# Using different Pitman-closeness techniques for the linear combination of multivariate forecasts

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**Abstract:** We specify the Pitman-closeness criterion for the evaluation of multivariate forecasts in three categories. This is done closely to the definition of covariance adjustment techniques analysed in other articles. We also apply the Pitman-closeness techniques to an example dealing with German economic data.

**Key words:** Pitman-closeness, multivariate forecasting methods, covariance adjustment.

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## 1 Introduction

We introduced the Pitman-closeness criterion for the evaluation of multivariate forecasts (see Wenzel, 1998b) and derived optimal weight matrices for the combination of forecasts. There we used real valued square matrices as weights, which is the most general case and is called ***Strong*** *Pitman-closeness*. Now we also analyse the situation where the weight matrices are assumed to be diagonal as well as the case where the weights for the combination of the multivariate forecasts are restricted to be scalar. For this we define the notions of ***Medium*** *Pitman-closeness* and the ***Weak*** *Pitman-closeness*. The denotations ***Strong***, ***Medium*** and ***Weak*** *Pitman-closeness* are close to those of the covariance adjustment techniques presented by Troschke (1999a). Finally we calculate the different combinations of forecasts for a German macro economic data set.

## 2 The Pitman-closeness criterion

First we give a description of the problem as in Wenzel (1998b).

Assume that

$\mathbf{Y} := (Y_1, \ldots, Y_k)'$ is a random vector to be forecasted (k≥2),

$\mathbf{F}_i := (F_{1i}, \ldots, F_{ki})'$ are unbiased multivariate forecasts (i=1,...,n) for **Y** and

$\mathbf{u}_i := (Y_1 - F_{1i}, \ldots, Y_k - F_{ki})'$ is the error vector of the i-th forecast method,

where $\mathbf{u} := \left(\mathbf{u}_1', \ldots, \mathbf{u}_n'\right)' \sim N_{n\cdot k}(\mathbf{0}, \Sigma)$ , $\Sigma$ p.d., and there exists a vector $\mathbf{u}_i$, without loss of generality $\mathbf{u}_i = \mathbf{u}_n$, so that $\mathrm{Cov}[((\mathbf{u}_1 - \mathbf{u}_n)', \ldots, (\mathbf{u}_{n-1} - \mathbf{u}_n)')']$ is p.d.

Furthermore we are again giving the definition of the component-by-component Pitman-closeness.

*Definition 1.* The forecast $\mathbf{F}_1$ is component-by-component Pitman-closer to a random vector **Y** than the forecast $\mathbf{F}_2$ $(\mathbf{F}_1 \neq \mathbf{F}_2)$ if and only if

$$P\left(\left|Y_j - F_{j1}\right| < \left|Y_j - F_{j2}\right|\right) > 0.5 \qquad \forall j \in \{1, \ldots, k\}, \text{ where } F_{j1} \neq F_{j2}.$$

The probability statement of *Definition 1* is equivalent to

$$P\left(\left|u_{j1}\right| < \left|u_{j2}\right|\right) > 0.5 \qquad \forall j \in \{1, \ldots, k\}, \text{ where } F_{j1} \neq F_{j2}.$$

In the following we specify the calculation procedure of optimal weights in three categories.

### 2.1 Strong Pitman-closeness

Here, the multivariate combination of forecasts are given as

$\mathbf{F}_A = \sum_{i=1}^{n} \mathbf{A}_i \mathbf{F}_i$ and $\mathbf{F}_B = \sum_{i=1}^{n} \mathbf{B}_i \mathbf{F}_i$ , where

$\mathbf{A}_i := \begin{pmatrix} a_{11}^{(i)} & \cdots & a_{1k}^{(i)} \\ \vdots & \ddots & \vdots \\ a_{k1}^{(i)} & \cdots & a_{kk}^{(i)} \end{pmatrix} \in \mathbb{R}^{k\times k}$ and $\mathbf{B}_i := \begin{pmatrix} b_{11}^{(i)} & \cdots & b_{1k}^{(i)} \\ \vdots & \ddots & \vdots \\ b_{k1}^{(i)} & \cdots & b_{kk}^{(i)} \end{pmatrix} \in \mathbb{R}^{k\times k}$ are matrices of weights,

$\sum_{i=1}^{n} \mathbf{A}_i = \mathbf{I}_k$ , $\sum_{i=1}^{n} \mathbf{B}_i = \mathbf{I}_k$ , and $\mathbf{I}_k$ denotes the identity matrix.

