The Desirability Index as an Instrument for Multivariate Process Control

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Abstract:
The Desirability Index (DI) is a method for multicriteria optimization in industrial quality control. By design of experiment methods and transferring the multivariate into a univariate optimization problem settings of influence factors are selected that lead to a process with simultaneously optimized quality measures. In this paper a new field of application for the DI is introduced — the field of process control. When a process was designed with the objective of reaching the optimal value of the DI, the DI therefore is the most appropriate measure to monitor this optimality over time. Based on the distribution of the DI control charts for individual measurements are presented and advantages compared to the traditional approaches are pointed out, especially caused by an innovative procedure for the interpretation of out-of-control signals.

1 Introduction

The Desirability Index (DI), which was introduced by [HAR65] and extended primarily by [DER80], by now has gained wide acceptance in practice in the course of multicriteria optimization in industrial quality management (e.g. [BAS02], [CAR01], [KOR02], [PAR02]). Harrington's desirability functions (DF) are based on exponential-type transformations of the quality measures considered onto a unitless scale between 0 and 1. The DI then combines the latter via the geometric mean or e.g. by taking the minimum of the DFs ([KIM00]). By optimizing the DI using functional relationships between the quality measures and the process influencing factors resulting from design of experiment methods, optimal levels of the influence factors are selected, which optimize all quality measures

simultaneously.
Once the quality of an industrial process has been initially optimized the ongoing process quality is of strong interest in order to detect undesired process changes. For this purpose so far separate univariate or multivariate control charts are used. A straightforward and more appropriate approach for quality control in this case though is the utilization of the DI not only for process optimization but also for quality control purposes. When the process was designed with the objective of reaching optimality regarding the DI, it is obviously the most appropriate measure to control its stability over time. In [WEB03] resp. [TRA04] the statistical distribution for different types of the DI was made available, which provide the basis for designing specific control charts for the DI. A review of these distributions is given in Chapter 2. Chapter 3 introduces control charts for individual measurements of the DI, and in Chapter 4 an innovative procedure for the interpretation of out-of-control-signals in DI control charts is presented. Afterwards a summary and an outlook on further research fields completes the results in Chapter 5.

2 Distribution of the Desirability Index

In [WEB03] the distributions of two types of DIs, namely the geometric mean and the minimum of the DFs based on Harrington's one-sided or two-sided DFs are derived. When using the geometric mean an approximative approach arises as the most suitable one for the one-sided case, whereas for the two-sided case the distribution of the DI is made available for two quality measures Y_i (i=1,2) with $n_i = 1$ where

$$d_i(Y_i') = e^{-|Y_i'|^{n_i}}, \qquad i = 1, \ldots, k;\ 0 < n_i < \infty \quad \text{with} \tag{1}$$

$$Y_i' = \frac{2Y_i - (USL_i + LSL_i)}{USL_i - LSL_i}, \qquad i = 1, \ldots, k, \tag{2}$$

LSL/USL : Lower / Upper Specification Limit, and in the one-sided case

$$d_i(Y_i') = e^{-e^{-Y_i'}}, \qquad i = 1, \ldots, k \quad \text{with} \tag{3}$$

$$Y_i' = b_{0i} + b_{1i} Y_i. \tag{4}$$

As control charts for the DI are based on its distribution function as a review only these are presented in the following.

Theorem 1 (DI Geometric Mean) *Given k independent quality measures $Y_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ $(i = 1, \ldots, k)$ with DFs d_i (1) resp. (3), the DI defined as $D := (\prod_{i=1}^k d_i)^{1/k}$ has the following distribution function:*

