autocorrelations* (with lag k) are linear serial rank statistics of the form

$$\underset{\sim}{r}^{(n)}_{J_1 J_2;k} := \left[(n-k)^{-1} \sum_{t=k+1}^{n} J_1\left(\frac{R_t^{(n)}}{n+1}\right) J_2\left(\frac{R_{t-k}^{(n)}}{n+1}\right) - m^{(n)}_{J_1 J_2}\right] \left(s^{(n)}_{J_1 J_2}\right)^{-1},$$

where J_1 and J_2 are (square-integrable) score functions, whereas $m^{(n)}_{J_1 J_2}$ and $s^{(n)}_{J_1 J_2} := s^{(n)}_{J_1 J_2;k}$ denote the exact mean of $J_1\left(\frac{R_t^{(n)}}{n+1}\right) J_2\left(\frac{R_{t-k}^{(n)}}{n+1}\right)$ and the exact standard error of $(n-k)^{-\frac{1}{2}}\sum_{t=k+1}^{n} J_1\left(\frac{R_t^{(n)}}{n+1}\right) J_2\left(\frac{R_{t-k}^{(n)}}{n+1}\right)$ under the assumption of i.i.d. $X_t^{(n)}$'s (exchangeable $R_t^{(n)}$'s), respectively; we refer to pages 186 and 187 of [16] for explicit formulas. *Signed-rank autocorrelation coefficients* are defined similarly; see [15] or [16].

Rank and signed-rank autocorrelations are measures of serial dependence offering rank-based alternatives to the usual autocorrelation coefficients, of the form

$$r_k^{(n)} := \sum_{t=k+1}^{n} X_t X_{t-k} / \sum_{t=1}^{n} X_t^2,$$

which consitute the Gaussian reference benchmark in this context. Of particular interest are

(i) the *van der Waerden autocorrelations* [14]

$$\underset{\sim}{r}_{\text{vdW};k}^{(n)} := \left[(n-k)^{-1} \sum_{t=k+1}^{n} \Phi^{-1}\left(\frac{R_t^{(n)}}{n+1}\right) \Phi^{-1}\left(\frac{R_{t-k}^{(n)}}{n+1}\right) - m_{\text{vdW}}^{(n)}\right] \left(s_{\text{vdW}}^{(n)}\right)^{-1},$$

(ii) the *Wilcoxon autocorrelations* [14]

$$\underset{\sim}{r}_{\text{W};k}^{(n)} := \left[(n-k)^{-1} \sum_{t=k+1}^{n} \left(\frac{R_t^{(n)}}{n+1} - \frac{1}{2}\right) \log\left(\frac{R_{t-k}^{(n)}}{n+1-R_{t-k}^{(n)}}\right) - m_{\text{W}}^{(n)}\right] \left(s_{\text{W}}^{(n)}\right)^{-1},$$

(iii) the *Laplace autocorrelations* [14]

$$\begin{aligned}\underset{\sim}{r}_{\text{L};k}^{(n)} := \Bigg[&(n-k)^{-1} \sum_{t=k+1}^{n} \text{sign}\left(\frac{R_t^{(n)}}{n+1} - \frac{1}{2}\right) \\ &\times \left[\log\left(2\frac{R_{t-k}^{(n)}}{n+1}\right) I\left[\frac{R_{t-k}^{(n)}}{n+1} \le \frac{1}{2}\right] - \log\left(2 - 2\frac{R_{t-k}^{(n)}}{n+1}\right) I\left[\frac{R_{t-k}^{(n)}}{n+1} > \frac{1}{2}\right]\right] \\ &- m_{\text{L}}^{(n)}\Bigg] \left(s_{\text{L}}^{(n)}\right)^{-1},\end{aligned}$$

(iv) the *Wald-Wolfowitz* or *Spearman autocorrelations* [35]

$$\underset{\sim}{r}_{\text{SWW};k}^{(n)} := \left[(n-k)^{-1} \sum_{t=k+1}^{n} R_t^{(n)} R_{t-k}^{(n)} - m_{\text{SWW}}^{(n)}\right] \left(s_{\text{SWW}}^{(n)}\right)^{-1},$$

(v) and the *Kendall autocorrelations* [4] (where explicit values of $m_{\rm K}^{(n)}$ and $s_{\rm K}^{(n)}$ are provided)

$$\underset{\sim}{r}{}^{(n)}_{{\rm K};k} := \left[1 - \frac{4D_k^{(n)}}{(n-k)(n-k-1)} - m_{\rm K}^{(n)}\right] \left(s_{\rm K}^{(n)}\right)^{-1}$$

with $D_k^{(n)}$ denoting the number of discordances at lag k, that is, the number of pairs $(R_t^{(n)}, R_{t-k}^{(n)})$ and $(R_s^{(n)}, R_{s-k}^{(n)})$ that satisfy either

$$R_t^{(n)} < R_s^{(n)} \quad \text{and} \quad R_{t-k}^{(n)} > R_{s-k}^{(n)}, \quad \text{or} \quad R_t^{(n)} > R_s^{(n)} \quad \text{and} \quad R_{t-k}^{(n)} < R_{s-k}^{(n)};$$

more specifically, $D_k^{(n)} := \sum_{t=k+1}^{n} \sum_{s=t+1}^{n} I(R_t^{(n)} < R_s^{(n)},\, R_{t-k}^{(n)} > R_{s-k}^{(n)})$.