Using the definitions of Wenzel (1998b), where

$$\Sigma := (\Sigma_{rs})_{r,s=1,\ldots,n} \qquad , \ \Sigma_{rs} := \mathrm{Cov}(\mathbf{u}_r, \mathbf{u}_s) \, ,$$

$$\mathbf{V}_{rs} := \Sigma_{rs} + \Sigma_{nn} - \Sigma_{rn} - \Sigma_{ns} \quad , \ r,s = 1,\ldots,n-1,$$

$$\mathbf{V} := (\mathbf{V}_{rs})_{r,s=1,\ldots,n-1} \, ,$$

$$\mathbf{w}_{ji} = (\Sigma_{nn} - \Sigma_{in})\mathbf{e}_j \ \text{ and therefore } \ \mathbf{w}_{ji}' := \mathbf{e}_j'(\Sigma_{nn} - \Sigma_{ni}) \ , \ j=1,..,k, \ i=1,..,n-1,$$

$$\mathbf{w}_j := \left(\mathbf{w}_{j1}', \ldots, \mathbf{w}_{j,n-1}'\right)',$$

we get the following theorem.

*Theorem 1.* The ***Strongly*** *Pitman-closest combination* (component-by-component Pitman-closest combination) of n multivariate forecasts for a random vector $\mathbf{Y}$ of dimension k ($k \geq 2$) is given by the matrix of weights:

$$\mathbf{A}^{opt,str} := [\mathbf{A}_1^{opt,str}, \ldots, \mathbf{A}_n^{opt,str}] = [\mathbf{W}'\mathbf{V}^{-1}, \mathbf{I}_k - \mathbf{W}'\mathbf{V}^{-1}\mathbf{I}_k^*]$$

where $\quad \mathbf{W} := (\mathbf{w}_1, \ldots, \mathbf{w}_k) \sim (n-1)\cdot k \times k$,

$$\mathbf{I}_k^* := [\mathbf{I}_k, \ldots, \mathbf{I}_k]' \sim (n-1)\cdot k \times k \, .$$

Proof: See Wenzel (1998b).

## 2.2 Medium Pitman-closeness

Here we use the restriction of diagonal matrices of weights, so

$$\mathbf{A}_i := \begin{pmatrix} a_{11}^{(i)} & & 0 \\ & \ddots & \\ 0 & & a_{kk}^{(i)} \end{pmatrix} \in \mathbb{R}^{k\times k} \text{ and } \mathbf{B}_i := \begin{pmatrix} b_{11}^{(i)} & & 0 \\ & \ddots & \\ 0 & & b_{kk}^{(i)} \end{pmatrix} \in \mathbb{R}^{k\times k} \text{ are the matrices of weights,}$$

i=1,...,n, and $\sum_{i=1}^{n} \mathbf{A}_i = \mathbf{I}_k \ , \ \sum_{i=1}^{n} \mathbf{B}_i = \mathbf{I}_k$ .

Using again the component-by-component Pitman-closeness definition we get the next theorem.

*Theorem 2.* The ***Medium*** *Pitman-closest combination* (where the matrices of weights are restricted to be diagonal matrices) of n multivariate forecasts for a random vector **Y** of dimension k (k≥2) is given by the matrices of weights:

$$\mathbf{A}_i^{opt,med} = \begin{pmatrix} a_{11}^{(i),opt,med} & & 0 \\ & \ddots & \\ 0 & & a_{kk}^{(i),opt,med} \end{pmatrix}, \quad i=1,...,n,$$

where $a_{jj}^{(i),opt,med}$ is the i-th component of the vector $\frac{\Sigma_j^{-1}\mathbf{1}}{\mathbf{1}'\Sigma_j^{-1}\mathbf{1}}$ (j=1,...,k) and $\Sigma_j$ denotes the covariance matrix of the individual forecast errors of the j-th component and $\mathbf{1} = (1,...,1)'$ is a vector of length n.

Proof: The proof is straightforward. Calculating the optimal weights for the j-th component is equivalent to calculating the optimal univariate combination for this component, which is described in Wenzel (1998a).

### 2.3 Weak Pitman-closeness

Here we concentrate on the combination of multivariate forecasts using scalar weights. Thus a forecast combination is given by $\mathbf{F}_a = \sum_{i=1}^{n} a_i \mathbf{F}_i$ , $a_i \in \mathbb{R}$, $\sum_{i=1}^{n} a_i = 1$. Obvious we cannot use the definition of component-by-component Pitman-closeness to calculate optimal weights for the combination. Therefore we define the ***Weak*** *Pitman-closeness*.