$$
\begin{aligned}
F_{\mathcal{D}}(D) &\approx 1 - \Phi\left[\frac{log(k) + log(-log(D)) - \mu^*}{\sigma^*}\right] \text{ with } \mu^* \text{ and } \sigma^{*2} \text{as defined in [SCH82]} \\
& \quad \text{resp. [WEB03], and in the two-sided case for } k=2 \text{ and } n_i = 1\,(i=1,2) \\
F_{\mathcal{D}}(D) &= \int_0^D f_{\mathcal{D}}(D)d(D) \text{ with} \\
f_{\mathcal{D}}(D) &= \frac{\sqrt{2}}{2D\sqrt{\pi(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}} \cdot \left(exp\left(-\frac{(-2log(D) - \tilde{\mu_1} - \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right)\right. \\
& \quad \cdot erf\left(\frac{((-2log(D))\tilde{\sigma_2}^2 - \tilde{\mu_1}\tilde{\sigma_2}^2 + \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right) \\
& \quad + exp\left(-\frac{(-2log(D) - \tilde{\mu_1} + \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right) erf\left(\frac{((-2log(D))\tilde{\sigma_2}^2 - \tilde{\mu_1}\tilde{\sigma_2}^2 - \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right) \\
& \quad + exp\left(-\frac{(-2log(D) + \tilde{\mu_1} - \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right) erf\left(\frac{((-2log(D))\tilde{\sigma_2}^2 + \tilde{\mu_1}\tilde{\sigma_2}^2 + \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right) \\
& \quad + exp\left(-\frac{(-2log(D) + \tilde{\mu_1} + \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right) erf\left(\frac{((-2log(D))\tilde{\sigma_2}^2 + \tilde{\mu_1}\tilde{\sigma_2}^2 - \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right) \\
& \quad + exp\left(-\frac{(-2log(D) - \tilde{\mu_1} - \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right) erf\left(\frac{((-2log(D))\tilde{\sigma_1}^2 + \tilde{\mu_1}\tilde{\sigma_2}^2 - \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right) \\
& \quad + exp\left(-\frac{(-2log(D) - \tilde{\mu_1} + \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right) erf\left(\frac{((-2log(D))\tilde{\sigma_1}^2 + \tilde{\mu_1}\tilde{\sigma_2}^2 + \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right) \\
& \quad + exp\left(-\frac{(-2log(D) + \tilde{\mu_1} - \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right) erf\left(\frac{((-2log(D))\tilde{\sigma_1}^2 - \tilde{\mu_1}\tilde{\sigma_2}^2 - \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right) \\
& \quad \left. + exp\left(-\frac{(-2log(D) + \tilde{\mu_1} + \tilde{\mu_2})^2}{2(\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2)}\right) erf\left(\frac{((-2log(D))\tilde{\sigma_1}^2 - \tilde{\mu_1}\tilde{\sigma_2}^2 + \tilde{\mu_2}\tilde{\sigma_1}^2)}{\tilde{\sigma_2}\tilde{\sigma_1}\sqrt{2}\sqrt{\tilde{\sigma_2}^2 + \tilde{\sigma_1}^2}}\right)\right); \\
\tilde{\mu_i} &= \frac{2}{USL_i - LSL_i} \cdot \mu_i - \frac{USL_i + LSL_i}{USL_i - LSL_i} \text{ and } \tilde{\sigma}_i^2 = (\frac{2}{USL_i - LSL_i})^2 \cdot \sigma_i^2; \\
erf(x) &= 2 \cdot \Phi(\sqrt{2}x) - 1 \quad \text{(Gaussian Error Function)}, \\
\Phi(x) &:= \text{Distribution function of } \mathcal{N}(0,1).
\end{aligned}
$$

Since 1965 modifications of Harrington's approach have been introduced, one type concerned with altered desirability functions (see [DER80] as the most important one), the other one aiming at different DIs e.g. the minimum of the DFs ([KIM00]):

Theorem 2 (DI Minimum DFs) *Given k independent quality measures $Y_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ $(i = 1, \ldots, k)$ with DFs d_i (1) resp. (3), the DI D defined as $D := \min_{i=1,\ldots,k} d_i$ has the following distribution function:*

$$
\begin{aligned}
F_{\mathcal{D}}(D) &= 1 - \prod_{i=1}^{k} \Phi\left[\frac{(log(-log(D)) - \tilde{\mu}_i)}{\tilde{\sigma}_i}\right] \quad (\textit{One-sided DFs}) \textit{ with} \\
& \quad\ \tilde{\mu}_i = -(b_{0i} + b_{1i} \cdot \mu_i) \textit{ and } \tilde{\sigma}_i^2 = (b_{1i})^2 \cdot \sigma_i^2, \\
F_{\mathcal{D}}(D) &= 1 - \prod_{i=1}^{k} \left(-1 + \Phi\left[\frac{((-log(D))^{1/n_i} - \tilde{\mu}_i)}{\tilde{\sigma}_i}\right] + \Phi\left[\frac{((-log(D))^{1/n_i} + \tilde{\mu}_i)}{\tilde{\sigma}_i}\right]\right) \\
& \quad\ (\textit{Two-sided DFs}) \textit{ with } \tilde{\mu}_i \textit{ and } \tilde{\sigma}_i^2 \textit{ as defined in Theorem 1.}
\end{aligned}
$$

3 Control Charts for the DI

In principle the selection process of a control chart for the DI follows the same procedure as for any univariate quality measure apart from some special challenging characteristics. Almost all well-known univariate control charts were designed for normally distributed quality measures, which cannot be assumed for the DI. So either a nonparametric control chart (see [CHA01] for a review) has to be chosen or the desired control chart type must be derived for the distributions of the DI given in Theorem 1 and 2. Furthermore the concerning choice is constricted to certain types of control charts:

In general control charts can be classified regarding the purpose of either monitoring the process expectation or the process variability where also composite control charts exist that simultaneously keep control of both parameters. With respect to the DI there is the special situation that changes in the distribution of a quality measure Y_i apart from very restricted cases lead to shifts of the expectation as well as the variance of the DI due to the nonlinear transformations of the quality measures in (1) and (3). This is an obvious fact because of the restricted domain of the DI. Assuming the expectation of the DI comes close to the interval borders a decrease of the variance must be the consequence in order to ensure that the values of the DI keep up with the domain restriction. This implies the necessity of simultaneously observing both process expectation and variance, and therefore either composite control charts or a control chart for individual measurements has to be chosen.

The latter will be the recommended one for most cases as the DI itself functions as an "average quality value" at each particular point of time. An additional timewise averaging often results in interpretation problems especially regarding the analysis of out-of-control signals. In general control charts for single measurements are suggested in the presence of long time spans between samples and in situations of high variability of the DI so that each realization of the DI has to be monitored separately. Based on the distribution of the DI a control chart for single measurements of the DI is defined as follows:

Definition 3 (Single-Measurements Control Chart) *Given a process characterized by k quality measures, which have been combined by a DI D, and known distribution functions $F_{\mathcal{D}}(D)$, the control (LCL, UCL)- and warning limits (LWL, UWL) of the single-measurements control chart are:*

$$\begin{aligned} LCL/UCL &= Q_{0.005}/Q_{0.995}, \quad LWL/UWL = Q_{0.025}/Q_{0.975} \textit{ with} \\ Q_\alpha &:= \alpha \cdot 100\%\textit{-quantile of } F_{\mathcal{D}}(D) \textit{ and centerline } Q_{0.5}. \end{aligned}$$

Assuming that the values of the quality measures emanate from equidistant samples taken from the ongoing process this grouping can be retained by using a control chart which is based on the extreme values of each sample — the extreme value control chart (analogously to [WEI99], p. 301).

Definition 4 (Extreme Value Control Chart) *Given a process characterized by k quality measures from samples of size g, which have been combined by a DI D, and known distribution functions $F_{\mathcal{D}}(D)$, the control (LCL, UCL)- and warning limits (LWL, UWL) of the extreme value control chart are:*

$$\begin{aligned} LCL/UCL &= Q_{(1\mp\sqrt[g]{0.99})/2}, \quad LWL/UWL = Q_{(1\mp\sqrt[g]{0.95})/2} \textit{ with} \\ Q_\alpha &:= \alpha \cdot 100\%\textit{-quantile of } F_{\mathcal{D}}(D) \textit{ and centerline } Q_{0.5} \textit{ resp. } E(D). \end{aligned}$$

The values of the quality measures are plotted one upon another at each point in time a sample is taken. The process therefore is deemed to be in control if all values of the current sample plot within the control limits, so that the decision is based on the extreme values of each sample. Despite of the advantage of the retained sample grouping this approach however leads to wider control limits than the single-measurements control chart. Which control chart type is recommended for use has to be decided individually on the basis of the considered process characteristics.

In comparison to existing control charts the introduced control charts prove to be superior when applied to the DI. A simulation study (30000 runs) was carried out to assess the performance of the single-measurements control chart (SMCC) in contrast to the Shewart-Single-Measurements- (SHCC)([WAD02]) and the Fence Control Chart (FCC) based on the theoretical distribution of the DI by a computation of their In- and Out-Of-Control Average Run Lengths (ARLs). A FCC ([WEI99], p. 303 f.) is based on the theoretical or empirical distribution of the quality measure at hand and shows analogies to the concept of boxplots. When interpreting a boxplot all points that plot outside its outer fences are denoted as outliers, i.e. lie in an abnormal distance from the other values considered. Utilizing this concept the control limits of the FCC are computed as

$$LCL/UCL = Q_{0.25} \mp 1.5 \cdot (Q_{0.75} - Q_{0.25}) \text{ or } LCL/UCL = q_{0.25} \mp 1.5 \cdot (q_{0.75} - q_{0.25}); \quad (5)$$

$$Q_\alpha / q_\alpha = \alpha \cdot 100\% - \text{Quantile of theoretical / empirical distribution of the quality measure.} \quad (6)$$

In each case two quality measures were selected and combined either by the geometric mean or the minimum of the DFs as a DI using one- and two-sided DFs. Table 1 shows exemplary simulation results, i.e. control limits and In-Control-ARLs, to facilitate the discussion of problems that can occur when applying the SHCC and FCC. Figures 1a)-d) each use one of the parameter settings of the simulation to visualize and compare the Out-Of-Control behaviour of the ARLs. For that purpose the control limits of the SHCC and the FCC were adjusted so that the resulting In-Control-ARL-values at least roughly equal the corresponding values of the SMCC. For Figures 1a)-d) the following parameter settings were used ($Y_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$):