Van der Waerden, Wilcoxon and Laplace autocorrelations are optimal—in the sense that they allow for *locally optimal* rank tests in the case of ARMA models with normal, logistic and double-exponential densities, respectively. The Spearman and Kendall autocorrelations are serial versions of Spearman's *rho* and Kendall's *tau*, respectively, and are asymptotically equivalent under the null hypothesis of independence; although they are never optimal under any density, they achieve excellent overall performance. Signed rank autocorrelations are defined in a similar way.

Denote by $\mathcal{F}_2$ the subclass of densities $f \in \mathcal{F}$ having finite moments of order two. Let J_i, $i = 1, \ldots, 4$ denote four square-summable score functions, and assume that they are monotone increasing, or the difference between two monotone increasing functions (that assumption tacitly will be made in the sequel each time AREs are to be computed). The asymptotic relative efficiency, under innovation density $f \in \mathcal{F}_2$, of the rank-based tests $\phi^r_{J_1J_2}$ involving the score functions J_1 and J_2 (that is, the autocorrelations $\underset{\sim}{r}{}^{(n)}_{J_1J_2;k}$) with respect to the rank-based tests $\phi^r_{J_3J_4}$ involving the score functions J_3 and J_4 (the autocorrelations $\underset{\sim}{r}{}^{(n)}_{J_3J_4;k}$) is

$$\begin{aligned}
&{\rm ARE}_f(\phi^r_{J_1J_2}/\phi^r_{J_3J_4}) \\
&= \frac{\mathcal{K}(J_3)}{\mathcal{K}(J_1)} \left(\frac{\int_0^1 J_1(v)\varphi_f(F^{-1}(v)){\rm d}v}{\int_0^1 J_3(v)\varphi_f(F^{-1}(v)){\rm d}v}\right)^2 \frac{\mathcal{K}(J_4)}{\mathcal{K}(J_2)} \left(\frac{\int_0^1 J_2(v)F^{-1}(v){\rm d}v}{\int_0^1 J_4(v)F^{-1}(v){\rm d}v}\right)^2 \\
&=: C_f(J_1, J_3)\, D_f(J_2, J_4) \qquad (3.1)
\end{aligned}$$

The ratio $C_f(J_1, J_3)$ has already been studied in Section 2, and the same conclusions apply here. As for the ratio $D_f(J_2, J_4)$, we can use similar tools to obtain an extension of Proposition 2.1 to the serial case. Denote by $\phi^r_{\rm vdW}$, $\phi^r_{\rm W}$, ... the tests based on $\underset{\sim}{r}{}^{(n)}_{{\rm vdW};k}$, $\underset{\sim}{r}{}^{(n)}_{{\rm W};k}$, etc.

Proposition 3.1. *Suppose that $f \in \mathcal{F}_2$ is a symmetric probability distribution function. If, furthermore, J_1 and J_2 are skew-symmetric about $1/2$ on $[0,1]$ and*

(i) convex on $(1/2, 1)$, then

$$\textit{ARE}_f(\phi^r_{\mathrm{SWW}}/\phi^r_{J_1J_2}) = \textit{ARE}_f(\phi^r_{\mathrm{K}}/\phi^r_{J_1J_2}) \leq 144\frac{\mathcal{K}(J_1)\mathcal{K}(J_2)}{(J_1'(1/2))^2(J_2'(1/2))^2};$$

(ii) concave on $(1/2, 1)$, then

$$\textit{ARE}_f(\phi^r_{J_1J_2}/\phi^r_{\mathrm{SWW}}) = \textit{ARE}_f(\phi^r_{\mathrm{K}}/\phi^r_{J_1J_2}) \leq \frac{1}{144}\frac{(J_1'(1/2))^2(J_2'(1/2))^2}{\mathcal{K}(J_1)\mathcal{K}(J_2)}.$$

Proof. Consider the ARE expression (3.1) with $J_3 = J_4 = J_{\mathrm{W}}$, with J_{W} the Wilcoxon score function. Clearly, $C_f(J_1, J_{\mathrm{W}})$ then coincides with the nonserial $\mathrm{ARE}_f(\phi_J/\phi_W)$ studied in the previous section and the results obtained there directly apply. As for $D_f(J_2, J_{\mathrm{W}})$, the same arguments on the convexity (resp., concavity) of J as in the proof of Proposition 2.1 entail that $D_f(J_2, J_{\mathrm{W}}) \geq J'(1/2)$ (resp., $D_f(J_2, J_{\mathrm{W}}) \leq J'(1/2)$); the claim follows. □

In particular, we deduce from Proposition 3.1 the following extension of Hodges and Lehmann's "$6/\pi$ result".