*Definition 2.* The forecast $\mathbf{F}_1$ is weakly Pitman-closer to the random vector **Y** than the forecast $\mathbf{F}_2$ $(\mathbf{F}_1 \neq \mathbf{F}_2)$ if and only if

$$P\left(\left|\sum_{j=1}^{k} Y_j - F_{j1}\right| < \left|\sum_{j=1}^{k} Y_j - F_{j2}\right|\right) > 0.5 \quad .$$

The probability statement of *Definition 2* is equivalent to

$$P\left(\left|\sum_{j=1}^{k} u_{j1}\right| < \left|\sum_{j=1}^{k} u_{j2}\right|\right) > 0.5 \quad .$$

With *Definition 2* in mind it is possible to calculate a ***Weakly*** *Pitman-closest* forecast.

*Theorem 3.* The ***Weakly*** *Pitman-closest combination* (where the weights are restricted to be scalar) of n multivariate forecasts for a random vector **Y** of dimension k (k≥2) is given by the vector of weights:

$$\mathbf{a}^{\text{opt,weak}} = (a_1^{\text{opt,weak}},\ldots,a_n^{\text{opt,weak}})' = \frac{\Sigma_{\text{sum}}^{-1}\mathbf{1}}{\mathbf{1}'\Sigma_{\text{sum}}^{-1}\mathbf{1}}, \text{ where}$$

$\Sigma_{\text{sum}} := \left(\tilde{\mathbf{1}}'\Sigma_{r,s}\tilde{\mathbf{1}}\right)_{r,s=1,\ldots,n}$ and $\tilde{\mathbf{1}}$ denotes the vector $(1,\ldots,1)'$ of length k.

Proof: If we compare two forecast combinations we have to find the vector **a** so that for any vector **b** $(\mathbf{a} \neq \mathbf{b}$ and $\sum_{i=1}^{n} b_i = 1)$ the inequality

$$P\left(\left|\sum_{j=1}^{k}\sum_{i=1}^{n} a_i u_{ji}\right| < \left|\sum_{j=1}^{k}\sum_{i=1}^{n} b_i u_{ji}\right|\right) > 0.5$$

$$\Leftrightarrow \qquad P\left(\left|\mathbf{a}'\begin{pmatrix} u_{11}+u_{21}+\ldots+u_{k1} \\ \vdots \\ u_{1n}+u_{2n}+\ldots+u_{kn}\end{pmatrix}\right| < \left|\mathbf{b}'\begin{pmatrix} u_{11}+u_{21}+\ldots+u_{k1} \\ \vdots \\ u_{1n}+u_{2n}+\ldots+u_{kn}\end{pmatrix}\right|\right) > 0.5 \quad \text{holds.}$$

The covariance matrix of the vector with the sums of the $u_{ji}$'s is $\Sigma_{\text{sum}}$ and therefore, using the conclusions as in Wenzel (1998a) we get the optimal weight vector given in the statement above.

## 2.4 Remarks

Consulting the paper of Troschke (1999a) we can see that the ***Strongly*** and ***Medium*** *Pitman-closest combinations* are equivalent to the combinations resulting by the corresponding covariance adjustment techniques. Therefore we use the same denotations, which are ***Strong*** *Pitman-closeness* and ***Medium*** *Pitman-closeness*. The ***Weakly*** *Pitman-closest combination* differs from the result of the weak covariance adjustment technique because of the problem of defining a closeness criterion for this case. We have to remark that ***Weak*** *Pitman-closeness* is a criterion using the absolute of the sum of errors, which is not really reasonable for the comparison of forecasts. Obviously, the ***Strongly*** *Pitman-closest combination* is component-by-component Pitman-closer to **Y** than the ***Medium*** *Pitman-closest combination* and the ***Weakly*** *Pitman-closest combination* and likewise the

***Medium*** *Pitman-closest combination* is component-by-component Pitman-closer to **Y** than the ***Weakly*** *Pitman-closest combination*.

## 3 Analysing German economic data

Klapper (1998) investigated German macro economic data and analysed different univariate combination techniques. In another article he also used a multivariate rank approach for the combination of the individual forecasts (Klapper, 1999). We derived the Pitman-closest combinations for this problem where we assumed that the errors are normally distributed with zero mean. Using only a ten years history for estimating the covariance matrix (by the usual ML-estimator) we have the problem of a singular matrix **V**. In the calculation of the ***Strongly*** *Pitman-closest combination* we therefore use the Moore-Penrose-Inverse $\mathbf{V}^{+}$ instead of $\mathbf{V}^{-1}$ but then the result depends on which of the individual forecasts is defined as $\mathbf{F}_{\mathrm{n}}$ . The order of the other forecasts does not influence the result. A comprehensive discussion of this can be found in Troschke (1999b).