Two-Sided DFs ($(LSL, USL, n)[\mu, \sigma]$):

$$\text{Fig. 2a)}: \quad d_1 : (3, 7, 1)[5, 0.3], \quad d_2 : (2, 9, 1)[3, 0.8]; \quad D := (\prod_{i=1}^{k} d_i)^{1/k}, \quad (7)$$

$$\text{Fig. 2c)}: \quad d_1 : (4, 6, 1.5)[4, 0.2], \quad d_2 : (3, 7, 1)[5, 0.5]; \quad D := \min_{i=1,\dots,k} d_i. \quad (8)$$

One-Sided DFs (DFs specified by (Y, d)):

$$\text{Fig. 2b)}: \quad d_1 : (3, 0.2), d_2 : (6, 0.6), \quad [\mu, \sigma] : [6.41, 0.2], [-0.98, 0.53]; \quad D := (\prod_{i=1}^{k} d_i)^{1/k}, \quad (9)$$

$$\text{Fig. 2d)}: \quad d_1 : (7, 0.4), d_2 : (10, 0.9) \text{ resp. } d_1 : (3, 0.2), d_2 : (6, 0.6), \quad [\mu, \sigma] : [9, 1], [6.1, 0.5]; \quad D := \min_{i=1,\dots,k} d_i. \quad (10)$$

The x-axis indicates a shift in the expectation of the DI, which is estimated using the arithmetic mean $\bar{D}$ of the DI resulting from all simulation runs, where the index (a, b)

	SMCC	SHCC	FCC	SMCC	SHCC	FCC
Control Limits \ Parameter of DFs	(3,7,1) [4,1] (3,7,1) [6,4]			(3,7,1)[5,0.3] (2,9,1)[3,0.8]		
LCL	0.03913	*-0.2349*	*-0.2514*	0.4772	0.4167	0.4431
UCL	0.9215	*1.01704*	*1.0138*	0.8982	0.9133	0.8802
In-Control-ARL	101.42	∞	∞	99.89	299.07	110.18

Table 1: Simulated In-Control-ARLs for different parameter settings (LSL,USL,n)[μ_i,σ_i] with $Y_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ based on two-sided DFs using the geometric mean as DI.

reflects the kind of shift in the distribution of the two underlying quality measures Y_1 and Y_2. Assuming that $Y_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ the index (a,b) is interpreted as

$$(a, b) \quad : \quad Y_1 \sim \mathcal{N}(\mu_1 + a \cdot \sigma_1, \sigma_1^2), Y_2 \sim \mathcal{N}(\mu_2 + b \cdot \sigma_2, \sigma_2^2). \tag{11}$$

The ARL-values are only connected for illustration purposes to facilitate the comparison of the different control charts.

When analyzing the simulation results the most important observations come out as follows:

- The usage of the SHCC and FCC may lead to control limits outside the domain of the DI as shown in the first example in Table 1. Despite on the one hand the In-Control-ARL value is very high or even equals ∞ this is also true for the Out-Of-Control-ARL, which does not result in an appropriate control chart for the DI. These situations can occur in the presence of high variances or skewed distributions of the DI as the SHCC acts on the assumption of a symmetric distribution — namely a normally distributed DI — and the FCC assumes symmetric distribution tails below and above its quartiles.

- The effect of shifts in the expectation of the quality measures onto the expectation as well as the variance of the DI becomes visible as the standard deviation $sd(D)$ is additionally plotted in Figures 2a)-d).

- The ARL behaviour of the SHCC and the FCC gets more and more problematic with increasing skewness of the distribution of the DI (see Figures 1a),c),d)). The corresponding density functions ([WEB03]) of the DI can be found in Fig. 2, where also a normal density with the expectation and the variance of the DI is added for illustration purposes of the skewness. Shifts of E(D) towards the steeper side of the