Corollary 3.1. *For all symmetric probability densities $f \in \mathcal{F}_2$,*

$$\textit{ARE}_f(\phi^r_{\mathrm{SWW}}/\phi^r_{\mathrm{vdW}}) = \textit{ARE}_f(\phi^r_{\mathrm{K}}/\phi^r_{\mathrm{vdW}}) \leq (6/\pi)^2, \tag{3.2}$$

and this bound is sharp.

Proof. The value $(6/\pi)^2$ as an upper bound immediately follows from the previous discussion, and all that remains to show is that it is sharp. Here the arguments provided in Section 2 no longer apply due to the fact that heavy-tailed distributions do not belong to $\mathcal{F}_2$. A little exploration, however, leads to a family of densities with finite second-order moments for which the bound in (3.2) constitutes a supremum: denoting by f_α the Weibull probability density function with scale one and shape parameter α, it is easily shown that

$$\lim_{\alpha\to\infty} C_{f_\alpha}(J_{\mathrm{W}}, J_{\mathrm{vdW}}) = \lim_{\alpha\to\infty} D_{f_\alpha}(J_{\mathrm{W}}, J_{\mathrm{vdW}}) = 6/\pi.$$ □

We conclude with a discussion of the properties of ratio $D_f(J_2, J_4)$ for more general score functions J_2 and J_4. Note that the integration by parts argument yielding the general ARE bound (2.7) in general no longer applies: in particular the score functions (appearing in the rank-based autocorrelation coefficients of Section 3) of the form $x \mapsto J_2(x) = F^{-1}(x)$ do not vanish at the edges of their supports. It is nevertheless possible to obtain non-trivial general bounds as follows. First, from Cauchy-Schwartz,

$$\Big(\int_0^1 J_2(u)F^{-1}(u)\mathrm{d}u\Big)^2 \leq \int_0^1 (J_2(u))^2\mathrm{d}u \int_0^1 (F^{-1}(u))^2\mathrm{d}u = \mathcal{K}(J_2)\ \mathrm{Var}_f(X)$$

where $\mathrm{Var}_f(X)$ is the variance of X under density f. Second, since the mappings $u \mapsto J_4(u)$ and $u \mapsto F^{-1}(u)$ both are non-decreasing over $(0, 1)$,

$$\int_{1/2}^1 J_4(u)F^{-1}(u)\mathrm{d}u \geq 2\int_{1/2}^1 J_4(u)\mathrm{d}u \int_{1/2}^1 F^{-1}(u)\mathrm{d}u = \int_{1/2}^1 J_4(u)\mathrm{d}u\ \mathrm{E}_f|X|,$$

via inequality (2.8) of [3]. This yields

$$D_f(J_2, J_4) \leq \frac{\mathcal{K}(J_2)}{\big(\int_{1/2}^1 J_4(u)\mathrm{d}u\big)^2}\frac{\mathrm{Var}_f(X)}{\big(\mathrm{E}_f|X|\big)^2}. \tag{3.3}$$

Because the function $u \mapsto F^{-1}(u)$ is typically unbounded as $u \to 1$, it is not possible, in general, to obtain bounds in the spirit of (2.7) for $D_f(J_2, J_4)$ without imposing restrictions on the class of reference densities f. As in the non-serial case (see the discussion at the end of Section 2), such restrictions provide an alternative source of interesting ARE bounds. For instance, restricting to densities f (with distribution function F) for which

$$J_2 \circ F(x) \geq J_4 \circ F(x)\ (\text{resp., } J_2 \circ F(x) \leq J_4 \circ F(x)) \quad \text{for all } x \in \mathbb{R}^+, \tag{3.4}$$

we get $D_f(J_2, J_4) \geq \mathcal{K}(J_4)/\mathcal{K}(J_2)$ (resp., $D_f(J_2, J_4) \leq \mathcal{K}(J_4)/\mathcal{K}(J_2)$). Obviously, if $J_2 = J_{\mathrm{W}}$ (the Wilcoxon score), a condition such as (3.4) holds uniformly over all distribution functions F with density $f \in \mathcal{F}_2$ as soon as J_4 is convex (resp., concave). More generally, ordering considerations (between J_2 and J_4), as in Loh (1984), clearly lead to new families of ARE inequalities.

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