In the following table we present the Root Mean Square Errors (RMSE) and the Mean Absolute Deviations (MAD) of the special combinations relative to the RMSE and the MAD of the simple average of the individual forecasts.

Table 1: RMSE and *MAD* of the special combinations relative to the RMSE and *MAD* of the simple average forecast.

| | GDP | Priv. Cons. | Publ. Cons. | Export | Import | Cons. Prices |
|---|---|---|---|---|---|---|
| $SPC_{OECD}$ | 1.37<br>*1.37* | 1.54<br>*1.70* | 1.11<br>*1.02* | 1.64<br>*1.60* | 1.13<br>***0.93*** | 2.50<br>*2.16* |
| $SPC_{WSI}$ | 1.37<br>*1.27* | 1.16<br>*1.16* | 1.43<br>*1.18* | 1.37<br>*1.31* | **0.85**<br>***0.83*** | 2.72<br>*2.32* |
| $SPC_{SVR}$ | 1.41<br>*1.44* | 1.27<br>*1.35* | 1.14<br>*1.13* | 1.36<br>*1.38* | **0.97**<br>*1.00* | 2.43<br>*2.29* |
| $SPC_{AWF}$ | 1.19<br>*1.15* | 1.36<br>*1.30* | 1.36<br>*1.12* | 1.38<br>*1.23* | **0.96**<br>***0.92*** | 2.62<br>*2.49* |
| $SPC_{IFW}$ | **0.79**<br>***0.79*** | 1.40<br>*1.47* | 1.30<br>*1.23* | 1.04<br>*1.01* | **0.72**<br>***0.70*** | 2.52<br>*2.26* |
| $SPC_{IFO}$ | 1.26<br>*1.14* | 1.01<br>***0.99*** | 1.00<br>***0.90*** | 1.36<br>*1.28* | **0.98**<br>***0.91*** | 2.86<br>*2.92* |
| $SPC_{DIW}$ | 1.61<br>*1.61* | 1.55<br>*1.84* | 1.34<br>*1.20* | 1.49<br>*1.46* | **0.99**<br>***0.96*** | 3.24<br>*3.29* |
| MPC | 1.83<br>*1.47* | 1.75<br>*1.92* | 2.51<br>*1.87* | 2.32<br>*1.95* | 1.45<br>*1.31* | 1.80<br>*1.68* |
| WPC | 2.71<br>*2.00* | 2.69<br>*2.61* | 4.35<br>*3.02* | 2.19<br>*1.93* | 1.54<br>*1.39* | 4.42<br>*3.92* |

$SPC_{name}$: ***Strongly*** *Pitman-closest combination* with $\mathbf{F}_{\mathrm{n}}$=*name*
MPC: ***Medium*** *Pitman-closest combination*
WPC: ***Weakly*** *Pitman-closest combination*

At first we can see that only in 7 of the 54 cases a Pitman-closeness combination performs better than the simple average (in the sense of the RMSE; 9 of 54 in the comparison of the MAD‘s). Therefore we can say that for this special data the multivariate combinations analysed here do not perform very well.

Furthermore the ***Strongly*** *Pitman-closest combinations* always outperform the ***Weakly*** Pitman-closest combination. Except for the variable Consumer Prices, where they perform worse in all cases, they always dominates the ***Medium*** *Pitman-closest combination* (there is only one further case where a ***Strongly*** *Pitman-closest combination* has a larger MAD). This indicates that one should use multivariate combination techniques instead of univariate combinations of forecasts. At last, we see that the ***Medium*** *Pitman-closest combination* is always better than the ***Weakly*** *Pitman-closest combination*.

## 5 Conclusions

We defined the Pitman-closeness in three categories, called ***Strong***, ***Medium*** and ***Weak*** *Pitman-closeness*. We compared the quality of these techniques by using German economic data. The theoretical conclusions that the ***Strongly*** *Pitman-closest combination* performs best and the ***Weakly*** *Pitman-closest combination* performs worst are underlined by this example. Finally, we have to remark that in some cases, especially for the variable *Import*, the ***Strongly*** *Pitman-closest combiantion*s performs very well in comparison to the simple average forecast, whereas for the variables *Consumer Prices* and *Export* the simple average outperforms all other combination techniques. Consequently further research is needed in which situations *Pitman-closeness* strategies are appropriate and when one should use simple combination techniques like e.g. the arithmetic mean.

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