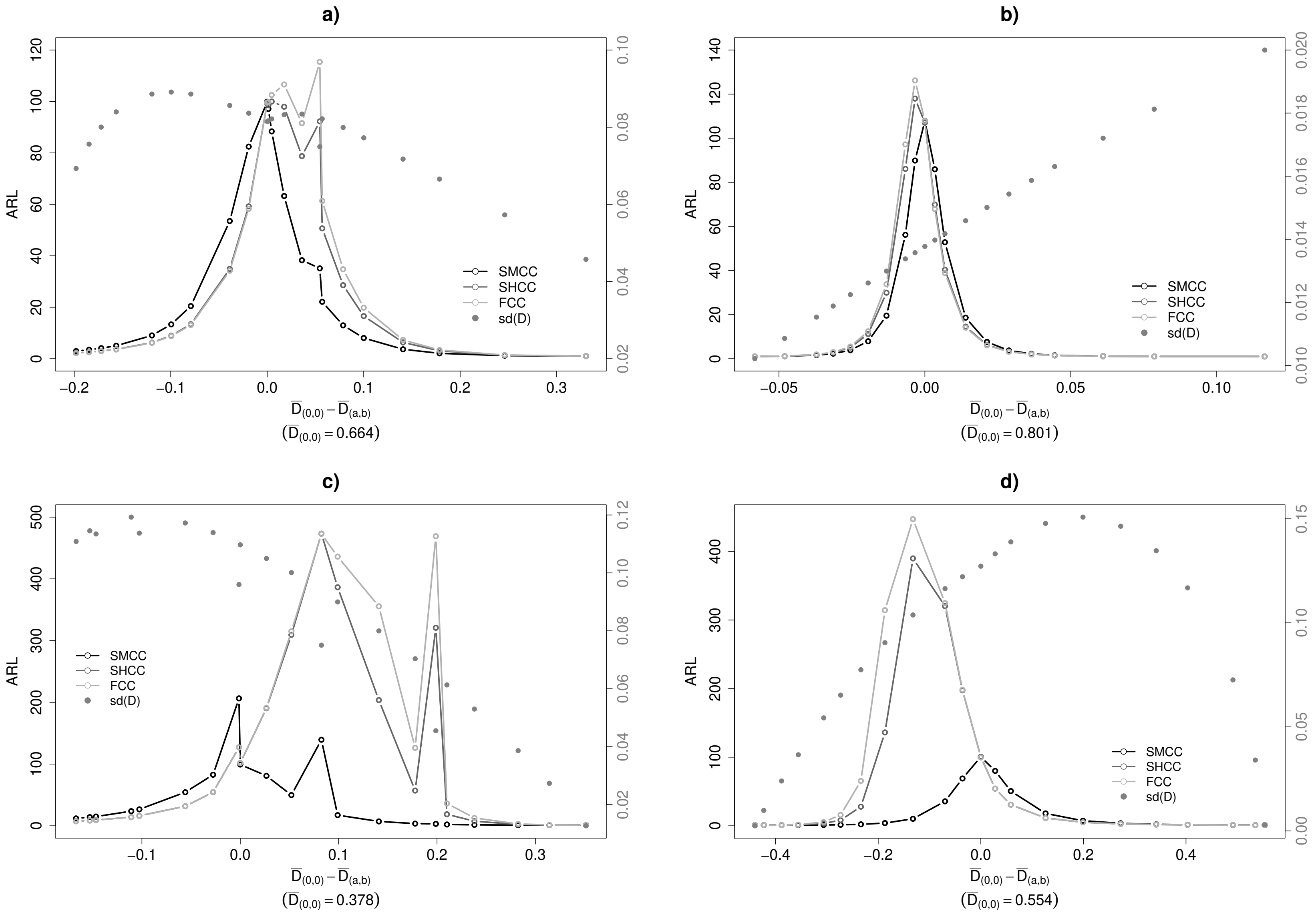


Figure 1: Out-Of-Control-ARLS of SMCC, SHCC, FCC. Parameter Settings are listed in (7)-(10).

density function of the DI result in very high values of the corresponding Out-Of-Control-ARL whereas the SMCC does not show this kind of extreme characteristic. Only the example in Fig. 1b), where two one-sided DFs were combined by the geometric mean, generates similar ARL shapes for all of the three control charts as in this case the distribution of the DI comes close to a symmetric one (see Fig. 2a)). In this specific case the SHCC and FCC are even able to detect a decrease in the expectation of the DI slightly sooner than the SMCC.

Summing up, the advantages of the SMCC in comparison to the SHCC and FCC become obvious especially in the presence of skewed distributions of the DI.
Analogous results concerning skewed distributions of the DI are obtained for the Extreme Value Control Chart (see Definition 4) in comparison to the Shewart-$\bar{x}$-Control Chart but are not presented in detail here (see [TRA04]).

4 Interpretation of Out-Of-Control Signals

By using a control chart for the DI undesired process changes can be detected at points in time when values of the DI plot outside the control limits, so that counteractions are required in order to get back to an in-control-situation. In order to decide which counteractions to carry out at first the cause of the specific out-of-control-signal has to be determined, which in general is a challenging and frequently discussed task in multivariate process control.

Though for multivariate control charts approaches like principal component analysis ([WEI99], p. 334 ff.), discriminant analyis ([CHU92]), special types of orthogonal decompositions ([MAS95]), regression fit ([MON01], p. 529 ff.) or exact simultaneous confidence intervals for parallel univariate control charts ([HAY94]) form means to facilitate this problem, especially when the T^2-chart is utilized, still no really satisfying general approach for this problem exists.

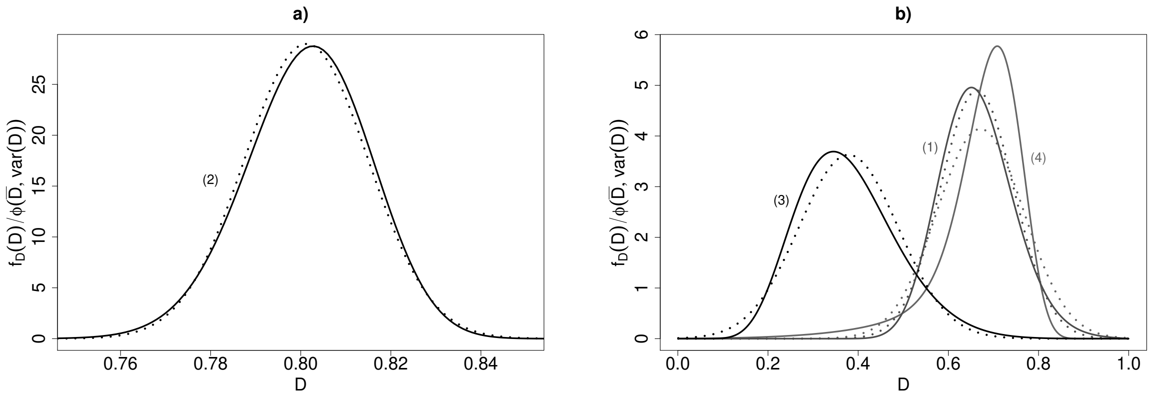


Figure 2: Densities of the DI and a normal distribution $\phi(\bar{D}, v\hat{a}r(D))$ (dotted) related to the examples in Figure 1a) ((1)) to 1d) ((4)).

In case the proposed single-measurements control chart (SMCC) or the Extreme Value Control Chart (EVCC) for the DI are used as defined in Definitions 3 and 4 the control limits for the DI can be transformed into control limits for the underlying desirability functions as well as the individual quality measures. Assuming the usage of the geometric mean as DI the control limit for a specific DF is dependent on the values of the remaining ones as the product of the DFs is taken. So low values of one DF may be compensated for by a high value of another DF. Therefore the resulting control limit does not comply with the usual horizontal line but is determined seperately for each realization of the DI, where only the lower control limit of the DI is of primary interest as "too high" values of a DI in principle do not exist.

Theorem 5 (Lower Control Limit for DF) *Given a process characterized by quality measures $Y_1, \ldots, Y_k$ and respective DFs $d_1, \ldots, d_k$ (1) or (3) as well as DI D, for which a lower control limit LCL was calculated as described in theorems 3 and 4, the lower control limit of a specific DF is determined as*

$$LCL_{d_i} = \frac{LCL^k}{\prod_{\substack{j=1 \\ j \neq i}}^k d_j}, \qquad i \in \{1, \ldots, k\}.$$

This relationship can directly be seen from the inequalities

$$\left(\prod_{i=1}^k d_i\right)^{1/k} \geq LCL \Leftrightarrow \prod_{i=1}^k d_i \geq LCL^k \Leftrightarrow d_i \geq \frac{LCL^k}{\prod_{\substack{j=1 \\ j \neq i}}^k d_j}. \qquad (12)$$

The interpretation of an out-of-control-signal in the SMCC or EVCC thus can be carried out in the resulting control chart for a specific DF. In case the realizations of the DF do not show any irregularities like trends, an increasing variance or extreme values in contrast to the lower control limit, the signal is mainly caused by another quality measure. So specific and suspicious DFs can be systematically monitored. In most applications it won't be necessary to set up all (maximal $(k-1)$) control charts for the DF.
Analogously control limits for the quality measures can be derived, where it depends on personal preferences what kind of control chart to analyze. Here attention should be paid to the fact that a lower control limit only results for one-sided DFs (3) with desired maximization of the quality measure Y_i, for minimization problems an upper control limit is determined. For two-sided DFs (1) one gets a lower control limit for realizations on the left hand side of the target value $T_i = (LSL_i + USL_i)/2$ and accordingly an upper one for values on the right hand side.

Theorem 6 (Control limits for quality measures) *Given a process characterized by quality measures $Y_1, \ldots, Y_k$ and respective DFs $d_1, \ldots, d_k$ (1) or (3) as well as DI D, for which a lower control limit LCL was calculated as described in theorems 3 and 4, the control limit for a specific quality measure is determined as*

1. One-sided DF:

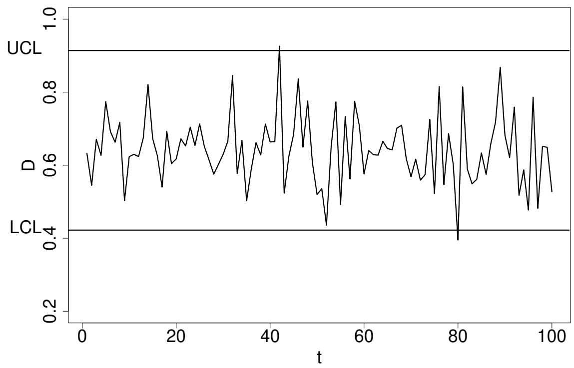
$$\frac{1}{b_{1i}}\left[-log\left(-log\left(\frac{LCL^k}{\prod_{\substack{j=1\\j\neq i}}^k d_j}\right)\right)-b_{0i}\right]=\begin{cases} UCL_{y_i} & for\ b_{1i}<0\\ LCL_{y_i} & for\ b_{1i}>0\end{cases}.$$

As for values $\frac{LCL^k}{\prod_{\substack{j=1\\j\neq i}}^k d_j} > 1$ the outer logarithm is undefined, for $b_{1i} < 0$ the control limit UCL_{y_i} is set to a constant value above the maximum realization of Y_i. For $b_{1i} > 0$ a constant value below the minimum realization of Y_i is assigned to LCL_{y_i}.

2. Two-Sided DF:

$$\begin{aligned} LCL_{y_i} &= -\frac{1}{2}\left[(USL_i - LSL_i)\cdot\left(-log\left(\frac{LCL^k}{\prod_{\substack{j=1\\j\neq i}}^k d_j}\right)\right)^{1/n_i} - (USL_i + LSL_i)\right],\ Y_i < T_i,\\ UCL_{y_i} &= \frac{1}{2}\left[(USL_i - LSL_i)\cdot\left(-log\left(\frac{LCL^k}{\prod_{\substack{j=1\\j\neq i}}^k d_j}\right)\right)^{1/n_i} + (USL_i + LSL_i)\right],\ Y_i \geq T_i.\end{aligned}$$

In the one-sided case the proposition results from a distinction subject to the sign of b_{1i}



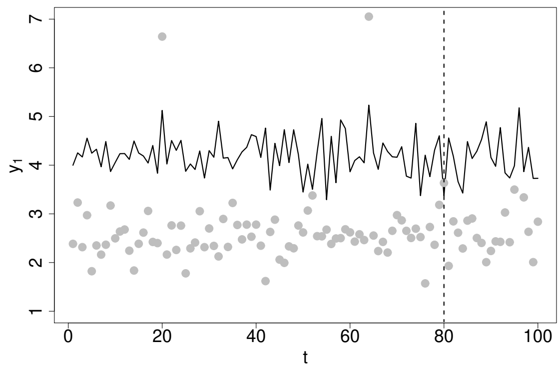


Figure 3: Example: Interpretation of an Out-Of-Control-signal using two-sided DFs (1)

using

$$exp\left[-exp(-(b_{0i} + b_{1i} \cdot y_i))\right] \geq \frac{LCL^k}{\prod_{\substack{j=1 \\ j \neq i}}^{k} d_j} \tag{13}$$

$$\Leftrightarrow \quad b_{1i} \cdot y_i \geq -log\left(-log\left(\frac{LCL^k}{\prod_{\substack{j=1 \\ j \neq i}}^{k} d_j}\right)\right) - b_{0i}, \tag{14}$$

and for two-sided DFs the distinction is carried out regarding the sign of $(2y_i - (USL_i + LSL_i))$, which is equivalent to the sign of $(y_i - T_i)$:

$$exp\left(-\left|\frac{2y_i - (USL_i + LSL_i)}{USL_i - LSL_i}\right|^{n_i}\right) \geq \frac{LCL^k}{\prod_{\substack{j=1 \\ j \neq i}}^{k} d_j} \tag{15}$$

$$\Leftrightarrow \quad |2y_i - (USL_i + LSL_i)| \leq (USL_i - LSL_i) \cdot \left(-log\left(\frac{LCL^k}{\prod_{\substack{j=1 \\ j \neq i}}^{k} d_j}\right)\right)^{1/n_i} \tag{16}$$

Fig. 3 shows an exemplary control chart for the DI resulting from two quality measures Y_1 and Y_2 with specified two-sided DFs as well as a control chart for the quality measure Y_1, for which the control limit was set up subject to Theorem 6. At $t = 80$ the DI plots below the lower control limit of the DI, which is reflected in the control chart for Y_1, as the realization of Y_1 does not exceed the lower control limit represented by grey dots. When interpreting this control chart it becomes obvious that Y_1 does not show any special irregularity at $t = 80$ but the lower control limit does by taking an extreme value. So it follows that the out-of-control-signal is primarily caused by Y_2.

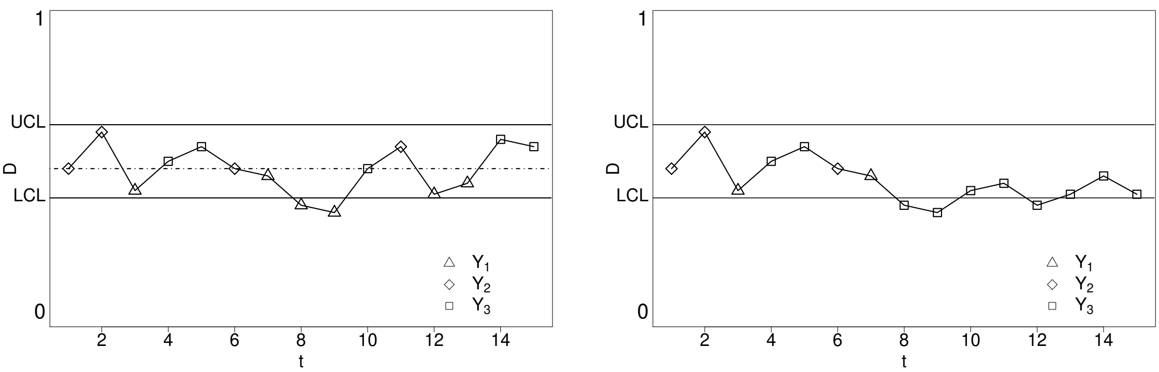


Figure 4: Exemplary control charts for the DI (Minimum DFs) with possible interpretation of Out-Of-Control-signals

The situation, in which the minimum of the DFs is applied as a DI, is addressed differently as in this case there is no possibility for compensating low values of individual quality measures. Focussed is rather the detection of the specific quality measure which is responsible for the value of the DI in presence of an out-of-control-signal at hand. For this purpose a symbol is chosen for each individual quality measure, which is used in the control chart of the DI for visualizing from which quality measure the displayed value of the DI originates (Fig. 4). The interpretation of an out-of-control-signal therefore can directly be carried out in the control chart of the DI and thus is independent of the type of DF.

Observing the left part of Fig. 4 the out-of-control-signals at points 8 and 9 can be assigned to Y_1. The right part however shows another important advantage of the approach proposed. Besides the analysis of the out-of-control-signals also trends or structural changes of the process become visible. From point 8 on all displayed values of the DI originate from Y_3 so that its desirability has permanently decreased.

For the EVCC the proposed approach for the geometric mean as well as the minimum of the DFs is also valid, merely the structure of the EVCC is retained when transforming the LCL of the DI into control limits for the DFs and the quality measures.

5 Summary and Outlook

In this paper multivariate process control as a new and very promising application field for the DI is introduced. So far the DI is only used for multicriteria optimization. Based on the statistical distribution of the DI provided in [WEB03], a Single-Measurement- as well as an Extreme Value Control Chart for two types of DIs, i.e. either the geometric mean or the minimum of the DFs, is presented. Compared to existing control charts for multivariate process control purposes many advantages become obvious, e.g. a complexity reduction compared to separate univariate control charts as only one control chart has to be monitored over time. This is also true for multivariate control charts though, for which however the interpretation of out-of-control signals is a challenging and problematic task. For the proposed control charts an appropriate approach for the analysis of out-of-control situations could be developed. Furthermore in comparison to existing univariate control charts as the Shewart-Single-Measurements- or the Fence Control Chart the ARL behaviour of the charts proposed shows superiority, especially with regard to very skewed distributions of the DI, as no symmetric or even normal distribution can be assumed. In addition the restricted domain of the DI has to be considered. Existing control charts in presence of high variances of the DI often consist of control limits outside the domain. A more detailed discussion is carried out in [TRA04].

For future work many perspectives exist. On the one hand other types of control charts for the DI can be developed, e.g. $\bar{X}$- or S-charts, but process expectation and process variance have to be monitored simultaneously by using composite control charts, as they are not independent for the DI. On the other hand the approach presented could be extended with regard to other types of DFs (e.g. [DER80]), where in the first stage the distribution of the DI has to be derived.

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